MATHEMATICS-II

UNIT-3 : ORDINARY DIFFERENTIAL EQUATIONS

Chapter 5

Higher Order Linear ODEs

The differential equation in which dependent variable and its derivatives occur only in first degree and are not multiplied together is called a *linear differential equation*. The standard form of an n^{th}-order linear ODE is

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + p_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = r(x), \tag{5.0.1}$$

where the *coefficients* $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ and $r(x)$ are functions of x. If $r(x) = 0$ for all x under consideration (usually in some open interval I), then equation (5.0.1) is called *homogeneous*. If $r(x) \neq 0$ for at least one x under consideration, then equation (5.0.1) is called *nonhomogeneous*.

5.1 Tutorial : Homogeneous Linear ODEs with Constant Coefficients

The standard form of an n^{th} order *homogeneous linear ODE with constant coefficients* is

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0, \tag{5.1.1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are constants.

Method of Solution

- Write the given homogeneous linear ODE

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0, \tag{5.1.2}$$

  in symbolic form as

$$(D^n + a_{n-1}D^{n-1} + a_{n-2}D^{n-2} + \cdots + a_0)y = 0, \quad \text{where } D = \frac{d}{dx}$$

- Write the auxiliary equation for (5.1.2) as

$$m^n + a_{n-1}m^{n-1} + a_{n-2}m^{n-2} + \cdots + a_0 = 0 \tag{5.1.3}$$

- Find the roots $m_1, m_2, \ldots, m_n$ of equation (5.1.3). The general solution of the differential equation (5.1.2) depends on the nature of these roots. We have following four possibilities for the roots:

  (1) Distinct real roots: If all the roots are real and distinct, then the general solution of the differential equation (5.1.2) is given by
  $$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

  (2) Equal real roots: If two roots are equal, say $m_1 = m_2$, then the general solution of the differential equation (5.1.2) is given by
  $$y(x) = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$
  Similarly, if three roots are equal, say $m_1 = m_2 = m_3$, then the general solution of the differential equation (5.1.2) is given by
  $$y(x) = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}.$$

  (3) One pair of roots is complex: If one pair of roots is complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution of the differential equation (5.1.2) is given by
  $$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

  (4) Two pairs of complex roots are equal: If two pairs of roots are complex and equal, say
  $$m_1 = m_2 = \alpha + i\beta \quad \text{and} \quad m_3 = m_4 = \alpha - i\beta,$$
  then the general solution of the differential equation (5.1.2) is given by
  $$y(x) = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$

Solved Examples

Example 5.1.1. *Solve the initial value problem* $y'' + y' - 2y = 0$; $y(0) = 4$ *and* $y'(0) = -5$.

Solution. The symbolic form of the given equation is
$$(D^2 + D - 2)y = 0, \quad \text{where } D = \frac{d}{dx}.$$
Therefore, the auxiliary equation is
$$m^2 + m - 2 = 0 \quad \Rightarrow \quad (m+2)(m-1) = 0 \quad \Rightarrow \quad m = -2, 1 \quad \textit{(distinct real roots)}.$$
Thus the general solution is
$$y(x) = c_1 e^{-2x} + c_2 e^{x}. \tag{5.1.4}$$
Differentiating equation (5.1.4), we get
$$y'(x) = -2c_1 e^{-2x} + c_2 e^{x}. \tag{5.1.5}$$
Since $y(0) = 4$, from (5.1.4) we obtain
$$c_1 + c_2 = 4. \tag{5.1.6}$$
Since $y'(0) = -5$, from (5.1.5) we obtain
$$-2c_1 + c_2 = -5. \tag{5.1.7}$$
Solving equations (5.1.6) and (5.1.7), we get $c_1 = 3$ and $c_2 = 1$. Hence, the required particular solution is
$$y(x) = 3e^{-2x} + e^{x}. \quad \blacksquare$$

Example 5.1.2. *Solve the initial value problem*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0, \quad y(0) = 1,\ y'(0) = 0.$$

Solution. The symbolic form of the given equation is

$$(D^2 - 6D + 9)y = 0, \quad \text{where } D = \frac{d}{dx}.$$

Therefore, the auxiliary equation is

$$m^2 - 6m + 9 = 0 \quad \Rightarrow \quad (m-3)^2 = 0 \quad \Rightarrow \quad m = 3, 3 \quad \textit{(equal real roots)}.$$

Thus the general solution of the given equation is

$$y(x) = (c_1 + c_2 x)e^{3x}. \tag{5.1.8}$$

Differentiating equation (5.1.8), we get

$$y'(x) = (c_1 + c_2 x)3e^{3x} + c_2 e^{3x}. \tag{5.1.9}$$

Since $y(0) = 1$, from (5.1.8) we obtain

$$c_1 = 1.$$

Since $y'(0) = 0$, from (5.1.9) we obtain

$$3c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -3c_1 = -3 \quad (\because c_1 = 1).$$

Hence, the required particular solution is

$$y(x) = (1 - 3x)e^{3x}. \quad \blacksquare$$

Example 5.1.3. *Find the general solution of* $16y'' - 8y' + 5y = 0$.

Solution. The symbolic form of the given equation is

$$(16D^2 - 8D + 5)y = 0, \quad \text{where } D = \frac{d}{dx}.$$

Therefore, the auxiliary equation is

$$16m^2 - 8m + 5 = 0$$

$$\Rightarrow \quad m = \frac{8 \pm \sqrt{64 - 320}}{32}$$

$$\Rightarrow \quad m = \frac{8 \pm \sqrt{-256}}{32}$$

$$\Rightarrow \quad m = \frac{8 \pm 16i}{32}$$

$$\Rightarrow \quad m = \frac{1}{4} \pm i\frac{1}{2} \quad \textit{(pair of complex roots)}.$$

Hence, the general solution is given by

$$y(x) = e^{\frac{x}{4}} \left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2} \right). \quad \blacksquare$$

Example 5.1.4. *Find the general solution of the differential equation*

$$\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} + 4y = 0.$$

Solution. The symbolic form of the given equation is

$$(D^4 + 4D^2 + 4)y = 0, \quad \text{where } D = \frac{d}{dx}.$$

Therefore, the auxiliary equation is

$$m^4 + 4m^2 + 4 = 0 \;\Rightarrow\; (m^2+2)^2 = 0 \;\Rightarrow\; m = \pm\sqrt{2}i, \pm\sqrt{2}i \quad \textit{(equal pairs of complex roots).}$$

Hence, the general solution is

$$y(x) = (c_1 + c_2x)\cos\sqrt{2}x + (c_3 + c_4x)\sin\sqrt{2}x. \quad \blacksquare$$

Exercises

Exercise 5.1.1. *Find the general solution of the differential equation*

$$y'' + 4y' - 12y = 0.$$

Exercise 5.1.2. *Solve the initial value problem* $y'' - 4y' + 4y = 0;\; y(0) = 3,\; y'(0) = 1.$

Exercise 5.1.3. *Find the general solution of* $\dfrac{d^4y}{dx^4} - 18\dfrac{d^2y}{dx^2} + 81y = 0.$

Exercise 5.1.4. *Solve* $(D^3 - 3D^2 + 3D - 1)y = 0.$

Exercise 5.1.5. *Solve* $y'' + 2y' + 2y = 0,\; y(0) = 1,\; y(\pi/2) = 0.$

Exercise 5.1.6. *Solve* $y''' - y'' + 100y' - 100y = 0,\; y(0) = 4,\; y'(0) = 11,\; y''(0) = -299.$

Additional Exercises

Exercise 5.1.7. *Find the general solution of* $(D^2 - 2D + 4)y = 0.$

Exercise 5.1.8. *Find the solution of differential equation* $y'' - 5y' + 6y = 0$ *with initial condition* $y(1) = e^2$ *and* $y'(1) = 3e^2.$

Viva Questions

Question 5.1.9. *What is meant by* D *?*

Question 5.1.10. *Define auxiliary equation.*

Question 5.1.11. *Find the general solution of the following differential equation:*

(i) $y'' + 5y' + 4y = 0$; (ii) $y'' - y = 0$; (iii) $(D^2 + 1)y = 0$.

