①

## Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$= e^{-t} \cdot \frac{t^n}{n}\Big|_0^\infty - \int_0^\infty \frac{t^n}{n} \cdot (-e^{-t})\, dt$$

$$= (0 - 0) + \frac{1}{n}\int_0^\infty e^{-t} t^n\, dt$$

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t} t^n\, dt$$

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1)$$

$$\Rightarrow \Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n(n-1)\Gamma(n-1)$$

$$\Gamma(n+1) = n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2) \cdots\cdots (n-\overline{n-1})\Gamma(1)$$

$$\Gamma(n+1) = n(n-1)(n-2) \cdots\cdots 1 \cdot \Gamma(1) \quad \text{—— ⓪}$$

$$n - \overline{n-1} = n-n+1 = 1$$

$$\Gamma(1) = \int_0^\infty e^{-t} \cdot t^{1-1}\, dt = \int_0^\infty e^{-t}\, dt = \left. \frac{e^{-t}}{-1} \right|_0^\infty$$

$$= -[0-1] = 1$$

by ⓪ $\Rightarrow$

$$\Gamma(n+1) = n(n-1)(n-2) \cdots\cdots 1$$

$$\boxed{\Gamma(n+1) = n!}$$

②

Find

Ex $\mathcal{L}\{t^n\}$

Sol: $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n \, dt$

Putting $st = u \Rightarrow t = u/s$

$s\,dt = du$

$dt = \frac{du}{s}$

when $t \to 0$, $u \to 0$

$t \to \infty$, $u \to \infty$

$$= \int_0^\infty e^{-u} \cdot \left(\frac{u}{s}\right)^n \cdot \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \cdot \frac{u^n}{s^{n+1}} \, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1).$$

$$= \frac{1}{s^{n+1}} n! = \frac{n!}{s^{n+1}}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$

$\int_0^1 t \, dt = \int_0^1 u \, du$

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**Ex 4** Let $f(t)=e^{at}$ where $t \geq 0$
a is const. Find $\mathcal{L}\{f(t)\} = ?$

**Sol:**

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty$$

$$= -\frac{1}{(s-a)} \cdot \left[0 - 1\right]$$

$$= \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

Similarly $\mathcal{L}\{e^{t}\} = \frac{1}{s-1}$

$$\mathcal{L}\{e^{-\frac{1}{2}t}\} = \frac{1}{(s+\frac{1}{2})}$$

## Linearity of Laplace Transform

**Theorem** The Laplace transform is a linear operator. That is for any function $f(t)$ and $g(t)$ whose Laplace transform exist and any constants $a$ and $b$

Then

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

**Proof:**

$$\mathcal{L}\{af(t)+bg(t)\} = \int_0^\infty e^{-st}\left[af(t)+bg(t)\right] dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}.$$

④

Laplace Transform of Sine & Cosine

Ex5 If $f(t) = \sin at$. Find $\mathcal{L}\{\sin at\} = ?$

Sol: $\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$

$$= \sin at \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(a\cos at)dt$$

$$= (0-0) + \frac{a}{s}\int_0^\infty e^{-st}\cos at \, dt$$

$$= \frac{a}{s}\left[\cos at \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot(-a\sin at)\,dt\right]$$

$$= \frac{a}{s}\left[\left(0+\frac{1}{s}\right) - \frac{a}{s}\int_0^\infty e^{-st}\sin at\right]$$

$$= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin at\}$$

$$\therefore \mathcal{L}\{\sin at\} + \frac{a^2}{s^2}\mathcal{L}\{\sin at\} = \frac{a}{s^2}$$

$$\mathcal{L}\{\sin at\}\left(1+\frac{a^2}{s^2}\right) = \frac{a}{s^2}$$

$$\left(\frac{s^2+a^2}{\cancel{s^2}}\right)\mathcal{L}\{\sin at\} = \frac{a}{\cancel{s^2}}$$

$$\boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}}$$

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Ex: Find Laplace transform of cosat.

Sol. $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$

Ans $\boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}}$

Ex 7 (i) If $f(t) = \cosh at$ Find $\mathcal{L}\{f(t)\}$.

Sol: $\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$\mathcal{L}\{\cosh at\} = \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$

$= \frac{1}{2}\cdot\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}$

$= \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$

$= \frac{s}{s^2-a^2}$

$\therefore$ $\boxed{\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}}$

(ii) $\boxed{\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}}$

(6)

First shifting theorem

(Replacement of s by s-a in the transform).

