MATHEMATICS-II

UNIT-4 : VECTOR SPACES

Chapter 6

Vector Spaces

Vector space is a nonempty set of objects that satisfies several axioms. These objects are called *vectors*. In this chapter, we mainely work with two operations: addition of two objects and multiplication between a scalar and an object.

6.1 Tutorial : Real Vector Spaces

Definition

Let V be a nonempty set of objects on which two operations are defined: addition and multiplication by a scalar. *Addition* is a rule that associates with each pair of objects u and v in V an object $u+v$; *scalar multiplication* is a rule that associates with each scalar k and each object u an object ku. Then V is called *vector space* if for all objects u, v, w in V and all scalars k and m, the following axioms are satisfied:

(1) V is closed under addition, i.e. $u, v \in V \Rightarrow u+v \in V$;

(2) $u+v = v+u$;

(3) $u+(v+w) = (u+v)+w$;

(4) There is an object $0 \in V$, called the *zero* for V, such that $u+0 = 0$ for all $u \in V$;

(5) For each $u \in V$, there is an object $-u \in V$, called the *negative* of u, such that $u+(-u) = 0$;

(6) V is closed under scalar multiplication, i.e. $u \in V \Rightarrow ku \in V$ for every scalar k;

(7) $k(u+v) = ku+kv$;

(8) $(k+m)u = ku+mu$;

(9) $k(mu) = (km)u$;

(10) $1u = u$.

Remark. In the above definition, scalars may be real numbers or complex numbers. Vector spaces in which the scalars are real numbers are called *real vector spaces*, and those in which the scalars are complex numbers are called *complex vector spaces*. We will discuss real vector spaces only. Complex vector spaces are beyond the scope of this book.

Examples of Standard Vector Spaces

(1) The set $\mathbb{R}^n = \{(u_1, u_2, \ldots, u_n) \mid u_1, u_2, \ldots u_n \in \mathbb{R}\}$ is a vector space under the operations defined as follows:

If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are in $\mathbb{R}^n$ and k is any scalar, then

$$u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \quad \text{and} \quad ku = (ku_1, ku_2, \ldots, ku_n).$$

These operations on $\mathbb{R}^n$ are called the *standard operations.*

(2) The set $M_{22} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ is a vector space with the operations defined as follows:

If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ are in M_{22} and k is any scalar, then

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} \quad \text{and} \quad kA = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}$$

These operations are standard matrix addition and scalar multiplication.

(3) The set $P_n = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$ is a vector space under the operations defined as follows:

If $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ and $q(x) = b_0 + b_1 x + \ldots + b_n x^n$ are in P_n and k is any scalar, then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$kp(x) = ka_0 + ka_1 x + \ldots + ka_n x^n.$$

Note: It is worth checking that above are vector spaces.

Solved Examples

Example 6.1.1. *Check whether the set $V = \mathbb{R}^2$ is a vector space under the operations*

$$(u_1, u_2) + (v_1, v_2) = (u_1 + u_2 - 1, v_1 + v_2 - 1) \quad \textit{and} \quad k(u_1, u_2) = (ku_1, ku_2).$$

Solution. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ be in V and k, m be any scalars.

Axiom 1. Observe that

$$u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 1, u_2 + v_2 - 1) \in V$$

because $u_1 + v_1 - 1,\ u_2 + v_2 - 1 \in \mathbb{R}$.

Axiom 2. Observe that

$$\begin{aligned} u + v &= (u_1, u_2) + (v_1, v_2) \\ &= (u_1 + v_1 - 1, u_2 + v_2 - 1) \\ &= (v_1 + u_1 - 1, v_2 + u_2 - 1) \quad (\because u_i, v_i, 1 \in \mathbb{R}) \\ &= (v_1, v_2) + (u_1, u_2) \end{aligned}$$

$$= v + u.$$

Axiom 3. Observe that

$$\begin{aligned}
u + (v + w) &= (u_1, u_2) + \big[(v_1, v_2) + (w_1, w_2)\big] \\
&= (u_1, u_2) + (v_1 + w_1 - 1, v_2 + w_2 - 1) \\
&= [u_1 + (v_1 + w_1 - 1) - 1, u_2 + (v_2 + w_2 - 1) - 1] \\
&= [(u_1 + v_1 - 1) + w_1 - 1, (u_2 + v_2 - 1) + w_2 - 1] \quad (\because u_i, v_i, w_i, 1 \in \mathbb{R}) \\
&= (u_1 + v_1 - 1, u_2 + v_2 - 1) + (w_1, w_2) \\
&= \big[(u_1, u_2) + (v_1, v_2)\big] + (w_1, w_2) \\
&= (u + v) + w.
\end{aligned}$$

Axiom 4. For any $u = (u_1, u_2) \in V$, let $0 = (1, 1)$. Then $0 \in V$. Also,

$$u + 0 = (u_1, u_2) + (1, 1) = (u_1 + 1 - 1, u_2 + 1 - 1) = (u_1, u_2) = u.$$

Thus $(1, 1)$ is the zero of V.

