CHAPTER 5

Laplace Transforms

The Laplace transform method solves differential equations and corresponding initial and boundary value problems. The process of solution consists of three main steps:

1st step. The given "hard" problem is transformed into a "simple" equation **(subsidiary equation).**

2nd step. The subsidiary equation is solved by purely algebraic manipulations.

3rd step. The solution of the subsidiary equation is transformed back to obtain the solution of the given problem.

In this way Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. This process is made easier by tables of functions and their transforms, whose role is similar to that of integral tables in calculus. Such a table is included at the end of the chapter.

This switching from operations of calculus to *algebraic* operations on transforms is called **operational calculus,** a very important area of applied mathematics, and for the engineer, the Laplace transform method is practically the most important operational method. It is particularly useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function, not merely a sine or cosine. (For another operational method, the Fourier transform, see Sec. 10.10.)

The Laplace transform also has the advantage that it solves problems directly, initial value problems without first determining a general solution, and nonhomogeneous differential equations without first solving the corresponding homogeneous equation.

In this chapter we consider Laplace transforms from a practical point of view and illustrate their use by important engineering problems, many of them related to *ordinary* differential equations.

Partial differential equations can also be treated by Laplace transforms, as we show in Sec. 11.12.

Section 5.8 contains a list of general formulas and Sec. 5.9 a list of transforms $F(s)$ and corresponding functions $f(t)$.

Prerequisite for this chapter: Chap. 2.
Sections that may be omitted in a very short course: 5.4–5.6.
References: Appendix 1, Part A.
Answers to problems: Appendix 2.

5.1 Laplace Transform. Inverse Transform Linearity. Shifting

Let $f(t)$ be a given function that is defined for all $t \geqq 0$. We multiply $f(t)$ by e^{-st} and integrate with respect to t from zero to infinity. Then, if the resulting integral exists (that is, has some finite value), it is a function of s, say, $F(s)$:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

This function $F(s)$ of the variable s is called the **Laplace transform**[1] of the original function $f(t)$, and will be denoted by $\mathcal{L}(f)$. Thus

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

So remember: the original function f depends on t and the new function F (its transform) depends on s.

The operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform.**

Furthermore, the original function $f(t)$ in (1) is called the *inverse transform* or **inverse** of $F(s)$ and will be denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F).$$

Notation

Original functions are denoted by ***lowercase letters*** *and their transforms by the same letters in* ***capitals,*** so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

EXAMPLE 1 Laplace transform

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad (s > 0).$$

[1]PIERRE SIMON MARQUIS DE LAPLACE (1749—1827), great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions, and probability theory. Napoléon Bonaparte was his student for a year. For Laplace's interesting political involvements, see Ref. [2], p. 260, listed in Appendix 1.

The powerful practical Laplace transform techniques were developed over a century later by the English electrical engineer OLIVER HEAVISIDE (1850—1925) and were often called "Heaviside calculus."

We shall drop variables when this simplifies formulas without causing confusion. For instance, in (1) we wrote $\mathcal{L}(f)$ instead of $\mathcal{L}(f)(s)$.

Our notation is convenient, but we should say a word about it. The interval of integration in (1) is infinite. Such an integral is called an **improper integral** and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt.$$

Hence our convenient notation means

$$\int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \left[-\frac{1}{s}e^{-st}\right]_0^T = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}e^0\right] = \frac{1}{s} \qquad (s > 0).$$

We shall use that notation throughout this chapter. ◀

EXAMPLE 2 Laplace transform of the exponential function

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathcal{L}(f)$.

Solution. Again by (1),

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}\,dt = \frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty;$$

hence, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}. \qquad ◀$$

Must we go on in this fashion and obtain the transform of one function after another directly from the definition? The answer is no. And the reason is that the Laplace transform has many general properties that are helpful for that purpose. Above all, the Laplace transform is a "linear operation," just as differentiation and integration are. By this we mean the following.

THEOREM 1 (Linearity of the Laplace transform)

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b,

$$\boxed{\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.}$$

PROOF. By the definition,

$$\begin{aligned}\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st}[af(t) + bg(t)]\,dt \\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.\end{aligned}$$ ◀

EXAMPLE 3 An application of Theorem 1: Hyperbolic functions

Let $f(t) = \cosh at = \frac{1}{2}(e^{at} + e^{-at})$. Find $\mathcal{L}(f)$.

Solution. From Theorem 1 and Example 2 we obtain

$$\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right);$$

and by taking the common denominator, when $s > a$ ($\geqq 0$),

$$\mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}.$$

Similarly, for the transform of the **hyperbolic sine** we obtain

$$\mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2} \qquad (s > a \geqq 0).$$ ◀

EXAMPLE 4 Cosine and Sine

In Example 2, if we set $a = i\omega$ with $i = \sqrt{-1}$, we obtain

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{(s - i\omega)(s + i\omega)} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i\frac{\omega}{s^2 + \omega^2}.$$

On the other hand, by Theorem 1 and $e^{i\omega t} = \cos \omega t + i \sin \omega t$ (Sec. 2.3),

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos \omega t + i \sin \omega t) = \mathcal{L}(\cos \omega t) + i\,\mathcal{L}(\sin \omega t).$$

Equating the real and imaginary parts of these two equations, we obtain the transforms of cosine and sine,

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

We shall use these formulas frequently. You may derive them without going into complex by using the definition (1) and integration by parts. Another derivation follows in the next section. ◀

First Shifting Theorem: Replacement of s by $s - a$ in the Transform

The Laplace transform has the following very useful general property.

THEOREM 2 (First Shifting theorem)

If $f(t)$ has the transform $F(s)$ (where $s > k$), then $e^{at}f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s - a)}$$

or, if we take the inverse on both sides,

$$\boxed{e^{at}f(t) = \mathcal{L}^{-1}\{F(s - a)\}.}$$

PROOF. We obtain $F(s - a)$ by replacing s by $s - a$ in the integral in (1), so that we get

$$F(s - a) = \int_0^\infty e^{-(s-a)t}f(t)\,dt = \int_0^\infty e^{-st}[e^{at}f(t)]\,dt = \mathcal{L}\{e^{at}f(t)\}.$$

If $F(s)$ exists (i.e., is finite) for s greater than some k, then our first integral exists for $s - a > k$. Now take the inverse on both sides to obtain the second formula in the theorem. ◀

EXAMPLE 5 Damped vibrations

From Example 4 and the first shifting theorem we immediately obtain the very useful formulas

$$\mathcal{L}(e^{at}\cos\omega t) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$\mathcal{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}.$$

For negative a these $f(t)$ are damped vibrations. ◀

A Short List of Important Transforms

Table 5.1 gives a short list of transforms that are basic. From these transforms we can obtain nearly all the other transforms that we shall need in this chapter, through the use of some simple general theorems that we shall consider in the next sections.

A more extensive list of functions and transforms follows in Sec. 5.9. See also Refs. [A4] and [A6] in Appendix 1.

PROOFS. Formulas 1, 2, and 3 in Table 5.1 are special cases of formula 4. We prove formula 4 by induction. It is true for $n = 0$ because of Example 1 and $0! = 1$. We now make the induction hypothesis that it holds for any positive integer n. From (1) we get by integration by parts

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st}t^{n+1}\,dt = -\frac{1}{s}e^{-st}t^{n+1}\Big|_0^\infty + \frac{(n+1)}{s}\int_0^\infty e^{-st}t^n\,dt.$$

Table 5.1
Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

1	1	$1/s$	7	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

The integral-free part is zero at $t = 0$ and for $t \to \infty$. The right side equals $(n + 1)\mathcal{L}(t^n)/s$. From this and the induction hypothesis we obtain

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s}\mathcal{L}(t^n) = \frac{(n+1)n!}{s \cdot s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

This proves formula 4.

$\Gamma(a + 1)$ in formula 5 is the so-called *gamma function* [(15) in Sec. 4.5 or (24) in Appendix A3.1]. We get formula 5 from (1), setting $st = x$:

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st}t^a\,dt = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a\,dx$$

where $s > 0$. The last integral is precisely that defining $\Gamma(a + 1)$, so we have $\Gamma(a + 1)/s^{a+1}$, as claimed.

Note that $\Gamma(n + 1) = n!$ for nonnegative integer n, so that formula 4 also follows from 5.

Formulas 6–12 were derived in Examples 2–5. ◀

EXAMPLE 6 First shifting theorem

From Table 5.1 and the first shifting theorem we immediately obtain another useful formula,

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}.$$

For instance, $\mathcal{L}(te^{at}) = 1/(s - a)^2$. ◀

Existence of Laplace Transforms

This is not a big *practical* problem, because in most cases we can check a solution of a differential equation by substitution without too much trouble. Of course, the existence of a transform is not always guaranteed because in (1) we integrate over an infinite interval. For a fixed s the integral in (1) will exist if the whole integrand $e^{-st}f(t)$ goes to zero fast enough as $t \to \infty$, say, at least like an exponential function with a negative exponent. This implies that $f(t)$ itself should not grow faster than, say, e^{kt}, for instance not like e^{t^2}. This motivates the inequality (2) below.

The function $f(t)$ need not be continuous. Sufficient is so-called piecewise continuity (a concept of general mathematical interest). This is of practical importance because discontinuous inputs (driving forces, electromotive forces) are just those for which the Laplace transform method becomes particularly useful.

By definition, a function $f(t)$ is **piecewise continuous** on a finite interval $a \leqq t \leqq b$ if $f(t)$ is defined on that interval and is such that the interval can be subdivided into finitely many intervals in each of which $f(t)$ is continuous and has finite limits as t approaches either endpoint of the interval of subdivision from the interior.

It follows from this definition that finite jumps are the only discontinuities that a piecewise continuous function may have; these are known as *ordinary discontinuities.* Figure 107 on the next page shows an example. Clearly, the class of piecewise continuous functions includes every continuous function.

Fig. 107. Example of a piecewise continuous function $f(t)$
(The dots mark the function values at the jumps.)

THEOREM 3 (Existence theorem for Laplace transforms)

Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geqq 0$ and satisfies

$$|f(t)| \leqq Me^{-kt} \qquad \text{for all } t \geqq 0 \tag{2}$$

and for some constants k and M. Then the Laplace transform of $f(t)$ exists for all $s > k$.

PROOF. Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From (2), assuming that $s > k$, we obtain

$$|\mathcal{L}(f)| = \left|\int_0^\infty e^{-st}f(t)\,dt\right| \leqq \int_0^\infty |f(t)|e^{-st}\,dt \leqq \int_0^\infty Me^{kt}e^{-st}\,dt = \frac{M}{s-k}$$

where the condition $s > k$ was needed for the existence of the last integral. This completes the proof. ◀

The conditions in Theorem 3 are sufficient for most applications, and it is easy to find out whether a given function satisfies an inequality of the form (2). For example,

$$\cosh t < e^t, \qquad t^n < n!\,e^t \quad (n = 0, 1, \cdots) \qquad \text{for all } t > 0, \tag{3}$$

and any function that is bounded in absolute value for all $t \geqq 0$, such as the sine and cosine functions of a real variable, satisfies that condition. An example of a function that does not satisfy a relation of the form (2) is the exponential function e^{t^2}, because, no matter how large we choose M and k in (2),

$$e^{t^2} > Me^{kt} \qquad \text{for all } t > t_0$$

where t_0 is a sufficiently large number, depending on M and k.

It should be noted that the conditions in Theorem 3 are sufficient rather than necessary. For example, the function $1/\sqrt{t}$ is infinite at $t = 0$, but its transform exists; in fact, from the definition and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [see (30) in Appendix A3.1] we obtain, setting $st = x$,

$$\mathcal{L}(t^{-1/2}) = \int_0^\infty e^{-st}t^{-1/2}\,dt = \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-1/2}\,dx = \frac{1}{\sqrt{s}}\,\Gamma(\tfrac{1}{2}) = \sqrt{\frac{\pi}{s}}.$$

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length,

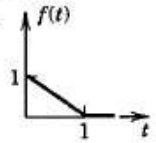
although they may differ at various isolated points (see Ref. [A8] in Appendix 1). Since this is of no importance in applications, we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical. Of course, this *is* of practical importance. Why? (Remember the introduction to the chapter.)

PROBLEM SET 5.1

Laplace Transforms. Find the Laplace transforms of the following functions. Show the details of your work. (a, b, c, ω, δ are constant.)

1. $2t + 6$ **2.** $a + bt + ct^2$ **3.** $\sin \pi t$ **4.** $\cos^2 \omega t$

5. e^{a-bt} **6.** $e^t \cosh 3t$ **7.** $\sin(\omega t + \delta)$ **8.** $\sin 2t \cos 2t$

9.


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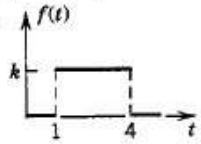

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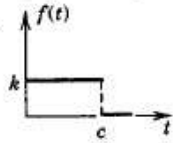

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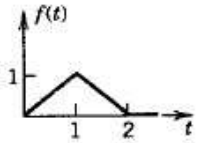

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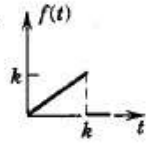

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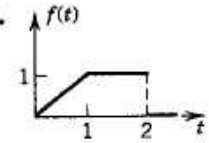

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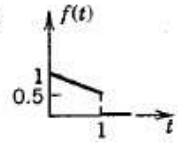

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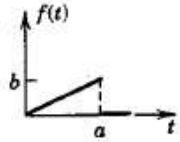


Inverse Laplace Transforms. Given $F(s) = \mathcal{L}(f)$, find $f(t)$. Show the details. (L, n, etc. are constant.)

17. $\dfrac{0.1s + 0.9}{s^2 + 3.24}$ **18.** $\dfrac{5s}{s^2 - 25}$ **19.** $\dfrac{-s - 10}{s^2 - s - 2}$

20. $\dfrac{s - 4}{s^2 - 4}$ **21.** $\dfrac{2.4}{s^4} - \dfrac{228}{s^6}$ **22.** $\dfrac{60 + 6s^2 + s^4}{s^7}$

23. $\dfrac{s}{L^2 s^2 + n^2 \pi^2}$ **24.** $\dfrac{1 - 7s}{(s - 3)(s - 1)(s + 2)}$ **25.** $\displaystyle\sum_{k=1}^{5} \frac{a_k}{s + k^2}$

26. $\dfrac{s^4 + 6s - 18}{s^5 - 3s^4}$ **27.** $\dfrac{1}{(s + \sqrt{2})(s - \sqrt{3})}$ **28.** $\dfrac{2s^3}{s^4 - 1}$

Applications of the First Shifting Theorem

Find the Laplace transform. (Show the details.)

29. $t^2 e^{-3t}$ **30.** $e^{-\alpha t} \cos \beta t$ **31.** $5e^{2t} \sinh 2t$

32. $2e^{-t} \cos^2 \frac{1}{2} t$ **33.** $\sinh t \cos t$ **34.** $(t + 1)^2 e^t$

Find the inverse transform. (Show the details.)

