Final Lecture 1

Laplace Transform

Defination: Let $f(t)$ be a given function that is defined for all $t \geq 0$ we multiply $f(t)$ by e^{-st} and integerate with respect to t from "0" to "infinity (∞)". Then if the resulting integral exists (that is has some finite value). It is the function of s say $F(s)$

$$\int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt = F(s)$$

$$\mathcal{L}\{f(t)\} = F(s)$$

$$f(t) = \mathcal{L}\{F(s)\}$$

Ex1: Let $f(t)=1$ when $t \geq 0$
Find $F(s)$.

Sol: $F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty$$

$$= \frac{-1}{s}[0-1]$$

$$= \frac{1}{s}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

Ex 2.

(i). $\mathcal{L}\{t\} = \int_0^\infty t e^{-st} dt$

$$= t\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt$$

$$= \frac{-1}{s}[0-0] + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$\mathcal{L}\{t\} = \frac{1}{s}\mathcal{L}\{1\}$$

$$= \frac{1}{s} \cdot \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

(ii). $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\, dt$

$$= t^2 \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 2t\, dt$$

$$= \frac{-1}{s}\left[0-0\right] + \frac{2}{s}\int_0^\infty e^{-st} \cdot t\, dt$$

$$= \frac{2}{s}\mathcal{L}\{t\}$$

$$= \frac{2}{s} \cdot \frac{1}{s}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

iii) $\mathcal{L}\{t^3\} = \int_0^\infty t^3 e^{-st}\,dt$

$$= t^3 \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, 3t^2\,dt$$

$$= \frac{-1}{s}[0-0] + \frac{3}{s}\int_0^\infty e^{-st}\, t^2\,dt$$

$$= \frac{3}{s}\cdot\mathcal{L}\{t^2\}$$

$$= \frac{3}{s}\cdot\frac{2}{s^3}$$

$$= \frac{6}{s^4}$$

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4}$$

Similarly

$$\mathcal{L}\{t^4\} = \frac{4!}{s^5}$$

$$\vdots \qquad \vdots$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Final Lecture 2

Gamma Function:

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\,dt$$

$$\Gamma(n) = e^{-t}\frac{t^n}{n}\Big|_0^\infty - \int_0^\infty \frac{t^n}{n}(-e^{-t})\,dt$$

$$= (0-0) + \frac{1}{n}\int_0^\infty t^n e^{-t}\,dt$$

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t}\, t^n\,dt$$

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1)$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(n+1) = n(n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\,\Gamma(n-2)$$

$$= n(n-1)(n-2)\cdots(n-\overline{n+1})\,\Gamma(1)$$

$$\Gamma(n+1) = n(n-1)(n-2)\cdots(1)\cdot\Gamma(1) \quad -(1)$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, t^{1-1}\,dt = \int_0^\infty e^{-t}\,dt = \left|\frac{e^{-t}}{-1}\right|_0^\infty$$

$$= -[0-1] = 1$$

eq. (1) $\Rightarrow$

$\Gamma(n+1) = n(n-1)(n-2)\ldots 1$

$\Gamma(n+1) = n!$

Ex: Find $\mathcal{L}\{t^n\}$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

putting $st = u \Rightarrow t = u/s$

$s\,dt = du$

$dt = \frac{du}{s}$

when $t \to 0,\ u \to 0$

$t \to \infty,\ u \to \infty$

$$= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$$\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}\, \Gamma(n+1)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Ex Let $f(t) = e^{at}$ where $t \geq 0$, a is constt. Find $\mathcal{L}\{f(t)\} = ?$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty$$

$$= \frac{-1}{(s-a)} [0-1]$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Similarly

$$\mathcal{L}\{e^{t}\} = \frac{1}{s-1}$$

$$\mathcal{L}\{e^{-\frac{1}{2}t}\} = \frac{1}{s+\frac{1}{2}}$$

Linearity of Laplace transform

Theorem

The laplace transform is a linear operator that is for any function $f(t)$ and $g(t)$ whose laplace transform exist and any constant a and b then