Theorem (First shifting theorem)

If a is a real number then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Proof.

$$\mathcal{L}\{e^{at} \cdot f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot f(t)\,dt$$

$$= F(s-a)$$

Ex7 (ii) $\mathcal{L}\{e^{at} \sin\omega t\} = ?$

(iii) $\mathcal{L}\{e^{at} \cos\omega t\} = ?$

$$\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}$$

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)}$$

$= e^{at} f(t)$

Ex7 (i) $\mathcal{L}\{e^{t} \cosh 3t\} = ?$

Sol: $\mathcal{L}\{e^{t} \cosh 3t\} = \mathcal{L}\{\cosh 3t\}_{s \to s-1}$

$$= \frac{s}{s^2-9}\Big|_{s \to s-1}$$

$$= \frac{s-1}{(s-1)^2-9}.$$

Ex 5.1

1) $2t+6$

Sol: $\mathcal{L}\{2t+6\} = 2\mathcal{L}\{t\} + 6\mathcal{L}\{1\}$

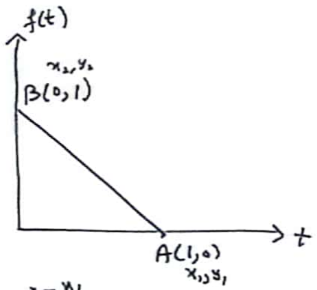
$= 2\frac{1}{s^2} + \frac{6}{s}$

$= \frac{2}{s^2} + \frac{6}{s}$

(1 – 8)

(7)

9)




Sol: $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0} = \frac{x-1}{0-1}$

$y = \frac{x-1}{-1}$

$y = -x+1$

$f(t) = -t+1 \quad 0 \le t \le 1$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^1 e^{-st}\cdot(-t+1)\,dt + \int_1^\infty e^{-st}\cdot 0\,dt$

$= -\int_0^1 te^{-st}\,dt + \int_0^1 e^{-st}\,dt$

$\boxed{9-16}$

$= -\left[t\cdot\frac{e^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\cdot 1\,dt\right] + \frac{e^{-st}}{-s}\Big|_0^1$

$= \frac{1}{s}\left[e^{-s} - 0\right] - \frac{1}{s}\int_0^1 e^{-st}\,dt$

$- \frac{1}{s}\left[e^{-s} - 1\right]$

$= \frac{e^{-s}}{s} - \frac{1}{s}\left|\frac{e^{-st}}{-s}\right|_0^1 - \frac{1}{s}e^{-s} + \frac{1}{s}$

$= \frac{e^{-s}}{s} + \frac{1}{s^2}\left[e^{-s} - 1\right] - \frac{1}{s}e^{-s} + \frac{1}{s}$

$=$

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$f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$\mathcal{L}^{-1}$ is also a linear operator.

Note 2

(i) $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

(ii) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(iv) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

(v) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

(vi) $\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$

(vii) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

17) $\frac{0.1s + 0.9}{s^2 + 3.24}$

Sol: $\mathcal{L}^{-1}\left\{\frac{0.1s+0.9}{s^2+3.24}\right\} = \mathcal{L}^{-1}\left\{\frac{0.1s}{s^2+(1.8)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{0.9}{s^2+(1.8)^2}\right\}$

$= 0.1\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+(1.8)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1.8}{s^2+(1.8)^2}\right\}$

$= 0.1 \cos 1.8t + \frac{1}{2}\sin 1.8t$

$\boxed{17-28}$

⑨

29) $t^2 e^{-3t}$

$$\mathcal{L}\{t^2 e^{-3t}\} = \mathcal{L}\{t^2\}_{s \to s+3}$$

$$= \frac{2!}{s^3}\Big|_{s \to s+3}$$

$$= \frac{2!}{(s+3)^3}.$$

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$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t) = e^{at} \mathcal{L}^{-1}\{F(s)\}.$$

3) $\frac{1}{(s+1)^2}$

Sol. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$= e^{-t} \cdot t.$

36, $\frac{12}{(s-3)^4}$

$$\mathcal{L}^{-1}\left\{\frac{12}{(s-3)^4}\right\} = e^{3t} \cdot \mathcal{L}^{-1}\left\{\frac{12}{s^4}\right\}$$

$$= 2e^{3t} \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}$$

$$= 2e^{3t} \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= 2e^{3t} \cdot t^3$$

35–40 //