Project

Q. a-f, Prove that

a. $\mathcal{L}\{t\cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$f(t) = t\cos\omega t$; $f(0) = 0$

$f'(t) = t(-\omega\sin\omega t) + \cos\omega t$; $f'(0) = 1$

$f''(t) = -\omega[t\cdot\omega\cos\omega t - \sin\omega t] - \omega\sin\omega t$

$= -\omega^2 t\cos\omega t - 2\omega\sin\omega t$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$\mathcal{L}\{-\omega^2 t\cos\omega t - 2\omega\sin\omega t\} = s^2\mathcal{L}\{t\cos\omega t\} - s(0) - 1$

$-\omega^2\mathcal{L}\{t\cos\omega t\} - 2\omega\mathcal{L}\{\sin\omega t\} = s^2\mathcal{L}\{t\cos\omega t\} - 1$

$1 - 2\omega\cdot\dfrac{\omega}{s^2+\omega^2} = (s^2+\omega^2)\mathcal{L}\{t\cos\omega t\}$

$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2} = \mathcal{L}\{t\cos\omega t\}$

Hence proved.

b. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega^3}[\sin\omega t - \omega t\cos\omega t]$

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{\omega^2}\mathcal{L}^{-1}\left\{\dfrac{\omega^2}{(s^2+\omega^2)^2}\right\}$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)^2}\right] = \frac{1}{w^2}\mathcal{L}^{-1}\left\{\frac{s^2-s^2+w^2}{(s^2+w^2)^2}\right\}$$

$$= \frac{1}{w^2}\mathcal{L}^{-1}\left\{\frac{s^2+w^2}{(s^2+w^2)^2}\right\} - \frac{1}{w^2}\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+w^2)^2}\right\}$$

$$= \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{1}{s^2+w^2}\right] - \frac{1}{w^2}\mathcal{L}^{-1}\left\{\frac{s^2+w^2-w^2}{(s^2+w^2)^2}\right\}$$

$$= \frac{1}{w^3}\mathcal{L}^{-1}\left[\frac{w}{s^2+w^2}\right] - \frac{1}{w^2}\mathcal{L}^{-1}\left\{\frac{s^2-w^2}{(s^2+w^2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s^2+w^2)^2}\right\}$$

$$2\mathcal{L}^{-1}\left[\frac{1}{s^2+w^2}\right] = \frac{1}{w^3}\sin wt - \frac{1}{w^2}t\cos wt$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+w^2}\right] = \frac{1}{2w^3}\left[\sin wt - wt\cos wt\right]$$

c. $$\mathcal{L}^{-1}\left[\frac{s}{(s^2+w^2)^2}\right] = \frac{1}{2w}t\sin wt$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+w^2)^2}\right\} = \frac{1}{2w}\mathcal{L}^{-1}\left\{\frac{2ws}{(s^2+w^2)}\right\}$$

$$= \frac{1}{2w}t\sin wt$$

Hence proved.

d). $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+w^2)^2}\right\} = \dfrac{1}{2w}\left[\sin wt + wt\cos wt\right]$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+w^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+w^2-w^2}{(s^2+w^2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s^2+w^2}{(s^2+w^2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{w^2}{(s^2+w^2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+w^2)}\right\} - \mathcal{L}^{-1}\left\{\frac{s^2-s^2+w^2}{(s^2+w^2)^2}\right\}$$

$$= \frac{1}{w}\mathcal{L}^{-1}\left\{\frac{w}{(s^2+w^2)}\right\} + \mathcal{L}^{-1}\left\{\frac{s^2-w^2}{(s^2+w^2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+w^2)^2}\right\}$$

$$2\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+w^2)^2}\right\} = \frac{1}{w}\sin wt + t\cos wt$$

$$= \frac{1}{2w}\left[\sin wt + t\cos wt\right]$$

e). $\mathcal{L}\{t\cosh at\} = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

