1

Unit step Function;

Second shifting theorem;

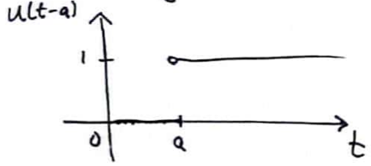
Dirac's Delta function;

Unit step function

By definition, $u(t-a)$ is "0" for $t<a$, has a jump of size 1 at $t=a$ and is "1" for $t>a$.

The function $u(t-a)$ is defined as

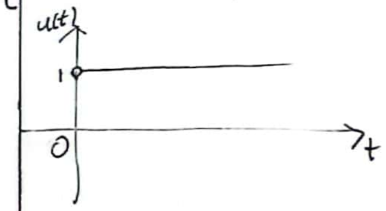
$$u(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t>a\end{cases}$$



$u(t-a)$.

when

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$



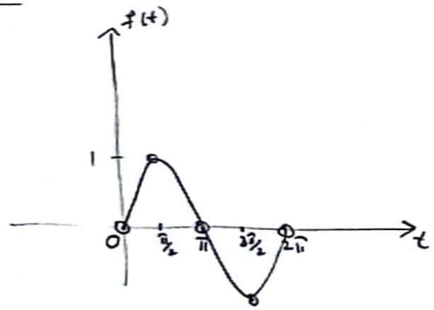
2

Unit step Function;

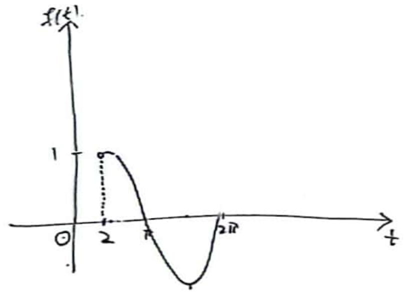
Second shifting theorem;

Dirac's Delta function;

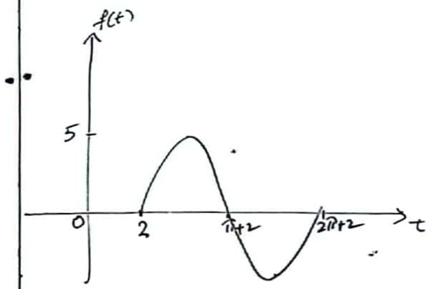
$f(t) = \sin t$



$f(t)\, u(t-2) = \sin t \; u(t-2).$



$f(t-2)\, u(t-2) = 5 \sin(t-2)\; u(t-2)$



3

Unit step Function;
Second shifting theorem;
Dirac's Delta function;

Second Shifting theorem (t-shifting theorem)

If $f(t)$ has the transform $F(s)$, then the shifted function

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

has the transform $e^{-as}F(s)$.

That is

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).$$

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\overbrace{u(t-a)}^{0}\,dt + \int_a^\infty e^{-st} f(t-a)\overbrace{u(t-a)}^{1}\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$\int_0 t\,dt \qquad \int_0^1 u\,du$

$t-a = u$ when $t=a \Rightarrow u=0$

$dt = du$ when $t=\infty \Rightarrow u=\infty$

$$= \int_0^\infty e^{-s(u+a)} \cdot f(u)\,du.$$

$$= \int_0^\infty e^{-su} \cdot e^{-sa} f(u)\,du$$

$$= e^{-as} \cdot \int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-as}\mathcal{L}\{f\} = e^{-as}F(s).$$

4

$$\boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\cdot F(s)}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\ u(t-a)}$$

<u>Ex</u> (use of step function)

Find the Laplace transform of the function

$$f(t) = \begin{cases} 2, & \text{if } 0<t<\pi \\ 0, & \text{if } \pi<t<2\pi \\ \sin t, & \text{if } t>2\pi \end{cases}$$

<u>Sol</u>:

we write $f(t)$ in terms of unit step functions. For $0<t<\pi$. we take $2u(t)$. For $t>\pi$ we want "zero". So we must subtract the step function $2u(t-\pi)$ with step at π. we have $2u(t)-2u(t-\pi)=0$ when $t>\pi$. This is fine untill we reach 2π, where we want $\sin t$ to come in. So we add

$\sin t\ u(t-2\pi)$

Together.

