Dynamic Programming

Chapter 1: Introduction

Thomas J. Sargent and John Stachurski

2023

Introduction

Summary of this lecture:

- Symbols and terminology
- 2 minute introduction to Julia
- Finite horizon job search
- Linear equations
- Fixed point theory
- Infinite horizon job search

Common Symbols

$\mathbb{1}\{P\}$	equals 1 if statement P true, 0 otherwise
$\alpha := 1$	α is defined as equal to 1
$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$	natural numbers, integers and real numbers
$\mathbb{C}$	complex numbers
$\mathbb{Z}_+$, $\mathbb{R}_+$, etc.	the nonnegative elements of $\mathbb{Z}$, $\mathbb{R}$, etc.
$\lvert x\rvert$ for $x \in \mathbb{R}$	absolute value of x
$\lvert\lambda\rvert$ for $\lambda \in \mathbb{C}$	modulus of λ
$a \vee b$	$\max\{a, b\}$
$a \wedge b$	$\min\{a, b\}$
$\lvert B\rvert$	the cardinality of set B
$\mathbb{R}^n$	all n-tuples of real numbers
$x \leqslant y \ \ (x, y \in \mathbb{R}^n)$	$x_i \leqslant y_i$ for $i = 1, \ldots n$ (pointwise partial order)
$x \ll y \ \ (x, y \in \mathbb{R}^n)$	$x_i < y_i$ for $i = 1, \ldots n$
$\mathcal{D}(F)$	the set of distributions (or pmfs) on F
$\mathbb{R}^M$	all functions from M to $\mathbb{R}$

Let M be any set

If $f\colon M \to \mathbb{R}$, then we call f a **real-valued function** on M

Let $\mathbb{R}^M$ be the **set of all real-valued functions on M**

If $f, g \in \mathbb{R}^M$ and $\alpha, \beta \in \mathbb{R}$, then

- $\alpha f + \beta g \in \mathbb{R}^M$ with $(\alpha f + \beta g)(x) := \alpha f(x) + \beta g(x)$
- $fg \in \mathbb{R}^M$ with $(fg)(x) := f(x)g(x)$
- $f \vee g \in \mathbb{R}^M$ with $(f \vee g)(x) := f(x) \vee g(x)$
- $f \wedge g \in \mathbb{R}^M$ with $(f \wedge g)(x) := f(x) \wedge g(x)$
- etc.

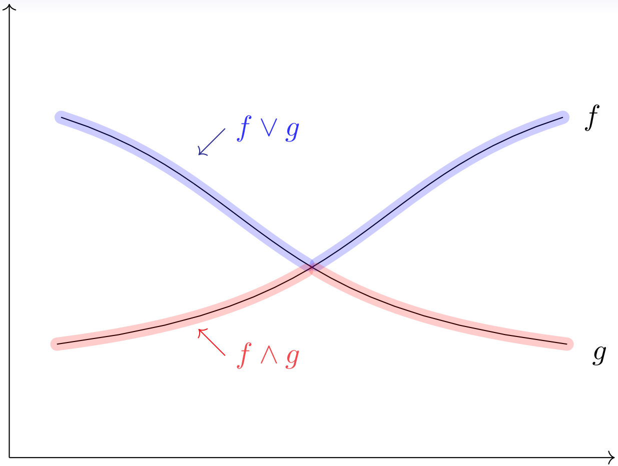


Figure: Functions $f \vee g$ and $f \wedge g$ when defined on a subset of $\mathbb{R}$

Important: Operations on real numbers such as $|\cdot|$ and $\vee$ are applied to vectors element-by-element

Example.

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \Longrightarrow \quad |a| = \begin{pmatrix} |a_1| \\ \vdots \\ |a_n| \end{pmatrix}$$

$$a \vee b = \begin{pmatrix} a_1 \vee b_1 \\ \vdots \\ a_n \vee b_n \end{pmatrix} \quad \text{and} \quad a \wedge b = \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix}$$

etc.

Topology in $\mathbb{R}^n$

Some quick reminders

A set $C \subset \mathbb{R}^n$ is called **closed** in $\mathbb{R}^n$ if

$$(u_m) \subset C \text{ and } u_m \to u \implies u \in C$$

A set G is called **open** if G^c is closed

$T\colon U \to V$ is called **continuous at** $u \in U$ if

$$(u_m) \subset U \text{ and } u_m \to u \implies Tu_m \to Tu$$

We call T **continuous on** U if T is continuous at every $u \in U$

If $M = \{x_1, \ldots, x_n\}$ = some finite set, then

$\mathbb{R}^M$ and $\mathbb{R}^n$ are the "same" set !

Indeed, $f \in \mathbb{R}^M$ is defined by its values $f(x_1), \ldots, f(x_n)$

Hence $\exists$ a one-to-one correspondence between $\mathbb{R}^M$ and $\mathbb{R}^n$:

$$\mathbb{R}^M \ni f \quad \longleftrightarrow \quad (f(x_1), \ldots, f(x_n)) \in \mathbb{R}^n$$

Example.

- We call $C \subset \mathbb{R}^M$ closed if C is closed in $\mathbb{R}^n$, etc.
- If $f \in \mathbb{R}^M$, then $\|f\| :=$ the norm of $(f(x_1), \ldots, f(x_n))$

Julia Syntax: Two Minute Introduction

- Install from `https://julialang.org/` (if you wish)
- To import `Library`, write `using Library`
- `f(x) = 2x` defines the function $f(x) = 2x$
- `cos.(x)` applies cos to each elements of vector x
- `x.^2` squares each element of vector x
- looping very similar to Python
- See also `https://julia.quantecon.org/intro.html`

```julia
# Defining functions, using conditions and loops

function f(x, y)                      # define a function
    if x < y                          # branch
        return sin(x + y)
    else
        return cos(x + y)
    end
end


function print_plurals(list_of_words)   # define a function
    for word in list_of_words           # loop
        println(word * "s")
    end
end
```

```julia
using LinearAlgebra          # import LinearAlgebra library

f(x) = 2x                    # simple function definition
f(x) = norm(x)               # norm defined in LinearAlgebra
g(x) = sum(x + x.^2)         # dot for pointwise operations
α, β = 2.0, -2.0             # unicode symbols

q(x) = sin(cos(x))           # another function

x = rand(5)
println(q(5))                  # OK
println(q.(x))                 # OK
println(q(x))                  # Error!
```

Let $f\colon A \to B$ and $g\colon B \to C$.

