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Transforms of Derivatives and integrals

Differential Equations

Theorem 1 [Laplace transform of the derivative of $f(t)$]

Suppose that $f(t)$ is continuous for all $t \geq 0$ and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geq 0$. Suppose $f(t)$ is of exponential order for $t > T$. Then Laplace transform of $f'(t)$ exists when $s > k$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad s > k.$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)\,dt$$

$$= e^{-st} \cdot f(t)\Big|_0^\infty - \int_0^\infty f(t) \cdot (-se^{-st})\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

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Theorem 2 Suppose that $f'(t)$ is continuous and $f''(t)$ is piecewise continuous for every finite interval in the range $t \geq 0$ and $f(t)$ & $f''(t)$ are of exponential order for $t > T$.

Then

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Sol:

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} \cdot f''(t)\,dt$$

$$= e^{-st} \cdot f'(t)\Big|_0^\infty - \int_0^\infty f'(t) \cdot (-se^{-st})\,dt$$

$$= \left[0 - f'(0)\right] + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s\left[s\mathcal{L}\{f(t)\} - f(0)\right]$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)}$$

$$\boxed{\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)}$$

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$$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots\cdots \underbrace{s^{n-n-1}}_{s} f^{n-2}(0) - \underbrace{s^{n-n}}_{1} f^{n-1}(0).$$

Ex 1 Find $\mathcal{L}\{f(t)\}$ $f(t)=t^2$

Sol.

$$\mathcal{L}\{f'(t)\} = s\underline{\mathcal{L}\{f(t)\}} - f(0)$$

$f(t)=t^2$ $f(0)=0$

$f'(t)=2t$

$$\mathcal{L}\{2t\} = s\mathcal{L}\{f(t)\} - 0$$

$$2\mathcal{L}\{t\} = s\mathcal{L}\{f(t)\}$$

$$\frac{2}{s^2} = s\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s^3}}$$

④

Ex 2 Initial value problem

Solve: $y'' - y = t \quad y(0) = 1 \quad y'(0) = 1$

Sol:

$$y'' - y = t$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0) - \mathcal{L}\{y\} = \frac{1}{s^2}$$

$$s^2 Y - s - 1 - Y = \frac{1}{s^2}$$

$$(s^2 - 1)Y = s + 1 + \frac{1}{s^2}$$

$$Y = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

$$Y = \frac{\cancel{s+1}}{\cancel{(s+1)}(s-1)} + \frac{1}{s^2(s+1)(s-1)}$$

$$Y = \frac{1}{s-1} + \frac{1}{s^2(s+1)(s-1)}$$

$$\mathcal{L}\{y\} = \frac{1}{s-1} + \left\{\frac{1}{s^2(s+1)(s-1)}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)(s-1)}\right\}$$

$$y = e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)(s-1)}\right\} \quad \text{—— ①}$$

Consider

$$\frac{1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$A = 0, \quad B = -1 \quad C = 1/2 \quad D = -1/2$$

$$\frac{1}{s^2(s+1)(s-1)} = \frac{-1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}$$

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$$y = e^t + \mathcal{L}^{-1}\left\{-\frac{1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\right\}$$

$$= e^t - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= e^t - t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} = e^t - t + \sinh t.$$

Ex3 Solve initial value problem

$$y'' + 2y' + y = e^{-t} \qquad y(0) = -1, \quad y'(0) = 1$$

Ex4 $f(t) = t^2$ Derive $\mathcal{L}\{f(t)\}$ from $\mathcal{L}\{1\}$.

Sol:

$f(t) = t^2$ $\quad f(0) = 0$

$f'(t) = 2t$ $\quad f'(0) = 0$

$f''(t) = 2$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{2\} = s^2\mathcal{L}\{f(t)\} - s(0) - 0$$

$$2\mathcal{L}\{1\} = s^2\mathcal{L}\{f(t)\}$$

$$2\,\frac{1}{s} = s^2\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s^3}}$$

Ex5 Find $\mathcal{L}\{\cos\omega t\}$.

