DE Final Lecture 3

Laplace Transform of Derivative and Integral :

Differential Equations

Theorem 1: [Laplace Transform of derivative of $f(t)$]

Suppose that $f(t)$ is continous for all $t \geq 0$ and has a derivative that exists $f'(t)$ that is piecewise continous for every finite interval of in the range $t \geq 0$. Suppose $f(t)$ is of exponential order for $t > T$. Then laplace transform of $f'(t)$ exists when $s > k$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad s > k$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t) \cdot -se^{-st}\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\mathcal{L}\{f(t)\} - f(0)$$

Theorem 2:

Suppose that $f'(t)$ is continuous and $f''(t)$ is piecewice continous for every finite interval int the range $t \geq 0$ and $f(t)$ & $f''(t)$ are of exponential order for $t > T$. Then

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty f'(t) - se^{-st}\,dt$$

$$= [0 - f'(0)] + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s[s\mathcal{L}\{f(t)\} - f(0)]$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

$$\vdots$$

$$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \ldots s^{n-\overline{n-1}} f^{n-2}(0) - s^{n-n} f^{n-1}(0)$$

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Ex: Find $\mathcal{L}\{f(t)\}$ $f(t) = t^2$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$f(t) = t^2 \qquad f(0) = 0$$

$$f''(t) = 2t$$

$$\mathcal{L}\{2t\} = s\mathcal{L}\{f(t)\} - 0$$

$$2\mathcal{L}\{t\} = s\mathcal{L}\{f(t)\}$$

$$s\mathcal{L}\{f(t)\} = 2\frac{1}{s^2}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3}$$

Ex Initial value problem

solve $y'' - y = t$ $\quad y(0) = 1 \quad y'(0) = 1$

$$y'' - y = t$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2\mathcal{L}\{y(t)\} - \cancel{s\mathcal{L}}\, sy(0) - y'(0) - \mathcal{L}\{y\} = \frac{1}{s^2}$$

$$s^2Y - s - 1 - Y = \frac{1}{s^2}$$

$$(s^2-1)Y = s + 1 + \frac{1}{s^2}$$

$$Y = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

$$= \frac{s+1}{(s+1)(s-1)} + \frac{1}{s^2(s+1)(s-1)}$$

$$Y = \frac{1}{s-1} + \frac{1}{s^2(s+1)(s-1)}$$

$$\mathcal{L}\{y\} = \frac{1}{s-1} + \frac{1}{s^2(s+1)(s-1)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)(s-1)}\right\}$$

$$y = e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)(s-1)}\right\} \quad -(1)$$

take

$$\frac{1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1} \quad -(a)$$

$$1 = A(s)(s+1)(s-1) + B(s+1)(s-1) + C(s^2)(s-1) + D(s^2)(s+ \quad -(b)$$

put $s = 0$ in b

$$1 = B(1)(-1)$$

$$B = -1$$

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put $s=1$ in (b)

$1 = D(1)(2)$

$D = \frac{1}{2}$

put $s=-1$ in (b)

$1 = C(1)(-2)$

$C = \frac{-1}{2}$

take (2) b

$\frac{1}{s^2(s+1)(s-1)}$

$1 = A(s)(s+1)(s-1) + B(s+1)(s-1) + C(s^2)(s-1) + D(s^2)(s+1)$

$1 = (A+C+D)s^3 + \cdots$

comparing coefficient of s^3 on b.s.

$A+C+D = 0$

$A - \frac{1}{2} + \frac{1}{2} = 0$

$A = 0$

put in (a)

$\frac{1}{s^2(s+1)(s-1)} = \frac{-1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}$

put in (1)

$y = e^t + \mathcal{L}^{-1}\left\{\frac{-1}{s^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\right\}$

$= e^t - \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$

$= e^t - t + \frac{e^t}{2} - \frac{e^{-t}}{2}$

$= e^t - t + \sinh t$

Ex : Solve the initial value problem

$y'' + 2y' + y = e^{-t}$ $\quad y(0) = -1$

$y'(0) = 1$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$

$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{y\} = \frac{1}{s+1}$

$s^2Y + s - 1 + 2sY + 2 + Y = \frac{1}{s+1}$

$(s^2+2s+1)Y + s+1 = \frac{1}{s+1}$

$(s+1)^2 Y = \frac{1}{(s+1)} - (s+1)$

$Y = \frac{1}{(s+1)^3} - \frac{1}{s+1}$

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$$\mathcal{L}\{y\} = \frac{1}{(s+1)^3} - \frac{1}{s+1}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - e^{-t}$$

$$= \frac{e^{-t}}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - e^{-t}$$

$$y = \frac{e^{-t}}{2}t^2 - e^{-t}$$

Ex: $f(t) = t^2$. Derive $\mathcal{L}\{f(t)\}$ from $\mathcal{L}\{1\}$.

