# An Introduction to Computational Macroeconomics

## Dynamic Programming: Chapter 4

John Stachurski

June – July 2022

# Valuation

To solve hard problems we must define objectives clearly

Objective of dynamic programs = max **lifetime rewards**

Examples.

- lifetime wages for a worker
- lifetime utility for a consumer
- net present value for a firm

We now lay the groundwork for DP by focusing on computation of lifetime rewards

# Fixed Discount Rates

A common task: compute $\mathbb{E}$ of discounted future rewards

These sums take the form

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) := \mathbb{E} \left[ \sum_{t \geqslant 0} \beta^t h(X_t) \mid X_0 = x \right]$$

Here

- $\beta \in \mathbb{R}_+$ and $h \in \mathbb{R}^\mathsf{X}$
- $(X_t)$ is $P$-Markov on finite set $\mathsf{X}$
- $\mathbb{E}_x$ indicates we are conditioning on $X_0 = x$

Example. Computing expected present value of a cash flow

**Lemma.** If $\beta \in (0, 1)$, then

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t)$$

is finite for all $x \in \mathsf{X}$

Moreover, the matrix $I - \beta P$ is invertible and

$$v = \sum_{t \geqslant 0} (\beta P)^t h = (I - \beta P)^{-1} h \tag{1}$$

<u>Proof</u>: Observe that

$$\mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) = \sum_{t \geqslant 0} \beta^t \mathbb{E}_x h(X_t) = \sum_{t \geqslant 0} \beta^t (P^t h)(x)$$

Result (1) holds because $r(\beta P) < 1$ — why?

# Application: Valuation of Firms

A firm receives profit stream $(\pi_t)_{t \geqslant 0}$ — a stochastic process

Valuation $=$ expected present value of its profit stream:

$$V_0 = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t \quad \text{with} \quad \beta := \frac{1}{1+r}$$

Assume $\pi_t = \pi(X_t)$ where $(X_t)_{t \geqslant 0}$ is $P$-Markov, set

$$v(x) := \mathbb{E}_x \sum_{t=0}^{\infty} \beta^t \pi_t$$

Now $r > 0$ implies $v = (I - \beta P)^{-1} \pi$

**Ex.** Suppose

- X is partially ordered
- $\pi \in i\mathbb{R}^{\mathsf{X}}$ and $P$ is monotone increasing

Prove that, under these conditions, $v$ **is increasing on** X

Proof 1: Let $\pi$ and $P$ satisfy the stated conditions

Given $t \in \mathbb{N}$

- $P$ monotone increasing implies $P^t$ monotone increasing
- Since $\pi \in i\mathbb{R}^{\mathsf{X}}$, we see that $P^t \pi \in i\mathbb{R}^{\mathsf{X}}$

Hence $v = \sum_{t \geqslant 0} \beta^t P^t \pi$ is also increasing

**Ex.** Suppose

- X is partially ordered
- $\pi \in i\mathbb{R}^{\mathsf{X}}$ and $P$ is monotone increasing

Prove that, under these conditions, $v$ **is increasing on** X

Proof 1: Let $\pi$ and $P$ satisfy the stated conditions

Given $t \in \mathbb{N}$

- $P$ monotone increasing implies $P^t$ monotone increasing
- Since $\pi \in i\mathbb{R}^{\mathsf{X}}$, we see that $P^t \pi \in i\mathbb{R}^{\mathsf{X}}$

Hence $v = \sum_{t \geqslant 0} \beta^t P^t \pi$ is also increasing

Proof 2: Let $\pi$ and $P$ satisfy the stated conditions

Rearranging $v = (I - \beta P)^{-1}\pi$ gives $v = \pi + \beta P v$

Hence $v$ is the fixed point in $\mathbb{R}^{\mathsf{X}}$ of $Tv = \pi + \beta P v$

Since $\pi \in i\mathbb{R}^{\mathsf{X}}$ and $P$ is monotone, $T$ is invariant on $i\mathbb{R}^{\mathsf{X}}$

The operator $T$ is a contraction of modulus $\beta$ (Ex: check it)

