$\dot{x}_1 = x_2$

$\dot{x}_2 = x_1 - x_2$

1a. Determine the state matrix A, where $\dot{x} = Ax$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

1b. Classify the equilibrium x=0 by examining the eigenvalues of A. Compute the eigenvalues by hand - you can check your answers with a computer/calculator

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

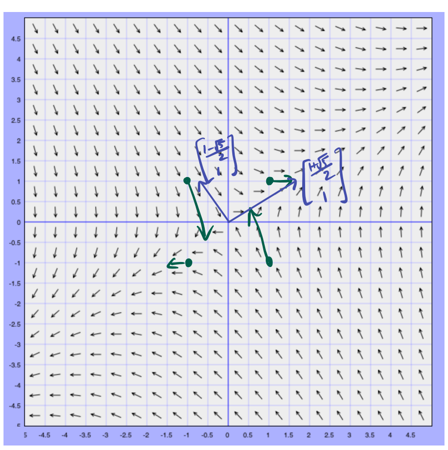
$$= (-\lambda)(-1-\lambda) - (1)(1)$$

$$= \lambda^2 + \lambda - 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

$\lambda_1 = \frac{-1-\sqrt{5}}{2} < 0 \longrightarrow$ saddle at x=0 $\longleftarrow \lambda_2 = \frac{-1+\sqrt{5}}{2} > 0$

1c. Plot phase line portraits. If eigenvectors real, draw on top of corresponding phase plane portrait. Pick 4 points and plot an arrow field in the direction of the vector field $f = (f_1, f_2)' = (\dot{x}_1, \dot{x}_2)$



$v_1: \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$ $\quad v_2: \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

v_1 corresponds to λ_1 and v_2 to λ_2

points: (-1, 1)

(1, 1)

(1, -1)

(-1, -1)

$\dot{x}_1 = x_2$

$\dot{x}_2 = -3x_1 - 4x_2$

2a. Determine the state matrix A, where $\dot{x} = Ax$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

2b. Classify the equilibrium $x=0$ by examining the eigenvalues of A. Compute the eigenvalues by hand - you can check your answers with a computer/calculator

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix}$$

$$= (-\lambda)(-4-\lambda) - (-3)(1)$$

$$= \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$\lambda_1 = -3 < 0$ $\qquad$ $\lambda_2 = -1 < 0$

stable node at $x=0$

2c. Plot phase line portraits. If eigenvectors real, draw on top of corresponding phase plane portrait. Pick 4 points and plot an arrow field in the direction of the vector field $f = (f_1, f_2)' = (\dot{x}_1, \dot{x}_2)$

$$\lambda_1 = -3 \Rightarrow \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \xrightarrow{R_1 = R_1/3} \begin{bmatrix} 1 & 1/3 \\ -3 & -1 \end{bmatrix}$$

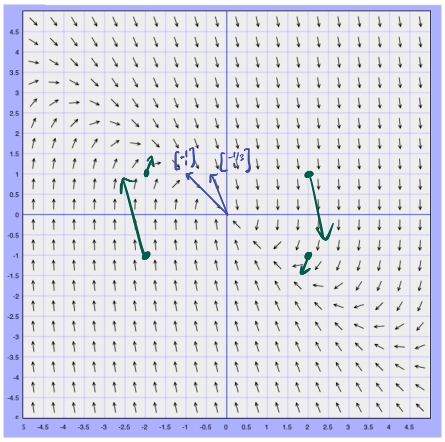
$$\xrightarrow{R_2 = R_2 + 3R_1} \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{take } x_2 = t \text{ then } x_1 = -t/3$$

so we get $\begin{bmatrix} -t/3 \\ t \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} t$ therefore $v_1 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

$$\lambda_1 = -1 \Rightarrow \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \xrightarrow{R_2 = R_2 + 3R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

take $x_2 = t$
then $x_1 = -t$

so we get $\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ therefore $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



