# MATHEMATICS-II

## *UNIT-2 : VECTOR CALCULUS*

Chapter 3

# Vector Calculus

Vector calculus, or vector analysis, is a branch of mathematics concerned with differentiation and integration of vectors. It plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

A *scalar field* associates a scalar value to every point in a space. A map $f : \mathbb{R}^n \to \mathbb{R}$ which assigns each $\overline{x}$, a *scalar function* $f(\overline{x})$. For example, the temperature distribution throughout space, the pressure distribution in a fluid.

A *vector field* associates a vector to every point in a space. A map $\overline{F} : \mathbb{R}^n \to \mathbb{R}^n$ which assigns each $\overline{x}$, a *vector function* $\overline{F}(\overline{x})$. For example, electric field, gravitational field.

## 3.1 Tutorial : Gradient, Divergence, Curl

A *vector differential operator* $\nabla$ is defined as

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

With the help of $\nabla$, the gradient, the divergence and the curl are defined as follows:

**Gradient**

The *gradient* $\text{grad} f$ of a scalar function $f(x, y, z)$ is a vector function defined by

$$\text{grad} f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}.$$

The gradient points in the direction of the greatest rate of the increase of the function and its magnitude is the slope of the graph in that direction.

If $f$ is describing the temperature at a point $(x, y, z)$ in the space, then $\text{grad} f$ gives the direction in which the temperature increasing most rapidly.

**Divergence**

The *divergence* of a vector function $\overline{F} = (F_1, F_2, F_3)$ is a scalar function defined by

$$\text{div}\ \overline{F} = \nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The divergence measures how much a vector field "spreads out" or diverges from a given point.

Imagine a fluid, with the vector field representing the velocity of the fluid at each point in the space. Its divergence measures the net flow of fluid out of a give point. If fluid is instead flowing into the point, the divergence will be negative.

A point or region with positive divergence is often referred to as a "source" while a point or region with negative divergence is a "sink".

A vector field $\overline{F}$ with divergence zero at all points, i.e. $\nabla \cdot \overline{F} = 0$ is called a *solenoidal vector field* (also known as an *incompressible vector field*).

## Curl

The *curl* of a vector function $\overline{F} = (F_1, F_2, F_3)$ is a vector function defined by

$$\begin{aligned} \text{curl } \overline{F} &= \nabla \times \overline{F} \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}, \end{aligned}$$

The curl is the circulation per unit area, circulation density, or rate of rotation (amount of twisting at a single point).

The angular velocity of rotation at any point is equal to the half of the curl of the linear velocity at that point of the body. So, if the curl of any field $\overline{F}$ is zero, there is no rotation and the field is called *irrotational*.

For a vector field $\overline{F}$, it can be shown that

$$\nabla \times \overline{F} = \overline{0} \quad \Leftrightarrow \quad \overline{F} = \nabla f,$$

for some scalar field $f$. In this case, corresponding scalar field $f$ is called *potential* and it is associated with potential energy.

## Solved Examples

**Example 3.1.1.** *Find the gradient of* $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1}(xz)$ *at* $(1, 1, 1)$.

**Solution.** Using definition

$$\begin{aligned} \nabla f &= \left[-6xz + \frac{1}{1 + x^2z^2} \cdot z\right]\hat{\imath} + [-6yz]\,\hat{\jmath} + \left[6z^2 - 3(x^2 + y^2) + \frac{1}{1 + x^2z^2}\right]\hat{k} \\ \Rightarrow \quad (\nabla f)_{(1,1,1)} &= \frac{-11}{2}\hat{\imath} - 6\hat{\jmath} + \frac{1}{2}\hat{k}. \quad \blacksquare \end{aligned}$$

