Differentiation and integration of Transform:

Differentiation of Transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds}[F(s)] = \int_0^\infty e^{-st}(-t) f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -\mathcal{L}\{t f(t)\}$$

$$\boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds}[F(s)]}$$

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$\mathcal{L}\{t^3 f(t)\} = (-1)^3 \frac{d^3}{ds^3} F(s)$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

$$\Rightarrow \mathcal{L}\{t f(t)\} = -F'(s)$$

$$t f(t) = -\mathcal{L}^{-1}\{F'(s)\}$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

Ex 1:

i) If $f(t) = \frac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$

then show that

$$\mathcal{L}\{f(t)\} = \frac{1}{(s^2+\beta^2)^2}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{2\beta^3}\left[\mathcal{L}\{\sin\beta t\} - \beta\mathcal{L}\{t\cos\beta t\}\right]$$

$$= \frac{1}{2\beta^3}\left[\frac{\beta}{(s^2+\beta^2)} - \beta\left[\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right)\right]\right]$$

$$= \frac{+1}{2\beta^3}\left[\frac{\beta}{s^2+\beta^2} - \beta\left(\frac{(s^2+\beta^2).1 - s(2s)}{(s^2+\beta^2)^2}\right)\right]$$

$$= \frac{1}{2\beta^2}\cdot\frac{1}{s^2+\beta^2} - \frac{1}{2\beta^2}\cdot\frac{-s^2+\beta^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}\{f(t)\} = \frac{1\, s^2+\beta^2-s^2+\beta^2}{2\beta^2(s^2+\beta^2)^2}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{(s^2+\beta^2)^2}$$

ii). $\mathcal{L}\left\{\frac{t}{2\beta}\sin\beta t\right\} = \frac{s}{(s^2+\beta^2)^2}$

$$\mathcal{L}\left\{\frac{t}{2\beta}\sin\beta t\right\} = \frac{1}{2\beta}\mathcal{L}\{t\sin\beta t\}$$

$$= \frac{1}{2\beta}\left[\frac{-d}{ds}\left(\frac{\beta}{(s^2+\beta^2)}\right)\right]$$

$$= -\frac{1}{2\beta}\left[\frac{(s^2+\beta^2).(0) - \beta(2s)}{(s^2+\beta^2)^2}\right]$$

$$= -\frac{1}{2\beta}\left[\frac{-2\beta s}{(s^2+\beta^2)^2}\right]$$

$$\mathcal{L}\left\{\frac{t}{2\beta}\sin\beta t\right\} = \frac{s}{(s^2+\beta^2)^2}$$

iii). $\mathcal{L}\left\{\frac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)\right\}$

$$\mathcal{L}\left\{\frac{1}{2\beta}\cancel{(\sin\beta t) + \beta t}\cos\beta t\right.$$

$$\mathcal{L}\left\{\frac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)\right\} = \frac{1}{2\beta}\mathcal{L}\{\sin\beta t\} + \frac{1}{2}\mathcal{L}\{t\cos\beta t\}$$

$$= \frac{1}{2\beta}\cdot\frac{\beta}{(s^2+\beta^2)} + \frac{1}{2}\cdot\left[\frac{-d}{ds}\left(\frac{s}{s^2+\beta^2}\right)\right]$$

$$= \frac{1}{2(s^2+\beta^2)} - \frac{1}{2}\left[\frac{(s^2+\beta^2)\cdot 1 - (s)(2s)}{(s^2+\beta^2)^2}\right]$$

$$= \frac{1}{2(s^2+\beta^2)} - \frac{1}{2}\frac{(-s^2+\beta^2)}{(s^2+\beta^2)^2}$$

$$= \frac{s^2+\beta^2+s^2-\beta^2}{2(s^2+\beta^2)^2}$$

$$= \frac{2s^2}{2(s^2+\beta^2)^2}$$

$$= \frac{s^2}{(s^2+\beta^2)^2}$$

Hence proved.

