FINITE DIFFERENCES

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Finite Difference Operators

Consider a tabulated function $f(x)$ i.e. $(x_i, f(x_i))$ given in tabular form as

Table 2.1: Table of tabulated values of a function $y = f(x)$

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$y = f(x)$	1.5	1.8936	2.2616	2.6136	2.9936	3.5000	4.2456

If we let,	then	But if we let	then
$x_0 = 0.0$	$f_0 = 1.5$	$x_{-2} = 0.0$	$f_{-2} = 1.5$
$x_1 = 0.2$	$f_1 = 1.8936$	$x_{-1} = 0.2$	$f_{-1} = 1.8936$
$\vdots$	$\vdots$	$x_0 = 0.4$	$f_0 = 2.2616$
$x_6 = 1.2$	$f_6 = 4.2456$	$x_1 = 0.6$	$f_1 = 2.6136$
		$\vdots$	$\vdots$
		$x_4 = 1.2$	$f_4 = 4.2456$

x_i are called ***tabular points*** or ***base points*** or ***arguments***.

f_i are called ***entries*** or ***tabular values***.

$$i = \cdots\cdots - 2, -1, 0, 1, 2, \cdots\cdots$$

$x_{i+1} - x_i = h$, is called the interval or step – length.

If we define the consecutive values of x to be $a, a+h, a+2h, a+3h, \cdots, a+nh$ differing by h, then the corresponding values of the function $f(x)$ are:

$$f(a), f(a+h), f(a+2h), f(a+3h), \cdots, f(a+nh)$$

We now introduce finite difference operators:

Note: An operator acts on a function to yield another function.

1) Identity Operator, *I*

Is defined as $If_i = f_i$.

The identity operator can also be defined by Δ^0, i.e. $\Delta^0 f(x) = f(x)$.

Δ^0 is the forward difference operator defined later in this section

e.g. in Table 2.1 above $If_2 = f_2$.

2) Shift Operator, *E* (Also called displacement operator)

The shift operator shifts the function value $f(x)$ to the nest higher value $f(x+h)$. E is defined as

$$Ef_i = f_{i+1}, \quad \text{or} \quad Ef(x) = f(x+h)$$

e.g. If $x_0 = 0.0$ in Table 2.1 above, then

$$Ef(x_0) = f(x_1) = 1.8936$$

Similarly, the inverse shift operator,

$$E^{-1}f(x) = f(x-h) \qquad or \qquad E^{-1}y_i = y_{i-h}$$

$$\therefore \qquad E^{-1}f(x_4) = f(x_3) = 2.6136$$

The second shift operator gives

$$E^2 f_i = E(Ef_i) = Ef_{i+1} = Ef_{i+2}$$

$$\therefore \qquad E^2 f_1 = E(Ef_i) = Ef_2 = f_3$$

In general then,

$$E^k f(x) = f(x + kh) \quad or \quad E^k y_i = y_{i+k}$$

where k is positive as well as negative rationals.

3) Forward difference Operator, Δ

Defined as

$$\Delta y_i = y_{i+1} - y_i \ \text{ or } \ \Delta f(x) = f(x + h) - f(x)$$

Δy_i is referred to as the ***first forward difference***.

i.e. $f(x_0 + h) - f(x_0) = \Delta f(x_0)$,

$f(x_0 + 2h) - f(x_0 + h) = \Delta f(x_0 + h) \equiv \Delta f(x_1)$,

$\vdots \qquad \vdots \qquad \vdots$

$f(x_0 + nh) - f(x_0 + (n-1)h) = \Delta f(x_0 + (n-1)h) \equiv \Delta f(x_{n-1})$

are called ***first forward differences***.

Illustration (Using Table 2.1)

If we let $x_0 = 0.0$, then $f(x_0) = 1.2$ and so on. Therefore,

$$\begin{aligned}\Delta f(x_0) &= f(x_0 + h) - f(x_0) = f(x_1) - f(x_0)\\ &= 1.8936 - 2.2616 = 0.368\end{aligned}$$

$$\Delta f(x_1) = f(x_2) - f(x_1) = 2.6136 - 2.2616 = 0.352$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta f(x_5) = f(x_6) - f(x_5) = 4.2456 - 3.50 = 0.7456$$

are the first forward differences of the data in the table.