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Proof:

$$\mathcal{L}\{af(t)+bg(t)\} = \int_0^\infty e^{-st}(af(t)+bg(t))\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Laplace Transform of sine & cosine

Ex5: If $f(t) = \sin at$. Find $\mathcal{L}\{\sin at\} = ?$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at$$

$$= \sin at\,\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,a\cos at\,dt$$

$$= \frac{-1}{s}(0-0) + \frac{a}{s}\int_0^\infty e^{-st}\cos at$$

$$= \frac{a}{s}\left[\cos at\,\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,-a\sin at\,dt\right]$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s}\left[\frac{-1}{s}(0-1) - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt\right]$$

$$= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin at\}$$

$$\mathcal{L}\{\sin at\} + \frac{a^2}{s^2}\mathcal{L}\{\sin at\} = \frac{a}{s^2}$$

$$\mathcal{L}\{\sin at\}\left(1 + \frac{a^2}{s^2}\right) = \frac{a}{s^2}$$

$$\mathcal{L}\{\sin at\}\left(\frac{s^2+a^2}{s^2}\right) = \frac{a}{s^2}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Ex : Find Laplace transform of $\cos at$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \cos at \left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} - a \sin at \, dt$$

$$= -\frac{1}{s}(0-1) - \frac{a}{s}\int_0^\infty e^{-st} \sin at \, dt$$

$$= \frac{1}{s} - \frac{a}{s}\mathcal{L}\{\sin at\}$$

$$= \frac{1}{s} - \frac{a}{s} \cdot \frac{a}{s^2+a^2}$$

$$= \frac{s^2+a^2-a^2}{s(s^2+a^2)}$$

$$= \frac{s^2}{s(s^2+a^2)}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}.$$

Ex : i). If $f(t) = \cosh at$

Find $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{\cosh t\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2} \cdot \frac{1}{s-a} + \frac{1}{2} \cdot \frac{1}{s+a}$$

$$= \frac{s+a+(s-a)}{2(s-a)(s+a)}$$

$$= \frac{2s}{2(s^2-a^2)}$$

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2-a^2}$$

ii). $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2} \cdot \frac{1}{s-a} - \frac{1}{2} \cdot \frac{1}{s+a}$$

$$\mathcal{L}\{\sinh at\} = \frac{s+a-(s-a)}{2(s+a)(s-a)}$$

$$= \frac{2a}{2(s^2-a^2)}$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

First shifting Theorem (Replacement of s by s-a in the transform)

Theorem:

If a is a real number then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Proof:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Ex: (i) $\mathcal{L}\{e^t \cosh 3t\}$

$$\mathcal{L}\{e^t \cosh 3t\} = \mathcal{L}\{\cosh 3t\}_{s \to s-1}$$

$$= \left.\frac{s}{s^2-9}\right|_{s \to s-1}$$

$$\mathcal{L}\{e^t \cosh 3t\} = \frac{s-1}{(s-1)^2-9}$$

ii) $\mathcal{L}\{e^{at} \sin \omega t\}$

$$\mathcal{L}\{e^{at} \sin \omega t\} = \mathcal{L}\{\sin \omega t\}_{s \to s-a}$$

$$= \left.\frac{\omega}{s^2-a^2}\right|_{s \to s-a}$$

$$\mathcal{L}\{e^{at} \sin \omega t\} = \frac{\omega}{(s+a)^2-\omega^2}$$

iii). $\mathcal{L}\{e^{at}\cos\omega t\}$

$$\mathcal{L}\{e^{at}\cos\omega t\} = \mathcal{L}\{\cos\omega t\}_{s\to s-a}$$

$$= \left.\frac{s}{s^2-\omega^2}\right|_{s\to s-a}$$

$$= \frac{(s-a)^2}{(s-a)^2+\omega^2}$$

Ex 5.1

1. $2t+6$

$$\mathcal{L}\{2t+6\} = 2\mathcal{L}\{t\} + 6\mathcal{L}\{1\}$$

$$= 2.\frac{1}{s^2} + 6.\frac{1}{s}$$

$$\mathcal{L}\{2t+6\} = \frac{2}{s^2} + \frac{6}{s}$$

2. $a+bt+ct^2$

$$\mathcal{L}\{a+bt+ct^2\} = a\mathcal{L}\{1\} + b\mathcal{L}\{t\} + c\mathcal{L}\{t^2\}$$

