- **Directional Derivative and Gradient**
  - **Scalar point function.** A function $\phi(x, y, z)$ is called a scalar point function if it associate a scalar at every point in space. The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of scalar point function.
  - **Vector Point function.** If a function F(x, y, z) defines a vector at every point of a region, then F(x,y,z) is called a vector point function. The velocity of a moving fluid, gravitational force are the example of vector point function.
  - **Vector Differential Operator Del.** The vector differential operator del (or nabla) is denoted by ∇. It is defined as

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

- **Gradient.** The gradient vector (gradient) of a scalar point function $f(x, y, z)$ is denoted by $\nabla f(grad\ f)$ and it is defined by

$$grad\ f = \nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

The $grad\ f$ is a vector normal to the surface $f(x, y, z) =$ constant and it has a magnitude equal to the rate of change of $f(x, y, z)$ along this normal.

Example-1. Find the gradient of $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + tan^{-1}(xz)$ at the point (1,1, 1).

Solution. Using definition of gradient

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \left[-6xz + \frac{1}{1+x^2z^2} \cdot z\right]\hat{\imath} + [-6yz]\hat{\jmath} + \left[6z^2 - 3(x^2 + y^2) + \frac{1}{1+x^2z^2}\right]\hat{k}$$

$$=> (\nabla f)_{(1,1,1)} = \frac{-11}{2}\hat{\imath} - 6\hat{\jmath} + \frac{1}{2}\hat{k}$$

- **Direction Derivative.** The directional derivative of a function $f(x, y, z)$ at a point $P(x, y, z)$ in the direction of vector $\bar{a}$ is given by

$$D_{\bar{a}}f = \left(\frac{df}{ds}\right)_{\bar{a},\ P} = (\nabla f)_P \cdot \hat{a}$$

Remark.

1. The function f increases most rapidly in the direction of the gradient vector ∇f or in the direction of $\frac{\nabla f}{|\nabla f|}$ at point P. The derivative in this direction is magnitude of ∇f $(i.e. |\nabla f|)$.
2. The function f decreases most rapidly in the direction of the gradient vector $-\nabla f$ or $-\frac{\nabla f}{|\nabla f|}$ at point P. The derivative in this direction is $-|\nabla f|$.

Example-1. Find the derivative of $f(x, y) = x^2 sin2y$ at the point $\left(1, \frac{\pi}{2}\right)$ in the direction of $\bar{v} = 3\hat{\imath} - 4\hat{\jmath}$.

Solution. The unit vector $\hat{v}$ is obtained by dividing $\bar{v}$ by its length

$$\hat{v} = \frac{\bar{v}}{|\bar{v}|} = \frac{3\hat{\imath} - 4\hat{\jmath}}{\sqrt{3^2 + (-4)^2}} = \frac{3\hat{\imath} - 4\hat{\jmath}}{5} = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$$

The gradient of f at $\left(1, \frac{\pi}{2}\right)$ is

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y}$$

$$= \hat{\imath}(2xsin2y) + \hat{\jmath}(2x^2cos2y)$$

$$\Rightarrow (\nabla f)_{\left(1,\ \frac{\pi}{2}\right)} = 0\hat{\imath} + 2\hat{\jmath} = 2\hat{\jmath}$$

The derivative of f in the direction of the vector $\bar{v}$ at the point P is given by

$$D_{\bar{v}}f = \left(\frac{df}{ds}\right)_{\bar{v},\ P} = (\nabla f)_P \cdot \hat{v}$$

$$= (2\hat{\jmath})\left(\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}\right)$$

$$= \frac{8}{5}$$

Example-2. Find the direction in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$,

(a) Increases most rapidly at the point $(1, 1)$

(b) Decreases most rapidly at the point $(1, 1)$.

(c) What are rates of change in these direction?

Solution. The gradient of $f(x, y)$ is

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y}$$

$$= \frac{2x}{2}\hat{\imath} + \frac{2y}{2}\hat{\jmath}$$

$$(\nabla f)_{(1,\ 1)} = \hat{\imath} + \hat{\jmath}$$

(a) The function f increases most rapidly in the direction of ∇f at $(1, 1)$.

$$u = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\imath} + \hat{\jmath}}{\sqrt{1^2 + 1^2}} = \frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}$$

The rate of change in this direction is $|\nabla f|$

$$|\nabla f| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

(b) The function f decreases most rapidly in the direction,

$$-u = -\frac{\nabla f}{|\nabla f|} = -\frac{\hat{\imath} + \hat{\jmath}}{\sqrt{1^2 + 1^2}} = -\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}}$$

The rate of change in this direction is $-|\nabla f|$

$$-|\nabla f| = -\sqrt{1^2 + 1^2} = -\sqrt{2}$$

Example-3. The temperature at any point in space is given by $T = xy + yz + zx$. Determine the derivative of T in the direction of the vector $3\hat{\imath} - 4\hat{k}$ at the point (1, 1, 1).

