MATHEMATICS-II

UNIT-1 : APPLICATIONS OF MATRICES

Chapter 1

Matrices and System of Linear Equations

Matrices are basic tools in linear algebra and system of linear equations arise in all sorts of applications to many different fields of study. In this chapter, we introduce the concepts of matrix and discuss an application to solve systems of linear equations.

1.1 Tutorial : Basics on Matrices

A *matrix* is a rectangular array of numbers or functions enclosed in brackets. These numbers or functions are called the *entries* of the matrix. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \end{bmatrix}, \quad \begin{bmatrix} e^x & e^{-x} \\ e^{4x} & e^{2x} \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Matrices are generally denoted by capital letters $A,\ B,\ C,\dots$ and their entries are denoted by corresponding small letters $a_{ij},\ b_{ij},\ c_{ij},\ \dots$.

The *size* of a matrix is defined in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. The size of the first matrix is 2×3 (read as 2 by 3). The size of the second matrix is 1×3 and so on.

A matrix having same number of rows and columns is called a *square matrix*. Thus third and fifth are square matrices.

In view of the last matrix in above examples (which is a square matrix), a_1, b_2, c_3 are called the *diagonal entries*.

Two matrices are said to be *equal* if they have the same size and their corresponding entries are equal.

Operations on Matrices

- **Sum:** For two matrices A and B be of same size, the *sum* $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A.

- **Product:** Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times r$ matrix. Then the *product* of A and B is the $m \times r$ matrix given by $AB = [c_{ik}]$, where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}.$$

- **Transpose:** The *transpose* of a matrix, denoted by A^T, is the matrix obtained by writing the rows of A, in order, as columns.

- **Conjugate:** The conjugate of a matrix A is denoted by $\overline{A}$ and is obtained on replacing all the entries of A by the corresponding complex conjugates.

- **Transposed Conjugate:** The *transposed conjugate* of a matrix A is obtained by taking conjugate of A^T. It is denoted by A^θ. Thus $A^\theta = \overline{A^T}$.

- **Trace:** Let $A = [a_{ij}]$ be any n-square matrix. Then the *trace* of A is denoted by $\operatorname{tr}(A)$ and is defined as the sum all the diagonal entries in A. Thus

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

- **Determinant:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2×2 matrix. Then the *determinant* of A, denoted by $\det(A)$ or $|A|$ is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a 3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, the determinant is defined as

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

Types of Matrices

- **Diagonal Matrix:** A square matrix is said to be *diagonal* if its nondiagonal entries are zero.

- **Identity Matrix:** A diagonal matrix having all diagonal entries 1 is called an *identity matrix* or *unit matrix*. It is denoted by I.

- **Symmetric Matrix:** A square matrix A is said to be *symmetric* if $A^T = A$.

- **Skew Symmetric Matrix:** A square matrix A is said to be *skew symmetric* if $A^T = -A$.

- **Hermitian Matrix:** A square matrix A is said to be *Hermitian* if $A^\theta = A$.

- **Skew Hermitian Matrix:** A square matrix A is called *skew Hermitian* if $A^\theta = -A$.

- **Invertible Matrix:** A square matrix A is *invertible* if there exists a square matrix B such that $AB = BA = I$. This matrix B is called the *inverse* of A.

- **Orthogonal Matrix:** A square matrix A is called *orthogonal* if $AA^T = A^TA = I$. Thus an orthogonal matrix is necessarily invertible, with $A^{-1} = A^T$.

- **Unitary Matrix:** A square matrix A is called *unitary* if $AA^\theta = A^\theta A = I$. Thus a unitary matrix is necessarily invertible, with $A^{-1} = A^\theta$.

- **Normal Matrix:** A real matrix A is said to be *normal* if $AA^T = A^T A$. Thus every symmetric, skew symmetric or orthogonal matrix is normal.
- **Singular and Nonsingular Matrices:** A square matrix A is called *singular* if $\det(A) = 0$ and *nonsingular* if $\det(A) \neq 0$.

Solved Examples

Example 1.1.1. *Which of the following matrices are symmetric or skew symmetric?*

$$\text{(i)} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 7 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 4 & 0 & 2 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 0 & 1 & 4 \\ 2 & 0 & -2 \\ 1 & 6 & 0 \end{bmatrix} \qquad \text{(iv)} \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$$

Solution. (i) Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 7 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 5 \\ 3 & 5 & 7 \end{bmatrix} = A$.

So, A is symmetric.

(ii) Let $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ 4 & 0 & 2 \end{bmatrix}$. Then $B^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 3 & 4 & 2 \end{bmatrix} \neq \pm B$.

Thus B is neither symmetric nor skew symmetric.

(iii) Let $C = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 0 & -2 \\ 1 & 6 & 0 \end{bmatrix}$. Then $C^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 6 \\ 4 & -2 & 0 \end{bmatrix} \neq \pm C$.

Thus C is neither symmetric nor skew symmetric.

(iv) Let $D = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$. Then

$$D^T = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = -D.$$

So, D is skew symmetric. ■

Example 1.1.2. *Determine whether the following matrix is Hermitian.*

$$A = \begin{bmatrix} 3 & i & 1+i \\ -i & 4 & -1+3i \\ 1-i & -1-3i & 5 \end{bmatrix}$$

Solution. Observe that

$$A^T = \begin{bmatrix} 3 & -i & 1-i \\ i & 4 & -1-3i \\ 1+i & -1+3i & 5 \end{bmatrix} \quad \Rightarrow \quad A^\theta = \begin{bmatrix} 3 & i & 1+i \\ -i & 4 & -1+3i \\ 1-i & -1-3i & 5 \end{bmatrix}$$

Since $A^\theta = A$, the matrix A is Hermitian.

Example 1.1.3. *Is* $A = \begin{bmatrix} 0 & -3+2i & -2+i \\ 3+2i & 3i & 3+5i \\ 2+i & -3+5i & 2i \end{bmatrix}$ *a skew Hermitian matrix ?*

Solution. Observe that

$$\begin{aligned} A^T &= \begin{bmatrix} 0 & 3+2i & 2+i \\ -3+2i & 3i & -3+5i \\ -2+i & 3+5i & 2i \end{bmatrix} \\ \Rightarrow\ A^\theta &= \begin{bmatrix} 0 & 3-2i & 2-i \\ -3-2i & -3i & -3-5i \\ -2-i & 3-5i & -2i \end{bmatrix} \\ \Rightarrow\ A^\theta &= -\begin{bmatrix} 0 & -3+2i & -2+i \\ 3+2i & 3i & 3+5i \\ 2+i & -3+5i & 2i \end{bmatrix} \\ \Rightarrow\ A^\theta &= -A. \end{aligned}$$

Thus A is skew Hermitian. ■

Example 1.1.4. *Find* $l,\ m,\ n,$ *if* $A = \begin{bmatrix} 5i & l & 3-2i \\ 4+i & 2i & m \\ n & 2+i & -6i \end{bmatrix}$ *is skew Hermitian.*

Solution. Observe that

$$A^T = \begin{bmatrix} 5i & 4+i & n \\ l & 2i & 2+i \\ 3-2i & m & -6i \end{bmatrix} \quad \Rightarrow \quad A^\theta = \begin{bmatrix} -5i & 4-i & \overline{n} \\ \overline{l} & -2i & 2-i \\ 3+2i & \overline{m} & 6i \end{bmatrix}$$

Since A is skew Hermitian, we have

$$A^\theta = -A \quad \Rightarrow \quad \begin{bmatrix} -5i & 4-i & \overline{n} \\ \overline{l} & -2i & 2-i \\ 3+2i & \overline{m} & 6i \end{bmatrix} = \begin{bmatrix} -5i & -l & -3+2i \\ -4-i & -2i & -m \\ -n & -2-i & 6i \end{bmatrix}$$