Since $i\mathbb{R}^{\mathsf{X}}$ is closed, nonempty and invariant for $T$, the fixed point of $T$ must lie in this set

In other words, $v$ is increasing on X

# Time-Varying Interest Rates

One limitation of previous model: discount rate is constant

This assumption is problematic

Interest rates are **stochastic and time-varying**

- even nominal rates for safe assets like US T-bills
- certainly real interest rates too

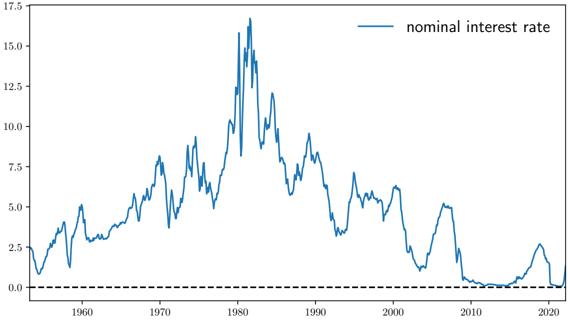


Figure: Nominal US interest rates

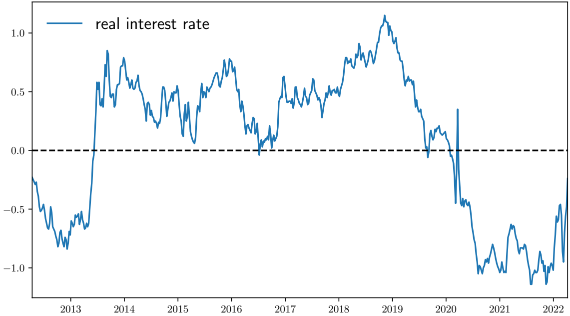


Figure: Real US interest rates

Example. When interest rates start to rise, the share prices of new and tech-heavy firms typically fall

Why?

Profit streams from such firms are usually biased towards the future

Dividends are

- initially low (profits reinvested)
- eventually high if the business model is successful

Rising interest rates imply such profit streams should be heavily discounted

Example. When interest rates start to rise, the share prices of new and tech-heavy firms typically fall

Why?

Profit streams from such firms are usually biased towards the future

Dividends are

- initially low (profits reinvested)
- eventually high if the business model is successful

Rising interest rates imply such profit streams should be heavily discounted

Consider a firm valuation problem where interest rate follow stochastic process $(r_t)_{t \geqslant 0}$

The time zero expected present value of time $t$ profit $\pi_t$ is

$$\mathbb{E}\left\{\beta_0 \cdots \beta_{t-1} \cdot \pi_t\right\} \quad \text{where} \quad \beta_t := \frac{1}{1+r_t}$$

The expected present value of the firm is

$$V_0 = \mathbb{E}\left\{\sum_{t=0}^{\infty}\left[\prod_{i=0}^{t-1}\beta_i\right]\pi_t\right\} \quad \text{where} \quad \prod_{i=0}^{-1}\beta_i := 1$$

- When is it finite?
- How can we compute it?

# Generalized Geometric Sums

Suppose

- $h \in \mathbb{R}^{\mathsf{X}}$ and $b \in \mathbb{R}^{\mathsf{X} \times \mathsf{X}}$
- $(X_t)_{t \geqslant 0}$ is $P$-Markov, $H_t = h(X_t)$ and $B_t = b(X_{t-1}, X_t)$
- $K$ is the matrix on $\mathsf{X}$ defined by $K(x, x') := b(x, x')P(x, x')$

**Theorem**. If $r(K) < 1$, then

$$v(x) = \mathbb{E}_x \left\{ \sum_{t=0}^{\infty} \left[ \prod_{i=1}^{t} B_i \right] H_t \right\} \qquad \text{with } \prod_{i=1}^{0} B_i := 1$$

is finite-valued

Moreover, $I - K$ is nonsingular and $v = (I - K)^{-1} h$

Sketch of proof:

$$v(x) = \mathbb{E}_x \left\{ \sum_{t=0}^{\infty} \left[ \prod_{i=1}^{t} B_i \right] H_t \right\} = \sum_{t=0}^{\infty} \mathbb{E}_x \prod_{i=1}^{t} b(X_{i-1}, X_i) h(X_t)$$