The differences of the first forward differences are called ***second forward differences***, i.e.

$$\begin{aligned}\Delta^2 f(x) &= \Delta\big(\Delta f(x)\big)\\ &= \Delta\big(f(x+h) - f(x)\big)\\ &= \Delta f(x+h) - \Delta f(x)\\ &= \big(f(x+2h) - f(x+h)\big)\\ &\qquad - \big(f(x+h) - f(x)\big)\\ &= f(x+2h) - 2f(x+h) + f(x)\end{aligned}$$

$$\begin{aligned}\Delta^2 f_i &= \Delta(\Delta f_i)\\ &= \Delta(f_{i+1} - f_i)\\ &= \Delta f_{i+1} - \Delta f_i\\ &= (f_{i+2} - f_{i+1}) - (f_{i+1} - f_i)\\ &= f_{i+2} - 2f_{i+1} + f_i\end{aligned}$$

In the second differences, the index 2 does not denote power 2 but indicates the second difference and in general the k^{th} differences are denoted as

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$$

Forward difference layout/table

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
x_{-2}	f_{-2}					
		Δf_{-2}				
x_{-1}	f_{-1}		$\Delta^2 f_{-2}$			
		Δf_{-1}		$\Delta^3 f_{-2}$		
x_0	f_0		$\Delta^2 f_{-1}$		$\Delta^4 f_{-2}$	
		Δf_0		$\Delta^3 f_{-1}$		$\Delta^5 f_{-2}$
x_1	f_1		$\Delta^2 f_0$		$\Delta^4 f_{-1}$	
		Δf_1		$\Delta^3 f_0$		
x_2	f_2		$\Delta^2 f_1$			
		Δf_2				
x_3	f_3					

A forward difference table is also called a ***Diagonal difference table*** (The entries along a forward diagonal are the forward differences of various order at the corresponding points in the table).

Thus in general
$$\begin{aligned}\Delta^k f_i &= \Delta^{k-1}(\Delta f_i)\\ &= \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i\end{aligned}$$

The values of the forward differences can be displayed in a horizontal difference table.

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
x_{-2}	f_{-2}	Δf_{-2}	$\Delta^2 f_{-2}$	$\Delta^3 f_{-2}$	$\Delta^4 f_{-2}$	$\Delta^5 f_{-2}$
x_{-1}	f_{-1}	Δf_{-1}	$\Delta^2 f_{-1}$	$\Delta^3 f_{-1}$	$\Delta^4 f_{-1}$	
x_0	f_0	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$		
x_1	f_1	Δf_1	$\Delta^2 f_1$			
x_2	f_2	Δf_2				
x_3	f_3					

Activity 2.1

i) What does *leading term* and *leading differences* mean?

ii) Using Table 2.1, construct a forward difference table. Taking $x_0 = 0.0$, find $\Delta f(x_1)$, $\Delta^2 f(x_3)$, $\Delta^3 f(x_3)$.

iii) What do you notice from the difference table?

4) Backward difference Operator, ∇

Defined as

$$\nabla f_i = f_i - f_{i-1} \quad \text{or} \quad \nabla f(x) = f(x) - f(x-h)$$

Backward difference layout/table

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
x_{-2}	f_{-2}					
		∇f_{-1}				
x_{-1}	f_{-1}		$\nabla^2 f_0$			
		∇f_0		$\nabla^3 f_1$		
x_0	f_0		$\nabla^2 f_1$		$\nabla^4 f_2$	
		∇f_1		$\nabla^3 f_2$		$\nabla^5 f_3$
x_1	f_1		$\nabla^2 f_2$		$\nabla^4 f_3$	
		∇f_2		$\nabla^3 f_3$		
x_2	f_2		$\nabla^2 f_3$			
		∇f_3				
x_3	f_3					

The entries along the backward diagonal are the backward differences of various orders at the corresponding source point in the table.

Consider the point f_1, $\Delta f_0 = \nabla \mathrm{f}_1$, $\Delta^2 f_{-1} = \nabla^2 \mathrm{f}_1$, $\Delta^3 f_{-2} = \nabla^3 \mathrm{f}_1$, $\Delta^4 f_{-3} = \nabla^4 \mathrm{f}_1$

The values of the backward differences can be displayed in a horizontal difference table.