Answers

5.1.1 $c_1e^{-6x} + c_2e^{2x}$ **5.1.2** $(3 - 5x)e^{2x}$ **5.1.3** $(c_1 + c_2x)e^{-3x} + (c_3 + c_4x)e^{3x}$

5.1.4 $(c_1 + c_2x + c_3x^2)e^x$ **5.1.5** $e^{-x}\cos x$ **5.1.6** $e^x + 3\cos 10x + \sin 10x$

5.1.7 $e^x(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$ **5.1.8** e^{3x-1}

5.1.11 (i) $c_1e^{-x} + c_2e^{-4x}$, (ii) $c_1e^x + c_2e^{-x}$, (iii) $c_1\cos x + c_2\sin x$

5.2 Tutorial : Nonhomogeneous Linear ODEs with Constant Coefficients

The standard form of an n^{th} order *nonhomogeneous linear ODE with constant coefficients* is

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1\frac{dy}{dx} + a_0 y = r(x), \tag{5.2.1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are constants and $r(x) \neq 0$ for at least one x under consideration.

Method of Solution

Consider a nonhomogeneous linear ODE with constant coefficients of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x). \tag{5.2.2}$$

- First find the general solution of the corresponding homogeneous equation

  $$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{5.2.3}$$

  by the usual method described in the section 3.2. This solution is called the *complementary function (C.F.)* of (5.2.2). It is denoted by Y_C.

- The symbolic form of (5.2.2) is

  $$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y = r(x) \quad \Rightarrow \quad f(D)y = r(x).$$

  Applying $\dfrac{1}{f(D)}$ (inverse of $f(D)$) on both sides, we obtain

  $$\frac{1}{f(D)}\big(f(D)y\big) = \frac{1}{f(D)}r(x) \quad \Rightarrow \quad y = \frac{1}{f(D)}r(x).$$

  This solution is called the *particular integral (P.I.)* of (5.2.2). It is denoted by Y_P.

- The general solution of (5.2.2) is given by

  $$y = C.F. + P.I. = Y_C + Y_P.$$

Direct Method For Finding Particular Integral

Consider the nonhomogeneous equation of the form

$$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y = r(x) \quad \text{or} \quad f(D)y = r(x). \tag{5.2.4}$$

Then the particular integral (P.I.) is given by

$$\text{P.I.} = Y_P = \frac{1}{f(D)}r(x).$$

The expression of Y_P depends on the nature of $r(x)$. The following are some special cases for $r(x)$:

Case-1. $r(x) = e^{ax}$

In this case,

$$Y_P = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \quad \text{provided } f(a) \neq 0.$$

If $f(a) = 0$, then

$$Y_P = \frac{1}{f(D)}e^{ax} = \frac{x}{f'(a)}e^{ax}, \quad \text{provided } f'(a) \neq 0.$$

If $f'(a) = 0$, then

$$Y_P = \frac{1}{f(D)}e^{ax} = \frac{x^2}{f''(a)}e^{ax}, \quad \text{provided } f''(a) \neq 0$$

and so on. If $f(D) = (D-a)^r e^{ax}$, then

$$Y_P = \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}.$$

Case-2. $r(x) = \cos(ax+b) \quad \text{or} \quad \sin(ax+b)$

In this case,

$$Y_P = \frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \quad \text{provided } f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, then

$$Y_P = \frac{1}{f(D^2)}\cos(ax+b) = \frac{x}{f'(-a^2)}\cos(ax+b), \quad \text{provided } f'(-a^2) \neq 0.$$

If $f''(-a^2) = 0$, then

$$Y_P = \frac{1}{f(D^2)}\cos(ax+b) = \frac{x^2}{f''(-a^2)}\cos(ax+b), \quad \text{provided } f''(-a^2) \neq 0$$

and so on. If $f(D^2) = (D^2+a^2)^2$, then

$$Y_P = \frac{1}{(D^2+a^2)^2}\cos ax = -\frac{1}{4a^2}\cdot\frac{x^2}{2!}\cos ax.$$

The method for $r(x) = \sin(ax+b)$ is similar.

Case-3. $r(x) = x^n$

In this case,

$$Y_P = \frac{1}{f(D)}x^n.$$

Take the constant, if not, then the lowest powered D (with sign) common from $f(D)$ and then expand $\frac{1}{f(D)}$ by either of the following binomial expansions:

$$\frac{1}{1-D} = 1 + D + D^2 + \dots \quad \text{or} \quad \frac{1}{1+D} = 1 - D + D^2 - \dots.$$

Operate the resulting expansion on x^n. We need to expand up to power D^n as higher derivatives vanish.

Case-4. $r(x) = e^{ax}\phi(x)$, where $\phi(x)$ is any function of x

In this case,

$$Y_P = \frac{1}{f(D)}\, e^{ax}\phi(x) = e^{ax}\frac{1}{f(D+a)}\phi(x).$$

Case-5. $r(x) = x\cos ax$ or $x\sin ax$

In this case,

$$Y_P = \frac{1}{f(D)}\, x\cos ax = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}\, \cos ax.$$

The method for $r(x) = x\sin ax$ is similar.