Recall that the **composition** of f and g is the map

$$g \circ f\colon A \to C, \quad A \ni a \mapsto g(f(a)) \in C$$

Example. $f(x) = x \wedge 0$ and $g(x) = x \vee 0$ implies $g \circ f \equiv 0$

In Julia we can compose as follows

```
f(x) = min(x, 0)
g(x) = max(x, 0)
h = f ∘ g        # type \circ and then hit tab
```

Dynamic program

```
an initial state X_0 is given
t ← 0
while t < T do
    observe current state X_t
    choose action A_t
    receive reward R_t based on (X_t, A_t)
    state updates to X_{t+1}
    t ← t + 1
end
```

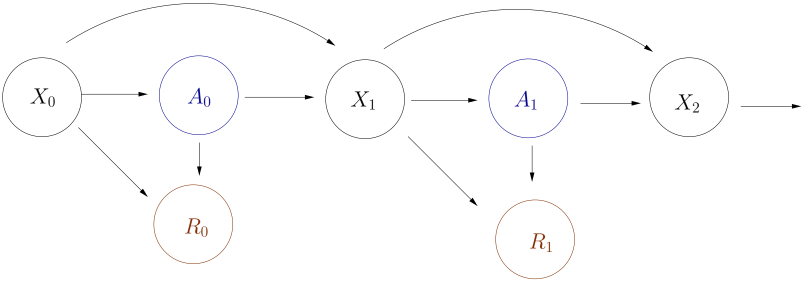


Figure: A dynamic program

Comments:

- Objective: maximize **lifetime rewards**
    - Some aggregation of $R_0, R_1, \ldots$
    - Example. $\mathbb{E}[R_0 + \beta R_1 + \beta^2 R_2 + \cdots]$ for some $\beta \in (0, 1)$
- If $T < \infty$ then the problem is called a **finite horizon** problem
- Otherwise it is called an **infinite horizon** problem
- The update rule can also depend on random elements:

$$X_{t+1} = F(X_t, A_t, \xi_{t+1})$$

Example. A retailer sets prices and manages inventories to maximize profits

- X_t measures
  - current business environment
  - the size of the inventories
  - prices set by competitors, etc.
- A_t specifies current prices and orders of new stock
- R_t is current profit π_t
- Lifetime reward is

$$\mathbb{E}\left[\pi_0 + \frac{1}{1+r}\pi_1 + \left(\frac{1}{1+r}\right)^2 \pi_2 + \cdots\right] = \text{EPV}$$

Flow of this lecture:

1. Begin with simple finite-horizon dynamic program
2. Introduce the recursive structure of dynamic programming
3. Shift to an infinite-horizon version
4. Show how the problem produces a system of nonlinear equations
5. Discuss how we can solve nonlinear equations
   - Fixed point theory

Finite-Horizon Job Search

A model of job search created by **John J. McCall**

We model the decision problem of an unemployed worker

Job search depends on

- current and likely future wage offers
- impatience, and
- the availability of unemployment compensation

We begin with a very simple version of the McCall model

(Later we consider extensions)

Set Up

An agent begins working life at time $t = 1$ without employment

Receives a new job offer paying wage W_t at each date t

She has two choices:

1. **accept** the offer and work permanently at W_t or
2. **reject** the offer, receive unemployment compensation c, and reconsider next period

Assume $\{W_t\}$ is $\overset{\text{IID}}{\sim}$ φ, where

- $\mathsf{W} \subset \mathbb{R}_+$ is a finite set of wage outcomes and
- $\varphi \in \mathcal{D}(\mathsf{W})$

The agent cares about the future but is **impatient**

Impatience is parameterized by a **time discount factor** $\beta \in (0, 1)$

- Present value of a next-period payoff of y dollars is βy

Trade off:

- $\beta < 1$ indicating some impatience
- hence the agent will be tempted to accept reasonable offers, rather than always waiting for a better one
- The key question is how long to wait

The Two Period Problem

Suppose that the working life is just two periods $(t = 1, 2)$

Let's start at $t = 2$, when W_2 is observed

backward induction – start at the end, work back

- If already employed, continue working at W_1
- If unemployed, take the max of c and W_2

Set $v_2(w) = \max\{c, w\}$ = the **time 2 value function**

- max value available for unemployed worker at $t = 2$ given w

Now we shift to $t = 1$

At $t = 1$, given W_1, the unemployed worker's options are

1. **accept** W_1 and receive it at $t = 1, 2$
2. **reject** it, receive compensation c, and then, at $t = 2$, get $v_2(W_2) = \max\{c, W_2\}$

Expected present value (**EPV**) of option 1 is $W_1 + \beta W_1$

- sometimes called the **stopping value**

EPV of option 2 is

$$h_1 := c + \beta \sum_{w' \in \mathsf{W}} v_2(w')\varphi(w'), \tag{1}$$

- sometimes called the **continuation value**

1. Look at EPV of two choices (accept, reject)
2. Choose the one with highest EPV

Let's label the actions

$$0 := \text{ reject}$$

$$1 := \text{ accept}$$

Then optimal choice is

$$\mathbb{1}\{W_1 + \beta W_1 \geqslant h_1\}$$

$$=: \mathbb{1}\{\text{stopping value} \geqslant \text{continuation value}\}$$

Let

$$w_1^* := \frac{h_1}{1+\beta} := \textbf{reservation wage}$$

We have

$$\begin{aligned} W_1 \geqslant w^* \quad &\iff \quad W_1 \geqslant \frac{h_1}{1+\beta} \\ &\iff \quad W_1 + \beta W_1 \geqslant h_1 \\ &\iff \quad \text{stopping value} \geqslant \text{continuation value} \end{aligned}$$