35. $\dfrac{1}{(s + 1)^2}$ **36.** $\dfrac{12}{(s - 3)^4}$ **37.** $\dfrac{3}{s^2 + 6s + 18}$

38. $\dfrac{4}{s^2 - 2s - 3}$ **39.** $\dfrac{s}{(s + \frac{1}{2})^2 + 1}$ **40.** $\dfrac{2}{s^2 + s + \frac{1}{2}}$

41. **(Growth)** Prove (3).
42. **(Inverse transform)** Prove that $\mathcal{L}^{-1}$ is linear. *Hint.* Use the fact that $\mathcal{L}$ is linear.
43. **(Inverse transform)** Rewrite Table 5.1, using $\mathcal{L}^{-1}$ (e.g., $\mathcal{L}^{-1}(1/s^3) = t^2/2$).
44. **(Replacement of t by ct)** If $\mathcal{L}(f(t)) = F(t)$ and c is any positive constant, show that $\mathcal{L}(f(ct)) = F(s/c)/c$. [*Hint.* Use (1).] Use this to obtain $\mathcal{L}(\cos \omega t)$ from $\mathcal{L}(\cos t)$.
45. **(Nonexistence)** Give simple examples of functions that have no Laplace transform. Indicate the reason.

5.2 Transforms of Derivatives and Integrals Differential Equations

The Laplace transform is a method of solving differential equations. The crucial idea is that ***the Laplace transform replaces operations of calculus by operations of algebra on transforms.*** Roughly, *differentiation* of $f(t)$ is replaced by *multiplication* of $\mathcal{L}(s)$ by s (see Theorems 1 and 2). *Integration* of $f(t)$ is replaced by *division* of $\mathcal{L}(f)$ by s (see Theorem 3).

THEOREM 1 [Laplace transform of the derivative of $f(t)$]

Suppose that $f(t)$ is continuous for all $t \geqq 0$, satisfies (2), Sec. 5.1, for some k and M, and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geqq 0$. Then the Laplace transform of the derivative $f'(t)$ exists when $s > k$, and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \qquad (s > k). \tag{1}$$

PROOF. We first consider the case when $f'(t)$ is continuous for all $t \geqq 0$. Then, by the definition and by integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt.$$

Since f satisfies (2), Sec. 5.1, the integrated portion on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$, the existence for $s > k$ being a consequence of Theorem 3 in Sec. 5.1. This proves that the expression on the right exists when $s > k$ and is equal to $-f(0) + s\mathcal{L}(f)$. Consequently, $\mathcal{L}(f')$ exists when $s > k$, and (1) holds.

If the derivative $f'(t)$ is merely piecewise continuous, the proof is quite similar. In this case, the range of integration in the original integral must be broken up into parts such that f' is continuous in each such part. ◀

REMARK This theorem may be extended to piecewise continuous functions $f(t)$, but in place of (1) we then obtain the formula (1*) in Project 10 in the problem set of this section.

By applying (1) to the second derivative $f''(t)$ we obtain

$$\begin{aligned}\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0)\\ &= s[s\mathcal{L}(f) - f(0)] - f'(0);\end{aligned}$$

that is,

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Similarly,

$$\mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2f(0) - sf'(0) - f''(0), \tag{3}$$

etc. By induction we thus obtain the following extension of Theorem 1.

THEOREM 2 (Laplace transform of the derivative of any order *n*)

Let $f(t)$ and its derivatives $f'(t), f''(t), \cdots, f^{(n-1)}(t)$ be continuous functions for all $t \geqq 0$, satisfying (2), Sec. 5.1, for some k and M, and let the derivative $f^{(n)}(t)$ be piecewise continuous on every finite interval in the range $t \geqq 0$. Then the Laplace transform of $f^{(n)}(t)$ exists when $s > k$ and is given by

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \tag{4}$$

EXAMPLE 1 Let $f(t) = t^2$. Derive $\mathcal{L}(f)$ from $\mathcal{L}(1)$.

Solution. Since $f(0) = 0$, $f'(0) = 0$, $f''(t) = 2$, and $\mathcal{L}(2) = 2\mathcal{L}(1) = 2/s$, we obtain from (2)

$$\mathcal{L}(f'') = \mathcal{L}(2) = \frac{2}{s} = s^2\mathcal{L}(f), \qquad \text{hence} \qquad \mathcal{L}(t^2) = \frac{2}{s^3},$$

in agreement with Table 5.1. The example is typical: it illustrates that in general there are several ways of obtaining the transforms of given functions. ◀

EXAMPLE 2 Derive the Laplace transform of $\cos \omega t$.

Solution. Let $f(t) = \cos \omega t$. Then $f''(t) = -\omega^2 \cos \omega t = -\omega^2 f(t)$. Also $f(0) = 1$, $f'(0) = 0$. Now we take the transform, $\mathcal{L}(f'') = -\omega^2\mathcal{L}(f)$. From this and (2),

$$-\omega^2\mathcal{L}(f) = \mathcal{L}(f'') = s^2\mathcal{L}(f) - s, \qquad \text{hence} \qquad \mathcal{L}(f) = \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Can you obtain $\mathcal{L}(\sin \omega t)$ by the present method? ◀

EXAMPLE 3 Let $f(t) = \sin^2 t$. Find $\mathcal{L}(f)$.

Solution. We have $f(0) = 0$, $f'(t) = 2 \sin t \cos t = \sin 2t$, and (1) gives

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2 + 4} = s\mathcal{L}(f) \qquad \text{or} \qquad \mathcal{L}(\sin^2 t) = \frac{2}{s(s^2 + 4)}.$$ ◀

EXAMPLE 4 Let $f(t) = t \sin \omega t$. Find $\mathcal{L}(f)$.

Solution. We have $f(0) = 0$ and

$$\begin{aligned} f'(t) &= \sin \omega t + \omega t \cos \omega t, \qquad f'(0) = 0, \\ f''(t) &= 2\omega \cos \omega t - \omega^2 t \sin \omega t \\ &= 2\omega \cos \omega t - \omega^2 f(t), \end{aligned}$$

so that by (2),

$$\mathcal{L}(f'') = 2\omega\mathcal{L}(\cos \omega t) - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f).$$

Using the formula for the Laplace transform of $\cos \omega t$, we thus obtain

$$(s^2 + \omega^2)\mathcal{L}(f) = 2\omega\mathcal{L}(\cos \omega t) = \frac{2\omega s}{s^2 + \omega^2}.$$

Hence the result is

$$\mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}.$$

◀

Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solves differential equations. We begin with an initial value problem

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1 \tag{5}$$

with constant a and b. Here $r(t)$ is the **input** (driving force) applied to the mechanical system and $y(t)$ is the **output** (response of the system). In Laplace's method we do three steps:

1st Step. We transform (5) by means of (1) and (2), writing $Y = \mathcal{L}(y)$ and $R = \mathcal{L}(r)$. This gives

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s).$$

This is called the **subsidiary equation.** Collecting Y-terms, we have

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

2nd Step. We solve the subsidiary equation ***algebraically*** for Y. Division by $s^2 + as + b$ and use of the so-called **transfer function**[2]

$$\boxed{Q(s) = \frac{1}{s^2 + as + b}} \tag{6}$$

gives the solution

$$\boxed{Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).} \tag{7}$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; thus Q is the quotient

$$\boxed{Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}}$$

and this explains the name of Q. Note that Q depends only on a and b, but neither on $r(t)$ nor on the initial conditions.

3rd Step. We reduce (7) (usually by partial fractions, as in calculus) to a sum of terms whose inverses can be found from the table, so that the solution $y(t) = \mathcal{L}^{-1}(Y)$ of (5) is obtained.

[2]Often denoted by H, but we need H much more frequently for other purposes.

EXAMPLE 5 Initial value problem: Explanation of the basic steps

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution. 1st Step. From (2) and Table 5.1 we get the subsidiary equation

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

2nd Step. The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s + 1)Q + \frac{1}{s^2}Q = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}$$

$$= \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

3rd Step. From this expression for Y and Table 5.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

The diagram in Fig. 108 summarizes our approach. ◀

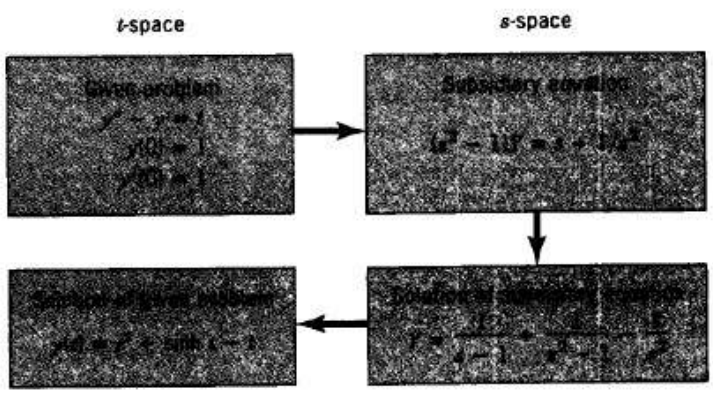


Fig. 108. Laplace transform method

EXAMPLE 6 Comparison with the usual method

Solve the initial value problem

$$y'' + 2y' + y = e^{-t}, \qquad y(0) = -1, \qquad y'(0) = 1.$$

Solution. From the formulas (1) and (2) for the transforms of the derivatives and from the initial conditions we obtain the subsidiary equation

$$(s^2Y + s - 1) + 2(sY + 1) + Y = \frac{1}{s + 1}.$$

Collecting Y-terms gives

$$(s^2 + 2s + 1)Y = (s + 1)^2Y = -s - 1 + \frac{1}{s + 1}.$$

Dividing by $(s + 1)^2$, we solve algebraically for Y and simplify:

$$Y = \frac{-s - 1}{(s + 1)^2} + \frac{1}{(s + 1)^3} = -\frac{1}{s + 1} + \frac{1}{(s + 1)^3}.$$

The first term on the right has the inverse $-e^{-t}$. For the other term on the right the first shifting theorem gives the inverse $\frac{1}{2}t^2e^{-t}$. Hence the solution of our initial value problem is

$$y = (\tfrac{1}{2}t^2 - 1)e^{-t}.$$

This agrees with the result in Example 3 of Sec. 2.9 and was obtained with much less work—take a look and compare. ◀

In practice, instead of justifying the use of formulas and theorems in this method, one simply checks at the end whether $y(t)$ satisfies the given equation and initial conditions.

Example 6 illustrates the **advantages of the method** compared to that in Chap. 2:

1. No determination of a general solution of the homogeneous equation.
2. No determination of values for arbitrary constants in a general solution.

Laplace Transform of the Integral of a Function

Differentiation and integration are inverse processes. Accordingly, since (roughly speaking) differentiation of a function corresponds to the multiplication of its transform by s, we expect integration of a function to correspond to division of its transform by s, because division is the inverse operation of multiplication:

THEOREM 3 **[Integration of $f(t)$]**

Let $F(s)$ be the Laplace transform of $f(t)$. If $f(t)$ is piecewise continuous and satisfies an inequality of the form (2), *Sec.* 5.1, *then*

(8) [redacted] $(s > 0,\ s > k)$

or, if we take the inverse transform on both sides of (8),

(9) [redacted]

PROOF. Suppose that $f(t)$ is piecewise continuous and satisfies (2), Sec. 5.1, for some k and M. Clearly, if (2) holds for some negative k, it also holds for positive k, and we may assume that k is positive. Then the integral

$$g(t) = \int_0^t f(\tau)\, d\tau$$

is continuous, and by using (2) in Sec. 5.1 we obtain for any positive t

$$|g(t)| \leqq \int_0^t |f(\tau)|\, d\tau \leqq M \int_0^t e^{k\tau}\, d\tau = \frac{M}{k}(e^{kt} - 1) \leqq \frac{M}{k} e^{kt} \qquad (k > 0).$$

This shows that $g(t)$ also satisfies an inequality of the form (2), Sec. 5.1. Also, $g'(t) = f(t)$, except for points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise

continuous on each finite interval, and, by Thereom 1,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) \qquad (s > k).$$

Here, clearly, $g(0) = 0$, so that $\mathcal{L}(f) = s\mathcal{L}(g)$. This implies (8), and (9) follows as indicated in the theorem. ◀

EXAMPLE 7 An application of Theorem 3

Let $\mathcal{L}(f) = \dfrac{1}{s(s^2 + \omega^2)}$. Find $f(t)$.

Solution. From Table 5.1 in Sec. 5.1 we have

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 + \omega^2}\right) = \frac{1}{\omega} \sin \omega t.$$

From this and Theorem 3 we obtain the answer

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2 + \omega^2}\right)\right\} = \frac{1}{\omega}\int_0^t \sin \omega\tau \, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

This proves formula 19 in the table in Sec. 5.9. ◀

EXAMPLE 8 Another application of Theorem 3

Derive formula 20 in the table in Sec. 5.9.

Solution. Applying Theorem 3 to the answer in Example 7, we obtain the desired formula

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{1}{s^2 + \omega^2}\right)\right\} = \frac{1}{\omega^2}\int_0^t (1 - \cos \omega\tau)\, d\tau = \frac{1}{\omega^2}\left(t - \frac{\sin \omega t}{\omega}\right).$$

This proves formula 20 in the table in Sec. 5.9. ◀

EXAMPLE 9 Shifted data problems

This is a short name for initial value problems with initial conditions referring to some later instant $t = t_0$ instead of $t = 0$. In this case, the conditions $y(0)$ and $y'(0)$ occurring in the Laplace transform approach cannot be used immediately. We may now proceed in two ways.

First Solution Method. Find the general solution by Laplace transform and from it find the solution of the problem as in the classical method.

Second Solution Method. Set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform becomes applicable throughout. We show this method for the problem

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$.

1st Step. Setting up the subsidiary equation. From (2) and Table 5.1 in Sec. 5.1 we obtain

$$s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0) + \tilde{Y} = \frac{2}{s^2} + \frac{\pi/2}{s}$$

where $\tilde{Y}$ is the transform of $\tilde{y}$.

2nd Step. Solution of the subsidiary equation. Solving algebraically, we have

$$\tilde{Y} = \frac{2}{(s^2+1)s^2} + \frac{\pi/2}{s(s^2+1)} + \tilde{y}(0)\frac{s}{s^2+1} + \tilde{y}'(0)\frac{1}{s^2+1}.$$

The first two terms on the right are those in Examples 7 and 8, so that

$$\tilde{y} = 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t}.$$

Substituting $\tilde{t} = t - \frac{1}{4}\pi$, canceling terms, and using $\cos\frac{1}{4}\pi = \sin\frac{1}{4}\pi = 1/\sqrt{2}$, we obtain the solution

$$y = 2t - \sin t + \cos t.$$

◀

PROBLEM SET 5.2

Initial Value Problems. Solve the following initial value problems by the Laplace transform. (Show the details of your work.)

