Improper Integral of First Kind

In the definite integral $\int_a^b f(x)\,dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded, then $\int_a^b f(x)\,dx$ is called an improper integral of the first kind.

For Example, $\int_1^{\infty} \frac{dx}{x}$, $\int_{-\infty}^{1} e^{2x}\,dx$, $\int_{-\infty}^{\infty} \frac{dx}{x^2+3x+1}$ are improper integrals of the first kind.

CASE:I

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, (t>a)$$

The improper integral $\int_a^{\infty} f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

CASE:II

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx, (b>t)$$

The improper integral $\int_{-\infty}^{b} f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

CASE:III

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t_1\to-\infty}\int_{t_1}^{b} f(x)\,dx + \lim_{t_2\to-\infty}\int_{b}^{t_2} f(x)\,dx,$$ where b is any real number

The improper $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Improper Integral of Second Kind

In the definite integral $\int_a^b f(x)\,dx$, if both a and b are finite so that the interval of integration is finite but f has one or more points of infinite discontinuity i.e., f is not bounded on $[a,b]$, then $\int_a^b f(x)\,dx$ is called an improper integral of the second kind.

For example, $\int_0^1 \frac{dx}{x^2}, \int_0^3 \frac{dx}{3-x}, \int_0^4 \frac{dx}{x(x-2)}$ are improper integrals of the second kind.

CASE:I

If $f(x)$ becomes infinite at $x=b$ only, we define $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+} \int_a^{b-\epsilon} f(x)\,dx$. The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

CASE:II

If $f(x)$ becomes infinite at $x=a$ only, we define $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+} \int_{a+\epsilon}^{b} f(x)\,dx$. The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely, otherwise it is said to be divergent.

CASE:III

If $f(x)$ becomes infinite at $x=c$ only where $a<c<b$, , we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\epsilon_1\to 0+} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to 0+} \int_{c+\epsilon_2}^{b} f(x)\,dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limit on the right hand side exists finitely and independent of each other, otherwise it is said to be divergent.

Evaluation of Improper Integral of First Kind

Determine if the following integral converges or diverges. If the integral converges determine its value.

Problem: $\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx$

Solution:

Improper because one of the limit of integral is infinite (Type I).

Let's do a u-substitution first.

Let $u = e^x$, then $du = e^x dx$.

When $x = 0, u = 1$ and when $x \to \infty, u \to \infty$:

$$\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx = \int_0^{\infty} \frac{e^x}{\left(e^x\right)^2+3}\,dx = \int_1^{\infty} \frac{1}{u^2+3}\,du = \lim_{t\to\infty}\int_1^{t} \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^t = \lim_{t\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{2} - \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

Problems:

(i) $\int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}\,dx$ (ii) $\int_{-\infty}^{0} 2^x\,dx$ (iii) $\int_{-\infty}^{\infty} \left(y^3 - 3y^2\right)dy$

(iv) $\int_{-\infty}^{\infty} \cos(\pi t)\,dt$ (v) $\int_{-\infty}^{-1} \frac{dx}{x^2+4x+5}$

Ans: (i) Diverges (ii) Converges, $\frac{1}{\ln 2}$ (iii) Diverges

(iv) Diverges (v) converges, $\frac{3\pi}{4}$

Evaluation of Improper Integral of Second Kind

Determine if the following integral converges or diverges. If the integral converges determine its value.

Problem: $\int_0^4 \frac{x}{x^2-9}\,dx$

Solution: Improper because integrand $\frac{x}{x^2-9}$ becomes infinite at $x=3$. 3 lies between the range of integration. (Type II).

We split up the integral at $x = 3$.

$$\int_0^4 \frac{x}{x^2-9}\,dx = \int_0^3 \frac{x}{x^2-9}\,dx + \int_3^4 \frac{x}{x^2-9}\,dx$$

$$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\int_0^t \frac{x}{x^2-9}\,dx + \lim_{t\to 3^+}\int_t^4 \frac{x}{x^2-9}\,dx$$

$$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Bigg|_0^t + \lim_{t\to 3^+}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Bigg|_t^4$$

$$= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right| - \frac{1}{2}\ln(9)\right) + \lim_{t\to 3^+}\left(\frac{1}{2}\ln(7) - \frac{1}{2}\ln\left|t^2-9\right|\right)$$

$$= \left[-\infty - \frac{1}{2}\ln(9)\right] + \left[\frac{1}{2}\ln(7) + \infty\right]$$

This indicates that the First integral tends to $-\infty$ whereas Second integral tends to ∞.
Therefore, each of these integrals is divergent. This in turn means that the integral diverges.

Problems:

(i) $\int_0^5 \frac{1}{\sqrt[3]{2-w}}\,dw$ (ii) $\int_1^2 \frac{1}{\sqrt{2-x}}\,dx$ (iii) $\int_1^4 \frac{1}{x^2+x-6}\,dx$

(iv) $\int_{-2}^2 \frac{dx}{x^3}$ (v) $\int_{-1}^4 \frac{x}{x^2-9}\,dx$

Ans: (i) Converges, $-\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)$ (ii) Converges, 2

(iii) Diverges (iv) Diverges (v) Diverges