Solved Examples

Example 5.2.1. *Solve the differential equation $y'' + 7y' + 10y = e^{-x}$.*

Solution. The symbolic form of given equation is

$$(D^2 + 7D + 10)y = e^{-x}.$$

First we find Y_C by solving the corresponding homogeneous equation

$$(D^2 + 7D + 10)y = 0.$$

The auxiliary equation is

$$m^2 + 7m + 10 = 0 \quad\Rightarrow\quad (m+2)(m+5) = 0 \quad\Rightarrow\quad m = -2, -5.$$

Thus

$$Y_C = c_1 e^{-2x} + c_2 e^{-5x}.$$

Now

$$\begin{aligned} Y_P &= \frac{1}{f(D)}\, r(x) \\ &= \frac{1}{D^2 + 7D + 10}\, e^{-x} \\ &= \frac{1}{(-1)^2 + 7(-1) + 10}\, e^{-x} \\ &= \frac{1}{4}\, e^{-x}. \end{aligned}$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad\Rightarrow\quad y = c_1 e^{-2x} + c_2 e^{-5x} + \frac{e^{-x}}{4}. \quad \blacksquare$$

Example 5.2.2. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}.$$

Solution. The symbolic form of given equation is

$$(D^2 - 3D + 2)y = e^{2x}.$$

Therefore, the auxiliary equation is

$$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1, 2.$$

Thus

$$Y_C = c_1 e^x + c_2 e^{2x}.$$

Now

$$\begin{aligned} Y_P &= \frac{1}{f(D)}\, r(x) \\ &= \frac{1}{D^2 - 3D + 2}\, e^{2x} \\ &= \frac{x}{2D - 3}\, e^{2x} \quad (\because f(2) = 0) \\ &= \frac{x}{2(2) - 3}\, e^{2x} \\ &= xe^{2x}. \end{aligned}$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1 e^x + c_2 e^{2x} + xe^{2x}. \quad \blacksquare$$

Example 5.2.3. *Find the general solution of the differential equation*

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \sin 2x.$$

Solution. The symbolic form of given equation is

$$(D^3 + D^2 - D - 1)y = \sin 2x.$$

Therefore, the auxiliary equation is

$$\begin{aligned} & m^3 + m^2 - m - 1 = 0 \\ \Rightarrow\ & m^2(m+1) - (m+1) = 0 \\ \Rightarrow\ & (m+1)(m^2 - 1) = 0 \\ \Rightarrow\ & (m+1)(m+1)(m-1) = 0 \\ \Rightarrow\ & m = -1, -1, 1. \end{aligned}$$

Thus

$$Y_C = (c_1 + c_2 x)e^{-x} + c_3 e^x.$$

Now,

$$Y_P = \frac{1}{f(D^2)}\, r(x)$$

$$
\begin{aligned}
&= \frac{1}{DD^2 + D^2 - D - 1}\ \sin 2x \\
&= \frac{1}{D(-2^2) + (-2^2) - D - 1}\ \sin 2x \\
&= \frac{1}{-4D - 4 - D - 1}\ \sin 2x \\
&= \frac{1}{-5D - 5}\ \sin 2x \\
&= -\frac{1}{5}\left[\frac{1}{D+1}\ \sin 2x\right] \\
&= -\frac{1}{5}\left[\frac{D-1}{(D-1)(D+1)}\ \sin 2x\right] \\
&= -\frac{1}{5}\left[\frac{D-1}{D^2-1}\ \sin 2x\right] \\
&= -\frac{1}{5}\left[\frac{D-1}{-2^2-1}\ \sin 2x\right] \\
&= \frac{1}{25}(D-1)\sin 2x \\
&= \frac{1}{25}(D\sin 2x - \sin 2x) \\
&= \frac{1}{25}(2\cos 2x - \sin 2x).
\end{aligned}
$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad \Rightarrow \quad y = (c_1 + c_2 x)e^{-x} + c_3 e^{x} + \frac{1}{25}(2\cos 2x - \sin 2x). \quad \blacksquare$$

Example 5.2.4. *Solve* $(D^4 + 2a^2D^2 + a^4)y = \cos ax$.

Solution. The auxiliary equation equation is

$$m^4 + 2a^2m^2 + a^4 = 0 \quad \Rightarrow \quad (m^2 + a^2)^2 = 0 \quad \Rightarrow \quad m = \pm ia, \pm ia.$$

Thus

$$Y_C = (c_1 + c_2 x)\cos ax + (c_3 + c_4 x)\sin ax.$$

Now,

$$Y_P = \frac{1}{D^4 + 2a^2D^2 + a^4}\cos ax = \frac{1}{(D^2 + a^2)^2}\cos ax = -\frac{1}{4a^2}\cdot\frac{x^2}{2!}\cos ax = -\frac{x^2}{8a^2}\cos ax.$$

Hence, the general solution is given by

$$y = Y_C + Y_P = (c_1 + c_2 x)\cos ax + (c_3 + c_4 x)\sin ax - \frac{x^2}{8a^2}\cos ax. \quad \blacksquare$$

Example 5.2.5. *Solve* $y'' + 2y' + 3y = 2x^2$.

Solution. The symbolic form of the given equation is

$$(D^2 + 2D + 3)y = 2x^2.$$

Therefore, the auxiliary equation is

$$m^2 + 2m + 3 = 0 \quad \Rightarrow \quad m = \frac{-2 \pm \sqrt{4 - 12}}{2} \quad \Rightarrow \quad m = \frac{-2 \pm 2\sqrt{2}i}{2} \quad \Rightarrow \quad m = -1 \pm \sqrt{2}i.$$

Thus

$$Y_C = e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x).$$

Now,

$$\begin{aligned}
Y_P &= \frac{1}{(D^2 + 2D + 3)} 2x^2 \\
&= \frac{1}{3\left[1 + \left(\frac{D^2+2D}{3}\right)\right]} 2x^2 \\
&= \frac{2}{3}\left[1 - \left(\frac{D^2 + 2D}{3}\right) + \left(\frac{D^2 + 2D}{3}\right)^2 - \cdots\right] x^2 \\
&= \frac{2}{3}\left[x^2 - \frac{1}{3}(D^2 + 2D)x^2 + \frac{1}{9}(D^4 + 4D^3 + 4D^2)x^2\right] \\
&= \frac{2}{3}\left[x^2 - \frac{1}{3}(2 + 4x) + \frac{1}{9}(0 + 0 + 8)\right] \\
&= \frac{2}{3}\left[x^2 - \frac{4}{3}x + \frac{2}{9}\right]
\end{aligned}$$

Hence, the general solution is given by

$$y = Y_C + Y_P = e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{2}{3}\left[x^2 - \frac{4}{3}x + \frac{2}{9}\right]. \quad \blacksquare$$

Example 5.2.6. *Solve the initial value problem*

$$y'' + 4y = 8e^{-2x} + 4x^2 + 2, \quad y(0) = 2, \;\; y'(0) = 2.$$

Solution. The symbolic form of the given equation is

$$(D^2 + 4)y = 8e^{-2x} + 4x^2 + 2.$$

Therefore, the auxiliary equation is

$$m^2 + 4 = 0 \quad \Rightarrow \quad m^2 = -4 \quad \Rightarrow \quad m = \pm 2i.$$

Thus

$$Y_C = c_1 \cos 2x + c_2 \sin 2x.$$

Now,

$$Y_P = \frac{1}{D^2 + 4}(8e^{-2x} + 4x^2 + 2)$$

$$
\begin{aligned}
&= 8\left[\frac{1}{D^2+4}\,e^{-2x}\right] + 4\left[\frac{1}{D^2+4}\,x^2\right] + 2\left[\frac{1}{D^2+4}\,1\right] \\
&= 8\left[\frac{1}{(-2)^2+4}\,e^{-2x}\right] + \left[\frac{1}{1+\frac{D^2}{4}}\,x^2\right] + 2\left[\frac{1}{D^2+4}\,e^{0x}\right] \\
&= 8\left[\frac{1}{8}\,e^{-2x}\right] + \left[1-\frac{D^2}{4}+\frac{D^4}{16}-\cdots\right]x^2 + 2\left[\frac{1}{0^2+4}\,e^{0x}\right] \\
&= e^{-2x} + \left[x^2 - \frac{1}{4}D^2(x^2) + \frac{1}{16}D^4(x^2)\right] + \frac{1}{2} \\
&= e^{-2x} + \left[x^2 - \frac{1}{2}\right] + \frac{1}{2} \\
&= e^{-2x} + x^2.
\end{aligned}
$$

