DE Final Lecture 2 (continue)

Problems 20 - 40 Ex 5.1

20. $\frac{s-4}{s^2-4}$

$$\mathcal{L}^{-1}\left\{\frac{s-4}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s^2-4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2-(2)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2-(2)^2}\right\}$$

$$= \cosh 2t - 2\sinh 2t$$

21. $\frac{2.4}{s^4} - \frac{228}{s^6}$

$$\mathcal{L}^{-1}\left\{\frac{2.4}{s^4} - \frac{228}{s^6}\right\} = \mathcal{L}^{-1}\left\{\frac{2.4}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{228}{s^6}\right\}$$

$$= 2.4\,\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} - 228\,\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\}$$

$$= \frac{2.4}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} - \frac{228}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\}$$

$$= \frac{2.4}{6}\,t^3 - \frac{228}{120}\,t^5$$

$$= \frac{2}{5}t^3 - \frac{19}{10}t^5$$

22. $\dfrac{60+6s^2+s^4}{s^7}$

$$\mathcal{L}^{-1}\left\{\frac{60+6s^2+s^4}{s^7}\right\} = \mathcal{L}^{-1}\left\{\frac{60}{s^7}\right\} + \mathcal{L}^{-1}\left\{\frac{6s^2}{s^7}\right\} + \mathcal{L}^{-1}\left\{\frac{s^4}{s^7}\right\}$$

$$= 60\mathcal{L}^{-1}\left\{\frac{1}{s^7}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{60}{6!}\mathcal{L}^{-1}\left\{\frac{6!}{s^7}\right\} + \frac{6}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} + \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$$

$$= \frac{60}{720}t^6 + \frac{6}{24}t^4 + \frac{1}{2}t^2$$

$$= \frac{1}{12}t^6 + \frac{1}{4}t^4 + \frac{1}{2}t^2$$

23. $\dfrac{1\cdot s}{L^2s^2+n^2\pi^2}$

$$\mathcal{L}^{-1}\left\{\frac{s}{L^2s^2+n^2\pi^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{L^2\left(s^2+\frac{n^2\pi^2}{L^2}\right)}\right\}$$

$$= \frac{1}{L^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\frac{n\pi}{L}\right)^2}\right\}$$

$$= \frac{1}{L^2}\cos\frac{n\pi}{L}t.$$

24. $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$

Take $\dfrac{1-7s}{(s-3)(s-1)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s-1} + \dfrac{C}{s+2}$ — (1)

$1-7s = A(s-1)(s+2) + B(s-3)(s+2) + C(s-3)(s-1)$ (2)

put $s=3$ in (2)

$1-7(3) = A(3-1)(3+2)$

$-20 = A(10) \Rightarrow A = -2$

put $s=1$ in (2)

$1-7(1) = B(1-3)(1+2)$

$-6 = B(-6) \Rightarrow B = 1$

put in (2) $s=-2$.

$1-7(-2) = C(-2-3)(-2-1)$

$15 = C(15) \Rightarrow C = 1$

put in (1)

$$\frac{1-7s}{(s-3)(s-1)(s+2)} = \frac{-2}{s-3} + \frac{1}{s-1} + \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{1-7s}{(s-3)(s-1)(s+2)}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -2e^{3t} + e^{t} + e^{-2t}$$

25. $\sum_{k=1}^{5} \frac{a_k}{s+k^2}$

$$\mathcal{L}^{-1}\left[\sum_{k=1}^{5} \frac{a_k}{s+k^2}\right] = \sum_{k=1}^{5} \mathcal{L}^{-1}\left[\frac{a_k}{s+k^2}\right]$$

$$= \sum_{k=1}^{5} a_k \mathcal{L}^{-1}\left[\frac{1}{s+k^2}\right]$$

$$= \sum_{k=1}^{5} a_k e^{-k^2 t}$$

26. $\frac{s^4+6s-18}{s^5+3s^4}$

$$\mathcal{L}^{-1}\left[\frac{s^4+6s-18}{s^5-3s^4}\right] = ?$$

Take

$$\frac{s^4+6s-18}{s^4(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s-3} \quad -(1)$$

$$s^4+6s-18 = A(s-3)s^3 + Bs^2(s-3) + C\,s(s-3) + D(s-3) + E(s^4) \quad -(2)$$

put $s = +3$

$(+3)^4 + 6(+3) - 18 = E(+3)^4$

$81 + 18 - 18 = E(81)$

$E = 1$

put $s = 0$ in (2)

$(0)^4 + 6(0) - 18 = D(0-3)$

$D = 6$

by eq (2)

$$s^4+6s-18 = A(s^4-3s^3) + B(s^3-3s^2) + C(s^2-3s) + D(s-3) + E(s^4)$$

$$s^4+6s-18 = (A+E)s^4 + Bs^3 + (-3B+C)s^2 + (D-3C)s + (-3D)$$

By comparing coefficients of s^4, s^3, s^2, s and absolute terms on b.s.