Axiom 5. For any $u = (u_1, u_2) \in V$, let $-u = (-u_1 + 2, -u_2 + 2) \in V$. Then,

$$u + (-u) = (u_1, u_2) + (-u_1 + 2, -u_2 + 2) = (u_1 - u_1 + 2 - 1, u_2 - u_2 + 2 - 1) = (1, 1) = 0.$$

Axiom 6. Observe that

$$ku = k(u_1, u_2) = (ku_1, ku_2) \in V$$

because $ku_1, ku_2 \in \mathbb{R}$.

Axiom 7. Observe that

$$\begin{aligned}
k(u + v) &= k\big[(u_1, u_2) + (v_1, v_2)\big] \\
&= k(u_1 + v_1 - 1, u_2 + v_2 - 1) \\
&= (ku_1 + kv_1 - k, ku_2 + kv_2 - k)
\end{aligned}$$

and

$$\begin{aligned}
ku + kv &= k(u_1, u_2) + k(v_1, v_2) \\
&= (ku_1, ku_2) + (kv_1, kv_2) \\
&= (ku_1 + kv_1 - 1, ku_2 + kv_2 - 1)
\end{aligned}$$

Thus

$$k(u + v) \neq ku + kv \quad \text{if} \quad k \neq 1.$$

Hence V is not a vector space. ■

Example 6.1.2. *Show that the set of all pairs of real numbers of the form $(1, y)$ is a not vector space with the standard operations.*

Solution. Let $V = \{(1, y) \mid y \in \mathbb{R}\}$ and let $u = (1, y_1)$ and $v = (1, y_1)$ in V. Then

$$u + v = (1, y_1) + (1, y_2) = (1 + 1, y_1 + y_2) = (2, y_1 + y_2).$$

Since the first coordinate in not 1, $u + v \notin V$. Thus V is not a vector space. ■

Example 6.1.3. *Show that the set of all* 2×2 *matrices of the form* $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ *is a vector space under the operations*

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix} \quad \text{and} \quad k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$$

[GTU- May 2012, June 2013]

Solution. Let $V = \left\{ A \in M_{22} \mid A = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \right\}$. We verify all the axioms for the vector space. For that, let $A_1 = \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix}$, $A_3 = \begin{bmatrix} a_3 & 1 \\ 1 & b_3 \end{bmatrix}$ in V and k, m be any scalars.

Axiom 1. Observe that

$$A_1 + A_2 = \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 1 \\ 1 & b_1+b_2 \end{bmatrix}$$

which is of the form given in V because $a_1 + a_2$, $b_1 + b_2 \in \mathbb{R}$. Thus $A_1 + A_2 \in V$.

Axiom 2. Observe that

$$\begin{aligned} A_1 + A_2 &= \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1+a_2 & 1 \\ 1 & b_1+b_2 \end{bmatrix} \\ &= \begin{bmatrix} a_2+a_1 & 1 \\ 1 & b_2+b_1 \end{bmatrix} \\ &= \begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix} + \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \\ &= A_2 + A_1. \end{aligned}$$

Axiom 3. Observe that

$$\begin{aligned} A_1 + (A_2 + A_3) &= \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + \left(\begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix} + \begin{bmatrix} a_3 & 1 \\ 1 & b_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2+a_3 & 1 \\ 1 & b_2+b_3 \end{bmatrix} \\ &= \begin{bmatrix} a_1+(a_2+a_3) & 1 \\ 1 & b_1+(b_2+b_3) \end{bmatrix} \\ &= \begin{bmatrix} (a_1+a_2)+a_3 & 1 \\ 1 & (b_1+b_2)+b_3 \end{bmatrix} \quad (\because a_i, b_i \in \mathbb{R}) \\ &= \begin{bmatrix} a_1+a_2 & 1 \\ 1 & b_1+b_2 \end{bmatrix} + \begin{bmatrix} a_3 & 1 \\ 1 & b_3 \end{bmatrix} \\ &= \left(\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix} \right) + \begin{bmatrix} a_3 & 1 \\ 1 & b_3 \end{bmatrix} \\ &= (A_1 + A_2) + A_3. \end{aligned}$$

Axiom 4. For any $A = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \in V$, let $0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $0 \in V$. Also

$$A + 0 = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a+0 & 1 \\ 1 & b+0 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = A.$$

Axiom 5. For any $A = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \in V$, let $-A = \begin{bmatrix} -a & 1 \\ 1 & -b \end{bmatrix}$. Then $-A \in V$. Also

$$A + (-A) = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} -a & 1 \\ 1 & -b \end{bmatrix} = \begin{bmatrix} a-a & 1 \\ 1 & b-b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0.$$

Axiom 6. Observe that

$$kA_1 = k\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} = \begin{bmatrix} ka_1 & 1 \\ 1 & kb_1 \end{bmatrix}$$

which is of the form given in V because $ka_1,\ ka_2 \in \mathbb{R}$. Thus $kA_1 \in V$.