$$\mathcal{L}\{t\cosh at\} = \mathcal{L}\left\{t\left(\frac{e^{at}}{2} + \frac{e^{-at}}{2}\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{te^{at}\} + \frac{1}{2}\mathcal{L}\{te^{-at}\}$$

$$= \frac{1}{2}\frac{1}{s^2}\Big|_{s\to s-a} + \frac{1}{2}\frac{1}{s^2}\Big|_{s\to s+a}$$

$$\mathcal{L}\{t\cosh at\} = \frac{1}{2}\cdot\frac{1}{(s-a)^2} + \frac{1}{2}\cdot\frac{1}{(s+a)^2}$$

$$= \frac{1}{2}\left[\frac{(s+a)^2+(s-a)^2}{(s-a)^2(s+a)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+a^2+2as+s^2+a^2-2as}{(s^2-a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s^2+2a^2}{(s^2-a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(s^2+a^2)}{(s^2-a^2)^2}\right]$$

$$= \frac{s^2+a^2}{(s^2-a^2)^2}$$

f). $\mathcal{L}\{t\sinh at\} = \dfrac{2as}{(s^2-a^2)^2}$

$$\mathcal{L}\{t\sinh at\} = \mathcal{L}\left\{t\cdot\left(\frac{e^{at}-e^{-at}}{2}\right)\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{te^{at}\} - \mathcal{L}\{te^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2}\Big|_{s\to s-a} - \frac{1}{s^2}\Big|_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)^2} - \frac{1}{(s+a)^2}\right]$$

$$\mathcal{L}\{t\sinh at\} = \frac{1}{2}\left[\frac{(s+a)^2-(s-a^2)}{(s-a)^2(s+a)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(s^2+a^2+2as-s^2-a^2+2as}{(s^2-a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{4as}{(s^2-a^2)^2}\right]$$

$$= \frac{2as}{(s^2-a^2)^2}$$

Hence proved.

Ex 5.7

Integral Equations: Using Laplace transforms. Solve the integral equations in Probs (27-33).

Show the detail of your work.

27 $y(t) = 1 + \int_0^t y(\tau)\,d\tau$

$$y(t) = 1 + \int_0^t 1 \cdot y(\tau)\,d\tau$$

$$y(t) = 1 + 1 * y(t)$$

$$Y(s) = \mathcal{L}\{1\} + \mathcal{L}\{1 * y(t)\}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s}Y(s)$$

$$\left(1 - \frac{1}{s}\right)Y(s) = \frac{1}{s}$$

$$\left(\frac{s-1}{s}\right) Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s-1}$$

$$y(t) = e^{t}$$

28. $y = 2t - 4\int_0^t y(\tau)(t-\tau)\,d\tau$

$$y = 2t - 4\int_0^t y(\tau)(t-\tau)\,d\tau$$

$$y = 2t - 4\,y(t) * t$$

$$Y = 2 \cdot \mathcal{L}\{t\} - 4\,\mathcal{L}\{y(t) * t\}$$

$$Y = \frac{2}{s^2} - 4Y \cdot \frac{1}{s^2}$$

$$\left(1 + \frac{4}{s^2}\right) Y = \frac{2}{s^2}$$

$$\frac{s^2+4}{s^2}\, Y = \frac{2}{s^2}$$

$$Y = \frac{2}{s^2+4}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \sin 2t$$