Comparing the corresponding entries, we obtain

$$\begin{aligned} &\overline{l} = -4-i,\ \ \overline{m} = -2-i,\ \ \overline{n} = -3+2i \\ \Rightarrow\ &l = -4+i,\ \ m = -2+i,\ \ n = -3-2i. \ ■ \end{aligned}$$

Example 1.1.5. *Show that every square matrix A can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.*

Solution. Observe that

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q,$$

where

$$P = \frac{1}{2}(A + A^T) \quad \text{and} \quad Q = \frac{1}{2}(A - A^T).$$

Now

$$P^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left[A^T + (A^T)^T\right] = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = P.$$

Thus P is symmetric. Also,

$$Q^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}\big[A^T - (A^T)^T\big] = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -Q.$$

Thus Q is skew symmetric. For the uniqueness, suppose that

$$A = M + N,$$

where M is symmetric and N is skew symmetric. Then

$$A^T = (M + N)^T = M^T + N^T = M - N.$$

Observe that

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\big[(M + N) + (M - N)\big] = M$$

and

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\big[(M + N) - (M - N)\big] = N.$$

Hence the representation $A = P + Q$ is unique. ■

Remark. Every square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix. It can be proved by expressing A as

$$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta).$$

Example 1.1.6. *Express the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 1 & 8 & 2 \end{bmatrix}$ *as the sum of a symmetric matrix and a skew symmetric matrix.*

Solution. It is known that every square matrix A can be expressed uniquely as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q, \tag{1.1.1}$$

where P is symmetric and Q is skew symmetric. Here

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 1 & 8 & 2 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 8 \\ 3 & 5 & 2 \end{bmatrix}$$

Now

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 1 & 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 8 \\ 3 & 5 & 2 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 4 \\ 1 & 8 & 13 \\ 4 & 13 & 4 \end{bmatrix}$$

and

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 1 & 8 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 8 \\ 3 & 5 & 2 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

Hence by (1.1.1),

$$A = P + Q = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ \frac{1}{2} & 4 & \frac{13}{2} \\ 2 & \frac{13}{2} & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & 1 \\ -\frac{3}{2} & 0 & -\frac{3}{2} \\ -1 & \frac{3}{2} & 0 \end{bmatrix}$$

where P is symmetric and Q is skew symmetric. ■

Example 1.1.7. *Show that the matrix* $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ *is orthogonal.*

Solution. Observe that

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

So we have

$$AA^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence A is orthogonal.

Example 1.1.8. *Show that the matrix A is unitary and hence find A^{-1}, where*

$$A = \frac{1}{2} \begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$$

Solution. Observe that

$$A^T = \frac{1}{2} \begin{bmatrix} 1 & i & 1+i \\ -i & 1 & -1+i \\ -1+i & 1+i & 0 \end{bmatrix} \quad \Rightarrow \quad A^\theta = \frac{1}{2} \begin{bmatrix} 1 & -i & 1-i \\ i & 1 & -1-i \\ -1-i & 1-i & 0 \end{bmatrix}$$

So we have

$$\begin{aligned} AA^\theta &= \frac{1}{4} \begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix} \begin{bmatrix} 1 & -i & 1-i \\ i & 1 & -1-i \\ -1-i & 1-i & 0 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1+1+2 & -i-i+2i & 1-i+i-1+0 \\ i+i-2i & 1+1+2 & i+1-1-i \\ 1+i-i-1+0 & -i+1-1+i & 2+2+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus $AA^\theta = I$. Hence A is unitary and

$$A^{-1} = A^\theta = \frac{1}{2} \begin{bmatrix} 1 & -i & 1-i \\ i & 1 & -1-i \\ -1-i & 1-i & 0 \end{bmatrix} \quad \blacksquare$$

Exercises

Exercise 1.1.1. *Determine whether the following matrices are symmetric or skew symmetric.*

$$(i)\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad (ii)\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad (iii)\begin{bmatrix} 5 & 4 & -7 \\ 4 & 2 & 6 \\ -7 & 6 & 3 \end{bmatrix}, \quad (iv)\begin{bmatrix} 1 & 4 & 8 \\ 3 & 0 & 4 \\ 5 & 6 & 3 \end{bmatrix}$$

Exercise 1.1.2. *Express the matrix* $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$ *as the sum of a symmetric matrix and a skew symmetric matrix.*

Exercise 1.1.3. *Verify whether the following matrices are Hermitian or skew Hermitian or neither. Give reason.*

$$(i) \quad \begin{bmatrix} a & c+id \\ c-id & b \end{bmatrix} \qquad (ii) \quad \begin{bmatrix} 2i & 1+i & -3+2i \\ -1+i & 0 & 2-i \\ 3+2i & -2-i & -3i \end{bmatrix}$$

Exercise 1.1.4. *Find k, l and m to make A, a Hermitian matrix*

$$A = \begin{bmatrix} -1 & k & -l \\ 3-5i & 0 & m \\ l & 2+4i & 2 \end{bmatrix}$$

Exercise 1.1.5. *Express the matrix* $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ *as the sum of a Hermitian matrix and a skew Hermitian matrix.*

Exercise 1.1.6. *Determine which of the following matrices are orthogonal. For those that are orthogonal, find the inverse.*

$$(i) \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad (ii) \begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Exercise 1.1.7. *Find l, m, n and A^{-1} if* $A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$ *is orthogonal.*

Exercise 1.1.8. *Is the following matrix unitary ? If yes find its inverse.*

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Exercise 1.1.9. *Prove that the matrix* $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$ *is a Hermitian and iA is a skew Hermitian matrix.*

Answers

1.1.1 (i)-none, (ii)-skew symmetric, (iii)-symmetric, (iv)-none

1.1.2 $P = \frac{1}{2}\begin{bmatrix} 2 & 4 & 15 \\ 4 & -4 & -2 \\ 15 & -2 & 26 \end{bmatrix}, \quad Q = \frac{1}{2}\begin{bmatrix} 0 & 6 & -1 \\ -6 & 0 & -6 \\ 1 & 6 & 0 \end{bmatrix}$

1.1.3 (i) Hermitian (ii) Skew Hermitian

1.1.4 $k = 3 + 5i, l = 0$ or purely imaginary $, m = 2 - 4i$

1.1.5 $P = \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}, \quad Q = \frac{1}{2}\begin{bmatrix} 6i & -5 & -1+3i \\ 5 & 2i & 11+i \\ 1+3i & -11+i & 0 \end{bmatrix}$

1.1.6 (i)

1.1.7 $l = \pm\frac{1}{\sqrt{2}}, m = \pm\frac{1}{\sqrt{6}}, n = \pm\frac{1}{\sqrt{3}}, \quad A^{-1} = \begin{bmatrix} 0 & \pm 1/\sqrt{2} & \pm 1/\sqrt{2} \\ \pm 2/\sqrt{6} & \pm 1/\sqrt{6} & \mp 1/\sqrt{6} \\ \pm 1/\sqrt{3} & \mp 1/\sqrt{3} & \pm 1/\sqrt{3} \end{bmatrix}$

1.1.8 Yes

1.2 Tutorial : Echelon Forms

Elementary Operations

(1) Interchange the i^{th} and j^{th} row: $R_i \leftrightarrow R_j$.

(2) Multiply the i^{th} row by a nonzero scalar k: $R_i \to kR_i$.

(3) Replace the i^{th} row by i^{th} row plus k times j^{th} row: $R_i \to R_i + kR_j$.

A matrix which can be obtained from the identity matrix by one single elementary row operation is called an *elementary matrix*. For example,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ etc.}$$

Row Equivalent Matrices

Matrix A is *row equivalent* to matrix B, if B can be obtained from A by a sequence of elementary row operations. Symbolically, it is written as $A \sim B$.