$$= a.\frac{1}{s} + b.\frac{1}{s^2} + c.\frac{2}{s^3}$$

$$\mathcal{L}\{a+bt+ct^2\} = \frac{a}{s} + \frac{b}{s^2} + \frac{2c}{s^3}$$

3. $\sin\pi t$

$$\mathcal{L}\{\sin\pi t\} = \frac{\pi}{s^2+\pi^2}$$

4. $\cos^2\omega t$

$$\mathcal{L}\{\cos^2\omega t\} = \mathcal{L}\left\{\frac{1+\cos 2\omega t}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cos 2\omega t\}$$

$$= \frac{1}{2}.\frac{1}{s} + \frac{1}{2}.\frac{s}{(s^2+(2\omega)^2)}$$

$$= \frac{s^2-4\omega^2+s^2}{2s(s^2+4\omega^2)}$$

$$\mathcal{L}\{\cos^2\omega t\} = \frac{s^2+2\omega^2}{s(s^2+4\omega^2)}$$

5. e^{a-bt}

$$\mathcal{L}\{e^{a}\cdot e^{-bt}\} = e^{a}\,\mathcal{L}\{e^{-bt}\}$$

$$= e^{a}\cdot\frac{1}{s+b}$$

$$\mathcal{L}\{e^{a-bt}\} = \frac{e^{a}}{s+b}$$

6. $e^{t}\cos 3t$

$$\mathcal{L}\{e^{t}\cos 3t\} = \mathcal{L}\{\cos 3t\}_{s\to s-1}$$

$$= \left.\frac{s}{s^2+3^2}\right|_{s\to s-1}$$

$$= \frac{s-1}{(s-1)^2+9}$$

7. $\sin(\omega t+\delta)$

$$\mathcal{L}\{\sin(\omega t+\delta)\} = \mathcal{L}\{\sin\omega t\cos\delta + \cos\omega t\sin\delta\}$$

$$= \cos\delta\,\mathcal{L}\{\sin\omega t\} + \sin\delta\,\mathcal{L}\{\cos\omega t\}$$

$$= \cos\delta\cdot\frac{\omega}{s^2+a^2} + \frac{\sin\delta\cdot s}{s^2+a^2}$$

$$\mathcal{L}\{\sin(\omega t+\delta)\} = \frac{\omega\cos\delta + s\sin\delta}{s^2+a^2}$$

8. $\sin 2t\cos 2t$

$$\mathcal{L}\{\sin 2t\cos 2t\} = \mathcal{L}\left\{\frac{1}{2}\,2\sin 2t\cos 2t\right\}$$

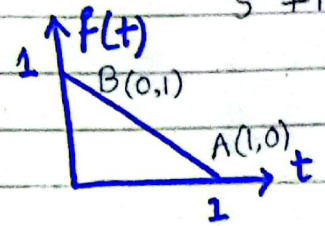
$$= \mathcal{L}\left\{\frac{1}{2}\cdot\sin 4t\right\}$$

$$= \frac{1}{2}\mathcal{L}\{\sin 4t\}$$

$$= \frac{1}{2}\cdot\frac{4}{(s^2+16)}$$

$$= \frac{2}{s^2+16}$$

9.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0} = \frac{x-1}{0-1}$$

$$y = 1-x$$

$$f(t) = 1-t \qquad 0\le t\le 1$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}(-t+1)\,dt + \int_1^\infty e^{-st}\cdot 0\,dt$$

$$= -\int_0^1 te^{-st}\,dt + \int_0^1 e^{-st}\,dt$$

$$= -\left[\frac{te^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt\right] + \frac{e^{-st}}{-s}\Big|_0^1$$