$$f(t) = 2u(t) - 2u(t-\pi) + \sin t\ u(t-2\pi).$$

$$f(t) = 2\left[u(t) - u(t-\pi)\right] + 0\left[u(t-\pi) - u(t-2\pi)\right]$$

$$+ \sin t\ u(t-2\pi)$$

$$= 2u(t) - 2u(t-\pi) + \sin t\ u(t-2\pi).$$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{u(t)\} - 2\mathcal{L}\{u(t-\pi)\} + \mathcal{L}\{\sin t \cdot u(t-2\pi)\}$$

$$= 2e^{-0s}\cdot\frac{1}{s} - 2e^{-\pi s}\frac{1}{s} + \mathcal{L}\{\sin(t+2\pi-2\pi)\,u(t-2\pi)\}$$

$$= \frac{2}{s} - 2\frac{e^{-\pi s}}{s} + e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$$

$$= 2/s - 2\frac{e^{-\pi s}}{s} + e^{-2\pi s}\mathcal{L}\{\sin t\}$$

$$= \frac{2}{s} - 2\frac{e^{-\pi s}}{s} + e^{-2\pi s}\frac{1}{s^2+1}.$$

$$\boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} \cdot F(s)}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\ u(t-a)}$$

Ex2 Find inverse Laplace transform $f(t)$ of

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{4e^{-\pi s}}{s^2+1}$$

$\mathcal{L}\{f(t)\}$

Sol:

$$f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s^2}\right\} - 4\mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{e^{-\pi s} \cdot \frac{1}{s^2+1}\right\}$$

$$f(t) = 2t - 2(t-2)u(t-2) - 4 \cdot 1 \cdot u(t-2) + 4\sin(t-\pi) \cdot u(t-\pi).$$

$\sin t\cos\pi - \cos t \sin\pi$

$$= 2t - 2t\,u(t-2) + \cancel{4\,u(t-2)} - \cancel{4\,u(t-2)} - 4\sin t\ u(t-\pi)$$

$$= 2t - 2t\,u(t-2) - 4\sin t\ u(t-\pi).$$

$$= 2t\,u(t) - 2t\,u(t-2) - 4\sin t \cdot u(t-\pi)$$

$$= \begin{cases} 2t & 0 < t < 2 \\ 2t - 2t & 2 < t < \pi \\ 2t - 2t - 4\sin t & t > \pi \end{cases}$$

$$= \begin{cases} 2t & 0 < t < 2 \\ 0 & 2 < t < \pi \\ -4\sin t & t > \pi \end{cases}$$

EX5.3

Q1 — 25

2) $t\,u(t-1)$

Sol: $\mathcal{L}\{t\,u(t-1)\}$

$= \mathcal{L}\{(t+1-1)\,u(t-1)\}$

$= e^{-1s}\,\mathcal{L}\{t+1\}$

$= e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right]$

3) $t^2 \quad (0<t<1)$

Sol: $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\cdot t^2\,dt$

$= \int_0^1 e^{-st}\,t^2\,dt + \int_1^\infty e^{-st}\cdot 0\,dt$

$= \int_0^1 e^{-st}\cdot t^2\,dt.$

Sol: $f(t) = t^2 \quad (0<t<1)$

$f(t) = t^2\,[u(t)-u(t-1)]$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u(t)\} - \mathcal{L}\{t^2 u(t-1)\}$

$= e^{-0s}\cdot\mathcal{L}\{t^2\} - \mathcal{L}\{(t+1-1)^2\,u(t-1)\}$

$= \frac{2!}{s^3} - e^{-1s}\,\mathcal{L}\{(t+1)^2\}$

$= \frac{2!}{s^3} - e^{-s}\,\mathcal{L}\{t^2+1+2t\}$

$= \frac{2!}{s^3} - e^{-s}\cdot\frac{2!}{s^3} - e^{-s}\cdot\frac{1}{s} - 2e^{-s}\frac{1}{s^2}.$

