8

Fourier Series

Solutions
- approximation
  - Taylor's series (continuous function)
  - Fourier series (continuous f. discontinuous functions also)
- exact

Periodic function

A function f(x) is called periodic ∀ real x ... if there is a +ve no. p such that

$$f(x+p) = f(x)$$

the number p is called period of f(x)

$f(x+2p) = f((x+p)+p) = f(x+p) = f(x)$

$f(x+3p) = f((x+2p)+p) = f(x+p) = f(x)$

⋮

$f(x+np) = f(x)$

Here $2p, 3p, \ldots, np$ are also periods of $f(x)$.

2

Fundamental period:

If a periodic function $f(x)$ has a smallest period $p>0$, this is often called Fundamental period of $f(x)$.

For $\sin x$ & $\cos x$, the Fundamental period is 2π

$\sin(x+2\pi) = \sin x$

$\cos(x+2\pi) = \cos x$.

$\sin 2x$.

$2x = 2\pi$

$\boxed{x = \pi}$

$\cos 3x$

$3x = 2\pi$

$\boxed{x = 2\pi/3}$

$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Ex 10.1

1, ...

$\sin 2\pi x$

$2\pi x = 2\pi$

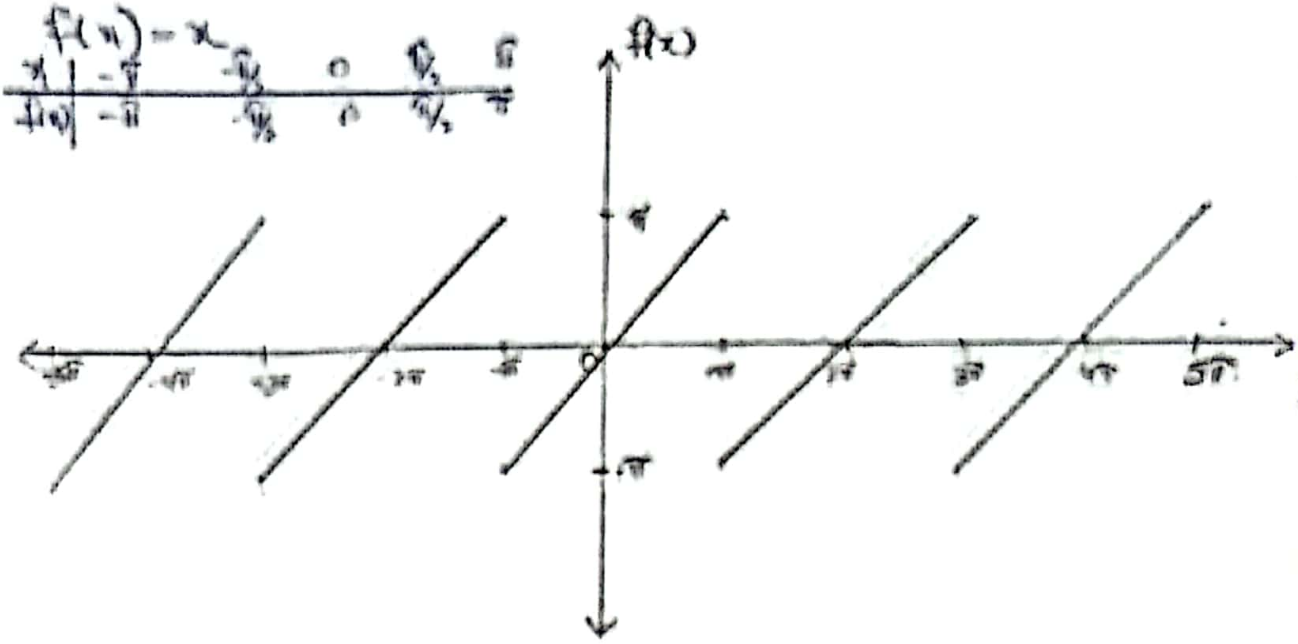
$\boxed{x = 1}$

Graphs of 2π periodic functions

Sketch or plot the following functions $f(x)$, which are assumed to be periodic with period 2π and for $-\pi < x < \pi$ are given by the formulae.