1. $y' + 3y = 10\sin t, \qquad y(0) = 0$
2. $y' - 5y = 1.5e^{-4t}, \qquad y(0) = 1$
3. $y' + 0.2y = 0.01t, \qquad y(0) = -0.25$
4. $y'' - y' - 2y = 0, \qquad y(0) = 8, \qquad y'(0) = 7$
5. $y'' + ay' - 2a^2y = 0, \qquad y(0) = 6, \qquad y'(0) = 0$
6. $y'' + y = 2\cos t, \qquad y(0) = 3, \qquad y'(0) = 4$
7. $y'' - 4y' + 3y = 6t - 8, \qquad y(0) = 0, \qquad y'(0) = 0$
8. $y'' + 0.04y = 0.02t^2, \qquad y(0) = -25, \qquad y'(0) = 0$
9. $y'' + 2y' - 3y = 6e^{-2t}, \qquad y(0) = 2, \qquad y'(0) = -14$

10. **PROJECT. Summary of Sec. 5.2. (a)** Compare the Laplace transform with the classical method of solving differential equations, explaining the advantages and illustrating the comparison with examples of your own.
    (b) Theorems 1 and 2 play a role different from that of Theorem 3. Explain the difference.
    (c) Extension of Theorem 1. Show that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at $t = a$ (> 0), the other conditions remaining the same as in Theorem 1, then (see Fig. 109)

    $$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a+0) - f(a-0)]e^{-as}. \tag{1*}$$

    (d) Using (1*), find the Laplace transform of $f(t) = t$ if $0 < t < 1$, $f(t) = 1$ if $1 < t < 2$, $f(t) = 0$ otherwise.
11. **Derivation by different methods** is possible for various formulas and is typical of Laplace transforms. Find $\mathcal{L}(\cos^2 t)$ (a) by using the result in Example 3, (b) by the method used in that example, (c) by expressing $\cos^2 t$ in terms of $\cos 2t$.

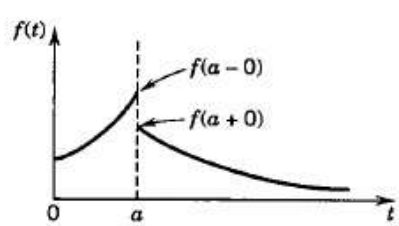


Fig. 109. Formula (1*)

12. PROJECT. Extension of Example 4. Extend the method of differentiation in Example 4 to obtain

(a) $\mathcal{L}(t \cos \omega t) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$.

From this and Example 4 derive

(b) $\mathcal{L}^{-1}\left(\dfrac{1}{(s^2 + \omega^2)^2}\right) = \dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$

(c) $\mathcal{L}^{-1}\left(\dfrac{s}{(s^2 + \omega^2)^2}\right) = \dfrac{1}{2\omega} t \sin \omega t$

(d) $\mathcal{L}^{-1}\left(\dfrac{s^2}{(s^2 + \omega^2)^2}\right) = \dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$.

Obtain similar formulas for hyperbolic functions, namely,

(e) $\mathcal{L}(t \cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

(f) $\mathcal{L}(t \sinh at) = \dfrac{2as}{(s^2 - a^2)^2}$.

New Inverse Transforms by Integration (Theorem 3). Given $\mathcal{L}(f)$, find $f(t)$. (Show the details of your work.)

13. $\dfrac{1}{s^2 + 4s}$ **14.** $\dfrac{4}{s^3 - 2s^2}$ **15.** $\dfrac{1}{s(s^2 + \omega^2)}$ **16.** $\dfrac{1}{s^5 + s^3}$

17. $\dfrac{1}{s^3 - s}$ **18.** $\dfrac{1}{s^2}\left(\dfrac{s - 1}{s + 1}\right)$ **19.** $\dfrac{9}{s^2}\left(\dfrac{s + 1}{s^2 + 9}\right)$ **20.** $\dfrac{\pi^5}{s^4(s^2 + \pi^2)}$

5.3 Unit Step Function Second Shifting Theorem Dirac's Delta Function

What state have we reached and what is our next goal? We know that differentiation of $f(t)$ roughly corresponds to multiplication of the transform $\mathcal{L}(f)$ by s (Theorems 1 and 2, Sec. 5.2) and that this property is essential in solving differential equations. Example 5 in Sec. 5.2 explains the three steps of this Laplace technique. But that equation can be solved easily by the usual methods, and our goal is to derive further properties of the Laplace transform in order to show the real power of this method in applications. We accomplish this essentially by defining two important functions, the unit step function and Dirac's delta function.

Unit Step Function $u(t - a)$

By definition, $u(t - a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave

it undefined) and is 1 for $t > a$:

(1) [redacted] $(a \geqq 0)$.

Figure 110 shows the special case $u(t)$, which has the jump at zero, and Fig. 111 the general case $u(t - a)$ for an arbitrary positive a. The unit step function is also called the **Heaviside function.**[3]

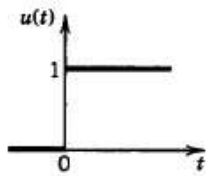


Fig. 110. Unit step function $u(t)$

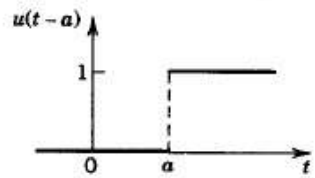
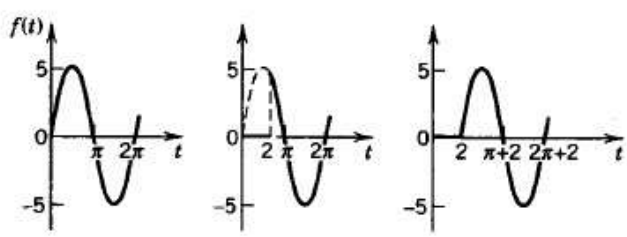


Fig. 111. Unit step function $u(t - a)$

The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on." Multiplying functions $f(t)$ with $u(t - a)$, we can produce all sorts of effects. The simple basic idea is illustrated in Figs. 112 and 113. In Fig. 112 the given function is shown in (a). In (b) it is switched off between $t = 0$ and $t = 2$ (because $u(t - 2) = 0$ when $t < 2$) and is switched on beginning at $t = 2$. In (c) it is shifted to the right by 2 seconds, so that it begins 2 seconds later in the same fashion as before. Figure 113 shows the effect of many unit step functions, three of them in (A) and infinitely many in (B); this is the effect of a rectifier that clips off the negative half-waves of a sinusoidal voltage. Before going on, make sure that you fully understand these figures, in particular the difference between parts (b) and (c) of Figure 112. Figure 112(c) will be applied next.



(a) $f(t) = 5 \sin t$ (b) $f(t)u(t - 2)$ (c) $f(t - 2)u(t - 2)$

Fig. 112. Effects of the unit step function: (a) Given function. (b) Switching off and on. (c) Shift.

[3]See footnote 1 in Sec. 5.1.

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 113. Use of many unit step functions

t-Shifting: Replacing t by $t - a$ in $f(t)$

If $f(t)$ has the transform $F(s)$, then $e^{at}f(t)$ has the transform $F(s-a)$. This is s-shifting (see Sec. 5.1). Now comes ***t*-shifting:**

THEOREM 1 **(Second shifting theorem; *t*-shifting)**

If $f(t)$ has the transform $F(s)$, then the ***"shifted function"***

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases} \tag{2}$$

has the transform $e^{-as}F(s)$. That is,

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s) \tag{3}$$

Or, if we take the inverse on both sides, we can write

$$f(t-a)u(t-a) = \mathcal{L}^{-1}[e^{-as}F(s)]. \tag{3*}$$

Practically speaking, if we know $F(s)$, we can obtain the transform of (2) by multiplying $F(s)$ by e^{-as}. In Fig. 112, the transform of $5 \sin t$ is $F(s) = 5/(s^2 + 1)$, hence $5 \sin(t-2)\,u(t-2)$ shown in Fig. 112(c) has the transform $e^{-2s}F(s) = 5e^{-2s}/(s^2+1)$.

PROOF. We prove Theorem 1. From the definition of the Laplace transform we have

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau.$$

Substituting $\tau + a = t$ in the integral, we obtain (note the lower limit of integration!)

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt.$$

We can write this as an integral from 0 to ∞ (as required for a Laplace transform!) if we make sure that the integrand is zero for all t from 0 to a. This is easy. We just multiply the integrand by the unit step function $u(t-a)$—do you now see why and how it comes

in? This gives (3) and completes the proof:

$$e^{-as}F(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt = \mathcal{L}\{f(t-a)u(t-a)\}. \qquad \blacktriangleleft$$

It is fair to say that we are already approaching the stage where we can attack problems for which the Laplace transform method is preferable to the usual method, as the examples in the next section will illustrate. In this connection we need the transform of the unit step function $u(t-a)$,

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}} \qquad (s > 0). \tag{4}$$

This formula follows directly from the definition because

$$\begin{aligned}\mathcal{L}\{u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)\,dt \\ &= \int_0^a e^{-st}0\,dt + \int_a^\infty e^{-st}1\,dt = -\frac{1}{s}e^{-st}\Big|_a^\infty.\end{aligned}$$

Let us consider two further examples. More applications follow in the problem set and in the next sections.

EXAMPLE 1 Application of Theorem 1. Use of unit step functions

Find the transform of the function (Fig. 114)

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ \sin t & \text{if } \quad t > 2\pi. \end{cases}$$

Solution. 1st Step. We write $f(t)$ in terms of unit step functions. For $0 < t < \pi$, we take $2u(t)$. For $t > \pi$ we want 0, so we must subtract the step function $2u(t-\pi)$ with step at π. Then we have $2u(t) - 2u(t-\pi) = 0$ when $t > \pi$. This is fine until we reach 2π where we want $\sin t$ to come in; so we add $u(t-2\pi)\sin t$. Together,

$$f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin t.$$

2nd Step. The last term equals $u(t-2\pi)\sin(t-2\pi)$ because of the periodicity, so that (4), (3), and Table 5.1 (Sec. 5.1) give

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}. \qquad \blacktriangleleft$$

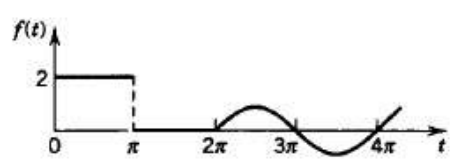


Fig. 114. Example 1

EXAMPLE 2 Application of Theorem 1. Inverse transform

Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{se^{-\pi s}}{s^2 + 1}$$

Solution. Without the exponential functions the four terms of $F(s)$ would have the inverses $2t$, $-2t$, -4, $\cos t$ (see Table 5.1 in Sec. 5.1). Hence by Theorem 1,

$$\begin{aligned} f(t) &= 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)\, u(t-\pi) \\ &= 2t - 2tu(t-2) - \cos t\, u(t-\pi) \end{aligned}$$

because $4u(t-2)$ and $-4u(t-2)$ cancel each other. This gives $2t$ if $0 < t < 2$, $2t - 2t = 0$ if $2 < t < \pi$, and $2t - 2t - \cos t = -\cos t$ if $t > \pi$. ◀

EXAMPLE 3 Response of an *RC*-circuit to a single square wave

Find the current $i(t)$ in the circuit in Fig. 115 if a single square wave with voltage V_0 is applied. The circuit is assumed to be quiescent before the square wave is applied.

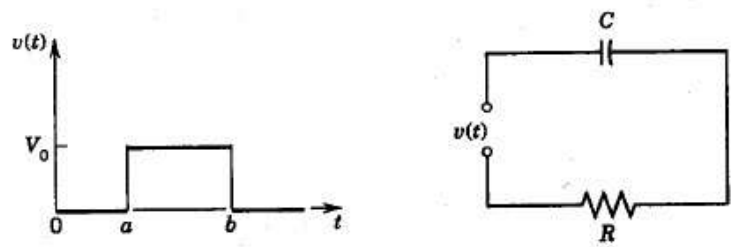


Fig. 115. Example 3

Solution. The equation of the circuit is (see Sec. 1.7)

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)\, d\tau = v(t)$$

where $v(t)$ can be represented in terms of two unit step functions:

$$v(t) = V_0[u(t-a) - u(t-b)].$$

Using Theorem 3 in Sec. 5.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right].$$

Solving this equation algebraically for $I(s)$, we get

$$I(s) = F(s)(e^{-as} - e^{-bs}) \qquad \text{where} \qquad F(s) = \frac{V_0/R}{s + 1/(RC)}.$$

From Table 5.1 in Sec. 5.1 we have

$$\mathcal{L}^{-1}(F) = \frac{V_0}{R}\, e^{-t/(RC)}.$$

Hence Theorem 1 yields the solution (Fig. 116)

$$i(t) = \mathcal{L}^{-1}(I) = \mathcal{L}^{-1}\{e^{-as}F(s)\} - \mathcal{L}^{-1}\{e^{-bs}F(s)\} = \frac{V_0}{R}\left[e^{-(t-a)/(RC)}u(t-a) - e^{-(t-b)/(RC)}u(t-b)\right];$$

that is, $i = 0$ if $t < a$, and

$$i(t) = \begin{cases} K_1 e^{-t/(RC)} & \text{if } a < t < b \\ (K_1 - K_2)e^{-t/(RC)} & \text{if } a > b \end{cases}$$

where $K_1 = V_0 e^{a/(RC)}/R$ and $K_2 = V_0 e^{b/(RC)}/R$. ◀

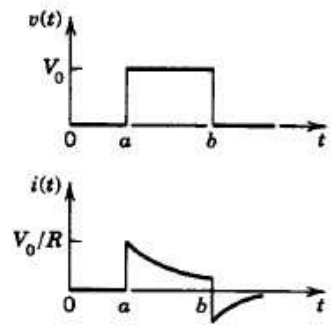


Fig. 116. Voltage and current in Example 3

Short Impulses. Dirac's Delta Function

Phenomena of an impulsive nature, such as the action of very large forces (or voltages) over very short intervals of time, are of great practical interest, since they arise in various applications. This situation occurs, for instance, when a tennis ball is hit, a system is given a blow by a hammer, an airplane makes a "hard" landing, a ship is hit by a high single wave, and so on. Our present goal is to show how to solve problems involving short impulses by Laplace transforms.

