- **Smooth Curve :**
A curve is said to be a smooth curve if it is given by
$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \quad a \le t \le b$$
where t is arclength of curve and $\bar{r}$ is continuous and has continuous first order derivative which is differentiable from zero vector for all $\bar{r}$.

- **Piecewise smooth curve :**
Curve is piecewise smooth if it consists of finite number of smooth curves.

- **Line Integral :**
Any integral which is to be evaluated along a curve is called a line integral.

For vector function $\bar{F}(\bar{r})$ along a curve C it is given by
$$\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\,\frac{d\bar{r}}{dt}$$

If $\bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{F} = (F_1, F_2, F_3)$
then
$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$
So,
$$\int_C \bar{F}\cdot d\bar{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

- If C is closed curve then line integral is denoted by $\oint_C$

☆ Remarks :

1) $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),\, y(t),\, z(t))\,dt$; $a \le t \le b$

2) $\int_C F(x,y,z)\,ds = \int_a^b F(x(t),\, y(t),\, z(t))\,\frac{ds}{dt}\,dt$

where $\frac{ds}{dt} = |v(t)|$ = arc length of curve

$= \left|\frac{dr}{dt}\right|$ $\quad (\because ds = \int_a^b |v(t)|\,dt)$

3) $\int_C \bar F(\bar r)\cdot d\bar r = \int_a^b \bar F(\bar r)\cdot\frac{d\bar r}{dt}\,dt = \int_a^b \left(f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right)dt$

Steps for evaluating line integral :

Step: 1 Let $\bar r = x\hat i + y\hat j + z\hat k$

then $d\bar r = dx\,\hat i + dy\,\hat j + dz\,\hat k$

Step: 2 Find $\bar F \cdot d\bar r$

Step: 3 From the equation of curve find dx, dy, dz

Step: 4 Evaluate $\int_C \bar F \cdot d\bar r$

Step: 1 Find a smooth parametrization of

$r(t) = x(t)\,\hat i + y(t)\,\hat j + z(t)\,\hat k$, $a \le t \le b$

Step: 2 Evaluate the integral as

$$\int_C F(x,y,z)\,ds = \int_a^b F(x(t),\, y(t),\, z(t))\,|v(t)|\,dt$$

Note: If F has the constant value 1 then the integral of F over C gives the length of C.

Application of line integral

1) <u>Work done</u>: If $\bar{F}$ is force acting on a partical moving along the arc AB of curve C, then line integral $\int_A^B \bar{F}\cdot d\bar{r}$ represents the work done in displacing particle from point A to B.

$\bar{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is force over a smooth curve $r(t)$ from $t=a$ to $t=b$ then

$$W = \int_a^b F\cdot T\, ds$$ where T is unit tangent vector

Different ways to write work done:

$$W = \int_a^b F\cdot T\, ds$$

$$= \int_a^b F\cdot dr \quad \left(\because T = \frac{dr}{ds}\right).$$

$$= \int_a^b \bar{F} \cdot \frac{dr}{dt}\, dt$$

$$\int_a^b (M, N, P)\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

$$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right) dt$$

$$= \int_a^b (M\,dx + N\,dy + P\,dz)$$

Steps for evaluating a work done along smooth curve $r(t)$, $a \le t \le b$.

1) Evaluate F on the curve as a f^n of t.
2) Find $\frac{dr}{dt}$ and $\bar{F} \cdot \frac{d\bar{r}}{dt}$
3) $\int \left(\bar{F} \cdot \frac{dr}{dt}\right) dt$ along $t=a$ to $t=b$.

II) Flow and Circulation

Consider smooth curve $\bar{r}(t) = (x(t), y(t), z(t))$ in the domain of continuous velocity field $\bar{F}(\bar{r}) = (F_1, F_2, F_3)$. Then flow along a curve from $t=a$ to $t=b$ is

$$\text{Flow} = \int \bar{F}(\bar{r}) \cdot d\bar{r}$$

$$= \int_a^b \bar{F} \cdot T\, ds$$

$$= \int_a^b \left(F \cdot \frac{dr}{dt}\right) dt$$

$$= \int_a^b (F_1\,dx + F_2\,dy + F_3\,dz)$$

The integral in this case is called flow integral.

Circulation : If the flow integral along the closed curve is called circulation around the curve.

If circulation is zero then $\bar{F}$ is irrotational.