Some algebra yields

$$\mathbb{E}_x \prod_{i=1}^{t} b(X_{i-1}, X_i) h(X_t) = (K^t h)(x)$$

Thus, with $r(K) < 1$, we have

$$v = \sum_{t=0}^{\infty} K^t h = (I - K)^{-1} h$$

## Back to the Firm Problem

Recall that

$$\mathbb{E}\left\{\beta_0 \cdots \beta_{t-1} \cdot \pi_t\right\} \quad \text{where} \quad \beta_t := \frac{1}{1+r_t}$$

and

$$v(x) = \mathbb{E}_x \left\{ \sum_{t=0}^{\infty} \left[ \prod_{i=0}^{t-1} \beta_i \right] \pi_t \right\}$$

Suppose

- $r_t = r(X_t)$ for $r \in \mathbb{R}^{\mathsf{X}}$
- Set $\beta(x) := 1/(1 + r(x))$

Let

$$K(x, x') := \beta(x) P(x, x') \qquad ((x, x') \in \mathsf{X} \times \mathsf{X})$$

**Proposition**. If $r(K) < 1$, then the firm valuation is finite and satisfies

$$v = (I - K)^{-1} \pi$$

Proof: Just apply the last theorem with

$$b(X_{t-1}, X_t) = \beta(X_{t-1}) = \frac{1}{1 + r(X_{t-1})}$$

and $h = \pi$

# Introduction to Asset Pricing

Consider an asset with payoff $G_{t+1}$ next period

What current price $\Pi_t$ should we assign?

**Risk neutral pricing** says

$$\Pi_t = \mathbb{E}_t \, \beta \, G_{t+1}$$

for some $\beta \in (0, 1)$

If the payoff is in $k$ periods, then the price is $\mathbb{E}_t \, \beta^k \, G_{t+k}$

Example. The time $t$ risk-neutral price of a **European call option** is

$$\Pi_t = \mathbb{E}_t \, \beta^k \max\{S_{t+k} - K, 0\}$$

where

- $S_t$ is the price of the underlying asset (e.g., stock)
- $K$ is the strike price
- $k$ is the duration
- $\beta = 1/(1+r)$ where $r$ is the discount rate

But assuming risk neutrality for all investors is **not consistent with the data**

Example. Consider the rate of return $r_{t+1} := (G_{t+1} - \Pi_t)/\Pi_t$

From $\Pi_t = \mathbb{E}_t \, \beta \, G_{t+1}$ we get

$$\mathbb{E}_t \, \beta \, \frac{G_{t+1}}{\Pi_t} = 1 \quad \iff \quad \mathbb{E}_t \, \beta (1 + r_{t+1}) = 1$$

Hence

$$\mathbb{E}_t \, r_{t+1} = \frac{1 - \beta}{\beta}$$

Risk neutrality implies that all assets have the same expected rate of return!

In fact riskier assets have usually higher average rates of return

- incentivize investors to bear risk

Example. The **risk premium** := expected rate of return minus the rate of return on a risk-free asset

If we assume risk-neutrality then, by the preceding discussion, the risk premium is zero for all assets

But calculations based on post-war US data show that

$$\text{average risk premium for equities} \approx 8\% \text{ per annum}$$

# Stochastic Discount Factors

Let's try a model containing one asset and one agent

Agent takes $\Pi_t$ as given and solves

$$\max_{0 \leqslant \alpha \leqslant 1} \{u(C_t) + \beta \mathbb{E}_t u(C_{t+1})\}$$

$$\text{subject to} \quad C_t = E_t - \Pi_t \alpha \quad \text{and} \quad C_{t+1} = E_{t+1} + \alpha G_{t+1}$$

Here

- $u$ is a flow utility function and $\beta$ measures impatience
- $G_{t+1}$ is the payoff of the asset and $\Pi_t$ is the time-$t$ price
- $E_t$ and $E_{t+1}$ are endowments and
- $\alpha$ is the share of the asset purchased by the agent