Hence, the general solution is given by

$$y = Y_C + Y_P = c_1\cos 2x + c_2\sin 2x + e^{-2x} + x^2. \tag{5.2.5}$$

Using the condition $y(0) = 2$, we get

$$c_1\cos 0 + c_2\sin 0 + e^0 + 0 = 2 \quad\Rightarrow\quad c_1 + 1 = 2 \quad\Rightarrow\quad c_1 = 1.$$

Differentiating (5.2.5) w. r. t. x, we get

$$y'(x) = -2c_1\sin 2x + 2c_2\cos 2x - 2e^{-2x} + 2x.$$

Using the condition $y'(0) = 2$, we get

$$-2c_1\sin 0 + 2c_2\cos 0 - 2e^0 + 2(0) = 2 \quad\Rightarrow\quad 2c_2 - 2 = 2 \quad\Rightarrow\quad c_2 = 2.$$

Thus the required particular solution is

$$\cos 2x + 2\sin 2x + e^{-2x} + x^2. \quad\blacksquare$$

Example 5.2.7. *Solve* $(D^3 - 3D + 2)y = xe^x$.

Solution. The auxiliary equation is

$$m^3 - 3m + 2 = 0 \quad\Rightarrow\quad (m-1)(m^2+m-2) = 0 \quad\Rightarrow\quad (m-1)(m-1)(m+2) = 0 \quad\Rightarrow\quad m = 1, 1, -2.$$

Thus

$$Y_C = (c_1 + c_2x)e^x + c_3e^{-2x}.$$

Now

$$
\begin{aligned}
Y_P &= \frac{1}{f(D)}\,r(x) \\
&= \frac{1}{D^3-3D+2}\,xe^x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(D-1)(D-1)(D+2)}\, xe^x \\
&= e^x \frac{1}{(D)(D)(D+1+2)}\, x \\
&= e^x \frac{1}{D^2(D+3)}\, x \\
&= e^x \frac{1}{D^2}\,\frac{1}{3}\left[\frac{1}{1+D/3}\right] x \\
&= \frac{e^x}{3}\,\frac{1}{D^2}\left[1-\frac{D}{3}+\cdots\right] x \\
&= \frac{e^x}{3}\,\frac{1}{D^2}\left[x-\frac{1}{3}(1)\right] \\
&= \frac{e^x}{3}\,\frac{1}{D}\left[\frac{x^2}{2}-\frac{x}{3}\right] \\
&= \frac{e^x}{3}\left[\frac{x^3}{6}-\frac{x^2}{6}\right] \\
&= \frac{x^2e^x}{18}(x-1).
\end{aligned}
$$

Hence, the general solution is

$$y = Y_C + Y_P \quad \Rightarrow \quad y = (c_1 + c_2x)e^x + c_3e^{-2x} + \frac{x^2e^x}{18}(x-1). \quad \blacksquare$$

Example 5.2.8. *Solve* $(D^2+1)y = x\sin 2x$.

Solution. The auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i.$$

Thus

$$Y_C = c_1\cos x + c_2\sin x.$$

Now,

$$
\begin{aligned}
Y_P &= \frac{1}{f(D)}\, x\sin 2x \\
&= \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}\, \sin 2x \\
&= \left[x - \frac{2D}{D^2+1}\right]\frac{1}{D^2+1}\, \sin 2x \\
&= \left[x - \frac{2D}{D^2+1}\right]\frac{1}{-2^2+1}\, \sin 2x \\
&= \left[x - \frac{2D}{D^2+1}\right]\left(-\frac{1}{3}\right) \sin 2x
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{3}\left[x\sin 2x - 2D\frac{1}{D^2+1}\sin 2x\right] \\
&= -\frac{1}{3}\left[x\sin 2x - 2D\frac{1}{-2^2+1}\sin 2x\right] \\
&= -\frac{1}{3}\left[x\sin 2x + \frac{2}{3}\frac{d}{dx}\sin 2x\right] \\
&= -\frac{1}{3}\left[x\sin 2x + \frac{4}{3}\cos 2x\right] \\
&= -\frac{1}{9}\left[3x\sin 2x + 4\cos 2x\right].
\end{aligned}
$$

Hence, the general solution is

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1\cos x + c_2\sin x - \frac{1}{9}\left(3x\sin 2x + 4\cos 2x\right). \quad \blacksquare$$

Exercises

Exercise 5.2.1. *Solve* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 12y = e^{6x}$.

Exercise 5.2.2. *Solve the non-homogeneous equation* $y'' - 3y' + 2y = e^x$.

Exercise 5.2.3. *Find the particular solution of* $y = \dfrac{1}{(D+1)^2}\cosh x$, *where* $D = \dfrac{d}{dx}$.

Exercise 5.2.4. *Solve* $y''' - 3y'' + 3y' - y = 4e^t$.

Exercise 5.2.5. *Find the general solution of* $\dfrac{d^4y}{dt^4} - 2\dfrac{d^2y}{dt^2} + y = \cos t + e^{2t} + e^t$.

Exercise 5.2.6. *Solve* $(D^2+1)y = \sin x \sin 2x$.

Exercise 5.2.7. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} - 9y = x^2.$$

Exercise 5.2.8. *Solve* $(D^3 - D^2 - 6D)y = x^2 + 1$.

Exercise 5.2.9. *Solve* $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 5y = e^x\cos 3x$.

Exercise 5.2.10. *Solve* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \dfrac{e^{2x}}{x^5}$.

Exercise 5.2.11. *Solve* $(D^2+4)y = x\sin x$.

Exercise 5.2.12. *Solve* $(D^2 - 4D + 3)y = \sin 3x\cos 2x$.

Exercise 5.2.13. *Find the particular solution of* $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$.

Exercise 5.2.14. *Find the general solution of* $(D^2-4)y = x^3e^{2x}$.

Exercise 5.2.15. *Solve the differential equation* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2e^x\cos\left(\dfrac{x}{2}\right)$.

Exercise 5.2.16. *Solve* $(D^2-4D+4)y = \dfrac{e^{2x}}{1+x^2}$, *where* $D = \dfrac{d}{dx}$.