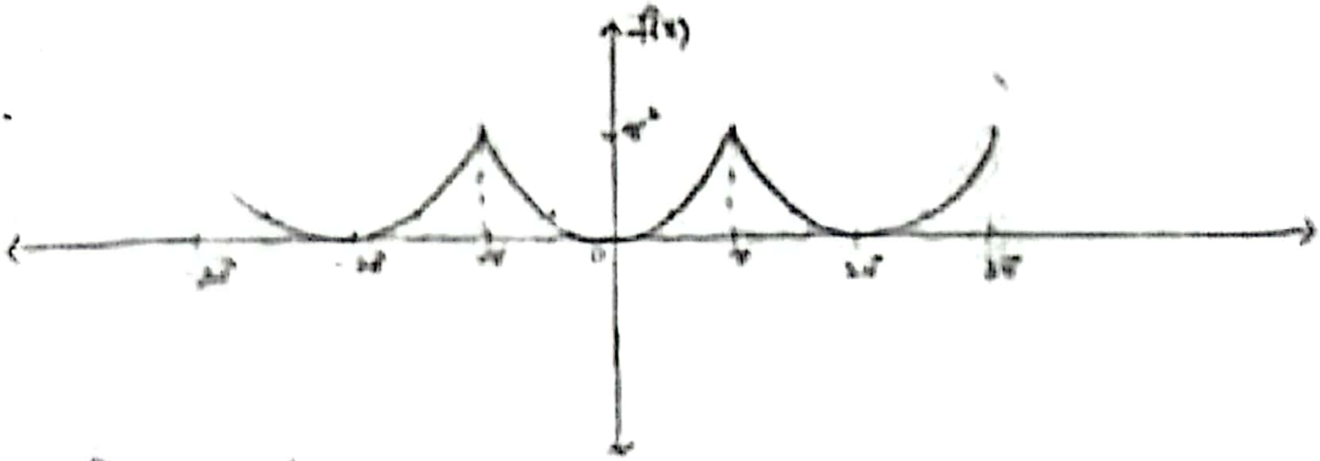
7) $f(x) = x$

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	$-\pi$	$-\pi/2$	0	$\pi/2$	π



8) $f(x) = x^2$

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	π^2	$\pi^2/4$	0	$\pi^2/4$	π^2