$v_1 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

v_1 corresponds to λ_1 and v_2 to λ_2

points: $(-2, 1)$

$(2, 1)$

$(2, -1)$

$(-2, -1)$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -4x_1 - 2x_2$

3a. Determine the state matrix A, where $\dot{x} = Ax$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$$

3b. Classify the equilibrium $x=0$ by examining the eigenvalues of A. Compute the eigenvalues by hand - you can check your answers with a computer/calculator

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -4 & -2-\lambda \end{vmatrix}$$

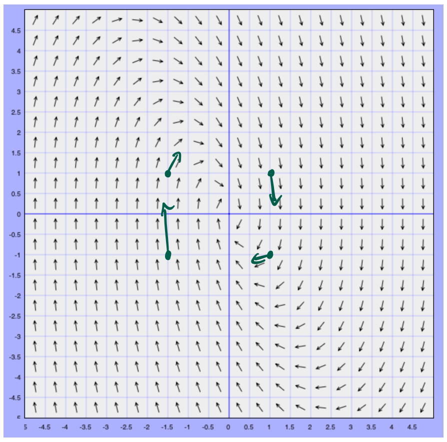
$$= (-\lambda)(-2-\lambda) - (-4)(1)$$

$$= \lambda^2 + 2\lambda + 4 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{-12}}{2} \Rightarrow \lambda_1 = -1 - \sqrt{3}\,i \quad \lambda_2 = -1 + \sqrt{3}\,i$$

stable spiral at $x=0$

3c. Plot phase line portraits. If eigenvectors real, draw on top of corresponding phase plane portrait. Pick 4 points and plot an arrow field in the direction of the vector field $f = (f_1, f_2)' = (\dot{x}_1, \dot{x}_2)$



$v_1: \begin{bmatrix} \frac{-1+\sqrt{3}i}{4} \\ 1 \end{bmatrix}$ $\quad v_2: \begin{bmatrix} \frac{-1-\sqrt{3}i}{4} \\ 1 \end{bmatrix}$ $\Rightarrow$ not real

v_1 corresponds to λ_1 and v_2 to λ_2

points: $(-1,1)$

$(1,1)$

$(1,-1)$

$(-1,-1)$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -2x_1 + 3x_2$

4a. Determine the state matrix A, where $\dot{x} = Ax$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

4b. Classify the equilibrium $x=0$ by examining the eigenvalues of A. Compute the eigenvalues by hand - you can check your answers with a computer/calculator

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= (-\lambda)(3-\lambda) - (-2)(1)$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

$$= (\lambda - 1)(\lambda - 2) = 0$$

$\lambda_1 = 1 \quad \lambda_2 = 2$

unstable node at $x=0$

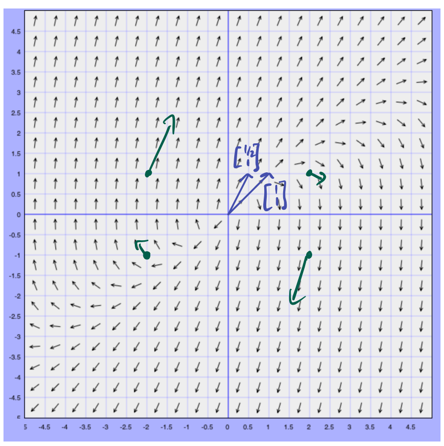
4c. Plot phase line portraits. If eigenvectors real, draw on top of corresponding phase plane portrait. Pick 4 points and plot an arrow field in the direction of the vector field $f=(f_1, f_2)' = (\dot{x}_1, \dot{x}_2)$

$$\lambda_1 = 1 \Rightarrow \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_1 = -R_1} \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{array}{l} \text{take } x_2 = t \\ \text{then } x_1 = t \end{array}$$

so we get $\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$ therefore $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\lambda_2 = 2 \Rightarrow \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \xrightarrow{R_1 = -\frac{1}{2}R_1} \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{array}{l} \text{take } x_2 = t \\ \text{then } x_1 = t/2 \end{array}$$

so we get $\begin{bmatrix} t/2 \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} t$ therefore $v_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$



