- **Chain Rule**
  - Let $z = f(u)$, where u is again a function of two variables x and y, i.e., $u = u(x, y)$. Then
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}$$
  - Let $z = f(x, y)$, where x and y are again functions of t, i.e., $x = x(t), y = y(t)$. Then
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
  Here $\frac{dz}{dt}$ is called the *total derivative* of z.
  - Let $= f(x, y)$, where x and y are again functions of two variables s and t, i.e., $x = x(s, t), y = y(s, t)$. Then
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example-1. For $z = xe^{xy}, x = t^2, y = t^{-1}$, compute $\frac{dz}{dt}$.

Solution. Using chain rule, we have
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (e^{xy} + xye^{xy})(2t) + x^2e^{xy}(-t^{-2}).$$
Putting the values of x and y in terms of t, we get
$$\frac{dz}{dt} = (2t + t^2)e^t.$$

Example-2. Let $z = e^{x^2y}$, where $x(u, v) = \sqrt{uv}$ and $y(u, v) = \frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Solution. Using chain rule.
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(2xye^{x^2y}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right) + \left(x^2e^{x^2y}\right)(0)$$
$$= 2\sqrt{uv}\cdot\frac{1}{v}e^{uv\cdot\frac{1}{v}}\cdot\frac{\sqrt{v}}{2\sqrt{u}} + uv\cdot e^{uv\cdot\frac{1}{v}}(0) = e^u.$$

Also,
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(2xye^{x^2y}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right) + \left(x^2e^{x^2y}\right)\left(-\frac{1}{v^2}\right)$$
$$= 2\sqrt{uv}\cdot\frac{1}{v}e^{uv\cdot\frac{1}{v}}\cdot\frac{\sqrt{u}}{2\sqrt{v}} + uv\cdot e^{uv\cdot\frac{1}{v}}\left(-\frac{1}{v^2}\right) = \frac{u}{v}e^u - \frac{u}{v}e^u = 0.$$

Example-3. If u is a function of x and y and x and y are functions of r and θ given by $x = e^r \cos\theta, y = e^r\, sin\, \theta$, then show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

Solution. Here $u = f(x,y),\ x = e^r\cos\theta, y = e^r\, sin\, \theta.$

Now

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}e^r\cos\theta + \frac{\partial u}{\partial y}e^r sin\,\theta = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \dots\dots\dots\dots\dots (1)$$

Also

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-e^r sin\,\theta) + \frac{\partial u}{\partial y}e^r cos\,\theta = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} \dots\dots\dots\dots\dots (2)$$

By equations (1) and (2), we get

$$\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 = x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2 + y^2\left(\frac{\partial u}{\partial x}\right)^2 - 2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + x^2\left(\frac{\partial u}{\partial y}\right)^2$$

$$= (x^2+y^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = e^{2r}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right].$$

Thus

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right].$$

Example-4. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Solution. Let $= \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$. Then $u = f(r,s,t)$. Using chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right) = \frac{1}{y}\frac{\partial u}{\partial r} - -\frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0) = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right) = -\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{x}{y}\frac{\partial u}{\partial r}-\frac{z}{x}\frac{\partial u}{\partial t}-\frac{x}{y}\frac{\partial u}{\partial r}+\frac{y}{z}\frac{\partial u}{\partial s}-\frac{y}{z}\frac{\partial u}{\partial s}+\frac{z}{x}\frac{\partial u}{\partial t}=0.$$

- **Homogeneous Function**
  A function f of two independent variables x and y is said to be *homogeneous* of degree n if for real number t we have
  $$f(tx,ty)=t^n f(x,y)$$
- **Euler's Theorem on Homogeneous Functions**
  If z is a smooth homogeneous function of x and y of degree n, then
  $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$
- **Corollary-1**
  If z is a smooth homogeneous function of x and y of degree n, then
  $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$
- **Corollary-2**
  If z is a homogeneous function of x and y of degree n and $z=f(u)$, then
  $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$
- **Corollary-3**
  If z is a homogeneous function of x and y of degree n and $z=f(u)$, then
  $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$
  where $g(u)=n\frac{f(u)}{f'(u)}$.

Example-1. If $u(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2}$, then find the value of

(a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ (b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$.

Solution. Here

$$u(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2}=u(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{x^2}\log\left(\frac{x}{y}\right).$$

Replacing x by tx and y by ty, we get

$$(tx,ty)=t^{-2}\left[\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{x^2}\log\left(\frac{x}{y}\right)\right]$$

Thus u is a homogeneous function of degree -2 in x and y. Hence by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u$$

and

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(-2)(-2-1)u=6u.$$

Example-2. If $u=tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-2sin^3ucos\,u.$$

Solution. Here

$$u=tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)\Rightarrow \tan u=\frac{x^2+y^2}{x+y}=f(u)\ \ (say).$$

Replacing x by tx and y by ty, we can see that $f(u)=\tan u$ is a homogeneous function of degree 1 in x and y. Hence by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=g(u)=1\frac{\tan u}{sec^2u}=\frac{1}{2}\sin 2u.$$

and

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g^{'}(u)-1]$$
$$=\frac{1}{2}\sin 2u(\cos 2u-1)=\sin u\cos u(-2sin^2u)=-2sin^3ucos\,u.$$