$A + E = 1$ — (a)

$B = 0$ — (b)

$-3B + C = 0$ — (c)

$D - 3C = 6$ — (d)

$-3D = -18$ — (e)

put $E = 1$ in (a)

$A + 1 = 1$

$A = 0$

put $B = 0$ in (c)

$C = 0$

put in (1)

$$\frac{s^4+6s-18}{s^5-3s^4} = \frac{6}{s^4} + \frac{1}{s-3}$$

$$\mathcal{L}^{-1}\left[\frac{s^4+6s-18}{s^5-3s^4}\right] = 6\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{6}{3!}\mathcal{L}^{-1}\left[\frac{3!}{s^4}\right] - e^{3t}$$

$$= t^3 - e^{3t}$$

27. $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$

Take

$$\frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{A}{s+\sqrt{2}} + \frac{B}{s-\sqrt{3}} \quad -(1)$$

$1 = A(s-\sqrt{3}) + B(s+\sqrt{2}) \quad -(2)$

put $s=\sqrt{3}$ in (2)

$1 = B(\sqrt{3}+\sqrt{2})$

$B = \frac{1}{\sqrt{2}+\sqrt{3}}$

put $s=-\sqrt{2}$ in (2)

$1 = A(-\sqrt{2}-\sqrt{3})$

$A = \frac{-1}{\sqrt{2}+\sqrt{3}}$

put in (1)

$$\frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{-1}{\sqrt{2}+\sqrt{3}} \cdot \frac{1}{s+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} \cdot \frac{1}{s-\sqrt{3}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}\right\} = \frac{-1}{\sqrt{2}+\sqrt{3}} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s+\sqrt{2}}\right\} + \frac{1}{\sqrt{2}+\sqrt{3}} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-\sqrt{3}}\right\}$$

$$= \frac{-e^{-\sqrt{2}t}}{\sqrt{2}+\sqrt{3}} + \frac{e^{\sqrt{3}t}}{\sqrt{2}+\sqrt{3}}$$

28. $\frac{2s^3}{s^4-1}$

$$\frac{2s^3}{s^4-1} = \frac{1}{(s^2-1)(s^2+1)} = \frac{1}{(s-1)(s+1)(s^2+1)}$$

$$\frac{2s^3}{s^4-1} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1} \quad -(1)$$

$2s^3 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1) \quad -(2)$

put $s=1$ in (2)

$2(1)^3 = A(1+1)(1^2+1)$

$A = \frac{1}{2}$

×

28. $\frac{2s^3}{s^4-1}$

take

$$\frac{2s^3}{s^4-1} = \frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1} \quad -(1)$$

$$2s^3 = (As+B)(s^2+1) + (Cs+D)(s^2-1)$$

$$2s^3 = (A+C)s^3 + (B+D)s^2 + (A-C)s + (B-D) \quad -(2)$$

Comparing coefficient of s^3, s^2, s and constant terms on b.s.

$A+C=2 \quad -(a)$

$B+D=0 \quad -(b)$

$A-C=0 \quad -(c)$

$B-D=0 \quad -(d)$

adding (a) and (c)

$2A=2$

$A=1$

put in (a)

$C=1$

adding (b) and (d)

$2B=0$

$B=0$

put in (d)

$D=0$

put in (1)

$$\frac{2s^3}{s^4-1} = \frac{s}{s^2-1} + \frac{s}{s^2+1}$$

Now

$$\mathcal{L}^{-1}\left\{\frac{2s^3}{s^4-1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \cosh t + \cos t$$

Application of First shifting Theorem

29. $t^2 e^{-3t}$

$$\mathcal{L}\{t^2 e^{-3t}\} = \mathcal{L}\{t^2\}_{s\to s+3}$$

$$= \left.\frac{2!}{s^3}\right|_{s\to s+3}$$

$$= \frac{2}{(s+3)^3}$$

30. $e^{-\alpha t}\cos\beta t$

$$\mathcal{L}\{e^{-at}\cos\beta t\} = \mathcal{L}\{\cos\beta t\}_{s\to s+\alpha}$$

$$= \left.\frac{s}{s^2+\beta^2}\right|_{s\to s+\alpha}$$

$$= \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

31. $5e^{2t}\sinh 2t$

$$\mathcal{L}\{5e^{2t}\sinh 2t\} = 5\mathcal{L}\{\sinh 2t\}_{s\to s-2}$$

$$= 5 \cdot \frac{2}{s^2 - (2)^2}\Big|_{s\to s-2}$$

$$= \frac{10}{(s-2)^2 - 4}$$

32. $2e^{-t}\cos^2 \frac{1}{2}t$

$$2e^{-t}\cos^2\frac{1}{2}t = 2e^{-t}\left(\frac{1+\cos t}{2}\right)$$

$$= e^{-t} + e^{-t}\cos t$$

$$\mathcal{L}\left\{2e^{-t}\cos^2\frac{1}{2}t\right\} = \mathcal{L}\{e^{-t}\} + \mathcal{L}\{\cos t\}_{s\to s+1}$$