**Example 3.1.2.** *Find divergence and curl of* $\overline{V} = xyz(x, y, z)$.

**Solution.** Here, $\overline{V} = x^2yz\hat{\imath} + xy^2z\hat{\jmath} + xyz^2\hat{k}$. Now,

$$\begin{aligned} \text{div } \overline{V} &= \nabla \cdot \overline{V} \\ &= \frac{\partial(x^2yz)}{\partial x} + \frac{\partial(xy^2z)}{\partial y} + \frac{\partial(xyz^2)}{\partial z} \end{aligned}$$

$$\begin{aligned} &= 2xyz + 2xyz + 2xyz \\ &= 6xyz. \end{aligned}$$

Also,

$$\begin{aligned} \text{curl } \overline{V} &= \nabla \times \overline{V} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\ &= \hat{i}(xz^2 - xy^2) + \hat{j}(x^2y - yz^2) + \hat{k}(y^2z - x^2z). \quad \blacksquare \end{aligned}$$

**Example 3.1.3.** *Find constants $a$, $b$ and $c$ so that $\overline{V} = (x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$ is irrotational.*

**Solution.** Observe that

$$\begin{aligned} \text{curl } \overline{V} &= \nabla \times \overline{V} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} \\ &= (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k}. \end{aligned}$$

Now, the vector $\overline{V}$ is irrotational if and only if

$$\text{curl } \overline{V} = 0 \quad \Leftrightarrow \quad c+1=0, \;\; a-4=0, \;\; b-2=0 \quad \Leftrightarrow \quad a=4, \;\; b=2, \;\; c=-1. \quad \blacksquare$$

**Example 3.1.4.** *A fluid motion is given by*

$$\overline{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}.$$

*Is the motion irrotational? If so, find velocity potential.*

**Solution.** Here,

$$\begin{aligned} \text{curl } \overline{V} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix} \\ &= (x\cos z + 2y - x\cos z - 2y)\hat{i} + (y\cos z - y\cos z)\hat{j} + (\sin z - \sin z)\hat{k} \\ &= \overline{0}. \end{aligned}$$

Thus the motion is irrotational. Hence,

$$\overline{V} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

Now,

$$\overline{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}.$$

Therefore,

$$\frac{\partial\phi}{\partial x} = y\sin z - \sin x, \quad \frac{\partial\phi}{\partial y} = x\sin z + 2yz, \quad \frac{\partial\phi}{\partial z} = xy\cos z + y^2. \tag{3.1.1}$$

Integrating the first equation of (3.1.1) with respect to $x$ (keeping $y$ and $z$ constant), we get

$$\phi = xy\sin z + \cos x + f_1(y,z). \tag{3.1.2}$$

Differentiating partially with respect to $y$, we get

$$\begin{aligned}
&\frac{\partial\phi}{\partial y} = x\sin z + \frac{\partial f_1(y,z)}{\partial y} \\
\Rightarrow\quad & x\sin z + 2yz = x\sin z + \frac{\partial f_1(y,z)}{\partial y} \quad \text{(using 2}^{\text{nd}}\text{ equation of (3.1.1))} \\
\Rightarrow\quad & \frac{\partial f_1(y,z)}{\partial y} = 2yz.
\end{aligned}$$

Integrating with respect to $y$ (keeping $z$ constant), we get

$$f_1(y,z) = y^2 z + f_2(z). \tag{3.1.3}$$

Substituting this value in (3.1.2), we get

$$\phi = xy\sin z + \cos x + y^2 z + f_2(z). \tag{3.1.4}$$

Differentiating partially with respect to $z$ and using third equation of (3.1.1), we get

$$\frac{\partial\phi}{\partial z} = xy\cos z + y^2 + \frac{\partial f_2}{\partial z} \;\Rightarrow\; xy\cos z + y^2 = xy\cos z + y^2 + \frac{\partial f_2}{\partial z} \;\Rightarrow\; \frac{\partial f_2}{\partial z} = 0 \;\Rightarrow\; f_2(z) = c. \tag{3.1.5}$$

Hence, from (3.1.4), the required velocity potential is $\phi = xy\sin z + y^2 z + \cos x + c$. ■

