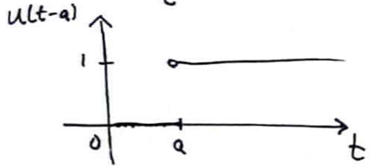
Unit step Function;
Second shifting theorem;
Dirac's Delta function;

Unit step function

By definition, $u(t-a)$ is "0" for $t<a$, has a jump of size 1 at $t=a$ and is "1" for $t>a$.

The function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$$



$u(t-a)$.

when

$$u(t) = \begin{cases} 0 \,, & t<0 \\ 1 \,, & t>0 \end{cases}$$

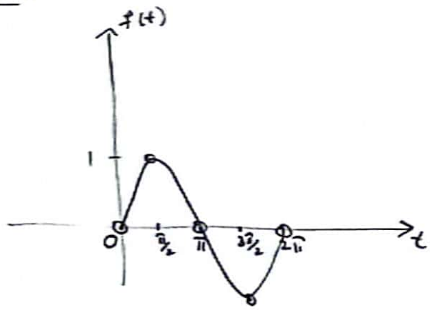
②

Unit step Function;

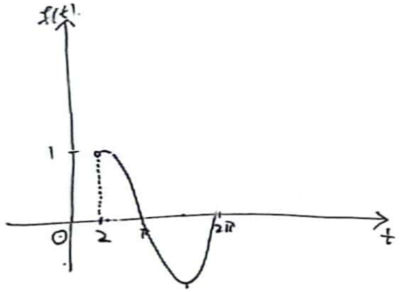
Second shifting theorem;

Dirac's Delta function;

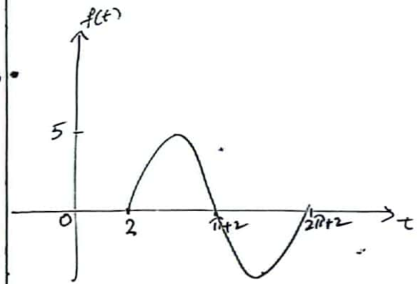
$f(t) = \sin t$



$f(t)\, u(t-2) = \sin t\; u(t-2).$



$f(t-2)\, u(t-2) = 5 \sin(t-2)\; u(t-2)$



③

Unit step Function;
Second shifting theorem;
Dirac's Delta function;

Second Shifting theorem (t- shifting theorem)

If $f(t)$ has the transform $F(s)$, then the shifted function

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

has the transform $e^{-as}F(s)$.

That is

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).$$

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)dt + \int_a^\infty e^{-st} f(t-a)u(t-a)dt$$

$\int_0 t\,dt \qquad \int_0^1 u\,du$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

$t-a = u$ when $t=a \Rightarrow u=0$

$dt = du$ when $t=\infty \Rightarrow u=\infty$

$$= \int_0^\infty e^{-s(u+a)} \cdot f(u)du.$$

$$= \int_0^\infty e^{-su} \cdot e^{-sa} f(u)du$$

$$= e^{-as} \cdot \int_0^\infty e^{-su} f(u)du$$

$$= e^{-as}\mathcal{L}\{f\} = e^{-as}F(s).$$

(4)

$$\boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\cdot F(s)}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\ u(t-a)}$$

Ex (Use of step function)

Find the Laplace transform of the function

$$f(t) = \begin{cases} 2, & \text{if } 0<t<\pi \\ 0, & \text{if } \pi<t<2\pi \\ \sin t, & \text{if } t>2\pi \end{cases}$$

Sol:

we write $f(t)$ in terms of unit step functions. For $0<t<\pi$. we take $2u(t)$. For $t>\pi$ we want "zero". So we must subtract the step function $2u(t-\pi)$ with step at π. we have $2u(t)-2u(t-\pi)=0$ when $t>\pi$. This is fine untill we reach 2π, where we want $\sin t$ to come in. So we add $\sin t\ u(t-2\pi)$

Together.

$$f(t) = 2u(t) - 2u(t-\pi) + \sin t\ u(t-2\pi).$$

$$f(t) = 2\left[u(t) - u(t-\pi)\right] + 0\left[u(t-\pi) - u(t-2\pi)\right] + \sin t\ u(t-2\pi)$$

$$= 2u(t) - 2u(t-\pi) + \sin t\ u(t-2\pi).$$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{u(t)\} - 2\mathcal{L}\{u(t-\pi)\} + \mathcal{L}\{\sin t\cdot u(t-2\pi)\}$$

$$= 2e^{-0s}\cdot\frac{1}{s} - 2e^{-\pi s}\frac{1}{s} + \mathcal{L}\{\sin(t+2\pi-2\pi)u(t-2\pi)\}$$

$$= \frac{2}{s} - 2\frac{e^{-\pi s}}{s} + e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$$

$$= 2/s - 2\frac{e^{-\pi s}}{s} + e^{-2\pi s}\mathcal{L}\{\sin t\}$$

$$= \frac{2}{s} - 2\frac{e^{-\pi s}}{s} + e^{-2\pi s}\frac{1}{s^2+1}.$$

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$$\boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\cdot F(s)}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\ u(t-a)}$$

Ex2 Find inverse Laplace transform $f(t)$ of

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{4e^{-\pi s}}{s^2+1}$$

$\mathcal{L}\{f(t)\}$

Sol:

$$f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2}\right\} - 4\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{s^2+1}\right\}$$

$$f(t) = 2t - 2(t-2)u(t-2) - 4\cdot 1\cdot u(t-2) + 4\sin(t-\pi)\cdot u(t-\pi).$$

$\sin t\cos\pi - \cos t\sin\pi$

$$= 2t - 2t\,u(t-2) + \cancel{4u(t-2)} - \cancel{4u(t-2)} - 4\sin t\ u(t-\pi)$$

$$= 2t - 2t\,u(t-2) - 4\sin t\ u(t-\pi).$$

$$= 2t\,u(t) - 2t\,u(t-2) - 4\sin t\cdot u(t-\pi)$$

$$= \begin{cases} 2t & 0<t<2 \\ 2t-2t & 2<t<\pi \\ 2t-2t-4\sin t & t>\pi \end{cases}$$

$$= \begin{cases} 2t & 0<t<2 \\ 0 & 2<t<\pi \\ -4\sin t & t>\pi \end{cases}$$

EX 5.3

Q1 — 25

2) $t\,u(t-1)$

Sol: $\mathcal{L}\{t\,u(t-1)\}$

$= \mathcal{L}\{(t+1-1)\,u(t-1)\}$

$= e^{-1s}\,\mathcal{L}\{t+1\}$

$= e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right]$

8) $t^2 \quad (0<t<1)$

Sol: $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\cdot t^2\,dt$

$= \int_0^1 e^{-st}\,t^2\,dt + \int_1^\infty e^{-st}\cdot 0\,dt$

$= \int_0^1 e^{-st}\cdot t^2\,dt.$

Sol: $f(t) = t^2 \quad (0<t<1)$

$f(t) = t^2[u(t)-u(t-1)]$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u(t)\} - \mathcal{L}\{t^2 u(t-1)\}$

$= e^{-0s}\cdot\mathcal{L}\{t^2\} - \mathcal{L}\{(t+1-1)^2\,u(t-1)\}$

$= \frac{2!}{s^3} - e^{-1s}\,\mathcal{L}\{(t+1)^2\}$

$= \frac{2!}{s^3} - e^{-s}\,\mathcal{L}\{t^2+1+2t\}$

$= \frac{2!}{s^3} - e^{-s}\cdot\frac{2!}{s^3} - e^{-s}\cdot\frac{1}{s} - 2e^{-s}\frac{1}{s^2}.$