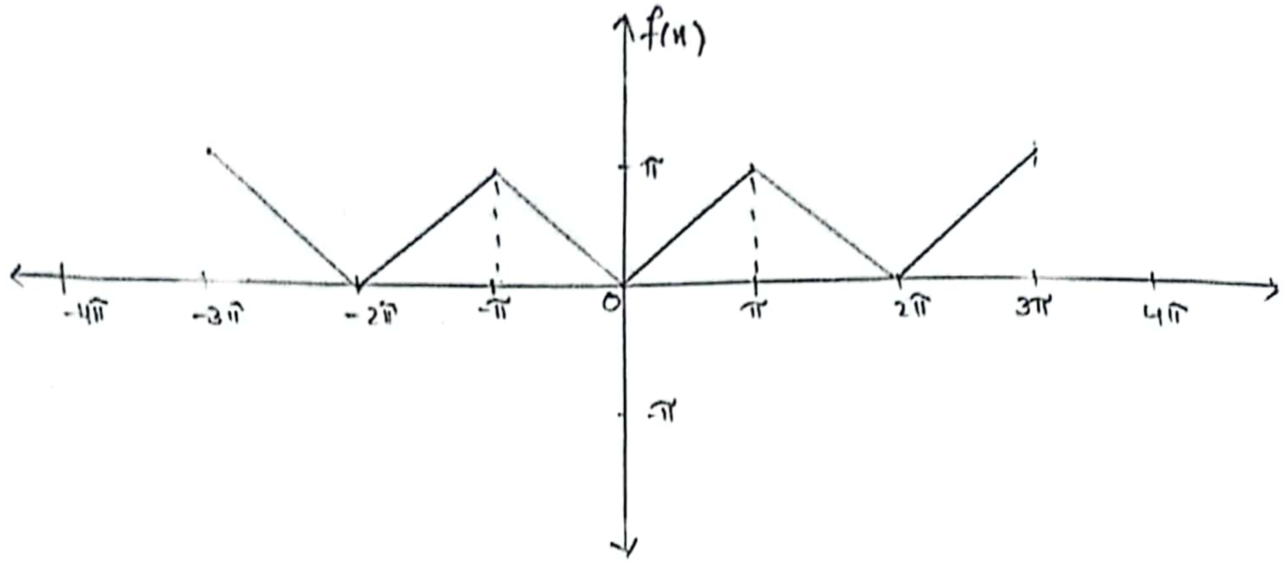


9) $f(x) = |x|$

$$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	π	$\pi/2$	0	$\pi/2$	π

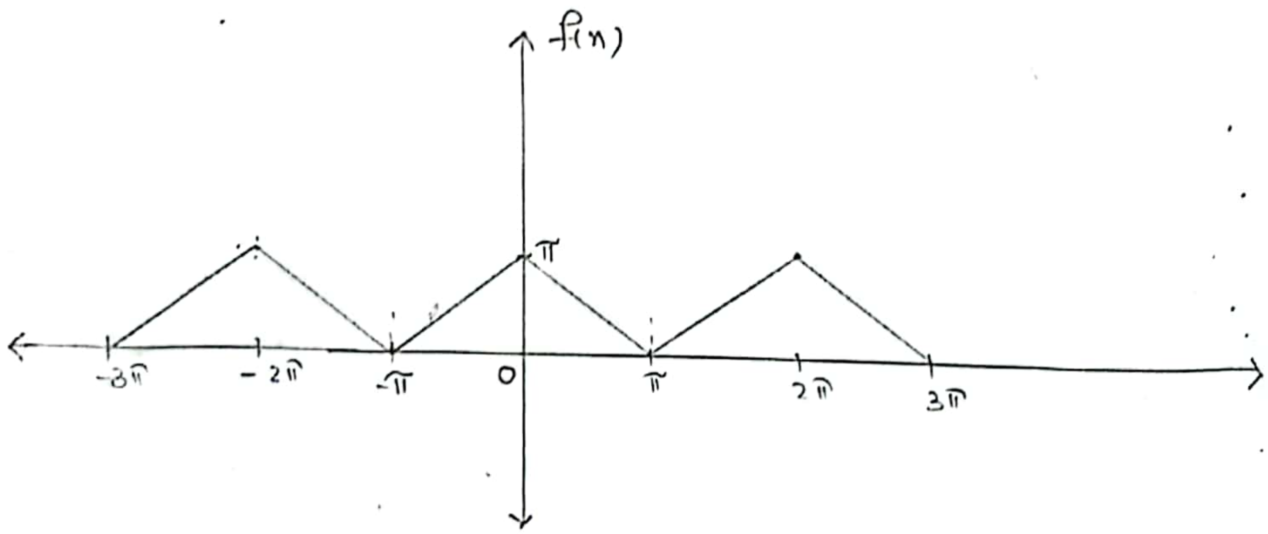
(3)



<u>10</u> $f(x) = \pi - |x|$

$$= \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$

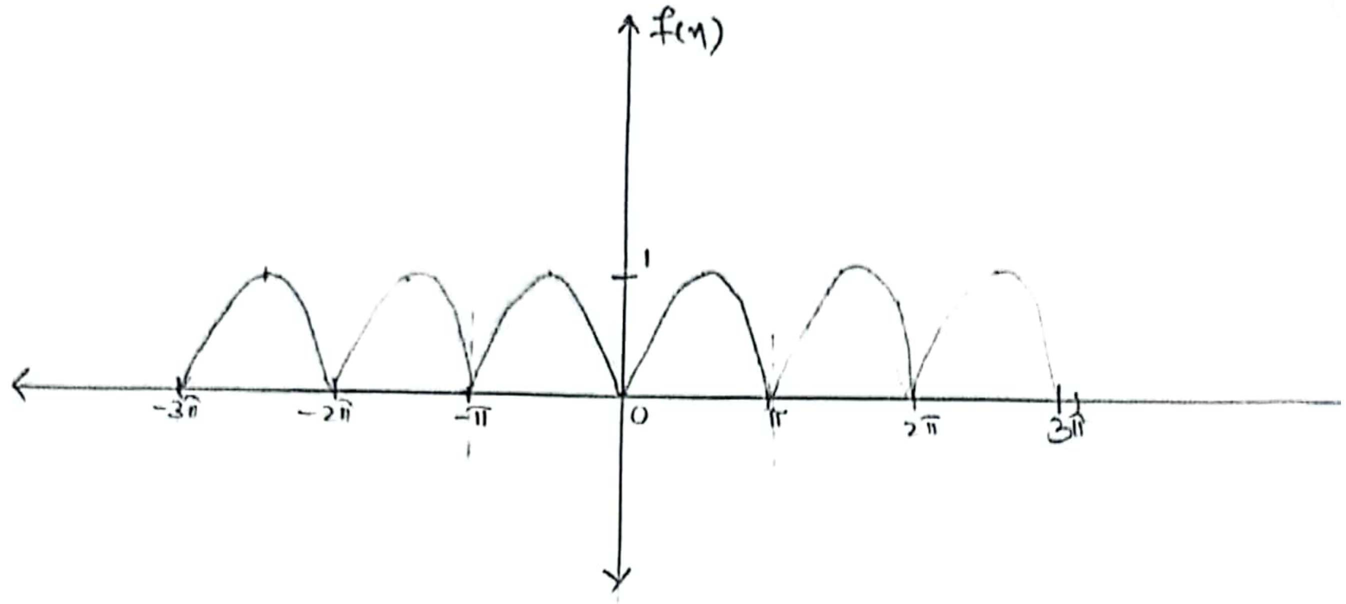
x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	0	$\pi/2$	π	$\pi/2$	0