## Exercises

**Exercise 3.1.1.** *Find div($\overline{v}$), where $\overline{v} = e^x\hat{i} + ye^{-x}\hat{j} + 2z\sinh x\hat{k}$.*

**Exercise 3.1.2.** *Is the vector $\overline{v} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$ solenoidal at a point $(1,2,1)$?*

**Exercise 3.1.3.** *Find the value of $\lambda$ such that the vector field*

$$\overline{F} = (2x^2y^2 + z^2)\hat{i} + (3xy^3 - x^2z)\hat{j} + (\lambda xy^2 z + xy)\hat{k}$$

*is solenoidal.*

**Exercise 3.1.4.** *Obtain curl $\overline{F}$ at the point $(2,0,3)$ for $\overline{F} = ze^{2xy}\hat{i} + 2xy\cos y\hat{j} + (x+2y)\hat{k}$.*

**Exercise 3.1.5.** *Show that the function is irrotational and hence find the scalar potential function* $f$ *for* $\overline{A} = y^2\hat{\imath} + 2xy\hat{\jmath} - z^2\hat{k}$ *such that* $A = \text{grad } f$.

**Exercise 3.1.6.** *Find the constants* $a$, $b$ *and* $c$ *so that* $(x+y+az)\hat{\imath}+(bx+2y-z)\hat{\jmath}+(-x+cy+2z)\hat{k}$ *is irrotational.*

**Exercise 3.1.7.** *If* $\overline{F} = (y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$, *show that* $\overline{F}$ *is the both solenoidal and irrotational.*

**Exercise 3.1.8.** *Prove that curl(grad* $f$*)* $= 0$.

**Answers**

**3.1.1** $e^x + e^{-x} + 2\sinh x$ **3.1.2** yes **3.1.3** $\lambda = -13$ **3.1.4** $2\hat{\imath} - 12\hat{k}$
**3.1.5** $xy^2 - \frac{z^3}{3} + C$; $C$-arbitrary constant **3.1.6** $a = -1$, $b = 1$, $c = -1$

## 3.2 Tutorial : Line Integrals

A line integral is an integral where the function to be integrated along a curve and is a generalization of a definite integral. Precisely, we have the following dentation.

The *line integral* of a vector function $\overline{F}(\overline{r})$ along a curve $C$ is defined by

$$\int_C \overline{F}\cdot d\overline{r} = \int_a^b \overline{F}(\overline{r}(t))\cdot \frac{d\overline{r}}{dt}. \tag{3.2.1}$$

In terms of components, with $\overline{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, formula (3.2.1) becomes

$$\int_C \overline{F}\cdot d\overline{r} = \int_C (F_1dx + F_2dy + F_3dz). \tag{3.2.2}$$

In equation (3.2.2) we integrate from a point $A$ to a point $B$ over a path $C$. The value of such an integral generally depends not only on $A$ and $B$, but also on the path $C$ along which we integrate. If $C$ is a closed curve then the line integral $\int\limits_C$ is denoted by $\oint\limits_C$.

**Independence of Path**

A line integral $\int\limits_C \overline{F}\cdot d\overline{r}$ is independent of path if its value remains same along all the curves $C$ with same end points $A$ and $B$. In this case, we write

$$\int_C \overline{F}\cdot d\overline{r} = \int_A^B \overline{F}\cdot d\overline{r}.$$

**Circulation**

If $\overline{F}$ represents the velocity of a fluid particle and $C$ is a closed curve, the line integral $\oint\limits_C \overline{F}\cdot d\overline{r}$ represents the *circulation* of $\overline{F}$ around the curve $C$. If circulation is zero, $\overline{F}$ is said to be *irrotational*.

