Ex # 5.7

System of Differential Equations.

1. $y_1' = -y_1 + y_2 \quad -(1)$ $\qquad y_1(0) = 1$

$y_2' = -y_1 - y_2 \quad -(2)$ $\qquad y_2(0) = 0$

eq 1 $\Rightarrow$

$\mathcal{L}\{y_1'\} = -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\}$

$s\mathcal{L}\{y_1\} - y_1(0) = -Y_1 + Y_2$

$sY_1 - 1 = -Y_1 + Y_2$

$(s+1)Y_1 - Y_2 - 1 = 0 \quad -(3)$

eq 2 $\Rightarrow$

$\mathcal{L}\{y_2'\} = -\mathcal{L}\{y_1\} - \mathcal{L}\{y_2\}$

$s\mathcal{L}\{y_2\} - y_2(0) = -Y_1 - Y_2$

$sY_2 - 0 = -Y_1 - Y_2$

$+Y_1 + (s+1)Y_2 = 0$ $\qquad (s+1)Y_1 - Y_2 - 1 = 0$

$+Y_1 + (s+1)Y_2 + 0 = 0 \quad -(4)$

$$\frac{Y_1}{s+1} = \frac{Y_2}{-(+1)} = \frac{1}{(s+1)^2+1}$$

$$Y_1 = \frac{s+1}{(s+1)^2+1}$$

$$\mathcal{L}\{y_1\} = \frac{s+1}{(s+1)^2+1}$$

$$y_1 = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\}$$

$$y_1 = e^{-t}\cos t$$

$$Y_2 = \frac{-1}{(s+1)^2+1}$$

$$\mathcal{L}\{y_2\} = \frac{-1}{(s+1)^2+1}$$

$$y_2 = -\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= -e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$= -e^{-t}\sin t$$

2. $y_1' = 6y_1 + 9y_2$ —(1) $\quad y_1(0) = -3$

$y_2' = y_1 + 6y_2$ —(2) $\quad y_2(0) = -3$

eq 1) ⇒

$$\mathcal{L}\{y_1'\} = 6\mathcal{L}\{y_1\} + 9\mathcal{L}\{y_2\}$$

$$s\mathcal{L}\{y_1\} - y_1(0) = 6Y_1 + 9Y_2$$

$$sY_1 + 3 = 6Y_1 + 9Y_2$$

$$(s-6)Y_1 - 9Y_2 + 3 = 0 \quad \text{—(3)}$$

eq 2) ⇒

$$\mathcal{L}\{y_2'\} = \mathcal{L}\{y_1\} + 6\mathcal{L}\{y_2\}$$

$$s\mathcal{L}\{y_2\} - y_2(0) = Y_1 + 6Y_2$$

$$sY_2 + 3 = Y_1 + 6Y_2 \qquad (s-6)Y_1 - 9Y_2 + 3 = 0$$

$$-Y_1 + (s-6)Y_2 + 3 = 0 \quad \text{—(4)}$$

$$\frac{Y_1}{-27-3(s-6)} = \frac{Y_2}{-[(s-6)\cdot 3+3]} = \frac{1}{(s-6)^2-9}$$

$$Y_1 = \frac{-3(s-6)-27}{(s-6)^2-(3)^2}$$

$$\mathcal{L}\{y_1\} = \frac{-3s+18-27}{(s-6)^2-(3)^2}$$

$$y_1 = \mathcal{L}^{-1}\left[\frac{-3(s-6)}{(s-6)^2-(3)^2} - \frac{27}{(s-6)^2-(3)^2}\right]$$

$$= -3\mathcal{L}^{-1}\left\{\frac{s-6}{(s-6)^2-(3)^2}\right\} - 9\mathcal{L}^{-1}\left[\frac{3}{(s-6)^2-(3)^2}\right]$$

$$= -3e^{6t}\mathcal{L}^{-1}\left\{\frac{s}{s^2-(3)^2}\right\} - 9e^{6t}\mathcal{L}^{-1}\left[\frac{3}{s^2-3^2}\right]$$

$$y_1 = -3e^{6t}\cosh 3t - 9e^{6t}\sinh 3t$$

$$Y_2 = \frac{-3(s-6)-3}{(s-6)^2-9}$$

$$\mathcal{L}\{y_2\} = \frac{-3(s-6)}{(s-6)^2-(3)^2} - \frac{3}{(s-6)^2-(3)^2}$$

$$y_2 = -3\mathcal{L}^{-1}\left[\frac{s-6}{(s-6)^2-3^2}\right] - \mathcal{L}^{-1}\left[\frac{3}{(s-6)^2-3^2}\right]$$