Axiom 7. Observe that

$$\begin{aligned}
k(A_1 + A_2) &= k\begin{bmatrix} a_1 + a_2 & 1 \\ 1 & b_1 + b_2 \end{bmatrix} \\
&= \begin{bmatrix} k(a_1 + a_2) & 1 \\ 1 & k(b_1 + b_2) \end{bmatrix} \\
&= \begin{bmatrix} ka_1 + ka_2 & 1 \\ 1 & kb_1 + kb_2 \end{bmatrix} \quad (\because k, a_1, a_2, b_1, b_2 \in \mathbb{R}) \\
&= \begin{bmatrix} ka_1 & 1 \\ 1 & kb_1 \end{bmatrix} + \begin{bmatrix} ka_2 & 1 \\ 1 & kb_2 \end{bmatrix} \\
&= k\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + k\begin{bmatrix} a_2 & 1 \\ 1 & b_2 \end{bmatrix} \\
&= kA_1 + kA_2.
\end{aligned}$$

Axiom 8. Observe that

$$\begin{aligned}
(k+m)A_1 &= (k+m)\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \\
&= \begin{bmatrix} (k+m)a_1 & 1 \\ 1 & (k+m)b_1 \end{bmatrix} \\
&= \begin{bmatrix} ka_1 + ma_1 & 1 \\ 1 & kb_1 + mb_1 \end{bmatrix} \quad (\because k, m, a_1, b_1 \in \mathbb{R}) \\
&= \begin{bmatrix} ka_1 & 1 \\ 1 & kb_1 \end{bmatrix} + \begin{bmatrix} ma_1 & 1 \\ 1 & mb_1 \end{bmatrix} \\
&= k\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} + m\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \\
&= kA_1 + mA_1.
\end{aligned}$$

Axiom 9. Observe that

$$\begin{aligned}(km)A_1 &= (km)\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} \\ &= \begin{bmatrix} (km)a_1 & 1 \\ 1 & (km)b_1 \end{bmatrix} \\ &= \begin{bmatrix} k(ma_1) & 1 \\ 1 & k(mb_1) \end{bmatrix} \quad (\because k, m, a_1, b_1 \in \mathbb{R}) \\ &= k\begin{bmatrix} ma_1 & 1 \\ 1 & mb_1 \end{bmatrix} \\ &= k\left(m\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix}\right) \\ &= k(mA_1).\end{aligned}$$

Axiom 10. Observe that

$$1A_1 = 1\begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} = \begin{bmatrix} 1a_1 & 1 \\ 1 & 1b_1 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & b_1 \end{bmatrix} = A_1.$$

Hence V is a vector space. ■

Exercises

Exercise 6.1.1. *Show that the set of all pairs of real numbers of the form* $(1, y)$ *is a vector space under the operations*

$$(1, y_1) + (1, y_2) = (1, y_1 + y_2) \quad \textit{and} \quad k(1, y) = (1, ky).$$

Remark. You must have observed that the same set was shown as not forming a vector space in Example 6.1.2. In fact, a set along with operations will make a vector space.

Exercise 6.1.2. *Show that the set of all triples of real numbers* (u_1, u_2, u_3) *is not a vector space with the operations*

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \textit{and} \quad k(u_1, u_2, u_3) = (0, 0, 0).$$

Exercise 6.1.3. *Show that the set* $V = \mathbb{R}^3$ *is not a vector space under the operations*

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \textit{and} \quad k(u_1, u_2, u_3) = (u_1, u_2, ku_3).$$

Exercise 6.1.4. *Determine whether the set of all* 2×2 *matrices of the form* $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ *is a vector space under the standard matrix addition and scalar multiplication.*

Exercise 6.1.5. *Show that the set of polynomials of the form* $a + bx$ $(a, b \in \mathbb{R})$ *is a vector space under the operations*

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x \quad \textit{and} \quad k(a_0 + a_1x) = (ka_0) + (ka_1)x.$$

Exercise 6.1.6. *Check whether $V = \mathbb{R}^2$ is a vector space with respect to the operations*

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 2, u_2 + v_2 - 3) \quad \textit{and} \quad \alpha(u_1, u_2) = (\alpha u_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3).$$

Clearly mention the axioms which are failed to be hold.

Exercise 6.1.7. *Check whether the set $V = \mathbb{R}^+$ is a vector space under the operations*

$$x + y = xy, \quad \text{and} \quad kx = x^k.$$

Exercise 6.1.8. *Check whether the set $V = \mathbb{R}^2$ is a vector space under the operations*

$$(x, y) + (x'y') = (x + x', 2y + y') \quad \textit{and} \quad \alpha(x, y) = (\alpha x, \alpha y).$$

Exercise 6.1.9. *Why $\mathbb{R}^2$ is not a vector space with the following operations? Justify.*

(a) $(x, y) + (x', y') = (x + x', y + y')$ and $k(x, y) = (k^2 x, k^2 y)$

(b) $(x, y) + (x', y') = (x + x', y + y')$ and $k(x, y) = (2kx, 2ky)$

(c) $(x, y) + (x', y') = (y + y', x + x')$ and $k(x, y) = (kx, ky)$

Answers

6.1.4 yes **6.1.6** no **6.1.7** yes **6.1.8** no

6.2 Tutorial : Subspaces

Definition

Let V be a vector space and W be any subset of V. Then W is called a *subspace* of V if W itself is a vector space under the the addition and scalar multiplication defined on V.

Necessary and Sufficient Conditions for Subspace

Let W be a subset of a vector space V. Then W is a subspace of V if and only if

(1) W is nonempty;

(2) W is closed under addition: $u,\, v \in W \Rightarrow u + v \in W$;

(3) W is closed scalar multiplication: $u \in W \Rightarrow ku \in W$ for every scalar k.