**Flux**

If $C$ is a smooth closed curve in the domain of a continuous vector field

$$\overline{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$$

in the plane, and if $\hat{n}$ is the outward-pointing unit normal vector on $C$, the *flux* of $\overline{F}$ across $C$ is

$$\text{Flux of } \overline{F} \text{ across } C = \int_C \overline{F}\cdot \hat{n}ds.$$

In terms of components,

$$\hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}.$$

Thus

$$\int_C \overline{F}\cdot\hat{n}ds = \int_C M\left(\frac{dy}{ds}\right) + N\left(\frac{dx}{ds}\right) = \oint_C Mdy - Ndx.$$

## Work Done

If $\overline{F}$ is the force acting upon a particle which is moving along curve $C$ then the line integral $\int\limits_C \overline{F}\cdot d\overline{r}$ represents the *work done* by the force.

## Conservative Vector Field

Let $\overline{F}$ be a conservative vector field in some domain $D$. Then $\overline{F}$ is said to be *conservative* if there exists some scalar function $\phi$ such that $\overline{F} = \nabla\phi$.

## Simply Connected Domain

A set $S$ is said to be *open* if every point $P$ of $S$ has a neighbourhood contained in $S$.

A set $S$ is said to be *connected* if any two points of $S$ can be joined by finitely many line segments joined end to end lying entirely in $S$.

A set which is open and connected is called a *domain*.

A domain $D$ is called *simply connected* if every closed curve in $D$ can be shrunk to any point in $D$ without leaving $D$.

## Theorems

(1) Let $F = P\hat{i} + Q\hat{j}$ be a vector field on a simply connected domain $D$. Suppose that $P$ and $Q$ have continuous first order partial derivatives in $D$. Then $F$ is conservative in $D$ if and only if throughout $D$, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

(2) Let $F$ be a continuous vector field on simply connected domain $D$. Then the following are equivalent:

(i) The line integral $\int_C \overline{F}\cdot d\overline{r}$ is independent of path;

(ii) $\overline{F}$ is conservative;

(iii) $\nabla\times\overline{F} = \overline{0}$.

(3) **(Fundamental Theorem of Vector Calculus)** Let $\overline{F}(x, y, z)$ be a conservative vector field. Then

$$\int_A^B \overline{F}\cdot d\overline{r} = \int_A^B \nabla\phi\cdot d\overline{r} = \int_A^B \left[\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right] = \int_A^B d\phi = \phi(B) - \phi(A).$$

(4) **(Divergence Theorem)** Let $\overline{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$ be a vector field and $R$ be a region enclosed by a simple closed curve $C$ traversed counterclockwise. Then the flux of $\overline{F}$ through $C$ can be calculated as

$$\int_C \overline{F}\cdot\hat{n}ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

**Solved Examples**

**Example 3.2.1.** *If* $\overline{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$, *evaluate* $\int\limits_C \overline{F}\cdot d\overline{r}$, *where* $C$ *is the arc of the parabola* $y = 2x^2$ *from* $(0,0)$ *to* $(1,2)$.

**Solution.** Here,

$$\overline{F}\cdot d\overline{r} = (3xy\hat{\imath} - y^2\hat{\jmath})\cdot(dx\hat{\imath} + dy\hat{\jmath}) = 3xy\ dx - y^2dy$$

and

$$y = 2x^2 \quad \Rightarrow \quad dy = 4xdx.$$

Now,

$$\overline{F}\cdot d\overline{r} = 3x(2x^2)dx - 4x^4(4x)dx = (6x^3 - 16x^5)dx.$$

Thus

$$\int_C \overline{F}\cdot d\overline{r} \;=\; \int_0^1 (6x^3 - 16x^5)dx \;=\; 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1 \;=\; \frac{3}{2} - \frac{8}{3} \;=\; -\frac{7}{6}. \quad \blacksquare$$

**Example 3.2.2.** *Find the work done when a force*

$$\overline{F} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$$

*moves a particle in the* $xy$*-plane from* $(0,0)$ *to* $(1,1)$ *along the parabola* $y^2 = x$. *Is the work done different when the path is the straight line* $y = x$?

