CHAPTER 5

Laplace Transforms

The Laplace transform method solves differential equations and corresponding initial and boundary value problems. The process of solution consists of three main steps:

1st step. The given "hard" problem is transformed into a "simple" equation **(subsidiary equation).**

2nd step. The subsidiary equation is solved by purely algebraic manipulations.

3rd step. The solution of the subsidiary equation is transformed back to obtain the solution of the given problem.

In this way Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. This process is made easier by tables of functions and their transforms, whose role is similar to that of integral tables in calculus. Such a table is included at the end of the chapter.

This switching from operations of calculus to *algebraic* operations on transforms is called **operational calculus,** a very important area of applied mathematics, and for the engineer, the Laplace transform method is practically the most important operational method. It is particularly useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function, not merely a sine or cosine. (For another operational method, the Fourier transform, see Sec. 10.10.)

The Laplace transform also has the advantage that it solves problems directly, initial value problems without first determining a general solution, and nonhomogeneous differential equations without first solving the corresponding homogeneous equation.

In this chapter we consider Laplace transforms from a practical point of view and illustrate their use by important engineering problems, many of them related to *ordinary* differential equations.

Partial differential equations can also be treated by Laplace transforms, as we show in Sec. 11.12.

Section 5.8 contains a list of general formulas and Sec. 5.9 a list of transforms $F(s)$ and corresponding functions $f(t)$.

Prerequisite for this chapter: Chap. 2.
Sections that may be omitted in a very short course: 5.4–5.6.
References: Appendix 1, Part A.
Answers to problems: Appendix 2.

5.1. Laplace Transform. Inverse Transform. Linerity. Shifting

1. $\mathcal{L}(2t+6) = 2\frac{1}{s^2} + 6\frac{1}{s}$

2. $\mathcal{L}(a+bt+ct^2) = a\frac{1}{s} + b\frac{1}{s^2} + c\frac{2}{s^3}$

3. $\frac{\pi}{s^2+\pi^2}$

4. $\cos^2 \omega t = \frac{1+\cos 2\omega t}{2}$
$\Rightarrow \mathcal{L}(\cos^2 \omega t) = \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+4\omega^2}$

5. $\mathcal{L}(e^{a-bt}) = e^a \frac{1}{s+b}$

6. $\mathcal{L}(e^t \cosh 3t) = \mathcal{L}(e^t \frac{e^{3t}+e^{-3t}}{2})$
$= \frac{1}{2}\mathcal{L}(e^{4t}+e^{-2t}) = \frac{1}{2}(\frac{1}{s-4} + \frac{1}{s+2})$

7. $\mathcal{L}(\sin(\omega t+\delta)) = \mathcal{L}(\sin\omega t\cos\delta + \cos\omega t \sin\delta)$
$= \cos\delta\frac{\omega}{s^2+\omega^2} + \sin\delta\frac{s}{s^2+\omega^2}$

8. $\mathcal{L}(\sin 2t + \cos 2t) = \mathcal{L}(\frac{\sin 4t}{2} = \frac{1}{2}\frac{4}{s^2+16}$

9. $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$
$= \int_0^1 e^{-st}(-t+1)dt$
$= \int_0^1 e^{-st}dt - \int_0^1 te^{st}dt$
$= -\frac{1}{s}(e^{-s}-1) - (-\frac{e^{-s}}{s} - \frac{e^{-s}}{s} + \frac{1}{s^2})$
$= \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{1}{s^2}$

10. $\mathcal{L}(f(t)) = \int_1^4 ke^{-st}dt$
$= k(-\frac{1}{s}e^{-st}|_1^4) = k(\frac{e^{-s}}{s} - \frac{e^{-4s}}{s})$

11. $\mathcal{L}(f(t)) = \int_0^c ke^{-st}dt$
$= k(-\frac{1}{s}e^{-st}|_0^c)$
$= k(\frac{1}{s} - \frac{e^{-sc}}{s})$

12. $\mathcal{L}(f(t)) = \int_0^1 e^{-st}tdt + \int_1^2 e^{-st}(-t+2)dt$
$= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s}$

13. $\mathcal{L}(f(t)) = \int_0^k e^{-st}ktdt$

14. $\mathcal{L}(f(t)) = \int_0^1 e^{-st}tdt + \int_1^2 e^{-st}dt$

15. $\mathcal{L}(f(t)) = \int_0^1 e^{-st}(-\frac{1}{2}+1)dt$

16. $\mathcal{L}(f(t)) = \int_0^a e^{-st}\frac{b}{a}tdt$

17. $\mathcal{L}^{-1}(\frac{0.1s+0.9}{s^2+3.24})$
$= \mathcal{L}^{-1}(\frac{0.1s}{s^2+3.4}) + \mathcal{L}^{-1}(\frac{0.9}{s^2+3.24})$
$= 0.1\cos(\sqrt{3.24}t) + \frac{0.9}{\sqrt{3.24}}\sin(\sqrt{3.24}t)$

18. $\mathcal{L}^{-1}(\frac{5s}{s^2-25}) = \mathcal{L}^{-1}(\frac{\frac{5}{2}}{s+5} - \frac{\frac{5}{2}}{s-5})$
$= \frac{5}{2}(e^{-5t} - e^{5t})$

19. $\mathcal{L}^{-1}(\frac{-s-2}{s^2-s-2}) = \mathcal{L}^{-1}(-\frac{4}{s-2} + \frac{3}{s+1})$
$= -4e^{2t} + 3^{-t}$

20. $\mathcal{L}^{-1}(\frac{s-4}{s^2-4}) = \mathcal{L}^{-1}(\frac{\frac{3}{2}}{s+2} - \frac{\frac{1}{2}}{s-2})$
$= \frac{3}{2}e^{-2t} - \frac{1}{2}e^{2t}$

21. $\mathcal{L}^{-1}(\frac{24}{s^4} - \frac{228}{s^6}) = \frac{24}{6}t^3 - \frac{228}{51}t^5$

22. $\mathcal{L}^{-1}(\frac{60+6s^2+s^4}{s^7}) = \frac{60}{6!}t^6 + \frac{6}{4!}t^4 + \frac{1}{2}t^2$

23. $\mathcal{L}^{-1}(\frac{s}{L^2s^2+n^2\pi^2}) = \frac{1}{L^2}\mathcal{L}^{-1}(\frac{s}{s^2+(\frac{n\pi}{L})^2})$
$= \frac{1}{L^2}\cos(\frac{n\pi}{L}t)$

24. $\mathcal{L}^{-1}(\frac{1-7s}{(s-3)(s-1)(s+2)})$
$= \mathcal{L}^{-1}(-\frac{2}{s-3} + \frac{1}{s-1} + \frac{1}{s+2})$
$= -2e^{3t} + e^t + e^{-2t}$

25. $\mathcal{L}^{-1}(\sum_{k=1}^5 \frac{a_k}{s+k^2}) = \sum_{k=1}^5 a_k e^{-k^2 t}$

26. $\mathcal{L}^{-1}(\frac{s^4+6s-18}{s^5-3s^4})$
$= \mathcal{L}^{-1}(\frac{1}{s-3} + \frac{6}{s^4-3s^3} - \frac{18}{s^5-3s^4})$
$= e^{3t} + \mathcal{L}^{-1}(\frac{\frac{2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{2}{s^3} + \frac{6}{s^4} - \frac{\frac{2}{9}}{s-3})$
$= e^{3t} + \frac{2}{9} + \frac{2}{3}t - t^2 - \frac{1}{9}e^{3t} + \frac{2}{9} + \frac{2}{3}t + t^2 + t^3 - \frac{2}{9}e^{3t}$
$= \frac{2}{3}e^{3t} + \frac{4}{9} + \frac{4}{3}t + t^3$

27. $\mathcal{L}^{-1}(\frac{1}{(s+\sqrt{2})(s-\sqrt{3})})$
$= \mathcal{L}^{-1}(-\frac{\frac{1}{\sqrt{2}+\sqrt{3}}}{s+\sqrt{2}} + \frac{\frac{1}{\sqrt{2}+\sqrt{3}}}{s-\sqrt{3}})$
$= -\frac{1}{\sqrt{2}+\sqrt{3}}e^{\sqrt{2}t} + \frac{1}{\sqrt{2}+\sqrt{3}}e^{\sqrt{3}t}$

28. $\mathcal{L}^{-1}(\frac{s}{s^2+1} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1})$
$= \sin t + \frac{1}{2}e^{-t} + \frac{1}{2}e^t$

29. $\mathcal{L}(t^2e^{-3t}) = \frac{2}{(s+3)^3}$

30. $\mathcal{L}(e^{\alpha t}\cos\beta t) = \int_0^\infty e^{-st}e^{\alpha t}\cos\beta t dt$
$= \int_0^\infty e^{(s+\alpha)t}\cos\beta t dt$

31. $\mathcal{L}(5e^{2t}\sinh 2t) = \mathcal{L}(5e^{2t}\frac{e^{2t}-e^{-2t}}{2})$
$= \mathcal{L}(\frac{5}{2}e^{4t} - \frac{5}{2})$
$= \frac{5}{3}\frac{1}{s-4} - \frac{5}{2}\frac{1}{s}$

32. $\mathcal{L}(2e^{-t}\cos^2\frac{1}{2}t) = \mathcal{L}(e^{-4}(1+\cos t))$
$= \frac{1}{s+1} + \frac{s+1}{(s+1)^2+1}$

33. $\mathcal{L}(\sinh t\cos t) = \mathcal{L}(\frac{e^t-e^{-t}}{2}\cos t)$
$= \frac{1}{2}\mathcal{L}(e^t\cos t) - \frac{1}{2}\mathcal{L}(e^{-t}\cos t)$
$= \frac{1}{2}\frac{s-1}{(s-1)^2+1} - \frac{1}{2}\frac{s+1}{(s+1)^2+1}$

34. $\mathcal{L}((t+1)^2e^t) = \mathcal{L}((t^2+2t+1)e^t)$
$= \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$

35. $\mathcal{L}^{-1}(\frac{1}{(s+1)^2}) = e^{-t}t$

36. $\mathcal{L}^{-1}(\frac{12}{(s-3)^4}) = e^{3t}t^3$

37. $\mathcal{L}^{-1}(\frac{3}{s^2+6s+18}) = \mathcal{L}^{-1}(\frac{3}{(s+3)^2+9})$

38. $\mathcal{L}^{-1}(\frac{4}{s^2-2s-3}) = \mathcal{L}^{-1}(\frac{4}{(s-1)^2-4})$
$= e^t(e^{2t}-e^{-2t})$

39. $\mathcal{L}^{-1}(\frac{s}{(s+\frac{1}{2})^2+1}) = e^{-\frac{1}{2}t}\cos t$

40. $\mathcal{L}^{-1}(\frac{2}{s^2+s+\frac{1}{2}}) = \mathcal{L}^{-1}(\frac{2}{(s+\frac{1}{2})^2+\frac{1}{4}})$
$= 4e^{-\frac{1}{2}t}\sin\frac{1}{2}t$

41. a) $\cosh t = \frac{e^t+e^{-t}}{2}$인데 $e^{-t} < e^t, t > 0$
$\Rightarrow \cosh t < e^t$
b)귀류법을 쓰면
$n = 0$일때 $1 < e^t, t > 0$이므로 성립
$n-1$ 때까지 가정하고 즉, $t^{n-1} < (n-1)!e^t$
$f(t) = n!e^t - t^n \Rightarrow f'(t) = n!e^t - nt^{n-1}$
$n((n-1)!e^t - t^{n-1}) > 0$이고 $f(0) = n! > 0$ 이면서 기울기가 양수이므로 $f(t)$는 항상 0보다 크다.

42. $\mathcal{L}^{-1}(F(s)+G(s))$,
where$\mathcal{L}(f(t)) = F(s), \mathcal{L}(g(t)) = G(s)$
$\mathcal{L}^{-1}(\mathcal{L}(f(t)) + \mathcal{L}(g(t)))$
$= \mathcal{L}^{-1}(\mathcal{L}(f(t)+g(t)))$
$= f(t) + g(t)$
$= \mathcal{L}^{-1}(F(s)) + \mathcal{L}^{-1}(G(s))$
$\mathcal{L}^{-1}(cF(s)) = \mathcal{L}^{-1}(x\mathcal{L}(f(t)))$
$= \mathcal{L}^{-1}(\mathcal{L}(cf(t))) = ct(t) = c\mathcal{L}^{-1}(F(s))$

43. $\frac{1}{s} \Rightarrow 1$
$\frac{1}{s-a} \Rightarrow e^{at}$
$\frac{n!}{s^{n+1}} \Rightarrow t^n$
$\frac{s}{s^2+\omega^2} \Rightarrow \cos\omega t$
$\frac{\omega}{s^2+\omega^2} \Rightarrow \sin\omega t$

44. $\mathcal{L}(f(ct)) = \int_0^\infty e^{-st}dt$
$t' = ct \Rightarrow \mathcal{L}(f(ct)) = \int_0^\infty e^{-\frac{s}{c}t'}f(t')\frac{1}{c}dt'$
$= \mathcal{L}(F(\frac{s}{c}))/c$
$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$
$\Rightarrow \mathcal{L}(\cos\omega t) = \frac{\frac{s}{\omega}}{(\frac{s}{\omega})^2+1}/\omega = \frac{s}{s^2+\omega^2}$

45. $|f(t)| \le Me^{-kt}$를 만족하지 않는 함수, 즉, $f(t) = e^{t^2}$이면 적분이 발산하게 되어 Laplace 값이 ∞가 된다.