In mechanics, the **impulse** of a force $f(t)$ over a time interval, say, $a \leqq t \leqq a + k$, is defined to be the integral of $f(t)$ from a to $a + k$. The analog for an electric circuit is the integral of the electromotive force applied to the circuit, integrated from a to $a + k$. Of particular practical interest is the case of a very short k (and its limit $k \to 0$), that is, the impulse of a force acting only for an instant. To handle the case, we consider the function

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a + k \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Fig. 117).} \tag{5}$$

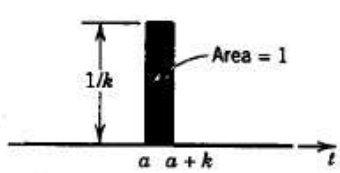


Fig. 117. The function $f_k(t - a)$ in (5)

Its impulse I_k is 1, since the integral evidently gives the area of the rectangle in Fig. 117:

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1. \tag{6}$$

We can represent $f_k(t-a)$ in terms of two unit step functions, namely,

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right].$$

From (4) we obtain the Laplace transform

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\,\frac{1-e^{-ks}}{ks}. \tag{7}$$

The limit of $f_k(t-a)$ as $k \to 0$ $(k > 0)$ is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

$\delta(t-a)$ is called the **Dirac delta function**[4] (sometimes the **unit impulse function**). The quotient in (7) has the limit 1 as $k \to 0$, as follows by l'Hôpital's rule (differentiate the numerator and also the denominator with respect to k). Hence the right side of (7) has the limit e^{-as}. This suggests defining the Laplace transform of $\delta(t-a)$ by this limit in (7), that is,

(8) [redacted]

We note that $\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but a so-called *"generalized function,"*[5] because (5) and (6) with $k \to 0$ imply

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-a)\,dt = 1,$$

but an ordinary function that is everywhere 0 except at a single point must have the integral 0. Nevertheless, in impulse problems it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function.

EXAMPLE 4 Response of a damped vibrating system to a single square wave and to a unit impulse

Determine the response of the damped mass–spring system (see Sec. 2.11) governed by

$$y'' + 3y' + 2y = r(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

[4]PAUL DIRAC (1902—1984), English physicist, was awarded the Nobel Prize [jointly with ERWIN SCHRÖDINGER (1887—1961)] in 1933 for his work in quantum mechanics.

[5]Or *"distribution."* A systematic theory of generalized functions was created in 1936 by the Russian mathematician, SERGEI L'VOVICH SOBOLEV (1908—1989) and in 1945, under wider aspects, by the French mathematician, LAURENT SCHWARTZ (born 1915).

where $r(t)$ is (A) the square wave

$$r(t) = u(t-1) - u(t-2) \qquad \text{(see Fig. 118)}$$

and (B) the unit impulse at time $t = 1$,

$$r(t) = \delta(t-1).$$

Solution (A). From (1) and (2) in Sec. 5.2 and (1) and (4) in this section we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}).$$

Algebraically solving for Y, we have

$$Y(s) = F(s)(e^{-s} - e^{-2s}) \qquad \text{where} \qquad F(s) = \frac{1}{s(s+1)(s+2)}.$$

In terms of partial fractions,

$$F(s) = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}.$$

Hence by Table 5.1 in Sec. 5.1,

$$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

Therefore, by Theorem 1 we have

$$\begin{aligned} y &= \mathcal{L}^{-1}(F(s)e^{-s} - F(s)e^{-2s}) \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \\ &= \begin{cases} 0 & (0 < t < 1) \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{-(t-2)} + \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2(t-2)} & (t > 2) \end{cases} \end{aligned}$$

(with $\frac{1}{2} - \frac{1}{2} = 0$ in the third line). This solution is shown in Fig. 118.

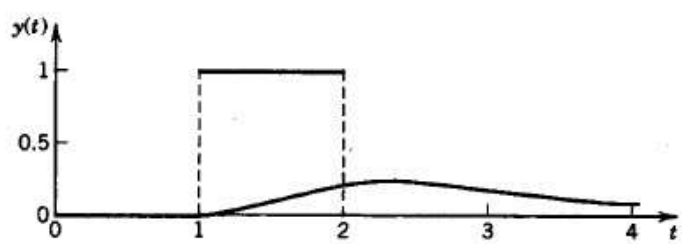


Fig. 118. Square wave and response in Example 4

Solution (B). With $r(t)$ as given in (B) we now obtain the subsidiary equation [see (8)]

$$s^2Y + 3sY + 2Y = e^{-s}.$$

Solving for Y, we have

$$Y = F(s)e^{-s} \qquad \text{where} \qquad F(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse transform of F, we obtain

$$f(t) = \mathcal{L}^{-1}(F) = e^{-t} - e^{-2t}.$$

Hence by Theorem 1 we have

$$y(t) = \mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)u(t-1) = \begin{cases} 0 & (0 \leqq t < 1) \\ e^{-(t-1)} - e^{-2(t-1)} & (t > 1). \end{cases}$$

Figure 119 shows the solution. Can you explain physically why this curve is steeper near $t = 1$ than the curve in Fig. 118? ◀

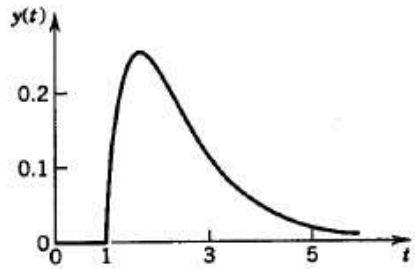


Fig. 119. Response to a hammer blow at $t = 1$ in Example 4

PROBLEM SET 5.3

1. **WRITING PROJECT. Shifting Theorems.** Explain and compare the different roles of the two shifting theorems. Use your own formulations and examples; do not copy phrases from the text.

Applications of the Second Shifting Theorem

Laplace transform. Sketch the following functions and find their Laplace transforms. (Show the details of your work.)

2. $tu(t-1)$ **3.** $(t-1)u(t-1)$ **4.** $(t-1)^2u(t-1)$
5. $t^2u(t-1)$ **6.** $e^{-2t}u(t-3)$ **7.** $4u(t-\pi)\cos t$

Laplace Transform. Sketch the given function, which is assumed to be zero outside the given interval. Find its Laplace transform. (Show the details of your work.)

8. $t^2 \quad (0 < t < 1)$ **9.** $\sin \omega t \quad (0 < t < \pi/\omega)$ **10.** $1 - e^{-t} \quad (0 < t < 2)$
11. $e^t \quad (0 < t < 1)$ **12.** $\sin t \quad (2\pi < t < 4\pi)$ **13.** $10 \cos \pi t \quad (1 < t < 2)$

Inverse Transform. Find and sketch the inverse Laplace transform. (Show the details of your work.)

14. $4(e^{-2s} - 2e^{-5s})/s$ **15.** e^{-3s}/s^3 **16.** $e^{-3s}/(s-1)^3$
17. $3(1 - e^{-\pi s})/(s^2 + 9)$ **18.** $e^{-2\pi s}/(s^2 + 2s + 2)$ **19.** $se^{-2s}/(s^2 + \pi^2)$

Initial Value Problems. Some with Discontinuous or Impulse Inputs. Using the Laplace transform, solve the following problems. (Show the details.)

20. $4y'' - 4y' + 37y = 0, \quad y(0) = 3, \quad y'(0) = 10.5$

21. $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, \quad y(0) = 0, \quad y'(0) = 0$

22. $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0, \quad y'(0) = 0$

23. $y'' + 9y = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0, \quad y'(0) = 4$

24. $y'' - 5y' + 6y = 4e^t$ if $0 < t < 2$ and 0 if $t > 2$; $y(0) = 1$, $y'(0) = -2$

25. $y'' + y' - 2y = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and $3 \sin 2t - \cos 2t$ if $t > 2\pi$; $y(0) = 1$, $y'(0) = 0$

26. $y'' + 16y = 4\delta(t - \pi)$, $y(0) = 2$, $y'(0) = 0$

27. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi)$, $y(0) = 0$, $y'(0) = 1$

28. $y'' + 4y' + 5y = \delta(t - 1)$, $y(0) = 0$, $y'(0) = 3$

29. $y'' + 2y' - 3y = 8e^{-t} + \delta(t - \frac{1}{2})$, $y(0) = 3$, $y'(0) = -5$

30. $y'' + 5y' + 6y = u(t - 1) + \delta(t - 2)$, $y(0) = 0$, $y'(0) = 1$

Models of Electric Circuits

***RL*-Circuit.** Using the Laplace transform (and showing the details of your work), find the current $i(t)$ in the circuit in Fig. 120, assuming $i(0) = 0$ and

31. $v(t) = t$ if $0 < t < 4\pi$ and 0 if $t > 4\pi$

32. $v(t) = \sin t$ if $0 < t < 2\pi$ and 0 otherwise

***LC*-Circuit.** Using the Laplace transform (and showing the details of your work), find the current $i(t)$ in the circuit in Fig. 121, assuming $L = 1$ henry, $C = 1$ farad, zero initial current and charge on the capacitor, and

33. $v(t) = t$ if $0 < t < 1$ and $v(t) = 1$ if $t > 1$

34. $v(t) = 1$ if $0 < t < a$ and 0 otherwise

35. $v(t) = 1 - e^{-t}$ if $0 < t < \pi$ and 0 otherwise

***RC*-Circuit.** Using the Laplace transform (and showing your work), find the current $i(t)$ in the circuit in Fig. 122 with $R = 100$ ohms, $C = 0.1$ farad, and $v(t)$ volts as follows. Assume that current and charge are zero at $t = 0$.

36. $v(t) = 10000$ if $1 < t < 1.01$ and 0 otherwise

37. $v(t) = 100$ if $1 < t < 2$ and 0 otherwise. Compare with Prob. 36.

38. $v(t) = 0$ if $t < 3$ and $50(t - 3)$ if $t > 3$

39. $v(t) = 0$ if $t < 2$ and e^{-t} if $t > 2$

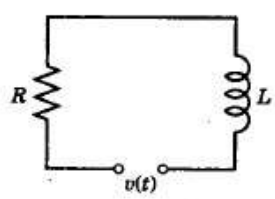


Fig. 120. Problems 31, 32

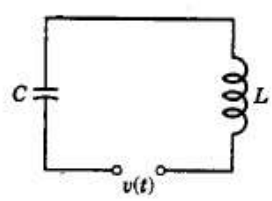


Fig. 121. Problems 33–35

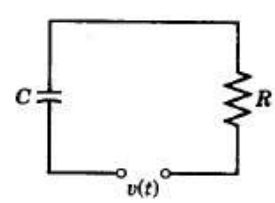


Fig. 122. Problems 36–39

40. CAS PROJECT. Limit of Square Wave. Effects of Impulse.

(a) In Example 4, take a square wave of area 1 from 1 to $1 + k$. Plot the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 119. *Hint.* If your CAS gives no solution for the differential equation involving k, take specific k's from the beginning.

(b) Experiment on the response of the equation in Example 4 (or of another equation of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting plots?

5.4 Differentiation and Integration of Transforms

The variety of methods for obtaining transforms or inverse transforms and their application in solving differential equations is surprisingly large. They include direct integration (Sec. 5.1), the use of linearity (Sec. 5.1), shifting (Secs. 5.1, 5.3), and differentiation or integration of original functions $f(t)$ (Sec. 5.2). But this is not all: in this section we consider differentiation and integration of *transforms* $F(s)$ and find out the corresponding operations for original functions $f(t)$.

Differentiation of Transforms

It can be shown that if $f(t)$ satisfies the conditions of the existence theorem in Sec. 5.1, then the derivative $F'(s)$ of its transform

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

with respect to s can be obtained by differentiating under the integral sign with respect to s (proof in Ref. [5] listed in Appendix 1); thus

$$F'(s) = -\int_0^\infty e^{-st}[tf(t)]\, dt.$$

Consequently, if $\mathcal{L}(f) = F(s)$, then

$$\mathcal{L}(tf(t)) = -F'(s). \tag{1}$$

differentiation of the transform of a function corresponds to the multiplication of the function by $-t$. Equivalently,

$$\mathcal{L}^{-1}\{F'(s)\} = -tf(t). \tag{1*}$$

This property enables us to get new transforms from given ones, as we show next.

EXAMPLE 1 Differentiation of transforms

We shall derive the following three formulas (formulas 21–23 in the table in Sec. 5.9):

	$f(t)$	$\mathcal{L}(f)$
(2)	$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
(3)	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
(4)	$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

Solution. From (1) and formula 8 (with $\omega = \beta$) in Table 5.1, Sec. 5.1, we obtain by differentiation (don't forget to apply the chain rule!)

$$\mathcal{L}(t \sin \beta t) = \frac{2\beta s}{(s^2 + \beta^2)^2}.$$

By dividing by 2β we obtain (3).

Formulas (2) and (4) are obtained as follows. From (1) and formula 7 (with $\omega = \beta$) in Table 5.1 we find

$$\mathcal{L}(t \cos \beta t) = -\frac{(s^2 + \beta^2) - 2s^2}{(s^2 + \beta^2)^2} = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2}. \tag{5}$$

From this and formula 8 (with $\omega = \beta$) in Table 5.1 we have

$$\mathcal{L}\left(t \cos \beta t \pm \frac{1}{\beta} \sin \beta t\right) = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2} \pm \frac{1}{s^2 + \beta^2}.$$

On the right we now take the common denominator. Then we see that for the plus sign the numerator becomes $s^2 - \beta^2 + s^2 + \beta^2 = 2s^2$, so that (4) follows by division by 2. Similarly, for the minus sign the numerator takes the form $s^2 - \beta^2 - s^2 - \beta^2 = -2\beta^2$, and we obtain (2). ◀

Integration of Transforms

Similarly, if $f(t)$ satisfies the conditions of the existence theorem in Sec. 5.1 and the limit of $f(t)/t$, as t approaches 0 from the right, exists, then

(6) [redacted] $(s > k)$;

in this manner, *integration of the transform of a function $f(t)$ corresponds to the division of $f(t)$ by t.* Equivalently,

(6*) [redacted]

In fact, from the definition it follows that

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)\, dt\right] d\tilde{s},$$

and it can be shown (see Ref. [5] in Appendix 1) that under the above assumptions we may reverse the order of integration, that is,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t} f(t)\, d\tilde{s}\right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-\tilde{s}t}\, d\tilde{s}\right] dt.$$

Integration of $e^{-\tilde{s}t}$ with respect to $\tilde{s}$ gives $e^{-\tilde{s}t}/(-t)$. Here the integral over $\tilde{s}$ on the right equals e^{-st}/t. Therefore,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\} \qquad (s > k). \blacktriangleleft$$

EXAMPLE 2 Integration of transforms

Find the inverse transform of the function $\ln\left(1 + \frac{\omega^2}{s^2}\right)$.