Viva Questions

Question 5.2.17. *When a linear differential equation is nonhomogeneous ?*

Question 5.2.18. *How to give the general solution of a non-homogeneous equation ?*

Question 5.2.19. *What will be P.I. for* $(D^2+2D+1)y = e^{-x}\sin x$ *?*

Question 5.2.20. *How to find P.I. when* $r(x) = e^{ax}$ *?*

Question 5.2.21. *How to find P.I. when* $r(x) = \sin ax$ *?*

Answers

5.2.1 $c_1e^{-4x} + c_2e^{3x} + \frac{e^{6x}}{30}$
5.2.2 $c_1e^{2x} + c_2e^x - xe^x$
5.2.3 $\frac{1}{8}e^x + \frac{1}{4}x^2e^{-x}$
5.2.4 $(c_1 + c_2t + c_3t^2)e^t + \frac{2}{3}t^3e^t$
5.2.5 $(c_1+c_2t)e^{-t} + (c_3+c_4t)e^t + \frac{\cos t}{4} + \frac{e^{2t}}{9} + \frac{t^2e^t}{8}$
5.2.6 $c_1\cos x + c_2\sin x + \frac{1}{2}\left[\frac{x\sin x}{2} + \frac{\cos 3x}{8}\right]$
5.2.7 $c_1e^{3x} + c_2e^{-3x} - \frac{x^2}{9} - \frac{2}{81}$
5.2.8 $c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}x^3 + \frac{1}{36}x^2 - \frac{25}{108}x$
5.2.9 $c_1e^{-x} + e^x(c_2\cos 2x + c_3\sin 2x) - \frac{e^x}{65}(3\sin 3x + 2\cos 3x)$
5.2.10 $(c_1+c_2x)e^{2x} + \frac{1}{12}\frac{e^{2x}}{x^3}$
5.2.11 $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$
5.2.12 $c_1e^x + c_2e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(2\cos x + \sin x)$
5.2.13 x^3
5.2.14 $c_1e^{2x} + c_2e^{-2x} + \frac{e^{2x}}{128}(8x^4 - 8x^3 + 6x^2 - 3x)$
5.2.15 $c_1e^x + c_2e^{2x} - \frac{8}{5}e^x\left(\cos\frac{x}{2} + 2\sin\frac{x}{2}\right)$
5.2.16 $(c_1+c_2x)e^{2x} + e^{2x}\left[x\tan^{-1}x - \frac{1}{2}\log(x^2+1)\right]$
5.2.19 $-e^{-x}\sin x$

5.3 Tutorial : Method of Undetermined Coefficients

This method can be applied when the function $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. In this method we will assume a form of Y_P similar to $r(x)$, but with unknown coefficients to be determined by substituting that Y_P and its derivatives into the given differential equation. The choice of Y_P depending on $r(x)$ and corresponding rules are as follows:

Term in r(x)	**Set of Solutions**	**Choice for $\mathbf{Y}_P$**
Ke^{ax}	$\{e^{ax}\}$	Ce^{ax}
Kx^n	$\{x^n, x^{n-1}, \ldots, x, 1\}$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$K\cos ax$	$\{\cos ax, \sin ax\}$	$K_1 \cos ax + K_2 \sin ax$
$K\sin ax$	$\{\cos ax, \sin ax\}$	$K_1 \cos ax + K_2 \sin ax$
$Ke^{ax}\cos ax$	$\{e^{ax}\cos ax, e^{ax}\sin ax\}$	$e^{ax}(K_1 \cos ax + K_2 \sin ax)$
$Ke^{ax}\sin ax$	$\{e^{ax}\cos ax, e^{ax}\sin ax\}$	$e^{ax}(K_1 \cos ax + K_2 \sin ax)$

Basic Rule

If $r(x)$ is one of the functions in the first column of above table, choose Y_P in the same line and determine its unknown coefficients by substituting Y_P and its derivatives into the given equation.

Modification Rule

If any member from the solution set of $r(x)$ occurs in Y_C corresponding to a simple root, then multiply each member of the set by x (or by x^2 if the member is corresponding to a double root and so on.)

Sum Rule

If $r(x)$ is a sum of functions in the first column of above table, choose for Y_P the sum of the functions in the corresponding lines of the third column.

Solved Examples

Example 5.3.1. *Using the method of undetermined coefficients, solve the differential equation*

$$y'' + 4y = 8x^2.$$

Solution. The symbolic form of the given equation is

$$(D^2 + 4)y = 8x^2.$$

First we find Y_C by solving

$$(D^2 + 4)y = 0.$$

The auxiliary equation is

$$m^2 + 4 = 0 \quad \Rightarrow \quad m^2 = -4 \quad \Rightarrow \quad m = \pm 2i.$$

Thus

$$Y_C = c_1 \cos 2x + c_2 \sin 2x.$$

Now, the solution set of $8x^2$ is $\{x^2, x, 1\}$. Therefore,

$$\begin{aligned} Y_P &= K_2 x^2 + K_1 x + K_0 \\ Y_P' &= 2K_2 x + K_1 \\ Y_P'' &= 2K_2. \end{aligned}$$

Substituting all these values in the given equation, we get

$$\begin{aligned} & 2K_2 + 4K_2 x^2 + 4K_1 x + 4K_0 = 8x^2 \\ \Rightarrow \quad & 4K_2 x^2 + 4K_1 x + (4K_0 + 2K_2) = 8x^2. \end{aligned}$$

Equating the corresponding coefficients on both the sides, we get

$$4K_2 = 8, \quad 4K_1 = 0, \quad 4K_0 + 2K_2 = 0.$$

Solving, we get

$$K_2 = 2. \quad K_1 = 0, \quad K_0 = -1.$$

Thus

$$Y_P = 2x^2 - 1.$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1 \cos 2x + c_2 \sin 2x + 2x^2 - 1. \quad \blacksquare$$

Example 5.3.2. *By the method of undetermined coefficients, find the general solution of*

$$y'' + y = 6 \cos 2x.$$

Solution. The symbolic form of the given equation is

$$(D^2 + 1)y = 6 \cos 2x.$$

Therefore, the auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m^2 = -1 \quad \Rightarrow \quad m = \pm i.$$

Thus

$$Y_C = c_1 \cos x + c_2 \sin x.$$

Now, the solution set of $6 \cos 2x$ is $\{\cos 2x, \sin 2x\}$. Therefore,

$$\begin{aligned} Y_P &= K_1 \cos 2x + K_2 \sin 2x \\ Y_P' &= -2K_1 \sin 2x + 2K_2 \cos 2x \\ Y_P'' &= -4K_1 \cos 2x - 4K_2 \sin 2x. \end{aligned}$$

Substituting all these values in the given equation, we get

$$-4K_1 \cos 2x - 4K_2 \sin 2x + K_1 \cos 2x + K_2 \sin 2x = 6 \cos 2x$$

$$\Rightarrow \quad -3K_1\cos 2x - 4K_2\sin 2x = 6\cos 2x.$$

Equating the corresponding coefficients on both the sides, we get

$$-3K_1 = 6 \quad \text{and} \quad -4K_2 = 0 \qquad \Rightarrow \qquad K_1 = -2 \quad \text{and} \quad K_2 = 0.$$

Thus

$$Y_P = -2\cos 2x.$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1\cos 2x + c_2\sin 2x - 2\cos 2x. \quad \blacksquare$$

Example 5.3.3. *Using the method of undetermined coefficients solve the differential equation*

$$\frac{d^2y}{dx^2} - 4y = e^{-2x} - 2x.$$

Solution. The symbolic form of the given equation is

$$(D^2 - 4)y = e^{-2x} - 2x.$$

Therefore, the auxiliary equation is

$$m^2 - 4 = 0 \quad \Rightarrow \quad (m-2)(m+2) = 0 \quad \Rightarrow \quad m = 2, -2.$$