$v_1 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad v_2 : \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

v_1 corresponds to λ_1 and v_2 to λ_2

points: $(-2, 1)$

$(2, 1)$

$(2, -1)$

$(-2, -1)$

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -2x_1 + x_2$$

5a. Determine the state matrix A, where $\dot{x} = Ax$

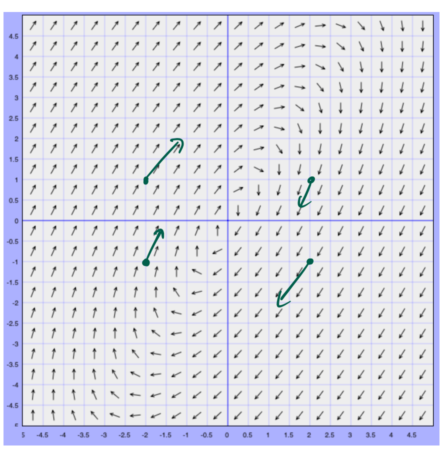
$$A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

5b. Classify the equilibrium $x=0$ by examining the eigenvalues of A. Compute the eigenvalues by hand - you can check your answers with a computer/calculator

$$|A - \lambda I| = \begin{vmatrix} -\lambda-1 & 1 \\ -2 & 1-\lambda \end{vmatrix}$$
$$= (-\lambda-1)(-\lambda+1) - (-2)(1)$$
$$= \lambda^2 - 1 + 2 = \lambda^2 + 1 = 0$$
$$\lambda_1 = -i \quad \lambda_2 = i$$

center at $x=0$

5c. Plot phase line portraits. If eigenvectors real, draw on top of corresponding phase plane portrait. Pick 4 points and plot an arrow field in the direction of the vector field $f = (f_1, f_2)' = (\dot{x}_1, \dot{x}_2)$



$$v_1: \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix} \qquad v_2: \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}$$

v_1 corresponds to λ_1 and v_2 to λ_2

points: $(-2, 1)$
$(2, 1)$
$(2, -1)$
$(-2, -1)$

6. Describe how the eigenvectors provide information about the shape of the phase plane portrait

When plotting a trajectory using eigenvectors we stay on that eigenvector and as $t \to \infty$ only the initial vector v_1 goes to $\binom{0}{0}$ and all others go to ∞.

$$\dot{x}_1 = x_1 x_2$$
$$\dot{x}_2 = -x_1^2 - x_2$$

7a. Identify the equilibrium point(s).

$\dot{x}_1 = x_1 x_2 = 0$ (1)

$x_1 = 0 \quad x_2 = 0$

$\dot{x}_2 = -x_1^2 - x_2 = 0 \Rightarrow x_2 = -x_1^2$

$x_2 = 0 \quad x_1 = 0$

So we find that eq. is at $(0,0)$

7b. Compute the linear approximation at each point

$$J = \begin{bmatrix} x_2, & x_1 \\ -2x_1, & -1 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0, & 0 \\ 0, & -1 \end{bmatrix}$$

$\lambda = 0, \quad \lambda = -1$

7c. Classify the behavior of the linearized system around each eq. pt.

We find that for eq. at $(0,0)$ we get $\lambda_1 = 0$ and $\lambda_2 = -1$ indicating no exponential stability

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 3x_2 + x_1^2 x_2$$

8a. Identify the equilibrium point(s).

$$\dot{x}_1 = x_2 = 0$$
$$x_2 = 0$$
$$\dot{x}_2 = -x_1 - 3x_2 + x_1^2 x_2 = 0$$

if $x_2 = 0$ then $x_1 = 0$ So we have eq. at $(0,0)$

8b. Compute the linear approximation at each point

$$J = \begin{bmatrix} 0 & , & 1 \\ 2x_1x_2 - 1 & , & x_1^2 - 3 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & , & 1 \\ -1 & , & -3 \end{bmatrix}$$

$$\lambda_1 = \frac{-3-\sqrt{5}}{2} \qquad \lambda_2 = \frac{-3+\sqrt{5}}{2}$$

8c. Classify the behavior of the linearized system around each eq. pt.