Remark. We will use these for checking subspaces.

Solved Examples

Example 6.2.1. *Determine whether the following sets are subspaces of* $\mathbb{R}^3$.

(a) all vectors of the form $(a, b, 0)$*;*

(b) all vectors of the form $(a, b, 1)$*;*

(c) all vectors of the form (a, b, c)*, where* $c = a + b$*;*

Solution. (a) Let $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$.

- Observe that $0 = (0, 0, 0) \in W$ since the third component of 0 is 0. So W is nonempty.
- For any $u = (a, b, 0), v = (c, d, 0) \in W$,

$$u + v = (a, b, 0) + (c, d, 0) = (a + c, b + d, 0) \in W$$

  because the third component of $u + v$ is 0 and $a + c, b + d \in \mathbb{R}$.

- For any $u = (a, b, 0) \in W$ and for any scalar k,

$$ku = k(a, b, 0) = (ka, kb, 0) \in W$$

  because the third component of ku is 0 and $ka, kb \in \mathbb{R}$.

Thus W is a subspace of $\mathbb{R}^3$.
(b) Let $W = \{(a, b, 1) : a, b \in \mathbb{R}\}$. If we take $u = (2, 3, 1), v = (-1, 2, 1) \in W$, then

$$u + v = (2, 3, 1) + (-1, 2, 1) = (1, 5, 2) \notin W$$

because the third component is not 1. Thus W is not a subspace of $\mathbb{R}^3$.
(c) Let $W = \{(a, b, c) \in \mathbb{R}^3 : c = a + b\}$.

- Observe that $0 = (0, 0, 0) \in W$ since $0 = 0 + 0$.

- Let $u = (a_1, b_1, c_1), v = (a_2, b_2, c_2) \in W$. Then $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$. Now

$$u + v = (a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2).$$

  Since $c_1 + c_2 = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$, we get $u + v \in W$.

- Let $u = (a, b, c) \in W$ and k be any scalar. Then $c = a + b$. Now

$$ku = k(a, b, c) = (ka, kb, kc).$$

  Since $kc = k(a + b) = ka + kb$, we get $ku \in W$.

Thus W is a subspace of $\mathbb{R}^3$. ■

Example 6.2.2. *Show that the set* $W = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ *is a subspace of* M_{22}.

Solution. We verify all the conditions for the subspace.

- Observe that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ because $0 = -0$.

- Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}, A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} \in W$. Then

$$\begin{aligned} A_1 + A_2 &= \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -a_1 - a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_1 + a_2) \end{bmatrix} \in W \end{aligned}$$

  because it of the form given in W.

- Let $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in W$ and k be any scalar. Then

$$kA = k \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & -ka \end{bmatrix} \in W$$

  because it of the form given in W.

Thus W is a subspace. ■

Example 6.2.3. *Check whether the set* $W = \{A \in M_{22} \mid \det(A) = 0\}$ *is a subspace of* M_{22}.

Solution. Consider the following matrices:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Since $\det(A_1) = 0$ and $\det(A_2) = 0$, we have $A_1, A_2 \in W$. Now

$$A_1 + A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

So, $\det(A_1 + A_2) = -1 \neq 0$. Thus $A_1 + A_2 \notin W$. Hence W is a not a subspace of M_{22}. ■

Example 6.2.4. *Check whether the set*

$$W = \{a_0 + a_1x + a_2x^2 + a_3x^3,\ \textit{where } a_0 + a_1 + a_2 + a_3 = 0; a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$$

is a subspace of P_3. **[GTU, May 2012]**

Solution. We verify all the conditions for the subspace.

- Observe that $0 = 0 + 0x + 0x^2 + 0x^3 \in W$ since $0 + 0 + 0 + 0 = 0$. Thus W is nonempty.
- Let $p(x), q(x) \in W$. Then

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3,\ \text{where } a_0 + a_1 + a_2 + a_3 = 0.$$

  and

$$q(x) = b_0 + b_1x + b_2x^2 + b_3x^3,\ \text{where } b_0 + b_1 + b_2 + b_3 = 0.$$

  Now

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3. \end{aligned}$$

  Observe that

$$\begin{aligned} & (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) \\ = \; & (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) \\ = \; & 0 + 0 = 0. \end{aligned}$$

  Thus $p(x) + q(x) \in W$.
- Let $p(x) \in W$ and k be any scalar. Then

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3,\ \text{where } a_0 + a_1 + a_2 + a_3 = 0.$$

  Now

$$kp(x) = k(a_0 + a_1x + a_2x^2 + a_3x^3) = ka_0 + ka_1x + ka_2x^2 + ka_3x^3.$$

  Observe that

$$ka_0 + ka_1 + ka_2 + ka_3 = k(a_0 + a_1 + a_2 + a_3) = k(0) = 0.$$

  Thus $kp(x) \in W$.

Hence W is a subspace of P_3 ■

Exercises

Exercise 6.2.1. *Determine whether the following sets are subspaces of* $\mathbb{R}^3$.