Integration of Transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left\{\int_0^\infty e^{-st} f(t)\,dt\right\} ds$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st}\,ds\right) f(t)\,dt$$

$$= \int_0^\infty \left.\frac{e^{-st}}{-t}\right|_s^\infty f(t)\,dt$$

$$= \int_0^\infty \frac{-1}{t}\left(0 - e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= \mathcal{L}\{f(t)/t\}$$

$$\int_s^\infty F(s)\,ds = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$$

$$f(t) = t\,\mathcal{L}^{-1}\left\{\int_0^\infty F(s)\,ds\right\}$$

$$f(t) = -\frac{1}{t}\,\mathcal{L}^{-1}\{F'(s)\}$$

Ex2: Find inverse Laplace Transform of.

$$F(s) = \ln\left(\frac{s^2+w^2}{s^2}\right)$$

$$= \ln(s^2+w^2) - \ln(s^2)$$

$$F'(s) = \frac{2s}{s^2+w^2} - \frac{2s}{s^2}$$

$$F'(s) = \frac{2s}{s^2+w^2} - \frac{2}{s}$$

$$f(t) = -\frac{1}{t}\,\mathcal{L}^{-1}\{F'(s)\}$$

$$= -\frac{1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+w^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right]$$

$$f(t) = -\frac{1}{t}\left[2\cos\omega t - 2\right]$$

$$= \frac{2\cos\omega t}{t} + \frac{2}{t}$$

Ex 3: Find Inverse Laplace Transform of

$$\left[\ln\left(1 - \frac{a^2}{s^2}\right)\right]$$

$$F(s) = \ln\left(\frac{s^2 - a^2}{s^2}\right)$$

$$= \ln(s^2 - a^2) - \ln s^2$$

$$F'(s) = \frac{2s}{s^2 - a^2} - \frac{2s}{s^2}$$

$$= \frac{2s}{s^2 - a^2} - \frac{2}{s}$$

$$f(t) = \frac{-1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

$$= \frac{-1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2 - a^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right]$$

$$= -\frac{1}{t}\left[2\cosh at - 2\right]$$

$$= \frac{-2}{t}\cosh at + \frac{2}{t}$$

Ex # 5.3

1. te^t

$$\mathcal{L}\{te^t\} = -\frac{d}{ds}\mathcal{L}\{e^t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$= -\left[\frac{(s-1).0-1(1)}{(s-1)^2}\right]$$

$$\mathcal{L}\{te^t = \frac{1}{(s-1)^2}$$

2. $3t\sinh 4t$

$$\mathcal{L}\{3t\sinh 4t\} = 3\left[-\frac{d}{ds}\mathcal{L}\{\sinh 4t\}\right]$$

$$= -3\frac{d}{ds}\left(\frac{4}{s^2-16}\right)$$

$$= -3\left[\frac{(s^2-16).0-4(2s)}{(s^2-16)^2}\right]$$

$$= \frac{24s}{(s^2-16)^2}$$

3. $t^2\cosh \pi t$

$$\mathcal{L}\{t^2\cosh \pi t\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\cosh \pi t\}$$

$$= 1.\frac{d}{ds}.\frac{d}{ds}\left(\frac{1\,s}{(s^2-\pi^2)}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2-\pi^2).1-s(2s)}{(s^2-\pi^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-\pi^2-s^2}{(s^2-\pi^2)^2}\right]$$

$$= \frac{(s^2-\pi^2)^2.(-2s)-(-\pi^2-s^2)[2(s^2-\pi^2).2s]}{((s^2-\pi^2)^2)^2}$$