Hence

$$\text{accept} \quad \iff \quad W_1 \geqslant w_1^*$$

The **time 1 value function** v_1 is

$$v_1(w_1) = \max \left\{ w_1 + \beta w_1, \; c + \beta \sum_{w' \in \mathsf{W}} v_2(w') \varphi(w') \right\}$$

$$= \max \{ w_1 + \beta w_1, \; h_1 \}$$

$$= \max \{ \text{stopping value}, \; \text{continuation value} \}$$

The maximum lifetime value available at $t = 1$ given

- currently unemployed
- current offer w_1

```julia
using Distributions

"Creates an instance of the job search model, stored as a NamedTuple."
function create_job_search_model(;
        n=50,        # wage grid size
        w_min=10.0,  # lowest wage
        w_max=60.0,  # highest wage
        a=200,       # wage distribution parameter
        b=100,       # wage distribution parameter
        β=0.96,      # discount factor
        c=10.0       # unemployment compensation
    )
    w_vals = collect(LinRange(w_min, w_max, n+1))
    ϕ = pdf(BetaBinomial(n, a, b))
    return (; n, w_vals, ϕ, β, c)
end

" Computes lifetime value at t=1 given current wage w_1 = w. "
function v_1(w, model)
    (; n, w_vals, ϕ, β, c) = model
    h_1 = c + β * max.(c, w_vals)'ϕ
    return max(w + β * w, h_1)
end

" Computes reservation wage at t=1. "
function res_wage(model)
    (; n, w_vals, ϕ, β, c) = model
    h_1 = c + β * max.(c, w_vals)'ϕ
    return h_1 / (1 + β)
end
```

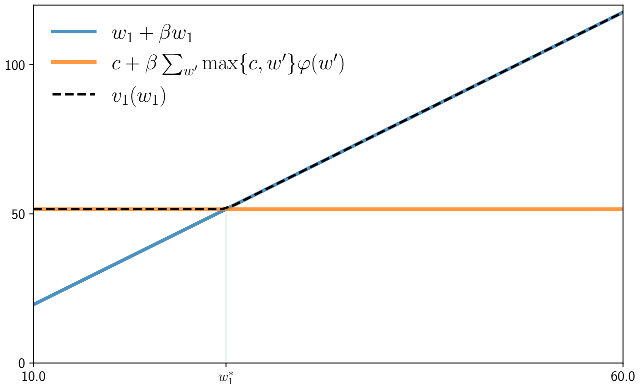


Figure: The value function v_1 and the reservation wage

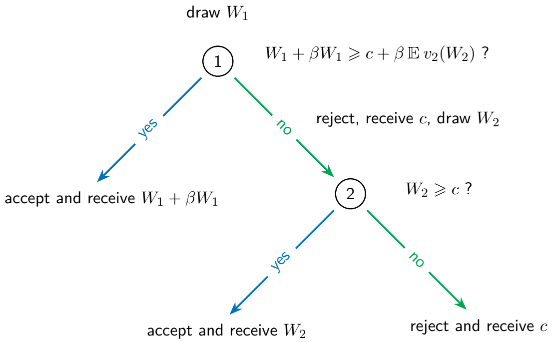


Figure: Decision tree for the two period problem

Three Period Problem

Now suppose we extend to three periods, with $t = 0, 1, 2$

At $t = 0$, the EPV of accepting W_0 is $W_0 + \beta W_0 + \beta^2 W_0$

Maximal EPV of rejecting is

1. unemployment compensation plus
2. max value we can expect from $t = 1$ when unemployed

That is,

$$\text{continuation value} = h_0 := c + \beta \sum_{w'} v_1(w') \varphi(w')$$

Putting it together,

$$v_2(w_2) = \max\{c, w_2\}$$

$$v_1(w_1) = \max\left\{w_1 + \beta w_1,\ c + \beta \sum_{w' \in \mathsf{W}} v_2(w')\varphi(w')\right\}$$

$$v_0(w_0) = \max\left\{w_0 + \beta\, w_0 + \beta^2\, w_0,\ c + \beta \sum_{w' \in \mathsf{W}} v_1(w')\varphi(w')\right\}$$

- **solve for all v_t** by **backward induction** (start from top)

From Values to Choices

Now we know the optimal values we can make optimal choices

At time $t = 0$

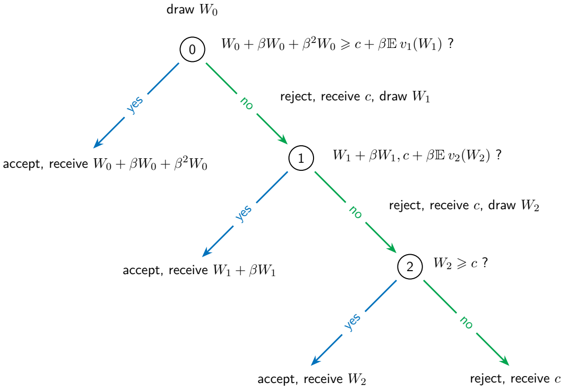
$$\text{action} \;\; = \mathbb{1}\left\{ w_0 + \beta\, w_0 + \beta^2\, w_0 \geqslant c + \beta \sum_{w' \in \mathsf{W}} v_1(w')\varphi(w') \right\}$$

At time $t = 1$, if still unemployed

$$\text{action} \;\; = \mathbb{1}\left\{ w_1 + \beta w_1 \geqslant c + \beta \sum_{w' \in \mathsf{W}} v_2(w')\varphi(w') \right\}$$

At time $t = 2$, if still unemployed

$$\text{action} \;\; = \mathbb{1}\{w_2 \geqslant c\}$$


draw W_0
0
$W_0 + \beta W_0 + \beta^2 W_0 \geqslant c + \beta \mathbb{E}\, v_1(W_1)$?
yes
no
reject, receive c, draw W_1
accept, receive $W_0 + \beta W_0 + \beta^2 W_0$
1
$W_1 + \beta W_1, c + \beta \mathbb{E}\, v_2(W_2)$?
yes
no
reject, receive c, draw W_2
accept, receive $W_1 + \beta W_1$
2
$W_2 \geqslant c$?
yes
no
accept, receive W_2
reject, receive c

Summary

We reduced the multi-stage problem to two period problems

- the **key idea** of dynamic programming!

The equation

$$v_0(w_0) = \max \left\{ w_0 + \beta\, w_0 + \beta^2\, w_0,\ c + \beta \sum_{w' \in \mathsf{W}} v_1(w')\varphi(w') \right\}$$

is an example of **Bellman's equation**

Similiar ideas easily extend to time T:

$$v_T(w_T) = \max\{c, w_T\}$$

and

$$v_t(w_t) =$$

$$\max \left\{ w_t + \beta\, w_t + \cdots + \beta^{T-t}\, w_t,\; c + \beta \sum_{w' \in \mathsf{W}} v_{t+1}(w') \varphi(w') \right\}$$

for $t = 0, \ldots, T-1$

Infinite Horizons

Now let us consider a worker who aims to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t Y_t \qquad (Y_t \text{ is income at time } t)$$

- offer process $\{W_t\} \stackrel{\text{IID}}{\sim} \varphi$ for $\varphi \in \mathcal{D}(\mathsf{W})$
- $\mathsf{W} \subset \mathbb{R}_+$ with $|\mathsf{W}| < \infty$
- $c, \beta > 0$ and $\beta < 1$
- jobs are permanent

What is max EPV of each option when lifetime is infinite?