(a) *all vectors of the form* $(a, 0, 0)$

(b) *all vectors of the form* $(a, 1, 1)$

(c) *all vectors of the form* (a, b, c), *where* $b = a + c + 1$ ***[GTU, June 2013]***

Exercise 6.2.2. *Show that* $V = \{(x, y) \mid x = 3y\}$ *is a subspace of* $\mathbb{R}^2$. *State all possible subspaces of* $\mathbb{R}^2$. **[GTU, June 2009]**

Exercise 6.2.3. *Check whether* $W = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0; a, b, c \in \mathbb{R}\}$ *is a subspace of* $\mathbb{R}^3$. **[GTU, July 2011]**

Exercise 6.2.4. *Check whether* $W = \{A \in M_{22} \mid \det(A) \neq 0\}$ *is a subspace of* M_{22}.

Exercise 6.2.5. *Check whether* $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid a + b + c + d = 1 \right\}$ *is a subspace of* M_{22}.

Exercise 6.2.6. *Check whether* $W = \{A \in M_n(\mathbb{R}) \mid tr(A) = 0\}$ *is a subspace of* $M_n(\mathbb{R})$.

Exercise 6.2.7. *Show that the set of all* $n \times n$ *symmetric matrices is a subspace of* $M_n(\mathbb{R})$.

Exercise 6.2.8. *Check whether* $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3 \mid a_0 = 0\}$ *is a subspace of* P_3.

Exercise 6.2.9. *Check whether the set*

$$W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3 \mid a_i's \text{ are integers for } i = 0, 1, 2, 3\}$$

is a subspace of P_3.

Exercise 6.2.10. *Determine which of the following sets are subspaces of* P_n?

(a) $W = \{p(x) \in P_n \mid \deg p(x) \leq 2\}$

(b) $W = \{p(x) \in P_n \mid \deg p(x) \geq 2\}$

(c) $W = \{p(x) \in P_n \mid p(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}\}$

Exercise 6.2.11. *Check whether* $W = \{f \in F(-\infty, \infty) \mid f(x) \leq 0, \forall x\}$ *is a subspace of* $F(-\infty, \infty)$. **[GTU, June 2010]**

Exercise 6.2.12. *Check whether* $W = \{f \in F(-\infty, \infty) \mid f(0) = 1\}$ *is a subspace of* $F(-\infty, \infty)$.

Answers

6.2.1 (a) yes **(b)** no **(c)** no **6.2.2** origin, a line through origin, $\mathbb{R}^2$ **6.2.3** yes
6.2.4 no **6.2.5** no **6.2.6** yes **6.2.7** yes **6.2.8** yes **6.2.9** no
6.2.10 (a) yes **(b)** no **(c)** yes **6.2.11** no **6.2.12** no

6.3 Tutorial : Linear Dependence & Independence

Definition

The vectors $v_1, v_2, \ldots, v_n$ are said to be *linear dependent* if there exist scalars $k_1, k_2, \ldots, k_n$, not all zero such that

$$k_1 v_1 + k_2 + \cdots + k_n v_n = 0$$

and *linearly independent* if

$$k_1 v_1 + k_2 + \cdots + k_n v_n = 0 \quad \Rightarrow \quad k_1 = k_2 = \cdots = k_n = 0.$$

Theorems on Linear Dependence & Independence

(1) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of n vectors. Then S is linearly dependent if and only if one of the vectors in S can be expressed as a linear combination of the other vectors in S.

(2) A set of two vectors is linearly dependent if and only if one vector is a scalar multiple of the other.

(3) A set containing zero vector is linearly dependent.

(4) If $v_1, v_2, \cdots, v_k$ are vectors in $\mathbb{R}^n$ and $k > n$, then the vectors are linearly dependent.

Solved Examples

Example 6.3.1. *Prove that the set of vectors* $\{(1,2,2),(2,1,2),(2,2,1)\}$ *is linearly independent in* $\mathbb{R}^3$. **[GTU, July 2011]**

Solution. Let $v_1 = (1,2,2), v_2 = (2,1,2), v_3 = (2,2,1)$. Suppose that

$$\begin{aligned} & k_1 v_1 + k_2 + \cdots + k_n v_n = 0 \\ \Rightarrow\ & k_1(1,2,2) + k_2(2,1,2) + k_3(2,2,1) = (0,0,0) \\ \Rightarrow\ & (k_1 + 2k_2 + 2k_3, 2k_1 + k_2 + 2k_3, 2k_1 + 2k_2 + k_3) = (0,0,0) \end{aligned}$$

Comparing the corresponding coefficients, we obtain

$$\begin{aligned} k_1 + 2k_2 + 2k_3 &= 0 \\ 2k_1 + k_2 + 2k_3 &= 0 \\ 2k_1 + 2k_2 + k_3 &= 0 \end{aligned}$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Observe that

$$\det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-4) + 2(4-2) = -3 + 4 + 4 = 5 \neq 0.$$

Therefore, the system has only the trivial solution (see Remark in Tutorial 1.5).

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0.$$

Hence v_1, v_2, v_3 are linearly independent. ■

Example 6.3.2. *Determine whether the vectors* $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$ *are linearly independent or dependent in* $\mathbb{R}^3$.