Thus

$$Y_C = c_1e^{2x} + c_2e^{-2x}.$$

Now, the solution sets of e^{-2x} and $-2x$ are $\{e^{-2x}\}$ and $\{x, 1\}$ respectively. Since e^{-2x} occurs in Y_C corresponding to a simple root, we have to modify the first solution set as $\{xe^{-2x}\}$. Thus

$$\begin{aligned} Y_P &= Axe^{-2x} + Bx + C \\ Y_P' &= -2Axe^{-2x} + Ae^{-2x} + B \\ Y_P'' &= 4Axe^{-2x} - 2Ae^{-2x} - 2Ae^{-2x}. \end{aligned}$$

Substituting all these values in the given equation, we get

$$\begin{aligned} & 4Axe^{-2x} - 4Ae^{-2x} - 4Axe^{-2x} - 4Bx - 4C = e^{-2x} - 2x \\ \Rightarrow \quad & -4Ae^{-2x} - 4Bx - 4C = e^{-2x} - 2x. \end{aligned}$$

Equating the corresponding coefficients on both the sides, we get

$$-4A = 1, \quad -4B = -2, \quad -4C = 0.$$

Solving, we get

$$A = -\frac{1}{4}, \quad B = \frac{1}{2}, \quad C = 0.$$

Therefore,

$$Y_P = -\frac{1}{4}xe^{-2x} + \frac{1}{2}x.$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}xe^{-2x} + \frac{1}{2}x. \quad \blacksquare$$

Exercises

Exercise 5.3.1. *Find the solution of differential equation $y'' + 4y = 2\sin 3x$ by the method of undetermined coefficients.*

Exercise 5.3.2. *Using the method of undetermined coefficients, find the general solution of the differential equation $y'' + 2y' + 10y = 25x^2 + 3$.*

Exercise 5.3.3. *Using the method of undetermined coefficients, find the general solution of*

$$y'' + 8y' + 16y = 64\cosh 4x.$$

Hint: Use the formula $\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$

Exercise 5.3.4. *Use the method of undetermined coefficients to solve the initial value problem*

$$y'' + 4y = 16\cos 2x, \quad y(0) = 0,\ y'(0) = 0.$$

Exercise 5.3.5. *Solve the initial value problem*

$$y'' + y' = 2 + 2x + x^2, \quad y(0) = 8,\ y'(0) = -1$$

using the method of undetermined coefficients.

Viva Questions

Question 5.3.6. *For which $r(x)$, the method of undetermined coefficients is applicable ?*

Question 5.3.7. *For the differential equation $y'' + y = x^2 + 4$, give the form of Y_P.*

Question 5.3.8. *How to modify the form of Y_P if one of the terms in Y_P is a solution of the corresponding homogeneous equation (i.e., the term occurs in Y_C) ?*

Question 5.3.9. *For the differential equation $y'' + 4y = e^{4x} + \sin 2x$, give the form of Y_P.*

Answers

5.3.1 $c_1\cos 2x + c_2\sin 2x - \frac{2}{5}\sin 3x$
5.3.3 $(c_1 + c_2x)e^{-4x} + \frac{1}{2}e^{4x} + 16x^2e^{-4x}$
5.3.4 $4x\sin 2x$
5.3.5 $3e^{-x} + 5 + 2x + \frac{1}{3}x^3$
5.3.7 $Ax^2 + Bx + C$
5.3.9 $Ae^{2x} + Bx\cos 2x + Cx\sin 2x$

5.4 Tutorial : Method of Variation of Parameters

The method of undetermined coefficients is restricted to functions $r(x)$ whose derivatives are of a form similar to $r(x)$ itself. Of course direct method is applicable almost in all cases but some times turns out to be complicated. So, it is required to establish a more general method. One such method, called the method of variation of parameters. We now discuss this method.

Method for Second Order Linear ODEs

Consider a nonhomogeneous linear ODEs of the form

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{(where } p(x) \text{ and } q(x) \text{ are continuous)}.$$

Suppose that

$$Y_C(x) = c_1y_1(x) + c_2y_2(x).$$

Then

$$Y_P(x) = -y_1(x)\int \frac{y_2(x)}{W(x)}r(x)dx + y_2(x)\int \frac{y_1(x)}{W(x)}r(x)dx,$$

where W is the Wronskian of y_1, y_2, i.e.,

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Method for Third Order Linear ODEs

Consider a nonhomogeneous linear ODE of the form

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = r(x) \quad \text{(where } p_0(x),\ p_1(x) \text{ and } p_2(x) \text{ are continuous)}.$$

Suppose that

$$Y_C(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x).$$

Then

$$Y_P(x) = y_1(x)\int \frac{W_1(x)}{W(x)}r(x)dx + y_2(x)\int \frac{W_2(x)}{W(x)}r(x)dx + y_3(x)\int \frac{W_3(x)}{W(x)}r(x)dx,$$

where

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = y_2y_3' - y_3y_2', \quad W_2 = y_3y_1' - y_1y_3', \quad W_3 = y_1y_2' - y_2y_1'.$$

Solved Examples

Example 5.4.1. *Using the method of variation of parameters solve the differential equation*

$$y'' + y = \operatorname{cosec} x.$$

Solution. The symbolic form of the given equation is

$$(D^2 + 1)y = \operatorname{cosec} x.$$

First we find Y_C by solving the corresponding homogeneous equation

$$(D^2 + 1)y = 0.$$

The auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m^2 = -1 \quad \Rightarrow \quad m = \pm i.$$

Thus

$$Y_C = c_1 \cos x + c_2 \sin x.$$

Comparing this with

$$Y_C = c_1 y_1 + c_2 y_2,$$

we obtain

$$y_1 = \cos x, \quad y_2 = \sin x.$$

Now

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

By the method of variation of parameters, we have

$$\begin{aligned} Y_P &= -y_1(x) \int \frac{y_2(x)}{W(x)} r(x) dx + y_2(x) \int \frac{y_1(x)}{W(x)} r(x) dx \\ &= -\cos x \int \frac{\sin x}{1} \operatorname{cosec} x dx + \sin x \int \frac{\cos x}{1} \operatorname{cosec} x dx \\ &= -\cos x \int dx + \sin x \int \cot x dx \\ &= -x \cos x + \sin x \cdot \ln|\sin x|. \end{aligned}$$

Hence, the general solution is

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \cdot \ln|\sin x|. \quad \blacksquare$$

Example 5.4.2. *Find the general solution of the differential equation*

$$y''' - 6y'' + 11y' - 6y = e^{-x}$$

using the method of variation of parameters.

