Successive Differentiation :

Introduction: Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are utmost importance in scientific and engineering applications.

Let $\boldsymbol{f(x)}$ be a differentiable function and let its successive derivatives be denoted by $\boldsymbol{f'(x), f''(x), f'''(x), \ldots\ldots.., f^{(n)}(x)}$.

❖ Common notations of higher order Derivatives of $\boldsymbol{y = f(x)}$

1st Derivative: $\boldsymbol{f'(x)}$ **or** $\boldsymbol{y'}$ **or** $\boldsymbol{y_1}$ **or** $\boldsymbol{\frac{dy}{dx}}$ **or** $\boldsymbol{Dy}$

2nd Derivative: $\boldsymbol{f''(x)}$ **or** $\boldsymbol{y''}$ **or** $\boldsymbol{y_2}$ **or** $\boldsymbol{\frac{d^2y}{dx^2}}$ **or** $\boldsymbol{D^2y}$

3rd Derivative: $\boldsymbol{f'''(x)}$ **or** $\boldsymbol{y'''}$ **or** $\boldsymbol{y_3}$ **or** $\boldsymbol{\frac{d^3y}{dx^3}}$ **or** $\boldsymbol{D^3y}$

$\vdots$

$\boldsymbol{n^{th}}$ **Derivatives:** $\boldsymbol{f^{(n)}(x)}$ **or** $\boldsymbol{y^{(n)}}$ **or** $\boldsymbol{y_n}$ **or** $\boldsymbol{\frac{d^ny}{dx^n}}$ **or** $\boldsymbol{D^ny}$**.**

❖ $\boldsymbol{n^{th}}$ derivatives of some standard Functions :

1) $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = e^{ax}}$**.**

$\Rightarrow$ Let $y = e^{ax}$ then

$$y_1 = a\, e^{ax}$$
$$y_2 = a^2 e^{ax}$$
$$y_3 = a^3 e^{ax}$$
$$\vdots$$
$$y_n = a^n\, e^{ax}\ .$$

2) $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = a^{bx}}$.

$\Rightarrow$ Let $y = a^{bx}$ then

$$y_1 = b\, a^{bx} \log a$$
$$y_2 = b^2\ a^{bx} (\log a)^2$$
$$y_3 = b^3 a^{bx} (\log a)^3$$
$$\vdots$$
$$y_n = b^n\, a^{bx} (\log a)^n\ .$$

3) $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = (ax + b)^m}$ **,** $\boldsymbol{m}$ **is a positive integer greater than** $\boldsymbol{n}$.

$\Rightarrow$ Let $y = (ax + b)^m$ then

$$y_1 = ma\,(ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2\,(ax + b)^{m-2}$$
$$y_3 = m(m-1)(m-2)a^3\,(ax + b)^{m-3}$$

$$\vdots$$

$$y_n = m(m-1)(m-2)\ldots\ldots(m-(n-1))a^n\,(ax+b)^{m-n}$$
$$= m(m-1)(m-2)\ldots\ldots(m-n+1)\,a^n\,(ax+b)^{m-n}$$
$$= \frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}.$$

Case (i): If $\boldsymbol{m}$ is a Positive integer and $\boldsymbol{m=n}$, then $y_n = n!\ a^n$.

Case (ii): If $\boldsymbol{m}$ is a Positive integer and $\boldsymbol{m<n}$, then $y_n = 0$.

Case(iii): If $\boldsymbol{m=-1}$ **,** i.e.$y = \frac{1}{ax+b}$ then $y_n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$.

4) $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \log(ax+b)}$**.**

$\Longrightarrow$ Let $y = \log(ax+b)$ then

$$y_1 = \frac{a}{ax+b}$$

$$y_2 = -\frac{a^2}{(ax+b)^2}$$

$$y_3 = \frac{2\,a^2}{(ax+b)^3} = \frac{2!\ a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}.$$

5) $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \sin(ax+b)}$

$\Rightarrow$ Let $y = \sin(ax+b)$ then

$$y_1 = a\ \cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\ \cos\left(ax+b+\frac{\pi}{2}\right) = a^2\ \sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = a^3\ \cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\ \sin\left(ax+b+\frac{3\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

Similarly if $\boldsymbol{y = \cos(ax+b)}$ then

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$$

6) $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = e^{ax}\sin(bx+c)}$**.**

$\Rightarrow$ Let $y = e^{ax}\sin(bx+c)$ then

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$
$$= e^{ax}(a\sin(bx+c) + b\ \cos(bx+c))$$