5.2. Transforms of Derivatives and Integrals. Differential Equations

1. $sY + 2Y = \frac{10}{s^2+1}$
$\Rightarrow Y = \frac{10}{(s+3)(s^2+1)}$
$\mathcal{L}^{-1}(\frac{-s+3}{s^2+1} + \frac{1}{s+3})$
$= -\cos t + 3\sin t + e^{-3t}$

2. $sY - 1 - 5Y = \frac{3}{2s+4}$
$Y = \frac{3}{(s-5)(2s+4)} + \frac{1}{s-5}$
$\Rightarrow y(t) = \frac{7}{6}e^{5t} - \frac{1}{6}e^{-4t}$

3. $sY + \frac{1}{4} + \frac{1}{5}Y = \frac{1}{100}\frac{1}{s}$
$100sY + 25 + 20Y = \frac{1}{s}$
$Y = \frac{1}{(100s+20)s} - \frac{25}{(100s+20)}$
$\Rightarrow y(t) - \frac{1}{4}e^{-\frac{1}{5}t} + \frac{1}{20} - \frac{1}{20}e^{-\frac{1}{5}t}$

4. $s^2Y - sy(0) - y'(0) - sY + y(0) - 2Y$
$= s^2Y - 8s - 7 - sY + 8 - 2Y$
$= (s^2 - s - 2)Y - 8s + 1$
$\Rightarrow Y = \frac{8s-1}{s^2-s-2}$
$\mathcal{L}^{-1}(Y) = y(t) = 3e^{-t} + 5e^{2t}$

5. $s^2Y - sy(0) - y'(0) + asY - ay(0) - 2a^2Y$
$= s^2Y - 6sasY - 6a - 2a^2Y$
$= (s^2 + as - 2a^2)Y - 6s - 6a$
$\Rightarrow Y = \frac{6s+6a}{(s+2a)(s-a)}$

$\mathcal{L}^{-1}(\frac{2}{s+2a}+\frac{4}{s-a}) = 2e^{-2at}+4e^{at}$

6. $s^2Y - sy(0) - y'(0) + Y$
$= s^2Y - 3s - 4 + Y$
$= (s^2+1)Y - 3s - 4 = \frac{2s}{s^2+1}$
$Y = \frac{2s}{(s^2+1)^2} + \frac{3s+4}{s^2+1}$
$\Rightarrow y(t) = 3\cos t + (4+t)\sin t$

7. $s^2Y - sy(0) - y'(0) - 4sY + 3Y$
$= (s^2-4s+3)Y = \frac{6}{s^2} - \frac{8}{5}$
$Y = \frac{6}{(s-3)(s-1)s^2} - \frac{8}{(s-3)(s-1)s}$
$\mathcal{L}^{-1}(\frac{\frac{8}{3}}{s} + \frac{2}{s^2} - \frac{\frac{26}{9}}{s-1} + \frac{\frac{2}{9}}{s-3})$
$- \mathcal{L}^{-1}(\frac{\frac{4}{3}}{s-3} - \frac{4}{s-1} + \frac{\frac{8}{3}}{s})$
$\Rightarrow y(t) = \frac{8}{3} + 2t - \frac{26}{9}e^t + \frac{2}{9}e^{3t} - \frac{4}{3}e^{3t} + 4e^t + 4e^t - \frac{8}{3}$
$= 2t + \frac{10}{9}e^t + \frac{10}{9}e^{3t}$

8. $s^2Y - sy(0) + \frac{1}{25}Y$
$= s^2Y + 25s + \frac{1}{25}Y = \frac{1}{2}25s^3$
$(s^2 + \frac{1}{25})Y = \frac{1}{25s^3} - 25s$
$Y = \frac{1}{25s^3(s^2+\frac{1}{25})} - \frac{25s}{s^2+\frac{1}{25}}$
$y(t) = -25 + \frac{1}{2}t^2$

9. $s^2Y - sy(0) - y'(0) + 25Y - 2y(0) - 3Y$
$= (s^2+2s-3)Y - 2s + 10 = \frac{6}{s+2}$
$Y = \frac{6}{(s+3)(s-1)(s+2)} + \frac{2s-10}{(s+3)(s-1)}$
$\mathcal{L}^{-1}(\frac{\frac{3}{2}}{s+3} + \frac{\frac{1}{2}}{s-1} - \frac{2}{s+2}) + \mathcal{L}^{-1}(\frac{4}{s+3} - \frac{2}{s-1})$
$\Rightarrow y(t) = \frac{11}{2}e^{-3t} - \frac{3}{2}e^t - 2e^{-2t}$

10. a) Laplace transform 은 미분방정식이 대수적인 방정식으로 바뀐다.
b) 정리 1,2 는 미분방정식이 대수식으로 바꾸는 역할을하고 정리 3은 Laplace 변환이나 역변환할때 도움을 준다.
c)
$\mathcal{L}(f') = \int_0^\infty e^{st}f'(t)dt$
$= \int_0^a e^{-st}f'(t)dt + \int_a^\infty e^{-st}f'(t)dt$
$= fe^{-st}|_0^a + s\int_0^a e^{-st}f(t)dt + f^{-st}|_a^\infty + s\int_a^\infty e^{-st}f(t)dt$
$= s\mathcal{L}(f) + f(a-0)e^{-sa} - f(0) - f(a+0)e^{-sa}$
$= s\mathcal{L}(f) - f(0) - e^{-as}(f(a+0) - f(a-0))$
d)
$f(2+0) - f(2-0) = -1, f(0) = 0$
$\Rightarrow \mathcal{L}(f') = \frac{1-e^{-s}}{s}$
$\Rightarrow \mathcal{L}(f) = (1 - e^{-s} - se^{-2s})/s^2$

11. a)
$\mathcal{L}(\cos^2 t) = \mathcal{L}(1-\sin^2 t) = \frac{1}{s} - \frac{2}{s(s^2+4)}$
b)
$f(t) = \cos^2 t, f(0) = 1,$
$f'(t) = -2\sin t\cos t = -\sin 2t$
$\mathcal{L}(f') = -\mathcal{L}(\sin 2t) = -\frac{2}{s^2+4}$
$= s\mathcal{L}(f) - f(0) = s\mathcal{L}(f) - 1$
$s\mathcal{L}(f) = 1 - \frac{2}{s^2+4}$
$\mathcal{L}(f) = -\frac{2}{s(s^2+4)} + \frac{1}{s}$
c)
$\mathcal{L}(\cos^2 t) = \mathcal{L}(\frac{1+\cos 2t}{2})$
$= \frac{1}{2s} + \frac{1}{2}(\frac{s}{s^2+4}) = \frac{1}{s} - \frac{2}{s(s^2+4)}$

12. a)
$f(t) = t\cos\omega t, f(0) = 0$
$f'(t) = \cos\omega t - \omega t\sin\omega t \Rightarrow f'(0) = 1$
$f''(t) = -\omega\sin\omega t - \omega\sin\omega t - \omega^2 t\cos\omega t$
$= -2\omega\sin\omega t - \omega^2 f(t)$
$\Rightarrow \mathcal{L}(f'') = s^2\mathcal{L}(f) - 1$
$= -2\omega\frac{\omega}{s^2+\omega^2} - \omega^2\mathcal{L}(f)$
$(s^2+\omega^2)\mathcal{L}(f) = 1 - \frac{2\omega^2}{s^2+\omega^2} = \frac{s^2-\omega^2}{s^2+\omega^2}$
$\mathcal{L}(f) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
b)
$\mathcal{L}(\sin\omega t - \omega t\cos\omega t)$
$= \frac{\omega}{s^2+\omega^2} - \omega\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$= \frac{2\omega^3}{(s^2+\omega^2)^2}$
$\Rightarrow \mathcal{L}^{-1}(\frac{1}{(s^2+\omega^2)^2} = \frac{1}{2\omega^2}(\sin\omega t - \omega t\cos\omega t)$
c)
$\mathcal{L}(f') = \mathcal{L}(\cos\omega t - \omega t\sin\omega t) = s\mathcal{L}(f)$
$\frac{s}{s^2+\omega^2} - \omega\mathcal{L}(ts\sin]\omega t) = \frac{s(s^2-\omega^2)}{(s^2+\omega^2)^2}$
$\Rightarrow \omega\mathcal{L}(t\sin\omega t) = \frac{s}{s^2+\omega^2} - \frac{s^3-s\omega^2}{(s^2+\omega^2)^2}$
$= \frac{\omega^2}{(s^2+\omega^2)^2}$
$\Rightarrow \mathcal{L}(t\sin\omega t) = \frac{s\omega^2}{(s^2+\omega^2)^2}$
$\mathcal{L}^{-1}(\frac{s}{(s^2+\omega^2)^2}) = \frac{1}{2\omega}t\sin\omega t$
d)
$\mathcal{L}(\sin\omega t + \omega t\cos\omega t)$
$= \frac{\omega}{w^2+\omega^2} + \omega\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$= \frac{2s^2\omega}{(s^2+\omega^2)^2}$

$\Rightarrow \mathcal{L}^{-1}(\frac{s^2}{(s^2+\omega^2)^2} = \frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

e)

$f(t) = t\cosh at, f(0) = 0$

$f'(t) = \cosh at + at\sinh at, f'(0) = 1$

$f''(t) = a\sinh at + a\sinh at + a^2 t\cosh at$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - 1$

$= a(\frac{1}{s-a} - \frac{1}{s+a}) + a^2\mathcal{L}(f)$

$\Rightarrow \mathcal{L}(f) = \frac{2a^2}{(s^2-\omega^2)^2} + \frac{1}{(s^2-a^2)}$

$= \frac{s^2+a^2}{(s^2-a^2)^2}$

f)

$\mathcal{L}(f') = \mathcal{L}(\cosh at) + \mathcal{L}(at\sinh at)$

$= \frac{1}{2}(\frac{1}{s-a} + \frac{1}{s+a}) + a\mathcal{L}(t\sinh at)$

$= s\mathcal{L}(f) = \frac{s(s^2+a^2)}{(s^2-a^2)}$

$\Rightarrow \mathcal{L}(t\sinh at) = \frac{1}{a}(\frac{s(s^2+a^2)}{(s^2-a^2)^2} - \frac{1}{2}(\frac{2s}{s^2-a^2}))$

$= \frac{2as}{(s^2-a^2)^2}$

13. $\mathcal{L}^{-1}(\frac{1}{s^2+4s})$

$= \mathcal{L}^{-1}(\frac{1}{s(s+4)})$

$= \int_0^t e^{-4\tau}d\tau$

$= \frac{1}{4} - \frac{1}{4}e^{-4t}$

14. $\frac{1}{s-2} \Rightarrow e^{2t}$

$\frac{1}{s(s-2)} \Rightarrow \int_0^t e^{2\tau}d\tau = \frac{1}{2}(e^{2t}-1)$

$\frac{1}{s^2(s-2)} \Rightarrow \int_0^t \frac{1}{2}(e^{2\tau}-1)d\tau = -\frac{1}{2}t + \frac{1}{4}e^{2t} - \frac{1}{4}$

$y(t) = -1 - 2t + e^{2t}$

15. $\frac{1}{\omega}\sin\omega t \Rightarrow \frac{1}{s^2+\omega^2}$

$\frac{1}{s(s^2\omega^2)} \Rightarrow \int_0^r \frac{1}{\omega}\sin\omega\tau d\tau = \frac{1}{\omega^2}(1-\cos\omega t)$

$\rightarrow \mathcal{L}^{-1}(\frac{1}{s(s^2+\omega^2)}) = \frac{1}{\omega^2}(1-\cos\omega t)$

16. $\frac{1}{s^2+1} \Rightarrow \sin t$

$\frac{1}{s(s^2+1)} \Rightarrow \int_0^t \sin\tau d\tau = 1 - \cos t$

$\frac{1}{s^2(s^2+1)} \Rightarrow \int_0^t 1 - \cos\tau d\tau = t - \sin t$

$\frac{1}{s^2(s^2+1)} \Rightarrow \int_0^t \tau - \sin\tau d\tau = \frac{1}{2}t^2 - 1 + \cos t$

$\rightarrow \mathcal{L}^{-1}(\frac{1}{s^5+s^3}) = -1 + \frac{1}{2}t^2 + \cos t$

17. $\frac{1}{s^2-1} \Rightarrow -\frac{1}{2}e^{-t} + \frac{1}{2}e^t$

$\frac{1}{s(s^2-1)} \Rightarrow \int_0^t(-\frac{1}{2}e^{-\tau} + \frac{1}{2}e^{\tau})d\tau = -1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t$

$\rightarrow \mathcal{L}^{-1}(\frac{1}{s(s^2-1)}) = -1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}$

18. $\frac{1}{s^2}\frac{s-1}{s+1} = \frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}$

$\frac{1}{s+1} \Rightarrow e^{-t}$

$\frac{1}{s(s+1)} \Rightarrow \int_0^t e^{-\tau}d\tau = 1 - e^{-t}$

$\frac{1}{s^2(s+1)} \Rightarrow \int_0^t 1 - e^{-\tau}d\tau = t + e^{-t} - 1$

$\rightarrow \mathcal{L}^{-1}(\frac{s-1}{s^2(s+1)}) = 2 - t - 2e^{-t}$

19. $\frac{9}{s^2}\frac{s+1}{s^2+9} = \frac{9}{s(s^2+9)} + \frac{9}{s^2(s^2+9)}$

$\frac{9}{s^2+9} \Rightarrow 3\sin 3t$

$\frac{9}{s(s^2+9)} \Rightarrow \int_0^t 3\sin 3\tau d\tau = 1 - \cos 3t$

$\frac{9}{s^2(s^2+9)} \Rightarrow \int_0^t 1 - \cos 3\tau d\tau = t - \frac{1}{3}\sin 3t$

$\rightarrow \mathcal{L}^{-1}(\frac{9}{s^2}\frac{s+1}{s^2+9}) = 1 - \cos 3t + t - \frac{1}{3}\sin 3t$.

