①

Ex 5.2

1) $y' + 3y = 10\sin t \qquad y(0) = 0$

Sol:

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 10\mathcal{L}\{\sin t\}$$

$$s\mathcal{L}\{y\} - y(0) + 3\mathcal{L}\{y\} = 10\,\frac{1}{s^2+1}$$

$$sY - 0 + 3Y = \frac{10}{s^2+1}$$

$$(s+3)Y = \frac{10}{s^2+1}$$

$$Y = \frac{10}{(s+3)(s^2+1)}$$

$$\mathcal{L}\{y\} = \frac{10}{(s+3)(s^2+1)}$$

$$y = 10\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s^2+1)}\right\} \text{ — } (1)$$

$$\frac{1}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s+3)$$

$A = 1/10, \quad B = -1/10$

$C = 3/10$

$$y = 10\mathcal{L}^{-1}\left\{\frac{1/10}{s+3} + \frac{-\frac{1}{10}s + \frac{3}{10}}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{-s+3}{s^2+1}\right\}$$

$$= e^{-3t} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-3t} - \cos t + 3\sin t.$$

$$\mathcal{L}\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Project:

(a) $\mathcal{L}\{t\cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega}\, t\sin\omega t$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

(e) $\mathcal{L}\{t\cosh at\} = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

(f) $\mathcal{L}\{t\sinh at\} = \dfrac{2as}{(s^2-a^2)^2}$

(a) $\mathcal{L}\{t\cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

Sol:

$f(t) = t\cos\omega t \quad \Rightarrow f(0) = 0$

$f'(t) = t[-\omega\sin\omega t] + \cos\omega t = -\omega t\sin\omega t + \cos\omega t \quad f'(0) = 1$

$f''(t) = -\omega[t\,\omega\cos\omega t + \sin\omega t \cdot 1] - \omega\sin\omega t$

$= -\omega^2 t\cos\omega t - \omega\sin\omega t - \omega\sin\omega t$

$= -\omega^2 t\cos\omega t - 2\omega\sin\omega t$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$\mathcal{L}\{-\omega^2 t\cos\omega t - 2\omega\sin\omega t\} = s^2\mathcal{L}\{t\cos\omega t\} - s(0) - 1$

$-\omega^2\mathcal{L}\{t\cos\omega t\} - 2\omega\mathcal{L}\{\sin\omega t\} = s^2\mathcal{L}\{t\cos\omega t\} - 1$

$1 - 2\omega\dfrac{\omega}{s^2+\omega^2} = (s^2+\omega^2)\mathcal{L}\{t\cos\omega t\}$

$1 - \dfrac{2\omega^2}{s^2+\omega^2} = (s^2+\omega^2)\mathcal{L}\{t\cos\omega t\}$

$\dfrac{s^2+\omega^2-2\omega^2}{s^2+\omega^2} = (s^2+\omega^2)\mathcal{L}\{t\cos\omega t\}$

$\mathcal{L}\{t\cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$$\boxed{\mathcal{L}\{t \sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}}$$

Project:

(a) $\mathcal{L}\{t \cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega} t\sin\omega t$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

(e) $\mathcal{L}\{t\cosh at\} = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

(f) $\mathcal{L}\{t\sinh at\} = \dfrac{2as}{(s^2-a^2)^2}$

(b) Sol:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{\omega^2}{(s^2+\omega^2)^2}\right\}$$

$$= \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{s^2-s^2+\omega^2}{(s^2+\omega^2)^2}\right\}$$

$$= \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{s^2+\omega^2}{(s^2+\omega^2)^2}\right\} - \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+\omega^2)^2}\right\}$$

$$= \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} - \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{s^2-\omega^2+\omega^2}{(s^2+\omega^2)^2}\right\}$$

$$= \frac{1}{\omega^3}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} - \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\} - \frac{1}{\omega^2}\mathcal{L}^{-1}\left\{\frac{\omega^2}{(s^2+\omega^2)^2}\right\}$$

$$= \frac{1}{\omega^3}\sin\omega t - \frac{1}{\omega^2}t\cos\omega t - \mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\}$$

$$2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{1}{\omega^3}(\sin\omega t - \omega t\cos\omega t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t).$$

$(S^2+\omega^2)$

Project:

(a) $\mathcal{L}\{t\cos\omega t\} = \dfrac{S^2-\omega^2}{(S^2+\omega^2)^2}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{(S^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$

(c) $\mathcal{L}^{-1}\left\{\dfrac{S}{(S^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega}\, t\sin\omega t$

(d) $\mathcal{L}^{-1}\left\{\dfrac{S^2}{(S^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

(e) $\mathcal{L}\{t\cosh at\} = \dfrac{S^2+a^2}{(S^2-a^2)^2}$

(f) $\mathcal{L}\{t\sinh at\} = \dfrac{2aS}{(S^2-a^2)^2}$

(c) Sol:

$\mathcal{L}^{-1}\left\{\dfrac{S}{(S^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega}\mathcal{L}^{-1}\left\{\dfrac{2\omega S}{(S^2+\omega^2)^2}\right\}$

$= \dfrac{1}{2\omega}\, t\sin\omega t$

(e) $\mathcal{L}\{t\cosh at\} = \mathcal{L}\left\{t\cdot\dfrac{e^{at}+e^{-at}}{2}\right\}$

$= \dfrac{1}{2}\mathcal{L}\{te^{at}\} + \dfrac{1}{2}\mathcal{L}\{te^{-at}\}$

$= \dfrac{1}{2}\dfrac{1}{S^2}\Big|_{S\to S-a} + \dfrac{1}{2}\dfrac{1}{S^2}\Big|_{S\to S+a}$

$= \dfrac{1}{2}\dfrac{1}{(S-a)^2} + \dfrac{1}{2}\dfrac{1}{(S+a)^2}$

$= \dfrac{1}{2}\left[\dfrac{(S+a)^2+(S-a)^2}{(S-a)^2(S+a)^2}\right]$

$= \dfrac{1}{2}\left[\dfrac{S^2+a^2+2aS+S^2+a^2-2aS}{(S^2-a^2)^2}\right] = \dfrac{1}{2}\left[\dfrac{2S^2+2a^2}{(S^2-a^2)^2}\right]$

13) $\frac{1}{s^2+4s} = \frac{1}{s(s+4)}$

Sol: $\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s+4}\right\}$

$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} d\tau$

$= \int_0^t e^{-4\tau} d\tau = \left| \frac{e^{-4\tau}}{-4} \right|_0^t$

$= -\frac{1}{4}\left[e^{-4t} - 1\right] = \frac{1}{4}\left[1 - e^{-4t}\right]$

Ex 5.2

Q1 — 20.