**Solution.** The parametric equations of $y^2 = x$ are

$$y = t,\ x = t^2 \quad (0 < t < 1).$$

Now, the work done along the parabola is

$$\begin{aligned}
\int_C \overline{F}\cdot d\overline{r} &= \int_C [(x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}]\cdot(dx\hat{\imath} + dy\hat{\jmath}) \\
&= \int_C [(x^2 - y^2 + x)dx - (2xy + y)dy] \\
&= \int_0^1 [(t^4 - t^2 + t^2)\cdot 2tdt - (2t^2\cdot t + t)dt] \\
&= \left[\frac{t^6}{3} - \frac{t^4}{2} - \frac{t^2}{2}\right] \\
&= -\frac{2}{3}.
\end{aligned}$$

Now, let $P = x^2 - y^2 + x$ and $Q = -2xy - y$. Then

$$\frac{\partial P}{\partial y} = -2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = -2y.$$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\overline{F}$ is conservative and $\int \overline{F}\cdot d\overline{r}$ is independent of path. Hence, the work done along the path $y = x$ is also $-\dfrac{2}{3}$. $\blacksquare$

**Example 3.2.3.** *Find the flux of the vector field* $\overline{F} = -y\hat{\imath} + x\hat{\jmath}$ *across the ellipse* $\overline{r}(t) = (\cos t)\hat{\imath} + (4\sin t)\hat{\jmath}, 0 \le t \le 2\pi$.

**Solution.** Here,

$$M = -y, \quad N = x, \quad \text{and} \quad x = \cos t, \quad y = 4\sin t.$$

Now, the flux of $\overline{F}$ across $C$ is

$$\int_C \overline{F}\cdot\hat{n}ds = \oint_C Mdy - Ndx = \int_0^{2\pi}(4\sin t\cdot 4\cos t + \cos t\cdot\sin t)dt = \frac{17}{2}\int_0^{2\pi}\sin 2t = 0 \text{ (verify!).} \blacksquare$$

**Example 3.2.4.** *Use the Divergence Theorem to find the flux of* $\overline{F}(x,y) = xy\hat{\imath} + 3\hat{\jmath}$ *through the rectangle with vertices* $(0,0)$, $(0,3)$, $(6,3)$, $(6,0)$ *traversed in clockwise direction.*

**Solution.** The path $C$ is a rectangle traced clockwise, not counterclockwise as is required by the Divergence Theorem. We can proceed, but must negate the final result to account for the clockwise movement. We have

$$\iint_R\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA = \int_0^6\int_0^2 ydydx = \int_0^6\left[\frac{y^2}{2}\right]_0^3 = \int_0^6\frac{9}{2}dx = \frac{9}{2}\cdot 6 = 27. \quad \blacksquare$$

Since the path around the rectangle is traced clockwise, we negate the result:

$$\int_C \overline{F}\cdot\hat{n}ds = -27. \quad \blacksquare$$

## Exercises

**Exercise 3.2.1.** *Evaluate* $\int\limits_C \overline{F}\cdot d\overline{r}$ *where* $\overline{F} = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$ *and* $C$ *is the arc of the curve* $\overline{r} = \cos t\hat{\imath} + \sin t\hat{\jmath} + t\hat{k}$ *from* $t = 0$ *to* $t = 2\pi$.

**Exercise 3.2.2.** *Find the total work done in moving a particle in a force field given by* $\overline{f} = 3xy\hat{\imath} - 5z\hat{\jmath} + 10x\hat{k}$ *along the curve* $x = t^2+1,\ y = 2t^2,\ z = t^3$ *from* $t = 1$ *to* $t = 2$.

**Exercise 3.2.3.** *Find the circulation of* $\overline{F}$ *around the curve* $C$ *where* $\overline{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ *and* $C:\ x^2+y^2 = 1,\ z = 0$.

**Exercise 3.2.4.** *Evaluate* $\int\limits_C \overline{F}\cdot d\overline{r}$ *where* $\overline{F} = (x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}$ *and* $C$ *is the boundary of the region enclosed by* $y = 0,\ x = 1,\ y = 2,\ x = 0$.