Ex # 5.3

2. $t\,u(t-1)$

$$\mathcal{L}\{t.u(t-1)\}$$

$$= \mathcal{L}\{(t+1-1)\,u(t-1)\}$$

$$= e^{-1.s}\,.\,\mathcal{L}\{t+1\}$$

$$= e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right]$$

3. $(t-1)u(t-1)$

$$\mathcal{L}\{(t-1)\,u(t-1)\}$$

$$= \cancel{\mathcal{L}}\; e^{-1.s}\,\mathcal{L}\{t\}$$

$$= \frac{e^{-s}}{s^2}$$

4. $(t-1)^2 u(t-1)$

$$\mathcal{L}\{(t-1)^2 u(t-1)\}$$

$$= e^{-s}\mathcal{L}\{t^2\}$$

$$= \frac{2e^{-s}}{s^3}$$

5. $t^2 u(t-1)$

$$\mathcal{L}\{t^2 u(t-1)\}$$

$$= \mathcal{L}\{(t^2-2t+1+2t-1)u(t-1)\}$$

$$= \mathcal{L}\{((t-1)^2+(2t-1))u(t-1)\}$$

$$= \mathcal{L}\{(t-1)^2 u(t-1)\} + \mathcal{L}\{(2t-1)u(t-1)\}$$

$$= e^{-s}\mathcal{L}\{t^2\} + 2\mathcal{L}\{t\,u(t-1)\} - \mathcal{L}\{u(t-1)\}$$

$$= \frac{2e^{-s}}{s^3} + 2\mathcal{L}\{(t+1-1)u(t-1)\} - e^{-s}\mathcal{L}\{1\}$$

$$= \frac{2e^{-s}}{s^3} + 2e^{-s}\mathcal{L}\{t+1\} - \frac{e^{-s}}{s}$$

$$= \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s} - \frac{e^{-s}}{s}$$

$$= \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

6. $e^{-2t} u(t-3)$

$$\mathcal{L}\{e^{-2t} u(t-3)\}$$

$$= \mathcal{L}\{e^{-2t+6-6} u(t-3)\}$$

$$= e^{-6}\mathcal{L}\{e^{-2(t-3)} u(t-3)\}$$

$$= e^{-6}\cdot e^{-3s}\cdot\mathcal{L}\{e^{-2t}\}$$

$$= \frac{e^{-6-3s}}{s+2}$$

7. $4u(t-\pi)\cos(t)$

$\mathcal{L}\{4u(t-\pi)\cos(t)\}$

$= 4\mathcal{L}\{\cos(t-\pi+\pi)\,u(t-\pi)\}$

$= -4\mathcal{L}\{\cos(t-\pi)u(t-\pi)\}$

$= -4e^{-\pi s}\mathcal{L}\{\cos t\}$

$= \dfrac{-4e^{-\pi s}.s}{s^2+1}$

8. $t^2 \quad 0<t<1$

$\mathcal{L}\{t^2\}$

$= \int_0^\infty e^{-st}.t^2\,dt$

$= \int_0^1 e^{-st}t^2dt + \int_1^\infty e^{-st}\,0\,dt$

$= \int_0^1 e^{-st}.t^2\,dt$

We will solve it using unit step functio
Solution on next page.

Sol: $f(t) = t^2 \quad 0<t<1$

$f(t) = t^2[u(t) - u(t-1)]$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2u(t)\} - \mathcal{L}\{t^2u(t-1)\}$

$= e^{-0.s}\mathcal{L}\{t^2\} - \mathcal{L}\{(t+1-1)^2u(t-1)\}$

$= \dfrac{2}{s^3} - e^{-1.s}\mathcal{L}\{(t+1)^2\}$

$= \dfrac{2}{s^3} - e^{-s}.\mathcal{L}\{t^2+2t+1\}$

$= \dfrac{2}{s^3} - e^{-s}\left[\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}\right]$

9. $\sin\omega t \quad (0<t<\pi/\omega)$

Sol:- $f(t) = \sin\omega t \quad (0<t<\pi/\omega)$

$f(t) = \sin\omega t\, u(t) - \sin\omega t\, u(t-\frac{\pi}{\omega})$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin\omega t\, u(t)\} - \mathcal{L}\{\sin\omega t\, u(t-\pi/\omega)\}$

$= e^{-0s}\mathcal{L}\left[\sin\omega t\right] - \mathcal{L}\left[\sin(\omega t-\pi+\pi)\, u(t-\pi/\omega)\right]$

$= \frac{\omega}{s^2+\omega^2} + \mathcal{L}\{\sin(\omega(t-\pi/\omega))\, u(t-\pi/\omega)\}$

$= \frac{\omega}{s^2+\omega^2} + e^{-\pi/\omega \cdot s}\mathcal{L}\{\sin\omega t\}$