What if we **accept** $w \in \mathsf{W}$ now?

$$\text{EPV} = \text{stopping value} = w + \beta w + \beta^2 w + \cdots = \frac{w}{1-\beta}$$

What if we **reject**?

$$\begin{aligned}\text{EPV} &= \text{continuation value} \\ &= c + \text{EPV of } \underline{\text{optimal}} \text{ choice in each future period}\end{aligned}$$

But what are optimal choices?!

Calculating optimal choice requires knowing optimal choice!

The Value Function

Let $v^*(w) :=$ max lifetime EPV given wage offer w

We call v^* the **value function**

Suppose that we know v^*

Then the (maximum) **continuation value** is

$$h^* := c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w')$$

The optimal choice is then

$$\mathbb{1}\,\{\text{stopping value} \geqslant \text{continuation value}\} = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant \; h^*\right\}$$

But how can we calculate v^*?

Key idea: We can use Bellman's equation to solve for v^*

Theorem. The value function v^* satisfies **Bellman's equation**

$$v^*(w) = \max\left\{\frac{w}{1-\beta},\, c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

Intuition:

- If accept, get $w/(1-\beta)$
- If reject, get c plus EPV of optimal future choices
- Max value today is max of these alternatives

Full proof coming later!

So how can we use Bellman's equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

to solve for v^*?

For this we need **fixed point theory**

Fixed point theory is used to solve equations

We start begin with the linear case

Linear Equations

Given one-dimensional equation $x = ax + b$, we have

$$|a| < 1 \quad \Longrightarrow \quad x^* = \frac{b}{1-a} = \sum_{k \geqslant 0} a^k b$$

How can we extend this beyond one dimension?

We define the **spectral radius** of square matrix A as

$$r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

Key idea:

- $r(A) < 1$ is a generalization of $|a| < 1$

Neumann Series Lemma

Suppose b is a column vector in $\mathbb{R}^n$ and A is $n \times n$

Let I be the $n \times n$ identity matrix

Theorem. If $r(A) < 1$, then

1. $I - A$ is nonsingular,
2. the sum $\sum_{k \geqslant 0} A^k$ converges,
3. $(I - A)^{-1} = \sum_{k \geqslant 0} A^k$, and
4. the vector equation $x = Ax + b$ has the unique solution

$$x^* := (I - A)^{-1} b = \sum_{k \geqslant 0} A^k b$$

Intuitive idea: with $S := \sum_{k \geqslant 0} A^k$, we have

$$I + AS = I + A(I + A + \cdots) = I + A + A^2 + \cdots = S$$

Rearranging $I + AS = S$ gives $S = (I - A)^{-1}$

The equation $x = Ax + b$ is equivalent to $(I - A)x = b$

Unique solution is $x^* = (I - A)^{-1} b = Sb$, as claimed

However, still need to show that

- $\sum_{k \geqslant 0} A^k$ converges
- the matrix $I - A$ is invertible

To complete the proof, we introduce the **matrix norm**

$$\|B\|_\infty := \max_{i,j} |b_{ij}|$$

Lemma. If B is any square matrix, then

- $r(B)^k \leqslant \|B^k\|_\infty$ for all $k \in \mathbb{N}$ and
- **Gelfand's formula** holds: $\|B^k\|_\infty^{1/k} \to r(B)$ as $k \to \infty$

Ex. Prove: $r(A) < 1 \implies \sum_{k \geqslant 0} A^k$ converges

- Hint 1: Suffices to show $\lim_{N \to \infty} \| \sum_{k \geqslant 0}^N A^k \|_\infty < \infty$
- Hint 2: Use triangle inequality and Cauchy's root test

To complete the proof, we introduce the **matrix norm**

$$\|B\|_\infty := \max_{i,j} |b_{ij}|$$

Lemma. If B is any square matrix, then

- $r(B)^k \leqslant \|B^k\|_\infty$ for all $k \in \mathbb{N}$ and
- **Gelfand's formula** holds: $\|B^k\|_\infty^{1/k} \to r(B)$ as $k \to \infty$

Ex. Prove: $r(A) < 1 \implies \sum_{k \geqslant 0} A^k$ converges

- Hint 1: Suffices to show $\lim_{N \to \infty} \| \sum_{k \geqslant 0}^{N} A^k \|_\infty < \infty$
- Hint 2: Use triangle inequality and Cauchy's root test

To complete the proof, we introduce the **matrix norm**

$$\|B\|_\infty := \max_{i,j} |b_{ij}|$$

Lemma. If B is any square matrix, then

- $r(B)^k \leqslant \|B^k\|_\infty$ for all $k \in \mathbb{N}$ and
- **Gelfand's formula** holds: $\|B^k\|_\infty^{1/k} \to r(B)$ as $k \to \infty$

Ex. Prove: $r(A) < 1 \implies \sum_{k \geqslant 0} A^k$ converges

- Hint 1: Suffices to show $\lim_{N \to \infty} \| \sum_{k \geqslant 0}^N A^k \|_\infty < \infty$
- Hint 2: Use triangle inequality and Cauchy's root test

Final step: Show that $(I - A)^{-1}$ exists:

- Suffices to show existence of a right inverse
    - See, e.g., §6.1.4.5 of networks.quantecon.org
- That is, we need an S such that $(I - A)S = I$
- Let $S = \sum_{k \geqslant 0} A^k$

We have

$$(I - A)S = I \sum_{k \geqslant 0} A^k - A \sum_{k \geqslant 0} A^k = \sum_{k \geqslant 0} A^k - \sum_{k \geqslant 1} A^k = I$$

Hence $(I - A)^{-1}$ exists and equals $\sum_{k \geqslant 0} A^k$

Fixed Points

To solve more complex equations we use **fixed point theory**

Recall that, if S is any set then

- T is a **self-map** on S if T maps S into itself
- $x^* \in S$ is called a **fixed point** of T in S if $Tx^* = x^*$