We find that for eq. at $(0,0)$ we get $\lambda_1 = \frac{-3-\sqrt{5}}{2}$ and $\lambda_2 = \frac{-3+\sqrt{5}}{2}$ indicating a node and locally exponential stability

$\dot{x}_1 = -x_1^3 - x_2$

$\dot{x}_2 = 2x_1 - x_2^3$

9a. Identify the equilibrium point(s).

$\dot{x}_1 = -x_1^3 - x_2 = 0$

$x_2 = -x_1^3 \qquad x_2 = 0, \; x_1 = 0$

$\dot{x}_2 = 2x_1 - x_2^3 = 0$

$x_1 = x_2^3/2 \qquad x_2 = 0, \; x_1 = 0$

So, we have eq. at $(0,0)$

9b. Compute the linear approximation at each point

$$J = \begin{bmatrix} -3x_1^2 & , & -1 \\ 2 & , & -3x_2^2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & , & -1 \\ 2 & , & 0 \end{bmatrix}$$

$\lambda_1 = -\sqrt{2}\,i \qquad \lambda_2 = \sqrt{2}\,i$

9c. Classify the behavior of the linearized system around each eq. pt.

We find that for eq. at $(0,0)$ we get $\lambda_1 = -\sqrt{2}i$ and $\lambda_2 = \sqrt{2}i$ indicating a center and no exponential stability

$\dot{x}_1 = -x_1 - x_2^3$

$\dot{x}_2 = -x_1^3 + x_2$

10a. Identify the equilibrium point(s).

$\dot{x}_1 = -x_1 - x_2^3 = 0$

$x_1 = -x_2^3$ so $x_1 = 0$, $x_2 = 0$

$\dot{x}_2 = -x_1^3 + x_2 = 0$

$x_2 = x_1^3$ so $x_2 = 0$, $x_1 = 0$

So, we have eq. at $(0,0)$

10b. Compute the linear approximation at each point

$$J = \begin{bmatrix} -1 & , & -3x_2^2 \\ -3x_1^2 & , & 1 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} -1 & , & 0 \\ 0 & , & 1 \end{bmatrix}$$

$\lambda_1 = 1 \quad \lambda_2 = -1$

10c. Classify the behavior of the linearized system around each eq. pt.

We find that for eq. at $(0,0)$ we get $\lambda_1 = 1$ and $\lambda_2 = -1$ indicating a saddle and unstability

$$\dot{x}_1 = -x_1(1 + x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2$$

11a. Identify the equilibrium point(s).

$$\dot{x}_1 = -x_1(1+x_2^2) = 0$$

$$-x_1 = 0 \Rightarrow x_1 = 0 \qquad 1+x_2^2 = 0 \Rightarrow x_2^2 = -1 \quad \cancel{x_2 = \pm i}$$

$$\dot{x}_2 = -x_1 + x_2 = 0$$

$$x_2 = x_1 \Rightarrow x_2 = 0$$

So, we have eq. at $(0,0)$

11b. Compute the linear approximation at each point

$$J = \begin{bmatrix} -x_2^2 - 1 & , & -2x_1x_2 \\ -1 & , & 1 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} -1 & , & 0 \\ -1 & , & 1 \end{bmatrix}$$

$\lambda_1 = -1 \qquad \lambda_2 = 1$

11c. Classify the behavior of the linearized system around each eq. pt.