$f(t) = t^2 \qquad f(0) = 0$

$f'(t) = 2t \qquad f'(0) = 0$

$f''(t) = 2$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{2\} = s^2\mathcal{L}\{f(t)\} - s(0) - 0$$

$$2\cdot\mathcal{L}\{1\} = s^2\mathcal{L}\{f(t)\}$$

$$2\cdot\frac{1}{s} = s^2\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3}$$

Ex 5. Find $\mathcal{L}\{\cos\omega t\}$

Sol: $f(t) = \cos\omega t \qquad f(0) = 1$

$f'(t) = -\omega\sin\omega t \qquad f'(0) = 0$

$f''(t) = -\omega^2\cos\omega t$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{-\omega^2\cos\omega t\} = s^2\mathcal{L}\{\cos\omega t\} - s - 0$$

$$-\omega^2\mathcal{L}\{\cos\omega t\} = s^2\mathcal{L}\{\cos\omega t\} - s$$

$$s = s^2\mathcal{L}\{\cos\omega t\} + \omega^2\mathcal{L}\{\cos\omega t\}$$

$$s = (s^2+\omega^2)\mathcal{L}\{\cos\omega t\}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

Ex 6: $f(t) = \sin^2 t$. Find $\mathcal{L}\{f(t)\}$

$f(t) = \sin^2 t \qquad f(0) = 0$

$f'(t) = 2\sin t\cos t = \sin 2t$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{\sin 2t\} = s\mathcal{L}\{f(t)\} - 0$$

$$\frac{2}{s^2+4} = s\mathcal{L}\{\sin^2 t\}$$

$$\mathcal{L}\{\sin^2 t\} = \frac{2}{s(s^2+4)}$$

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Ex 7.

Let $f(t) = t\sin\omega t$. Find $\mathcal{L}\{f(t)\}$.

$f(t) = t\sin\omega t \qquad f(0) = 0$

$f'(t) = \cancel{-t\cos\omega t}\ \omega t\cos\omega t + \sin\omega t \qquad f'(0) = 0$

$f''(t) = -\omega^2 t\sin\omega t + \omega\cos\omega t + \omega\cos\omega t$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$-\omega^2\mathcal{L}\{t\sin\omega t\} + 2\omega\mathcal{L}\{\cos\omega t\} = s^2\mathcal{L}\{t\sin\omega t\}$

$2\omega\cdot\left(\frac{s}{s^2+\omega^2}\right) = (s^2+\omega^2)\mathcal{L}\{t\sin\omega t\}$

$\frac{2\omega s}{(s^2+\omega^2)^2} = \mathcal{L}\{t\sin\omega t\}$

Laplace Transform of integral of a function

Theorem: Let $F(s)$ be the laplace transform of $f(t)$. If $f(t)$ is piecewise continuous for $t \geq 0$ and of exponential order then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

Proof: We make use of facts

$$\int_a^a f(\tau)\,d\tau = 0 \quad \text{and} \quad \frac{d}{dt}\int_0^t f(\tau)\,d\tau = f(t)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \int_0^t f(\tau)\,d\tau\,\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\frac{d}{dt}\left(\int_0^t f(\tau)\,d\tau\right)dt$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = (0-0) + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s}F(s) \qquad \text{Hence proved.}$$

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau$$

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Ex 8. $\mathcal{L}\{f(t)\} = \dfrac{1}{s^2(s^2+\omega^2)}$, $f(t) = ?$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s^2+\omega^2)}\right\}$$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} d\tau$$

$$= \frac{1}{\omega}\int_0^t \left\{\frac{\omega}{s^2+\omega^2}\right\} d\tau$$

$$= \frac{1}{\omega}\int_0^t \sin\omega\tau \, d\tau$$

$$= \frac{1}{\omega}\left[\frac{\cos\omega\tau}{-\omega}\right]_0^t$$

$$= \frac{-1}{\omega^2}(\cos\omega t - 1)$$

$$f(t) = \frac{1}{\omega^2}[1-\cos\omega t]$$

Ex 9: $\mathcal{L}\{f(t)\} = \dfrac{1}{s^2(s^2+\omega^2)}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+\omega^2)}\right\}$$

$$f(t) = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} d\tau \quad \text{—①}$$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+\omega^2}\right\} d\tau$$

$$= \int_0^t \int_0^t \frac{1}{s^2+\omega^2}\, d\tau \cdot d\tau$$

$$= \int_0^t \frac{1}{\omega}\left|\frac{\cos\omega\tau}{-\omega}\right|_0^t d\tau$$

$$= \int_0^t \frac{-1}{\omega^2}(\cos\omega t - 1)\, d\tau$$

$$= \int_0^t +\frac{1}{\omega^2}(1-\cos\omega\tau)\, d\tau$$

$$= \frac{1}{\omega^2}\int_0^t 1\, d\tau - \frac{1}{\omega^2}\int_0^t \cos\omega\tau \, d\tau$$

$$= \frac{1}{\omega^2}\left|\tau\right|_0^t - \frac{1}{\omega^2}\left|\frac{\sin\omega\tau}{\omega}\right|_0^t$$