$\boldsymbol{x}$	$\boldsymbol{f}$	$\boldsymbol{\nabla f}$	$\boldsymbol{\nabla^2 f}$	$\boldsymbol{\nabla^3 f}$	$\boldsymbol{\nabla^4 f}$	$\boldsymbol{\nabla^5 f}$
$\boldsymbol{x_{-2}}$	f_{-2}					
$\boldsymbol{x_{-1}}$	f_{-1}	∇f_{-1}				
$\boldsymbol{x_0}$	f_0	∇f_0	$\nabla^2 f_0$			
$\boldsymbol{x_1}$	f_1	∇f_1	$\nabla^2 f_1$	$\nabla^3 f_1$		
$\boldsymbol{x_2}$	f_2	∇f_2	$\nabla^2 f_2$	$\nabla^3 f_2$	$\nabla^4 f_2$	
$\boldsymbol{x_3}$	f_3	∇f_3	$\nabla^2 f_3$	$\nabla^3 f_3$	$\nabla^4 f_3$	$\nabla^4 f_3$

Activity 2.2

i) Using Table 2.1 construct a backward difference table. Taking $x_0 = 0.0$ in difference table, determine (x_0), $\nabla^2 f(x_4)$. $\nabla^3 f(x_6)$.

ii) How do the backward and forward difference tables compare?

5) Central Difference Operator, δ

Is defined as

$$\delta f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \quad or \quad \delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$e.g.\ \delta f_1 = f_{1+\frac{1}{2}} - f_{1-\frac{1}{2}}$$

$$= f_{\frac{3}{2}} - f_{\frac{1}{2}}$$

Only the even central differences use the tabular point values (x_i, y_i).

6) Average Operator, μ

Defined as

$$\mu f_i = \frac{f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}}{2} \quad or \quad \mu f(x) = \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2}$$

Properties of the operator Δ.

1. If c is a constant then $\Delta c = 0$.
2. Δ is distributive , i.e. $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$.
3. If c is a constant, then $\Delta[cf(x)] = c\Delta f(x)$.
4. If m and n are positive integers then $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.
5. $\Delta[f_1(x) + f_2(x) + \cdots + f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \cdots + \Delta f_n(x)$
6. $\Delta[f(x)g(x)] = f(x)\Delta g(x) + g(x)\Delta f(x)$
7. $\Delta \frac{f(x)}{g(x)} = \frac{f(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

Error Propagation in a Difference Table.

Worked example (Relationship between difference operators)

1. Show that $\nabla = \mathrm{I} - \mathrm{E}^{-1}$.

   Solution

   By definition,

$$\begin{aligned} \nabla f(x) &= f(x) - f(x+h) \\ &= f(x) - Ef(x) \\ &= (I - E)f(x) \end{aligned}$$

$$\therefore \quad \nabla = \mathrm{I} - \mathrm{E}$$

2. Show that $\Delta = E - I$

   Solution

   By definition,

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ &= Ef(x) - If(x) \\ &= (E - I)f(x) \end{aligned}$$

$$\therefore \quad \Delta = \mathrm{E} - \mathrm{I}$$

3. If Δ and ∇ are the forward and backward differences of a function $f(x)$, show that $(\Delta - \nabla) = \Delta\nabla$.

   Solution

   By definition, $\Delta f(x) = f(x+h) - f(x)$ and $\nabla f(x) = f(x) - f(x-h)$

   where h is the interval of differencing,

   Therefore,

$$\begin{aligned} (\Delta - \nabla)f(x) &= \Delta f(x) - \nabla f(x) \\ &= \Delta f(x) - [f(x) - f(x-h)] \\ &= \Delta f(x) - \nabla f(x) \\ &= \Delta f(x) - \Delta f(x-h) \\ &= \Delta\big(f(x) - f(x-h)\big) \\ &= \Delta\nabla f(x) \end{aligned}$$

Exercises

1. Show that $E\nabla = \nabla E = \Delta$.
2. Prove that $(I + \Delta)(I - \nabla) = I$
3. Prove that $\Delta\nabla = \Delta - \nabla$.