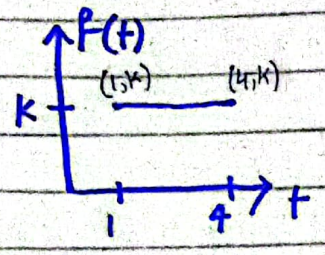
$$= -\left[-\frac{1}{s}\left(e^{-s}-0\right) + \frac{1}{s}\int_0^1 e^{-st}\,dt\right] + \left(\frac{-1}{s}\right)\left(e^{-s}-1\right)$$

$$= \frac{1}{s}e^{-s} - \frac{1}{s}\left|\frac{e^{-st}}{-s}\right|_0^1 - \frac{1}{s}e^{-s} + \frac{1}{s}$$

$$= -\frac{1}{s}e^{-s} - \left(\frac{-1}{s}\right)\left(e^{-s}-1\right) + \frac{1}{s}$$

$$= +\frac{1}{s^2}e^{-s} - \frac{1}{s^2} + \frac{1}{s}$$

10.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-k}{k-k} = \frac{x-1}{4-1}$$

$$3(y-k) = 0(x-1)$$

$$y = k$$

$$f(t) = k \qquad 1 \le t \le 4$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot 0\,dt + \int_1^4 e^{-st}\cdot k\,dt + \int_4^\infty e^{-st}\,0\,dt$$

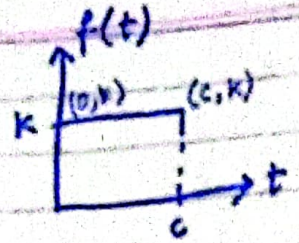
$$= \int_1^4 e^{-st}\cdot k\,dt$$

$$= k\cdot\left|\frac{e^{-st}}{-s}\right|_1^4$$

$$= -\frac{k}{s}\left(e^{-4s} - e^{-1}\right)$$

$$= -\frac{k}{se^{4s}} + \frac{k}{se}$$

11.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-k}{k-k} = \frac{x-0}{c-0}$$

$$y-k = 0\cdot\left(\frac{x}{c}\right)$$

$$y = k$$

$$f(t) = k \quad 0 \le t \le c$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

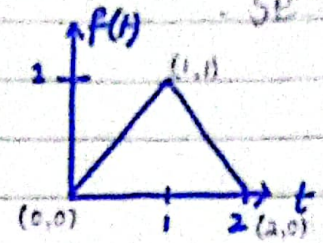
$$= \int_0^c e^{-st}\cdot k\,dt - \int_c^\infty e^{-st}\cdot 0\,dt$$

$$= k\cdot\left[\frac{e^{-st}}{-s}\right]_0^c$$

$$= -\frac{k}{s}\left(e^{-sc} - e^0\right)$$

$$\mathcal{L}\{f(t)\} = \frac{-k}{se^{sc}} + \frac{k}{s}$$

12.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$y = x$$

$$f(t) = t \quad 0 \le t \le 1$$

$$\frac{y-y_2}{y_3-y_2} = \frac{x-x_2}{x_3-x_2}$$

$$\frac{y-1}{0-1} = \frac{x-1}{2-1}$$

$$y-1 = -(x-1)$$

$$y-1 = 1-x$$

$$y = 2-x$$

$$f(t) = 2-t \quad 1 \le t \le 2$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\, t\,dt + \int_1^2 e^{-st}(2-t)\,dt + \int_2^\infty e^{-st}\, 0\,dt$$

$$= \left[\frac{te^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt\right] + 2\int_1^2 e^{-st}\,dt - \int_1^2 e^{-st}\, t\,dt$$

$$= \left[\frac{-1}{s}(e^{-s}-0) + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1\right] + 2\left|\frac{e^{-st}}{-s}\right|_1^2$$

$$- \left[t\frac{e^{-st}}{-s}\Big|_1^2 - \int_1^2 \frac{e^{-st}}{-s}\,dt\right]$$

$$= \left[\frac{-1}{s}e^{-s} - \frac{1}{s^2}(e^{-s}-1)\right] + \frac{2}{s}(e^{-2s}-e^{-s}) -$$

$$\left[\frac{-1}{s}(2e^{-2s}-e^{-s}) + \frac{1}{s}\left|\frac{e^{-st}}{-s}\right|_1^2\right]$$