**Exercise 3.2.5.** *Evaluate* $\int\limits_C \overline{F}\cdot d\overline{r}$ *where* $\overline{F} = [2z, x, -y]$ *and* $C:\overline{r}(t) = [\cos t, \sin t, 2t]$ *from* $(0,0,0)$ *to* $(1,0,4\pi)$.

**Exercise 3.2.6.** *Check whether the line integral* $\int\limits_C \overline{F}d\overline{r}$ *is independent of path, where*

(a) $\overline{F}(x,y) = (2x+y^2)\hat{\imath} + (3y-x^2)\hat{\jmath}$, (b) $\overline{F}(x,y) = (x^2+y^2)\hat{\imath} + (3y+2xy)\hat{\jmath}$.

**Exercise 3.2.7.** *Show that the line integral* $\int\limits_C \overline{F}\cdot d\overline{r}$ *is independent of path, where*

$$\overline{F}(x,y,z) = (2xy+z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k}.$$

*Hence, evaluate the integral when* $C$ *is any path joining* $A(1,-2,1)$ *to* $B(3,1,4)$.

**Exercise 3.2.8.** *Find the flux of the field* $\overline{F} = xy\hat{i} - y^2\hat{j}$ *across the parabola* $\overline{r}(t) = t\hat{i} + t^2\hat{j}$, *where* $0 \leq t \leq 1$.

**Exercise 3.2.9.** *Find the flux of the field* $\overline{F} = (x+y)\hat{i} - (x^2+y^2)\hat{j}$ *outward across the triangle with vertices* $(1,0), (0,1), (-1,0)$.

**Answers**

**3.2.1** $3\pi$ **3.2.2** 303 **3.2.3** $-\pi$ **3.2.4** $-8$ **3.2.5** $9\pi$ **3.2.6** (b) independent of path

**3.2.7** 202 **3.2.8** $\frac{3}{5}$ **3.2.9** 0

## 3.3 Tutorial : Green's Theorem

Green's theorem gives the relationship between a line integral and a double integral. The theorem is as follow:

*If $M(x,y)$, $N(x,y)$ and their partial derivatives $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are continuous in some region $R$ of xy-plane bounded by a closed curve $C$, then*

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

- From this theorem it is derived that the area $A$ of a region $R$ bounded by a closed curve $C$ is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx).$$

- In polar coordinates it is given by

$$A = \frac{1}{2}\oint_C r^2 d\theta.$$

**Solved Examples**

**Example 3.3.1.** *Using Green's theorem, evaluate the integral $\oint_C \overline{F} \cdot \overline{dr}$, where*

$$\overline{F} = (y^2 - 7y)\hat{\imath} + (2xy + 2x)\hat{\jmath} \quad \text{and} \quad C:\ x^2 + y^2 = 1.$$

**Solution.** Observe that

$$\oint_C \overline{F} \cdot dr = \oint_C \left[(y^2 - 7y)dx + (2xy + 2x)dy\right].$$

By Green's theorem,

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Here, $M = y^2 - 7y,\ N = 2xy + 2x$. Therefore,

$$\frac{\partial N}{\partial x} = 2y + 2 \quad \text{and} \quad \frac{\partial M}{\partial y} = 2y - 7.$$

Hence,

$$\oint_C \overline{F} \cdot \overline{dr} = \iint_R [2y + 2 - 2y + 7]dxdy = \iint_R 9dxdy = 9 \times \text{Area of } C = 9\pi. \quad \blacksquare$$

**Example 3.3.2.** *Using Green's theorem obtain the area of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

**Solution.** The parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are

$$x = a\cos t,\ y = b\sin t \qquad (0 \le t \le 2\pi).$$

By Green's theorem the area $A$ of a region $R$ bounded by a closed curve $C$ is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_C [(a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt] \\
&= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)dt \\
&= \frac{1}{2}ab\big[t\big]_0^{2\pi} \\
&= \pi ab. \quad \blacksquare
\end{aligned}
$$

## Exercises

**Exercise 3.3.1.** *State Green's theorem and also evaluate the integral*

$$\oint_C [(6y + x)dx + (y + 2x)dy]$$

*where* $C$*:* $(x-2)^2 + (y-3)^2 = 4$.