Solution. Let $\bar{a} = 3\hat{\imath} - 4\hat{k}$. Then

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{3\hat{\imath} - 4\hat{k}}{\sqrt{3^2 + (-4)^2}} = \frac{3\hat{\imath} - 4\hat{k}}{5}.$$

Also, $\nabla T = \hat{\imath}\frac{\partial T}{\partial x} + \hat{\jmath}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$

$$= (y + z)\hat{\imath} + (x + z)\hat{\jmath} + (x + y)\hat{k}$$

⇨ $(\nabla T)_{(1,1,1)} = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

The derivative of T in the direction of the vector $\bar{a}$ at the point P is given by

$$D_{\bar{a}}T = \frac{dT}{ds} = (\nabla T)_P \cdot \hat{a} = (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) \cdot \left(\frac{3\hat{\imath} - 4\hat{k}}{5}\right) = -\frac{2}{5}.$$

- **Tangent plane and Normal line**

Let the equation of the surface be $f(x, y, z) = 0$. The equation of tangent plane at

$P(x_1, y_1, z_1)$ to the surface is

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_P + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_P + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_P = 0$$

The equations of normal line are

$$\frac{(x - x_1)}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{(y - y_1)}{\left(\frac{\partial f}{\partial y}\right)_P} = \frac{(z - z_1)}{\left(\frac{\partial f}{\partial z}\right)_P}$$

Example-1. Find the equations of the tangent plane and normal line at the point (-2, 1, -3) to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution. Let $f(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3$

$f_x(x, y, z) = \frac{x}{2}$ $\quad$ $f_x(-2, 1, -3) = -1$

$f_y(x, y, z) = 2y$ $\quad$ $f_y(-2, 1, -3) = 2$

$f_z(x, y, z) = \frac{2z}{9}$ $\quad$ $f_z(-2, 1, -3) = -\frac{2}{3}$

Hence, the equation of the tangent plane at (-2, 1, -3) is

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

$$3x - 6y + 2z = -18$$

The equations of normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-2/3}$$

Example-2. Find the plane tangent to the surface z =$1 - \frac{1}{10}(x^2 + 4y^2)$, at $\left(1, 1,\ ,\frac{1}{2}\right)$.

Solution. Let $f(x, y, z) = \mathrm{z} - 1 + \frac{1}{10}(x^2 + 4y^2)$,

$f_x(x, y, z) = \frac{x}{5}$ $\quad$ $f_x\left(1, 1,\ ,\frac{1}{2}\right) = \frac{1}{5}$

$f_y(x, y, z) = \frac{4}{5}y$ $\quad$ $f_y\left(1, 1, \frac{1}{2}\right) = \frac{4}{5}$

$f_z(x, y, z) = 1$ $\quad$ $f_z\left(1, 1,\ ,\frac{1}{2}\right) = 1$

Hence, the equation of the tangent plane at $(1, 1, 1/2)$ is

$$\frac{1}{5}(x-1)+\frac{4}{5}(y-1)+1\left(z-\frac{1}{2}\right)=0$$

$$\frac{1}{5}x+\frac{4}{5}y+z-\frac{3}{2}=0$$

$$2x+8y+10z=15$$

- **Local Maximum and Local Minimum:**

  Let $f(x,y)$ be defined on a region R containing the point (a,b). Then

1. $f(a,b)$ is a local maximum value of f if $f(a,b)\geq f(x,y)$ for all domain point (x,y) in an open disk connected at (a,b)
2. $f(a,b)$ is a local minimum value of f if $f(a,b)\leq f(x,y)$ for all domain point (x,y) in an open disk connected at (a,b)

- **First Derivative Test for Local Extreme Values:** If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first order partial derivative exist then $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}=0$ and $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}=0$.

- **Critical point:** An interior point of the domain of a function $f(x,y)$ where both first order partial derivatives are zero or where one or both of the first order partial derivative do not exist is a critical point of f.

- **Saddle point:** A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are points (x,y) in the domain where $f(x,y)>f(a,b)$ and domain points (x,y) where $f(x,y)<f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is called a saddle point of the surface.