Example. Every x in set S is fixed under the **identity map**

$$I \colon x \mapsto x$$

Example. If $S = \mathbb{N}$ and $Tx = x + 1$, then T has no fixed point

Example. If $S = \mathbb{R}$ and $Tx = x^2$, then T has fixed points at $0, 1$

Example. If $S = \mathbb{R}^n$ and $Tx = Ax + b$, then

$$r(A) < 1 \implies x^* := (I - A)^{-1} b \text{ is the unique f.p. of } T \text{ in } S$$

Example. If $S \subset \mathbb{R}$, $Tx = x \iff T$ meets the 45 degree line

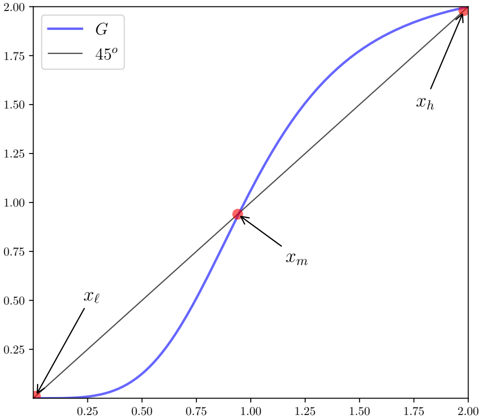


Figure: Graph and fixed points of $G\colon x \mapsto 2.125/(1 + x^{-4})$

Given self-map T on S, common to

- write Tx instead of $T(x)$ and
- call T an **operator** rather than a function

Key idea:

$$\text{solving equation } x = Tx \iff \text{finding fixed points of } T$$

Example. If $S = \mathbb{R}^n$ and $Tx = Ax + b$, then

$$x^* \text{ solves equation } x = Ax + b \iff x^* \text{ is a fixed point of } T$$

(But fixed point theory is mainly for nonlinear equations)

Point on notation:

- $T^2 = T \circ T$
- $T^3 = T \circ T \circ T$
- etc.

Example. $Tx = Ax + b$ implies $T^2x = A(Ax + b) + b$

Lemma Let S be any set and let T be a self-map on S. If

$$\exists\, \bar{x} \in S,\ m \in \mathbb{N} \ \text{ s.t. }\ T^k x = \bar{x} \text{ for all } x \in S \text{ and } k \geqslant m$$

then $\bar{x}$ is the unique fixed point of T in S.

<u>Proof</u> of uniqueness:

Let x and y be any two two fixed points of T in S

Since $T^m x = \bar{x}$ and $T^m y = \bar{x}$, we have $T^m x = T^m y$

But x and y are fixed points, so

$$x = T^m x \text{ and } y = T^m y$$

We conclude that $x = y$, so uniqueness holds

Proof of existence:

We claim that $\bar{x}$ is a fixed point

To see this, recall that

$$T^k x = \bar{x} \text{ for } k \geqslant m \text{ and all } x \in S$$

Hence $T^m \bar{x} = \bar{x}$ and $T^{m+1} \bar{x} = \bar{x}$

But then

$$T\bar{x} = T(T^m \bar{x}) = T^{m+1} \bar{x} = \bar{x}$$

That is, $\bar{x}$ is a fixed point of T

Let T be a self-map on $S \subset \mathbb{R}^d$

Ex. Prove the following: If

1. $T^m u \to u^*$ as $m \to \infty$ for some pair $u, u^* \in S$ and
2. T is continuous at u^*

then u^* is a fixed point of T

Answer: Assume hypotheses and let $u_m := T^m u$ for all $m \in \mathbb{N}$

By continuity and $u_m \to u^*$ we have $Tu_m \to Tu^*$

But $(Tu_m)_{m \geqslant 1}$ is just $(u_2, u_3, \ldots)$

Since $u_m \to u^*$, we just have $Tu_m \to u^*$

Limits are unique, so $u^* = Tu^*$

Self-map T is called **globally stable** on S if

1. T has a unique fixed point x^* in S and
2. $T^k x \to x^*$ as $k \to \infty$ for all $x \in S$

Example. If $S = \mathbb{R}^n$ and $Tx = Ax + b$, then

$$T^k x = A^k x + A^{k-1} b + A^{k-2} b + \cdots + Ab + b \qquad (x \in S, \; k \in \mathbb{N})$$

If $r(A) < 1$, then $A^k x \to 0$ and $\sum_{i=0}^{k} A^i \to (I - A)^{-1}$, so

$$\lim_{k \to \infty} T^k x = \lim_{k \to \infty} \left[A^k x + \sum_{i=0}^{k} A^{i-1} b \right] = (I - A)^{-1} b = x^*$$

Example. Consider Solow–Swan growth dynamics

$$k_{t+1} = g(k_t) := sAk_t^\alpha + (1-\delta)k_t, \qquad t = 0, 1, \ldots,$$

where

- k_t is capital stock per worker,
- $A, \alpha > 0$ are production parameters, $\alpha < 1$
- $s > 0$ is a savings rate, and
- $\delta \in (0, 1)$ is a rate of depreciation

Iterating with g from k_0 generates a time path for capital stock

The map g is globally stable on $(0, \infty)$

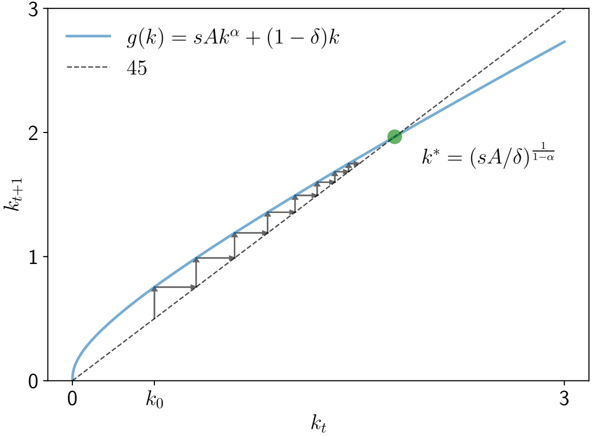


Figure: Global stability for the Solow–Swan model

Note from last slide

- If g is flat near k^*, then $g(k) \approx k^*$ for k near k^*
- A flat function near the fixed point $\implies$ fast convergence

Conversely

- If g is close to the 45 degree line near k^*, then $g(k) \approx k$
- Close to 45 degree line means high persistence, slow convergence

Let T be a self-map on $S \subset \mathbb{R}^n$.