Solution. By differentiation,

$$-\frac{d}{ds}\ln\left(1 + \frac{\omega^2}{s^2}\right) = -\frac{1}{1 + \frac{\omega^2}{s^2}} \cdot (-2)\frac{\omega^2}{s^3} = \frac{2\omega^2}{s(s^2 + \omega^2)} = \frac{2}{s} - 2\frac{s}{s^2 + \omega^2},$$

where the last equality can be readily verified by direct calculation. This is our present $F(s)$. It is the derivative of the given function (times -1), so that the latter is the integral of $F(s)$ from s to ∞. From Table 5.1 in Sec. 5.1 we obtain

$$f(t) = \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left\{\frac{2}{s} - 2\frac{s}{s^2 + \omega^2}\right\} = 2 - 2\cos\omega t.$$

This function satisfies the conditions under which (6) holds. Therefore,

$$\mathcal{L}^{-1}\left\{\ln\left(1 + \frac{\omega^2}{s^2}\right)\right\} = \mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}.$$

Our result is

$$\mathcal{L}^{-1}\left\{\ln\left(1 + \frac{\omega^2}{s^2}\right)\right\} = \frac{2}{t}(1 - \cos\omega t).$$

This proves formula 42 in the table in Sec. 5.9. ◀

EXAMPLE 3 Integration of transforms

Reasoning as in Example 2, we obtain (see formula 43 in the table in Sec. 5.9)

$$\mathcal{L}^{-1}\left\{\ln\left(1 - \frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1 - \cosh at).$$ ◀

Differential Equations with Variable Coefficients

From (1) with $f = y'$ dy/dt and $\mathcal{L}(y') = sY - y(0)$ (see Sec. 5.2) and subsequent product differentiation we obtain

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}, \tag{7}$$

Similarly, by (1) with $f = y''$ and (2), Sec. 5.2, we find

$$\mathcal{L}(ty'') = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2\frac{dY}{ds} + y(0). \tag{8}$$

Hence if a differential equation has coefficients such as $at + b$, we get a first-order differential equation for Y, which is sometimes simpler than the given equation. But if the latter has coefficients $at^2 + bt + c$, we get, by two applications of (1), a second-order differential equation for Y, and this shows that the Laplace transform method works well only for very special equations with *variable* coefficients. We illustrate it for an important equation in the following example.

EXAMPLE 4 Laguerre's differential equation, Laguerre polynomials

Laguerre's differential equation is

$$ty'' + (1 - t)y' + ny = 0. \tag{9}$$

We determine a solution of (9) with $n = 0, 1, 2, \cdots$. From (7)–(9) we get

$$\left[-2sY - s^2 \frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0.$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken zero), and taking exponentials, we get

$$\frac{dY}{Y} = -\frac{n + 1 - s}{s - s^2}\,ds = \left(\frac{n}{s - 1} - \frac{n + 1}{s}\right)ds \qquad \text{and} \qquad Y = \frac{(s - 1)^n}{s^{n+1}}. \tag{10*}$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and show that

$$l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots. \tag{10}$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (see Problem Set 4.7; but we conform to our convention of reserving capital letters for transforms). We prove (10). By Table 5.1 and the first shifting theorem,

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s + 1)^{n+1}}.$$

By (4) in Sec. 5.2, since the derivatives are zero at 0,

$$\mathcal{L}\{(t^n e^{-t})^{(n)}\} = \frac{n! s^n}{(s + 1)^{n+1}}.$$

Now make another shift and divide by $n!$ to get [see (10) and (10*)]

$$\mathcal{L}(l_n) = \frac{(s - 1)^n}{s^{n+1}} = Y. \qquad ◀$$

PROBLEM SET 5.4

Transforms by Differentiation. Find the Laplace transform. (Show the details of your work.)

1. te^t **2.** $3t \sinh 4t$ **3.** $t^2 \cosh \pi t$ **4.** $te^{-t} \cos t$

5. $t \cos \omega t$ **6.** $t^2 \sin 2t$ **7.** $te^{-t} \sin t$ **8.** $t^2 \cos \omega t$

Inverse Transforms by Differentiation or Integration. Using (6) or (1), find the inverse transform. (Show your work.)

9. $\dfrac{1}{(s - 3)^3}$ **10.** $\dfrac{s}{(s^2 - 9)^2}$ **11.** $\dfrac{s^2 - \pi^2}{(s^2 + \pi^2)^2}$ **12.** $\dfrac{2s + 6}{(s^2 + 6s + 10)^2}$

13. $\ln \dfrac{s^2 + 1}{(s - 1)^2}$ **14.** $\ln \dfrac{s + a}{s + b}$ **15.** $\dfrac{s}{(s^2 + 4)^2}$ **16.** $\operatorname{arc\,cot} \dfrac{s}{\pi}$

17. **(Shifting)** Can you solve Probs. 1 and 3 by the first shifting theorem?
18. **(Differentiation)** Find $\mathcal{L}(t^n e^{at})$ by repeated application of (1), choosing $f(t) = e^{at}$
19. **WRITING PROJECT. Differentiation and Integration of Functions and Transforms.** Make a short draft on these four operations from memory. Then compare your notes with the text and write an essay of 2–3 pages on these operations and their significance in applications.
20. **CAS PROJECT. Laguerre Polynomials. (a)** Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

    (b) Show that

$$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n}{m} t^m$$

    and calculate $l_0, \cdots, l_{10}$ from this formula.

    (c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

$$(n + 1)l_{n+1} = (2n + 1 - t)l_n - nl_{n-1}.$$

CAUTION! Sometimes the functions $\tilde{l}_n = n!l_n$ are also called *Laguerre polynomials.* Their recursion is different! (Such differences in normalization are typical of several special functions—and a possible source of errors.)

5.5 Convolution. Integral Equations

Another important general property of the Laplace transform has to do with products of transforms. It often happens that we are given two transforms $F(s)$ and $G(s)$ whose inverses $f(t)$ and $g(t)$ we know, and we would like to calculate the inverse of the product $H(s) = F(s)G(s)$ from those known inverses $f(t)$ and $g(t)$. This inverse $h(t)$ is written $(f * g)(t)$, which is a standard notation, and is called the **convolution** of f and g. How can we find h from f and g? This is stated in the following theorem. Since the situation and task just described arise quite often in applications, this theorem is of considerable practical importance.

THEOREM 1 (Convolution theorem)

Let $f(t)$ and $g(t)$ satisfy the hypothesis of the existence theorem (Sec. 5.1). Then the product of their transforms $F(s) = \mathcal{L}(f)$ and $G(s) = \mathcal{L}(g)$ is the transform $H(s) = \mathcal{L}(h)$ of the ***convolution*** *$h(t)$ of $f(t)$ and $g(t)$, which is denoted by $(f * g)(t)$ and defined by*

(1) [redacted]

(The proof follows after Example 3.)

EXAMPLE 1 Convolution

Using convolution, find the inverse $h(t)$ of

$$H(s) = \frac{1}{(s^2 + 1)^2} = \frac{1}{s^2 + 1} \cdot \frac{1}{s^2 + 1}.$$

Solution. We know that each factor on the right has the inverse $\sin t$. Hence by the convolution theorem and by (11) in Appendix A3.1, we get

$$\begin{aligned} h(t) = \mathcal{L}^{-1}(H) &= \sin t * \sin t \\ &= \int_0^t \sin \tau \sin (t - \tau)\, d\tau \\ &= \frac{1}{2}\int_0^t -\cos t\, d\tau + \frac{1}{2}\int_0^t \cos (2\tau - t)\, d\tau \\ &= -\tfrac{1}{2} t \cos t + \tfrac{1}{2} \sin t. \end{aligned}$$

◀

EXAMPLE 2 Convolution

$1/s^2$ has the inverse t and $1/s$ has the inverse 1, and the convolution theorem confirms that $1/s^3 = (1/s^2)(1/s)$ has the inverse

$$t * 1 = \int_0^t \tau \cdot 1\, d\tau = \frac{t^2}{2}.$$

◀

EXAMPLE 3 Convolution

Let $H(s) = 1/[s^2(s - a)]$. Find $h(t)$.

Solution. From Table 5.1 in Sec. 5.1 we know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$

Using the convolution theorem and integrating by parts, we get the answer

$$\begin{aligned} h(t) = t * e^{at} &= \int_0^t \tau e^{a(t-\tau)}\, d\tau = e^{at}\int_0^t \tau e^{-a\tau}\, d\tau \\ &= \frac{1}{a^2}(e^{at} - at - 1). \end{aligned}$$

◀

PROOF. Theorem 1 can be proved as follows. By the definition of $G(s)$ and the second shifting theorem, for each fixed τ ($\tau \geqq 0$) we have

$$\begin{aligned} e^{-s\tau}G(s) &= \mathcal{L}\{g(t - \tau)u(t - \tau)\} \\ &= \int_0^\infty e^{-st} g(t - \tau)u(t - \tau)\, dt \\ &= \int_\tau^\infty e^{-st} g(t - \tau)\, dt \end{aligned}$$

where $s > k$. From this and the definition of $F(s)$ we obtain

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau)G(s)\, d\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t - \tau)\, dt\, d\tau$$

where $s > k$. Here we integrate over t from τ to ∞ and then over τ from 0 to ∞; this corresponds to the colored wedge-shaped region extending to infinity in the $t\tau$-plane shown in Fig. 123 on p. 281. Our assumptions on f and g are such that the order of integration can be reversed. (A proof requiring the knowledge of uniform convergence is included in Ref. [A2] listed in Appendix 1.) We then integrate first over τ from 0 to t (see Fig. 123)

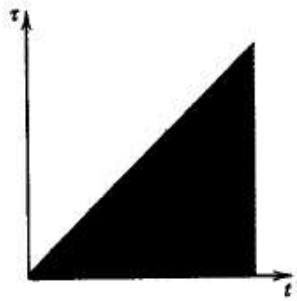


Fig. 123. Region of integration in the $t\tau$-plane in the proof of Theorem 1

and then over t from 0 to ∞; thus

$$F(s)G(s) = \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)\,d\tau\,dt$$
$$= \int_0^\infty e^{-st}h(t)\,dt = \mathcal{L}(h)$$

where h is given by (1). This completes the proof. ◀

Using the definition, the reader may show that the convolution $f * g$ has the properties

$$f * g = g * f \qquad \text{(commutative law)}$$
$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)}$$
$$(f * g) * v = f * (g * v) \qquad \text{(associative law)}$$
$$f * 0 = 0 * f = 0,$$

just as for numbers. But $f * 1 \neq f$ in general, as Example 2 shows. Another unusual property is that $(f * f)(t) \geqq 0$ may not hold, as we can see from Example 1.

Very useful applications of convolution occur in a natural way in the solution of differential equations, as we shall now discuss.

Differential Equations

From Sec. 5.2 we recall that the differential equation

$$y'' + ay' + by = r(t) \tag{2}$$

has the subsidiary equation $(s^2 + as + b)Y = (s + a)y(0) + y'(0) + \mathcal{L}(r)$. The solution of the latter is

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s) \tag{3}$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function. Hence for the solution $y(t)$ of (2) satisfying $y(0) = y'(0) = 0$ we have $Y = RQ$ in (3) and obtain from the

convolution theorem the integral representation

(4) [redacted] $q(t) = \mathcal{L}^{-1}(Q).$

EXAMPLE 4 Response of a damped system to a single square wave

We reconsider the model in Sec. 5.3, Example 4 (Fig. 118):

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}, \qquad y(0) = y'(0) = 0.$$

We solve it by the convolution technique in order to see how it works for ***inputs that act for some time only.***

Solution. We have

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the integral (4) is (except for the limits of integration)

$$\int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau = e^{-(t-\tau)} - \tfrac{1}{2}e^{-2(t-\tau)}$$

Now comes an important point. $r(t) = 1$ if $1 < t < 2$. Hence if $t < 1$, the integral is 0. If $1 < t < 2$, we have to integrate from 1 to t. This gives

$$e^{-0} - e^{-(t-1)} - \tfrac{1}{2}(e^{-0} - e^{-2(t-1)}) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}$$

If $t > 2$, we have to integrate from 1 to 2 only (not to t). This gives

$$e^{-(t-2)} - e^{-(t-1)} - \tfrac{1}{2}\left[e^{-2(t-2)} - e^{-2(t-1)}\right].$$

All three results agree with those in Example 4(A) in Sec. 5.3. ◀

Integral Equations

Convolution also helps in solving certain **integral equations,** that is, equations in which the unknown function $y(t)$ appears under the integral (and perhaps also outside of it). This concerns only very special ones (those whose integral is of the form of a convolution), so that it suffices to consider a typical example and a handful of problems, but we do this because integral equations are practically important and often difficult to solve.

EXAMPLE 5 Integral equation

Solve the integral equation

$$y(t) = t + \int_0^t y(\tau) \sin(t - \tau)\, d\tau.$$

Solution. 1st Step. Equation in terms of convolution. We see that the given equation can be written

$$y = t + y * \sin t.$$

2nd Step. Application of the convolution theorem. We write $Y = \mathcal{L}(y)$. By the convolution theorem,

$$Y(s) = \frac{1}{s^2} + Y(s)\frac{1}{s^2 + 1}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{s^2 + 1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}.$$

3rd Step. Taking the inverse transform. This gives the solution

$$y(t) = t + \tfrac{1}{6}t^3.$$

The reader may check this by substitution and evaluating the integral by repeated integration by parts (which will need patience). ◀

PROBLEM SET 5.5

Calculation of Convolutions by Integrating (1). Do this for the following. (Show the details of your work.)

1. $1 * 1$ **2.** $1 * \sin \omega t$ **3.** $e^t * e^{-t}$

4. $\cos \omega t * \cos \omega t$ **5.** $\sin \omega t * \cos \omega t$ **6.** $e^{at} * e^{bt} \quad (a \neq b)$

7. $t * e^t$ **8.** $u(t - 1) * t^2$ **9.** $u(t - 3) * e^{-2t}$

Inverse Transforms by Convolution. Find $h(t)$ by the convolution theorem from the given $H(s) = \mathcal{L}(h)$. (Show the details of your work.)

10. $\dfrac{6}{s(s + 3)}$ **11.** $\dfrac{1}{s^2(s - 1)}$ **12.** $\dfrac{1}{(s - a)^2}$

13. $\dfrac{1}{s(s^2 + 4)}$ **14.** $\dfrac{s^2}{(s^2 + \omega^2)^2}$ **15.** $\dfrac{s}{(s^2 + \pi^2)^2}$

16. $\dfrac{e^{-as}}{s(s - 2)}$ **17.** $\dfrac{\omega}{s^2(s^2 + \omega^2)}$ **18.** $\dfrac{1}{(s + 3)(s - 2)}$

Initial Value Problems. Applying convolution, find and sketch or plot the solution. First guess what the solution might look like. (Show the details of your work.)

19. $y'' + y = 3 \cos 2t; \quad y(0) = 0, \quad y'(0) = 0$

20. $y'' + y = t; \quad y(0) = 0, \quad y'(0) = 0$

21. $y'' + 4y = r(t)$, $r(t) = 1$ if $0 < t < 1$ and 0 if $t > 1$; $\quad y(0) = 1, \quad y'(0) = 0$

22. $y'' + 3y' + 2y = r(t)$, $r(t)$ as in Prob. 21; $\quad y(0) = 0, \quad y'(0) = 1$

23. $y'' + 2y' + 2y = r(t)$, $r(t) = 5u(t - 2\pi) \sin t$; $\quad y(0) = 1, \quad y'(0) = 0$

24. $y'' - 5y' + 6y = r(t)$, $r(t) = 4e^t$ if $0 < t < 2$ and 0 if $t > 2$; $\quad y(0) = 1, \quad y'(0) = -2$

25. $y'' + y = r(t)$, $r(t) = t$ if $1 < t < 2$ and 0 otherwise; $\quad y(0) = 0, \quad y'(0) = 0$

26. $y'' + 3y' + 2y = r(t)$, $r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $\quad y(0) = 0; \quad y'(0) = 0$

Integral Equations. Using Laplace transforms, solve the integral equations in Probs. 27–33. (Show the details of your work.)