20. $\frac{\pi^5}{s^2\pi^2} \Rightarrow \pi^4\sin\pi t$

$\frac{\pi^5}{s(s^2+\pi^2)} \Rightarrow \int_0^t \pi^4\sin\pi\tau d\tau = \pi^3(1-\cos\pi t)$

$\frac{\pi^5}{s^2(s^2+\pi^2)} \Rightarrow \int_0^t \pi^3(1-\cos\pi\tau)d\tau = \pi^3 - \pi^2\sin\pi t$

$\frac{\pi^5}{s^3(s^2+\pi^2)} \Rightarrow \int_0^t \pi^3 - \pi^2\sin\pi\tau d\tau \frac{\pi^3}{2}t^2 + \pi\cos\pi t - \pi$

$\frac{\pi^5}{s^4(s^2+\pi^2)} \Rightarrow \int_0^t \frac{\pi^3}{2}\tau^2 + \pi\cos\pi\tau - \pi d\tau = -\pi t + \frac{\pi^3}{6}t^3 + \sin\pi t$

5.3. Unit Step Function,Second Shifting Theorem,Dirac's Delta Function

2. Second shifting Theorem 을 쓰기위해 변형해보면
$tu(t-1) = (t-1)u(t-1) + u(t-1)$
그러므로 라플라스 변환은
$e^{-s}(\frac{1}{s^2} + \frac{1}{s})$
3. $\frac{2e^{-s}}{s^2}$
4. $\frac{2e^{-s}}{s^3}$
5. $e^{-s}(2s^{-3} + 2s^{-2} + s^{-1})$
6. $e^{-2t}u(t-3) = e^{-2(t-3)}e^{-6}u(t-3)$
을 이용하면
$\frac{e^{-3s-6}}{s+2}$
7. $\frac{-4e^{-\pi s}s}{s^2+1}$
8. $t^2(1-u(t-1)) = t^2 - [(t-1)^2 + 2(t-1) + 1]u(t-1)$
를 이용하면
$2s^{-3} - e^{-s}(2s^{-3} + 2s^{-2} + s^{-1})$
9. $\frac{\omega(1+e^{-\frac{\pi s}{\omega}})}{s^2+\omega^2}$
10. $(1-e^{-t})[1-u(t-2)]$
$= 1 - e^{-t} - (1 - e^{(t-2)}e^{-2})u(t-2)$ 를 이용하면
$\frac{1}{s} - \frac{1}{s+1} - e^{-2s}(\frac{1}{s} - \frac{e^{-2}}{s+1})$
11. $\frac{(1-e^{1-s})}{s-1}$
12. 주어진 함수는 $[u(t-2\pi) - u(t-4\pi)]\sin t$
$= u(t-2\pi)\sin(t-2\pi) - u(t-4\pi)\sin(t-4\pi)$ 이다.
그러므로 답은 $\frac{1}{s^2+1}(e^{-2\pi s} - e^{-4\pi s})$이다.
13. $-10s\frac{(e^{-s}+e^{-2s})}{(s^2+\pi^2)}$
14. $4u(t-2) - 8u(t-5)$
15. $\frac{1}{2}(t-3)^2u(t-3)$
16. s^{-3}의 역변환은 $\frac{t^2}{2}$이다.그러므로 $(s-1)^{-3}$의 역변환은 $e^t\frac{t^2}{2}$이고 $\frac{e^{-3s}}{(s-1)^3}$의 역변환은
$\frac{1}{2}e^{t-3}(t-3)^2u(t-3)$이다.
17. $[1+u(t-\pi)]\sin 3t$
18. $s^2+2s+2 = (s+1)^2+1$이므로, 이것의 역수의 역변환은 $e^{-t}\sin t$이다. second shifting 정리를 쓰면
$e^{-(t-2\pi)}\sin t\, u(t-2\pi)$
19. $\cos(\pi t)u(t-2)$
20. $y = 3e^{\frac{t}{2}}(\cos 3t + \sin 3t)$
21. $\frac{1}{3}(e^t-1)^3e^{-5t}$
22. 우변을 함수 u 로 나타내면
$r(t) = 4t[1-u(t-1)] + 8u(t-1)$
$= 4t - [4(t-1)-4]u(t-1)$ 그러므로
$y =$
$$\begin{cases} 4e^{-t} - e^{-2t} + 2t - 3 & if\ \ 0<t<1 \\ (4-8e)e^{-t} + (3e^2-1)e^{-2t} + 4 & if\ \ t>1 \end{cases}$$
23. $$\begin{cases} \sin 3t + \sin t & if\ \ 0<t<\pi \\ \frac{4}{3}\sin 3t & if\ \ t>\pi \end{cases}$$
24. $r(t) = 4e^t[1-u(t-2)] = 4e^t - 4e^2e^{t-2}u(t-2)$
$y =$
$$\begin{cases} 2e^t + e^{2t} - 2e^{3t} & if\ \ 0<t<2 \\ (1+4e^{-2})e^{2t} + (-2-2e^{-4})e^{3t} & if\ \ t>2 \end{cases}$$
25. $$y = \begin{cases} e^t - \sin t & if\ \ 0<t<2\pi \\ e^t - \frac{1}{2}\sin 2t & if\ \ t>2\pi \end{cases}$$
26. $y = 2\cos 4t + u(t-\pi)\sin 4(t-\pi)$
27. $$y = \begin{cases} \sin t & if\ \ 0<t<\pi \\ 0 & if\ \ \pi<t<2\pi \\ -\sin t & if\ \ t>2\pi \end{cases}$$
28. $$y = \begin{cases} 3e^{-2t}\sin t & if\ \ 0<t<1 \\ e^{-2t}(3\sin t + e^2\sin(t-1)) & if\ \ t>1 \end{cases}$$
29. $y =$
$$\begin{cases} 3e^{-3t} - 2e^t + 2e^t & if\ \ 0<t<\frac{1}{2} \\ 3e^{-3t} - 2e^{-t} + 2e^t & \\ +\frac{1}{4}(-e^{-3t+\frac{3}{2}} + e^{t-\frac{1}{2}}) & if\ \ t>\frac{1}{2} \end{cases}$$
30. $y =$
$$\begin{cases} e^{-2t} - e^{-3t} & if\ \ 0<t<1 \\ \frac{1}{6} + (1-\frac{1}{2}e^2)e^{-2t} & \\ -(1-\frac{1}{3}e^3)e^{-3t} & if\ \ 1<t<2 \\ \frac{1}{6} + (1-\frac{1}{2}e^2+e^4)e^{-2t} & \\ -(1-\frac{1}{3}e^3+e^6)e^{-3t} & if\ \ t>2 \end{cases}$$
31. $$\begin{cases} L\frac{(e^{-\frac{Rt}{L}}-1)}{R^2} + \frac{t}{R} & if\ \ 0<t<4\pi \\ L\frac{e^{-\frac{Rt}{L}}}{R^2} + (\frac{4\pi}{R} - \frac{L}{R^2})e^{-\frac{(t-4\pi)R}{L}} & if\ \ t>4\pi \end{cases}$$
32. $$y = \begin{cases} \frac{-L\cos t + R\sin t + e^{-\frac{Rt}{L}}}{L^2+R^2} & if\ \ 0<t<2\pi \\ Le^{\frac{-Rt}{L}}\frac{1-e^{\frac{2\pi R}{L}}}{L^2+R^2} & if\ \ t>2\pi \end{cases}$$
33. $sI + \frac{I}{s} = \frac{(1-e^{-s})}{s^2}$ 이므로
$$y = \begin{cases} 1-\cos t & if\ \ 0<t<1 \\ \cos(t-1) - \cos t & if\ \ t>1 \end{cases}$$
34. $v = 1-u(t-a)$이므로 주어진 식은
$sI + \frac{I}{s} = \frac{1}{s} - \frac{e^{-as}}{s}$이다.그러므로 정답은
$$i = \begin{cases} \sin t & \text{if}\ \ 0<t<a \\ \sin t - \sin(t-a) & \text{if}\ \ t>a \end{cases}$$
35. $$y = \begin{cases} \frac{1}{2}(e^{-t} - \cos t + \sin t) & if\ \ 0<t<\pi \\ \frac{1}{2}[-(1+e^{-\pi})\cos t & \\ +(3-e^{-\pi})\sin t] & if\ \ t>\pi \end{cases}$$
36. $I = 100\frac{e^{-t}-e^{-1.01s}}{s+0.1}$ 이므로
$i =$
$$\begin{cases} 0 & if\ \ t<1 \\ 100e^{-0.1(t-1)} & if\ \ 0<t<1.01 \\ 100[e^{-0.1(t-1)} - e^{-0.1(t-1.01)}] & \\ = -0.1106e^{-0.1t} & if\ \ t>1.01 \end{cases}$$
37. $$\begin{cases} 0 & if\ \ t<1 \\ e^{-0.1(t-1)} & if\ \ 1<t<2 \\ e^{-0.1(t-1)} - e^{-0.1(t-2)} & if\ \ t>2 \end{cases}$$
38. $$i = \begin{cases} 0 & if\ \ t<3 \\ 5 - 5e^{-0.1(t-3)} & \\ = 5(1-1.3499e^{-0.1t}) & if\ \ t>3 \end{cases}$$
39. $$\begin{cases} 0 & if\ \ t<2 \\ \frac{(10e^{(t-2)} - e^{-0.1(t-2)})}{900e^2} & if\ \ t>2 \end{cases}$$

5.4. Differentiation and Integration of Transforms

1. $\frac{1}{(s-1)^2}$
2. $-(\frac{12}{s^2-16})' = \frac{24s}{(s^2-16)^2}$
3. $\frac{2s^3+6\pi^2 s}{(s^2-\pi^2)^3}$
4. $-(\frac{s+1}{(s+1)^2+1})' = -\frac{s^2+2s+2-(s+1)(2s+2)}{(s^2+2s+2)^2}$
   $= \frac{s^2+2s}{(s^2+2s+2)^2}$
5. $\frac{s^2-w^2}{(s^2+w^2)^2}$
6. $\frac{d^2}{ds^2}(\frac{2}{s^2+4}) = \frac{d}{ds}(\frac{-4s}{(s^2+4)^2})$
   $= \frac{-4(s^2+4)^2+16s^2(s^2+4)}{(s^2+4)4} = \frac{12s^2-16}{(s^2+4)^3}$
7. $\frac{2s+2}{(s^2+2s+2)^2}$
8. $\frac{d^2}{ds^2}(\frac{s}{s^2+\omega^2}) = \frac{d}{ds}\frac{-s^2+\omega^2}{(s^2+\omega^2)^2} = \frac{2s^3-6\omega^2 s}{(s^2+\omega^2)^3}$
9. $\frac{1}{2}t^2e^{3t}$
10. $(\frac{3}{s^2-9})' = -\frac{6s}{(s^2-9)^2}$ 이므로
    $\frac{1}{6}t\sinh 3t$
11. $t\cos\pi t$
12. (6)에 의해
    $\int_s^\infty \frac{2\tilde{s}+6}{(\tilde{s}^2+6\tilde{s}+10)^2}d\tilde{s} = \frac{1}{s^2+6s+10} = \mathcal{L}(\frac{f}{t})$
    이다.이 적분의 역변환은 $e^{-3t}\sin t$이므로, $te^{-3t}\sin t$
13. $2t^{-1}(e^t - \cos t)$
14. $[ln(s+a) - ln(s+b)]' = \frac{1}{s+a} - \frac{1}{s+b}$ 의 역변환은
    $e^{-at} - e^{-bt}$이다.그러므로 (1)에 의해
    $\frac{e^{-bt}-e^{-at}}{t}$
15. $\frac{1}{4}t\sin 2t$
16. $arccot\frac{s}{\pi} = \int_s^\infty \frac{\pi}{\tilde{s}^2+\pi^2}d\tilde{s}$
    이므로,적분기호안의 함수의 역변환은 $\sin\pi t$ 이다. 그러므로 (6)으로부터
    $\frac{\sin\pi t}{t}$
18. $\frac{n!}{(s-a)^{n+1}}$
20. (b)
    $l_2 = 1 - 2t + \frac{1}{2}t^2$
    $l_3 = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$
    $l_4 = 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4$
    $l_5 = 1 - 5t + 5t^2 - \frac{5}{3}t^3 + \frac{5}{24}t^4 - \frac{1}{120}t^5$