We call $C \subset S$ **invariant** for T if

$$u \in C \implies Tu \in C$$

Lemma. If T is globally stable on $S \subset \mathbb{R}^n$ with fixed point u^* and C is nonempty, closed and invariant for T, then $u^* \in C$

Proof: Let the stated hypotheses hold and fix $u \in C$

By global stability we have $T^k u \to u^*$

Since T is invariant on C we have $(T^k u)_{k \in \mathbb{N}} \subset C$

Since C is closed, this implies that the limit is in C

Hence $u^* \in C$, as claimed

Given a self-map T on S, we typically ask

- Does T have at least one fixed point on S (existence)?
- Does T have at most one fixed point on S (uniqueness)?
- How can we compute fixed points of T?

For the last question, we seek an algorithm

Then we investigate its properties

Successive Approximation

A natural algorithm for approximating the fixed point in S:

---

fix x_0 and $k = 0$
while *some stopping condition fails* **do**
  $x_{k+1} \leftarrow Tx_k$
  $k \leftarrow k + 1$
end
return x_k

<u>If</u> T is globally stable on S, then $(x_k) = (T^k x_0)$ converges to x^*

hence output $\approx x^*$

The algorithm just described is called **successive approximation**

```julia
function successive_approx(T,                     # Operator (callable)
                           u_0;                   # Initial condition
                           tolerance=1e-6,        # Error tolerance
                           max_iter=10_000,       # Max iteration bound
                           print_step=25)         # Print at multiples
    u = u_0
    error = Inf
    k = 1

    while (error > tolerance) & (k <= max_iter)
        u_new = T(u)
        error = maximum(abs.(u_new - u))
        if k % print_step == 0
            println("Completed iteration $k with error $error.")
        end
        u = u_new
        k += 1
    end

    if error <= tolerance
        println("Terminated successfully in $k iterations.")
    else
        println("Termination Warning: Error is greater than tolerance.")
    end

    return u
end
```

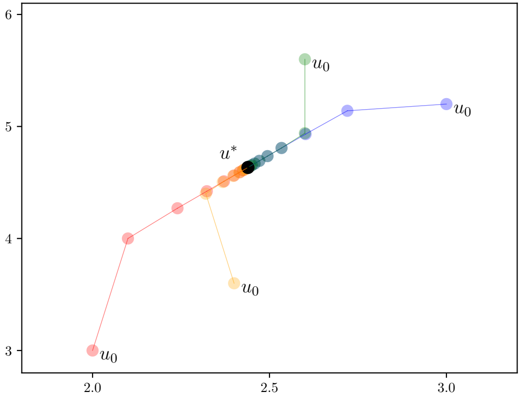


Figure: Successive approximation from different initial conditions

Newton's Method

Let h be a differentiable real-valued function on $(a, b) \subset \mathbb{R}$

We seek a **root** of h, which is an x^* such that $h(x^*) = 0$

We start with guess x_0 and then update it

To do this we use $h(x_1) \approx h(x_0) + h'(x_0)(x_1 - x_0)$

Setting the RHS $= 0$ and solving for x_1 gives

$$x_1 = x_0 - \frac{h(x_0)}{h'(x_0)}$$

Continuing in the same way, we set

$$x_{k+1} = q(x_k) \quad \text{where} \quad q(x) := x - \frac{h(x)}{h'(x)},$$

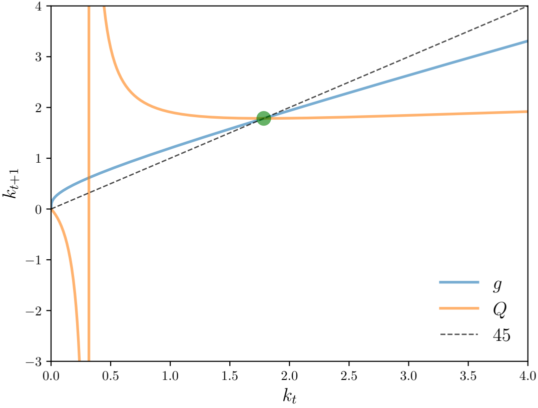


Figure: Successive approximation vs Newton's method

Comments:

- The map q is flat close to the fixed point k^*
- Hence Newton's method converges quickly <u>near</u> k^*
- But Newton's method is not globally convergent
- Successive approximation is slower but more robust

Key ideas

- There is almost always a trade-off between robustness and speed
- Speed requires assumptions, and assumptions can fail

Newton's method extends naturally to **multiple dimensions**

When h is a map from $S \subset \mathbb{R}^n$ to itself, we use

$$x_{k+1} = x_k - [J(x_k)]^{-1} h(x_k)$$

Here $J_h(x_k) :=$ the Jacobian of h evaluated at x_k

Comments

- Typically faster but less robust
- Matrix operations can be parallelized
- Automatic differentiation can be helpful

We want to use fixed point theory in $\mathbb{R}^n$

For this purpose it will be helpful to study alternative norms on $\mathbb{R}^n$

A function $\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}$ is called a **norm** on $\mathbb{R}^n$ if, for any $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$,

(a) $\|u\| \geqslant 0$

(b) $\|u\| = 0 \iff u = 0$

(c) $\|\alpha u\| = |\alpha| \|u\|$ and

(d) $\|u + v\| \leqslant \|u\| + \|v\|$

Example. The **Euclidean norm** $\|u\| := \sqrt{\langle u, u \rangle}$ obeys (a)–(d)

Example. The ℓ_1 **norm** of a vector $u \in \mathbb{R}^n$ is defined by

$$u = (u_1, \ldots, u_n) \mapsto \|u\|_1 := \sum_{i=1}^{n} |u_i|$$

Example. The **supremum norm**, defined by

$$\|u\|_\infty := \max_{i=1}^{n} |u_i|$$

is also a norm on $\mathbb{R}^n$

Ex. Verify that

1. the ℓ_1 norm on $\mathbb{R}^n$ satisfies (a)–(d) above
2. the supremum norm on $\mathbb{R}^n$ satisfies (a)–(d) above

Equivalence of Norms

Let u and $(u_m) := (u_m)_{m \in \mathbb{N}}$ be elements of $\mathbb{R}^n$

We say that (u_m) **converges** to u and write $u_m \to u$ if

$$\|u_m - u\| \to 0 \text{ as } m \to \infty \text{ for some norm } \|\cdot\| \text{ on } \mathbb{R}^n$$

Do we need to say "convergence with respect to $\|\cdot\|$"?