$$= \frac{t}{\omega^2} - \frac{1}{\omega^3}(\sin\omega t)$$

$$f(t) = \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)$$

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Exercise 5.2

1. $y' + 3y = 10\sin t \qquad y(0) = 0$

$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 10\mathcal{L}\{\sin t\}$

$s\mathcal{L}\{y\} - y(0) + 3\mathcal{L}\{y\} = 10 \cdot \frac{1}{s^2+1}$

$sY - 0 + 3Y = \frac{10}{s^2+1}$

$(s+3)Y = \frac{10}{s^2+1}$

$Y = \frac{10}{(s+3)(s^2+1)}$

$\mathcal{L}\{y\} = \frac{10}{(s+3)(s^2+1)}$

$y = \mathcal{L}^{-1}\left\{\frac{10}{(s+3)(s^2+1)}\right\} \quad -(1)$

Take

$\frac{10}{(s+3)(s^2+1)} = \frac{A}{(s+3)} + \frac{Bs+C}{s^2+1} \quad -(2)$

$10 = A(s^2+1) + (Bs+C)(s+3) \quad -(3)$

$10 = (A+B)s^2 + (3B+C)s + (A+3C)$

Comparing coefficients on b.s

$A + B = 0 \quad -4$

$3B + C = 0 \quad -5$

$A + 3C = 10 \quad -6$

In eq 3 put $s = -3$

$10 = A((-3)^2+1)$

$10 = 10A$

$A = 1$

put in (6

$1 + 3C = 10$

$\Rightarrow C = 3$

put $A = 1$ in (4

$B = -1$

put in (2

$\frac{10}{(s+3)(s^2+1)} = \frac{1}{(s+3)} + \frac{-s+3}{s^2+1}$

eq 1 $\Rightarrow$

$y = \mathcal{L}^{-1}\left\{\frac{1}{s+3} + \frac{-s+3}{s^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= e^{-3t} - \cos t + 3\sin t$

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2. $y' - 5y = 1.5e^{-4t}$

$\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = 1.5\,\mathcal{L}\{e^{-4t}\}$

$s\mathcal{L}\{y\} - y(0) - 5\mathcal{L}\{y\} = \frac{3}{2}\frac{1}{s+4}$

$sY - 1 - 5Y = \frac{3}{2(s+4)}$

$(s-5)Y = \frac{3}{2(s+4)} + 1$

$(s-5)Y = \frac{2s+11}{2(s+4)}$

$Y = \frac{2s+11}{2(s+4)(s-5)}$

$\mathcal{L}\{y\} = \frac{2s+11}{2(s+4)(s-5)}$

$y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2s+11}{(s+4)(s-5)}\right\}$ — (1)

Take

$\frac{2s+11}{(s+4)(s-5)} = \frac{A}{(s+4)} + \frac{B}{(s-5)}$ — (2)

$2s+11 = A(s-5) + B(s+4)$ — (3)

put $s=-4$ in (3)

$2(-4)+11 = A(-4-5)$

$3 = -9A$

$A = \frac{-1}{3}$

put $s=5$ in (3)

$2(5)+11 = B(5+4)$

$21 = 9B$

$B = \frac{7}{3}$

eq 2 $\Rightarrow$

$\frac{2s+11}{(s+4)(s-5)} = \frac{-1/3}{s+4} + \frac{7/3}{s-5}$

eq (1) $\Rightarrow$

$y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{-1/3}{s+4} + \frac{7/3}{s-5}\right\}$

$= \frac{1}{2}\cdot\frac{-1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{1}{2}\cdot\frac{7}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$

$= \frac{-1}{6}e^{-4t} + \frac{7}{6}e^{5t}$

$y = \frac{7e^{5t} - e^{-4t}}{6}$

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3. $y' + 0.2y = 0.01t \qquad y(0) = -0.25$

$$y' + \frac{2}{10}y = \frac{1}{100}t$$

$$\mathcal{L}\{y'\} + \frac{2}{10}\mathcal{L}\{y\} = \frac{1}{100}\mathcal{L}\{t\}$$

$$s\mathcal{L}\{y\} - y(0) + \frac{2}{10}\mathcal{L}\{y\} = \frac{1}{100s^2}$$

$$sY - \left(-\frac{25}{100}\right) + \frac{2}{10}Y = \frac{1}{100s^2}$$

$$\left(s + \frac{2}{10}\right)Y = \frac{1}{100s^2} - \frac{25}{100}$$

$$\left(\frac{10s+2}{10}\right)Y = \frac{1-25s^2}{100s^2}$$

$$Y = \frac{10}{(10s+2)} \cdot \frac{1-25s^2}{100s^2}$$

$$\mathcal{L}\{y\} = \frac{1-25s^2}{10(10s+2)s^2}$$

$$y = \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{1-25s^2}{(5s+1)\cdot s^2}\right\} \quad -(1)$$

Take

$$\frac{1-25s^2}{(5s+1)\cdot s^2} = \frac{A}{5s+1} + \frac{B}{s} + \frac{C}{s^2} \quad -(2)$$

$$1-25s^2 = A(s^2) + B(5s+1)(s) + C(5s+1) \quad -3$$

put $s=0$ in 3

$$1-25(0)^2 = C(5(0)+1)$$

$$C = 1$$

put $s = \frac{-1}{5}$ in 3

$$1 - 25\left(\frac{-1}{5}\right)^2 = A\left(\frac{-1}{5}\right)^2$$

$$1 - 25 \cdot \frac{1}{25} = A \cdot \frac{1}{25}$$

$$A = 0$$

By eq. 3

$$1-25s^2 = \cancel{As^2 +} (A+5B)s^2 + (B+5C)s + C$$

Comparing coefficient of s^2 on Both sides

$$A + 5B = -25$$

$$5B = -25$$

$$B = -5$$

putting values in (2)

$$\frac{1-25s^2}{(5s+1)\cdot s^2} = \frac{-5}{s} + \frac{1}{s^2}$$

eq. (1) $\Rightarrow$

$$y = \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{-5}{s} + \frac{1}{s^2}\right\}$$