Solution. Here the number of vectors is 4 and the vector space is $\mathbb{R}^3$. Since $4 > 3$, the vectors are linearly dependent. ■

Example 6.3.3. *Show that* $S = \{2x^2 - x + 7, x^2 + 4x + 2, x^2 - 2x + 4\}$ *is linearly dependent in* P_2.

Solution. Let $p_1(x) = 2x^2 - x + 7$, $p_2(x) = x^2 + 4x + 2$, $p_3(x) = x^2 - 2x + 4$. Suppose

$$\begin{aligned} & k_1(2x^2 - x + 7) + k_2(x^2 + 4x + 2) + k_3(x^2 - 2x + 4) = 0x^2 + 0x + 0 \\ \Rightarrow \quad & (2k_1 + k_2 + k_3)x^2 + (-k_1 + 4k_2 - 2k_3)x + (7k_1 + 2k_2 + 4k_3) = 0x^2 + 0x + 0 \end{aligned}$$

Equating the corresponding coefficients of x^2, x, 1 on both sides, we get

$$\begin{aligned} 2k_1 + k_2 + k_3 &= 0 \\ -k_1 + 4k_2 - 2k_3 &= 0 \\ 7k_1 + 2k_2 + 4k_3 &= 0 \end{aligned}$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 4 & -2 \\ 7 & 2 & 4 \end{bmatrix}$$

Observe that

$$\det(A) = 2(16 + 4) - 1(-4 + 14) + 1(-2 - 28) = 40 - 10 - 30 = 0.$$

Consequently, the system will have a nontrivial solution. Thus there exist k_1, k_2, k_3, not all zero such that $k_1p_1(x) + k_2p_2(x) + k_3p_3(x) = 0$. Hence $p_1(x)$, $p_2(x)$, $p_3(x)$ are linearly dependent. ■

Example 6.3.4. *Determine whether the following vectors are linearly dependent or independent:*

(i) $A_1 = \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}$; $A_2 = \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$ *in* M_{22}

(ii) $p_1 = 1 - x^2$; $p_2 = 6 + 3x - 4x^2$ *in* P_2

Solution. It is known that two vectors are linearly dependent if and only if one is a scalar multiple of the other.

(i) Observe that $A_1 = -A_2$. Thus A_1 and A_2 are linearly dependent.

(ii) Since $p_1 \neq kp_2$ for any value of k, p_1 and p_2 are linearly independent. ■

Exercises

Exercise 6.3.1. *Show that the set of vectors* $\{(2,1,1),(1,2,2),(1,1,1)\}$ *is linearly dependent in* $\mathbb{R}^3$. **[GTU, July 2011]**

Exercise 6.3.2. *Check whether the vectors* $(0,0,2,2),(3,3,0,0),(1,1,0,-1)$ *are linearly independent in* $\mathbb{R}^4$ *or not.*

Exercise 6.3.3. *Check whether* $S=\{(2,2,2),(-1,3,4),(0,0,1),(3,0,0)\}$ *is linearly dependent or independent in* $\mathbb{R}^3$.

Exercise 6.3.4. *Show that* $S=\{1-t-t^3,-2+3t+t^2+2t^3,1+t^2+5t^3\}$ *is linearly independent in* P_3. **[GTU, June 2010]**

Exercise 6.3.5. *Show that the set* $S=\{1,x,e^x\}$ *is linearly independent in* $C^2(-\infty,\infty)$.

Exercise 6.3.6. *Show that the set* $S=\{e^x,xe^x,x^2e^x\}$ *in* $C^2(-\infty,\infty)$ *is linearly independent.* ***[GTU, June 2010]***

Answers

6.3.2 yes **6.3.3** linearly dependent

6.4 Tutorial : Basis and Dimension

Basis

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space V. Then S is called a *basis* for V if it satisfies the following conditions:

(1) S is linearly independent;

(2) S spans V.

Some Standard Bases

(1) The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is the standard basis for $\mathbb{R}^3$.

(2) The set $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is the standard basis for M_{22}.

(3) The set $\{1, x, x^2, \ldots, x^n\}$ is the standard basis for P_n.

Basis for span(S)

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent set in vector space V. Then S is a basis for the subspace span(S) since the set S spans span(S) by definition of span(S).

Dimension

Let V be a vector space and S be any basis for V. Then the number of vectors in S is called the *dimension* of V. It is denoted by $\dim(V)$.

Finite Dimensional Vector Space

A nonzero vector space V is called *finite-dimensional* if it contains a finite set of vectors $\{v_1, v_2, \ldots, v_n\}$ that forms a basis. If no such set exists, then V is called *infinite-dimensional*.

For example, the vector spaces $\mathbb{R}^3$, M_{22} and P_n are finite-dimensional while $F(-\infty,\infty)$, $C(-\infty,\infty)$, $C^n(-\infty,\infty)$ are infinite-dimensional.

Theorems on Basis and Dimension

(1) Let V be a vector space with $\dim(V) = n$.

  (a) If a set has more than n vectors, then it is linearly dependent.

  (b) If a set has fewer than n vectors, then it does not span V.

(2) All bases for a finite dimensional vector space have the same number of vectors.