Row-Echelon Form

A matrix is said to be in *row-echelon form* if it satisfies the following properties:

(1) If the matrix has any zero rows, then they are at the bottom of the matrix.

(2) In any nonzero row, the first nonzero entry is 1. We call this a *leading 1*.

(3) Each leading 1 is to the right of the leading 1 in the previous row.

For Example, the matrices

$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are in row-echelon form.

Reduced Row-Echelon Form

A matrix is said to be in *reduced row-echelon form* if

(1) It is in row-echelon form.

(2) Each leading 1 is the only nonzero entry in its column.

For example, the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 9 \end{bmatrix}$$

are in reduced row-echelon form.

By means of a finite sequence of elementary row operations any matrix can be transformed to row echelon form. The resulting echelon form is not unique; for example, any multiple by a scalar of a matrix in echelon form is also an echelon form of the same matrix. However, every matrix has a unique reduced row echelon form.

Solved Examples

Example 1.2.1. *Determine whether the following matrices are in row-echelon form, reduced row-echelon form, both, or neither.*

$$(i)\ \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (ii)\ \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}, \qquad (iii)\ \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Solution. (i) The matrix in is in both row-echelon form and reduced row-echelon form.

(ii) The matrix in is neither in row-echelon form nor in reduced row-echelon form because leading 1 in the third row is not to the right of the leading 1 in the second row.

(iii) The matrix in is in row-echelon form but not in reduced row-echelon form because the leading 1 in the second row is not the only nonzero entry in its column. ■

Example 1.2.2. *Obtain a row echelon form of the following matrix:*

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 4 & 2 & 4 & 3 & 3 \\ 1 & 3 & 2 & -2 & 0 & 0 \end{bmatrix}$$

Solution. Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 0 & 4 & 2 & 4 & 3 & 3 \\ 0 & 0 & 0 & -2 & -1 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & -2 & -1 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 + R_3$, we get

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & -4 & 1 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{2}R_2$, $R_3 \to \frac{1}{2}R_3$ and $R_4 \to -\frac{1}{4}R_4$, we get

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \end{bmatrix}$$

which is in row-echelon form. ■

Example 1.2.3. *Obtain the reduced row-echelon form of the matrix*

$$A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix}$$

Solution. Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 3R_1$, we obtain

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -2 & 2 \end{bmatrix}$$

Applying $R_2 \to (-1)R_2$, we obtain

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -2 & 2 \end{bmatrix}$$

Applying $R_3 \to R_3 + 2R_2$ and $R_4 \to R_4 + 2R_2$ we obtain

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - 2R_3$, we obtain

$$\begin{bmatrix} 1 & 3 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the reduced row-echelon form of A. $\blacksquare$

Exercises

Exercise 1.2.1. *Is* $\begin{bmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *in row-echelon form or reduced row-echelon form ?*

Exercise 1.2.2. *Is* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *row-echelon or reduced row-echelon form ?*

Exercise 1.2.3. *Is the matrix* $\begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *in row echelon or reduced row-echelon form ?*

Exercise 1.2.4. *Is* $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ *row-echelon or reduced row-echelon form ?*

Exercise 1.2.5. *Obtain a row-echelon form of the matrix*

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$$

Exercise 1.2.6. *Reduce the following matrix into reduced row-echelon form*

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Answers

1.2.1 only in row-echelon form **1.2.2** both **1.2.3** no **1.2.4** both

1.2.5 one form: $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **1.2.6** $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.3 Tutorial : System of Linear Equations

A system of m linear equations in n unknowns can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

We write this system in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

That is, $A\mathbf{x} = \mathbf{b}$. Here, A is called the *coefficient matrix* of the system.

If $\mathbf{b} = \mathbf{0}$, i.e., the right side constants $b_1, b_2, \ldots, b_n$ are all zero, then the system is called *homogeneous*, otherwise it is called *nonhomogeneous*.

We can capture all the information contained in the system in a single matrix as

$$[A : \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

This matrix is called the *augmented matrix* of the system. To solve such type of system, we will discuss following two methods:

Gauss Elimination Method

- Write a system of linear equations as an augmented matrix.
- Perform the elementary row operations to put the matrix into row-echelon form.
- Convert the matrix back into a system of linear equations.
- Use back substitution to obtain all the answers.

Gauss-Jordan Elimination Method

- Write a system of linear equations as an augmented matrix.
- Perform the elementary row operations to put the matrix into reduced row-echelon form.
- Convert the matrix back into a system of linear equations.
- No back substitution is necessary.

Types of Solutions

There are three types of solutions which are possible when solving a system of linear equations:

(1) Independent

- Consistent.
- Unique Solution.
- A row-echelon form has the same number of non-zero rows as variables.

(2) Dependent

- Consistent.
- Infinitely many solutions.
- A row-echelon form has more variables than non-zero rows.

(3) Inconsistent

- No Solution.
- A row-echelon form has a zero row on the left side, but the right hand side is not zero.

Every system of linear equations has either no solutions, or has exactly one solution, or has infinitely many solutions.

Solved Examples

Example 1.3.1. *Solve the following system using Gaussian elimination:*

$$\begin{aligned} x + y - 3z &= 4 \\ 2x + y - z &= 2 \\ 3x + 2y - 4z &= 6 \end{aligned}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & -4 & 6 \end{array}\right]$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ we get

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -1 & 5 & -6 \end{array}\right]$$

Applying $R_3 \to R_3 - R_1$ we get

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_2 \to (-1)R_2$, we get

$$\left[\begin{array}{ccr|c} 1 & 1 & -3 & 4 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row-echelon form. Here number of non-zero rows are less than the number of variables. So, given system has infinitely many solutions. The system corresponding to the last matrix is

$$\begin{aligned} x + y - 3z &= 4 \\ y - 5z &= 6 \end{aligned}$$

Let $z = t$. Then $y = 5t + 6$ and $x = -2 - 2t$. Hence solution set is given by

$$\{(-2 - 2t, 5t + 6, t) | t \in \mathbb{R}\}. \quad \blacksquare$$

Example 1.3.2. *Use Gaussian elimination to solve the system of linear equations*

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

Applying $R_3 \to R_1 + R_3$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

Applying $R_2(-1)$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

Applying $R_3 \to R_3 - 2R_2$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

Applying $R_3 \to (-1)R_3$, we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

which is in row-echelon form. Here number of the numbers non-zero rows is same as the number of unknowns, so that system has a unique solution. The system corresponding to the last matrix is

$$x_1 - 2x_2 - 3x_3 = 0, \qquad x_2 + x_3 = -3, \qquad x_3 = 2.$$

Using back substitution, we get

$$x_1 = -4, \quad x_2 = -5, \quad x_3 = 2. \quad \blacksquare$$

Example 1.3.3. *Use Gaussian elimination to solve the system of linear equations*

$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= 12 \\ 2x_1 + 4x_2 + 12x_3 &= -17 \\ x_1 - 4x_2 - 12x_3 &= 22 \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{array}\right]$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 12 \\ 0 & 8 & 24 & -41 \\ 0 & -2 & -6 & 10 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 8 & 24 & -41 \end{array}\right]$$

Applying $R_3 \to R_3 + 4R_2$ we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Applying $R_2 \to \left(-\frac{1}{2}\right) R_2$ and $R_3 \to (-1)R_3$, we get

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which is in row-echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= 12 \\ x_2 + 3x_3 &= -5 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

From the last equation, we get $0 = -1$, which is not possible. So, given system is inconsistent and has no solution. $\blacksquare$

Example 1.3.4. *Solve the following system of linear equations by Gauss-Jordan elimination method:*

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

Applying the operations $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

Applying $R_2 \to (-1)R_2$, we get

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

Applying $R_3 \to R_3 + 10R_2$, we get

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

Applying $R_3 \to \left(-\frac{1}{52}\right) R_3$, we get

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Applying $R_2 \to R_2 + 5R_3$ and $R_1 \to R_1 - 2R_3$, we get