5.5. Convolution. Integral Equations

1. t
2. $1 * \sin\omega t = \int_0^t \sin\omega\tau d\tau = -\frac{\cos\omega\tau}{\omega}|_0^t = \frac{1-\cos\omega t}{\omega}$
3. $\sinh t$
4. 예제(Example)1과 비슷하다.
   $\int_0^t \cos\omega\tau\sin(\omega t - \omega\tau)d\tau$
   $= \frac{1}{2}\int_0^t[\cos\omega t + \cos(2\omega\tau - \omega t)]d\omega$
   $= \frac{1}{2}[t\cos\omega t + \frac{\sin\omega t - \sin(-\omega t)}{2\omega}]_0^t$
   $= \frac{1}{2}t\cos\omega t + \frac{1}{2\omega}\sin\omega t$
5. $\frac{1}{2}t\sin wt$
6. $e^{at} * e^{bt} = \int_0^t e^{a\tau}e^{b(t-\tau)}d\tau = e^{bt}\int_0^t e^{(a-b)\tau}d\tau$
   $= \frac{e^{at}-e^{bt}}{a-b}$
7. $e^t - t - 1$
8. $\int_0^t (t-\tau)^2 u(\tau-1)d\tau = \int_1^t (t-\tau)^2 d\tau$
   $= -\frac{1}{3}(t-\tau)^3|_1^t = \frac{1}{3}(t-1)^3 \ \ if \ \ t > 1$
   $0 \ \ if \ \ t < 1$
9. $\frac{1}{2}(1 - e^{-2(t-3)})u(t-3)$
10. $6 * e^{-3t} = \int_0^t 6e^{-3\tau}d\tau = -2e^{-3t} + 2$
11. $e^t - t - 1$
12. $e^{at} * e^{at} = \int_0^t e^{a\tau}e^{a(t-\tau)}d\tau = e^{at}\int_0^t d\tau = te^{at}$
13. $\frac{1}{4} - \frac{1}{4}\cos 2t$
14. $\cos\omega t * \cos\omega t$를 문제 4와 같이 전개하면
    $\frac{1}{2}t\cos\omega t + \frac{1}{2\omega}\sin\omega t$
15. $\frac{t\sin\pi t}{2\pi}$
16. $u(t-a) * e^{2t} = \int_a^t e^{2(t-\tau)}d\tau = e^{2t}\int_a^t e^{-2\tau}d\tau$
    $= \frac{1}{2}(e^{2(t-a)} - 1) \ \ if \ \ t > a$
    $0 \ \ if \ \ t < a$
17. $\frac{wt - \sin wt}{w^2}$
18. $e^{-3t} * e^{2t} = \int_0^t e^{-3\tau}e^{2(t-\tau)}d\tau = e^{2t}\int_0^t e^{-5\tau}d\tau$
    $= \frac{1}{5}(e^{2t} - e^{-3t})$
19. $y = \cos t - \cos 2t$
20. 주어진 식이
    $s^Y + Y = s^{-2}$ 이므로
    $y = t - \sin t$
21. $y = \frac{1}{4}(1 + 3\cos 2t) + \frac{1}{4}[\cos(2t-2) - 1]u(t-1)$
22. 주어진 식이
    $(s^2 + 3s + 2)Y = 1 + \frac{1-e^{-s}}{s}$ 이므로
    $Y = \frac{s+1-e^{-s}}{s(s^2+3s+2)}$이다.
    $y = \frac{1}{2}(1-e^{-2t}) + \frac{1}{2}(2e^{-(t-1)} - e^{-2(t-1)} - 1)u(t-1)$
23. $y = e^{-t}(\cos t + \sin t) + [-2\cos t + \sin t + e^{-(t-2\pi)}(2\cos t + \sin t)]u(t-2\pi)$
24. $q = e^{3t} - e^{2t}$,
    $\mathcal{L}^{-1}\{[(s+a)y(0) + y'(0)]Q\} = -4e^{3t} + 5e^{2t}$ 이므로
    $$\begin{cases} r * q = 2e^{3t} - 4e^{2t} + 2e^t & if \ \ 0 < t < 2 \\ y = -2e^{3t} + 2^{2t} + 2e^t & if \ \ 0 < t < 2 \\ r * q = 2(1 - e^{-4})e^{3t} & \\ +4(e^{-2} - 1)e^{2t} & if \ \ t > 2 \\ y = -(2 + 2e^{-4})e^{3t} & \\ +(1 + 4e^{-2})e^{2t} & if \ \ t > 2 \end{cases}$$
25. $y = [t - \cos(t-1) - \sin(t-1)]u(t-1) + [-t + 2\cos(t-2) + \sin(t-2)]u(t-2\pi)$
26. $q = e^{-t} - e^{-2t}$,
    $r * q =$
    $$\begin{cases} 4e^{-t} - e^{-2t} + 2t - 3 & if \ \ 0 < t < 1 \\ \int_0^1 \tau q(t-\tau)d\tau + \int_1^t 8q(t-\tau)d\tau & \\ = (4 - 8e)e^{-t} + (3e^2 - 1)e^{-2t} + 4 & if \ \ t > 1 \end{cases}$$
27. e^t
28. $Y = 2s^{-2} - 4s^{-2}Y, Y = \frac{2}{s^2+4}, y = \sin 2t$
29. $\cos t$
30. $Y = \frac{2}{s^2+4} + \frac{2Y}{s^2+4}, Y = \frac{2}{s^2+2}, y = \sqrt{2}\sin\sqrt{2}t$
31. $\sinh t$

32. $Y = \frac{1}{s^2+1} + \frac{Y}{s^2+1}, Y = \frac{1}{s^2}, y = t$

33. $\cosh t$

34. (a)
$t - \tau = p$로 놓으면, $d\tau = -dp$이다.τ를 p로 치환하면
$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$
$= \int_t^0 g(p)f(t-p)(-dp) = \int_0^t g(p)f(t-p)dp = g * f$
(b)
(a)를 이용하면
$(f * g) * v = v * (f * g)$
$= \int_0^t v(p) \int_0^{t-p} f(\tau)g(t-p-\tau)d\tau dp$
$= \int_0^t f(\tau) \int_0^{t-\tau} g(t-\tau-p)v(p)dpd\tau = f * (g * v)$
(c)
적분의 덧셈에 관한 결과이다.
(d)
5.3절의 (8)에 의해 $\mathcal{L}(\delta) = 1$이므로
$\mathcal{L}(\delta * f) = \mathcal{L}(\delta)\mathcal{L}(f) = 1 \cdot \mathcal{L}(f) = \mathcal{L}(f)$
(e)
$t > k$라 하자. 0과 k사이에 다음 식을 만족하는 $\tilde{t}$가 존재한다.
$(f_k * f)(t) = \int_0^k \frac{1}{k} f(t-\tau)d\tau = f(t-\tilde{t})$
이제 $k \to 0$이라 하면,
$\tilde{t} \to 0, f_k(t-\tilde{t}) \to \delta(t)$
이므로 주어진 식이 성립한다.
(f)
$s^2Y - sy(0) - y'(0) + \omega^2 Y = \mathcal{L}(r)$의 해는
$Y = \frac{1}{\omega}(\frac{\omega}{s^2+\omega^2})\mathcal{L}(r) + y(0)\frac{s}{s^2+\omega^2} + \frac{y'(0)}{\omega}\frac{\omega}{s^2+\omega^2}$

5.6. Partial Fractions. Differential Equations

1. $\dfrac{6}{(s+2)(s-4)} = -\dfrac{1}{s+2} + \dfrac{1}{s-4}$
$\Rightarrow f(t) = -e^{-2t} + e^{4t}$

2. $\dfrac{s^3+2s^2+2}{s^3(s^2+1)} = \dfrac{2}{s^3} + \dfrac{s}{s^2+1} \to 2t + \cos t$
$\dfrac{s^3+2s^2+2}{s^3(s^2+1)} \to \int_0^t 2\tau + \cos\tau d\tau$
$\to t^2 + \sin t$

3. $\dfrac{s^2+9s-9}{s^3-9s} = \dfrac{s^2+9s-9}{s(s+3)(s-3)}$
$= \dfrac{1}{s} - \dfrac{3}{2}\dfrac{1}{s+3} + \dfrac{3}{2}\dfrac{1}{s-3}$
$f(t) = 1 - \dfrac{3}{2}e^{-3t} + \dfrac{3}{2}e^{3t} = 1 + 3\sinh 3t$

4. $\dfrac{s}{(s+1)^2} = \dfrac{1}{s+1} - \dfrac{1}{(s+1)^2}$
$f(t) = e^{-t} - e^{-t}t$

5. $\dfrac{2s^3}{s^4-81} = \dfrac{2s^3}{(s^2+9)(s+3)(s-3)}$
$= \dfrac{1}{2}\dfrac{1}{s+3} + \dfrac{1}{2}\dfrac{1}{s-3} + \dfrac{s}{s^2+9}$
$\Rightarrow f(t) = \dfrac{1}{2}e^{-3t} + \dfrac{1}{2}e^{3t} + \cos 3t$

6. $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$
$= \dfrac{s}{s^2-2s+2} - \dfrac{1}{s^2-2s+2} + \dfrac{2s-2}{(s^2-2s+2)^2}$
$= \dfrac{s-1}{(s-1)^2+1} + \dfrac{2(s-1)}{((s-1)^2+1)^2}$
$\dfrac{s}{s^2+1} \to \cos t$
$\dfrac{1}{s^2+1} \to \sin t$
$\dfrac{2s}{(s^2+1)^2} \to t\sin t$
$\Rightarrow f(t) = e^t\cos t + e^t t\sin t$

7. $\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3} = \dfrac{1}{(s+1)^3} + \dfrac{3}{s^4}$
$\Rightarrow f(t) = e^{-t}\dfrac{t^2}{2} + \dfrac{1}{2}t^3$

8. 조립제법을 이용해 분자를 $s-2$의 내림차순으로 정리하면,
$s^3 - 7s^2 + 14s - 9 = (s-2)^3 - (s-2)^2 - 2(s-2) - 1$
$= (s-2)^3 - (s-2+1)^2$
$F(s) = \frac{1}{(s-1)^2} - \frac{1}{(s-2)^3} \Rightarrow f(t) = te^t - \frac{1}{2}t^2e^{2t}$

9. $\dfrac{s^3+6s^2+14s}{(s+2)^4}$
$= \dfrac{1}{s+2} + \dfrac{2}{(s+2)^3} - \dfrac{12}{(s+2)^4}$
$\Rightarrow f(t) = e^{-2t} + e^{-2t}t^2 - 2e^{-2t}t^3$
$= e^{-2t}(1 + t^2 - 2t^3)$

10. $\dfrac{1}{s^4+4a^4}$
$= \dfrac{1}{s^4+2a^2s^2+4a^2-4a^2s^2}$
$= \dfrac{1}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$
$= \dfrac{\frac{1}{8a^3}s + \frac{1}{4a^2}}{(s^2+2as+2a^2)} + \dfrac{-\frac{s}{8a^3} + \frac{1}{4a^2}}{s^2-2as+2a^2}$
$= \dfrac{1}{8a^3}\left[\dfrac{s+2a}{(s+a)^2+a^2} + \dfrac{-s+2a}{(s-a)^2+a^2}\right]$
$\Rightarrow f(t) =$
$\dfrac{1}{8a^3}(e^{-at}(\cos at + \sin at) - e^{at}(\cos at - \sin at))$
$\dfrac{1}{4a^3}(\cos at\dfrac{e^{-at}-e^{at}}{2} + \sin at\dfrac{e^{-at}+e^{at}}{2})$
$\dfrac{1}{4a^3}(\cosh at\sin at - \sinh at\cos at)$

11. $\dfrac{s}{s^4+4a^4}$
$= -\dfrac{1}{4a}\dfrac{1}{s^2+2as+2a^2} + \dfrac{1}{4a}\dfrac{1}{s^2-2as+2a^2}$
$= \dfrac{1}{4a}(-\dfrac{1}{(s+a)^2+a^2} + \dfrac{1}{(s-a)^2+a^2})$
$\Rightarrow$
$f(t) = \dfrac{1}{4a}(-e^{-at}\dfrac{\sin at}{a} + e^{at}\dfrac{\sin at}{a})$

$= \frac{1}{2a^2}\sin at(\frac{e^{at}-e^{-at}}{2})$
$= \frac{1}{2a^2}\sin at \sinh at$

12. $\frac{s^2}{s^4+4a^4}$
$= -\frac{s}{4a}\frac{1}{s^2+2as+2a^2} + \frac{s}{4a}\frac{1}{s^2-2as+2a^2}$
$= \frac{1}{4a}(-\frac{s}{(s+a)^2+a^2} + \frac{s}{(s-a)^2+a^2})$
$\Rightarrow$
$f(t) = \frac{1}{4a}(-e^{-at}(\cos at - \sin at) + e^{at}(\cos at + \sin at))$
$= \frac{1}{2a}(\cos at\frac{e^{at}-e^{-at}}{2} + \sin at\frac{e^{at}+e^{-at}}{2})$
$= \frac{1}{2a}(\cos at \sinh at + \sin at \cosh at)$