(3) If $S = \{v_1, v_2, \ldots v_n\}$ is a basis for a vector space V, then every vector $v \in V$ can be uniquely represented in the form

$$v = k_1v_1 + k_2v_2 + \cdots + k_nv_n.$$

(4) *Plus/Minus Theorem:* Let S be a nonempty subset of a vector space V and v be any vector in V.

(a) If S is linearly independent and $v \notin \text{Span}(S)$, then the set $S \cup \{v\}$ is also linearly independent.

(b) If $v \in S$ and it can be expressed as a linear combination of other vectors in S, then S and $S - \{v\}$ span the same space, i.e.,

$$\text{span}(S) = \text{span}(S - \{v\}).$$

(5) Let V be a vector space with $\dim(V) = n$. Then a set S in V with exactly n vectors is a basis for V if either S is linearly independent or S spans V.

(6) Let V be a finite dimensional vector space and S be a finite subset of V.

(a) If S spans V but it is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is linearly independent but it is not a basis for V, then S can be extended to a basis for V by inserting appropriate vectors into S.

(7) Let W be a subspace of a finite-dimensional vector space V. Then $\dim(W) \leq \dim(V)$. Further, if $\dim(W) = \dim(V)$, then $W = V$.

Coordinate Vector Relative to a Basis

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V and

$$v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$$

be the representation of a vector v in terms of the basis S. Then the scalars $k_1, k_2, \ldots, k_n$ are called the *coordinates* of v relative to the basis S and the vector

$$(v)_S = (k_1, k_2, \ldots, k_n)$$

is called the *coordinate vector* of v relative to S.

Solved Examples

Example 6.4.1. *Show that* $S = \{(1,3,4), (-1,0,1), (4,1,2)\}$ *forms a basis for* $\mathbb{R}^3$.

Solution. Since S has 3 vectors and $\dim(\mathbb{R}^3) = 3$, it is enough to show that S is linearly independent. Suppose that

$$\begin{aligned} k_1(1,3,4) + k_2(-1,0,1) + k_3(4,1,2) &= (0,0,0) \\ (3k_1 - k_2 + 4k_3, 3k_1 + k_3, 4k_1 + k_2 + 2k_3) &= (0,0,0) \end{aligned}$$

Comparing the corresponding components on both sides, we get

$$\begin{aligned} 3k_1 - k_2 + 4k_3 &= 0 \\ 3k_1 + 0k_2 + k_3 &= 0 \\ 4k_1 + k_2 + 2k_3 &= 0 \end{aligned}$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 3 & 0 & 4 \\ 4 & 1 & 2 \end{bmatrix}$$

Observe that

$$\det(A) = 3(0-4) + 1(6-16) + 4(3-0) = -12 - 10 + 12 = -10 \neq 0.$$

Therefore, the system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0.$$

Thus S is linearly independent and hence forms a basis for $\mathbb{R}^3$. ■

Example 6.4.2. *Check whether the set $S = \{(2,2,2), (1,-1,-1), (0,1,1)\}$ forms a basis for $\mathbb{R}^3$ or not.*

Solution. Since S has 3 vectors and $\dim(\mathbb{R}^3) = 3$, it is enough to check whether S is linearly independent or not. Suppose that

$$\begin{aligned} k_1(2,2,2) + k_2(1,-1,-1) + k_3(0,1,1) &= (0,0,0) \\ (2k_1 + k_2, 2k_1 - k_2 + k_3, 2k_1 - k_2 + k_3) &= (0,0,0) \end{aligned}$$

Comparing the corresponding components on both sides, we get

$$\begin{aligned} 2k_1 + k_2 + 0k_3 &= 0 \\ 2k_1 - k_2 + k_3 &= 0 \\ 2k_1 - k_2 + k_3 &= 0 \end{aligned}$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

Observe that

$$\det(A) = 2(-1+1) - 1(2-2) = 0 - 0 = 0.$$

Consequently, the system will have a nontrivial solution. Thus S is linearly dependent and hence does not form a basis for $\mathbb{R}^3$. ■

Example 6.4.3. *Check whether the following polynomials form a basis for P_2 or not.*

$$p_1(x) = 5 + x^2, \quad p_2(x) = 5 - x + 2x^2, \quad p_3(x) = -x + x^2$$

Solution. Since the number of polynomials is 3 and $\dim(P_2) = 3$, it is enough to check whether the polynomials are linearly independent or not. Suppose that

$$\begin{aligned} & k_1p_1(x) + k_2p_2(x) + k_3p_3(x) = 0 \\ \Rightarrow\ & k_1(5 + x^2) + k_2(5 - x + 2x^2) + k_3(-x + x^2) = 0 + 0x + 0x^2 \\ \Rightarrow\ & (5k_1 + 5k_2) + (-k_2 - k_3)x + (k_1 + 2k_2 + k_3)x^2 = 0 + 0x + 0x^2 \end{aligned}$$

Comparing the corresponding coefficients of 1, x and x^2 on both sides, we get

$$\begin{aligned} 5k_1 + 5k_2 + 0k_3 &= 0 \\ 0k_1 - k_2 - k_3 &= 0 \\ k_1 + 2k_2 + k_3 &= 0 \end{aligned}$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 5 & 5 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Observe that

$$\det(A) = 5(-1+2) - 5(0+1) = 5 - 5 = 0.$$

Consequently, the system will have a nontrivial solution. Thus $p_1(x)$, $p_2(x)$, $p_3(x)$ are linearly dependent and hence do not form a basis for P_2. ■