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$$y = \frac{-5}{20}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{-1}{4}(1) + \frac{1}{20}t$$

$$y = \frac{t}{20} - \frac{1}{4}$$

4. $y'' - y' - 2y = 0 \qquad y(0) = 8,\ y'(0) =$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - s\mathcal{L}\{y\} + y(0) - 2\mathcal{L}\{y\} =$$

$$s^2Y - s(8) - 7 - sY + 8 - 2Y = 0$$

$$(s^2 - s - 2)Y = 8s - 1$$

$$Y = \frac{8s-1}{s^2-s-2}$$

$$\mathcal{L}\{y\} = \frac{8s-1}{(s-2)(s+1)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{8s-1}{(s-2)(s+1)}\right\} \quad -(1)$$

Take

$$\frac{8s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \quad -(2)$$

$$8s - 1 = A(s+1) + B(s-2) \quad -(3)$$

put $s = -1$ in (3)

$$8(-1) - 1 = B(-1-2)$$

$$\Rightarrow B = 3$$

put $s = 2$ in (3)

$$8(2) - 1 = A(2+1)$$

$$\Rightarrow A = 5$$

putting values in (2)

$$\cancel{y =} \frac{8s-1}{(s-2)(s+1)} = \frac{5}{(s-2)} + \frac{3}{(s+1)}$$

eq (1) $\Rightarrow$

$$y = \mathcal{L}^{-1}\left[\frac{5}{s-2} + \frac{3}{s+1}\right]$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$y = 5e^{2t} + 3e^{-t}$$

5. $y'' + ay' - 2a^2y = 0 \qquad y(0) = 6,\ y'(0) = 0$

$$\mathcal{L}\{y''\} + a\mathcal{L}\{y'\} - 2a^2\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + a\cdot s\mathcal{L}\{y\} - ay(0) - 2a^2\mathcal{L}\{y\} = 0$$

$$s^2Y - 6s - 0 + asY - 6a - 2a^2Y = 0$$

$$(s^2 + as - 2a^2)Y = 6s + 6a$$

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$$Y = \frac{6(s+a)}{s^2+as-2a^2}$$

$$\mathcal{L}\{y\} = \frac{6(s+a)}{s^2+2as-as-2a^2}$$

$$y = 6\mathcal{L}^{-1}\left[\frac{(s+a)}{(s+2a)(s-a)}\right] \quad -(1)$$

Take $\frac{s+a}{(s+2a)(s-a)} = \frac{A}{s+2a} + \frac{B}{s-a}$ $-(2)$

$$s+a = A(s-a) + B(s+2a) \quad -(3)$$

put $s=a$ in (3)

$$a+a = B(a+2a)$$

$$\Rightarrow B = \frac{2}{3}$$

put $s=-2a$ in (3)

$$-2a+a = A(-2a-a)$$

$$A = \frac{1}{3}$$

putting values in (2)

$$\frac{s+a}{(s+2a)(s-a)} = \frac{1/3}{s+2a} + \frac{2/3}{s-a}$$

eq (1) $\Rightarrow$

$$\mathcal{L}\left[\cancel{s+2a}\ y = 6\mathcal{L}^{-1}\left\{\frac{1/3}{s+2a} + \frac{2/3}{s-a}\right\}\right.$$

$$y = \cancel{6}\,\frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+2a}\right] + \cancel{6}\,\frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s-a}\right]$$

$$= 2e^{-2at} + 4e^{at}$$

$$y = 2e^{-2at} + 4e^{at}$$

$y'' + y = 2\cos t \qquad y(0) = 3,\ y'(0) = 4$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 2\mathcal{L}\{\cos t\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = 2\frac{s}{s^2+1}$$

$$s^2Y - 3s - 4 + Y = \frac{2s}{s^2+1}$$

$$(s^2+1)Y = \frac{2s}{s^2+1} + 3s + 4$$

$$\mathcal{L}\{y\} = \frac{2s}{(s^2+1)^2} + \frac{3s}{s^2+1} + \frac{4}{s^2+1}$$

$$y = \mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2} + \frac{3s}{s^2+1} + \frac{4}{s^2+1}\right]$$

$$y = \mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] + 3\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\boxed{\mathcal{L}\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}}$$

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$y = t\sin t + 3\cos t + 4\sin t$

7. $y'' - 4y' + 3y = 6t - 8$; $y(0) = 0$, $y'(0) = 0$

$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 6\mathcal{L}\{t\} - 8\mathcal{L}\{1\}$

$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 4s\mathcal{L}\{y\} + 4y(0) + 3\mathcal{L}\{y\} = \frac{6}{s^2} - \frac{8}{s}$

$s^2Y - s(0) - 0 - 4sY + 4(0) + 3Y = \frac{6-8s}{s^2}$

$(s^2 - 4s + 3)Y = \frac{6-8s}{s^2}$

$Y = \frac{6-8s}{s^2(s-3)(s-1)}$

$\mathcal{L}\{y\} = \frac{6-8s}{s^2(s-3)(s-1)}$

$y = \mathcal{L}^{-1}\left\{\frac{6-8s}{s^2(s-3)(s-1)}\right\}$ — ①

Pr Take, $\frac{6-8s}{s^2(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1}$ — ②