13. $\frac{s^3}{s^4+a^4}$
$= \frac{1}{2}\frac{s+a}{s^2+2as+2a^2} + \frac{1}{2}\frac{s-a}{s^2-2as+2a^2}$
$= \frac{1}{2}\frac{s+a}{(s+a)^2+a^2} + \frac{1}{2}\frac{s-a}{(s-a)^2+a^2}$
$\Rightarrow$
$f(t) = \frac{1}{2}e^{-at}\cos at + \frac{1}{2}e^{at}\cos at$
$= \cos at \cosh at$

14. $s^2Y + \omega_0^2 Y = \frac{Kp}{(s^2+\omega_0^2)(s^2+p^2)}$
$= Kp\frac{1}{p^2-\omega_0^2}(\frac{1}{s^2+\omega_0^2} - \frac{1}{s^2+p^2})$
$\Rightarrow$
$y(t) = \frac{Kp}{p^2-\omega_0^2}(\frac{\sin\omega_0 t}{\omega_0} - \frac{\sin pt}{p})$

15. a) $\frac{F(s)}{G(s)} = \frac{A}{s-a}$ + further fractions($K(s)$) 인데 simple root 이기 때문에 $K(a) \neq 0$이다. 양변에 $s-a$를 곱하고 $s \to a$로 보내면
$(s-a)\frac{F(s)}{G(s)} = A + (s-a)K(s)$
$\Rightarrow A = \lim_{s\to a}\frac{(s-a)F(s)}{G(s)}$
b)
$\frac{F(s)}{G(s)} = \frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a} + K(s)$
$\frac{(s-a)^m F(s)}{G(s)} = A_m + (s-a)A_{m-1} + \cdots$
$+ (s-a)^{m-1}A_1 + (s-a)^m K(s)$
$\Rightarrow A_m = \lim_{s\to a}\frac{(s-a)^m F(s)}{G(s)}$
$A_{m-1} = \lim_{s\to a}\frac{d}{ds}(\frac{(s-a)^m F(s)}{G(s)})$
마찬가지로 계산하면
$A_k = \frac{1}{(m-k)!}\lim_{s\to a}\frac{d^{m-k}}{ds^{m-k}}\left[\frac{(s-a)^m F(s)}{G(s)}\right]$

16. a)
$\mathcal{L}(f)$
$= \int_0^\infty e^{-st}f(t)dt$
$= \int_0^p e^{-st}f(t)dt + \int_p^{2p} e^{-st}f(t)dt + \cdots$
$= \int_0^p e^{-st}f(t)dt + \int_0^p e^{-s(t+p)}f(t)dt + \cdots$
$= \int_0^p e^{-st}f(t)dt + e^{-sp}\int_0^p e^{-st}f(t)dt$
$+ e^{-2sp}\int_0^p e^{-st}f(t)dt + \cdots$
$= \int_0^p e^{-st}f(t)dt(1 + e^{-sp} + e^{-2sp} + \cdots)$
$= \int_0^p e^{-st}f(t)dt(\frac{1}{1-e^{-sp}})$
b)
$\mathcal{L}(f) = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\sin\omega t dt$
$\int_0^{\pi/\omega} e^{-st}\sin\omega t dt = \mathrm{Im}\int_0^{\pi/\omega} e^{-st}e^{i\omega t}dt$
$= \mathrm{Im}(\frac{-s-i\omega}{s^2+\omega^2}(-e^{-s\pi/\omega}-1))$
$\Rightarrow$
$\mathcal{L}(f) = \frac{\omega(e^{-s\pi/\omega}+1)}{s^2+\omega^2}$
c)
$\sin\omega t$를 0 에서 π/ω 까지 $-\sin\omega t$를 π/ω에서 $2\pi/\omega$까지 잡고 주기가 $2\pi/\omega$인 변환은
$\frac{\omega}{s^2+\omega^2}\frac{1+e^{\pi s/\omega}}{e^{\pi s/\omega}-1} = \frac{\omega}{s^2+\omega^2}\frac{2^{-\pi s/2\omega}+e^{\pi s/2\omega}}{e^{\pi s/2\omega}-e^{-\pi s/2\omega}}$
$= \frac{\omega}{s^2+\omega^2}\frac{\cosh(\pi s/2\omega)}{\sinh(\pi s/2\omega)}$
d)
$f(t) = \frac{k}{p}t$ if $0 < t < p, f(t+p) = f(t)$
$\int_0^p e^{-st}tdt = -\frac{t}{s}e^{-st}|_0^p + \frac{1}{s}\int_0^p e^{-st}dt$
$= -\frac{p}{s}e^{-sp} - \frac{1}{s^2}(e^{-sp}-1)$
$\Rightarrow \mathcal{L}(f) = \frac{k}{ps^2} - \frac{ke^{-ps}}{s(1-e^{-ps})}$
e)
$\mathcal{L}(\frac{kt}{p}) = \frac{k}{ps^2}$이므로 $\mathcal{L}(f) = \frac{ke^{-ps}}{s(1-e^{-ps})}$

5.7. Systems of Differential Equations

1. $sY_1 = -Y_1 + Y_2$
$sY_2 = -Y_1 - Y_2$
$\Rightarrow Y_2 = -\dfrac{1}{(s+1)^2+1}$
$Y_1 = \dfrac{s+1}{(s+1)^2+1}$
$\Rightarrow y_1(t) = e^{-t}\cos t,\ y_2(t) = -e^{-t}\sin t$

2. $sY_1 + 3 = 6Y_1 + 9Y_2$
$sY_2 + 3 = Y_1 + 6Y_2$
$\Rightarrow$
$(s-6)Y_1 = 9Y_2 - 3$
$(s-6)Y_2 = Y_1 - 3$
$\Rightarrow$
$(s-6)((s-6)Y_2+3) = (s-6)^2Y_2+3(s-6) = 9Y_2-3$
$\Rightarrow$
$Y_2 = \dfrac{-3s+15}{(s-6)^2-9} = -\dfrac{2}{s-9} + \dfrac{3}{s-3}$
$Y_1 = -\dfrac{6}{s-9} + \dfrac{3}{s-3}$
$\Rightarrow$
$y_2(t) = -2e^{9t} - e^{3t},\ y_1(t) = -6e^{9t} + 3e^{3t}$

3. $sY_1 - 3 = -Y_1 + 4Y_2$
$sY_2 - 4 = 3Y_1 - 2Y_2$
$\Rightarrow$
$(s+1)Y_1 - 4Y_2 = 3$
$-3Y_1 + (s+2)Y_2 = 4$
$\Rightarrow$
$Y_2 = \dfrac{4s+13}{s^2+3s-10} = \dfrac{4s+13}{(s-2)(s+5)}$
$= \dfrac{3}{s-2} + \dfrac{1}{s+5}$
$Y_1 = \dfrac{1}{3}(\dfrac{(s+2)(4s+13)}{s^2+3s-10} - 4)$
$= \dfrac{3s+22}{s^2+3s-10} = \dfrac{4}{s-2} - \dfrac{1}{s+5}$
$\Rightarrow$
$y_1(t) = 4e^{2t} - e^{-5t},\ y_2(t) = 3e^{2t} + e^{-5t}$

4. $sY_1 + 3 = 5Y_1 + Y_2$
$sY_2 - 7 = Y_1 + 5Y_2$
$\Rightarrow$
$y_1(t) = 2e^{6t} - 5e^{4t},\ y_2(t) = 2e^{6t} + 5e^{4t}$

5. $sY_1 + Y_2 = \dfrac{2s}{s^2+1}$
$Y_1 + sY_2 = 1$
$\Rightarrow$
$Y_2 = \dfrac{s}{s^2-1} - \dfrac{2s}{(s^2-1)(s^2+1)}$
$= \dfrac{s^3-s}{(s^2-1)(s^2+1)} = \dfrac{s}{s^2+1}$
$Y_1 = 1 - sY_2 = \dfrac{1}{s^2+1}$
$\Rightarrow$
$y_1(t) = \sin t,\ y_2(t) = \cos t$

6. $s^2Y_1 - s - 1 + Y_2 = \dfrac{-5s}{s^2+4}$
$s^2Y_2 + s - 1 + Y_1 = \dfrac{5s}{s^2+4}$
$\Rightarrow$
$y_1(t) = \sin t + \cos 2t,\ y_2(t) = \sin t - \cos 2t$

7. $s^2Y_1 - 2s - 3 = Y_1 + 3Y_2$
$s^2Y_2 - s - 2 = 4Y_1 - \dfrac{4}{s-1}$
$(s^2-1)Y_1 - 3Y_2 = 2s+3$
$-4Y_1 + s^2Y_2 = s + 2 - \dfrac{4}{s-1}$
$\Rightarrow$
$Y_2 = \dfrac{(s^2-1)(s+2-\frac{4}{s-1})}{s^2(s^2-1)-12} + \dfrac{8s+12}{s^2(s^2-1)-12}$
$= \dfrac{1}{s-2}$
$Y_1 = -\dfrac{1}{4}(-\dfrac{s^2}{s-2} + s + 2 - \dfrac{4}{s-1})$
$= -\dfrac{1}{4}(-\dfrac{4}{s-2} - \dfrac{4}{s-1})$
$\Rightarrow$
$y_1(t) = e^{2t} + e^t,\ y_2(t) = e^{2t}$

8. $s^2Y_1 - 3s = -5Y_1 2Y_2$
$s^2Y_2 - s = 2Y_1 - 2Y_2$
$\Rightarrow$
$(s^2+5)Y_1 - 2Y_2 = 3s$
$-2Y_1 + (s^2+2)Y_2 = s$
$\Rightarrow$
$Y_1 = \dfrac{3s^3+8s}{(s^2+1)(s^2+6)} = \dfrac{s}{s^2+1} + \dfrac{2s}{s^2+6}$
$Y_2 = \dfrac{s^3+11s}{(s^2+1)(s^2+6)} = \dfrac{2s}{s^2+1} - \dfrac{s}{s^2+6}$
$\Rightarrow$
$y_1(t) = \cos t + 2\cos\sqrt{6}t$
$y_2(t) = 2\cos t - \cos\sqrt{6}t$

9. $sY_1 - 2 + sY_2 = \dfrac{2}{s^2-1}$
$sY_2 - 1 + sY_3 = \dfrac{1}{s-1}$
$sY_3 + sY_1 - 1 = \dfrac{2}{s-1} + \dfrac{1}{s+1}$
$\Rightarrow$
$Y_3 = \dfrac{3}{2}\dfrac{1}{s(s-1)} + \dfrac{1}{2s(s+1)} - \dfrac{1}{s(s^2-1)}$
$= \dfrac{2s}{s(s^2-1)} = \dfrac{2}{s^2-1}$
$Y_2 = \dfrac{1}{s} + \dfrac{1}{s(s-1)} - \dfrac{2}{s^2-1} = \dfrac{1}{s+1}$
$Y_1 = \dfrac{2}{s} - \dfrac{1}{s+1} + \dfrac{2}{s(s^2-1)} = \dfrac{1}{s-1}$
$\Rightarrow$
$y_3(t) = 2\sinh t,\ y_2(t) = e^{-t},\ y_1(t) = e^t$

10. $2sY_1 - sY_2 - sY_3 = 0$
$sY_1 + sY_2 = \dfrac{4}{s^2} + \dfrac{2}{s}$
$sY_2 + Y_3 = \dfrac{2}{s^3} + \dfrac{2}{s}$

$\Rightarrow$
$y_1(t) = t^2,\ y_2(t) = t^2 + 2t,\ y_3(t) = t^2 - 2t$

11. $sY_1 = -Y_2 + \frac{1}{s} - \frac{e^{-s}}{s}$
$sY_2 = Y_1 + \frac{1}{s} - \frac{e^{-s}}{s}$
$\Rightarrow$
$Y_1 = \frac{1}{1+s^2}(1 - e^{-s} - \frac{1}{s} + \frac{e^{-s}}{s})$
$= (1 - e^{-s})\frac{1}{1+s^2}(1 - \frac{1}{s})$
$\mathcal{L}^{-1}\left[\frac{1}{1+s^2}(1 - \frac{1}{s})\right] = \sin t - \int_0^t \sin t dt$
$= \sin t + \cos t - 1$ 이므로
$y_1 = \sin t + \cos t - 1 - u(t-1)[\sin(t-1) + \cos(t-1) - 1]$

$Y_2 = \frac{1}{s}(1 - e^{-s}) - sY_1 = (1 - e^{-s})(\frac{1}{s} - \frac{s-1}{1+s^2})$
$y_2 = 1 - \cos t + \sin t - u(t-1)[1 - \cos(t-1) + \sin(t-1)]$

12. $sY_1 + Y_2 = \frac{2s}{s^2+1} - e^{-2\pi s}\frac{2s}{s^2+1}$
$= \frac{2s}{s^2+1}(1 - e^{-2\pi s})$
$Y_1 + sY_2 = 1$
$\Rightarrow$
$Y_1 = \frac{1}{s^2+1} - \frac{2s^2 e^{-2\pi s}}{s^4-1}$
$= \frac{1}{s^2+1} - e^{-2\pi s}[\frac{1}{s^2+1} + \frac{1}{s^2-1}]$
$Y_2 = \frac{s}{s^2+1} + \frac{2se^{-2\pi s}}{s^4-1}$
$= \frac{s}{s^2+1} + e^{-2\pi s}[\frac{s}{s^2-1} - \frac{s}{s^2+1}]$
$\Rightarrow$
$y_1(t) = \sin t - u(t-2\pi)[\sin t + \sinh(t-2\pi)]$
$= \begin{cases} \sin t & \text{if } 0 \le t \le 2\pi \\ -\sinh(t-2\pi) & \text{if } t > 2\pi \end{cases}$
$y_2(t) = \cos t + u(t-2\pi)[\cosh(t-2\pi) - \cos t]$
$= \begin{cases} \cos t & \text{if } 0 \le t \le 2\pi \\ \cosh(t-2\pi) & \text{if } t > 2\pi \end{cases}$