Example 6.4.4. *Show that* $S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$ *is a basis for* M_{22}. **[GTU, June 2010]**

Solution. Since the number of vectors in S is 4 and $\dim(M_{22}) = 4$, it is enough to show that S is linearly independent. Suppose that

$$k_1 \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1 & 2k_1 - k_2 + 2k_3 \\ k_1 - k_2 + 3k_3 - k_4 & -2k_1 + k_3 + 2k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing the corresponding entries in both matrices, we obtain

$$\begin{aligned} k_1 &= 0 \\ 2k_1 - k_2 + 2k_3 &= 0 \\ k_1 - k_2 + 3k_3 - k_4 &= 0 \\ -2k_1 + k_3 + 2k_4 &= 0 \end{aligned}$$

The coefficient matrix of the system is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{bmatrix}$$

Observe that

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 2 & 0 \\ -1 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix} = -1(6+1) - 2(-2+0) = -3 \neq 0.$$

Therefore, the system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0 \quad k_4 = 0.$$

Thus S is linearly independent and hence forms a basis for M_{22}. ■

Example 6.4.5. *Find the coordinate vector of* p *relative to the basis* $S = \{p_1, p_2, p_3\}$, *where* $p = 2 - x + x^2$, $p_1 = 1 + x$, $p_2 = 1 + x^2$, $p_3 = x + x^2$. ***[GTU, May 2012]***

Solution. To find the coordinate vector of p relative to the basis $S = \{p_1, p_2, p_3\}$, we have to find k_1, k_2, k_3 such that

$$\begin{aligned} & p = k_1p_1 + k_2p_2 + k_3p_3 \\ \Rightarrow\ & 2 - x + x^2 = k_1(1+x) + k_2(1+x^2) + k_3(x+x^2) \\ \Rightarrow\ & 2 - x + x^2 = (k_1+k_2) + (k_1+k_3)x + (k_2+k_3)x^2 \end{aligned}$$

Comparing the corresponding coefficients of 1, x and x^2 on both sides, we get

$$\begin{aligned} k_1 + k_2 &= 2 \\ k_1 + k_3 &= -1 \\ k_2 + k_3 &= 1 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Applying $R_3 \to R_3 + R_2$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

Applying $R_2 \to (-1)R_2$ and $R_3 \to \frac{1}{2}R_3$, we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The system corresponds to the last matrix is

$$k_1 + k_2 = 2 \quad k_2 - k_3 = 3 \quad k_3 = -1$$

Using back substitution, we obtain

$$k_1 = 0, \quad k_2 = 2, \quad k_3 = -1.$$

Thus the coordinate vector of p relative to S is $(p)_S = (0, 2, -1)$. ■

Exercises

Exercise 6.4.1. *Show that* $S = \{(3,1,-4),(2,5,6),(1,4,8)\}$ *is a basis for* $\mathbb{R}^3$.

Exercise 6.4.2. *Show that* $S = \{(1,6,4),(2,4,-1),(-1,2,5)\}$ *is not a basis for* $\mathbb{R}^3$.

Exercise 6.4.3. *Let* $v_1 = 1 - 3x + 2x^2$, $v_2 = 1 - x + 4x^2$, $v_3 = 1 - 7x$. *Show that the set* $S = \{v_1, v_2, v_3\}$ *is a basis for* P_2. ***[GTU, June 2013]***

Exercise 6.4.4. *Check whether the following vectors form a basis for* M_{22}*:*

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

Exercise 6.4.5. *Find a standard basis vector that can be added to the set* $S = \{(1,0,3),(2,1,4)\}$ *to produce a basis of* $\mathbb{R}^3$. **[GTU, May 2012]**

Exercise 6.4.6. *Determine the dimension and basis for the solution space of the following system.*

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

Exercise 6.4.7. *Find basis and dimension of*

$$W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1 + a_2 = 0,\ a_2 + a_3 = 0,\ a_3 + a_4 = 0\}.$$

[GTU, June 2010]

Exercise 6.4.8. *Find the dimension of the subspace*

$$W = \{(a,b,c,d) \in \mathbb{R}^4 \mid d = a + b,\ c = a - b\}.$$

Exercise 6.4.9. *If* $v_1 = (1,2,1)$, $v_2 = (2,9,0)$, $v_3 = (3,3,4)$, *show that* $S = \{v_1, v_2, v_3\}$ *is a basis for* $\mathbb{R}^3$. *Find the coordinate vector of* $v = (5,-1,9)$ *w.r.t.* S.

Exercise 6.4.10. *Find the coordinate vector of* $p = 4 - 3x + x^2$ *relative to the standard basis* $p_1 = 1, p_2 = x, p_3 = x^2$ *of* P_2.

Answers

6.4.4 yes **6.4.5** $(1,0,0)$

6.4.6 $\{(3,1,0),(-1,0,1)\}$, 2-dimensional **6.4.7** $\{(-1,1,-1,1)\}$, 1-dimensional

6.4.8 2-dimensional **6.4.9** $(1,-1,2)$ **6.4.10** $(4,-3,1)$