27. $y(t) = 1 + \displaystyle\int_0^t y(\tau)\, d\tau$ **28.** $y = 2t - 4\displaystyle\int_0^t y(\tau)(t - \tau)\, d\tau$

29. $y(t) = 1 - \displaystyle\int_0^t (t - \tau)y(\tau)\, d\tau$ **30.** $y(t) = \sin 2t + \displaystyle\int_0^t y(\tau) \sin 2(t - \tau)\, d\tau$

31. $y(t) = te^t - 2e^t \int_0^t e^{-\tau} y(\tau)\, d\tau$

32. $y(t) = \sin t + \int_0^t y(\tau) \sin(t - \tau)\, d\tau$

33. $y(t) = 1 - \sinh t + \int_0^t (1 + \tau) y(t - \tau)\, d\tau$

34. TEAM PROJECT. Properties of Convolution. Prove:

(a) Commutativity, $f * g = g * f$

(b) Associativity, $(f * g) * v = f * (g * v)$

(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$

(d) Dirac's delta. Using the convolution theorem and treating $\delta(t)$ (Sec. 5.3) as though it were an ordinary function, show that

$$(\delta * f)(t) = f(t).$$

(e) Derive the formula in (d) by using f_k with $a = 0$ (Sec. 5.3) and applying the mean value theorem for integrals.

(f) Unspecified driving force. Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \qquad y(0) = K_1, \qquad y'(0) = K_2$$

with unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega} \sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega} \sin \omega t \qquad (\omega \neq 0).$$

5.6 Partial Fractions Differential Equations

The solution $Y(s)$ of a subsidiary equation of a differential equation (see Sec. 5.2) usually comes out as a quotient of two polynomials,

$$Y(s) = \frac{F(s)}{G(s)}.$$

Hence we can often determine its inverse by writing $Y(s)$ as a sum of **partial fractions** (as in calculus) and obtain the inverse of the latter from a table and the first shifting theorem (Sec. 5.1). This is an easy matter, which we shall now explain by typical examples in the context of Laplace transforms, where partial fraction representations and their inverse transforms are sometimes called **Heaviside expansions.** The form of the partial fractions depends on the kind of factors in the product form of $G(s)$. Of practical interest are:

(Case 1) Unrepeated factors $s - a$
(Case 2) Repeated factors $(s - a)^m$
(Case 3) Complex factors $(s - a)(s - \bar{a})$
(Case 4) Repeated complex factors $[(s - a)(s - \bar{a})]^2$.

For determining coefficients use your own favorite method from calculus and any shortcut you can discover.

Case 1. Unrepeated Factor $s - a$

EXAMPLE 1 Unrepeated factors. An initial value problem

Solve the initial value problem

$$y'' + y' - 6y = 1, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Solution. From $\mathcal{L}(1) = 1/s$ and the formulas (1), (2) in Sec. 5.2 for the derivatives, we obtain the subsidiary equation

$$(s^2 + s - 6)Y = 1 + \frac{1}{s} = \frac{s+1}{s}.$$

Now $s^2 + s - 6 = (s-2)(s+3)$. These are unrepeated factors. Hence the solution $Y(s)$ and its partial fraction representation are

$$Y(s) = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}.$$

Next we determine A_1, A_2, and A_3. Multiplication by the common denominator $s(s-2)(s+3)$ gives

$$s + 1 = (s-2)(s+3)A_1 + s(s+3)A_2 + s(s-2)A_3.$$

Taking $s = 0$, $s = 2$, $s = -3$, we obtain

$$\begin{aligned} 1 &= -2 \cdot 3A_1 \\ 3 &= 2 \cdot 5A_2 \\ -2 &= -3(-5)A_3. \end{aligned}$$

Hence $A_1 = -1/6$, $A_2 = 3/10$, and $A_3 = -2/15$. The answer is

$$\mathcal{L}^{-1}(Y) = -\tfrac{1}{6} + \tfrac{3}{10}e^{2t} - \tfrac{2}{15}e^{-3t}.$$

◀

Case 2. Repeated Factor $(s - a)^m$

Repeated factors $(s-a)^2$, $(s-a)^3$, etc., require partial fractions

$$(1) \qquad \frac{A_2}{(s-a)^2} + \frac{A_1}{s-a}, \qquad \frac{A_3}{(s-a)^3} + \frac{A_2}{(s-a)^2} + \frac{A_1}{s-a}, \text{ etc.,}$$

respectively.

EXAMPLE 2 Repeated factor. An initial value problem

Solve the initial value problem

$$y'' - 3y' + 2y = 4t \qquad y(0) = 1, \qquad y'(0) = -1.$$

Solution. From Table 5.1, Sec. 5.1, and (1) and (2) in Sec. 5.2 we get the subsidiary equation

$$s^2Y - s + 1 - 3(sY - 1) + 2Y = \frac{4}{s^2}.$$

Collecting the terms in Y on the left and the others on the right, we have

$$(s^2 - 3s + 2)Y = \frac{4}{s^2} + s - 4 = \frac{4 + s^3 - 4s^2}{s^2}$$

Since s^2 is a double factor and $s^2 - 3s + 2 = (s - 2)(s - 1)$ has simple factors, we obtain the partial fraction representation

$$(2^*) \qquad Y(s) = \frac{s^3 - 4s^2 + 4}{s^2(s - 2)(s - 1)} = \frac{A_2}{s^2} + \frac{A_1}{s} + \frac{B}{s - 2} + \frac{C}{s - 1}$$

Multiplication with $s^2(s - 2)(s - 1)$ gives

$$(2) \qquad s^3 - 4s^2 + 4 = A_2(s - 2)(s - 1) + A_1 s(s - 2)(s - 1) + Bs^2(s - 1) + Cs^2(s - 2).$$

For $s = 1$ this is $1 = C(-1)$, hence $C = -1$. For $s = 2$ it is $-4 = 4B$, hence $B = -1$. This was nothing new. New are the double root $s = 0$ and the two fractions A_2/s^2 and A_1/s. For $s = 0$ we get $4 = 2A_2$, hence $A_2 = 2$. Differentiation of (2) gives

$$3s^2 - 8s = A_2(2s - 3) + A_1(s - 2)(s - 1) + \text{further terms all containing a factor } s.$$

For $s = 0$ this is $0 = -3A_2 + 2A_1$, hence $A_1 = 3A_2/2 = 3$. Alternatively, you could get A_1 by taking the terms in s in (2), $0 = -3A_2 + 2A_1$. Substituting all these constants into (2*), we obtain from Table 5.1 in Sec. 5.1 the answer

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{3}{s} - \frac{1}{s - 2} - \frac{1}{s - 1}\right\}$$
$$= 2t + 3 - e^{2t} - e^t.$$

The first two terms result from the "driving force" $4t$ and the exponential terms from the general solution of the homogeneous equation. ◀

For a triple root we would proceed similarly, obtaining A_3, A_2, A_1 in (1) from the analog of (2) and its first *and second* derivatives, respectively.

Case 3. Unrepeated Complex Factors $(s - a)(s - \bar{a})$

Such factors occur, for instance, in connection with vibrations. If $s - a$ with complex $a = \alpha + i\beta$ is a factor of $G(s)$, so is $s - \bar{a}$ with $\bar{a} = \alpha - i\beta$ the conjugate. To $(s - a)(s - \bar{a}) = (s - \alpha)^2 + \beta^2$ there corresponds the partial fraction

$$(3) \qquad \frac{As + B}{(s - a)(s - \bar{a})} \qquad \text{or} \qquad \frac{As + B}{(s - \alpha)^2 + \beta^2}.$$

EXAMPLE 3 Unrepeated complex factors. Damped forced vibrations

Solve the initial value problem

$$y'' + 2y' + 2y = r(t),\ r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1,\ \ y'(0) = -5.$$

This is a damped mass–spring system (Fig. 124) with a sinusoidal driving force acting during the interval $0 < t < \pi$ only.

Solution. From Table 5.1 in Sec. 5.1, formulas (1), (2) in Sec. 5.2, and the second shifting theorem in Sec. 5.3, we obtain the subsidiary equation

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\,\frac{2}{s^2 + 4}\,(1 - e^{-\pi s}).$$

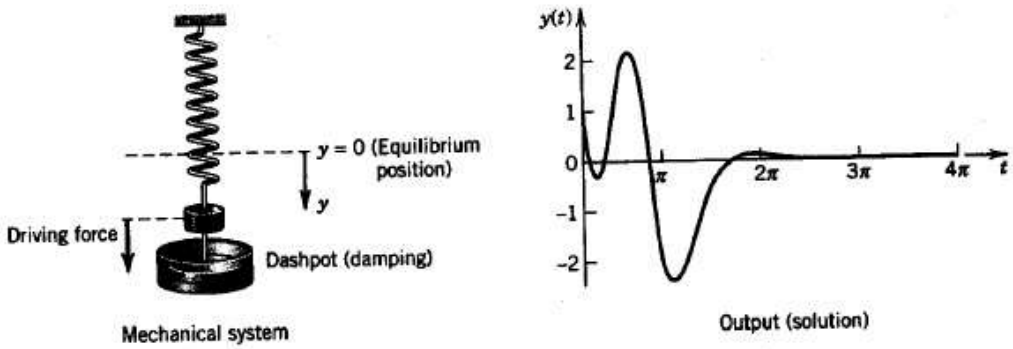


Fig. 124. Example 3

We take $-s + 5 - 2 = -s + 3$ to the right, collect the Y-terms to get $(s^2 + 2s + 2)Y$, and solve the equation for Y,

$$Y = \frac{20}{(s^2 + 4)(s^2 + 2s + 2)} - \frac{20e^{-\pi s}}{(s^2 + 4)(s^2 + 2s + 2)} + \frac{s - 3}{s^2 + 2s + 2}. \tag{4}$$

For the last fraction we get from Table 5.1 and the first shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{s + 1 - 4}{(s + 1)^2 + 1}\right\} = e^{-t}(\cos t - 4 \sin t). \tag{5}$$

In the first fraction in (4) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2 + 4)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 4} + \frac{Ms + N}{s^2 + 2s + 2}.$$

Multiplication by the common denominator gives

$$20 = (As + B)(s^2 + 2s + 2) + (Ms + N)(s^2 + 4).$$

We determine A, B, M, N. Equating the coefficients of each power of s on both sides gives the four equations

$$\begin{aligned}
&\text{(a)} \quad [s^3]: \quad 0 = A + M \\
&\text{(b)} \quad [s^2]: \quad 0 = 2A + B + N \\
&\text{(c)} \quad [s]: \quad 0 = 2A + 2B + 4M \\
&\text{(d)} \quad [s^0]: \quad 20 = 2B + 4N.
\end{aligned}$$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (4) has the representation

$$\frac{-2s - 2}{s^2 + 4} + \frac{2(s + 1) + 6 - 2}{(s + 1)^2 + 1}.$$

By Table 5.1 and the first shifting (Sec. 5.1), the inverse transform is

$$-2 \cos 2t - \sin 2t + e^{-t}(2 \cos t + 4 \sin t). \tag{6}$$

The sum of (5) and (6) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$y(t) = 3e^{-t} \cos t - 2 \cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi. \tag{7}$$

In the second fraction in (4) taken with the minus sign we have the factor $e^{-\pi s}$, so that from (6) and the second shifting theorem (Sec. 5.3) we get the inverse transform

$$+2\cos(2t-2\pi)+\sin(2t-2\pi)-e^{-(t-\pi)}[2\cos(t-\pi)+4\sin(t-\pi)]$$
$$=2\cos 2t+\sin 2t+e^{-(t-\pi)}(2\cos t+4\sin t).$$

The sum of this and (7) is the solution for $t>\pi$,

$$y(t)=e^{-t}[(3+2e^{\pi})\cos t+4e^{\pi}\sin t] \qquad \text{if } t>\pi. \tag{8}$$

Figure 124 shows (7) (for $0<t<\pi$) and (8) (for $t>0$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t=\pi$. ◀

Case 4. Repeated Complex Factors $[(s-a)(s-\bar{a})]^2$

In this case the partial fractions are of the form

$$\frac{As+B}{[(s-a)(s-\bar{a})]^2}+\frac{Ms+N}{(s-a)(s-\bar{a})}. \tag{9}$$

This case is important, for instance, in connection with resonance.

EXAMPLE 4 Repeated complex factors. Resonance

In an undamped mass–spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system. Then the model is (see Sec. 2.11)

$$y''+\omega_0^2 y=K\sin\omega_0 t$$

where $\omega_0^2=k/m$, k is the spring constant, and m is the mass of the body attached to the spring. We assume $y(0)=0$ and $y'(0)=0$, for simplicity. Then the subsidiary equation is

$$s^2Y+\omega_0^2Y=\frac{K\omega_0}{s^2+\omega_0^2}.$$

Hence

$$Y=\frac{K\omega_0}{(s^2+\omega_0^2)^2}.$$

The denominator is a repeated complex factor with double roots $s=i\omega_0$ and $-i\omega_0$. Hence our Y consists of this single partial fraction. Its inverse can be obtained by convolution. This is similar to Example 1 in Sec. 5.5. Using the convolution theorem and (11) in Appendix A3.1, we obtain

$$y(t)=\mathcal{L}^{-1}(Y)=\frac{K}{\omega_0}\sin\omega_0 t*\sin\omega_0 t$$
$$=\frac{K}{\omega_0}\int_0^t \sin\omega_0\tau\sin(\omega_0 t-\omega_0\tau)\,d\tau$$
$$=\frac{K}{2\omega_0}\left[\int_0^t -\cos\omega_0 t\,d\tau+\int_0^t\cos(2\omega_0\tau-\omega_0 t)\,d\tau\right]$$
$$=\frac{K}{2\omega_0^2}(-\omega_0 t\cos\omega_0 t+\sin\omega_0 t).$$

This is the solution of our problem. The first term grows without bound. It is clear that in the case of resonance such a term must occur. (See also Fig. 58 in Sec. 2.11, which shows a similar kind of solution.) ◀

PROBLEM SET 5.6

Inverse Transforms. Find the function $f(t)$ for a given transform $\mathcal{L}(f)$ by partial fraction reduction or any other method that you think is simplest or fastest. Indicate the method used and show the details of your work.

1. $\dfrac{6}{(s+2)(s-4)}$
2. $\dfrac{s^3+2s^2+2}{s^3(s^2+1)}$
3. $\dfrac{s^2+9s-9}{s^3-9s}$
4. $\dfrac{s}{(s+1)^2}$
5. $\dfrac{2s^3}{s^4-81}$
6. $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$
7. $\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3}$
8. $\dfrac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^3}$
9. $\dfrac{s^3+6s^2+14s}{(s+2)^4}$

Inverse Transforms. Derive the following formulas, showing the details of your work.

10. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4+4a^4}\right\} = \dfrac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$

11. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\} = \dfrac{1}{2a^2}\sinh at \sin at$

12. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{s^4+4a^4}\right\} = \dfrac{1}{2a}(\cosh at \sin at + \sinh at \cos at)$

13. $\mathcal{L}^{-1}\left\{\dfrac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$

14. **(Undamped system, no resonance)** What happens in Example 4 if you replace the driving force by $K \sin pt$ with $p^2 \neq \omega_0{}^2$? Find the solution by the Laplace transform. (Show the details of your work.)

15. **PROJECT. Heaviside Formulas. (a)** Show that for a simple root a and fraction $A/(s-a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s-a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s-a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m-k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s-a)^m F(s)}{G(s)}\right], \qquad k = 1, \cdots, m-1.$$

16. TEAM PROJECT. Laplace Transform of Periodic Functions

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

(10) [redacted] $(s > 0)$.

Prove this theorem. *Hint.* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n-1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (10), show that the half-wave rectification of $\sin \omega t$ in Fig. 125 has the Laplace transform

$$\mathcal{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 126.)

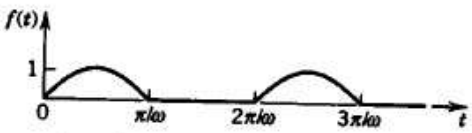


Fig. 125. Half-wave rectification

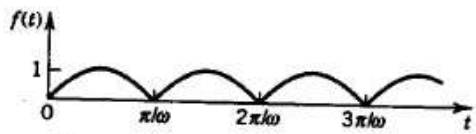


Fig. 126. Full-wave rectification

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}$$

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 127

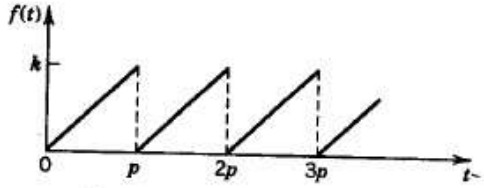


Fig. 127. Saw-tooth wave

(e) Staircase function. Find the Laplace transform of the staircase function in Fig. 128 by noting that it is the difference of kt/p and the function in (d).

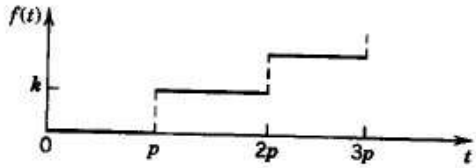


Fig. 128. Staircase function

5.7 Systems of Differential Equations

The Laplace transform method may also be used for solving *systems* of differential equations. We shall explain this in terms of typical applications.

For a first-order linear system (as discussed in Sec. 3.1),

$$(1)\qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t), \end{aligned}$$

writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) in Sec. 5.2 the subsidiary equations

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s) \end{aligned}$$

or, by collecting the Y_1- and Y_2-terms,

$$(2)\qquad \begin{aligned} (a_{11} - s)Y_1 + \quad a_{12}Y_2 \quad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \quad + (a_{22} - s)Y_2 &= -y_2(0) - G_2(s). \end{aligned}$$

This must be solved algebraically for $Y_1(s)$ and $Y_2(s)$. The solution of the given system is then obtained if we take the inverse $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$.

EXAMPLE 1 Mixing problem involving two tanks

Tank T_1 in Fig. 129 contains initially 100 gal of pure water. Tank T_2 contains initially 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1. The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

Solution. The model is obtained in the form of two equations

$$\text{Time rate of change} = \text{Inflow/min} - \text{Outflow/min}$$

for the two tanks (see Sec. 3.1). Thus,

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6, \qquad y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2.$$

The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this we see that the subsidiary equations (2) are

$$\begin{aligned} (-0.08 - s)Y_1 + \quad 0.02Y_2 \quad &= -\frac{6}{s} \\ 0.08Y_1 \quad + (-0.08 - s)Y_2 &= -150. \end{aligned}$$

We solve these algebraically for Y_1 and Y_2 by Cramer's rule (Sec. 6.6) or by elimination, and we write the solutions in terms of partial fractions,

$$Y_1 = \frac{9s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} - \frac{62.5}{s + 0.12} - \frac{37.5}{s + 0.04}$$

$$Y_2 = \frac{150s^2 + 12s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} + \frac{125}{s + 0.12} - \frac{75}{s + 0.04}.$$

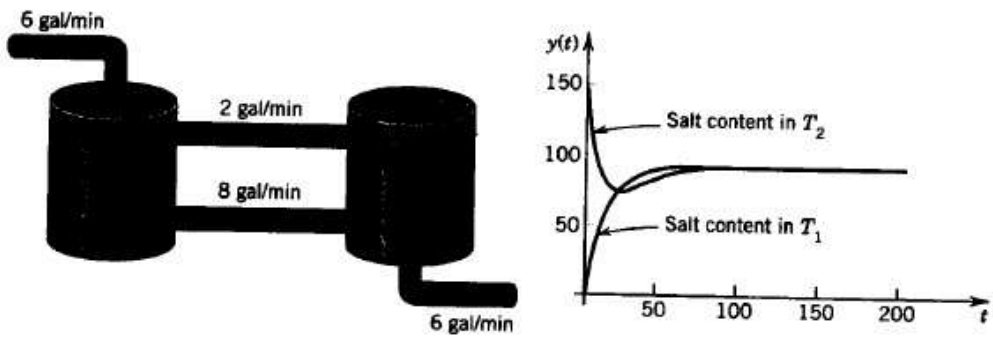


Fig. 129. Mixing problem in Example 1

By taking the inverse transform we arrive at the solution

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$
$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}$$

Figure 129 shows the interesting plot of these functions. Can you give physical explanations for their main features? Why do they have the limit 100? Why is y_2 not monotone, whereas y_1 is? Why is y_1 from some time on all of a sudden larger than y_2? Etc. ◀

Other systems of differential equations of practical importance can be solved by the Laplace transform method in a similar way, and eigenvalues and eigenvectors as we had to determine them in Chap. 3 will come out automatically, as we have seen in Example 1.

EXAMPLE 2 Electrical network

Find the currents $i_1(t)$ and $i_2(t)$ in the network in Fig. 130 with L and R measured in terms of the usual units (see Sec. 1.7), $v(t) = 100$ volts if $0 \leqq t \leqq 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.

Solution. The model of the network is obtained from Kirchhoff's voltage law as in Secs. 1.7 and 2.12,

$$\begin{aligned} 0.8i_1' + 1(i_1 - i_2) + 1.4i_1 &= 100\left[1 - u(t - \tfrac{1}{2})\right] \\ 1 \cdot i_2' + 1(i_2 - i_1) &= 0. \end{aligned}$$

Division by 0.8 and ordering gives

$$\begin{aligned} i_1' + 3i_1 - 1.25i_2 &= 125\left[1 - u(t - \tfrac{1}{2})\right] \\ i_2' - i_1 + i_2 &= 0. \end{aligned}$$

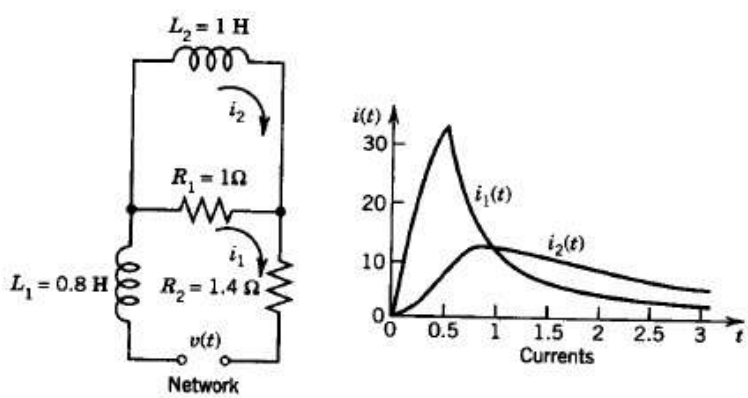


Fig. 130. Electrical network in Example 2

With $i_1(0) = 0$, $i_2(0) = 0$ we obtain from (1) in Sec. 5.2 and the second shifting theorem the subsidiary equations

$$\begin{aligned} (s+3)I_1 - 1.25I_2 &= 125\left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right) \\ -I_1 + (s+1)I_2 &= 0. \end{aligned}$$

Solving algebraically for I_1 and I_2 gives

$$I_1 = \frac{125(s+1)}{s(s+\frac{1}{2})(s+\frac{7}{2})}(1 - e^{-s/2})$$

$$I_2 = \frac{125}{s(s+\frac{1}{2})(s+\frac{7}{2})}(1 - e^{-s/2}).$$

The right sides without the factor $1 - e^{-s/2}$ have the partial fraction expansions

$$\frac{500}{7s} - \frac{125}{3(s+\frac{1}{2})} - \frac{625}{21(s+\frac{7}{2})}, \qquad \frac{500}{7s} - \frac{250}{3(s+\frac{1}{2})} + \frac{250}{21(s+\frac{7}{2})},$$

respectively. The inverse transform of this gives the solution for $0 \leqq t \leqq \frac{1}{2}$,

$$\begin{aligned} i_1(t) &= -\frac{125}{3}e^{-t/2} - \frac{625}{21}e^{-7t/2} + \frac{500}{7} \\ i_2(t) &= -\frac{250}{3}e^{-t/2} + \frac{250}{21}e^{-7t/2} + \frac{500}{7} \end{aligned} \qquad (0 \leqq t \leqq \tfrac{1}{2}).$$

According to the second shifting theorem, the solution for $t > \frac{1}{2}$ is obtained if we subtract from this $i_1(t - \frac{1}{2})$ and $i_2(t - \frac{1}{2})$, respectively. This gives

$$\begin{aligned} i_1(t) &= -\frac{125}{3}(1 - e^{1/4})e^{-t/2} - \frac{625}{21}(1 - e^{7/4})e^{-7t/2} \\ i_2(t) &= -\frac{250}{3}(1 - e^{1/4})e^{-t/2} + \frac{250}{21}(1 - e^{7/4})e^{-7t/2} \end{aligned} \qquad (t > \tfrac{1}{2}).$$

Can you explain physically why both currents eventually go to zero, and why $i_1(t)$ has a sharp cusp whereas $i_2(t)$ has a continuous tangent direction at $t = \frac{1}{2}$? ◀

Systems of differential equations of higher order can be solved by the Laplace transform method in a similar fashion. As an important application, typical of many similar mechanical systems, we consider coupled vibrating masses on springs.

EXAMPLE 3 Model of two masses on springs (Fig. 131)

The mechanical system in Fig. 131 consists of two bodies of mass 1 on three springs and is governed by the differential equations

$$\begin{aligned} y_1'' &= -ky_1 + k(y_2 - y_1) \\ y_2'' &= -k(y_2 - y_1) - ky_2, \end{aligned} \tag{3}$$

where k is the spring constant of each of the three springs, and y_1 and y_2 are the displacements of the bodies from their positions of static equilibrium; the masses of the springs and damping are neglected. The equations follow from ***Newton's second law,*** *Mass* × *Acceleration* = *Force,* as in Sec. 2.5 for a single body. In the equations, $-ky_1$ is the force of the upper spring on the first body and $k(y_2 - y_1)$ that of the middle spring, where $y_2 - y_1$ is the net change in spring length—think this over before going on. Also, $-k(y_2 - y_1)$ is the force of the middle spring on the lower body and $-ky_2$ that of the lower spring.

We shall determine the solution corresponding to the initial conditions $y_1(0) = 1$, $y_2(0) = 1$, $y_1'(0) = \sqrt{3k}$, $y_2'(0) = -\sqrt{3k}$. Let $Y_1 = \mathcal{L}(y_1)$ and $Y_2 = \mathcal{L}(y_2)$. Then from (2) in Sec. 5.2 and the initial conditions we obtain the subsidiary equations

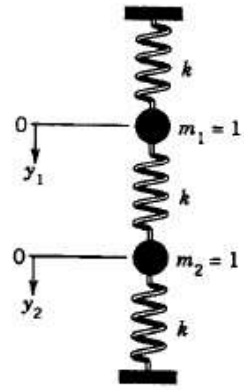


Fig. 131. Example 3

$$s^2Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2.$$

This system of linear algebraic equations in the unknowns Y_1 and Y_2 may be written

$$\begin{aligned} (s^2 + 2k)Y_1 - \quad kY_2 \quad &= s + \sqrt{3k} \\ -kY_1 \quad + (s^2 + 2k)Y_2 &= s - \sqrt{3k}. \end{aligned}$$

Cramer's rule (Sec. 6.6) or elimination yields the solution, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s + \sqrt{3k})(s^2 + 2k) + k(s - \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$
$$Y_2 = \frac{(s^2 + 2k)(s - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}.$$

Hence the solution of our initial value problem is

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$
$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" and a "rapid" oscillation. ◀

Problem Set 5.7

Systems of Differential Equations. In Probs. 1–14 solve the given initial value problem by means of Laplace transforms. (Show the details of your work.)

1. $y_1' = -y_1 + y_2, \quad y_2' = -y_1 - y_2, \quad y_1(0) = 1, \quad y_2(0) = 0$
2. $y_1' = 6y_1 + 9y_2, \quad y_2' = y_1 + 6y_2, \quad y_1(0) = -3, \quad y_2(0) = -3$
3. $y_1' = -y_1 + 4y_2, \quad y_2' = 3y_1 - 2y_2, \quad y_1(0) = 3, \quad y_2(0) = 4$
4. $y_1' = 5y_1 + y_2, \quad y_2' = y_1 + 5y_2, \quad y_1(0) = -3, \quad y_2(0) = 7$
5. $y_1' + y_2 = 2\cos t, \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1$

6. $y_1'' + y_2 = -5\cos 2t, \quad y_2'' + y_1 = 5\cos 2t,$
$y_1(0) = 1, \quad y_1'(0) = 1, \quad y_2(0) = -1, \quad y_2'(0) = 1$

7. $y_1'' = y_1 + 3y_2, \quad y_2'' = 4y_1 - 4e^t,$
$y_1(0) = 2, \quad y_1'(0) = 3, \quad y_2(0) = 1, \quad y_2'(0) = 2$

8. $y_1'' = -5y_1 + 2y_2, \quad y_2'' = 2y_1 - 2y_2,$
$y_1(0) = 3, \quad y_1'(0) = 0, \quad y_2(0) = 1, \quad y_2'(0) = 0$

9. $y_1' + y_2' = 2\sinh t, \quad y_2' + y_3' = e^t, \quad y_3' + y_1' = 2e^t + e^{-t},$
$y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0$

10. $2y_1' - y_2' - y_3' = 0, \quad y_1' + y_2' = 4t + 2, \quad y_2' + y_3 = t^2 + 2,$
$y_1(0) = y_2(0) = y_3(0) = 0$

11. $y_1' = -y_2 + 1 - u(t-1), \quad y_2' = y_1 + 1 - u(t-1), \quad y_1(0) = 0, \quad y_2(0) = 0$

12. $y_1' + y_2 = 2[1 - u(t - 2\pi)]\cos t, \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1$

13. $y_1' = 2y_1 - 4y_2 + u(t-1)e^t, \quad y_2' = y_1 - 3y_2 + u(t-1)e^t, \quad y_1(0) = 3, \quad y_2(0) = 0$

14. $y_1' = 2y_1 + 4y_2 + 64tu(t-1), \quad y_2' = y_1 + 2y_2, \quad y_1(0) = -4, \quad y_2(0) = -4$

15. **TEAM PROJECT. First-Order Linear Systems of Differential Equations.**
   (a) **Models.** Solve the models in Examples 1 and 2 of Sec. 3.1 by the Laplace transform method and compare the work with that in Sec. 3.1.
   (b) **Homogeneous Systems.** Solve the systems (8), (11)–(13) in Sec. 3.3 by the Laplace transform method.
   (c) **Nonhomogeneous Systems.** Solve the systems (3) and (4) in Sec. 3.6 by the Laplace transform method.
   In (a)–(c) always show the details of your work.