<u>11</u> $f(x) = |\sin x|$

$$= \begin{cases} -\sin x & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

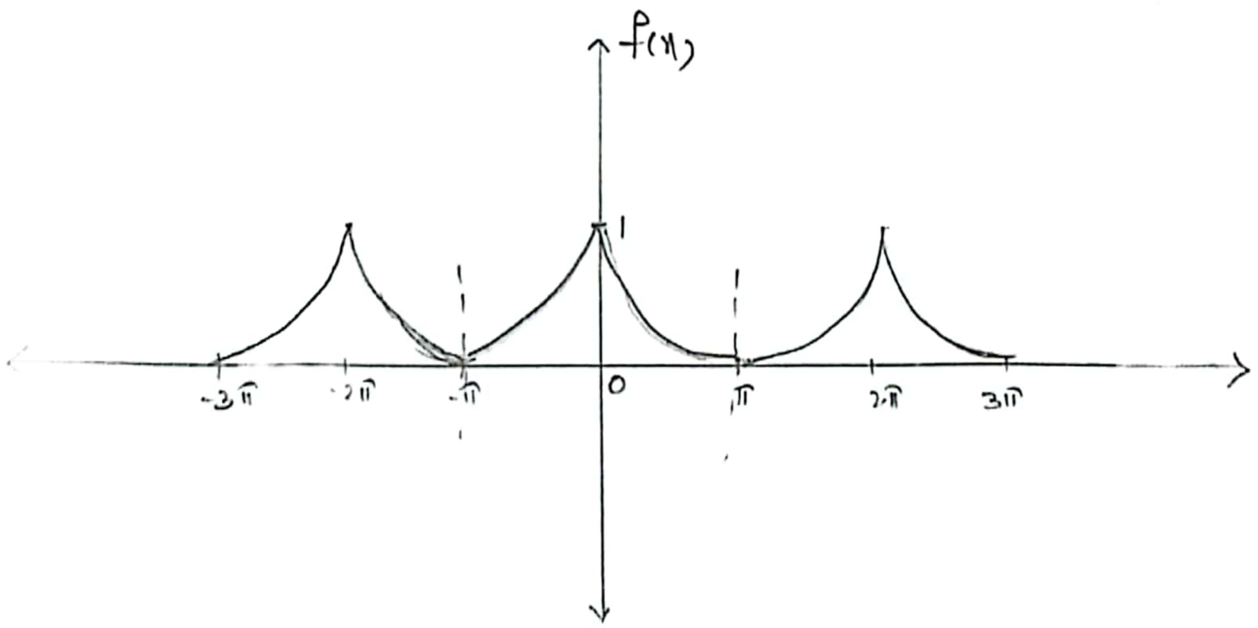
x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	0	1	0	1	0



12 $f(x) = e^{-|x|}$

$$= \begin{cases} e^{x} & -\pi < x < 0 \\ e^{-x} & 0 < x < \pi \end{cases}$$

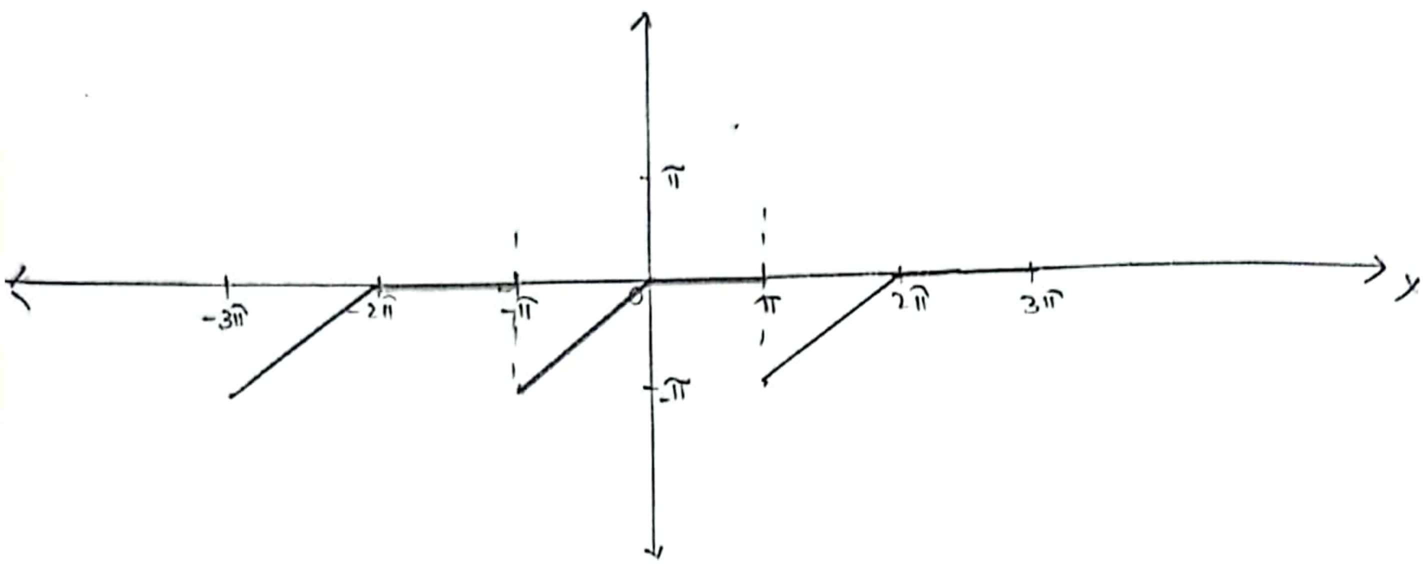
x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	0.043		1		0.042