- **Second Derivative Test for Local Extreme Values:** Suppose that $f(x,y)$ and its first and second derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b)=f_y(a,b)=0$.

  At (a,b) let $r=f_{xx},\ s=f_{xy},\ t=f_{yy.}$

i. f has a local maximum at (a,b) if $rt-s^2>0$ and $r<0$.

ii. f has a local minimum at (a,b) if $rt-s^2>0$ and $r>0$.

iii. f has a saddle point at (a,b) if $rt - s^2 < 0$.

iv. The test is inconclusive at (a,b) if $rt - s^2 = 0$

Example 1. Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Solution : Here $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

For stationary values,

$$\frac{\partial f}{\partial x} = 0$$

$$\therefore 3x^2 - 3 = 0$$

$$x = \pm 1$$

And

$$\frac{\partial f}{\partial y} = 0$$

$$\therefore 3y^2 - 12 = 0$$

$$x = \pm 2$$

Thus stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$;

$$r = \frac{\partial^2 f}{\partial x^2} = 6x\,, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0\,, \quad t = \frac{\partial^2 f}{\partial y^2} = 6y$$

(x,y)	$rt - s^2$	r	Max/Min/Saddle Point
$(1,2)$	72 > 0 r > 0	6	local minimum
$(1,-2)$	$-72 < 0$	6	neither max nor min
$(-1,2)$	$-72 < 0$	-6	neither max nor min
$(-1,-2)$	72 > 0	-6	local maximum

Example 2: Find the maximum and minimum values of $2(x^2 - y^2) - x^4 + y^4$.

Solution: Let $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

$$\frac{\partial f}{\partial x} = 4x - 4x^3$$

$$\frac{\partial f}{\partial y} = -4y + 4y^3$$

$$r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$$

For maxima or minima,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$\therefore 4x - 4x^3 = 0$$

$$\therefore 4x\,(1 - x^2) = 0$$

$$\therefore x = 0 \quad or \quad x = \pm 1$$

Also

$$-4y + 4y^3 = 0$$

$$-4y\,(1 - y^2) = 0$$

$$\therefore y = 0 \quad or \quad y = \pm 1$$

The likely points where $f(x, y)$ has maxima or minima are

$$(0,0), (0, \pm 1), (\pm 1, 0)$$

The results for these points are in the following table:

Points	$rt - s^2$	r	Max/Min/Saddle Point
(0,0)	-16 < 0	0	Saddle Point
(0,1)	32 > 0	4 > 0	Minimum Value -1
$(0, -1)$	32 > 0	4 > 0	Minimum Value -1
(1,0)	32 > 0	-8 < 0	Maximum Value 1
$(-1,0)$	32 > 0	-8 < 0	Maximum Value 1

Example 3. (Solving a volume problem with a constraint) A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

Solution: Let x, y and z represent the length, width and height of the rectangular box, respectively. Then the girth is $2x + 2z$. We want to maximize the volume $V = xyz$ of the box satisfying $x + 2y + 2z = 108$ (the largest box accepted by the delivery company). Thus, we can write the volume of the box as a function of two variables.

$$V(y, z) = (108 - 2y - 2z)yz$$

$$V(y, z) = 108yz - 2y^2z - 2yz^2$$

Setting the first partial derivatives equal to zero,

$$V_y(y, z) = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0$$

$$V_z(y, z) = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0$$

The critical points are $(0,0), (0,54), (54,0)$ *and* $(18, 18)$. The volume is zero at$(0,0), (0,54), (54,0)$, which are not maximum values. At the point$(18, 18)$, we apply the second derivative test.

$$r = V_{yy} = -4z, \qquad t = V_{zz} = -4y, \qquad s = V_{yz} = 108 - 4y - 4z$$

Then $\qquad rt - s^2 = V_{yy}V_{zz} - {V_{yz}}^2 = 16yz - 16(27 - y - z)^2$

Thus $\qquad r = V_{yy}(18,18) = -4(18) < 0$

And at point $(18, 18)$; $\quad rt - s^2 = V_{yy}V_{zz} - {V_{yz}}^2 = 16(18)(18) - 16(-9)^2 > 0$

At point (18,18), the function has maximum volume. The dimensions of the package are

$$x = 108 - 2(18) - 2(18) = 36\ in.$$

$$y = 18\ in. \text{ and } z = 18\ in.$$

The maximum volume is $V = (36)(18)(18) = 11{,}664\ in.^3$