$6 - 8s = As(s-3)(s-1) + B(s-3)(s-1) + Cs^2(s-1) + Ds^2$ — ③

put $s = 0$ in ③

$6 - 8(0) = B(0-3)(0-1)$

$\Rightarrow B = 2$

put $s = 3$ in ③

$6 - 8(3) = C(3)^2(3-1)$

$\Rightarrow C = -1$

put $s = 1$ in ③

$6 - 8(1) = D(1)^2(1-3)$

$\Rightarrow D = 1$

By eq ③

$6 - 8s = \underset{(A+C+D)}{A}s^3 + (-4A + B - C - 3D)s^2 + (3A - 4B)s + 3B$

comparing coefficients of s^3 on b.s

$A + C + D = 0$

$A - 1 + 1 = 0$

$A = 0$

putting values in ②

$\frac{6-8s}{s^2(s-3)(s-1)} = \frac{2}{s^2} - \frac{1}{s-3} + \frac{1}{s-1}$

eq ① $\Rightarrow$

$y = \mathcal{L}^{-1}\left\{\frac{2}{s^2} - \frac{1}{s-3} + \frac{1}{s-1}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$= 2t - e^{3t} + e^{t}$

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$8y'' + 0.04y = 0.02t^2$; $y(0) = -25$, $y'(0) = 0$

$$\mathcal{L}\{y''\} + \frac{4}{100}\mathcal{L}\{y\} = \frac{2}{100}\mathcal{L}\{t^2\}$$

$$s\mathcal{L}\{y\} - sy(0) - y'(0) + \frac{4}{100}\mathcal{L}\{y\} = \frac{2}{100}\cdot\frac{2}{s^3}$$

$$sY + 25s + \frac{4}{100}Y = \frac{4}{100s^3}$$

$$s + \frac{4}{100}Y = \frac{4}{100s^3} - 25s$$

$$\mathcal{L}\{y\} = \frac{100}{100s+4}\left(\frac{4 - 2500s^4}{100s^3}\right)$$

$$y = \mathcal{L}^{-1}\left\{\frac{4 - 2500s^4}{(100s+4)\cdot s^3}\right\} \quad -(1)$$

Take

$$\frac{4 - 2500s^4}{(100s+4)s^3} = \frac{A}{100s+4} + \frac{B}{s} + \frac{C}{s^2} + \frac{D}{s^3} \quad -(2)$$

$$4 - 2500s^4 = As^3 + Bs^2(100s+4) + Cs(100s+4) + D(100s+4) \quad -(3)$$

put $s = 0$ in (3)

$$4 - 2500(0)^4 = D(100(0) + 4)$$

$$\Rightarrow D = 1$$

put $s = \frac{-4}{100}$ in (3)

$$4 - 2500\left(\frac{-4}{100}\right)^4 = A\left(\frac{-4}{100}\right)^3$$

$$4 - 2500$$

$$y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(2+50s^2)(2-50s^2)}{(2+50s)s^3}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(2+50s^2)(1+25s)(1-25s)}{(1+25s)s^3}\right\}$$

$$y = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{(2+50s^2)(1-25s)}{s^3}\right\} \quad -(1)$$

Take

$$\frac{(2+50s^2)(1-25s)}{s^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3}$$

$$y = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2 - 50s + 50s^2 - 1250s^3}{s^3}\right\}$$

$$y = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} - \frac{50}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{50}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + -1250\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$y = \frac{1}{4}t^2 - 22.5t + 22.5 -$$

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8.

$$y'' + \frac{4}{100}y = \frac{2}{100}t^2$$

$$y'' + \frac{1}{25}y = \frac{1}{50}t^2$$

$$\mathcal{L}\{y''\} + \frac{1}{25}\mathcal{L}\{y\} = \frac{1}{50}\mathcal{L}\{t^2\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \frac{1}{25}\mathcal{L}\{y\} = \frac{1}{50}\cdot\frac{2}{s^3}$$

$$s^2Y + 25s - 0 + \frac{1}{25}Y = \frac{1}{25s^3}$$

$$\left(s^2 + \frac{1}{25}\right)Y = \frac{1}{25s^3} - 25s$$

$$\left(\frac{25s^2+1}{25}\right)Y = \frac{1-(25)^2s^4}{25s^3}$$

$$\mathcal{L}\{y\} = \frac{25}{1+25s^2}\left(\frac{(1+25s^2)(1-25s^2)}{25s^3}\right)$$

$$= \frac{1-25s^2}{25s^3}$$

$$\mathcal{L}\{y\} = \frac{1}{25s^3} - \frac{25}{s}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^3} - \frac{25}{s}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - 25\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - 25$$

$$= \frac{t^2}{2} - 25$$

9. $y'' + 2y' - 3y = 6e^{-2t}$; $y(0) = 2$, $y'(0) = -14$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 6\mathcal{L}\{e^{-2t}\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - y(0) - 3\mathcal{L}\{y\} = \frac{6}{s+2}$$

$$s^2Y - 2s + 14 + 2sY - 2 - 3Y = \frac{6}{s+2}$$

$$(s^2+2s-3)Y - 2s + 12 = \frac{6}{s+2}$$

$$\cancel{(s^2+2s-3)}$$

$$(s^2+2s-3)Y = \frac{6}{s+2} + 2s - 12$$

$$(s+3)(s-1)Y = \frac{6+(2s-12)(s+2)}{(s+2)}$$

$$Y = \frac{6+2s^2-8s-24}{(s+2)(s+3)(s-1)}$$

$$\mathcal{L}\{y\} = \frac{2s^2-8s-18}{(s+2)(s+3)(s-1)}$$

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8 $y = \mathcal{L}^{-1}\left\{\frac{2s^2 - 8s - 14}{(s+2)(s+3)(s-1)}\right\}$ — ①