$= \frac{\omega}{s^2+\omega^2} + \frac{e^{-\frac{\pi s}{\omega}}\cdot\omega}{s^2+\omega^2}$

10. $1-e^{-t} \quad 0<t<2$

$f(t) = 1-e^{-t} \quad 0<t<2$

$= (1-e^{-t})u(t) - (1-e^{-t})u(t-2)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(1-e^{-t})u(t)\} - \mathcal{L}\{(1-e^{-t})u(t-2)\}$

$= e^{-0.s}\mathcal{L}\left[1-e^{-t}\right] - \mathcal{L}\left[(1-e^{-t+2-2})u(t-2)\right]$

$= \frac{1}{s} - \frac{1}{s+1} - \mathcal{L}\left[(1-e^{-(t-2)-2})u(t-2)\right]$

$= \frac{1}{s} - \frac{1}{s+1} - e^{-2s}\mathcal{L}\left[1-e^{-t-2}\right]$

$= \frac{1}{s} - \frac{1}{s+1} - e^{-2s}\left[\frac{1}{s} - \frac{e^{-2}}{s+1}\right]$

11. $e^t \quad (0<t<1)$

$f(t) = e^t \quad (0<t<1)$

$= e^t u(t) - e^t u(t-1)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^t u(t)\} - \mathcal{L}\{e^{+t}\cdot u(t-1)\}$

$= e^{-0s}\cdot\mathcal{L}\{e^t\} - \mathcal{L}\{e^{+t+1-1}u(t-1)\}$

$= \frac{1}{s-1} - \mathcal{L}\{e^{+(t-1)+1}u(t-1)\}$

$= \frac{1}{s-1} - e^{-1.s}\mathcal{L}\{e^{+t+1}\}$

$= \frac{1}{s-1} - e^{-s}\left[\frac{e^{+1}}{s-1}\right]$

12. sin t $(2\pi < t < 4\pi)$

$f(t) = \sin t \quad (2\pi < t < 4\pi)$

~~f(t) = sin t u(t) + sin~~

$f(t) = \sin t\, u(t-2\pi) - \sin t\, u(t-4\pi)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t \cdot u(t-2\pi)\} - \mathcal{L}\{\sin t \cdot u(t-4\pi)\}$

$= \mathcal{L}\{\sin(t-2\pi+2\pi)u(t-2\pi)\} - \mathcal{L}\{\sin(t-4\pi+4\pi)u(t-4\pi)\}$

$= \mathcal{L}\{\sin(t-2\pi)u(t-2\pi)\} - \mathcal{L}\{\sin(t-4\pi)u(t-4\pi)\}$

$= e^{-2\pi s} \cdot \mathcal{L}\{\sin t\} - e^{-4\pi s} \cdot \mathcal{L}\{\sin t\}$

$= \dfrac{e^{-2\pi s}}{s^2+1} - \dfrac{e^{-4\pi s}}{s^2+1}$

13. 10 cos πt $(1 < t < 2)$

$f(t) = 10\cos\pi t \quad (1 < t < 2)$

$= 10\cos\pi t \cdot u(t-1) - 10\cos\pi t \cdot u(t-2)$

$\mathcal{L}\{f(t)\} = 10\mathcal{L}\{\cos\pi t \cdot u(t-1)\} - 10\mathcal{L}\{\cos\pi t\, u(t-2)\}$

$= 10\mathcal{L}\{\cos(\pi t+\pi-\pi)u(t-1)\} - 10\mathcal{L}\{\cos(\pi t+2\pi-2\pi)u(t-2)\}$

$= 10\mathcal{L}\left\{\cos(\pi(t-1)+\pi)u(t-1)\right\} - 10\mathcal{L}\{\cos(\pi(t-2)+2\pi)u(t-2)\}$

$= -10\mathcal{L}\{\cos(\pi(t-1))u(t-1)\} - 10\mathcal{L}\{\cos(\pi(t-2))u(t-2)\}$

$= -10e^{-1\cdot s} \cdot \mathcal{L}\{\cos\pi t\} - 10e^{-2s}\mathcal{L}\{\cos\pi t\}$