(5) 15

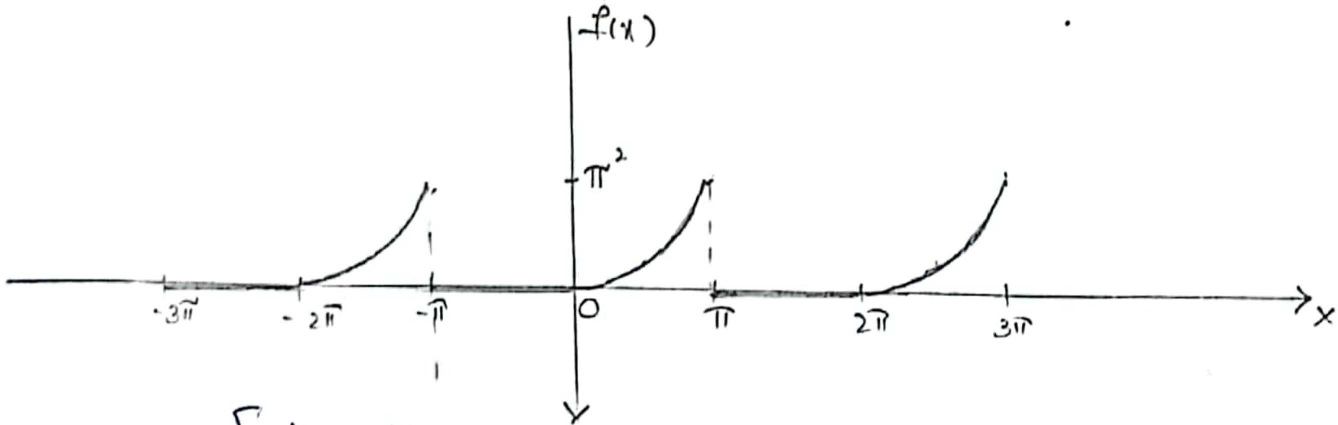
$$f(x) = \begin{cases} x & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$$

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	$-\pi$	$-\pi/2$	0	0	0



14) $$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$$

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	0	0	0	$\pi^2/4$	π^2



15 $$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$f(x)$	-1	-1		1	1

Euler Formulas for the fourier Co-efficients

Let us assume that $f(x)$ is a periodic function of Period 2π that can be represented by a trignometric Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad (1)$$

that is, we assume that this series converges and has $f(x)$ as its sum. Given such a function $f(x)$, we can determine the co-efficients a_n & b_n of eq (1) which are represented as

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n=1,2,3,\ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n=1,2,3,\ldots \end{aligned}\right\} \quad (2)$$

These are called fourier co-efficients of the fourier series (1) of a periodic function of Period 2π.

Note:

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Derivation of Co-efficients:

Determination of the Constant term a_0

To find a_0, integrate both sides of eq (1) from $-\pi$ to π, we get.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0\,dx + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]dx$$

$$= \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos nx\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n\sin nx\,dx$$

$$= a_0\, x\Big|_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n \left|\frac{\sin nx}{n}\right|_{-\pi}^{\pi} + \sum_{n=1}^{\infty} b_n\left(-\frac{\cos nx}{n}\right|_{-\pi}^{\pi}$$

$$= a_0(\pi+\pi) + \sum_{n=1}^{\infty}\frac{a_n}{n}(\sin n\pi + \sin n\pi) + {}$$

$$- \sum_{n=1}^{\infty}\frac{b_n}{n}(\cos n\pi - \cos n\pi)$$

$$= a_0(\pi+\pi) + 0 - 0$$

$$= 2\pi a_0$$

$$\Rightarrow \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

Determination of the co-efficients a_n of the cosine terms.

Multiply eq (1) by $\cos mx$, where m is any fixed positive integer and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} a_0\cos mx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx$$

$$+ \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\cos mx\sin nx\,dx$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0 + a_n\pi + 0$$

$$\Rightarrow \quad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \qquad m=1,2,3,\ldots$$

Determination of b_n of the sine terms.

Multiply eq (1) by $\sin mx$, where m is any fixed positive integer and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} a_0 \sin mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx\,dx$$

$$= 0 + 0 + b_n(\pi)$$

$$\Rightarrow \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx. \qquad m=1,2,3,\ldots$$

Summary of these calculations:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\cancel{2}\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n=1,2,3,\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n=1,2,3,\ldots$$

These are called Euler's formulas.