Rewrite as

$$\max_{\alpha}\{u(E_t - \Pi_t\alpha) + \beta\mathbb{E}_t u(E_{t+1} + \alpha G_{t+1})\}$$

Differentiating wrt $\alpha$ leads to first order condition

$$u'(E_t - \Pi_t\alpha)\Pi_t = \beta\mathbb{E}_t u'(E_{t+1} + \alpha G_{t+1})G_{t+1}$$

Rearranging gives us

$$\Pi_t = \mathbb{E}_t M_{t+1} G_{t+1} \quad \text{where} \quad M_{t+1} := \beta\frac{u'(C_{t+1})}{u'(C_t)}$$

- $M_{t+1}$ is called the **stochastic discount factor** (SDF)

Example. In the CRRA case $u(c) = c^{1-\gamma}/(1-\gamma)$ we get

$$M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

Alternatively,

$$M_{t+1} = \beta \exp(-\gamma g_{t+1}) \quad \text{where} \quad g_{t+1} := \ln(C_{t+1}/C_t)$$

Applies

- heavier discounting to payoffs in states of the world where consumption growth is high
- lower discounting to payoffs in states of the world where consumption growth is low

Favors assets that hedge against the risk of low consumption states

The general theory of asset pricing: $\exists$ and $M_{t+1}$ such that

$$\Pi_t = \mathbb{E}_t\, M_{t+1}\, G_{t+1}$$

- same SDF can price **any** asset — see Hansen, Kreps, etc.

Suppose $(X_t)_{t \geqslant 0}$ is $P$-Markov on X,

$$M_{t+1} = m(X_t, X_{t+1}) \quad \text{and} \quad G_{t+1} = g(X_t, X_{t+1})$$

With $\pi(x) := \mathbb{E}_x M_{t+1}\, G_{t+1}$, we have

$$\pi(x) = \sum_{x' \in \mathsf{X}} m(x, x') g(x, x') P(x, x')$$

# Pricing Dividend Streams

Consider the price of a claim on dividend stream $(D_t)_{t \geqslant 0}$

Let the price at time $t$ be $\Pi_t$

Buying at $t$ and selling at $t+1$ pays $\Pi_{t+1} + D_{t+1}$

Hence the price sequence $(\Pi_t)_{t \geqslant 0}$ must obey

$$\Pi_t = \mathbb{E}_t \, M_{t+1} (\Pi_{t+1} + D_{t+1})$$

Current price depends on future price — how can we solve it?

Recall the key equation

$$\Pi_t = \mathbb{E}_t \, M_{t+1}(\Pi_{t+1} + D_{t+1})$$

Let

- $D_t = d(X_t)$ where $(X_t)_{t \geqslant 0}$ is $P$-Markov
- $\pi(x) =$ current price given $X_t = x$

We get

$$\pi(x) = \sum_{x'} m(x, x')(\pi(x') + d(x'))P(x, x') \qquad (x \in \mathsf{X})$$

Rewrite the last expression as

$$\pi = A\pi + Ad$$

where

$$A(x, x') := m(x, x')P(x, x')$$

Neumann series lemma: $r(A) < 1 \implies$ the unique solution is

$$\pi^* = (I - A)^{-1}Ad$$

- $\pi^*$ is called an **equilibrium price function**
- $A$ is called the **Arrow–Debreu discount operator**

# Nonstationary Dividends

A more realistic model is one where dividends grow over time

A standard model of dividend growth is

$$\ln \frac{D_{t+1}}{D_t} = \kappa(X_t, \eta_{t+1}) \qquad t = 0, 1, \ldots,$$

Here

- $\kappa$ is a fixed function
- $(X_t)$ is $P$-Markov on finite set $\mathsf{X}$
- $(\eta_t)$ is $\textsc{iid}$ with density $\varphi$
- $M_{t+1} = m(X_t, X_{t+1})$ for some positive function $m$