Putting $a = \mathrm{r}\cos\alpha$, $b = r\sin\alpha$ we get,

$$y_1 = e^{ax}(\mathrm{r}\cos\alpha\ \sin(bx+c) + r\ \sin\alpha\cos(bx+c))$$

$$= r\ e^{ax}(\cos\alpha\ \sin(bx+c) + \sin\alpha\cos(bx+c))$$

$$= re^{ax}\ \sin(bx+c+\alpha)$$

Similarly $y_2 = r^2 e^{ax}\sin(bx+c+2\alpha)$

$$y_3 = r^3 e^{ax}\sin(bx+c+3\alpha)$$

$$\vdots$$

$$y_n = r^n e^{ax}\sin(bx+c+n\alpha).$$

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right).$$

Where $r = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$.

Similarly n^{th} Derivative of $y = e^{ax}\cos(bx+c)$ is,

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right).$$

❖ Summary :

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n\ e^{ax}$
$y = a^{bx}$	$y_n = b^n\ a^{bx}\ (\log a)^n$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!}\ a^n\ (ax+b)^{m-n}, & m>0, m>n \\ 0, & m>0,\ m<n \\ n!\ a^n, & m>0,\ m=n \\ \frac{(-1)^n n!\ a^n}{(ax+b)^{n+1}}, & m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1}\frac{(n-1)!\ a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = \cos(ax+b)$	$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$
$y = e^{ax}\, sin(bx+c)$	$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right)$
$y = e^{ax}\, cos(bx+c)$	$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right)$

Example-1: Find the n^{th} derivative of the function $y = \frac{1}{1-5x+6x^2}$.

Solution : Here $y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$

$\therefore \frac{1}{(2x-1)(3x-1)} = \frac{A}{2x-1} + \frac{B}{3x-1}$ ....(1)

$\therefore 1 = A(3x-1) + B(2x-1)$

If $x = 1/2$ then $A = 2$

If $x = 1/3$ then $B = -3$.

$\therefore$ From equation (1) , we get

$y = \frac{2}{2x-1} - \frac{3}{3x-1}$ So the n^{th} derivative of the given function is,

$y_n = \frac{2\,(-1)^n\, n!\, 2^n}{(2x-1)^{n+1}} - \frac{3\,(-1)^n\, n!\, 3^n}{(3x-1)^{n+1}}$

$= (-1)^n\, n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right]$.

Example-2 : Find the n^{th} derivative of the function $y = \frac{2x-1}{(x^2-5x+6)}$.

Solution : Here $y = \frac{2x-1}{(x^2-5x+6)} = \frac{2x-1}{(x-2)(x-3)}$.

Let $y = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$.

$\therefore 2x-1 = A(x-3) + B\,(x-2)$.

If $x = 2$, then $A = -3$.

If $x = 3$, then $B = 5$.

$\therefore y = \frac{5}{x-3} - \frac{3}{x-2}$

$\therefore y_n = \frac{5\,(-1)^n\, n!}{(x-3)^{n+1}} - \frac{3\,(-1)^n\, n!}{(x-2)^{n+1}}$.

Example-3 : Find the n^{th} derivative of the function $y = \frac{x^4}{x^2-3x+2}$.

Solution : Here $y = \frac{x^4}{x^2-3x+2} = \frac{x^4}{(x-2)(x-1)}$

$y = \frac{x^4}{(x-2)(x-1)} = x^2 + 3x + 7 + \frac{15x-14}{(x-2)(x-1)}$

$= x^2 + 3x + 7 + \frac{16}{x-2} - \frac{1}{x-1}$

$\therefore y_n = 0 + \frac{16(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}$.

Example-4 : Find the n^{th} derivative of the function $y = \sin 6x\ \cos 4x$.

Solution : Here $y = \sin 6x\ \cos 4x = \frac{1}{2}\,(\sin 10x + \sin 2x)$ $(\because s + s = 2\,s\,c)$

$\therefore y_n = \frac{1}{2}\left(10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n\ \sin\left(2x + \frac{n\pi}{2}\right)\right)$.

Example-5 : Find the n^{th} derivative of the function $y = sin^4 x$.

Solution : Here $y = sin^4 x = (sin^2 x)^2 = \left(\frac{1-\cos}{2}\right)^2$

$$= \frac{1}{4}\ (1 - 2\cos 2x + cos^2 2x) = \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos}{2}\right)$$

$$= \frac{1}{8}\ (3 - 4\cos 2x + \cos 4x)$$

$$\therefore y_n = \frac{1}{8}\left(0 - 4\ 2^n\ \cos\left(2x + \frac{n\pi}{2}\right) + \ 4^n\ \cos\left(4x + \frac{n\pi}{2}\right)\right).$$

Example-6 : Find the n^{th} derivative of the function $y = e^{2x}\ \cos 2x\ \cos x$.