Solution. The symbolic form of the given equation is

$$(D^3 - 6D^2 + 11D - 6)y = e^{-x}.$$

First we find Y_C by solving the corresponding homogeneous equation

$$(D^3 - 6D^2 + 11D - 6)y = 0.$$

The auxiliary equation is

$$\begin{aligned} & m^3 - 6m^2 + 11m - 6 = 0 \\ \Rightarrow\ & (m-1)(m^2 - 5m + 6) = 0 \\ \Rightarrow\ & (m-1)(m-2)(m-3) = 0 \\ \Rightarrow\ & m = 1, 2, 3. \end{aligned}$$

Thus

$$Y_C = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Comparing this with

$$Y_C = c_1 y_1 + c_2 y_2 + c_3 y_3,$$

we obtain

$$\begin{aligned} & y_1 = e^x, \quad y_2 = e^{2x}, \qquad y_3 = e^{3x} \\ \Rightarrow\ & y_1' = e^x, \quad y_2' = 2e^{2x}, \qquad y_3' = 3e^{3x} \\ \Rightarrow\ & y_1'' = e^x, \quad y_2'' = 4e^{2x}, \qquad y_3'' = 9e^{3x}. \end{aligned}$$

By the method of variation of parameters, we have

$$Y_P(x) = y_1(x)\int \frac{W_1(x)}{W(x)} r(x)dx + y_2(x)\int \frac{W_2(x)}{W(x)} r(x)dx + y_3(x)\int \frac{W_3(x)}{W(x)} r(x)dx. \tag{5.4.1}$$

Now

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ 0 & e^{2x} & 2e^{3x} \\ 0 & 3e^{2x} & 8e^{3x} \end{vmatrix} = e^x(8e^{5x} - 6e^{5x}) = 2e^{6x}; \\ W_1 &= y_2 y_3' - y_3 y_2' = \left(e^{2x}\right)\left(3e^{3x}\right) - \left(e^{3x}\right)\left(2e^{2x}\right) = 3e^{5x} - 2e^{5x} = e^{5x}; \\ W_2 &= y_3 y_1' - y_1 y_3' = \left(e^{3x}\right)\left(e^{x}\right) - \left(e^{x}\right)\left(3e^{3x}\right) = e^{4x} - 3e^{4x} = -2e^{4x}; \\ W_3 &= y_1 y_2' - y_2 y_1' = \left(e^{x}\right)\left(2e^{2x}\right) - \left(e^{2x}\right)\left(e^{x}\right) = 2e^{3x} - e^{3x} = e^{3x}. \end{aligned}$$

Substituting all these values in Equation (5.4.1), we get

$$\begin{aligned} Y_P &= e^x \int \frac{e^{5x}}{2e^{6x}} e^{-x} dx + e^{2x} \int \frac{-2e^{4x}}{2e^{6x}} e^{-x} dx + e^{3x} \int \frac{e^{3x}}{2e^{6x}} e^{-x} dx \\ &= \frac{1}{2} e^x \int e^{-2x} dx - e^{2x} \int e^{-3x} dx + \frac{1}{2} e^{3x} \int e^{-4x} dx \\ &= \frac{1}{2} e^x \left(\frac{e^{-2x}}{-2}\right) - e^{2x}\left(\frac{e^{-3x}}{-3}\right) + \frac{1}{2} e^{3x}\left(\frac{e^{-4x}}{-4}\right) \\ &= -\frac{1}{4} e^{-x} + \frac{1}{3} e^{-x} - \frac{1}{8} e^{-x} \\ &= -\frac{1}{24} e^{-x}. \end{aligned}$$

Hence, the general solution is

$$y = Y_C + Y_P \quad \Rightarrow \quad y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{24} e^{-x}. \quad \blacksquare$$

Exercises

Exercise 5.4.1. *Solve* $y'' + 9y = \sec 3x$ *by the method of variation of parameters.*

Exercise 5.4.2. *Solve* $(D^2 - 3D + 2)y = \frac{e^x}{1+e^x}$ *by method of variation of parameters.*

Exercise 5.4.3. *Use the method of variation of parameters to find the general solution of*

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}.$$

Exercise 5.4.4. *Solve* $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$ *by method of variation of parameters.*

Exercise 5.4.5. *Solve the differential equation* $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = \operatorname{cosec} x$ *by method of variation of parameters.*

Exercise 5.4.6. *Using the method of variation of parameters find the general solution of the differential equation*

$$(D^2 - 2D + 1)y = 3x^{3/2} e^x.$$

Exercise 5.4.7. *Solve* $(D^2 + 2D + 2)y = 4e^{-x} \sec^3 x$ *using the method of variation of parameters.*

Viva Questions

Question 5.4.8. *When should we use the method of variation of parameters ?*

Question 5.4.9. *In the method of variation of parameters, what is the form of* Y_P *for a second order linear ODE ?*

Question 5.4.10. *In the method of variation of parameters, what is the form of* Y_P *for a third order linear ODE ?*

Answers

5.4.1 $c_1 \cos 3x + c_2 \sin 3x + \frac{1}{9} \cos 3x \log \cos 3x + \frac{1}{3} x \sin 3x$
5.4.2 $c_1 e^x + c_2 e^{2x} + e^x \log(1 + e^{-x}) - e^x + e^{2x} \log(1 + e^{-x})$
5.4.3 $(c_1 + c_2 x)e^{2x} + (\ln x - 1)xe^{2x}$
5.4.4 $c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$
5.4.5 $c_1 + c_2 \cos x + c_3 \sin x + \log(\operatorname{cosec} x - \cot x) - \cos x \log \sin x - x \sin x$
5.4.6 $(c_1 + c_2 x)e^x + \frac{12}{35} x^{7/2} e^x$
5.4.7 $e^{-x}(c_1 \cos x + c_2 \sin x) + 2e^{-x} \sin^2 x \sec x$

5.5 Tutorial : Euler-Cauchy Equations

Definition

An equation of the form

$$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y^{(1)} + a_0 y = r(x), \tag{5.5.1}$$

where $a_0,\ a_1, \cdots, a_{n-1}$ are constants, is called an *Euler-Cauchy Equation* of order n.

Method of Solution

- For a given Euler-Cauchy equation, substitute $x = e^z$ or $z = \ln x$. Then we obtain
$$xy' = Dy, \quad x^2 y'' = D(D-1)y, \quad x^3 y''' = D(D-1)(D-2)y \quad \left(\text{where } D = \frac{d}{dz}\right)$$
and so on.
- Substitute all these values in the given equation. Now the equation becomes a linear differential equation with constant coefficients. Solve it by the usual methods discussed in the previous chapter. Here the solution will be in the form $y \equiv y(z)$.
- Replace z by $\ln x$ or e^z by x to obtain the general solution of the given equation.

Solved Examples

Example 5.5.1. *Solve* $x^3 \dfrac{d^3y}{dx^3} + 7x\dfrac{dy}{dx} - 27y = 0$.