$$= -3e^{6t}\mathcal{L}^{-1}\left[\frac{s}{s^2-3^2}\right] - e^{6t}\mathcal{L}^{-1}\left[\frac{3}{s^2-3^2}\right]$$

$$= -3e^{6t}\cosh 3t - e^{6t}\sinh 3t$$

3. $y_1' = -y_1 + 4y_2$ — (1) $y_1(0) = 3$

$y_2' = 3y_1 - 2y_2$ — (2) $y_2(0) = 4$

$$\mathcal{L}\{y_1'\} = -\mathcal{L}\{y_1\} + 4\mathcal{L}\{y_2\}$$

$$s\mathcal{L}\{y_1\} - y_1(0) = -Y_1 + 4Y_2$$

$$sY_1 - 3 = -Y_1 + 4Y_2$$

$$(s+1)Y_1 - 4Y_2 - 3 = 0 \quad \text{— (3)}$$

$$\mathcal{L}\{y_2'\} = 3\mathcal{L}\{y_1\} + -2\mathcal{L}\{y_2\}$$

$$s\mathcal{L}\{y_2\} - y_2(0) = 3Y_1 - 2Y_2$$

$$sY_2 - 4 = 3Y_1 - 2Y_2$$

$$(s+1)Y_1 - 4Y_2 - 3 = 0 \quad \text{— (3)}$$

$$-3Y_1 + (s+2)Y_2 - 4 = 0 \quad \text{— (4)}$$

$$\frac{Y_1}{16+3(s+2)} = \frac{Y_2}{-[-4(s+1) - 9]} = \frac{1}{(s+1)(s+2)-12}$$

$$Y_1 = \frac{3s+22}{s^2+3s-10}$$

$$\mathcal{L}\{y_1\} = \frac{3s+22}{s^2+5s-2s-10}$$

$$y_1 = \mathcal{L}^{-1}\left[\frac{3s+22}{(s-2)(s+5)}\right] \quad \text{— (5)}$$

take

$$\frac{3s+22}{(s-2)(s+5)} = \frac{A}{s-2} + \frac{B}{s+5}$$

$$3s+22 = A(s+5) + B(s-2) \quad \text{— (a)}$$

put $s = +2$ in (a)

$$28 = 7A$$

$$\Rightarrow A = 4$$

put $s = -5$ in (a)

$$3(-5)+22 = B(-5-2)$$

$$\Rightarrow B = -1$$

eq (5) $\Rightarrow$

$$y_1 = \mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{1}{s+5}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= 4e^{2t} - e^{-5t}$$

$$Y_2 = \frac{4s+13}{(s-2)(s+5)}$$

$$\mathcal{L}\{y_2\} = \frac{4s+13}{(s-2)(s+5)}$$

$$y_2 = \mathcal{L}^{-1}\left\{\frac{4s+13}{(s-2)(s+5)}\right\} \quad -(6)$$

take

$$\frac{4s+13}{(s-2)(s+5)} = \frac{A}{s-2} + \frac{B}{s+5}$$

$$4s+13 = A(s+5) + B(s-2) \quad -(6)$$

put $s = -5$ in (6)

$$4(-5)+13 = B(-5-2)$$

$$-20+13 = -7B$$

$$\Rightarrow B = 1$$

put $s = 2$ in (6)

$$21 = 7A$$

$$A = 3$$

eq 6 $\Rightarrow$

$$y_2 = \mathcal{L}^{-1}\left\{\frac{3}{s-2} + \frac{1}{s+5}\right\}$$

$$y_2 = 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$y_2 = 3e^{2t} + e^{-5t}$$