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Applying $R_1 \to R_1 - R_2$, we get

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

which is in reduced row-echelon form. The system corresponds to the last matrix is

$$x_1 = 3, \quad x_2 = 1, \quad x_3 = 2.$$

Hence the solution of the system is $x_1 = 3,\ x_2 = 1,\ x_3 = 2.$ ■

Example 1.3.5. *Solve the following homogeneous system of linear equations using Gauss-Jordan elimination:*

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{rrrrr|r} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_4$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

Applying $R_3 \to R_3 - 3R_1$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

Applying $R_3 \to \left(-\frac{1}{3}\right)$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

Applying $R_4 \to R_4 + 3R_3$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - R_3$, we get

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_1 \to R_1 + 2R_1$, we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is in reduced row-echelon form. The system corresponds to the last matrix is

$$x_1 + x_2 + x_5 = 0, \qquad x_3 + x_5 = 0, \qquad x_4 = 0.$$

Let $x_2 = s$ and $x_5 = t$, we get the solution of the system as

$$x_1 = -s - t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = 0, \quad x_5 = t. \quad \blacksquare$$

Exercises

Exercise 1.3.1. *Solve the following system by Gauss elimination method:*

$$\begin{aligned} 2x + y - z &= 2 \\ x - 3y + z &= 1 \\ 2x + y - 2z &= 6 \end{aligned}$$

Exercise 1.3.2. *Solve the following system by Gauss Jordan elimination method:*

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

Exercise 1.3.3. *Solve the system by Gaussian elimination.*

$$\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

Exercise 1.3.4. *Solve the following system by Gauss Jordan elimination method:*

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -15 \\ 5x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 11 \\ -6x_1 - 4x_2 + 2x_3 &= 30 \end{aligned}$$

Exercise 1.3.5. *Solve the following system by Gauss elimination method:*

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 9 \\ 3x_1 - x_2 + x_3 &= 6 \\ 4x_1 - x_2 + 2x_3 &= 7 \\ -x_1 + x_2 - x_3 &= 4 \end{aligned}$$

Exercise 1.3.6. *Solve the following homogeneous system of linear equations by Gauss Jordan elimination method*

$$\begin{aligned} v+3w-2x &= 0\\ 2u+v-4w+3x &= 0\\ 2u+3v+2w-x &= 0\\ -4u-3v+5w-4x &= 0 \end{aligned}$$

Exercise 1.3.7. *Solve the following homogeneous system of linear equations by Gauss Jordan elimination method*

$$\begin{aligned} x_1+3x_2+x_4 &= 0\\ x_1+4x_2+2x_3 &= 0\\ -2x_2-2x_3-x_4 &= 0\\ 2x_1-4x_2+x_3+x_4 &= 0\\ x_1-2x_2-x_3+x_4 &= 0 \end{aligned}$$

Exercise 1.3.8. *Solve the following system for x, y and z:*

$$-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30,\quad \frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9,\quad \frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10 \quad (x\neq 0, y\neq 0, z\neq 0)$$

Exercise 1.3.9. *Solve the following nonlinear system for the unknown angles α, β and γ, where $0\leq\alpha\leq 2\pi$, $0\leq\beta\leq 2\pi$, $0\leq\gamma<\pi$:*

$$\begin{aligned} 2\sin\alpha-\cos\beta+3\tan\gamma &= 3\\ 4\sin\alpha+2\cos\beta-2\tan\gamma &= 2\\ 6\sin\alpha-3\cos\beta+\tan\gamma &= 9 \end{aligned}$$

Exercise 1.3.10. *For which values of a will the following system have no solution ? Exactly one solution ? Infinitely many solutions ?*

$$\begin{aligned} x+2y-3z &= 4\\ 3x-y+5z &= 2\\ 4x+y+(a^2-14)z &= a+2 \end{aligned}$$

Additional Exercises

Exercise 1.3.11. *Solve the following system of linear equations by any method:*

(i) $2x_1 + x_2 + 3x_3 = 0$
$x_1 + 2x_2 = 0$
$x_2 + x_3 = 0$

(ii) $2x - y - 3z = 0$
$-x + 2y - 3z = 0$
$x + y + 4z = 0$

(iii) $3x_1 + x_2 + x_3 + x_4 = 0$
$5x_1 - x_2 + x_3 - x_4 = 0$

(iv) $2x + 2y + 4z = 0$
$w - y - 3z = 0$
$2w + 3x + y + z = 0$
$-2w + x + 3y - 2z = 0$

(v) $z_3 + z_4 + z_5 = 0$
$-z_1 - z_2 + 2z_3 - 3z_4 + z_5 = 0$
$z_1 + z_2 - 2z_3 - z_5 = 0$
$2z_1 + 2z_2 - z_3 + z_5 = 0$

Exercise 1.3.12. *Show that the following nonlinear system has 18 solutions if* $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, $0 \le \gamma \le 2\pi$*:*

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

Exercise 1.3.13. *Solve the system*

$$2x_1 - x_2 = \lambda x_1, \qquad 2x_1 - x_2 + x_3 = \lambda x_2, \qquad -2x_1 + 2x_2 + x_3 = \lambda x_3$$

for x_1, x_2 *and* x_3 *in two cases* $\lambda = 1$ *and* $\lambda = 2$.

Answers

1.3.1 $x = -\frac{1}{7},\ y = -\frac{12}{7},\ z = -4$
1.3.2 $x = t - 1,\ y = 2s,\ z = s,\ w = t$ **1.3.3** Inconsistent
1.3.4 $x_1 = -4,\ x_2 = 2,\ x_3 = 7$ **1.3.5** Inconsistent
1.3.6 $u = 7s - 5t,\ v = -6s + 4t,\ w = 2s,\ x = 2t$
1.3.7 $x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$ **1.3.8** $x = \frac{1}{2},\ y = \frac{1}{4},\ z = \frac{1}{5}$
1.3.9 $\alpha = \frac{\pi}{2},\ \beta = \pi,\ \gamma = 0$
1.3.10 no solution for $a = -4$, exactly one solution for $a \neq \pm 4$, infinitely many solutions for $a = 4$

1.4 Tutorial : Inverse of a Matrix

Let A be a square matrix. If there exists another square matrix B such that $AB = BA = I$, then A is said to be *invertible* and B is called the *inverse* of A.

Minors and Cofactors

Let $A = (a_{ij})$ be a square matrix. Then the *minor of entry* a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A.

The *cofactor of entry* a_{ij} is denoted by C_{ij} and is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

Adjoint of a Matrix

Let $A = (a_{ij})$ be any $n \times n$ matrix. If C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the *matrix of cofactors from A*. The transpose of this matrix is called the *adjoint of A* and is denoted by adj(A).

In particular, for 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of a Matrix by Adjoint (Determinant Method)

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{ adj}(A).$$

The inverse of a matrix A exists if and only if $\det(A) \neq 0$.