$$= \frac{-2s(s^2-\pi^2)^2+4s(s^2-\pi^2)(s^2+\pi^2)}{(s^2-\pi^2)^4}$$

$$= \frac{s(s^2-\pi^2)[-2(s^2-\pi^2)+4(s^2+\pi^2)]}{(s^2-\pi^2)^4}$$

$$= \frac{s(2s^2+6\pi^2)}{(s^2-\pi^2)^3}$$

$$= \frac{2s^3+6\pi^2 s}{(s^2-\pi^2)^3}$$

4. $te^{-t}\cos t$

$$\mathcal{L}\{te^{-t}\cos t\} = -\frac{d}{ds}\mathcal{L}\{e^{-t}\cos t\}$$

$$= -\frac{d}{ds}\mathcal{L}\{\cos t\}_{s\to s+1}$$

$$= -\frac{d}{ds}\left.\frac{s}{s^2+1}\right|_{s\to s+1}$$

$$= -\frac{d}{ds}\frac{s+1}{(s+1)^2+1}$$

$$= -\left[\frac{((s+1)^2+1).1-(s+1)[2(s+1)]}{[(s+1)^2+1]^2}\right]$$

$$= -\left[\frac{s^2+2s+2-2(s+1)^2}{(s^2+2s+2)^2}\right]$$

$$= -\left[\frac{s^2+2s+2-2s^2-2-4s}{(s^2+2s+2)^2}\right]$$

$$= \frac{s^2+2s}{(s^2+2s+2)^2}$$

5. $t\cos wt$

$$\mathcal{L}\{t\cos wt\} = -\frac{d}{ds}\mathcal{L}\{\cos wt\}$$

$$= -\frac{d}{ds}\frac{s}{s^2+w^2}$$

$$= -\left[\frac{(s^2+w^2).1-s(2s)}{(s^2+w^2)^2}\right]$$

$$= \frac{s^2-w^2}{(s^2+w^2)^2}$$

6. $t^2\sin 2t$

$$\mathcal{L}\{t^2\sin 2t\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\sin 2t\}$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left[\frac{2}{s^2+4}\right]\right]$$

$$= \frac{d}{ds}\left(\frac{(s^2+4).0-2(+2s)}{(s^2+4)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right)$$

$$= \frac{(s^2+4)^2.(-4)-(-4s)[4s(s^2+4)]}{(s^2+4)^4}$$

$$= \frac{(s^2+4)[-4(s^2+4)+16s^2]}{(s^2+4)^4}$$

$$\mathcal{L}\{t^2 \sin 2t\} = \frac{16s^2 - 4s^2 - 16}{(s^2+4)^3}$$

$$= \frac{12s^2 - 16}{(s^2+4)^3}$$

7. $te^{-t}\sin t$

$$\mathcal{L}\{te^{-t}\sin t\} = -\frac{d}{ds}\mathcal{L}\{e^{-t}\sin t\}$$

$$= -\frac{d}{ds}\mathcal{L}\{\sin t\}_{t \to t+1}$$

$$= -\frac{d}{ds}\frac{1}{s^2+1}\Big|_{s \to s+1}$$

$$= -\frac{d}{ds}\frac{1}{(s+1)^2+1}$$

$$= -\left[\frac{[(s+1)^2+1].0 - 1[2(s+1)]}{((s+1)^2+1)^2}\right]$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2}$$

8. $t^2 \cos wt$

$$\mathcal{L}\{t^2\cos wt\} = (-1)^2\frac{d}{ds}\left(\frac{d}{ds}\mathcal{L}\{\cos wt\}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{s^2+w^2}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+w^2).1 - s(2s)}{(s^2+w^2)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{w^2-s^2}{(s^2+w^2)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+w^2)^2.(-2s) - (w^2-s^2)[4s(s^2+w^2)]}{(s^2+w^2)^4}\right)$$

$$= \frac{(s^2+w^2)[(s^2+w^2)(-2s) + (s^2-w^2)4s]}{(s^2+w^2)^4}$$

$$= \frac{2s^3 - 6w^2s}{(s^2+w^2)^3}.$$

Inl

9. $\frac{1}{(s+3)^3}$

$$F(s) = \frac{1}{(s+3)^3}$$

$$F'(s) = \frac{-3}{(s+3)^4}$$

$$f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{-3}{(s+3)^4}\right\}$$

$$= \frac{3}{t}\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^4}\right\}$$

$$= \frac{3}{3!\,t}\mathcal{L}^{-1}\left\{\frac{3!}{(s+3)^4}\right\}$$

$$= \frac{1}{2t}e^{-3t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= \frac{1}{2t}e^{-3t}\cdot t^3$$