13. $sY_1 - 3 = 2Y_1 - 4Y_2 + e^{-s+1}\frac{1}{s-1}$
$sY_2 = Y_1 - 3Y_2 + e^{-s+1}\frac{1}{s-1}$
변변 빼면,
$s(Y_1 - Y_2) - 3 = Y_1 - Y_2$
$Y_1 = Y_2 + \frac{3}{s-1}$
이것을 두번째 식에 대입하면,
$Y_2 = \frac{3}{(s+2)(s-1)} + \frac{e^{-s+1}}{(s+2)(s-1)}$
$= \frac{1}{s-1} - \frac{1}{s+2} + e^{-s}(\frac{e/3}{s-1} - \frac{e/3}{s+2})$
$y_2(t) = e^t - e^{-2t} + \frac{1}{3}u(t-1)[e^t - e^{-2t+3}]$
$Y_1 = Y_2 + \frac{3}{s-1}$ 이므로 $y_1 = y_2 + 3e^t$

14. $sY_1 + 4 = 2Y_1 + 4Y_2 + 64(\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s})$
$sY_2 + 4 = Y_1 + 2Y_2$
$\Rightarrow$
$Y_1 = \frac{-4s-8}{s(s-4)} - \frac{64(2+s-s^2)e^{-s}}{s^3(s-4)}$
$Y_2 = \frac{-4s+4}{s(s-4)} + \frac{64(1+s)e^{-s}}{s^3(s-4)}$
$\Rightarrow$
$y_1(t) = -6e^{4t} + 2 + u(t-1)(-18 + 10e^{4t-4} - 8t + 16t^2)$
$y_2(t) = -3e^{4t} - 1 + u(t-1)(7 + 5e^{4t-4} - 4t - 8t^2)$

15. (a) example 1;
$sY_1 = -0.02Y_1 + 0.02Y_2$
$sY_2 = 0.02Y_1 - 0.02Y_2 + 150$
두번째 식에서 $Y_2 = \frac{1}{s+0.02}(0.02Y_1 + 150)$
첫번째 식에 대입하면, $Y_1 = \frac{0.02 \cdot 150}{(s+0.02)^2 - 0.02^2}$
$y_1(t) = 0.02 \cdot 150e^{-0.02t}\frac{1}{0.02}\sinh 0.02t$
$= 150\frac{1 - e^{-0.04t}}{2}$
$Y_2 = \frac{150(s+0.02)}{(s+0.02)^2 - 0.02^2}$
$y_2(t) = 150e^{-0.02t}\cosh 0.02t = 150\frac{1 + e^{-0.04t}}{2}$

example 2;
$sY_1 = -4Y_1 + 4Y_2 + \frac{12}{s}$
$sY_2 = -1.6Y_1 + 1.2Y_2 + \frac{4.8}{s}$
상수항을 소거하면,
$4.8sY_1 - 12sY_2$
$= 4.8(-4Y_1 + 4Y_2) - 12(-1.6Y_1 + 1.2Y_2) = 4.8Y_2$
$Y_2 = \frac{4.8s}{12s + 4.8}Y_1$
첫번째 식에 대입하면,
$sY_1 = -4Y_1 + \frac{4.8s}{3s+1.2}Y_1 + \frac{12}{s}$
$Y_1 = \frac{12(3s+1.2)}{s(3s+2.4)(s+2)} = \frac{3}{s} + \frac{15}{3s+2.4} - \frac{8}{s+2}$
$\Rightarrow$
$I_1(t) = 3 + 5e^{-0.8t} - 8e^{-2t}$

$Y_2 = \frac{4.8 \cdot 12}{4(3s+2.4)(s+2)} = \frac{12}{3s+2.4} - \frac{4}{s+2}$
$\Rightarrow$
$I_2(t) = 4e^{-0.8t} - 4e^{-2t}$
3.1절의 방법은 행렬의 eigenvector을 구해야 하고 Laplace 변환을 이용하는 방법은 행렬의 역행렬을 구해야 한다.

(b) 3.3절 (8);
$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, y_1(0)=a, y_2(0)=b$
$\begin{bmatrix} s+3 & -1 \\ -1 & s+3 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$
$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{(s+3)^2 - 1}\begin{bmatrix} s+3 & 1 \\ 1 & s+3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$
$= \frac{1}{(s+3)^2 - 1}\begin{bmatrix} a(s+3) + b \\ b(s+3) + a \end{bmatrix}$
$\Rightarrow$
$y_1 = ae^{-3t}\cosh t + be^{-3t}\sinh t$

$y_2 = be^{-3t}\cosh t + ae^{-3t}\sinh t$

(c) 3.6절 (3);

$$\begin{bmatrix} s-2 & 4 \\ -1 & s+3 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{s^3}+\frac{10}{s^2}+a \\ \frac{2}{s^3}+\frac{9}{s^2}+\frac{3}{s}+b \end{bmatrix}.$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} =$$

$$\frac{1}{s^2+s-2}\begin{bmatrix} s+3 & -4 \\ 1 & s-2 \end{bmatrix}\begin{bmatrix} (as^3+10s+4)/s^3 \\ (bs^3+3s^2+9s+2)/s^3 \end{bmatrix}$$

$$= \frac{1}{s^3(s^2+s-2)}\begin{bmatrix} as^4+(3a-4b)s^3-2s^2-2s+4 \\ bs^4+(3+a-2b)s^3+3s^2-6s \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+s-2}\right] = \frac{1}{3}(e^t+2e^{-2t})$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+s-2}\right] = \frac{1}{3}(e^t-e^{-2t})$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+s-2)}\right] = \frac{1}{3}\int_0^t e^t - e^{-2t}dt$$

$$= \frac{1}{3}(e^t+\frac{1}{2}e^{-2t}-\frac{3}{2})$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+s-2)}\right] = \frac{1}{3}\int_0^t e^t+\frac{1}{2}e^{-2t}-\frac{3}{2}dt$$

$$= \frac{1}{3}(e^t-\frac{1}{4}e^{-2t}-\frac{3}{2}t-\frac{3}{4})$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+s-2)}\right] = \frac{1}{3}\int_0^t e^t-\frac{1}{4}e^{-2t}-\frac{3}{2}t-\frac{3}{4}dt$$

$$= \frac{1}{3}(e^t+\frac{1}{8}e^{-2t}-\frac{3}{4}t^2-\frac{3}{4}t-\frac{9}{8})$$

$\Rightarrow$

$$y_1(t) = \frac{a}{3}(e^t+2e^{-2t}) + \frac{3a-4b}{3}(e^t-e^{-2t})$$
$$-\frac{2}{3}(e^t+\frac{1}{2}e^{-2t}-\frac{3}{2}) - \frac{2}{3}(e^t-\frac{1}{4}e^{-2t}-\frac{3}{2}t-\frac{3}{4})$$
$$+\frac{4}{3}(e^t+\frac{1}{8}e^{-2t}-\frac{3}{4}t^2-\frac{3}{4}t-\frac{9}{8})$$

$$y_2(t) = \frac{b}{3}(e^t+2e^{-2t}) + \frac{3+a-2b}{3}(e^t-e^{-2t})$$
$$+(e^t+\frac{1}{2}e^{-2t}-\frac{3}{2}) - 2(e^t-\frac{1}{4}e^{-2t}-\frac{3}{2}t-\frac{3}{4})$$

16. 시간이 두배 빨리 지나가는 효과이다.

$$y_1' = -\frac{16}{100}y_1 + \frac{4}{100}y_2 + 12$$
$$y_2' = \frac{16}{100}y_1 - \frac{16}{100}y_2$$

$\Rightarrow$

$$\begin{bmatrix} s+0.16 & -0.04 \\ -0.16 & s+0.16 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 12/s \\ 150 \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} =$$

$$\frac{1}{(s+0.16)^2-0.08^2}\begin{bmatrix} s+0.16 & 0.04 \\ 0.16 & s+0.16 \end{bmatrix}\begin{bmatrix} 12/s \\ 150 \end{bmatrix}$$

$$= \frac{1}{(s+0.16)^2-0.08^2}\begin{bmatrix} 18+1.92/s \\ 1.92/s+150s+6 \end{bmatrix}$$

$\Rightarrow$

$$y_1 = 18(e^{-0.16t}\frac{1}{0.08}\sinh 0.08t)$$
$$+1.92\int_0^t e^{-0.16t}\frac{1}{0.08}\sinh 0.08t dt$$
$$= \frac{18}{0.16}(e^{-0.08t}-e^{-0.24t})$$
$$+\frac{1.92}{0.16}\int_0^t e^{-0.08t}-e^{-0.24t}dt$$
$$= 100-37.5e^{-0.08t}-62.5e^{-0.24t}$$

마찬가지로 하면,

$$y_2 = 100-75e^{-0.08t}+125e^{-0.24t}$$

17. $y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6\sin^2 t$

$$y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2$$

$\Rightarrow$

$$\begin{bmatrix} s+0.08 & -0.02 \\ -0.08 & s+0.08 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3(\frac{1}{s}-\frac{s}{s^2+4}) \\ 150 \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} =$$

$$\frac{1}{(s+0.08)^2-0.04^2}\begin{bmatrix} s+0.08 & 0.02 \\ 0.08 & s+0.08 \end{bmatrix}\begin{bmatrix} \frac{12}{s(s^2+4)} \\ 150 \end{bmatrix}$$

$$= \frac{1}{(s+0.08)^2-0.04^2}\begin{bmatrix} \frac{12}{s^2+4}+\frac{0.96}{s(s^2+4)}+3 \\ \frac{0.96}{s(s^2+4)}+150(s+0.08) \end{bmatrix}$$

$$T_1 = \mathcal{L}^{-1}\left[\frac{1}{[(s+0.08)^2-0.04^2](s^2+4)}\right]$$
$$= \mathcal{L}^{-1}\left[-\frac{3.1138}{s+0.12}+\frac{3.1237}{s+0.04}-\frac{0.0099s+0.2487}{s^2+4}\right]$$
$$= -3.1138e^{-0.12t}+3.1237e^{-0.04t}$$
$$-0.0099\cos 2t - 0.1244\sin 2t,$$

$$T_2 = \mathcal{L}^{-1}\left[\frac{1}{[(s+0.08)^2-0.04^2]s(s^2+4)}\right] = \int_0^t T_1 dt$$
$$= 25.9483e^{-0.12t}-78.0925e^{-0.04t}$$
$$-0.0049\sin 2t + 0.0622\cos 2t + 52.083,$$

$$T_3 = \mathcal{L}^{-1}\left[\frac{1}{(s+0.08)^2-0.04^2}\right]$$
$$= 25e^{-0.08t}\sinh 0.04t,$$

$$T_4 = \mathcal{L}^{-1}\left[\frac{s+0.08}{(s+0.08)^2-0.04^2}\right] = e^{-0.08t}\cosh 0.04t.$$

$\Rightarrow$

$$y_1(t) = 12T_1+0.96T_2+3T_3$$
$$= 50-49.9553e^{-0.12t}+0.015e^{-0.04t}$$
$$-0.060\cos 2t - 1.497\sin 2t,$$
$$y_2(t) = 0.96T_2+150T_4$$
$$= 50+99.910e^{-0.12t}+0.030e^{-0.04t}$$
$$+0.060\cos 2t - 0.005\sin 2t.$$

18. $y_1'' = -3y_1+3(y_2-y_1)+8\sin t$

$$y_2'' = -3(y_2-y_1)-3y_2-8\sin t$$

$\Rightarrow$

$$s^2Y_1-s-3 = -3Y_1+3(Y_2-Y_1)+\frac{8}{s^2+1}$$
$$s^2Y_2-s+3 = -3(Y_2-Y_1)-3Y_2-\frac{8}{s^2+1}$$

변변 더하면, $Y_1+Y_2 = \frac{2s}{s^2+3}$.