Inverse of a matrix Using Row Operations (Guass Jordan Method)

Let A be a given square matrix with $\det(A) \neq 0$. Then A^{-1} can be obtained as follows:

- Form a new matrix by augmenting the identity matrix I to the right side of A as

$$[A \mid I].$$

- By applying appropriate row operations, reduce the left side to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$[I \mid A^{-1}].$$

Solved Examples

Example 1.4.1. *Find the inverse of matrix* $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Solution. Observe that
$$\det(A) = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0.$$
Therefore, A^{-1} exists and is given by
$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \quad \blacksquare$$

Example 1.4.2. *Find the inverse of the following matrix by Gauss Jordan method if it is invertible:*
$$A = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Solution. First we have to check about the existence of A^{-1}. It is known that A^{-1} exists if and only if $\det(A) \neq 0$. Here,
$$\det(A) = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 1 \\ 3 & 2 & 0 \end{vmatrix} = 2(0-2) - 1(0-3) + 5(-2-0) = -4 + 3 - 10 = -11 \neq 0.$$
Therefore, A^{-1} exists. Now
$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 5 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$
Applying the operation $R_1 \leftrightarrow R_2$, we get
$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 5 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$
Applying the operations $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + 3R_1$, we get
$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 7 & 1 & 2 & 0 \\ 0 & 2 & 3 & 0 & 3 & 1 \end{array}\right]$$
Applying the operation $R_3 \to R_3 - 2R_2$, we get
$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 7 & 1 & 2 & 0 \\ 0 & 0 & -11 & -2 & -1 & 1 \end{array}\right]$$
Applying the operation $R_3 \to \left(-\frac{1}{11}\right) R_3$, we get
$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 7 & 1 & 2 & 0 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & -\frac{1}{11} \end{array}\right]$$

Applying the operations $R_2 \to R_2 - 7R_3$ and $R_1 \to R_1 - R_3$, we get

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -\frac{2}{11} & \frac{10}{11} & \frac{1}{11} \\ 0 & 1 & 0 & -\frac{3}{11} & \frac{15}{11} & \frac{7}{11} \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & -\frac{1}{11} \end{array}\right]$$

Applying the operation $R_1 \to (-1)R_1$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{11} & -\frac{10}{11} & -\frac{1}{11} \\ 0 & 1 & 0 & -\frac{3}{11} & \frac{15}{11} & \frac{7}{11} \\ 0 & 0 & 1 & \frac{2}{11} & \frac{1}{11} & -\frac{1}{11} \end{array}\right]$$

Since the left side of the last matrix is I, the right side will be A^{-1}. Thus

$$A^{-1} = \begin{bmatrix} \frac{2}{11} & -\frac{10}{11} & -\frac{1}{11} \\ -\frac{3}{11} & \frac{15}{11} & \frac{7}{11} \\ \frac{2}{11} & \frac{1}{11} & -\frac{1}{11} \end{bmatrix} \qquad \blacksquare$$

Example 1.4.3. *Find the inverse of the matrix A if it is invertible.*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution. First we have to check about the existence of A^{-1}. It is known that A^{-1} exists if and only if $\det(A) \neq 0$. Here,

$$\det(A) = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 1(0+1) + 1(-1-0) = 1 - 1 = 0.$$

Therefore, A^{-1} does not exist. $\blacksquare$

Example 1.4.4. *Find A^{-1} using row operations if* $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Solution. Consider the matrix

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Applying the operation $R_2 \to R_2 + R_1$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Applying the operation $R_3 \to R_3 - R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

Applying the operation $R_3 \to \left(-\frac{1}{2}\right) R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Applying the operations $R_2 \to R_2 - 2R_3$ and $R_1 \to R_1 - R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Since the left side of the last matrix is I, the right side will be A^{-1}. Thus

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \blacksquare$$

Exercises

Exercise 1.4.1. *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Exercise 1.4.2. *Find* A^{-1} *using row operations, if* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Exercise 1.4.3. *Find the inverse of the matrix* A *using Gauss Jordan method, if*

$$A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$$

Exercise 1.4.4. *Find the inverse of the matrix* A *if it is invertible, where*

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 5 & 1 & 1 \\ -3 & 2 & 0 \end{bmatrix}$$

Exercise 1.4.5. *Find the inverse of the matrix* A *if it exists, where*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

Additional Exercises

Exercise 1.4.6. *Find the inverse of each of the following 4×4 matrices, k_1, k_2, k_3, k_4 and k are all nonzero.*

$$\text{(i)} \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$$

Exercise 1.4.7. *Show that*

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

Answers

1.4.1 $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ **1.4.2** $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

1.4.3 $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$ **1.4.4** not invertible **1.4.5** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$

1.5 Tutorial : Linear System and Invertibility

We have studied two general methods for solving system of linear equations: Gauss elimination and Gauss Jordan elimination. In this section, we will discuss two other methods for solving certain linear systems.

Matrix Inversion Method

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix b, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x}= A^{-1}\mathbf{b}$. Equivalently, the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$ if and only if $\det(A) \neq 0$.

Remark. For the homogenous system $A\mathbf{x} = \mathbf{0}$, if $\det(A) \neq 0$, then the system has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Solved Examples

Example 1.5.1. *Solve the following system by inverting the coefficient matrix:*

$$\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$$

Solution. The matrix form the system is $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

Observe that

$$\det(A) = 1(1+4) - 1(1-16) + 1(1+4) = 5 + 15 + 5 = 25 \neq 0.$$

Therefore, A^{-1} exists and we can apply matrix inversion method. Now

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying the operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 4R_1$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$$

Applying the operation $R_2 \leftrightarrow R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

Applying the operation $R_3 \to \left(-\frac{1}{5}\right) R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

Applying the operations $R_2 \to R_2 - 5R_3$ and $R_1 \to R_1 - R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

Applying the operation $R_2 \to \left(\frac{1}{5}\right) R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

Applying the operation $R_1 \to R_1 - R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix}$$

Hence

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5}\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

Thus the solution is

$$x_1 = 1, \quad x_2 = 5, \quad x_3 = -1. \quad \blacksquare$$

Exercises

Exercise 1.5.1. *Solve the following system by inverting the coefficient matrix*

$$\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$$

Exercise 1.5.2. *What condition must b_1, b_2 and b_3 satisfy in order for the system*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

Exercise 1.5.3. *What condition must b_1, b_2 and b_3 satisfy in order for the following system to be consistent?*

$$5x_1 + 2x_2 + 9x_3 = b_1, \quad 3x_1 + x_2 + 4x_3 = b_2, \quad -x_1 + x_3 = b_3$$

Additional Exercises

Exercise 1.5.4. *For which values of λ does the system of equations*

$$\begin{aligned} (\lambda - 3)x + y &= 0 \\ x + (\lambda - 3)y &= 0 \end{aligned}$$

have nontrivial solutions?

Exercise 1.5.5. *Find conditions that the b's must satisfy for the system to be consistent.*

$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

Exercise 1.5.6. *Solve the following matrix equation for X:*

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

Exercise 1.5.7. *Determine (without solving) whether the homogeneous system has a nontrivial solution; then state whether the corresponding coefficient matrix is invertible.*

$$\begin{aligned} 2x_1 + x_2 - 3x_3 + x_4 &= 0 \\ 5x_2 + 4x_3 + 3x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ 3x_4 &= 0 \end{aligned}$$

Exercise 1.5.8. *Determine (without solving) whether the homogeneous system has a nontrivial solution; then state whether the corresponding coefficient matrix is invertible.*

$$\begin{aligned} 5x_1 + x_2 + 4x_3 + x_4 &= 0 \\ 2x_3 - x_4 &= 0 \\ x_3 + x_4 &= 0 \\ 7x_4 &= 0 \end{aligned}$$

Exercise 1.5.9. *What restrictions must be placed on x and y for the following matrices to be invertible?*

(i) $\begin{bmatrix} x & y \\ x & x \end{bmatrix}$ (ii) $\begin{bmatrix} x & 0 \\ y & y \end{bmatrix}$ (iii) $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$

Answers

1.5.1 $x_1 = 1,\ x_2 = -11,\ x_3 = 16$ **1.5.2** no condition is required **1.5.3** $b_3 = b_1 - 2b_2$

MATHEMATICS-II

UNIT-1 : APPLICATIONS OF MATRICES

Chapter 2

Eigenvalues, Eigenvectors

2.1 Tutorial : Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an *eigenvalue* of A if there exists a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

This vector $\mathbf{x}$ is called an *eigenvector* of A corresponding to λ.