Take

$$\frac{2s^2 - 8s - 14}{(s+2)(s+3)(s-1)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s-1}$$

$$2s^2 - 8s - 14 = A(s+3)(s-1) + B(s+2)(s-1) + C(s+2)(s+3)$$

put $s=-2$ in ③

$$2(-2)^2 - 8(-2) - 14 = A(-2+3)(-2-1)$$

$$8 + 16 - 14 = A(1)(-3)$$

$$\Rightarrow A = -2$$

put $s=-3$ in ③

$$2(-3)^2 - 8(-3) - 14 = B(-3+2)(-3-1)$$

$$\Rightarrow B = 6$$

put $s=1$ in ③

$$2(1)^2 - 8(1) - 14 = C(1+2)(1+3)$$

$$-24 = C(12)$$

$$C = -2$$

putting values in ②

$$\frac{2s^2 - 8s - 14}{(s+2)(s+3)(s-1)} = \frac{-2}{s+2} + \frac{6}{s+3} + \frac{-2}{s-1}$$

Q ⑥ ⇒

$$y'' + 2y' - 3y = 6e^{-2t}; \quad y(0) = 2, \; y'(0) = 4$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 6\mathcal{L}\{e^{-2t}\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) - 3\mathcal{L}\{y\} = 6 \cdot \frac{1}{s+2}$$

$$s^2Y - 2s + 14 + 2sY - 4 - 3Y = \frac{6}{s+2}$$

$$(s^2 + 2s - 3)Y = \frac{6}{s+2} + 2s - 10$$

$$\mathcal{L}\{y\} = \frac{6 + (2s-10)(s+2)}{(s+2)(s^2+2s-3)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{6 + 2s^2 - 6s - 20}{(s+2)(s+3)(s-1)}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2s^2 - 6s - 14}{(s+2)(s+3)(s-1)}\right\} \quad — ①$$

Take $\frac{2s^2 - 6s - 14}{(s+2)(s+3)(s-1)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s-1}$

$$2s^2 - 6s - 14 = A(s+3)(s-1) + B(s+2)(s-1) + C(s+2)(s+3)$$

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put $s=-2$ in (b)

$$2(-2)^2-6(-2)-14 = A(-2+3)(-2-1)$$

$$\Rightarrow A = -2$$

put $s=-3$ in (b)

$$2(-3)^2-6(-3)-14 = B(-3+2)(-3-1)$$

$$B = \frac{11}{2}$$

put $s=1$ in (b)

$$2(1)^2-6(1)-14 = C(1+2)(1+3)$$

$$-18 = C(12)$$

$$C = \frac{-3}{2}$$

put in (a)

$$\frac{2s^2-6s-14}{(s+2)(s+3)(s-1)} = \frac{-2}{s+2} + \frac{11}{2}\cdot\frac{1}{s+3} - \frac{3}{2}\cdot\frac{1}{s-1}$$

eq. (1) $\Rightarrow$

$$y = \mathcal{L}^{-1}\left\{\frac{-2}{s+2} + \frac{11/2}{s+3} - \frac{3/2}{s-1}\right\}$$

$$= -2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{11}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$y = 2e^{-2t} + \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t}$$

2. Project.

a). $\mathcal{L}\{t\cos\omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

let $f(t) = t\cos\omega t \Rightarrow f(0)=0$

~~$f'(t) = -\omega t\sin t + \cos t \Rightarrow f'(0) = 1$~~

$f'(t) = t[-\omega\sin\omega t] + \cos\omega t \Rightarrow f'(0) = 1$

$$f''(t) = -\omega[t\cdot\omega\cos\omega t - \sin\omega t] - \omega\sin\omega t$$
$$= -\omega^2 t\cos\omega t - 2\omega\sin\omega t$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{-\omega^2 t\cos\omega t - 2\omega\sin\omega t\} = s^2\mathcal{L}\{t\cos\omega t\} - 1$$

$$-\omega^2\mathcal{L}\{t\cos\omega t\} - 2\omega\mathcal{L}\{\sin\omega t\} = s^2\mathcal{L}\{t\cos\omega t\} - 1$$

$$1 - 2\omega\cdot\frac{\omega}{s^2+\omega^2} = (s^2+\omega^2)\mathcal{L}\{t\cos\omega t\}$$

$$\frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \mathcal{L}\{t\cos\omega t\}$$

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b). $\mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)^2}\right] = \frac{1}{2w^3}\left[\sin wt - wt\cos\right.$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)^2}\right] = \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{w^2}{(s^2+w^2)^2}\right]$$

$$= \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{s^2-s^2+w^2}{(s^2+w^2)^2}\right]$$

$$= \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{s^2+w^2}{(s^2+w^2)^2}\right] - \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right]$$

$$= \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)}\right] - \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{s^2+w^2-w^2}{(s^2+w^2)^2}\right]$$

$$= \frac{1}{w^3}\mathcal{L}^{-1}\left[\frac{w}{s^2+w^2}\right] - \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{s^2-w^2}{(s^2+w^2)^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)}\right]$$

$$2\mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)^2}\right] = \frac{1}{w^3}\sin wt - \frac{1}{w^2}t\cos wt$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+w^2)^2}\right] = \frac{1}{2w^2}\left[\sin wt - wt\cos wt\right]$$