$$= \frac{1}{s+1} + \frac{s}{s^2+1^2}\Big|_{s\to s+1}$$

$$= \frac{1}{s+1} + \frac{s+1}{(s+1)^2+1}$$

33. $\sinh t \cos t$

$$\sinh t\cos t = \left(\frac{e^t - e^{-t}}{2}\right)\cos t$$

$$= \frac{1}{2}e^t\cos t - \frac{1}{2}e^{-t}\cos t$$

$$\mathcal{L}\{\sinh t\cos t\} = \frac{1}{2}\mathcal{L}\{\cos t\}_{s\to s-1} - \frac{1}{2}\mathcal{L}\{\cos t\}_{s\to s+1}$$

$$= \frac{1}{2}\cdot\frac{s}{s^2+1^2}\Big|_{s\to s-1} - \frac{1}{2}\cdot\frac{s}{s^2+1}\Big|_{s\to s+1}$$

$$= \frac{s-1}{2[(s-1)^2+1]} - \frac{s+1}{2[(s+1)^2+1]}$$

34. $(t+1)^2 e^t$

$$(t+1)^2e^t = (t^2+2t+1)e^t$$

$$= t^2e^t + 2te^t + e^t$$

$$\mathcal{L}\{(t+1)^2e^t\} = \mathcal{L}\{t^2\}_{s\to s-1} + 2\mathcal{L}\{t\}_{s\to s-1} + \mathcal{L}\{e^t\}$$

$$= \frac{2!}{s^3}\Big|_{s\to s-1} + \frac{2\cdot 1!}{s^2}\Big|_{s\to s-1} + \frac{1}{s-1}$$

$$= \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)}$$

Inverse Transform.

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

35. $\dfrac{1}{(s+1)^2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-t}.t$$

36. $\dfrac{12}{(s-3)^4}$

$$\mathcal{L}^{-1}\left\{\frac{12}{(s-3)^4}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\frac{12}{s^4}\right\}$$

$$= 2e^{3t}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}$$

$$= 2e^{3t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= 2e^{3t}\,t^3$$

37. $\dfrac{3}{s^2+6s+18}$

$$\frac{3}{s^2+6s+18} = \frac{3}{(s^2+2(s)(3)+3^2)+9}$$

$$= \frac{3}{(s+3)^2+3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+6s+18}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{(s+3)^2+3^2}\right\}$$

$$= e^{-3t}.\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= e^{-3t}.\sin 3t$$

38. $\dfrac{4}{s^2-2s-3}$

$$\frac{4}{s^2-2s-3} = \frac{4}{(s^2-2s+1)-1-3}$$

$$= \frac{4}{(s-1)^2-(2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{(s-1)^2-(2)^2}\right\}$$

$$= 2e^{t}\mathcal{L}^{-1}\left\{\frac{2}{s^2-(2)^2}\right\}$$

$$= 2e^{t}\sinh 2t$$

39. $\dfrac{s}{\left(s+\frac{1}{2}\right)^2+1}$

$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s+\frac{1}{2}\right)^2+1}\right\} = e^{-1/2t}\,\mathcal{L}^{-1}\left[\quad\right]$$

$$\frac{s}{(s+1/2)^2+1} = \frac{s+1/2-1/2}{(s+1/2)^2+1}$$

$$= \frac{s+1/2}{(s+1/2)^2+1^2} - \frac{1/2}{(s+1/2)^2+1^2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1/2)^2+1}\right] = \mathcal{L}^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+1^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1/2)^2+1^2}\right\}$$

$$= e^{-1/2t}\,\mathcal{L}^{-1}\left[\frac{s}{s^2+1^2}\right] - \frac{1}{2}e^{-1/2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$= e^{-1/2t}\cos t - \frac{1}{2}e^{-1/2t}\sin t$$

40. $\dfrac{2}{s^2+s+1/2}$

$$\frac{2}{s^2+s+1/2} = \frac{2}{s^2+2s\frac{1}{2}+\left(\frac{1}{2}\right)^2+\frac{1}{4}}$$

$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

$$= \frac{2}{(s+1/2)^2+1/4}$$

$$\mathcal{L}^{-1}\left[\frac{2}{s^2+s+1/2}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{(s+1/2)^2+(1/2)^2}\right]$$

$$= 2e^{-\frac{1}{2}t}\,\mathcal{L}^{-1}\left[\frac{1}{s^2+(1/2)^2}\right]$$

$$= 2e^{-1/2t}\cdot 2\,\mathcal{L}^{-1}\left[\frac{1/2}{s^2+(1/2)^2}\right]$$

$$= 4e^{-1/2t}\cdot\sin\frac{1}{2}t$$