Computation of Eigenvalues

Let A be an $n \times n$ matrix. Then the eigenvalues of A can be obtained by solving the equation

$$\det(A - \lambda I) = 0.$$

This equation is called the *characteristic equation* of A. The expansion of $\det(A - \lambda I)$ is a polynomial $p(\lambda)$ of degree n, called the *characteristic polynomial* of A. Thus every $n \times n$ matrix has exactly n eigenvalues. The eigenvalues may be real or may be complex. In the present book, we shall confine ourself to real eigenvalues.

Computation of Eigenvectors

Let A be an $n \times n$ matrix and λ be an eigenvalue of A. Then the eigenvectors of A corresponding to λ are the nonzero vectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$. Thus all such eigenvectors can be obtained by finding the solution space of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. This solution space is called the *eigenspace* of A corresponding to λ.

Theorems on Eigenvalues and Eigenvectors

(1) If A is a 2×2 matrix, then the characteristic equation is given by

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0.$$

(2) If A is a 3×3 matrix, then the characteristic equation is given by

$$\lambda^3 - \mathrm{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0,$$

where M_{11}, M_{22}, M_{33} are the minors of the diagonal entries in A.

(3) If A is an $n \times n$ triangular matrix, then the eigenvalues of A are the diagonal entries of A.

(4) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

(5) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector.

(6) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $\lambda - k$ is an eigenvalue of $A - kI$ and $\mathbf{x}$ is a corresponding eigenvector.

(7) If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $k\lambda$ is an eigenvalue of kA and $\mathbf{x}$ is a corresponding eigenvector.

(8) A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

(9) The constant term of the characteristic polynomial of A is given by $(-1)^n \det(A)$.

(10) Let A and B be similar matrices. Then

  (a) A and B have the same characteristic polynomials.
  (b) A and B have the same eigenvalues.
  (c) If λ is an eigenvalue of A and B, then the eigenspaces of A and B corresponding to λ have the same dimension.

(11) If A is a square matrix, then A and A^T have the same eigenvalues but may not have the same eigenvectors.

Solved Examples

Example 2.1.1. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

Solution. The characteristic equation of A is

$$\begin{aligned} & \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0 \\ \Rightarrow\ & \lambda^2 - 5\lambda + 6 = 0 \\ \Rightarrow\ & (\lambda - 2)(\lambda - 3) = 0 \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 2, 3$. To find the eigenvectors $\mathbf{x}$ corresponding to λ, we have to solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.1.1}$$

- For $\lambda = 2$ in (2.1.1), we get

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equations of this system are

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

Solving, we get

$$x_1 = \frac{s}{2}, \quad x_2 = s \quad (s \in \mathbb{R}).$$

Thus the eigenvectors corresponding to $\lambda = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad (s \in \mathbb{R}).$$

- For $\lambda = 3$ in (2.1.1), we get

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equations of this system are

$$\begin{aligned} x_1 - x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \end{aligned}$$

Solving, we get

$$x_1 = t, \quad x_2 = t \quad (t \in \mathbb{R}).$$

Thus the eigenvectors corresponding to $\lambda = 3$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}). \quad \blacksquare$$

Example 2.1.2. *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution. The characteristic equation of A is

$$\begin{aligned} & \lambda^3 - \operatorname{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0 \\ \Rightarrow\; & \lambda^3 - 5\lambda^2 + (6 + 2 + 0)\lambda - 4 = 0 \\ \Rightarrow\; & \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \\ \Rightarrow\; & (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0 \\ \Rightarrow\; & (\lambda - 1)(\lambda - 2)^2 = 0 \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 1, 2, 2$. To find the eigenvectors corresponding to λ, we have to solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2 - \lambda & 1 \\ 1 & 0 & 3 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.1.2}$$

- For $\lambda = 1$ in (2.1.2), we obtain

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

Applying $R_1 \to (-1)R_1$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Solving, we get

$$x_1 = -2s, \quad x_2 = s, \quad x_3 = s.$$

Thus the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad (s \in \mathbb{R}).$$

- For $\lambda = 2$ in (2.1.2), we obtain

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

Applying $R_1 \to \left(-\frac{1}{2}\right) R_1$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$x_1 + x_3 \;=\; 0$$

Solving, we get

$$x_1 = -t_2, \quad x_2 = t_1, \quad x_3 = t_2.$$

Thus the eigenvectors corresponding to $\lambda = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (t_1, t_2 \in \mathbb{R}). \quad \blacksquare$$

Example 2.1.3. *Find the eigenvalues and bases for the eigenspaces of the matrix*

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution. The characteristic equation of A is

$$\begin{aligned} & \lambda^3 - \mathrm{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0 \\ \Rightarrow\; & \lambda^3 - 6\lambda^2 + (1 + 6 + 4)\lambda - 6 = 0 \\ \Rightarrow\; & \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \\ \Rightarrow\; & (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0 \\ \Rightarrow\; & (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 1, 2, 3$. To find the eigenvectors corresponding to λ, we have to solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.1.3}$$

- For $\lambda = 1$ in (2.1.3), we obtain

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$, we obtain

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \to -\left(\frac{1}{2}\right) R_1$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 + 2R_1$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 &= 0 \\ x_3 &= 0 \end{aligned}$$

From this, we get

$$x_1 = 0, \quad x_2 = s, \quad x_3 = 0.$$

Thus the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (s \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 1$.

- For $\lambda = 2$ in (2.1.3), we obtain

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we obtain

$$\left[\begin{array}{rrr|r} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_1 \to \left(\frac{1}{2}\right) R_1$ and $R_2 \to (-1)R_2$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 + \frac{1}{2}x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Solving, we get

$$x_1 = -\frac{1}{2}t, \quad x_2 = t, \quad x_3 = t.$$

Thus the eigenvectors corresponding to $\lambda = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} (t \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 2$.

- For $\lambda = 3$ in (2.1.3), we obtain

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + 2R_1$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_1 \to \left(-\frac{1}{2}\right) R_1$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$x_1 + x_3 = 0$$

$$x_2 - x_3 = 0.$$

Solving, we get

$$x_1 = -r, \quad x_2 = r, \quad x_3 = r.$$

Thus the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad (r \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 3$. ■

Example 2.1.4. *If* $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, *then find the eigenvalues of* A^T *and* $5A$.

Solution. Here the matrix A is an upper triangular matrix. So the eigenvalues of A are the diagonal entries, namely,

$$\lambda = 1, 2, 2.$$

Since A and A^T have the same eigenvalues, the eigenvalues of A^T are also

$$\lambda = 1, 2, 2.$$

It is known that, if λ is an eigenvalue of a matrix A, then $k\lambda$ is an eigenvalue of kA. Thus the eigenvalues of $5A$ are

$$\lambda = 5(1), 5(2), 5(2) \quad \Rightarrow \quad \lambda = 5, 10, 10. \quad ■$$

Example 2.1.5. *Find the eigenvalues of* A^9 *for* $A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution. Here the matrix A is an upper triangular matrix. So the eigenvalues of A are the diagonal entries, namely,

$$\lambda = 1, \frac{1}{2}, 0, 2.$$

It is known that, if λ is an eigenvalue of a matrix A, then λ^k is an eigenvalue of A^k. Thus the eigenvalues of A^9 are

$$\lambda = 1^9, \left(\frac{1}{2}\right)^9, 0^9, 2^9 \quad \Rightarrow \quad \lambda = 1, \frac{1}{512}, 0, 512. \quad ■$$

Exercises

Exercise 2.1.1. *Find the eigenvalues of the matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Exercise 2.1.2. *Find the eigenvalues and bases for the eigenspaces of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Exercise 2.1.3. *Find bases for the eigenspace of* $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Exercise 2.1.4. *Find the eigenvalues of the matrix* A^T*. Is* A *invertible ?*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

Exercise 2.1.5. *If* $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$*, then find the eigenvalues of* A^2 *and* A^{-1}*.*