**Exercise 3.3.2.** *Use Green's theorem to evaluate the integral*

$$\oint_C (y^2 dx + x^2 dy),$$

*where* $C$*: the triangle bounded by* $x = 0,\ x + y = 1,\ y = 0$.

**Exercise 3.3.3.** *Use Green's theorem to find the work done by* $\overline{F} = (4x - 2y)\hat{\imath} + (2x - 4y)\hat{\jmath}$ *in moving a particle once clockwise around the circle* $(x-2)^2 + (y-2)^2 = 4$.

**Exercise 3.3.4.** *Verify Green's theorem for the function* $\overline{F} = (x+y)\hat{\imath} + 2xy\hat{\jmath}$ *and* $C$ *is the rectangle in the* $xy$*-plane bounded by* $x = 0, y = 0, x = a, y = b$.

## Answers

**3.3.1** $-16\pi$ **3.3.2** $0$ **3.3.3** $16\pi$

## 3.4 Tutorial : Review Exercises

**Exercise 3.4.1.** *Find div($\overline{v}$) for the following vectors $\overline{v}$:*

(a) $e^x(\cos y\hat{i} + \sin y\hat{j})$

(b) $(x^2+y^2)^{-1}(-y\hat{i}+x\hat{j})$

(c) $(x^2+y^2+z^2)^{-3/2}(x\hat{i}+y\hat{j}+z\hat{)}$

(d) $v_1(y,z)\hat{i}+v_2(z,x)\hat{j}+v_3(z,y)\hat{k}$

**Exercise 3.4.2.** *Find curl($\overline{v}$) for the following vectors $\overline{v}$:*

(a) $[\sin y, \cos z, 0]$

(b) $\frac{1}{2}(x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$

(c) $(x^2+y^2+z^2)^{-3/2}(x\hat{i}+y\hat{j}+z\hat{)}$

(d) $xyz(x\hat{i}+y\hat{j}+z\hat{k})$

**Exercise 3.4.3.** *In each case, the velocity vector $\overline{v}$ is given. Is the flow irrotational? Incompressible? Find the paths of particles:*

(a) $[2y^2, 0, 0]$

(b) $[\sec x, \operatorname{cosec} x, 0]$

(c) $x^3\hat{x}$

(d) $x\hat{i}+y\hat{j}-z\hat{k}$

**Exercise 3.4.4.** *Show that the vector function $\overline{f}$ defined by*

$$\overline{f} = (\sin y + z\cos x)\hat{i} + (x\cos y + \sin z)\hat{j} + (y\cos z + \sin x)\hat{k}$$

*is irrotational and find a function $\phi$ such that $\overline{f} = \nabla\phi$.*

**Exercise 3.4.5.** *Which curves are represented by the following parametric representations?*

(a) $[3\cos t, 4\sin t, t]$

(b) $[a+2\cos 2t, b-2\sin 2t, 0]$

**Exercise 3.4.6.** *Represent the curve $4x^2-3y^2=12,\ z=1$ parametrically and sketch it.*

**Exercise 3.4.7.** *Find the arc length of the curve $\overline{r}(t) = [\cos t, t+\sin t]$, $0 \le t \le \pi$.*

**Exercise 3.4.8.** *Find the arc length of the curve $\overline{r}(t) = a\cos^3 t\hat{i} + a\sin^3 t\hat{j}$, total length.*

**Exercise 3.4.9.** *If $\overline{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \overline{F}\cdot d\overline{r}$, where $C$ is the curve in the $xy$-plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.*

**Exercise 3.4.10.** *Evaluate the line integral $\int_C [(x^2+xy)dx + (x^2+y^2)dy]$, where $C$ is the square formed by the lines $y=\pm 1$ and $x=\pm 1$.*

**Exercise 3.4.11.** *Compute the line integral $\int_C (y^2dx - x^2dy)$ about the triangle whose vertices are $(1,0)$, $(0,1)$ and $(-1,0)$.*