$= \dfrac{-10e^{-s} \cdot s}{s^2+\pi^2} - \dfrac{10e^{-2s} \cdot s}{s^2+\pi^2}$

Inverse transform

14. $4(e^{-2s} - 2e^{-5s})/s$

$F(s) = \dfrac{4e^{-2s}}{s} - \dfrac{8e^{-5s}}{s}$

$f(t) = 4\mathcal{L}^{-1}\left\{e^{-2s} \cdot \dfrac{1}{s}\right\} - 8\mathcal{L}\left\{e^{-5s} \cdot \dfrac{1}{s}\right\}$

$= 4.1.\, u(t-2) - 8.1.\, u(t-5)$

$= 4u(t-2) - 8u(t-5)$

$$f(t) = \begin{cases} 4 & 2 < t < 5 \\ 4-8 & t > 5 \end{cases}$$

15. $\dfrac{e^{-3s}}{s^3}$

$$F(s) = \frac{e^{-3s}}{s^3}$$

$$f(t) = \mathcal{L}^{-1}\left\{ e^{-3s} \cdot \frac{1}{s^3} \right\}$$

$$= \frac{1}{2!}\mathcal{L}^{-1}\left[e^{-3s} \cdot \frac{2!}{s^3} \right]$$

$$f(t) = \frac{1}{2}(t-3)^2 u(t-3)$$

$$f(t) = \begin{cases} 0 & t < 3 \\ \dfrac{(t-3)^2}{2} & t > 3 \end{cases}$$

16. $\cancel{\dfrac{e^{-3s}}{(s-1)^3}}$

$$\cancel{F(s) = \frac{e^{-3s}}{s^3 - 3s^2 + 3s - 1}}$$

$$\cancel{= \frac{e^{-3s}}{s^3} - \frac{e^{-3s}}{3s^2}}$$

16. $\dfrac{e^{-3s}}{(s-1)^3}$

$$F(s) = \frac{e^{-3s}}{(s-1)^3}$$

$$f(t) = \mathcal{L}\left\{ \frac{e^{-3s}}{(s-1)^3} \right\}$$

$$= \mathcal{L}\left\{ \frac{e^{-3s+3-3}}{(s-1)^3} \right\}$$

$$= \mathcal{L}\left\{ \frac{e^{-3(s-1)-3}}{(s-1)^3} \right\}$$

$$= e^{t}\mathcal{L}\left\{ \frac{e^{-3s-3}}{s^3} \right\}$$

$$= e^{t} \cdot e^{-3} \cdot \frac{1}{2!}\mathcal{L}^{-1}\left[e^{-3s} \cdot \frac{2!}{s^3} \right]$$

$$= \frac{1}{2} e^{t-3} \cdot (t-3)^2 u(t-3)$$

17. $\dfrac{3(1-e^{-\pi s})}{s^2+9}$

$$F(s) = \frac{3}{s^2+9} - \frac{3e^{-\pi s}}{s^2+9}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s} \cdot \frac{3}{s^2+9}\right\}$$

$$= \sin 3t - \sin 3(t-\pi)\,u(t-\pi)$$

$$= \sin 3t - \sin(3t-3\pi)\,u(t-\pi)$$

$$= \sin 3t + \sin 3t\,u(t-\pi)$$

18. $e^{-2\pi s}/(s^2+2s+2)$

$$F(s) = \frac{e^{-2\pi s}}{s^2+2s+\cancel{1}+1}$$

$$= \frac{e^{-2\pi s}}{(s+1)^2+1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s+2\pi-2\pi}}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-2\pi(s+1)+2\pi}}{(s+1)^2+1}\right\}$$

$$= e^{-t}\,\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s+2\pi}}{s^2+1}\right\}$$

$$= e^{-t}\cdot e^{2\pi}\,\mathcal{L}^{-1}\left\{e^{-2\pi s}\cdot\frac{1}{s^2+1}\right\}$$

$$= e^{2\pi-t}\cdot\sin(t-2\pi)\cdot u(t-2\pi)$$

$$= e^{2\pi-t}\cdot\sin t\,u(t-2\pi)$$

19. $\dfrac{se^{-2s}}{s^2+\pi^2}$

$$F(s) = \frac{e^{-2s}\cdot s}{s^2+\pi^2}$$

$$f(t) = \mathcal{L}^{-1}\left\{e^{-2s}\,\frac{s}{s^2+\pi^2}\right\}$$

$$= \cos\pi(t-2)\,.\,u(t-2)$$

$$= \cos(\pi t-2\pi)\,u(t-2)$$

$$= \cos\pi t\,.\,u(t-2).$$