Exercise 2.1.6. *Find the eigenvalues and bases for the eigenspaces of* A^{25} *and* $A + 2I$*, where*

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Exercise 2.1.7. *Show that if* $0 < \theta < \pi$*, then* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *has no real eigenvalues and consequently no eigenvector.*

Additional Exercises

Exercise 2.1.8. *Find the eigenvalues and bases for the eigenspaces of the matrix*

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answers

2.1.1 $\lambda = 3, 2, 5$ **2.1.2** $\lambda = -2 : \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}; \lambda = 2 : \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \lambda = 1 : \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

2.1.3 $\lambda = \frac{3}{2} + \frac{\sqrt{17}}{2} : \begin{bmatrix} -\frac{3}{2} + \frac{\sqrt{17}}{2} \\ 1 \\ 1 \end{bmatrix}; \lambda = 2 : \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \frac{3}{2} - \frac{\sqrt{17}}{2} : \begin{bmatrix} -\frac{3}{2} - \frac{\sqrt{17}}{2} \\ 1 \\ 1 \end{bmatrix}$

2.1.4 $1, 0, 9$; no **2.1.5** $1, 4$; $1, \frac{1}{2}$ **2.1.6** $\lambda = 3^{25} : \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$; $\lambda = -1 : \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2.1.8 $\lambda = -2 : \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \lambda = -1 : \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}; \lambda = 1 : \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \lambda = 1 : \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

2.2 Tutorial : Cayley-Hamilton Theorem

Cayley-Hamilton Theorem

Every square matrix A satisfies its characteristic equation.

Solved Examples

Example 2.2.1. *Verify Cayley-Hamilton theorem for the matrix A and hence find A^4, where*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

Solution. The characteristic equation of A is

$$\begin{aligned} & \lambda^3 - \operatorname{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0 \\ \Rightarrow\ & \lambda^3 - 4\lambda^2 + (-3 + 1 + 0)\lambda + 3 = 0 \\ \Rightarrow\ & \lambda^3 - 4\lambda^2 - 2\lambda + 3 = 0. \end{aligned}$$

We have to verify that

$$A^3 - 4A^2 - 2A + 3I = \mathbf{0}. \tag{2.2.1}$$

Now

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 8 \\ 1 & 3 & 3 \\ 4 & 9 & 14 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 3 & 6 & 8 \\ 1 & 3 & 3 \\ 4 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 24 & 36 \\ 4 & 9 & 14 \\ 18 & 42 & 59 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} & A^3 - 4A^2 - 2A + 3I \\ = & \begin{bmatrix} 11 & 24 & 36 \\ 4 & 9 & 14 \\ 18 & 42 & 59 \end{bmatrix} - \begin{bmatrix} 12 & 24 & 32 \\ 4 & 12 & 12 \\ 16 & 36 & 56 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 2 \\ 2 & 6 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence, Cayley-Hamilton theorem is verified. Also, from (2.2.1), we have

$$\begin{aligned} & A^3 = 4A^2 + 2A - 3I \\ \Rightarrow\ & A^4 = 4A^3 + 2A^2 - 3A \\ \Rightarrow\ & A^4 = \begin{bmatrix} 44 & 96 & 144 \\ 16 & 36 & 56 \\ 72 & 168 & 236 \end{bmatrix} + \begin{bmatrix} 6 & 12 & 16 \\ 2 & 6 & 6 \\ 8 & 18 & 28 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ 0 & 0 & 3 \\ 3 & 9 & 9 \end{bmatrix} \\ \Rightarrow\ & A^4 = \begin{bmatrix} 47 & 108 & 154 \\ 18 & 42 & 59 \\ 77 & 177 & 253 \end{bmatrix} \quad \blacksquare \end{aligned}$$

Example 2.2.2. *Use Cayley-Hamilton theorem to find* A^{-1} *for* $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Solution. The characteristic equation of A is

$$\begin{aligned} & \lambda^3 - \operatorname{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0 \\ \Rightarrow\ & \lambda^3 - 4\lambda^2 + (-4 - 6 - 10)\lambda - 35 = 0 \\ \Rightarrow\ & \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0. \end{aligned}$$

By the Cayley-Hamilton theorem,

$$A^3 - 4A^2 - 35I \quad = \quad \mathbf{0}$$

Multiplying both sides by A^{-1}, we get

$$\begin{aligned} & A^{-1}(A^3 - 4A^2 - 20A - 35I) = A^{-1}(\mathbf{0}) \\ \Rightarrow\ & A^2 - 4A - 20I - 35A^{-1} = \mathbf{0} \\ \Rightarrow\ & A^{-1} = \frac{1}{35}(A^2 - 4A - 20I) \end{aligned} \tag{2.2.2}$$

Now

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

Hence by (2.2.2), we obtain

$$\begin{aligned} A^{-1} &= \frac{1}{35}\left(\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} \right) \\ &= \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} \quad \blacksquare \end{aligned}$$

Exercises

Exercise 2.2.1. *Verify Cayley-Hamilton theorem for the matrix A and hence find A^4, where*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

Exercise 2.2.2. *Verify Cayley-Hamilton theorem for the matrix*

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and hence find A^{-1}.

Exercise 2.2.3. *If* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, *then prove that*

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Answers

2.2.1 $\begin{bmatrix} 3 & -8 & 6 \\ 6 & -15 & 10 \\ 10 & -24 & 15 \end{bmatrix}$ **2.2.2** $A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

2.3 Tutorial : Diagonalization

In the last section of previous chapter, we have studied about the similarity of the matrices. In this section, we will study how to find a matrix P for certain matrices such that the matrices are similar to diagonal matrices. The process of finding such P is called the *diagonalization*.

Definition

A square matrix A is called *diagonalizable* if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. The matrix P is said to *diagonalize* A.

Algebraic and Geometric Multiplicities

Let λ_0 be an eigenvalue of an $n \times n$ matrix A. Then the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the *algebraic multiplicity* of λ_0. The dimension of the eigenspace corresponding to λ_0 is called the *geometric multiplicity* of λ_0.

Theorems on Diagonalization

(1) Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

(2) The eigenvectors of an $n \times n$ matrix A corresponding to distinct eigenvalues are linearly independent.

(3) An $n \times n$ matrix A having n distinct eigenvalues is diagonalizable.

(4) Let A be an $n \times n$ matrix. Then

 (a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

 (b) A is diagonalizable if and only if, for every eigenvalue, the geometric multiplicity is equal to the algebraic multiplicity.

Method for Diagonalization

Let A be an $n \times n$ matrix. Then following is the procedure for diagonalizing A.

- Find the eigenvalues of A.
- Find a basis for the eigenspace of A corresponding to each eigenvalue.
- If there are total n basis vectors, then the matrix is diagonalizable, otherwise not.
- Form the matrix P having the n basis vectors as its columns.
- Find the inverse of P.
- Find the matrix $P^{-1}AP$ which will be a diagonal matrix having the eigenvalues of A as its diagonal entries.

Computation of Powers of a matrix

Let A be an $n \times n$ diagonalizable matrix. Then there exists an invertible matrix P such that

$$P^{-1}AP = D, \tag{2.3.1}$$

where D is a diagonal matrix. Now,

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P.$$

In general,

$$(P^{-1}AP)^k = P^{-1}A^kP.$$

Using (2.3.1), we get

$$P^{-1}A^kP = D^k \quad \Rightarrow \quad A^k = PD^kP^{-1}.$$

The last equation expresses the k^{th} power of A in terms of the k^{th} power of the diagonal matrix D and D^k can be easily obtained by taking the k^{th} power of the diagonal entries of D.