Sol:

$f(t) = \cos\omega t$ $\quad f(0) = 1$

$f'(t) = -\omega\sin\omega t$ $\quad f'(0) = 0$

$f''(t) = -\omega^2\cos\omega t$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{-\omega^2\cos\omega t\} = s^2\mathcal{L}\{\cos\omega t\} - s(1) - 0$$

$$-\omega^2\mathcal{L}\{\cos\omega t\} = s^2\mathcal{L}\{\cos\omega t\} - s$$

$$s = (s^2 + \omega^2)\mathcal{L}\{\cos\omega t\}$$

$$\boxed{\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}}$$

Ex6 Let $f(t) = \sin^2 t$ find $\mathcal{L}\{f(t)\}$

Sol:

$f(t) = \sin^2 t$. $\quad f(0) = 0$

$f'(t) = 2\sin t\cos t = \sin 2t$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{\sin 2t\} = s\mathcal{L}\{\sin^2 t\} - 0$$

$$\frac{2}{s^2+4} = s\mathcal{L}\{\sin^2 t\}$$

$$\mathcal{L}\{\sin^2 t\} = \frac{2}{s(s^2+4)}$$

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Ex7 Let $f(t) = t\sin\omega t$ Find $\mathcal{L}\{f(t)\}$. $\mathcal{L}\{t\sin\omega t\}$

Sol:

$f(t) = t\sin\omega t$ $\qquad f(0) = 0$

$f'(t) = t\,\omega\cos\omega t + \sin\omega t \cdot 1$

$= \omega t\cos\omega t + \sin\omega t$ $\qquad f'(0) = 0$

$f''(t) = \omega\left[t(-\omega\sin\omega t) + \cos\omega t \cdot 1\right] + \omega\cos\omega t$

$= -\omega^2 t\sin\omega t + \omega\cos\omega t + \omega\cos\omega t$

$= -\omega^2 t\sin\omega t + 2\omega\cos\omega t$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$\mathcal{L}\{-\omega^2 t\sin\omega t + 2\omega\cos\omega t\} = s^2\mathcal{L}\{t\sin\omega t\}$

$-\omega^2\mathcal{L}\{t\sin\omega t\} + 2\omega\mathcal{L}\{\cos\omega t\} = s^2\mathcal{L}\{t\sin\omega t\}$

$2\omega\left(\frac{s}{s^2+\omega^2}\right) = (s^2+\omega^2)\,\mathcal{L}\{t\sin\omega t\}$

$$\boxed{\mathcal{L}\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}}$$

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Laplace transform of the integral of a function

Theorem: Let $F(s)$ be the Laplace transform of $f(t)$. If $f(t)$ is piecewise continuous for $t \geq 0$ and of exponential order then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s).$$

Proof:

We make use of the facts.

$$\int_a^a f(\tau)\,d\tau = 0 \quad \text{and} \quad \frac{d}{dt}\int_0^t f(\tau)\,d\tau = f(t).$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,d\tau\right)dt$$

$$= \int_0^t f(\tau)\,d\tau \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,\frac{d}{dt}\left[\int_0^t f(\tau)\,d\tau\right]dt$$

$$= (0-0) + \frac{1}{s}\int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$= \frac{1}{s}\,\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s}F(s)$$

Hence proved.

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$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau.}$$

Ex8 $\mathcal{L}\{f(t)\} = \underset{F(s)}{\frac{1}{s(s^2+\omega^2)}}$. find $f(t)$.

Sol.

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+\omega^2}\right\}$$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\}d\tau$$

$$= \int_0^t \frac{1}{\omega}\sin\omega\tau\,d\tau$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t.$$

$$= \frac{1}{\omega}\left|\frac{-\cos\omega\tau}{\omega}\right|_0^t$$

$$= -\frac{1}{\omega^2}\left[\cos\omega t - 1\right]$$

$$= \frac{1}{\omega^2}\left[1-\cos\omega t\right].$$

Ex9 $\mathcal{L}\{f(t)\} = \frac{1}{s^2(s^2+\omega^2)}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+\omega^2)}\right\}$$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\}d\tau \quad \text{—— (1)}$$

Ex9 $\mathcal{L}\{f(t)\} = \dfrac{1}{s^2(s^2+\omega^2)}$

Sol:

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \cdot \frac{1}{s^2+\omega^2} \right\}$$

$$= \int_0^t \int_0^t \left(\frac{1}{\omega} \sin\omega\tau \right) d\tau \, d\tau$$

$$= \frac{1}{\omega} \int_0^t \left| -\frac{\cos\omega\tau}{\omega} \right|_0^t d\tau$$

$$= -\frac{1}{\omega^2} \int_0^t (\cos\omega t - 1) \, d\tau$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2+\omega^2} \right\} = \frac{1}{\omega} \sin\omega t$$

$$= + \frac{1}{\omega^2} \int_0^t (1 - \cos\omega\tau) \, d\tau$$

$$= \frac{1}{\omega^2} \left| \tau - \frac{\sin\omega\tau}{\omega} \right|_0^t$$

$$= \frac{1}{\omega^2} \left[t - \frac{\sin\omega t}{\omega} - 0 \right]$$

$$= \frac{1}{\omega^2} \left[\frac{\omega t - \sin\omega t}{\omega} \right]$$

$$= \frac{1}{\omega^3} \left[\omega t - \sin\omega t \right].$$