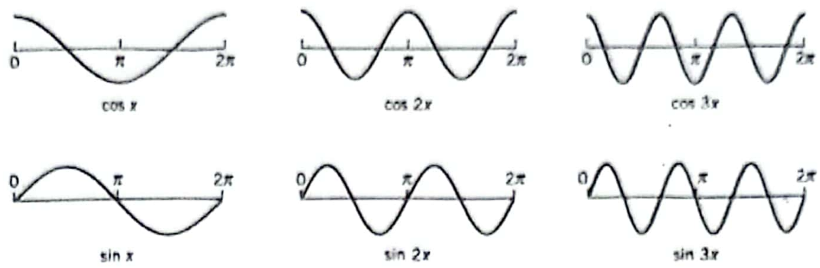


Fig. 237. Cosine and sine functions having the period 2π

These functions have the period 2π. Figure 237 shows the first few of them.
The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots, \tag{4}$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants. Such a series is called a **trigonometric series,** and the a_n and b_n are called the **coefficients** of the series. Using the summation sign,[3] we may write this series

$$\boxed{a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).} \tag{4}$$

The set of functions (3) from which we have made up the series (4) is often called the **trigonometric system,** to have a short name for it.

We see that each term of the series (4) has the period 2π. Hence *if the series* (4) *converges, its sum will be a function of period* 2π.

The point is that trigonometric series can be used for representing any practically important periodic function f, simple or complicated, of any period p. (This series will then be called the *Fourier series* of f.)

Problem Set 10.1

Fundamental Period. Find the smallest positive period p of

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$
2. $\cos nx$, $\sin nx$, $\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$, $\cos \dfrac{2\pi n x}{k}$, $\sin \dfrac{2\pi n x}{k}$
3. **(Vector space)** If $f(x)$ and $g(x)$ have period p, show that $h = af + bg$ (a, b constant) has the period p. Thus all functions of period p form a vector space.
4. **(Integer multiples of period)** If p is a period of $f(x)$, show that np, $n = 2, 3, \cdots$, is a period of $f(x)$.

[3] And inserting parentheses; from a convergent series this gives again a convergent series with the same sum, as can be proved.

5. **(Constant)** Show that the function $f(x) = const$ is a periodic function of period p for every positive p.
6. **(Change of scale)** If $f(x)$ is a periodic function of x of period p, show that $f(ax)$, $a \neq 0$, is a periodic function of x of period p/a, and $f(x/b)$, $b \neq 0$, is a periodic function of x of period bp. Verify these results for $f(x) = \cos x$, $a = b = 2$.

Graphs of 2π-Periodic Functions

Sketch or plot the following functions $f(x)$, which are assumed to be periodic with period 2π and, for $-\pi < x < \pi$, are given by the formulas

7. $f(x) = x$
8. $f(x) = x^2$
9. $f(x) = |x|$
10. $f(x) = \pi - |x|$
11. $f(x) = |\sin x|$
12. $f(x) = e^{-|x|}$
13. $f(x) = \begin{cases} x & \text{if } -\pi \leqq x \leqq 0 \\ 0 & \text{if } 0 \leqq x \leqq \pi \end{cases}$
14. $f(x) = \begin{cases} 0 & \text{if } -\pi \leqq x \leqq 0 \\ x^2 & \text{if } 0 \leqq x \leqq \pi \end{cases}$
15. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$
16. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$
17. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ e^{-x} & \text{if } 0 < x < \pi \end{cases}$
18. $f(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0 \\ -x^2 & \text{if } 0 < x < \pi \end{cases}$
19. **CAS PROJECT. Plotting Periodic Functions. (a)** Write a program for plotting periodic functions $f(x)$ of period 2π given for $-\pi < x \leqq \pi$. Using your program, plot the functions in Probs. 7–12 for $-10\pi \leqq x \leqq 10\pi$. Also plot some functions of your own choice.
    (b) Extend your program to 2π-periodic functions given on two subintervals of the same length, as in Probs. 13–18. Apply your program to those problems with $-10\pi \leqq x \leqq 10\pi$.
20. **CAS PROJECT. Partial Sums of Trigonometric Series. (a)** Write a program that prints a partial sum[4] of a trigonometric series (4). Applying it, list all partial sums of up to five nonzero terms of each of the series

$$\frac{1}{3}\pi^2 - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right)$$

$$\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$

$$2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right).$$

(b) Plot the partial sums in (a) (for each series on common axes). Guess what periodic function the series might represent.

Page ①

10.2 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series (Sec. 10.1) whose coefficients are determined from $f(x)$ by the "Euler formulas" [(6), below], which we shall derive first. Afterwards we shall take a look at the theory of Fourier series.

