6

Dirac's Delta function

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$$

26. $y'' + 16y = 4\,\delta(t-\pi)$, $y(0)=2$, $y'(0)=0$

Sol:

$$\mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = 4\mathcal{L}\{\delta(t-\pi)\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 16\mathcal{L}\{y\} = 4e^{-\pi s}$$

$$s^2Y - 2s - 0 + 16Y = 4e^{-\pi s}$$

$$(s^2+16)Y = 2s + 4e^{-\pi s}$$

$$Y = \frac{2s + 4e^{-\pi s}}{s^2+16}$$

$$\mathcal{L}\{y\} = \frac{2s}{s^2+16} + \frac{4e^{-\pi s}}{s^2+16}$$

$$y = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + \mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{4}{s^2+4^2}\right\}$$

$$y = 2\cos 4t + \sin 4(t-\pi)\cdot u(t-\pi)$$

$$= 2\cos 4t\, u(t) + \sin 4t\, u(t-\pi)$$

$$= \begin{cases} 2\cos 4t & 0 < t < \pi \\ 2\cos 4t + \sin 4t & t > \pi \end{cases}$$

5.3, questions, 26, 27, 28

Differentiation and Integration of Transform

Differentiation of Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d[F(s)]}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}(-t)\cdot f(t)\,dt$$

$$\frac{d}{ds}[F(s)] = -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$\frac{d}{ds}[F(s)] = -\mathcal{L}\{t f(t)\}$$

$$\boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds}[F(s)]} \implies$$

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

$$\mathcal{L}\{t^3 f(t)\} = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]}$$

$$\mathcal{L}\{t f(t)\} = -F'(s)$$

$$t f(t) = -\mathcal{L}^{-1}\{F'(s)\}$$

$$\boxed{f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}}$$

Ex1 (i) If $f(t)=\frac{1}{2\beta^3}(\sin\beta t-\beta t\cos\beta t)$

Then show that $\mathcal{L}\{f(t)\}=\frac{1}{(s^2+\beta^2)^2}$

Sol:

$$\mathcal{L}\{f(t)\}=\frac{1}{2\beta^3}\left[\mathcal{L}\{\sin\beta t\}-\beta\,\mathcal{L}\{t\cos\beta t\}\right]$$

$$=\frac{1}{2\beta^3}\cdot\frac{\beta}{(s^2+\beta^2)}-\frac{1}{2\beta^2}\left[-\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right)\right]$$

$$=\frac{1}{2\beta^2}\cdot\frac{1}{s^2+\beta^2}+\frac{1}{2\beta^2}\left[\frac{(s^2+\beta^2)\cdot 1-s(2s)}{(s^2+\beta^2)^2}\right]$$

$$=\frac{1}{2\beta^2}\left[\frac{1}{s^2+\beta^2}+\frac{\overbrace{s^2+\beta^2-2s^2}^{\beta^2-s^2}}{(s^2+\beta^2)^2}\right]$$

$$=\frac{1}{2\beta^2}\left[\frac{\cancel{s^2}+\beta^2+\beta^2-\cancel{s^2}}{(s^2+\beta^2)^2}\right]$$

$$=\frac{1}{(s^2+\beta^2)^2}$$

(ii) $\mathcal{L}\left\{\frac{t}{2\beta}\sin\beta t\right\}=\frac{s}{(s^2+\beta^2)^2}$

(iii) $\mathcal{L}\left\{\frac{1}{2\beta}(\sin\beta t+\beta t\cos\beta t)\right\}=\frac{s^2}{(s^2+\beta^2)^2}$

Integration of transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left\{ \int_0^\infty e^{-st} f(t)\,dt \right\} ds$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st}\,ds \right) f(t)\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{-t} \Big|_s^\infty f(t)\,dt$$

$$= \int_0^\infty \left[-\frac{1}{t}(0 - e^{-st}) \right] f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{f(t)}{t}\,dt$$

$$= \mathcal{L}\left\{ \frac{f(t)}{t} \right\}$$

$$\boxed{\int_s^\infty F(s)\,ds = \mathcal{L}\left\{ \frac{f(t)}{t} \right\}}$$

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left\{ \int_s^\infty F(s)\,ds \right\}$$

$$f(t) = t\,\mathcal{L}^{-1}\left\{ \int_s^\infty F(s)\,ds \right\}$$

$$\boxed{f(t) = t\,\mathcal{L}^{-1}\left\{ \int_s^\infty F(s)\,ds \right\}}$$

$$\boxed{f(t) = -\frac{1}{t}\,\mathcal{L}^{-1}\left\{ F'(s) \right\}}$$

Find inverse transform of

Ex2 $\ln\left(1+\frac{\omega^2}{s^2}\right)$

Sol:

$$F(s) = \ln\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$F(s) = \ln|s^2+\omega^2| - \ln s^2$$