Growing dividends implies share prices will grow

- should not seek a $\pi$ such that $\Pi_t = \pi(X_t)$ for all $t$

Instead we try to solve for the **price-dividend ratio** $V_t := \Pi_t / D_t$

**Ex.** Show that $\Pi_t = \mathbb{E}_t \left[ M_{t+1} (D_{t+1} + \Pi_{t+1}) \right]$ implies

$$V_t = \mathbb{E}_t \, \left[ M_{t+1} \exp(\kappa(X_t, \eta_{t+1})) \, (1 + V_{t+1}) \right]$$

Conditioning on $X_t = x$,

$$v(x) = \sum_{x' \in \mathsf{X}} m(x, x') \int \exp(\kappa(x, \eta)) \varphi(\mathrm{d}\eta) \left[ 1 + v(x') \right] P(x, x')$$

Let

$$A(x, x') := m(x, x') \int \exp(\kappa(x, \eta)) \varphi(\mathrm{d}\eta) P(x, x')$$

Now we see a $v$ that solves

$$v(x) = \sum_{x' \in \mathsf{X}} \left[1 + v(x')\right] A(x, x')$$

Equivalent:

$$v = A\mathbb{1} + Av$$

If $r(A) < 1$, then the unique solution is

$$v^* = (I - A)^{-1} A\mathbb{1}$$

Example. Dividend growth is

$$\kappa(X_t, \eta_{d,t+1}) = \mu_d + X_t + \sigma_d \, \eta_{d,t+1} \quad \text{where} \quad (\eta_{d,t})_{t \geqslant 0} \overset{\text{IID}}{\sim} N(0,1)$$

Consumption growth is given by

$$\ln \frac{C_{t+1}}{C_t} = \mu_c + X_t + \sigma_c \, \eta_{c,t+1} \quad \text{where} \quad (\eta_{c,t})_{t \geqslant 0} \overset{\text{IID}}{\sim} N(0,1)$$

We use the Lucas CRRA SDF, implying that

$$M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} = \beta \exp(-\gamma(\mu_c + X_t + \sigma_c \eta_{c,t+1}))$$

```julia
using QuantEcon, LinearAlgebra

"Creates an instance of the asset pricing model with Markov state."
function create_asset_pricing_model(;
        n=200,              # state grid size
        ρ=0.9, ν=0.2,       # state persistence and volatility
        β=0.99, γ=2.5,      # discount and preference parameter
        μ_c=0.01, σ_c=0.02, # consumption growth mean and volatility
        μ_d=0.02, σ_d=0.1)  # dividend growth mean and volatility
    mc = tauchen(n, ρ, ν)
    x_vals, P = exp.(mc.state_values), mc.p
    return (; x_vals, P, β, γ, μ_c, σ_c, μ_d, σ_d)
end
```

```julia
" Build the discount matrix A. "
function build_discount_matrix(model)
    (; x_vals, P, β, γ, μ_c, σ_c, μ_d, σ_d) = model
    e = exp.(μ_d - γ*μ_c + (γ^2*σ_c^2 + σ_d^2)/2 .+ (1-γ)*x_vals)
    return β * e .* P
end

"Compute the price-dividend ratio associated with the model."
function pd_ratio(model)
    (; x_vals, P, β, γ, μ_c, σ_c, μ_d, σ_d) = model
    A = build_discount_matrix(model)
    @assert maximum(abs.(eigvals(A))) < 1 "Requires r(A) < 1."
    n = length(x_vals)
    return (I - A) \ (A * ones(n))
end
```

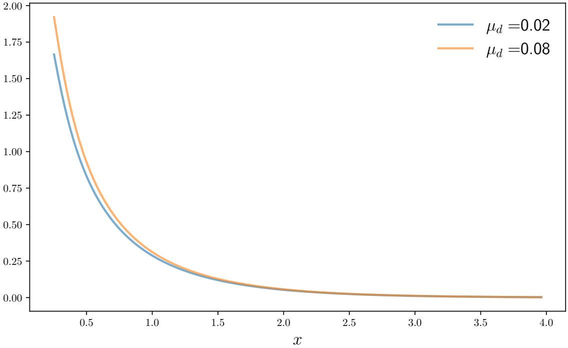


Figure: Price-dividend ratio as a function of $x$