Gamma and Beta Functions

Chapter Outline

3.1 INTRODUCTION

There are some special functions which have importance in mathematical analysis, functional analysis, physics or other applications. In this chapter, we will study two special functions, *gamma* and *beta functions*. The beta function is also called the *Euler integral of the first kind*. The gamma function is an extension of the factorial function to real and complex numbers and is also known as *Euler integral of the second kind*. Gamma function is a component in various probability distribution functions. It also appears in various areas such as asymptotic series, definite integration, number theory, etc.

3.2 GAMMA FUNCTION

Gamma function is defined by the improper integral $\int_0^\infty e^{-x}x^{n-1}\mathrm{d}x$, $n > 0$ and is denoted by Γn.

Hence, $$\Gamma n = \int_0^\infty e^{-x}x^{n-1}\mathrm{d}x,\ n > 0$$

Alternate form of gamma function

$$\Gamma n = 2\int_0^\infty e^{-x^2}x^{2n-1}\mathrm{d}x$$

Proof: By definition,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Let $x = t^2$, $\quad dx = 2t\,dt$

$$\Gamma n = \int_0^\infty e^{-t^2} \cdot t^{2n-2} \cdot 2t\,dt$$

$$= 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\,dt$$

Changing the variable t to x,

$$\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} dx$$

3.3 PROPERTIES OF GAMMA FUNCTION

Property 1: $\Gamma(n+1) = n\,\Gamma n$

Proof: $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

Integrating by parts,

$$\Gamma(n+1) = \left|-e^{-x} x^n\right|_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1} dx$$

$$= n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\,\Gamma n$$

Hence, $\Gamma(n+1) = n\,\Gamma n$

This is known as recurrence or reduction formula for Gamma function.

Note:

(i) $\Gamma(n+1) = n!$ if n is a positive integer

(ii) $\Gamma(n+1) = n\,\Gamma n$ if n is a positive real number

(iii) $\Gamma n = \dfrac{\Gamma(n+1)}{n}$ if n is a negative fraction

(iv) $\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Property 2: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: By alternate form of Gamma function,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} x^{2\left(\frac{1}{2}\right)-1} dx = 2\int_0^\infty e^{-x^2} dx$$

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx \cdot 2\int_0^\infty e^{-y^2} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

Changing to polar coordinates, $x = r\cos\theta$, $\quad y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

Limits of x	$x = 0$	to	$x \to \infty$
Limits of y	$y = 0$	to	$y \to \infty$

This shows that the region of integration is the first quadrant.

Draw an elementary radius vector in the region which starts from the pole and extends up to ∞.

Limits of r	$r = 0$	to	$r \to \infty$
Limits of θ	$\theta = 0$	to	$\theta = \dfrac{\pi}{2}$

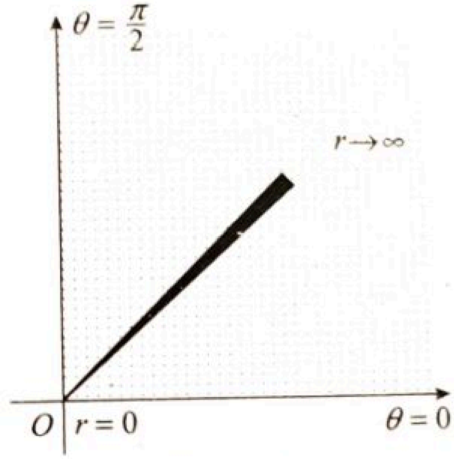

Fig. 3.1

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = 4\int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \cdot r \,\mathrm{d}r \,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \cdot \int_0^{\infty} \left(-\frac{1}{2}\right) e^{-r^2} (-2r)\,\mathrm{d}r$$

$$= \frac{4}{-2} \left|\theta\right|_0^{\frac{\pi}{2}} \left|e^{-r^2}\right|_0^{\infty} \qquad \left[\because \int e^{f(r)} \cdot f'(r)\mathrm{d}r = e^{f(r)}\right]$$

$$= -2 \cdot \frac{\pi}{2}(0-1)$$

$$= \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 1

Find the value of $\Gamma\left(-\dfrac{5}{2}\right)$.

Solution

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right)$$

$$= -\frac{2}{5} \cdot \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{4}{15}\Gamma\left(-\frac{1}{2}\right)$$

$$= \frac{4}{15} \cdot \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -\frac{8}{15}\Gamma\left(\frac{1}{2}\right) = -\frac{8\sqrt{\pi}}{15}$$

Example 2

Given $\Gamma\left(\frac{8}{5}\right) = 0.8935$, *find the value of* $\Gamma\left(-\frac{12}{5}\right)$.