Solution. Let $x = e^z$ or $z = \ln x$. Then

$$x\frac{dy}{dx} = Dy, \quad x^3 \frac{d^2y}{dx^3} = D(D-1)(D-2)y \quad \left(\text{where } D = \frac{d}{dz}\right).$$

Substituting all these values in the given equation, we get

$$\begin{aligned} & [D(D-1)(D-2) + 7D - 27]y = 0 \\ \Rightarrow\ & [D(D^2 - 3D + 2) + 7D - 27]y = 0 \\ \Rightarrow\ & (D^3 - 3D^2 + 9D - 27)y = 0 \end{aligned}$$

which is a linear differential equation with constant coefficients. The auxiliary equation is

$$\begin{aligned} & m^3 - 3m^2 + 9m - 27 = 0 \\ \Rightarrow\ & m^2(m-3) + 9(m-3) = 0 \\ \Rightarrow\ & (m-3)(m^2+9) = 0 \\ \Rightarrow\ & m = 3, \pm 3i. \end{aligned}$$

Hence, the general solution is

$$y = c_1 e^{3z} + c_2 \cos 3z + c_3 \sin 3z.$$

Replacing z by $\ln x$ or e^z by x, we obtain

$$y = c_1 x^3 + c_2 \cos(3 \ln x) + c_3 \sin(3 \ln x). \quad \blacksquare$$

Example 5.5.2. *Solve* $x^2y'' + 3xy' + y = x^2 \log x$.

Solution. Let $x = e^z$ or $z = \ln x$. Then

$$xy' = Dy, \quad x^2y'' = D(D-1)y \quad \left(\text{where } D = \frac{d}{dz}\right).$$

Substituting all these values in the given equation, we get

$$\begin{aligned} [D(D-1) + 3D + 1]y &= e^{2z}z \\ \Rightarrow \quad (D^2 + 2D + 1)y &= e^{2z}z. \end{aligned}$$

which is a linear differential equation with constant coefficients. The auxiliary equation is

$$m^2 + 2m + 1 = 0 \quad \Rightarrow \quad (m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1.$$

Thus

$$Y_C = (c_1 + c_2 z)e^{-z}.$$

Now

$$\begin{aligned} Y_P &= \frac{1}{(D^2 + 2D + 1)^2} e^{2z} z \\ &= \frac{1}{(D+1)^2} e^{2z} z \\ &= e^{2z} \frac{1}{(D+3)^2} z \\ &= e^{2z} \frac{1}{D^2 + 6D + 9} z \\ &= \frac{e^{2z}}{9} \left(\frac{1}{1 + \frac{D^2+6D}{9}} \right) z \\ &= \frac{e^{2z}}{9} \left(1 - \frac{D^2 + 6D}{9} + \cdots \right) z \\ &= \frac{e^{2z}}{9} \left(z - \frac{2}{3} \right) \\ &= \frac{e^{2z}}{27} (3z - 2). \end{aligned}$$

Hence, the general solution is given by

$$y = Y_C + Y_P \quad \Rightarrow \quad y = (c_1 + c_2 z)e^{-z} + \frac{e^{2z}}{27}(3z - 2).$$

Replacing z by $\ln x$ or e^z by x, we obtain

$$y = \frac{c_1 + c_2 \ln x}{x} + \frac{x^2}{27}(3 \ln x - 2). \quad \blacksquare$$

Exercises

Exercise 5.5.1. *Solve the Euler-Cauchy equation* $x^2y'' - 7xy' + 16y = 0$.

Exercise 5.5.2. *Solve* $(x^2D^2 - 3xD + 4)y = 0,\ y(1) = 0,\ y'(1) = 3$.

Exercise 5.5.3. *Solve* $(x^2D^2 + xD - 9)y = 48x^5$.

Exercise 5.5.4. *Find the general solution of the equation* $(x^2D^2 - 2xD + 2)y = x^3 \cos x$.
[**Hint:** Use method of variation of parameters to find P.I.]

Exercise 5.5.5. *Find the general solution of the differential equation*

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$

Exercise 5.5.6. *Solve* $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = 3\log x - 4$.

Viva Questions

Question 5.5.7. *What is the standard form of an Euler-Cauchy equation of* n^{th} *order ?*

Question 5.5.8. *How to solve an Euler-Cauchy equation ?*

Question 5.5.9. *Which of the following are Euler-Cauchy equations ?*

(i) $x^3y''' - 5xy' + 6y = \sin x$

(ii) $xy'' + 10y' = 12x^7$

(iii) $xy''' + y' = 0$

(iv) $xy''' + 3y'' = e^x$

Answers

5.5.1 $(c_1 + c_2 \ln x)x^4$
5.5.2 $3x^2 \log x$
5.5.3 $c_1x^3 + c_2x^{-3} + 3x^5$
5.5.4 $c_1x + c_2x^2 - x\cos x$
5.5.5 $\frac{c_1}{x} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + \frac{2}{x}\log x$
5.5.6 $c_1x + c_2x^3 + \log x$
5.5.9 (i), (ii), (iv)

5.6 Tutorial : Legendre's Linear Equation

An equation of the form

$$(ax+b)^n y^{(n)} + a_{n-1}(ax+b)^{n-1}y^{(n-1)} + \cdots + a_1(ax+b)y^{(1)} + a_0 y = r(x), \tag{5.6.1}$$

where $a_0,\ a_1, \cdots, a_{n-1}$ are constants, is called an *Euler-Cauchy Equation* of order n.

Method of Solution

- For a given Legendre's equation, substitute $(ax+b) = e^z$ or $z = \ln(ax+b)$. Then we obtain
  $$(ax+b)y' = aDy, \quad (ax+b)^2y'' = a^2D(D-1)y, \quad (ax+b)^3y''' = a^3D(D-1)(D-2)y, \ldots$$
  where $D = \dfrac{d}{dz}$.
- Substitute all these values in the given equation. Now the equation becomes a linear differential equation with constant coefficients. Solve it by the usual methods discussed earlier. Here the solution will be in the form $y \equiv y(z)$.
- Replace z by $\ln(ax+b)$ or e^z by $ax+b$ to obtain the general solution of the given equation.

Solved Examples

Example 5.6.1. *Solve* $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$.

Solution. The given equation is a Legendre's equation. Let

$$1+x = e^t \quad \Rightarrow \quad z = \ln(1+x).$$

Then

$$(1+x)\frac{dy}{dx} = Dy, \quad (1+x)^2\frac{d^2y}{dx^2} = D(D-1)y, \text{where } D = \frac{d}{dz}.$$

So the given equation becomes

$$D(D-1)y + Dy + y = 2\sin t \quad \Rightarrow \quad (D^2+1)y = 2\sin t$$

which is a linear equation with constant coefficients. It's a.e. is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i.$$

Thus

$$C.F. = c_1\cos z + c_2\sin z.$$

Also

$$P.I. = 2\frac{1}{D^2+1}\sin z = 2z\frac{1}{2D}\sin z = -z\cos z.$$

Hence, the general solution is given

$$y = C.F. + P.I. = c_1\cos z + c_2\sin z - z\cos z$$

Replacing z by $\log(1+x)$, we get

$$y = C.F. + P.I. = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)].$$

Exercises

Exercise 5.6.1. *Solve* $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$

Exercise 5.6.2. *Solve* $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Exercise 5.6.3. *Solve* $(2x+3)^2\dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x$

Answers

5.6.1 $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)]$

5.6.2 $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \dfrac{1}{108}[(3x+2)^2\log(3x+3)]$

5.6.3 $y = c_1(2x+3)^a + c_2(2x+3)^b - \dfrac{3}{14}(2x+3) + \dfrac{3}{4}$, where $a, b = \dfrac{3\pm\sqrt{57}}{4}$