We find that for eq. at $(0,0)$ we get $\lambda_1 = 1$ and $\lambda_2 = 1$ indicating a saddle and anstability

$\dot{x}_1 = -x_1(1+x_2^2)$

$\dot{x}_2 = -x_2 - x_1^2 x_2$

12. Using $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ specify whether eq. at $(0,0)$ is stable, asymptotically stable, globally asymptotically stable, or unstable.

$$\dot{V}(x_1, x_2) = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1(-x_1(1+x_2^2)) + x_2(-x_2 - x_1^2 x_2)$$
$$= x_1(-x_1 - x_1x_2^2) - x_2^2 - x_1^2x_2^2$$
$$= -x_1^2 - x_1^2x_2^2 - x_2^2 - x_1^2x_2^2$$
$$= -2x_1^2x_2^2 - x_2^2 - x_1^2$$

1. $V(0,0) = 0 + 0 = 0$
2. $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0$ for $x_1, x_2 \neq 0$
3. $\dot{V}(0,0) = 0 - 0 - 0 = 0$ at $x = x^*$
4. $\dot{V}(x_1, x_2) = -2x_1^2x_2^2 - x_2^2 - x_1^2 < 0$ for all $x \neq 0$

We find that $V(x)$ is positive-semidefinite, $\dot{V}(x_1, x_2)$ is negative-semidefinite, so $(0,0)$ is globally asymptotically stable since $V(x)$ is also radially unbounded.

$$\dot{x}_1 = x_1 x_2^2 - x_1^3$$

$$\dot{x}_2 = -x_1^2 x_2 - x_2^3$$

13. Using $V(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$ specify whether eq. at $(0,0)$ is stable, asymptotically stable, globally asymptotically stable, or unstable.

$$\begin{aligned}\dot{V}(x_1, x_2) &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1 (x_1 x_2^2 - x_1^3) + x_2(-x_1^2 x_2 - x_2^3) \\ &= x_1^2 x_2^2 - x_1^4 - x_1^2 x_2^2 - x_2^4 \\ &= -x_1^4 - x_2^4 - 2x_1^2 x_2^2\end{aligned}$$

1. $V(0,0) = 0$
2. $V(x) > 0$ for $x \neq 0$
3. $\dot{V}(0,0) = 0 - 0 - 0 = 0$ at $x = x^*$
4. $\dot{V}(x_1, x_2) = -x_1^4 - x_2^4 - 2x_1^2 x_2^2 < 0$ for all $x \neq 0$

We find that $V(x)$ is positive-semidefinite, $\dot{V}(x_1, x_2)$ is negative-semidefinite, so $(0,0)$ is globally asymptotically stable since $V(x)$ is also radially unbounded.

$\dot{x}_1 = x_1^2 x_2 + 2x_1 x_2^2 + x_1^3$

$\dot{x}_2 = -x_1^3 + x_2^3$

14. Using $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ specify whether eq. at $(0,0)$ is stable, asymptotically stable, globally asymptotically stable, or unstable.

$$\dot{V}(x_1, x_2) = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1(x_1^2x_2 + 2x_1x_2^2 + x_1^3) + x_2(-x_1^3 + x_2^3)$$
$$= \cancel{x_1^3x_2} + 2x_1^2x_2^2 + x_1^4 \cancel{- x_1^3x_2} + x_2^4$$
$$= x_1^4 + x_2^4 + 2x_1^2x_2^2$$

1. $V(0,0) = 0$
2. $V(x) > 0$ for $x \neq 0$
3. $\dot{V}(0,0) = 0 - 0 - 0 = 0$ at $x = x^*$
4. $\dot{V}(x_1, x_2) = x_1^4 + x_2^4 + 2x_1^2x_2^2 > 0$ for all $x \neq 0$

We find that $V(x)$ is positive-semidefinite, $\dot{V}(x_1, x_2)$ is positive-semidefinite, so $(0,0)$ is unstable.