첫번째 식에 대입하면,

$$Y_1 = \frac{s+3}{s^2+9}+\frac{6s}{(s^2+9)(s^2+3)}+\frac{8}{(s^2+9)(s^2+1)}$$
$$= \frac{s}{s^2+3}+\frac{2}{s^2+9}+\frac{1}{s^2+1}$$

$\Rightarrow$

$$y_1(t) = \cos\sqrt{3}t+\frac{2}{3}\sin 3t+\sin t$$

$Y_1+Y_2 = \frac{2s}{s^2+3}$ 이므로 $y_1+y_2 = 2\cos\sqrt{3}t$,

$$y_2(t) = \cos\sqrt{3}t-\frac{2}{3}\sin 3t-\sin t$$

19. $2i_1+4(i_1-i_2)+i_1' = 195\sin t$

$$4(i_2-i_1)+4i_2+2i_2' = 0$$

$\Rightarrow$

$$\begin{bmatrix} 6+s & -4 \\ -4 & 8+2s \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{195}{s^2+1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} =$$

$$\frac{1}{2(s^2+10s+16)} \begin{bmatrix} 8+2s & 4 \\ 4 & 6+s \end{bmatrix} \begin{bmatrix} \dfrac{195}{s^2+1} \\ 0 \end{bmatrix}$$

$$= \frac{195}{2(s^2+10s+16)} \begin{bmatrix} \dfrac{8+2s}{s^2+1} \\ \dfrac{4}{s^2+1} \end{bmatrix}$$

$$T_1 = \mathcal{L}^{-1}\left[\frac{1}{(s^2+10s+16)(s^2+1)}\right]$$

$$= \frac{1}{65}\mathcal{L}^{-1}\left[\frac{2s+17}{s^2+10s+16} - \frac{2s-3}{s^2+1}\right]$$

$$= \frac{1}{65}\mathcal{L}^{-1}\left[\frac{2(s+5)+7}{(s+5)^2-3^2} - \frac{2s-3}{s^2+1}\right]$$

$$= \frac{1}{65}[e^{-5t}(2\cosh 3t + \frac{7}{3}\sinh 3t) - (2\cos t - 3\sin t)]$$

$$= \frac{1}{65}(\frac{13}{6}e^{-2t} - \frac{1}{6}e^{-8t} - 2\cos t + 3\sin t)$$

$\Rightarrow$

$$i_2(t) = 2\cdot 195T_1 = 13e^{-2t} - e^{-8t} - 12\cos t + 18\sin t$$

$T_1(0) = 0$ 이므로

$$T_2 = \mathcal{L}^{-1}\left[\frac{s}{(s^2+10s+16)(s^2+1)}\right] = T_1',$$

$$i_1(t) = 4\cdot 195T_1 + 195T_2$$
$$= 13e^{-2t} + 2e^{-8t} - 15\cos t + 42\sin t$$

20. input source를 $195\sin t - 195u(t-2\pi)\sin t$ 로 생각할 수 있다.
문제 19번의 해를 각각 y_1, y_2라 하면, second shifting 정리에 의해,

$$i_1(t) = y_1(t) - u(t-2\pi)y_1(t-2\pi) =$$
$$\begin{cases} 13e^{-2t} + 2e^{-8t} - 15\cos t + 42\sin t & \text{if } 0 \le t \le 2\pi \\ 13(1-e^{4\pi})e^{-2t} + 2(1-e^{16\pi})e^{-8t} & \text{if } t > 2\pi \end{cases}$$

$$i_2(t) = y_2(t) - u(t-2\pi)y_2(t-2\pi) =$$
$$\begin{cases} 13e^{-2t} - e^{-8t} - 12\cos t + 18\sin t & \text{if } 0 \le t \le 2\pi \\ 13(1-e^{4\pi})e^{-2t} - (1-e^{16\pi})e^{-8t} & \text{if } t > 2\pi \end{cases}$$

Chapter5. Review

1. Laplace transform을 쓰면 미분방정식을 대수식으로 바꿀 수 있다.

2. 1번 외에 연속이 아니거나 주기적으로 주어진 함수가 Laplace transform 에 의해 미분성이 좋은 함수로 변한다.

3. $\mathcal{L}(c_1 f + c_2 g) = \int_0^\infty e^{-st}(c_1 f + c_2 g)dt$
$= c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_-^\infty e^{-st} g(t)dt$
$= c_1 \mathcal{L}(f) + c_2 \mathcal{L}(g)$

4. 2번에서처럼 비제차 미방의 함수가 비연속적일 때나 편비분방정식의 특수한 경우에 사용한다.

5. 주어진 미방을 Laplace 변환해서 얻는 대수방정식.

6. Laplace transform이 의미를 가지려면 $\int_0^\infty e^{-st} f(t)dt$의 값을 정할수 있어야 한다. 따라서, $|f(t)| < Me^{ct}$ 정도의 경계가 필요하다. 만약 $f(t) = e^{t^2}$으로 잡으면 이때 $\int \infty_0 e^{-st} e^{t^2} dt$의 적분값은 존재하지 않는다.

7. 불연속 함수를 나타낼 때 단위 계단 함수는 매우 유용하다.

8. 한점에서만 1을 정의하는 함수로 convolution함수의 표현에 많이 쓴다.

9. $\mathcal{L}(1) = \frac{1}{s}, \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \mathcal{L}(e^{at}) = \frac{1}{s-a},$
$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}, \cdots$

10. $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$
$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1}\mathcal{L}(f) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$

11. Laplace변환은 적분으로 표현되기 때문에 불연속 함수라도 적분구간을 나눔으로서 잘 정의된다.

12. $\tan t$함수의 경우 piecewise 연속이긴 하지만 Laplace 변환이 성립하기 위한 유한조건 $|f(t)| < Me^{ct}$을 만족하는 c 을 $(0, \frac{\pi}{2})$사이 에서 잡을수 없다.

13. $F(s) \to f(t)$
$\frac{F(s)}{s} \to \int_0^t f(\tau)d\tau = g(t)$
$\frac{F(s)}{s^2} \to \int_0^t g(\tau)d\tau$을 만족한다.

14. $\mathcal{L}(f(t)g(t)) \neq \mathcal{L}(f(t))\mathcal{L}(g(t))$이고,
$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$이다.

15. $F(s)$의 Laplace 역변환을 알 때, first shifting 정리는 $F(s-a)$의 역변환을 구할 때 유용하고, second shifting 정리는 $e^{-as}F(s)$의 역변환을 구할 때 유용하다.

16. $\sin \pi t \to \frac{\pi}{s^2 + \pi^2}$
$e^{-t} \sin \pi t \to \frac{\pi}{(s+1)^2 + \pi^2}$

17. $\cos^2 t = \frac{1 + \cos 2t}{2} \to \frac{1}{2}(\frac{1}{s} + \frac{s}{s^2+4})$

18. $\sin^2(\frac{\pi t}{2}) = \frac{1 - \cos \pi t}{2} \to \frac{1}{2}(\frac{1}{s} - \frac{s}{s^2 + \pi^2})$

19. $\mathcal{L}(e^t u(t-2)) = e^{-2s} \frac{e^2}{s-1}$

20. $\mathcal{L}(t^2 u(t - \frac{1}{4})) = e^{-\frac{s}{4}} \mathcal{L}((t + \frac{1}{4})^2)$
$= e^{-\frac{s}{4}}(\frac{2}{s^3} + \frac{1}{2s^2} + \frac{1}{16s})$
$= e^{-\frac{s}{4}}(\frac{32 + 8s + s^2}{16s^3})$

21. $\mathcal{L}(t * e^{-3t}) = \mathcal{L}(t)\mathcal{L}(e^{-3t})$
$= \frac{1}{s^2}\frac{1}{s+3}$

22. $\mathcal{L}(e^{2t} * \cos 4t) = \mathcal{L}(e^{2t})\mathcal{L}(\cos 4t)$
$= \frac{1}{s-2}\frac{s}{s^2+16}$

23. $\cosh \frac{1}{10} t \to \frac{s}{s^2 - 0.1^2}$

24. $\mathcal{L}(tf(t)) = -F'(s)$
$\Rightarrow \mathcal{L}(t \cos t) = -\mathcal{L}(\cos t)'$
$= -(\frac{s}{s^2+1})'$
$= \frac{s^2 - 1}{(s^2+1)^2}$
$\mathcal{L}(t \cos t + \sin t) = \frac{s^2 - 1}{(s^2+1)^2} + \frac{1}{s^2+1}$

25. $\frac{s}{s^2+9} \to \cos 3t$
$\frac{3}{s^2+9} \to \sin 3t$
$\Rightarrow$
$\mathcal{L}^{-1}(\frac{s+3}{s^2+9}) = \cos 3t + \sin 3t$

26. $\frac{1}{s^2 - 2s - 8} = -\frac{1}{6}\frac{1}{s+2} + \frac{1}{6}\frac{1}{s-4}$
$\Rightarrow$
$\mathcal{L}^{-1}(\frac{1}{s^2 - 2s - 8}) = -\frac{1}{6}e^{-2t} + \frac{1}{6}e^{4t}$

27. $\frac{1}{s} \to 1$
$\frac{1}{s^2} \to t$
$\Rightarrow$
$\mathcal{L}^{-1}(e^{-s}\frac{s+1}{s^2}) = u(t-1)t$

28. $\frac{6(s+1)}{s^4} = \frac{6}{s^3} + \frac{6}{s^4}$
$\Rightarrow$
$\mathcal{L}^{-1}(\frac{6(s+1)}{s^4}) = 3t^2 + t^3$

29. $\dfrac{s^2-6s+4}{s^3-3s^2+2s} = \dfrac{s^2-6s+4}{s(s^2-3s+2)}$
$= \dfrac{s^2-6s+4}{s(s-2)(s-1)}$
$= \dfrac{2}{s} + \dfrac{1}{s-1} - \dfrac{2}{s-2}$
$\Rightarrow$
$f(t) = 2 + e^t - 2e^{2t}$

30. $\dfrac{2s^2-3s+4}{(s^2+4)(s-3)} = \dfrac{s}{s^2+4} + \dfrac{1}{s-3}$
$\Rightarrow$
$f(t) = e^{3t} + \cos 2t$

31. $\dfrac{s^3-s^2-s+4}{s^4-5s^2+4} = \dfrac{s^3-s^2-s+4}{(s+2)(s-2)(s+1)(s-1)}$
$= \dfrac{1}{2}(-\dfrac{1}{s-1} + \dfrac{1}{s-2} + \dfrac{1}{s+1} + \dfrac{1}{s+2})$
$\Rightarrow$
$f(t) = \dfrac{1}{2}(-e^t + e^{2t} + e^{-t} + e^{-2t})$

32. $f(t) = \cos\theta \sin\omega t + \sin\theta\cos\omega t = \sin(\omega t + \theta)$

33. $\dfrac{3s+4}{s^2+4s+5} = \dfrac{3(s+2)-2}{(s+2)^2+1}$
$\Rightarrow f(t) = 3e^{-2t}\cos t - 2e^{-2t}\sin t$

34. $s^2Y - 2.5s + Y = e^{-2s}$
$Y = \dfrac{e^{-2s}+2.5s}{s^2+1}$
$\Rightarrow\ y = u(t-2)\sin(t-2) + 2.5\cos t$

35. $s^2Y - s + 4Y = \dfrac{e^{-3s}}{s}$
$Y = \dfrac{s}{s^2+4} + \dfrac{e^{-3s}}{s(s^2+4)}$
$\dfrac{1}{s(s^2+4)} = \dfrac{1}{4}(\dfrac{1}{s} - \dfrac{s}{s^2+4})$
$\rightarrow \dfrac{1}{4}(1-\cos 2t) = \dfrac{1}{2}\sin^2 t$
$\Rightarrow\ y = \cos 2t + \dfrac{1}{2}u(t-3)\sin^2(t-3)$

36. $s^2Y - 16s + 16 - 2sY + 32 + 2Y = 8\dfrac{s+1}{(s+1)^2+1}$
$Y = \dfrac{16s-48}{s^2-2s+2} + \dfrac{8s+8}{(s^2-2s+2)(s^2+2s+2)}$
$= 16\dfrac{s-1-2}{(s-1)^2+1} + \dfrac{-s+4}{s^2-2s+2} + \dfrac{s}{s^2+2s+2}$
$\Rightarrow$
$y(t) = 16e^t(\cos t - 2\sin t) - e^t(\cos t - 3\sin t)$
$\quad + e^{-t}(\cos t - \sin t)$
$\quad = e^t(15\cos t - 29\sin t) + e^{-t}(\cos t - \sin t)$

37. $y'' + 9y = 6\sin t - 6u(t-\pi)\sin t$
$s^2Y + 9Y = \dfrac{6}{s^2+1} + \dfrac{6e^{-\pi s}}{s^2+1}$
$Y = 6\dfrac{1+e^{-\pi s}}{(s^2+9)(s^2+1)}$,
$\dfrac{1}{(s^2+9)(s^2+1)} = \dfrac{1}{8}(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+9})$
$\Rightarrow$
$y = \dfrac{3}{4}[\sin t - \dfrac{1}{3}\sin 3t - u(t-\pi)(\sin t - \dfrac{1}{3}\sin 3t)]$
$= \begin{cases} \dfrac{3}{4}(\sin t - \dfrac{1}{3}\sin 3t) = \sin^3 t & \text{if } 0 \le t \le \pi \\ 0 & \text{if } t > \pi \end{cases}$

38. $s^2Y + 3sY + 2Y = \dfrac{2e^{-2s}}{s}$
$Y = \dfrac{2e^{-2s}}{s(s^2+3s+2)}$
$\dfrac{2}{s(s^2+3s+2)} = \dfrac{1}{s} - \dfrac{s+3}{s^2+3s+2}$
$\rightarrow 1 - e^{-\frac{3}{2}t}(\cosh\frac{1}{2}t + 3\sinh\frac{1}{2}t) = 1 - 2e^{-t} + e^{-2t}$
$\Rightarrow\ y = u(t-2)(1 - 2e^{-t+2} + e^{-2t+4})$

39. $sY_1 = -Y_2 + 1,\ sY_2 = Y_1$
$Y_1 = \dfrac{s}{s^2+1},\ Y_2 = \dfrac{1}{s^2+1}$
$\Rightarrow\ y_1 = \cos t,\ y_2 = \sin t$