[4]That is, $a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$ for $N = 1, 2, 3, \cdots$.

"Book Pages"

(8)

Similarly, $|b_n| < 2\,M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete. (Readers already familiar with uniform convergence will see that, by the Weierstrass test in Sec. 14.5, under our present assumptions the Fourier series converges uniformly, and our derivation of (6) by integrating term by term is then justified by Theorem 3 of Sec. 14.5.)

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (7) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C9]. ◀

EXAMPLE 2 Convergence at a jump as indicated in Theorem 1

The square wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k (Fig. 238). Hence the average of these limits is 0. The Fourier series (8) of the square wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 1. ◀

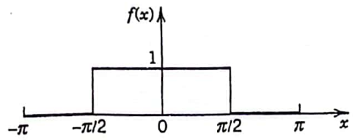
Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (7) with coefficients given by the Euler formulas (6). Theorem 1 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

PROBLEM SET 10.2

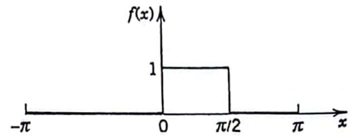
Fourier Series

Showing the details of your work, find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the first three partial sums, where $f(x)$ equals

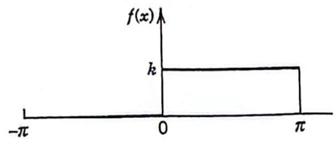
1.



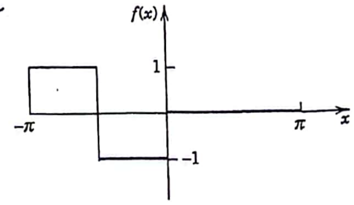
2.



3.



4.



5. $f(x) = x \quad (-\pi < x < \pi)$

6. $f(x) = x \quad (0 < x < 2\pi)$

7. $f(x) = x^2 \quad (-\pi < x < \pi)$

8. $f(x) = x^2 \quad (0 < x < 2\pi)$

9. $f(x) = x^3 \quad (-\pi < x < \pi)$

10. $f(x) = x + |x| \quad (-\pi < x < \pi)$

(4)

In a similar fashion we find from (2c) that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

PROBLEM SET 10.3

Fourier Series for Period $p = 2L$

Find the Fourier series of the periodic function $f(x)$, of period $p = 2L$, and sketch $f(x)$ and the fir three partial sums. (Show the details of your work.)

1. $f(x) = -1 \quad (-1 < x < 0), \quad f(x) = 1 \quad (0 < x < 1), \quad p = 2L = 2$
2. $f(x) = 1 \quad (-1 < x < 0), \quad f(x) = -1 \quad (0 < x < 1), \quad p = 2L = 2$
3. $f(x) = 0 \quad (-2 < x < 0), \quad f(x) = 2 \quad (0 < x < 2), \quad p = 2L = 4$
4. $f(x) = |x| \quad (-2 < x < 2), \quad p = 2L = 4$
5. $f(x) = 2x \quad (-1 < x < 1), \quad p = 2L = 2$
6. $f(x) = 1 - x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
7. $f(x) = 3x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
8. $f(x) = \frac{1}{2} + x \quad (-\frac{1}{2} < x < 0), \quad f(x) = \frac{1}{2} - x \quad (0 < x < \frac{1}{2}), \quad p = 2L = 1$
9. $f(x) = 0, \quad (-1 < x < 0), \quad f(x) = x \quad (0 < x < 1), \quad p = 2L = 2$
10. $f(x) = x \quad (0 < x < 1), \quad f(x) = 1 - x \quad (1 < x < 2), \quad p = 2L = 2$
11. $f(x) = \pi \sin \pi x \quad (0 < x < 1), \quad p = 2L = 1$
12. $f(x) = \pi x^3/2 \quad (-1 < x < 1), \quad p = 2L = 2$

13. **(Periodicity)** Show that each term in (1) has the period $p = 2L$.
14. **(Rectifier)** Find the Fourier series of the periodic function that is obtained by passing the voltag $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier.
15. **(Transformation)** Obtain the Fourier series in Prob. 1 from that in Example 1, Sec. 10.2.
16. **(Transformation)** Obtain the Fourier series in Prob. 7 from that in Prob. 7, Sec. 10.2.
17. **(Transformation)** Obtain the Fourier series in Prob. 3 from that in Example 1, Sec. 10.2.
18. **(Interval of Integration)** Show that in (2) the interval of integration may be replaced by an other interval of length $p = 2L$.
19. **CAS PROJECT. Fourier Series of $2L$-Periodic Functions.** (a) Write a program for obtainin any partial sum of a Fourier series (1).
    (b) Apply the program to Probs. 5–7, plotting the first few partial sums of each of the thre series on common axes. Choose the first five or more partial sums until they approximat the given function reasonably well.
20. **CAS PROJECT. Gibbs Phenomenon.** The partial sums $s_n(x)$ of a Fourier series show oscillations near a discontinuity point. These do not disappear as n increases but instead becom sharp "spikes." They were explained mathematically by J. W. Gibbs.[10] Plot $s_n(x)$ in Prob. 5 When $n = 20$, say, you will see those oscillations quite distinctly. Consider two other Fourie series of your choice in a similar way.