:). $\mathcal{L}^{-1}\left[\frac{s}{(s^2+w^2)^2}\right] = \frac{1}{2w}t\sin wt$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+w^2)^2}\right] = \frac{1}{2w}\mathcal{L}^{-1}\left[\frac{2ws}{(s^2+w^2)^2}\right]$$

$$= \frac{1}{2w}t\sin wt$$

d). $\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right] = \frac{1}{2w}(\sin wt + wt\cos wt)$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{s^2+w^2-w^2}{(s^2+w^2)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s^2+w^2}{(s^2+w^2)^2}\right] - \mathcal{L}^{-1}\left[\frac{w^2}{(s^2+w^2)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2+w^2}\right] + \mathcal{L}^{-1}\left[\frac{s^2-s^2-w^2}{(s^2+w^2)^2}\right]$$

$$= \frac{1}{w}\mathcal{L}^{-1}\left[\frac{w}{s^2+w^2}\right] + \mathcal{L}^{-1}\left[\frac{s^2-w^2}{(s^2+w^2)^2}\right] - \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right] = \frac{1}{w}\sin wt + t\cos wt$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+w^2)^2}\right] = \frac{1}{2w}\left[\sin wt + wt\cos wt\right]$$

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e). $\mathcal{L}\{t\cosh at\} = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

$$\mathcal{L}\{t\cosh at\} = \mathcal{L}\left\{t\cdot\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{te^{at}\} + \frac{1}{2}\mathcal{L}\{te^{-at}\}$$

$$= \frac{1}{2}\,\frac{1}{s^2}\Big|_{s\to s-a} + \frac{1}{2}\,\frac{1}{s^2}\Big|_{s\to s+a}$$

$$= \frac{1}{2}\cdot\frac{1}{(s^2-a^2)^2} + \frac{1}{2}\cdot\frac{1}{(s+a)^2}$$

$$= \frac{1}{2}\left[\frac{(s+a)^2+(s-a)^2}{(s-a)^2(s+a)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+a^2+2as+s^2+a^2-2as}{(s^2-a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s^2+2a^2}{(s^2-a^2)^2}\right]$$

$$= \frac{s^2+a^2}{(s^2-a^2)^2}$$

f). $\mathcal{L}\{t\sinh at\} = \dfrac{+2as}{(s^2-a^2)^2}$

$$\mathcal{L}\{t\sinh at\} = \mathcal{L}\left\{t\cdot\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{te^{at}\} - \frac{1}{2}\mathcal{L}\{te^{-at}\}$$

$$= \frac{1}{2}\,\frac{1}{s^2}\Big|_{s\to s-a} - \frac{1}{2}\,\frac{1}{s^2}\Big|_{s\to s+a}$$

$$= \frac{1}{2}\,\frac{1}{(s-a)^2} - \frac{1}{2}\,\frac{1}{(s+a)^2}$$

$$= \frac{1}{2}\left[\frac{(s+a)^2-(s-a)^2}{(s-a)^2(s+a)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+a^2+2as-s^2-a^2+2as}{(s^2-a^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{4as}{(s^2-a^2)^2}\right]$$

$$= \frac{2as}{(s^2-a^2)^2}$$

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13. $\dfrac{1}{s^2+4s}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+4}\right]$$

$$= \int_0^t \left[\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]\right] d\tau$$

$$= \int_0^t e^{-4\tau}\, d\tau$$

$$= \left|\frac{e^{-4\tau}}{-4}\right|_0^t$$

$$= \frac{-1}{4}\left[e^{-4t} - 1\right]$$

$$= \frac{1}{4}\left[1 - e^{-4t}\right]$$

14. $\dfrac{4}{s^3-2s^2}$

$$\mathcal{L}^{-1}\left[\frac{4}{s^3-2s^2}\right] = 4\mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s-2}\right]$$

$$= 4\int_0^t\int_0^t \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] d\tau\, d\tau$$

$$= 4\int_0^t\int_0^t e^{2\tau}\, d\tau\, d\tau$$

$$= 4\int_0^t \left|\frac{e^{2\tau}}{2}\right|_0^t d\tau$$

$$= 2\int_0^t (e^{2t} - 1)\, d\tau$$

$$= 2\int_0^t e^{2\tau}\, d\tau - 2\int_0^t 1\, d\tau$$

$$= 2\left|\frac{e^{2\tau}}{2}\right|_0^t - 2|\tau|_0^t$$

$$= e^{2t} - 1 - 2t$$

15. $\dfrac{1}{s(s^2+w^2)}$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+w^2)}\right] = \int_0^t \mathcal{L}^{-1}\left[\frac{1}{s^2+w^2}\right] d\tau$$

$$= \int_0^t \frac{1}{w}\mathcal{L}^{-1}\left[\frac{w}{s^2+w^2}\right] d\tau$$

$$= \int_0^t \frac{1}{w}\sin w\tau\, d\tau$$

$$= \frac{1}{w}\left|\frac{-\cos w\tau}{w}\right|_0^t$$

$$= \frac{-1}{w^2}(\cos wt - 1)$$

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$$s+1 \overline{\big)\, s-1 \,}\ \ -1 \qquad \frac{-s-1}{-2} \qquad -1 - \frac{2}{s+1}$$

$$s-1 \overline{\big)\, s+1 \,}\ \ -1 \qquad \frac{-s+1}{2} \qquad -1 \frac{2}{s-1}$$

$$-1 + \frac{2}{s+1}$$

16. $\dfrac{1}{s^5+s^3}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^5+s^3}\right] = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s^2+1}\right\}$$

$$= \int_0^t\int_0^t\int_0^t \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] d\tau\, d\tau\, d\tau$$