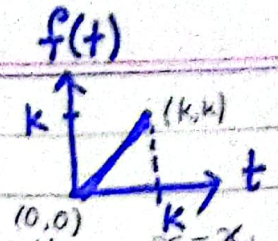
$$= \cancel{\frac{-e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \cancel{\frac{2e^{-2s}}{s}} + \cancel{\frac{2e^{-s}}{s}} + \cancel{\frac{2e^{-2s}}{s}} - \cancel{\frac{e^{-s}}{s}} + \frac{1}{s^2}(e^{-2s}-e^{-s})$$

$$= \frac{-e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$= \frac{-2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2}(e^{-s}-1)^2$$

13.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{k-0} = \frac{x-0}{k-0}$$

$$\frac{y}{k} = \frac{x}{k}$$

$$y = x$$

$$f(t) = t \qquad 0 \le t \le k$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^k e^{-st}\, t\,dt + \int_k^\infty e^{-st}\, 0\,dt$$

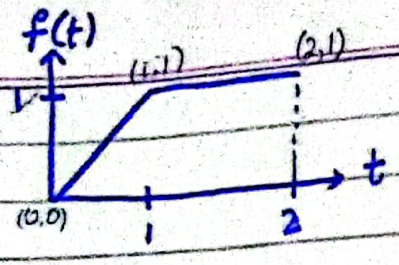
$$= t\frac{e^{-st}}{-s}\Big|_0^k - \int_0^k \frac{e^{-st}}{-s}\,dt$$

$$= \frac{-1}{s}(ke^{-sk}-0) + \frac{1}{s}\left|\frac{e^{-st}}{-s}\right|_0^k$$

$$= \frac{-k}{s}e^{-sk} - \frac{1}{s^2}(e^{-sk}-1)$$

$$\mathcal{L}\{f(t)\} = \frac{-k}{s}e^{-sk} - \frac{1}{s^2}e^{-sk} + \frac{1}{s^2}$$

14.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$y = x$$

$$f(t) = t \qquad 0 \le t \le 1$$

$$\frac{y-y_2}{y_3-y_2} = \frac{x-x_2}{x_3-x_2}$$

$$\frac{(y-1)}{1-1} = \frac{x-1}{2-1}$$

$$y-1 = 0\left(\frac{x-1}{1}\right)$$

$$y = 1$$

$$f(t) = 1 \qquad 1 \le t \le 2$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}\cdot 1\,dt + \int_2^\infty e^{-st}\cdot 0\,dt$$

$$= \left\{ \left.\frac{e^{-st}}{-s}\right|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt + \left.\frac{e^{-st}}{-s}\right|_1^2 \right.$$

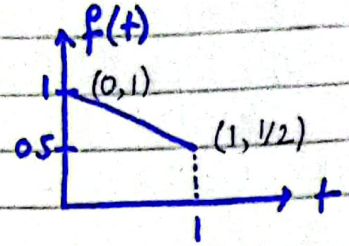
$$= \frac{-1}{s}\left(e^{-s}-0\right) + \frac{1}{s}\left|\frac{e^{-st}}{-s}\right|_0^1 - \frac{1}{s}\left(e^{-2s}-e^{-s}\right)$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}\left(e^{-s}-1\right) - \frac{e^{-2s}}{s} + \frac{e^{-s}}{s}$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{e^{-2s}}{s} + \frac{e^{-s}}{s}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-s}}{s^2}$$

15.



$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-1}{1/2-1} = \frac{x-0}{1-0}$$

$$y-1 = -\frac{1}{2}x$$

$$y = 1 - \frac{1}{2}x$$

$$f(t) = 1 - \frac{1}{2}t \qquad 0 \le t \le 1$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^1 e^{-st} \cdot (1 - 1/2\, t)\, dt + \int_1^\infty e^{-st} \cdot 0\, dt$$