[10] JOSIAH WILLARD GIBBS (1839—1903), American mathematician, professor of mathematical physics at Yale from 1871, one of the founders of vector calculus [another being O. Heaviside (see Sec. 5.1)], mathematical thermodynamics, and statistical mechanics. His work was of great importance to the development of mathematical physics.

(3)

Euler Formulas for the Fourier series (If function $f(x)$ is periodic of period 2π). $\left[\begin{array}{l} -\pi < x < \pi \\ 0 < x < 2\pi \\ 2\pi < x < 4\pi \end{array}\right]$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

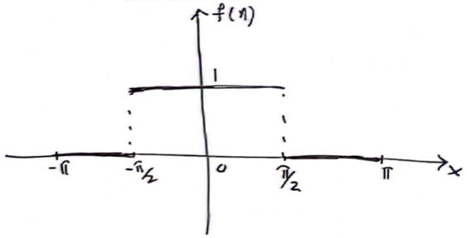
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Ex 10.2

Q 1



$\cancel{f(x) = x \quad 0 < x < 2\pi}$

Sol:

$$f(x) = \begin{cases} 0 & -\pi < x < -\pi/2 \\ 1 & -\pi/2 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$

The required fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Sol.

$$f(x)=\begin{cases}0 & -\pi<x<-\pi/2\\ 1 & -\pi/2<x<\pi/2\\ 0 & \pi/2<x<\pi\end{cases}$$

The required fourier series is

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \quad \text{—— (i)}$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_0=\frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2}0\,dx+\int_{-\pi/2}^{\pi/2}1\,dx+\int_{\pi/2}^{\pi}0\,dx\right] \quad \text{(ii)}$$

$$=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}dx=\frac{1}{2\pi}\Big|x\Big|_{-\pi/2}^{\pi/2}=\frac{1}{2\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$$=\frac{1}{2\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=\frac{1}{2\pi}\pi=\frac{1}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx.$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}0\cos nx\,dx+\int_{-\pi/2}^{\pi/2}1\cdot\cos nx\,dx+\int_{\pi/2}^{\pi}0\cos nx\,dx\right]$$

$$a_n=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left|\frac{\sin nx}{n}\right|_{-\pi/2}^{\pi/2}$$

$$a_n=\frac{1}{n\pi}\left[\sin\frac{n\pi}{2}-\sin n\left(-\frac{\pi}{2}\right)\right]$$

$$=\frac{1}{n\pi}\left[2\sin\frac{n\pi}{2}\right]=\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n=0$$

in (i) =>

$$\boxed{f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx}$$

Ex 10.2

Q5 $f(x) = x \quad (-\pi < x < \pi)$

Q7 $f(x) = x^2 \quad (-\pi < x < \pi)$

Sol: $f(x) = x^2 \quad (-\pi < x < \pi)$

The required fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi} \left| \frac{x^3}{3} \right|_{-\pi}^{\pi} = \frac{1}{6\pi} \left[\pi^3 - (-\pi)^3 \right]$$

$$= \frac{1}{6\pi} \left[\pi^3 + \pi^3 \right] = \frac{2\pi^3}{6\pi} = \boxed{\frac{\pi^2}{3}}$$

$$\boxed{a_0 = \pi^2/3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

x^2	$+$	$\cos nx$
$2x$	$-$	$\frac{\sin nx}{n}$
2	$+$	$-\frac{\cos nx}{n^2}$
0		$-\frac{\sin nx}{n^3}$
		$-\frac{\cos nx}{n^4}$

$$= \frac{1}{\pi} \left[x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(\pi^2 \frac{\cancelto{0}{\sin n\pi}}{n} + \frac{2\pi \cos n\pi}{n^2} - 2 \frac{\cancelto{0}{\sin n\pi}}{n^3} \right) - \left(\pi^2 \frac{\cancelto{0}{\sin n(-\pi)}}{n} - 2\pi \frac{\cos n(-\pi)}{n^2} - 2 \frac{\cancelto{0}{\sin n(-\pi)}}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{2\pi \cos n\pi}{n^2} + \frac{2\pi \cos n\pi}{n^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi \cos n\pi}{n^2} \right.$$

$$= \frac{4 \cos n\pi}{n^2} = \frac{4}{n^2} (-1)^n$$

$b_n =$ ______