29. $y(t) = 1 - \int_0^t (t-\tau)\, y(\tau)\, d\tau$

$$y(t) = 1 - \int_0^t (t-\tau)\, y(\tau)\, d\tau$$

$$y(t) = 1 - t * y(t)$$

$$Y(s) = \mathcal{L}\{1\} - \mathcal{L}\{t * y(t)\}$$

$$= \frac{1}{s} - \frac{1}{s^2} \cdot Y(s)$$

$$\left(1 + \frac{1}{s^2}\right) Y(s) = \frac{1}{s}$$

$$\left(\frac{s^2+1}{s^2}\right) Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{s}{s^2+1} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\}$$

$$y(t) = \cos t$$

30. $y(t) = \sin 2t + \int_0^t y(\tau) \sin 2(t-\tau)\, d\tau$

$$y(t) = \sin 2t + \int_0^t y(\tau) \sin 2(t-\tau)\, d\tau$$

$$y(t) = \sin 2t + y(t) * \sin 2t$$

$$Y(s) = \mathcal{L}\{\sin 2t\} + \mathcal{L}\{y(t) * \sin 2t\}$$

$$Y(s) = \frac{2}{s^2+4} + Y(s) \frac{2}{s^2+4}$$

$$\left(1 - \frac{2}{s^2+4}\right) Y(s) = \frac{2}{s^2+4}$$

$$\left(\frac{s^2+4-2}{s^2+4}\right) Y(s) = \frac{2}{s^2+4}$$

$$Y(s) = \frac{2}{s^2+2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+2}\right\}$$

$$= \sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+\sqrt{2}^2}\right\}$$

$$y(t) = \sqrt{2}\sin\sqrt{2}t$$

31. $y(t) = te^t - 2e^t \int_0^t e^{-\tau} y(\tau)\, d\tau$

$$y(t) = te^t - 2e^t \int_0^t e^{t-t-\tau} y(\tau)\, d\tau$$

$$\cancel{Y(s)} = te^t - 2e^t \cdot e^{t-t} * y(t)$$

$$y(t) = te^t - 2e^t * y(t)$$

$$Y(s) = \mathcal{L}\{te^t\} - 2\mathcal{L}\{e^t * y(t)\}$$

$$Y(s) = \left.\frac{1}{s^2}\right|_{s \to s-1} - 2 \cdot \frac{1}{s-1} \cdot Y(s)$$

$$\left(1 + \frac{2}{s-1}\right) Y(s) = \frac{1}{(s-1)^2}$$

$$\left(\frac{s-1+2}{s-1}\right) Y(s) = \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{1}{(s+1)(s-1)}$$

$$Y(s) = \frac{1}{s^2 - 1^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1^2}\right\}$$

$$y(t) = \sinh t$$

32. $y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

$$y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$$

$$y(t) = \sin t + y(t) * \sin t$$

$$Y(s) = \mathcal{L}\{\sin t\} + \mathcal{L}\{y(t) * \sin t\}$$

$$Y(s) = \frac{1}{s^2+1} + Y(s)\cdot\frac{1}{s^2+1}$$

$$\left(1 - \frac{1}{s^2+1}\right)Y(s) = \frac{1}{s^2+1}$$

$$\left(\frac{s^2+1-1}{s^2+1}\right)Y(s) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$y(t) = t$$

33. $y(t) = 1 - \sinh t + \int_0^t (1+\tau)\, y(t-\tau)\, d\tau$

$$y(t) = 1 - \sinh t + \int_0^t (1+\tau)\, y(t-\tau)\, d\tau$$

$$y(t) = 1 - \sinh t + (1+t) * y(t)$$

$$Y(s) = \mathcal{L}\{1\} - \mathcal{L}\{\sinh t\} + \mathcal{L}\{(1+t) * y(t)\}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s^2 - 1^2} + \left[\frac{1}{s} + \frac{1}{s^2}\right] Y(s)$$

$$\left(1 - \frac{1}{s} - \frac{1}{s^2}\right) Y(s) = \frac{s^2 - 1 - s}{s\cancel{(s+1)}(s^2-1)}$$

$$\left(\frac{s^2 - s - 1}{s^2}\right) Y(s) = \frac{s^2 - s - 1}{s(s^2-1)}$$

$$Y(s) = \frac{s}{s^2 - 1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 1^2}\right\}$$

$$y(t) = \cancel{s}\ \cosh t$$