$$= \int_0^1 e^{-st}\, dt - \frac{1}{2}\int_0^1 t e^{-st}\, dt$$

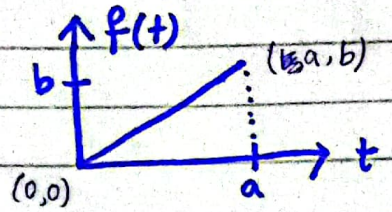
$$= \left| \frac{e^{-st}}{-s} \right|_0^1 - \frac{1}{2}\left[\left| \frac{t e^{-st}}{-s} \right|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\, dt \right]$$

$$= \frac{-1}{s}(e^{-s} - 1) - \frac{1}{2}\left[\frac{-1}{s}(e^{-s} - 0) + \frac{1}{s}\left| \frac{e^{-st}}{-s} \right|_0^1 \right]$$

$$= \frac{-1}{s}(e^{-s} - 1) + \frac{1}{2s}e^{-s} + \frac{1}{2s^2}(e^{-s} - 1)$$

$$\mathcal{L}\{f(t)\} \quad \frac{1}{2}\left[\frac{e^{-s} - 1}{s^2} - \frac{e^{-s} - 2}{s} \right]$$

16.



$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 0}{a - 0}$$

$$y = \frac{b}{a}x$$

$$f(t) = \frac{b}{a}t \qquad 0 \le t \le a$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^a e^{-st} \cdot \frac{b}{a} t\, dt + \int_a^\infty e^{-st} \cdot 0\, dt$$

$$= \frac{b}{a}\int_0^a e^{-st} \cdot t\, dt$$

$$= \frac{b}{a}\left[\left| \frac{t e^{-st}}{-s} \right|_0^a - \int_0^a \frac{e^{-st}}{-s}\, dt \right]$$

$$= \frac{b}{a}\left[\frac{-1}{s}(a e^{-as} - 0) - \frac{1}{s^2}(e^{-as} - 1) \right]$$

$$= \frac{-b}{as} \cdot a e^{-as} - \frac{b}{as^2}(e^{-as} - 1)$$

$$= \frac{-b e^{-as}}{s} - \frac{b}{as^2}(e^{-as} - 1)$$

Inverse laplace transform:

Note 1: $\mathcal{L}^{-1}$ is also a linear operator

Note 2:

i). $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

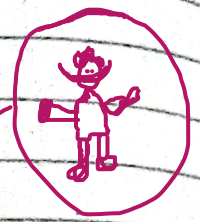
ii). $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

iii). $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

iv). $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

v). $\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$

vi). $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$



vii). $\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

17. $\frac{0.1s + 0.9}{s^2 + 3.24}$

$$\mathcal{L}^{-1}\left\{\frac{0.1s+0.9}{s^2+3.24}\right\} = \mathcal{L}^{-1}\left\{\frac{0.1s}{s^2+(1.8)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}(1.8)}{s^2+(1.8)^2}\right\}$$

$$= 0.1\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+(1.8)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1.8}{s^2+(1.8)^2}\right\}$$

$$= 0.1\cos 1.8t + \frac{1}{2}\sin 1.8t$$

18. $\frac{5s}{s^2-25}$

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2-25}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-(5)^2}\right\}$$

$$= 5\cosh 5t$$

19. $\frac{-s-10}{s^2-s-2}$

$$\mathcal{L}^{-1}\left\{\frac{-s-10}{s^2-s-2}\right\} = -\mathcal{L}^{-1}\left\{\frac{+s+10}{s^2-s-2}\right\} \quad \text{--- (1)}$$

$$\frac{s+10}{s^2-s-2} = \frac{A}{s-2} + \frac{B}{s+1} \quad \text{--- (a)}$$

$$s+10 = A(s+1) + B(s-2) \quad \text{--- (b)}$$

put $s=-1$ in (b) $\quad 9 = B(-1-2) \Rightarrow B = -3$

put $s=2$ in (b) $\quad 2+10 = A(2+1) \Rightarrow A = 4$

put in a). $\frac{s+10}{s^2-s-2} = \frac{4}{s-2} + \frac{-3}{s+1}$

eq (1) $\Rightarrow \mathcal{L}^{-1}\left\{\frac{-s-10}{s^2-s-2}\right\} = -4\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$$f(t) = -4e^{2t} + 3e^{-t}$$