**Exercise 3.4.12.** *If $\overline{F} = [\exp(y^{3/2}), -\exp(x^{3/2})]$, evaluate $\int_C \overline{F}\cdot d\overline{r}$, where $C : y = x^{3/2}$ is the curve from $(0,0)$ to $(1,1)$.*

**Exercise 3.4.13.** *If $\overline{F} = 2z\hat{i} + x\hat{j} - y\hat{k}$, evaluate $\int_C \overline{F}\cdot d\overline{r}$, where $C : \overline{r}(t) = \cos t\hat{i} + \sin t\hat{j} + 2t\hat{k}$ is the curve from $(0,0,0)$ to $(1,0,4\pi)$.*

**Exercise 3.4.14.** *If* $\overline{F} = [\cosh x, \sinh y, e^z]$*, evaluate* $\int\limits_C \overline{F} \cdot d\overline{r}$*, where* $C : \overline{r}(t) = [t, t^2, t^3]$ *is the curve from* $(0, 0, 0)$ *to* $(2, 4, 8)$*.*

**Exercise 3.4.15.** *Find the work done in moving a particle in the force field* $\overline{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$ *along the straight line from* $(0, 0, 0)$ *to* $(2, 1, 3)$*.*

**Exercise 3.4.16.** *Show that the form under the integral sign is exact and evaluate the integral*

$$\int_{(0,\pi)}^{(3,\pi/2)} e^x(\cos y dx - \sin y dy).$$

**Exercise 3.4.17.** *Show that the form under the integral sign is exact and evaluate the integral*

$$\int_{(-1,5)}^{(4,3)} (3z^2 dx + 6xzdz).$$

**Exercise 3.4.18.** *Find the flux of* $\overline{F} = (2, 0)$ *through the line segment from* $(3, 0)$ *to* $(0, 3)$*.*

**Exercise 3.4.19.** *Find the flux of* $\overline{F}(x, y) = [3xy, x - y]$ *through the parabolic arc* $y = x^2$ *between* $(-1, 1)$ *and* $(4, 16)$*.*

**Exercise 3.4.20.** *Find the flux of* $\overline{F}(x, y) = [x, y]$ *through the line connecting* $(0, 0)$ *to* $(a, b)$*.*

**Exercise 3.4.21.** *Use the Divergence Theorem to find the flux of* $\overline{F}(x, y) = 3x\hat{\imath} + 5y\hat{\jmath}$ *through the circle* $x^2 + y^2 = 1$*, traversed counterclockwise.*

**Exercise 3.4.22.** *Using Green's theorem, evaluate* $\int\limits_C \overline{F} \cdot d\overline{r}$*, where* $\overline{F} = [3x^2, x - y^4]$ *and* $C$ *is the square with vertices* $(1, 1), (-1, 1), (-1, -1), (1, -1)$*.*

**Exercise 3.4.23.** *Using Green's theorem, evaluate* $\int\limits_C \overline{F} \cdot d\overline{r}$*, where* $\overline{F} = [y, -x]$ *and* $C$ *is the circle* $x^2 + y^2 = 1/4$*.*

**Exercise 3.4.24.** *Using Green's theorem, evaluate* $\int\limits_C \overline{F} \cdot d\overline{r}$*, where* $\overline{F} = [\sin y, \cos x]$ *and* $C$ *is the triangle with vertices* $(0, 0), (\pi, 0), (\pi, 1)$*.*

**Exercise 3.4.25.** *Evaluate* $\oint\limits_C \left[(x^2 + 2y)dx + (4x + y^2)dy\right]$ *by Green's theorem where* $C$ *is the boundary of the region bounded by* $y = 0$*,* $y = 2x$*, and* $x + y = 3$*.*

**Exercise 3.4.26.** *Find the area of the region bounded by the parabola* $y = x^2$ *and the line* $y = x + 2$*.*

**Exercise 3.4.27.** *Find the area of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$*.*