Solution : Here $y = e^{2x}\ \cos 2x\ \cos x = \frac{1}{2}\ e^{2x}(\ 2\cos 2x\ \cos x)$

$$= \frac{1}{2}\ e^{2x}(\cos 3x + \cos x)\ \ (\because c + c = 2\ c\ c)$$

$$= \frac{1}{2}\ (e^{2x}\ \cos 3x + \ e^{2x}\ \cos x)$$

$$= \frac{1}{2}\left[(13)^{\frac{n}{2}}\ e^{2x}\ \cos\left(3x + n\tan^{-1}\frac{3}{2}\right) + \ (5)^{\frac{n}{2}} e^{2x}\ \cos\left(x + n\tan^{-1}\frac{1}{2}\right)\right].$$

Example-7 : If $y = e^{ax}\ \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)\ y = 0$.

Solution : Here $y = e^{ax}\ \sin bx$ then $y_1 = \ e^{ax}\ b\cos bx + ae^{ax}\ \sin bx$

$\therefore y_1 = e^{ax}\ b\ \cos bx + a\ y.$

$\Longrightarrow y_1 - ay = e^{ax}\ b\ \cos bx$, again differentiating with respect to x we get,

$y_2 - ay_1 = ae^{ax}\ b\ \cos bx - \ b^2 e^{ax}\ \sin bx$

$\Longrightarrow y_2 - ay_1 = a(\ y_1 - ay) - b^2\ y = a\ y_1 - a^2\ y - b^2 y$

$\therefore \therefore y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

Example-8 : If $x = a(\cos t + t\ \sin t)$, $y = a(\sin t - t\ \cos t)$, find $\frac{d^2y}{dx^2}$.

Solution : We have $\frac{dx}{dt} = a(-\sin t + \sin t + t\ \cos t) = a\ t\ \cos t$

and $\frac{dy}{dt} = a(\cos t - \cos t + t\ \sin t) = a\ t\ \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\ t\ sin\ t}{a\ t\ cos\ t} = \ \tan t\ , \frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t)\ \frac{dt}{dx}$$

$$\therefore\ \ \frac{d^2y}{dx^2} = sec^2 t\ \frac{1}{at\ \cos t} = \frac{sec^3 t}{at}.$$

Problems :

1. If $a\,x^2 + 2hxy + b\,y^2 = 1$, prove that $\frac{d^2y}{dx^2} = \frac{h^2-ab}{hx+by}$.
2. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x\,\frac{dy}{dx} + y\,cos^2\,x = 0$.
3. Find $\frac{d^2y}{dx^2}$, when $x = a\,cos^3\theta\,, y = b\,sin^3\theta$.
4. If $x = \sin t\,, y = \sin pt$, prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$.
5. If $x = 2\cos t - \cos 2t,\ \ y = 2\sin t - \sin 2t$, find the value of $\frac{d^2y}{dx^2}$ when $t = {}^{\pi}/_{2}$.
6. If $x^3 + y^3 = 3axy$ then prove that $\frac{d^2y}{dx^2} = -\frac{2\,a^2\,xy}{(y^2-ax)^3}$.
7. If $y = \tan^{-1}(\sin hx)$, prove that $\frac{d^2y}{dx^2} + \tan y\left(\frac{dy}{dx}\right)^2 = 0$.
8. If $y = e^{-kt}\cos(lt+c)$, show that $\frac{d^2y}{dx^2} + 2k\,\frac{dy}{dx} + n^2y = 0$, where $n^2 = k^2 + l^2$.

9. Find the n^{th} derivatives of the following functions **:**

(i) $y = cos\,x\,cos\,2x\,cos\,3x$
(ii) $y = e^{2x}\,cos^2x\,\sin x$
(iii) $y = \frac{x}{(x-1)\,(2x+3)}$
(iv) $y = e^{-x}sin^2x$
(v) $y = \frac{x^2-4x+1}{x^3+2x^2-x-2}$
(vi) $y = sin^2x\,cos^3\,x$
(vii) $y = e^{-x}\,sin^2\,x$
(viii) $y = \log(ax+b)(cx+d)$
(ix) $y = cos^6x$.

LEIBNITZ'S THEOREM :

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n.$$

Where u_r and v_r represents r^{th} derivatives of u and v respectively.