$$= \int_0^t\int_0^t\int_0^t \sin\tau\, d\tau\, d\tau\, d\tau$$

$$= \int_0^t\int_0^t -\left|\cos\tau\right|_0^t d\tau\, d\tau$$

$$= \int_0^t\int_0^t (1-\cos\tau)\, d\tau\, d\tau$$

$$= \int_0^t \left[\int_0^t 1\, d\tau - \int_0^t \cos\tau\, d\tau\right] d\tau$$

$$= \int_0^t \left|\tau\right|_0^t - \left|\sin\tau\right|_0^t\, d\tau$$

$$= \int_0^t \tau - \sin\tau\, d\tau$$

$$= \int_0^t \tau\, d\tau - \int_0^t \sin\tau\, d\tau$$

$$= \left|\frac{\tau^2}{2}\right|_0^t + \left|\cos\tau\right|_0^t$$

$$= \frac{t^2}{2} + \cos t - 1$$

7. $\dfrac{1}{s^3-s}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3-s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2-1}\right]$$

$$= \int_0^t \mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] d\tau$$

$$= \int_0^t \sinh\tau\, d\tau$$

$$= \left|\cosh\tau\right|_0^t$$

$$= \cosh t - 1$$

8. $\dfrac{1}{s^2}\left(\dfrac{s-1}{s+1}\right)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s}\cdot\left(\frac{s-1}{s+1}\right)\right\}$$

$$= \int_0^t \mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{s-1}{s+1}\right] d\tau \quad -(1)$$

Take

$$\frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$s-1 = A(s+1) + Bs$$

put $s=-1$

$$-1-1 = B(-1) \Rightarrow B = 2$$

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put $s=0$

$-1 = A$

$$\frac{s-1}{s(s+1)} = \frac{2}{s+1} - \frac{1}{s}$$

eq. ① $\Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s}\right\} d\tau$$

$$= \int_0^t \left\{2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right\} d\tau$$

$$= \int_0^t 2e^{-\tau} - 1 \, d\tau$$

$$= 2\int_0^t e^{-\tau} d\tau - \int_0^t 1 \, d\tau$$

$$= 2\left|\frac{e^{-\tau}}{-1}\right|_0^t - |\tau|_0^t$$

$$= -2(e^{-t} - 1) - t$$

$$= 2 - 2e^{-t} - t$$

19. $\frac{9}{s^2}\left(\frac{s+1}{s^2+9}\right)$

$$\mathcal{L}^{-1}\left\{\frac{9}{s^2}\left(\frac{s+1}{s^2+9}\right)\right\} = 9\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s}\left(\frac{s+1}{s^2+9}\right)\right\}$$

$$= 9\int_0^t\int_0^t \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} d\tau \, d\tau$$

$$= 9\int_0^t\int_0^t \left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}\right] d\tau \, d\tau$$

$$= 9\int_0^t\int_0^t \left(\cos 3\tau + \frac{1}{3}\sin 3\tau\right) d\tau \, d\tau$$

$$= 9\int_0^t \left|\frac{\sin 3\tau}{3} - \frac{1}{9}\cos 3\tau\right|_0^t d\tau$$

$$= 9\int_0^t \frac{\sin 3\tau}{3} - \frac{\cos 3\tau}{9} + \frac{1}{9} \, d\tau$$

$$= 9\left|\frac{-\cos 3\tau}{9} - \frac{\sin 3\tau}{27} + \frac{\tau}{9}\right|_0^t$$

$$= 9\left[\frac{-\cos 3t}{9} + \frac{1}{9} - \frac{\sin 3t}{27} + \frac{t}{9}\right]$$

$$= -\cos 3t + 1 - \frac{\sin 3t}{3} + t$$

20. $\frac{\pi^5}{s^4(s^2+\pi^2)}$

$$\mathcal{L}^{-1}\left\{\frac{\pi^5}{s^4(s^2+\pi^2)}\right\} = \pi^4\int_0^t\int_0^t\int_0^t\int_0^t \left[\frac{\mathcal{L}^{-1}\pi}{(s^2+\pi^2)}\right] d\tau \, d\tau \, d\tau \, d\tau$$

$$= \pi^4\int_0^t\int_0^t\int_0^t\int_0^t \sin \pi t \, d\tau \, d\tau \, d\tau \, d\tau$$

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$$= \pi^{1}\int_0^t\int_0^t\int_0^t \cancel{\sin\pi}\left(-\frac{\cos\pi\tau}{\pi}\right)_0^t d\tau\, d\tau\, d\tau$$

$$= \pi^{3}\int_0^t\int_0^t\int_0^t (1-\cos\pi\tau)\, d\tau\, d\tau\, d\tau$$

$$= \pi^{3}\int_0^t\int_0^t \left|\tau - \frac{\sin\pi\tau}{\pi}\right|_0^t d\tau\, d\tau$$

$$= \pi^{2}\int_0^t\int_0^t (\pi\tau - \sin\pi\tau)\, d\tau\, d\tau$$

$$= \pi^{2}\int_0^t \left|\frac{\pi\tau^2}{2} + \frac{\cos\pi\tau}{\pi}\right|_0^t d\tau$$

$$= \pi\int_0^t \frac{\pi^2\tau^2}{2} + \cos\pi\tau - 1\, d\tau$$

$$= \pi\left[\frac{\pi^2\tau^3}{6} + \frac{\sin\pi\tau}{\pi} - \tau\right]_0^t$$

$$= \frac{\pi^3 t^3}{6} + \sin\pi t - \pi t$$

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