Further Applications

16. **(Mixing problem)** What will happen in Example 1 if you double all flows, leaving the size of the tanks and the initial conditions as before. First guess, then calculate. Can you relate the new solution to the old one?

17. **(Sinusoidal inflow from the outside)** What will happen in Example 1 if you assume that the total salt content of the inflowing 6 gal/min varies between 0 and 6 according to $6\sin^2 t$? Why is $y_2(t)$ much less wavy than $y_1(t)$?

18. **(Forced vibrations of two masses)** Solve the model in Example 3 with $k = 3$ and initial conditions $y_1(0) = 1$, $y_2(0) = 1$, $y_1'(0) = 3$, $y_2'(0) = -3$ under the assumption that the force $8\sin t$ acts on the first body and $-8\sin t$ acts on the second.

19. **(Electrical network)** Using the Laplace transform method, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 132, where $v(t) = 195\sin t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state? Guess what the little curve in Fig. 132 is.

20. **(Single sine wave)** Solve Prob. 19 when the electromotive force acts from 0 to 2π only. Can you obtain the solution from that in Prob. 19 practically without calculation?

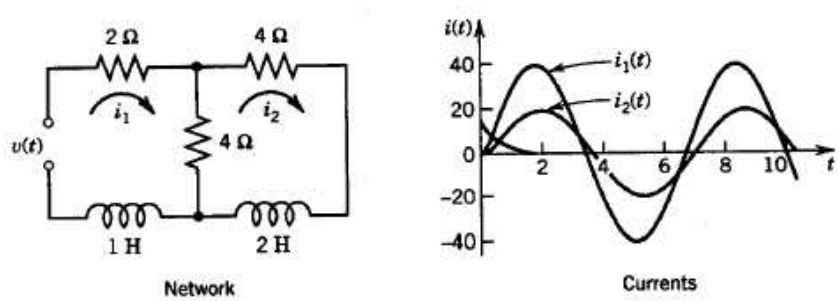


Fig. 132. Electrical network and currents in Problem 19

5.8 Laplace Transform: General Formulas

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	5.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	5.1
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)	5.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	5.2
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (Second Shifting Theorem)	5.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Differentiation of Transform Integration of Transform	5.4
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ $= \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	5.5
$\mathcal{L}(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p	5.6 Project 16

5.9 Table of Laplace Transforms

For more extensive tables, see Refs.* [A4] *and* [A6] *in Appendix 1.

1	$1/s$	1	5.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	5.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!} t^{n-1}e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$	
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\sin \omega t$	5.1
14	$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$	
15	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$	
16	$\dfrac{s}{s^2-a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}e^{at}\sin \omega t$	
18	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos \omega t$	
19	$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$	5.2
20	$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$	
21	$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	5.4

(continued)

Table of Laplace Transforms (*continued*)

22	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\sin\omega t$	5.4
23	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin\omega t+\omega t\cos\omega t)$	
24	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}\quad(a^2\neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at-\cos bt)$	
25	$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt\cos kt-\cos kt\sinh kt)$	5.6
26	$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2}\sin kt\sinh kt$	
27	$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt-\sin kt)$	
28	$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt-\cos kt)$	
29	$\sqrt{s-a}-\sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2}I_0\left(\dfrac{a-b}{2}t\right)$	4.6
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	4.5
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	
33	$\dfrac{1}{(s^2-a^2)^k}\quad(k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2}I_{k-1/2}(at)$	4.6
34	e^{-as}/s	$u(t-a)$	5.3
35	e^{-as}	$\delta(t-a)$	
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	4.5
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}}\quad(k>0)$	$\dfrac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$	
40	$\dfrac{1}{s}\ln s$	$-\ln t-\gamma\quad(\gamma\approx 0.5772)$	4.6
41	$\ln\dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt}-e^{at})$	5.4

(*continued*)

Table of Laplace Transforms (*continued*)

42	$\ln \dfrac{s^2 + \omega^2}{s^2}$	$\dfrac{2}{t}(1 - \cos \omega t)$	5.4
43	$\ln \dfrac{s^2 - a^2}{s^2}$	$\dfrac{2}{t}(1 - \cosh at)$	
44	$\arctan \dfrac{\omega}{s}$	$\dfrac{1}{t} \sin \omega t$	
45	$\dfrac{1}{s} \operatorname{arc\,cot} s$	$\mathrm{Si}(t)$	App. A3.1

CHAPTER 5 REVIEW

1. Compared to the usual method, what are the advantages of the Laplace transform in solving differential equations?
2. What is the crucial property of the Laplace transform that makes it suitable for solving differential equations?
3. What do we mean by saying that the Laplace transform is a *linear* operation? Why is this practically important?
4. For what problems would you prefer the Laplace transform over the usual method? Give a reason.
5. What is the subsidiary equation? How is it used?
6. Does every continuous function have a Laplace transform? Give a reason or a counterexample.
7. What is the unit step function? Why is it important?
8. What is Dirac's delta function? How did we use it?
9. State the Laplace transforms of a few simple functions from memory.
10. State the formula for the Laplace transform of the nth derivative of a function $f(t)$ from memory.
11. Can a discontinuous function have a Laplace transform? (Give a reason for your answer.)
12. Does $\tan t$ have a Laplace transform? Is it piecewise continuous?
13. If you know $f(t) = \mathcal{L}^{-1}\{F(s)\}$, how would you find $\mathcal{L}^{-1}\{F(s)/s^2\}$?
14. Is $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Or what?
15. What is the difference in the shifting by the first shifting theorem and by the second shifting theorem?

Laplace Transform. Find the Laplace transform of the given function. (Show the details of your work.)

16. $e^{-t} \sin \pi t$ **17.** $\cos^2 t$ **18.** $\sin^2 (\pi t/2)$

19. $e^t u(t - 2)$ **20.** $t^2 u(t - \frac{1}{4})$ **21.** $t * e^{-3t}$

22. $e^{2t} * \cos 4t$ **23.** $\cosh \frac{1}{10} t$ **24.** $t \cos t + \sin t$

Inverse Laplace Transform. In Probs. 25–33 find the inverse Laplace transform of the given function. (Show your work.)

25. $\dfrac{s + 3}{s^2 + 9}$ **26.** $\dfrac{1}{s^2 - 2s - 8}$ **27.** $\dfrac{s + 1}{s^2} e^{-s}$

28. $\dfrac{6(s+1)}{s^4}$

29. $\dfrac{s^2-6s+4}{s^3-3s^2+2s}$

30. $\dfrac{2s^2-3s+4}{(s^2+4)(s-3)}$

31. $\dfrac{s^3-s^2-s+4}{s^4-5s^2+4}$

32. $\dfrac{\omega\cos\theta+s\sin\theta}{s^2+\omega^2}$

33. $\dfrac{3s+4}{s^2+4s+5}$

Differential Equations. Systems of Differential Equations. Solve the given initial value problems by Laplace transforms. Show the details of your work.

34. $y''+y=\delta(t-2),\quad y(0)=2.5,\quad y'(0)=0$
35. $y''+4y=u(t-3),\quad y(0)=1,\quad y'(0)=0$
36. $y''-2y'+2y=8e^{-t}\cos t,\quad y(0)=16,\quad y'(0)=-16$
37. $y''+9y=6\sin t$ if $0\leqq t\leqq\pi$ and 0 if $t>\pi$, $\quad y(0)=0,\quad y'(0)=0$
38. $y''+3y'+2y=2u(t-2),\quad y(0)=0,\quad y'(0)=0$
39. $y_1'=-y_2,\quad y_2'=y_1,\quad y_1(0)=1,\quad y_2(0)=0$
40. $y_1'=2y_1+4y_2,\quad y_2'=y_1+2y_2,\quad y_1(0)=-4,\quad y_2(0)=-4$
41. $y_1'=2y_1-4y_2,\quad y_2'=y_1-3y_2,\quad y_1(0)=3,\quad y_2(0)=0$
42. $y_1'=-2y_1+3y_2,\quad y_2'=4y_1-y_2,\quad y_1(0)=4,\quad y_2(0)=3$

Models of Mass–Spring Systems, Circuits, Networks

Solve the following problems by Laplace transforms. (Show your work.)

43. Show that the model of the mechanical system in Fig. 133 (no friction, no damping) is

$$m_1y_1''=-k_1y_1+k_2(y_2-y_1)$$
$$m_2y_2''=-k_2(y_2-y_1)-k_3y_2.$$

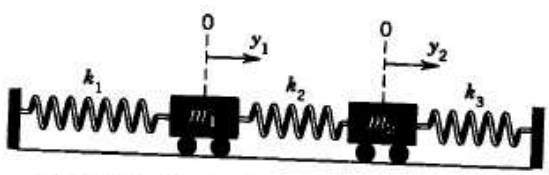


Fig. 133. System in Problems 43 and 44

44. In Prob. 43, let $m_1=m_2=10$ kg, $k_1=k_3=20$ kg/sec^2, $k_2=40$ kg/sec^2. Find the solution satisfying the initial conditions $y_1(0)=y_2(0)=0$, $y_1'(0)=1$ meter/sec, $y_2'(0)=-1$ meter/sec.
45. Find the current $i(t)$ in the RC-circuit in Fig. 134, where $R=10$ ohms, $C=0.1$ farad, $e(t)=10t$ volts if $0<t<4$, $v(t)=40$ volts if $t>4$, and the initial charge on the capacitor is 0.

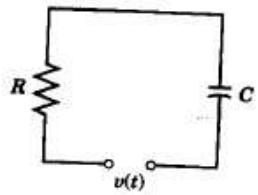


Fig. 134. *RC*-circuit

46. Find the charge $q(t)$ and the current $i(t)$ in the LC-circuit in Fig. 135, assuming $L = 1$ henry, $C = 1$ farad, $v(t) = 1 - e^{-t}$ if $0 < t < \pi$, $e(t) = 0$ if $t > \pi$, and zero initial current and charge.

47. Find the current $i(t)$ in the RLC-circuit in Fig. 136, where $R = 160$ ohms, $L = 20$ henrys, $C = 0.002$ farad, $v(t) = 37 \sin 10t$ volts, assuming zero initial current and charge.

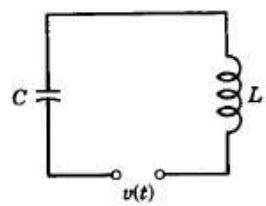


Fig. 135. LC-circuit

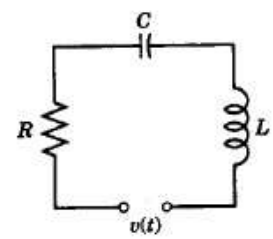


Fig. 136. RLC-circuit

48. Show that, by Kirchhoff's voltage law (Sec. 1.7), the currents in the network in Fig. 137 are obtained from the system

$$Li_1' + R(i_1 - i_2) = v(t)$$
$$R(i_2' - i_1') + \frac{1}{C} i_2 = 0.$$

Solve the system, assuming that $R = 10$ ohms, $L = 20$ henrys, $C = 0.05$ farad, $v = 20$ volts, $i_1(0) = 0$, $i_2(0) = 2$ amperes.

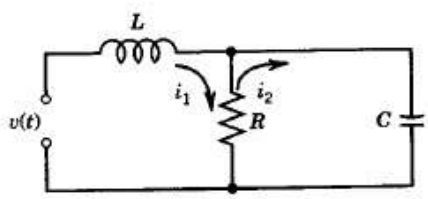


Fig. 137. Network in Problems 48 and 49

49. Solve the system in Prob. 48 when $R = 0.8$ ohm, $L = 1$ henry, $C = 0.25$ farad, $v = \frac{4}{5}t + \frac{21}{25}$ volt, $i_1(0) = 1$ ampere, and $i_2(0) = -3.8$ ampere. What do the initial conditions mean physically?

50. Set up the model of the network in Fig. 138 and find the solution, assuming that all charges and currents are 0 when the switch is closed at $t = 0$. Find the limits of $i_1(t)$ and $i_2(t)$ as $t \to \infty$, (i) from the solution, (ii) directly from the given network.

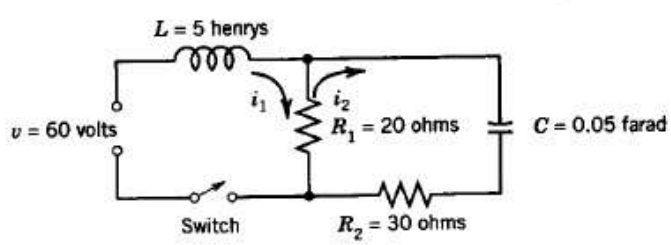


Fig. 138. Network in Problem 50

SUMMARY OF CHAPTER 5. LAPLACE TRANSFORMS

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

(Sec. 5.1)

This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$\begin{aligned} \mathcal{L}(f') &= s\mathcal{L}(f) - f(0), \\ \mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0), \end{aligned} \tag{2}$$

(Sec. 5.2)

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \tag{3}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subsidiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0). \tag{4}$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the small table in Sec. 5.1 or the larger table in Sec. 5.9. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the inverse transform $y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem. This is generally the hardest step, and in it we may again use one of those two tables. $Y(s)$ will often be a rational function, so that we can obtain the inverse $\mathcal{L}^{-1}(Y)$ by partial fraction reduction (Sec. 5.6) if we see no simpler way.

The Laplace method avoids the determination of a general solution of the homogeneous equation, and we also need not determine values of arbitrary constants in a general solution from initial conditions; instead, we can insert the latter directly into (4). Two further facts account for the practical importance of the Laplace transform. First, it has some basic properties and resulting techniques that simplify the determination of transforms and inverses. The most important of these properties are listed in Sec. 5.8, together with references to the corresponding sections. More on the use of unit step functions and Dirac's delta can be found in Sec. 5.3, and more on convolution in Sec. 5.5. Second, due to these properties, the present method is particularly suitable for handling right sides $r(t)$ given by different expressions over different intervals of time, for instance, when $r(t)$ is a square wave or an impulse or of a form such as $r(t) = \cos t$ if $0 \leqq t \leqq 4\pi$ and 0 elsewhere.

The application of the Laplace transform to systems of differential equations is shown in Sec. 5.6. (The application to partial differential equations follows in Sec. 11.12.)