Example-1 Find the n^{th} derivative of $x \log x$.

Solution : Let $u = \log x$ and $v = x$.

$$\text{Then } u_n = (-1)^{n-1}\frac{(n-1)!}{x^n},\; u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}} \text{ and } v_1 = 1,\; v_2 = 0.$$

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n$$

$$\Rightarrow (x\log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n}\,x + n\,(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\,1 + 0$$

$$\Rightarrow (x\log x)_n = (-1)^{n-1}\frac{(n-1)(n-2)!}{x^{n-1}} + n\;(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= -(-1)^{n-2}\frac{(n-1)(n-2)!}{x^{n-1}} + \;n\;(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\,[-(n-1)+n]$$

$$= = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\,1.$$

Example-2 Find the n^{th} derivative of $x^2 e^{3x}\,\sin 4x$.

Solution : Let $u = e^{3x}\sin 4x$ and $v = x^2$.

$$\text{Then } u_n = (25)^{\frac{n}{2}}\,e^{3x}\,\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right) = 5^n e^{3x}\,\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right),$$

$$u_{n-1} = 5^{n-1}e^{3x}\,\sin\left(4x + (n-1)\,\tan^{-1}\frac{4}{3}\right) \text{ and}$$

$$v_1 = 2x, v_2 = 2, v_3 = 0.$$

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n$$

$$\Rightarrow (x^2 e^{3x}\,\sin 4x)_n = 5^n e^{3x}\,\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right)x^2$$

$$+\, n\,5^{n-1}e^{3x}\sin\left(4x + (n-1)\,\tan^{-1}\frac{4}{3}\right)\,(2x)$$

$$+\,\frac{n(n-1)}{2}5^{n-2}e^{3x}\,\sin\left(4x + (n-2)\,\tan^{-1}\frac{4}{3}\right)2 + \,0\,.$$

$$= e^{3x}\,5^n[x^2\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\,\tan^{-1}\frac{4}{3}\right)$$

$$+\,\frac{n(n-1)}{25}\sin\left(4x + (n-2)\,\tan^{-1}\frac{4}{3}\right)]\;.$$

Example-3 Find the n^{th} derivative of $e^x\,(2x+3)^3$.

Solution : Let $u = e^x$ and $v = (2x+3)^3$.

Then $u_n = e^x$, for all integer values of n, and

$v_1 = 6(2x+3)^2\,, v_2 = 24(2x+3), v_3 = 48, v_4 = 0.$

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n$$

$$\Longrightarrow (e^x\,(2x+3)^3)_n = e^x(2x+3)^3 + n\,e^x 6(2x+3)^2 + \frac{n(n-1)}{2}\,e^x 24(2x+3)$$

$$+\frac{n(n-1)(n-2)}{6}e^x 48\ + 0$$

$$= e^x\{\,(2x+3)^3 + 6n\,(2x+3)^2 + 12n\,(n-1)(2x+3) + 8n(n-1)(n-2)\}.$$

Example-4 If $y = (\sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$

Solution : Here If $y = (\sin^{-1}x)^2$ then differentiating with respect to x we get,

$$y_1 = \frac{2(\sin^{-1}x)}{\sqrt{1-x^2}} \text{ or } (1-x^2){y_1}^2 = 4(\sin^{-1}x)^2 = 4y$$

Again differentiating , we get

$$(1-x^2)2\,y_1\,y_2 - 2x{y_1}^2 = 4\,y_1 \text{ or } (1-x^2)\,y_2 - xy_1 - 2 = 0$$

Differentiating it n times by Leibnitz's theorem,

$$(1-x^2)y_{n+2} + n\,(-2x)\,y_{n+1} + \frac{n(n-1)}{2}\,(-2)y_n - [xy_{n+1} + n\,y_n] = 0$$

$$\Longrightarrow (1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - x\,y_{n+1} - n\;\;y_n = 0$$

$$\Longrightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2y_n = 0.$$

Which is the required result.

Problems :

1. If $y^{1/m} + y^{-1/m} = 2\,x$, prove that
   $(x^2-1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0.$
2. If $y = e^{m\cos^{-1}x}$, prove that (i) $(1-x^2)y_2 - xy_1 = m^2y$
   (ii) $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+m^2) = 0.$
3. If $y = \tan^{-1}x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n\,(n-1)y_{n-1} = 0.$
4. Find the nth derivative of the following functions **:**
   (i) $x^2\log 3x$
   (ii) $x^2\cos x$
   (iii) x^2e^x