Solved Examples

Example 2.3.1. *Find a matrix P that diagonalizes A and determine $P^{-1}AP$, where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution. The characteristic equation of A is

$$\begin{aligned} & \lambda^3 - \operatorname{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0 \\ \Rightarrow\ & \lambda^3 - 3\lambda^2 + (0 + 1 + 1)\lambda - 0 = 0 \\ \Rightarrow\ & \lambda^3 - 3\lambda^2 + 2\lambda = 0. \\ \Rightarrow\ & \lambda(\lambda^2 - 3\lambda + 2) = 0 \\ \Rightarrow\ & \lambda(\lambda - 1)(\lambda - 2) = 0 \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 0, 1, 2$. To find the eigenvectors corresponding to λ, we have to solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.3.2}$$

- For $\lambda = 0$ in (2.3.2), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_3 \leftrightarrow R_3 - R_2$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

From this, we get

$$x_1 = 0, \quad x_2 = -t, \quad x_3 = t.$$

Thus the eigenvectors corresponding to $\lambda = 0$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 0$.

- For $\lambda = 1$ in (2.3.2), we obtain

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$, we obtain

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

So, the general solution of the system is given by

$$x_1 = s, \quad x_2 = 0, \quad x_3 = 0.$$

Thus the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ (s \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 1$.

- For $\lambda = 2$ in (2.3.2), we obtain

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Applying $R_1 \to (-1)R_1$ and $R_2 \to (-1)R_2$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Solving, we get

$$x_1 = 0, \quad x_2 = r, \quad x_3 = r.$$

Thus the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad (r \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 2$.

Since there are total three basis vectors for the eigenspace, the matrix A is diagonalizable and

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

To find P^{-1}, form the block matrix

$$[P \mid I] = \left[\begin{array}{rrr|rrr} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$, we obtain

$$[P \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$, we obtain

$$[P \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

Applying $R_2 \to R_3$, we obtain

$$[P \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

Applying $R_3 \to \left(\frac{1}{2}\right) R_3$, we obtain

$$[P \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Applying $R_1 \to R_1 - R_3$, we obtain

$$[P \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Thus

$$P^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Also,

$$P^{-1}AP = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \blacksquare$$

Example 2.3.2. *Find the algebraic and geometric multiplicity of each of the eigenvalues of the matrix* $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution. The characteristic equation of A is

$$\begin{aligned}
& \lambda^3 - \operatorname{tr}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0 \\
\Rightarrow\ & \lambda^3 - 0\lambda^2 + (-1 - 1 - 1)\lambda - 2 = 0 \\
\Rightarrow\ & \lambda^3 - 3\lambda - 2 = 0 \\
\Rightarrow\ & (\lambda + 1)(\lambda^2 - \lambda - 2) = 0 \\
\Rightarrow\ & (\lambda + 1)(\lambda + 1)(\lambda - 2) = 0.
\end{aligned}$$

Thus the eigenvalues of A are $\lambda = -1, -1, 2$. Since the factors $\lambda + 1$ and $\lambda - 2$ appear two times and one time respectively, the algebraic multiplicity of $\lambda = -1$ is 2 and that of $\lambda = 2$ is 1.

To find the geometric multiplicity, we have to find the dimension of the eigenspace corresponding to each λ. Consider the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.3.3}$$

- For $\lambda = -1$ in (2.3.3), we obtain

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponding to the last matrix is

$$x_1 + x_2 + x_3 = 0.$$

Solving, we get

$$x_1 = -t_1 - t_2, \quad x_2 = t_2, \quad x_3 = t_1.$$

Thus the eigenvectors corresponding to $\lambda = -1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 \\ t_2 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad (t_1, t_2 \in \mathbb{R}).$$

Since the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent, they form a basis for the eigenspace of A corresponding to $\lambda = -1$. Thus the geometric multiplicity of $\lambda = -1$ is 2.

- For $\lambda = 2$ in (2.3.3), we obtain

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_3$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_2 \to \left(-\frac{1}{3}\right) R_2$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_1 \to R_2 - R_1$, we obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in reduced row echelon form. The system corresponding to the last matrix is

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0. \end{aligned}$$

Solving, we get

$$x_1 = s, \quad x_2 = s, \quad x_3 = s.$$

Thus the eigenvectors corresponding to $\lambda = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (s \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 2$. Thus the geometric multiplicity of $\lambda = 2$ is 1. ■

Example 2.3.3. *Show that the matrix* $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ *is not diagonalizable.*

Solution. Observe that the matrix A is a lower triangular matrix. So, the eigenvalues of A are the diagonal entries, namely

$$\lambda = 3, 2, 2.$$

Thus the algebraic multiplicity of $\lambda = 3$ is 1 and that of 2 is 2.

It is known that, a matrix A is diagonalizable if and only if, for every eigenvalue, the geometric multiplicity is equal to the algebraic multiplicity.

Let us find the geometric multiplicity of $\lambda = 2$. Consider the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 2$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_3$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in row echelon form. The system corresponds to the last matrix is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

So, the general solution of the system is given by

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = t.$$

Thus the eigenvectors of A corresponding to $\lambda = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ (t \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 2$. So, the geometric multiplicity of $\lambda = 2$ is 1.

Thus for $\lambda = 2$, the geometric multiplicity is not equal to the algebraic multiplicity. Hence the matrix A is not diagonalizable. ■

Example 2.3.4. *Determine whether the matrix* $A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ *is diagonalizable.*

Solution. Observe that the matrix A is a lower triangular matrix. So, the eigenvalues of A are the diagonal entries, namely

$$\lambda = 4, 2, 1.$$

Since the eigenvectors of A corresponding to the distinct eigenvalues are linearly independent, the matrix A will have three linearly independent eigenvectors. So, A is diagonalizable. ■

Example 2.3.5. *Find a matrix* P *that diagonalizes the matrix* $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ *and hence find* A^{13}.

Solution. Observe that the matrix A is a lower triangular matrix. So, the eigenvalues of A are the diagonal entries, namely

$$\lambda = 1, 2.$$

We find the eigenvectors $\mathbf{x}$ corresponding to λ by solving the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.3.4}$$

- For $\lambda = 1$ in (2.3.4), we get

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equations of this system are

$$\begin{aligned} 0x_1 + 0x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

Solving, we get

$$x_1 = t, \quad x_2 = t \quad (t \in \mathbb{R}).$$

Thus the eigenvectors corresponding to $\lambda = 1$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (t \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 1$.

- For $\lambda = 2$ in (2.3.4), we get

$$\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equations of this system are

$$\begin{aligned} -x_1 &= 0 \\ -x_1 &= 0 \end{aligned}$$

So, the general solution of the system is given by

$$x_1 = 0, \quad x_2 = s \quad (s \in \mathbb{R}).$$

Thus the eigenvectors corresponding to $\lambda = 2$ are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (s \in \mathbb{R}).$$

Since the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace of A corresponding to $\lambda = 2$.

Let

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Then

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

So, we obtain

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Now,

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{13} & 0 \\ 0 & 2^{13} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8191 & 8192 \end{bmatrix} \quad \blacksquare$$

Exercises

Exercise 2.3.1. *Find a matrix P that diagonalizes $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$. Also determine $P^{-1}AP$.*

Exercise 2.3.2. *Find a matrix P that diagonalizes $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, and determine $P^{-1}AP$.*

Exercise 2.3.3. *Determine the algebraic and geometric multiplicity of each of the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$*

Exercise 2.3.4. *Show that the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable.*

Exercise 2.3.5. *Determine whether the matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is diagonalizable.*

Exercise 2.3.6. *Find a matrix P that diagonalizes the matrix $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}$ and hence find A^{11}.*

Answers

2.3.1 $P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **2.3.2** $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2.3.3 $\lambda = 1 : 1, 1$; $\lambda = 2 : 2, 1$ **2.3.5** A is diagonalizable

2.3.6 $P = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$; $A^{11} = \begin{bmatrix} 0 & 0 & 0 \\ -1022 & 2046 & -4094 \\ -1023 & 2047 & -4095 \end{bmatrix}$