No because any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ are **equivalent**

That is, for any such pair, $\exists\, M, N$ such that

$$M\|u\|_a \leqslant \|u\|_b \leqslant N\|u\|_a \quad \text{for all } u \in \mathbb{R}^n$$

- See, e.g., Kreyszig (1978)

Hence convergence is independent of the norm

Ex. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be any two norms on $\mathbb{R}^n$

Given u in $\mathbb{R}^n$ and a sequence (u_m) in $\mathbb{R}^n$, confirm that

$$\|u_m - u\|_a \to 0 \text{ implies } \|u_m - u\|_b \to 0 \text{ as } m \to \infty$$

Proof: Let $\|\cdot\|_a$, $\|\cdot\|_b$, u and (u_m) be as stated

We can find an $M \in \mathbb{R}$ with

$$0 \leqslant \|u_m - u\|_b \leqslant M\|u_m - u\|_a \text{ for all } m \in \mathbb{N}$$

Since $\|u_m - u\|_a \to 0$, we also have $\|u_m - u\|_b \to 0$

Contractions

Let

- U be a nonempty subset of $\mathbb{R}^n$,
- $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and
- T be a self-map on U

T is called a **contraction** on U with respect to $\|\cdot\|$ if

$$\exists\, \lambda < 1 \text{ such that } \|Tu - Tv\| \leqslant \lambda \|u - v\| \quad \text{for all} \quad u, v \in U$$

Example. $Tx = ax + b$ is a contraction on $\mathbb{R}$ with respect to $|\cdot|$ if and only if $|a| < 1$

Indeed,

$$|Tx - Ty| = |ax + b - ay - b| = |a||x - y|$$

Ex. Prove: If T is a contraction on U with respect to any norm, then

1. T is continuous on U and
2. T has at most one fixed point in U

Let's check part 2 under the stated hypotheses

If u, v are fixed points of T in U, then

$$\|u - v\| = \|Tu - Tv\| \leqslant \lambda \|u - v\| \qquad \text{for some } \lambda < 1$$

$$\therefore \quad \|u - v\| = 0$$

$$\therefore \quad u = v$$

Banach's Contraction Mapping Theorem

Theorem If

1. U is closed in $\mathbb{R}^n$ and
2. T is a contraction of modulus λ on U with respect to some norm $\| \cdot \|$ on $\mathbb{R}^n$,

then T has a unique fixed point u^* in U and

$$\|T^n u - u^*\| \leqslant \lambda^n \|u - u^*\| \quad \text{for all } n \in \mathbb{N} \text{ and } u \in U$$

In particular, T is globally stable on U

Proof: See the course notes

Infinite-Horizon Job Search

Let's now return to the job search problem

Recall that that the value function v^* solves Bellman's equation

That is,

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

The infinite-horizon **continuation value** is defined as

$$h^* := c + \beta \sum_{w'} v^*(w') \varphi(w')$$

Key question: how to solve for v^*?

We introduce the **Bellman operator**, defined at $v \in \mathbb{R}^{\mathsf{W}}$ by

$$(Tv)(w) = \max\left\{\frac{w}{1-\beta},\, c + \beta \sum_{w' \in \mathsf{W}} v(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

By construction, $Tv = v$ iff v solves Bellman's equation

Proposition. T is a contraction $\mathbb{R}^{\mathsf{W}}$ with respect to $\| \cdot \|_\infty$

In the proof, we use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Fixing f, g in $\mathbb{R}^{\mathsf{W}}$ fix any $w \in \mathsf{W}$, we have

$$|(Tf)(w) - (Tg)(w)| \leqslant \left| \beta \sum_{w'} f(w')\varphi(w') - \beta \sum_{w'} g(w')\varphi(w') \right|$$

$$= \beta \left| \sum_{w'} [f(w') - g(w')]\varphi(w') \right|$$

Applying the triangle inequality,

$$|(Tf)(w) - (Tg)(w)| \leqslant \beta \sum_{w'} |f(w') - g(w')|\varphi(w') \leqslant \beta \|f - g\|_\infty$$

$$\therefore \quad \|Tf - Tg\|_\infty \leqslant \beta \|f - g\|_\infty$$

As a consequence

1. T has a unique fixed point $\bar{v}$ in $\mathbb{R}^{\mathsf{W}}$
2. $T^k v \to \bar{v}$ as $k \to \infty$ for all $v \in \mathbb{R}^{\mathsf{W}}$

Moreover, we know that $v^* \in \mathbb{R}^{\mathsf{W}}$ and v^* solves Bellman's equation

Hence $\bar{v} = v^*$

Summary: We can compute v^* by successive approximation:

1. choose any initial $v \in \mathbb{R}^{\mathsf{W}}$
2. iterate with T to obtain $T^k v \approx v^*$ (k large)

Optimal Policies

Recall: The optimal decision facing current offer w is

$$\mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\} \quad \text{where } h^* := c + \beta \sum_{w'} v^*(w')\varphi(w')$$

Let's try to write this in the language of dynamic programming

Dynamic programming centers around the problem of finding optimal **policies**

In general, for a dynamic program, choices = sequence $(A_t)_{t \geqslant 0}$

- specifies how the agent acts at each t

Agents are not clairvoyant: A_t cannot depend on future events

Hence, for some function σ_t,

$$A_t = \sigma_t(X_t, A_{t-1}, X_{t-1}, A_{t-2}, X_{t-2}, \ldots A_0, X_0)$$

In dynamic programming, σ_t is called a **policy function**

Key idea Design the state such that X_t is

- sufficient to determine the optimal current action
- but not so large as to be unmanagable

Finding the state is an art!

Example. Recall retailer who chooses stock orders and prices in each period

What to include in the current state?

- level of current inventories
- interest rates and inflation?
- competitors prices?