$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$

$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$

$$= -\frac{2}{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} + \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= -\frac{2}{t}\cos\omega t + \frac{2}{t}\cdot 1$$

$$= -\frac{2}{t}\cos\omega t + \frac{2}{t}.$$

Ex3 Find inverse Laplace transform of

$$\left[\ln\left(1-\frac{a^2}{s^2}\right)\right]$$

Sol:

$$F(s) = \ln\left(\frac{s^2-a^2}{s^2}\right)$$

$$= \ln|s^2-a^2| - \ln|s^2|$$

Differential equations with variable co-efficients

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}\mathcal{L}\{y'\}$$

$$= -\frac{d}{ds}\left[s\mathcal{L}\{y\} - y(0)\right]$$

$$= -\frac{d}{ds}\left[sY - y(0)\right]$$

$$= -\frac{d(sY)}{ds} + \frac{d[y(0)]}{ds}$$

$$= -\left[s\frac{dY}{ds} + Y\cdot 1\right]$$

$$\boxed{\mathcal{L}\{ty'\} = -s\frac{dY}{ds} - Y}$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\mathcal{L}\{y''\}$$

$$= -\frac{d}{ds}\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right]$$

$$= -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right]$$

$$= -\frac{d(s^2Y)}{ds} + \frac{d[sy(0)]}{ds} + \frac{d[y'(0)]}{ds}$$

$$= -\left[s^2\frac{dY}{ds} + Y\cdot 2s\right] + y(0) + 0$$

$$\boxed{\mathcal{L}\{ty''\} = -s^2\frac{dY}{ds} - 2sY + y(0)}$$

Ex 5.4

a) te^t

$$\mathcal{L}\{te^t\} = -\frac{d}{ds}\mathcal{L}\{e^t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$= -\left[\frac{d}{ds}(s-1)^{-1}\right]$$

$$= -(-1)(s-1)^{-2}$$

$$= \frac{1}{(s-1)^2}$$

b) $t^2 \sin 2t$

Sol $\mathcal{L}\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{\sin 2t\}$

$$= \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= 2\frac{d^2}{ds^2}\left[(s^2+4)^{-1}\right]$$

$$= 2\frac{d}{ds}\left[-1(s^2+4)^{-2}(2s)\right]$$

$$= 2\frac{d}{ds}\left[\frac{-2s}{(s^2+4)^2}\right]$$

$$= -4\frac{d}{ds}\left(\frac{s}{(s^2+4)^2}\right)$$

$$= -4\left[\frac{(s^2+4)^2 \cdot 1 - s\,2(s^2+4)(2s)}{(s^2+4)^4}\right]$$

9) $\frac{1}{(s+3)^3}$

Sol: $F(s) = \frac{1}{(s+3)^3}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\}$

$= e^{-3t}\cdot \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

$= e^{-3t}\cdot \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$

$= \frac{e^{-3t}}{2} t^2$

$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$

2nd method:

$F(s) = (s+3)^{-3}$

$F'(s) = -3(s+3)^{-4} = -\frac{3}{(s+3)^4}$

$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{3}{(s+3)^4}\right\}$

$= +\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{3}{(s+3)^4}\right\}$

$= \frac{1}{2t}\mathcal{L}^{-1}\left\{\frac{3\times 2}{(s+3)^4}\right\}$

$= \frac{1}{2t}\mathcal{L}^{-1}\left\{\frac{3!}{(s+3)^4}\right\}$

$= \frac{1}{2t}\cdot e^{-3t}\cdot \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$

$= \frac{1}{2t} e^{-3t}\cdot t^3$

$= \frac{t^2 e^{-3t}}{2}$.

16) arc $\cot(\frac{s}{\pi})$

Sol: $F(s) = \cot^{-1}(\frac{s}{\pi})$

$F'(s) = \frac{1}{1+\frac{s^2}{\pi^2}}\cdot\frac{1}{\pi}$

$= \frac{\pi^2}{\pi^2+s^2}\cdot\frac{1}{\pi}$

$= \frac{\pi}{s^2+\pi^2}$

$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$

$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$

$= -\frac{1}{t}\sin \pi t$.