4. $y_1' = 5y_1 + y_2 \quad -(1) \qquad y_1(0) = -3$

$y_2' = y_1 + 5y_2 \quad -(2) \qquad y_2(0) = 7$

$$\mathcal{L}\{y_1'\} = 5\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\}$$

$$s\mathcal{L}\{y_1\} - y_1(0) = 5Y_1 + Y_2$$

$$sY_1 + 3 = 5Y_1 + Y_2$$

$$(s-5)Y_1 - Y_2 + 3 = 0 \quad -(3)$$

$$\mathcal{L}\{y_2'\} = \mathcal{L}\{y_1\} + 5\mathcal{L}\{y_2\}$$

$$s\mathcal{L}\{y_2\} - y_2(0) = Y_1 + 5Y_2$$

$$sY_2 - 7 = Y_1 + 5Y_2$$

$$-Y_1 + (s-5)Y_2 - 7 = 0 \quad -(4)$$

$$(s-5)Y_1 - Y_2 + 3 = 0 \quad -(3)$$
$$-Y_1 + (s-5)Y_2 - 7 = 0 \quad -(4)$$

$$\frac{Y_1}{(7-3(s-5))} = \frac{Y_2}{-[-7(s-5)+3]} = \frac{1}{(s-5)^2 \ldots}$$

$$Y_1 = \frac{7-3(s-5)}{(s-5)^2-1}$$

$$y_1 = \mathcal{L}^{-1}\left[\frac{7}{(s-5)^2-1} - \frac{3(s-5)}{(s-5)^2-1}\right]$$

$$= 7\mathcal{L}^{-1}\left[\frac{1}{(s-5)^2-1^2}\right] - 3\mathcal{L}^{-1}\left[\frac{s-5}{(s-5)^2-1^2}\right]$$

$$= 7e^{5t}\mathcal{L}^{-1}\left[\frac{1}{s^2-1^2}\right] - 3e^{5t}\mathcal{L}^{-1}\left[\frac{s}{s^2-1^2}\right]$$

$$y_1 = 7e^{5t}\sinh t - 3e^{5t}\cosh t$$

$$Y_2 = \frac{7(s-5)-3}{(s-5)^2-1^2}$$

$$y_2 = \mathcal{L}^{-1}\left[\frac{7(s-5)}{(s-5)^2-1^2} - \frac{3}{(s-5)^2-1^2}\right]$$

$$= 7\mathcal{L}^{-1}\left[\frac{(s-5)}{(s-5)^2-1^2}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{(s-5)^2-1^2}\right]$$

$$y_2 = 7e^{5t}\mathcal{L}^{-1}\left[\frac{s}{(s)^2-1^2}\right] - 3e^{5t}\mathcal{L}^{-1}\left[\frac{1}{s^2-1^2}\right]$$

$$= 7e^{5t}\cosh t - 3e^{5t}\sinh t$$

5. $y_1' + y_2 = 2\cos t \quad -(1) \qquad y_1(0) = 0$

$y_1 + y_2' = 0 \quad -(2) \qquad y_2(0) = 1$

eq (1) $\Rightarrow$

$$y_1' = 2\cos t - y_2$$
$$\mathcal{L}\{y_1'\} = 2\mathcal{L}\{\cos t\} - \mathcal{L}\{y_2\}$$
$$s\mathcal{L}\{y_1\} - y_1(0) = \frac{2s}{s^2+1} - Y_2$$
$$sY_1 - 0 + Y_2 - \frac{2s}{(s^2+1)} = 0$$
$$s(s^2+1)Y_1 + (s^2+1)Y_2 - 2s = 0 \quad -(3)$$

eq (2) $\Rightarrow$

$$y_2' = -y_1$$
$$\mathcal{L}\{y_2'\} = -\mathcal{L}\{y_1\}$$
$$s\mathcal{L}\{y_2\} - y_2(0) = -Y_1$$
$$+Y_1 + sY_2 - 1 = 0 \quad -(4)$$

$$\frac{Y_1}{-(s^2+1)+2s^2} = \frac{Y_2}{-[-s(s^2+1)+2s]} = \frac{1}{s^2(s^2+1) - (s^2+1)}$$

$$Y_1 = \frac{2s^2 - s^2 - 1}{s^4 + s^2 - s^2 - 1}$$

$$y_1 = \mathcal{L}^{-1}\left\{\frac{s^2-1}{s^4-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s^2-1}{(s^2+1)(s^2-1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y_1 = \sin t$$

$$Y_2 = \frac{s(s^2+1) - 2s}{s^4 + 2s^2 - s^2 - 1}$$

$$\mathcal{L}\{y_2\} = \frac{s^3 + s - 2s}{s^4 - 1}$$

$$y_2 = \mathcal{L}^{-1}\left\{\frac{s(s^2-1)}{(s^2+1)(s^2-1)}\right\}$$

$$y_2 = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \cos t$$