40. $\begin{bmatrix} s-2 & -4 \\ -1 & s-2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$
$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \dfrac{1}{(s-2)^2-4}\begin{bmatrix} s-2 & 4 \\ 1 & s-2 \end{bmatrix}\begin{bmatrix} -4 \\ -4 \end{bmatrix}$
$= \dfrac{-4}{(s-2)^2-4}\begin{bmatrix} s+2 \\ s-1 \end{bmatrix}$
$\Rightarrow$
$y_1 = -4e^{2t}(\cosh 2t + 2\sinh 2t) = 2(1-3e^{4t})$
$y_2 = -4e^{2t}(\cosh 2t + \frac{1}{2}\sinh 2t) = -3e^{4t} - 1$

41. $\begin{bmatrix} s-2 & 4 \\ -1 & s+3 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \dfrac{1}{s^2+s-2}\begin{bmatrix} s+3 & -4 \\ 1 & s-2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}$
$= \dfrac{3}{s^2+s-2}\begin{bmatrix} s+3 \\ 1 \end{bmatrix}$
$\Rightarrow$
$y_1 = 3e^{-\frac{1}{2}t}(\cosh\frac{3}{2}t + \frac{5}{3}\sinh\frac{3}{2}t) = 4e^t - e^{-2t}$
$y_2 = 3e^{-\frac{1}{2}t}(\frac{2}{3}\sinh\frac{3}{2}t) = e^t - e^{-2t}$

42. $\begin{bmatrix} s+2 & -3 \\ -4 & s+1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$
$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \dfrac{1}{s^2+3s-10}\begin{bmatrix} s+1 & 3 \\ 4 & s+2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
$= \dfrac{1}{s^2+3s-10}\begin{bmatrix} 4s+13 \\ 3s+22 \end{bmatrix}$
$\Rightarrow\ y_1 = 3e^{2t} + e^{-5t},\ y_2 = 4e^{2t} - e^{-5t}$

43. 왼쪽에서 오른쪽 방향을 양의 방향으로 잡으면, m_1, m_2가 각각 평형위치에서 y_1, y_2의 위치에 있을 때, k_1, k_2, k_3 스프링이 늘어난 길이는 각각 $y_1, y_2 - y_1, -y_2$ 이다.
m_1에는 Hooke 법칙에 의해, k_1스프링에 의한 힘 k_1y_1이 음의 방향으로, k_2스프링에 의한 힘 $k_2(y_2-y_1)$이 양의 방향으로 작용하므로, m_1에 작용하는 힘은 $-k_1y_1 + k_2(y_2-y_1)$ 이다. 그러므로 뉴턴 제 2법칙에 의해 $m_1y_1'' = -k_1y_1 + k_2(y_2-y_1)$ 이 성립한다.
마찬가지로 m_2에는 k_2스프링에 의한 힘 $k_2(y_2-y_1)$이 음의 방향으로, k_3스프링에 의한 힘 $-k_3y_2$가 양의 방향으로 작용하므로 m_2에 작용하는 힘은 $-k_2(y_2 -$

$y_1) - k_3y_2$이어서 $m_2y_2'' = -k_2(y_2 - y_1) - k_3y_2$가 성립한다.

44. $y_1'' = -6y_1 + 4y_2$
$y_2'' = 4y_1 - 6y_2$
$y_1(0) = y_2(0) = 0,\ y_1'(0) = 1,\ y_2'(0) = -1$
$\Rightarrow$
$s^2Y_1 - 1 = -6Y_1 + 4Y_2$
$s^2Y_2 + 1 = 4Y_1 - 6Y_2$
변변 더하면 $Y_1 + Y_2 = 0$
첫번째 식에 대입하면 $Y_1 = \dfrac{1}{s^2+10}$
$\Rightarrow\ y_1 = \dfrac{1}{\sqrt{10}}\sin\sqrt{10}t,\ y_2 = -\dfrac{1}{\sqrt{10}}\sin\sqrt{10}t$

45. $Q(0) = 0$ 이므로 방정식은 $10i + 10\int_0^t i\,dt = v(t)$.
$t = 0$ 을 대입하면, $i(0) = \dfrac{e(0)}{10} = 0$,
미분하면,
$10i' + 10i = v' = 10 - 10u(t-4)$
위의 적분방정식을 보면 v가 연속이므로 i도 연속임을 알 수 있다. 따라서 i에 관한 일차미방은 Laplace 변환할 수 있다.
$\Rightarrow$
$sY + Y = \dfrac{1}{s} - \dfrac{e^{-4s}}{s}$
$Y = \dfrac{1}{(s+1)s} - e^{-4s}\dfrac{1}{(s+1)s}$
$\dfrac{1}{(s+1)s} = \dfrac{1}{s} - \dfrac{1}{s+1} \to 1 - e^{-t}$
$\Rightarrow$
$i = 1 - e^{-t} - u(t-4)(1 - e^{-t+4})$
$= \begin{cases} 1 - e^{-t} & \text{if } 0 \le t \le 4 \\ e^{-t}(e^4 - 1) & \text{if } t > 4 \end{cases}$

46. 방정식은 $i' + \int_0^t i\,dt = v$
v가 연속이 아니므로 i'는 연속이 아니다. 따라서 i에 관한 2차미방은 Laplace 변환할 수 없으므로 q에 대한 미방으로 바꾸어 q를 먼저 구한다.
$q'' + q = v = 1 - e^{-t} - u(t-\pi)(1 - e^{-t})$,
$q(0) = 0, q'(0) = i(0) = 0$
$\Rightarrow$
$(s^2+1)Y = \dfrac{1}{s} - \dfrac{1}{s+1} - e^{-\pi s}(\dfrac{1}{s} - \dfrac{e^{-\pi}}{s+1})$
$\dfrac{1}{(s^2+1)s} = \dfrac{1}{s} - \dfrac{s}{s^2+1} \to 1 - \cos t$
$\dfrac{1}{(s^2+1)(s+1)} = \dfrac{1}{2}(\dfrac{1}{s+1} - \dfrac{s-1}{s^2+1})$
$\to \dfrac{1}{2}(e^{-t} - \cos t + \sin t)$
$\Rightarrow$
$q(t) = 1 - \cos t - \dfrac{1}{2}(e^{-t} - \cos t + \sin t)$
$- u(t-\pi)[1 + \cos t - \dfrac{1}{2}e^{-\pi}(e^{-t+\pi} + \cos t - \sin t)]$
$= \begin{cases} 1 - \dfrac{1}{2}(e^{-t} + \cos t + \sin t) & \text{if } 0 \le t \le \pi \\ \dfrac{1}{2}[(e^{-\pi} - 3)\cos t - (e^{-\pi} + 1)\sin t] & \text{if } t > \pi \end{cases}$
$i(t) = q'(t)$
$= \dfrac{1}{2}\begin{cases} e^{-t} + \sin t - \cos t & \text{if } 0 < t < \pi \\ (3 - e^{-\pi})\sin t - (e^{-\pi} + 1)\cos t & \text{if } t > \pi \end{cases}$

47. $160i + 500\int_0^t i\,dt + 20i' = 37\sin 10t$
$t = 0$일 때, $160i(0) + 20i'(0) = 0$ 으로부터 $i'(0) = 0$.
미분하면, $20i'' + 160i' + 500i = 370\cos 10t$
$\Rightarrow$
$2s^2Y + 16sY + 50Y = \dfrac{37s}{s^2 + 10^2}$
$Y = \dfrac{37s}{2(s^2+8s+25)(s^2+100)}$
$= \dfrac{1}{26}(\dfrac{3s-8}{s^2+8s+25} - \dfrac{3s-32}{s^2+100})$
$= \dfrac{1}{26}(\dfrac{3(s+4)-20}{(s+4)^2+9} - \dfrac{3s-32}{s^2+100})$
$\Rightarrow$
$i = \dfrac{1}{26}[e^{-4t}(3\cos 3t - \frac{20}{3}\sin 3t)$
$- (3\cos 10t - \frac{32}{10}\sin 10t)]$
$= e^{-4t}(\frac{3}{26}\cos 3t - \frac{10}{39}\sin 3t) - \frac{3}{26}\cos 10t + \frac{8}{65}\sin 10t$

48. Kirchhoff 법칙에 의해 매순간 각 폐회로에서의 전압 강하의 합은 0이므로
$Li_1' + R(i_1 - i_2) - v = 0$,
$R(i_2 - i_1) + \dfrac{1}{C}\int_a^t i_2 dt = 0$ 이 성립한다.
두번째 식을 미분하면 i_1, i_2에 관한 미분 방정식을 얻을 수 있다.

$20i_1' + 10(i_1 - i_2) = 20$
$10(i_2' - i_1') + 20i_2 = 0$
$\Rightarrow$
$\begin{bmatrix} 2s+1 & -1 \\ -s & s+2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2/s \\ 2 \end{bmatrix}$
$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \dfrac{1}{2(s+1)^2}\begin{bmatrix} s+2 & 1 \\ s & 2s+1 \end{bmatrix}\begin{bmatrix} 2/s \\ 2 \end{bmatrix}$
$= \dfrac{1}{(s+1)^2}\begin{bmatrix} 2 + \dfrac{2}{s} \\ 2s+2 \end{bmatrix}$
$\Rightarrow$
$i_1 = 2e^{-t}t + 2\int_0^t e^{-t}t\,dt = 2(1 - e^{-t}),\ i_2 = 2e^{-t}$

49. $i_1' + 0.8(i_1 - i_2) = \dfrac{4}{5}t + \dfrac{21}{25}$
$0.8(i_2' - i_1') + 4i_2 = 0$
$\Rightarrow$
$\begin{bmatrix} s+0.8 & -0.8 \\ -0.8s & 0.8s+4 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5s^2} + \frac{21}{25s} + 1 \\ -96/25 \end{bmatrix}$
$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} =$
$\dfrac{1}{s^2+5s+4}\begin{bmatrix} s+5 & 1 \\ s & \frac{5}{4}s+1 \end{bmatrix}\begin{bmatrix} \frac{4}{5s^2} + \frac{21}{25s} + 1 \\ -96/25 \end{bmatrix}$
$= \dfrac{1}{s^2+5s+4}\begin{bmatrix} s + 2 + \frac{5}{s} + \frac{4}{s^2} \\ -\frac{19}{5}s - 3 + \frac{4}{5s} \end{bmatrix}$
$T_1 = \mathcal{L}^{-1}\left[\dfrac{s}{s^2+5s+4}\right]$
$= e^{-\frac{5}{2}t}(\cosh\frac{3}{2}t - \frac{5}{3}\sinh\frac{3}{2}t) = \frac{4}{3}e^{-4t} - \frac{1}{3}e^{-t}$
$T_2 = \mathcal{L}^{-1}\left[\dfrac{1}{s^2+5s+4}\right] = \int_0^t T_1\,dt$
$= -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t}$

$T_3 = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+5s+4)}\right] = \int_0^t T_2 dt$
$= \frac{1}{12}e^{-4t} - \frac{1}{3}e^{-t} + \frac{1}{4}$
$T_4 = \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+5s+4)}\right] = \int_0^t T_3 dt$
$= -\frac{1}{48}e^{-4t} + \frac{1}{3}e^{-t} + \frac{1}{4}t - \frac{5}{16}$
$\Rightarrow$
$i_1 = T_1 + 2T_2 + 5T_3 + 4T_4 = e^{-4t} + t$
$i_2 = -\frac{19}{5}T_1 - 3T_2 + \frac{4}{5}T_3 = -4e^{-4t} + \frac{1}{5}$

50. $5i_1' + 20(i_1 - i_2) = 60$
$30i_2' + 20(i_2' - i_1') + 20i_2 = 0$

$$\begin{bmatrix} 5s+20 & -20 \\ -20s & 50s+20 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 60/s \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{50(5s^2+14s+8)}\begin{bmatrix} 50s+20 & 20 \\ 20s & 5s+20 \end{bmatrix}\begin{bmatrix} \frac{60}{s} \\ 0 \end{bmatrix}$$

$$= \frac{1}{(5s+4)(s+2)}\begin{bmatrix} 60+24/s \\ 24 \end{bmatrix}$$

$$\frac{1}{(5s+4)(s+2)} = \frac{1}{6}\left(\frac{5}{5s+4} - \frac{1}{s+2}\right)$$
$$\to \frac{1}{6}(e^{-0.8t} - e^{-2t})$$

$\Rightarrow$
$i_1 = 10(e^{-0.8t} - e^{-2t}) + 4\int_0^t e^{-0.8t} - e^{-2t} dt$
$= 5e^{-0.8t} - 8e^{-2t} + 3$
$i_2 = 4(e^{-0.8t} - e^{-2t})$

(i) 해로부터 $\lim_{t\to\infty} i_1 = 3$, $\lim_{t\to\infty} i_2 = 0$임을 알 수 있다.
(ii) v가 상수로서 일정하므로 평형상태의 전류는 상수일 것이고 시간이 지남에 따라 평형상태에 가까워져 갈 것이다. 평형상태의 전류는 미분방정식을 만족하는 상수해를 구하면, $20(i_1 - i_2) = 60$, $20i_2 = 0$으로부터 $i_1 = 3$, $i_2 = 0$