$$= \frac{t^2}{2}e^{-3t}$$

10. $\dfrac{s}{(s^2-9)^2}$

$$F(s) = \frac{1.2s}{2(s^2-9)^2}$$

$$\int_s^\infty F(s)\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2-9)^2}\,ds$$

$$= \frac{1}{2}\left[-\frac{1}{s^2-9}\right]_s^\infty$$

$$= -\frac{1}{2}\left(0 - \frac{1}{s^2-9}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s^2-9}\right)$$

$$f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{s^2-9}\right)\right\}$$

$$= \frac{1}{6}\cdot t\,\mathcal{L}^{-1}\left\{\frac{3}{s^2-(3)^2}\right\}$$

$$= \frac{t}{6}\cdot \sinh 3t$$

11. $\dfrac{s^2-\pi^2}{(s^2+\pi^2)^2}$

12. $\dfrac{2s+6}{(s^2+6s+10)^2}$

$$F(s) = \frac{2s+6}{(s^2+6s+10)}$$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \frac{2s+6}{(s^2+6s+10)^2}\,ds$$

$$= \left| \frac{-1}{(s^2+6s+10)} \right|_s^\infty$$

$$= -\left(0 - \frac{1}{s^2+6s+10}\right)$$

$$\int F(s)\,ds = \frac{1}{s^2+6s+10}$$

$$f(t) = t\mathcal{L}^{-1}\left\{\int_s^\infty \frac{1}{s^2+6s+10}\right\} \quad \text{— (1)}$$

$$\frac{1}{s^2+6s+10} = \frac{1}{s^2+6s+9+1}$$

$$= \frac{1}{(s+3)^2+1}$$

(1) $\Rightarrow$

$$\int F(s)\,ds = t\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1^2}\right\}$$

$$= te^{-3t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= te^{-3t}\sin t$$

13. $\ln \dfrac{s^2+1}{(s-1)^2}$

$$F(s) = \ln(s^2+1) - \ln(s-1)^2$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2(s-1)}{(s-1)^2}$$

$$F'(s) = 2\cdot\frac{s}{s^2+1} - \frac{2}{(s-1)}$$

$$f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left\{F'(s)\right\}$$

$$= \frac{-1}{t}\left[2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\right]$$

$$= \frac{-1}{t}\left\{2\cos t - 2e^{t}\right\}$$

$$= \frac{-2\cos t}{t} + \frac{2e^{t}}{t}$$

14. $\ln \frac{s+a}{s+b}$

$F(s) = \ln(s+a) - \ln(s+b)$

$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$f(t) = -\frac{1}{t} \mathcal{L}^{-1}\{F'(s)\}$

$= -\frac{1}{t}\left[\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+b}\right]\right]$

$= -\frac{1}{t}(e^{-at} - e^{-bt})$

15. $\frac{s}{(s^2+4)^2}$

$F(s) = \frac{1}{2} \cdot \frac{2s}{(s^2+4)^2}$

$\int_s^\infty F(s)\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+4)^2}\,ds$

$= \frac{1}{2}\left|\frac{-1}{(s^2+4)}\right|_s^\infty$

$= \frac{-1}{2}\left(0 - \frac{1}{s^2+4}\right)$

$\int_s^\infty F(s)\,ds = \frac{1}{2} \cdot \frac{1}{s^2+4}$

$f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$

$= t\,\mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{1}{s^2+4}\right\}$

$= \frac{1}{4} t\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= \frac{t}{4}\sin 2t$

16. $\operatorname{arc\,cot} \frac{s}{\pi}$

$F(s) = \cot^{-1}\frac{s}{\pi}$

$F'(s) = \frac{1}{1+\frac{s^2}{\pi^2}} \cdot \frac{1}{\pi}$

$= \frac{\pi}{s^2+\pi^2}$

$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$

$$f(t) = \frac{-1}{t} \mathcal{L}^{-1} \left\{ \frac{\pi}{s^2+\pi^2} \right\}$$

$$f(t) = \frac{-1}{t} \sin \pi t$$