So suppose state X_t determines the current action A_t

Then we can write $A_t = \sigma(X_t)$ for some function σ

Note that we dropped the time subscript on σ

No loss of generality: can include time in the current state

- i.e., expand X_t to $\hat{X}_t = (t, X_t)$

Depends on the problem at hand

- For the job search model with finite horizon, the date matters
- For the infinite horizon version of the problem, however, the agent always looks forward toward an infinite horizon

For job search model,

- state $=$ current wage offer and
- possible actions are accept (1) or reject (0)

A policy is a map σ from W to $\{0, 1\}$

Let Σ be the set of all such maps

For each $v \in \mathbb{R}^{\mathsf{W}}$, let us define a **$v$-greedy policy** to be a $\sigma \in \Sigma$ satisfying

$$\sigma(w) = \mathbb{1}\left\{ \frac{w}{1-\beta} \geqslant c + \beta \sum_{w' \in \mathsf{W}} v(w')\varphi(w') \right\} \quad \text{for all } w \in \mathsf{W}$$

Accepts iff $w/(1-\beta) \geqslant$ continuation value computed using v

Optimal choice:

- agent should adopt a v^*-greedy policy
- Sometimes called **Bellman's principle of optimality**

We can also express a v^*-greedy policy via

$$\sigma^*(w) = \mathbb{1}\{w \geqslant w^*\} \quad \text{where} \quad w^* := (1-\beta)h^* \tag{2}$$

The term w^* in (2) is called the **reservation wage**

- Same ideas as before, different language
- We prove optimality more carefully later

Computation

Since T is globally stable on $\mathbb{R}^{\mathsf{W}}$, we can compute an approximate optimal policy by

1. applying successive approximation on T to compute v^*
2. calculate a v^*-greedy policy

In dynamic programming, this approach is called **value function iteration**

input $v_0 \in \mathbb{R}^{\mathsf{W}}$, an initial guess of v^*
input τ, a tolerance level for error
$\varepsilon \leftarrow \tau + 1$
$k \leftarrow 0$
while $\varepsilon > \tau$ **do**
    for $w \in \mathsf{W}$ **do**
        $v_{k+1}(w) \leftarrow (Tv_k)(w)$
    end
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
Compute a v_k-greedy policy σ
return σ

```julia
include("two_period_job_search.jl")
include("s_approx.jl")

" The Bellman operator. "
function T(v, model)
    (; n, w_vals, ϕ, β, c) = model
    return [max(w / (1 - β), c + β * v'ϕ) for w in w_vals]
end

" Get a v-greedy policy. "
function get_greedy(v, model)
    (; n, w_vals, ϕ, β, c) = model
    σ = w_vals ./ (1 - β) .>= c .+ β * v'ϕ  # Boolean policy vector
    return σ
end

" Solve the infinite-horizon IID job search model by VFI. "
function vfi(model=default_model)
    (; n, w_vals, ϕ, β, c) = model
    v_init = zero(model.w_vals)
    v_star = successive_approx(v -> T(v, model), v_init)
    σ_star = get_greedy(v_star, model)
    return v_star, σ_star
end
```

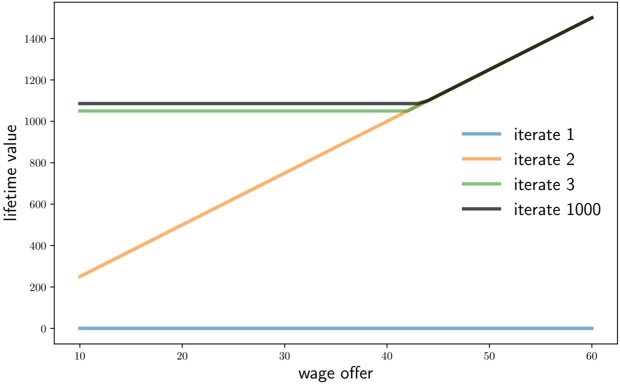


Figure: A sequence of iterates of the Bellman operator

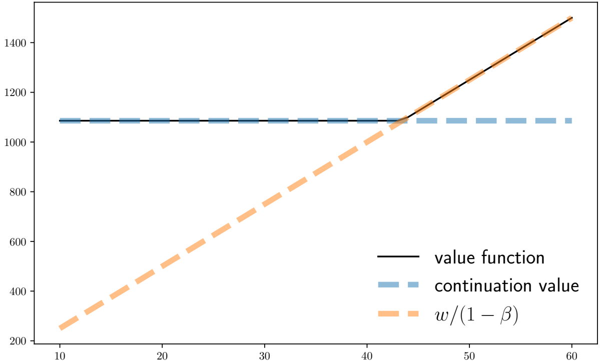


Figure: The approximate value function for job search

Computing the Continuation Value Directly

We used a standard dynamic programming approach to solve this problem

Sometimes we can find more efficient ways to solve particular problems

For the infinite horizon job search problem, a more efficient way exists

The idea is to compute the continuation value directly

This shifts the problem from n-dimensional to one-dimensional

Method: Recall that

$$v^*(w) = \max\left\{\frac{w}{1-\beta},\, c + \beta \sum_{w'} v^*(w')\varphi(w')\right\} \qquad (w \in \mathsf{W})$$

Using the definition of h^*, we can write

$$v^*(w') = \max\left\{w'/(1-\beta),\, h^*\right\} \qquad (w' \in \mathsf{W})$$

Take expectations, multiply by β and add c to obtain

$$h^* = c + \beta \sum_{w'} \max\left\{\frac{w'}{1-\beta},\, h^*\right\} \varphi(w')$$

How to find h^* from the equation

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h^* \right\} \varphi(w') \tag{3}$$

We introduce the map $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h \right\} \varphi(w')$$

By construction, h^* solves (3) if and only if h^* is a fixed point of g

Ex. Show that g is a contraction map on $\mathbb{R}_+$

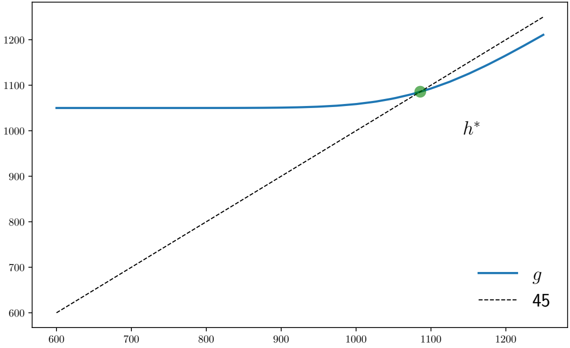


Figure: Computing the continuation value as the fixed point of g

New algorithm:

1. Compute h^* via successive approximation on g

    - Iteration in $\mathbb{R}$, not $\mathbb{R}^n$

2. Optimal policy is

$$\sigma^*(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\}$$