Solution:

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{12}{5}\right) = \frac{\Gamma\left(-\frac{12}{5}+1\right)}{-\frac{12}{5}} = -\frac{5}{12}\cdot\frac{\Gamma\left(-\frac{7}{5}+1\right)}{-\frac{7}{5}} = \frac{25}{84}\cdot\frac{\Gamma\left(-\frac{2}{5}+1\right)}{-\frac{2}{5}}$$

$$= -\frac{125}{168}\cdot\frac{\Gamma\left(\frac{3}{5}+1\right)}{\frac{3}{5}} = -\frac{625}{504}\Gamma\left(\frac{8}{5}\right) = -\frac{625}{504}(0.8935) = -1.108$$

Example 3

Evaluate $\int_0^\infty e^{-x^3}\,dx$.

Solution

Let $\quad x^3 = t,\ x = t^{\frac{1}{3}},\ dx = \frac{1}{3}t^{-\frac{2}{3}}dt$

When $\quad x = 0, \quad t = 0$

When $\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty e^{-x^3}dx = \int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-\frac{2}{3}}dt = \frac{1}{3}\int_0^\infty e^{-t}\,t^{\frac{1}{3}-1}dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

Example 4

Evaluate $\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}\,dx$.

Solution

Let $\quad \sqrt{x} = t,\ x = t^2,\ dx = 2t\,dt$

When $\quad x = 0, \quad t = 0$

When $\quad x \to \infty, \quad t \to \infty$

$$\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}dx = \int_0^\infty e^{-t}(t^2)^{\frac{1}{4}}\,2t\,dt$$

$$= 2\int_0^\infty e^{-t}t^{\frac{3}{2}}dt = 2\Gamma\left(\frac{5}{2}\right) = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

Example 5

Evaluate $\int_0^\infty (x^2+4)e^{-2x^2}\,dx.$

Solution

Let $\qquad 2x^2 = t,\ x = \left(\frac{t}{2}\right)^{\frac{1}{2}},\ dx = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}t^{-\frac{1}{2}}dt = \frac{t^{-\frac{1}{2}}}{2\sqrt{2}}dt$

When $\qquad x = 0, \qquad t = 0$

When $\qquad x \to \infty, \qquad t \to \infty$

$$\int_0^\infty (x^2+4)\,e^{-2x^2}\,dx = \int_0^\infty \left(\frac{t}{2}+4\right)e^{-t}\cdot\frac{t^{-\frac{1}{2}}}{2\sqrt{2}}\,dt$$

$$= \frac{1}{4\sqrt{2}}\int_0^\infty e^{-t}t^{\frac{1}{2}}\,dt + \frac{2}{\sqrt{2}}\int_0^\infty e^{-t}t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{4\sqrt{2}}\left\lceil\frac{3}{2}\right. + \frac{2}{\sqrt{2}}\left\lceil\frac{1}{2}\right. = \frac{1}{4\sqrt{2}}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right. + \frac{2}{\sqrt{2}}\left\lceil\frac{1}{2}\right.$$

$$= \frac{1}{8\sqrt{2}}\sqrt{\pi} + \frac{2}{\sqrt{2}}\sqrt{\pi} = \frac{17\sqrt{\pi}}{8\sqrt{2}}$$

Example 6

Evaluate $\int_0^\infty x^n e^{-\sqrt{ax}}\,dx.$

Solution

Let $\qquad \sqrt{ax} = t,\ x = \frac{t^2}{a},\ dx = \frac{2t}{a}dt$

When $\qquad x = 0, \qquad t = 0$

When $\qquad x \to \infty, \qquad t \to \infty$

$$\int_0^\infty x^n e^{-\sqrt{ax}}\,dx = \int_0^\infty \left(\frac{t^2}{a}\right)^n e^{-t}\cdot\frac{2t}{a}\,dt$$

$$= \frac{2}{a^{n+1}}\int_0^\infty e^{-t}t^{2n+1}\,dt \qquad = t^{2(n+1)+1}$$

$$= \frac{2}{a^{n+1}}\left\lceil 2n+2\right.$$

Example 7

Evaluate $\int_0^\infty \sqrt{x}\,e^{-x^2}\,dx\cdot\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx.$

Solution

Let $x^2 = t,\ x = t^{\frac{1}{2}},\ dx = \frac{1}{2}t^{-\frac{1}{2}}dt$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty \sqrt{x}\, e^{-x^2}dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}dx = \int_0^\infty t^{\frac{1}{4}}e^{-t} \cdot \frac{1}{2}t^{-\frac{1}{2}}dt \cdot \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{4}}} \cdot \frac{1}{2}t^{-\frac{1}{2}}dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}t^{-\frac{1}{4}}dt \cdot \int_0^\infty e^{-t}t^{-\frac{3}{4}}dt$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4} \cdot \pi\sqrt{2} = \frac{\pi}{2\sqrt{2}}$$

Example 8

Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx \cdot \int_0^\infty x^4 e^{-x^6}dx.$

Solution

Let $x^3 = t,\ x = t^{\frac{1}{3}},\ dx = \frac{1}{3}t^{-\frac{2}{3}}dt$

When $x = 0, \quad t = 0$

When $x \to \infty, \quad t \to \infty$

$$\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx \cdot \int_0^\infty x^4 e^{-x^6}dx = \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{6}}} \cdot \frac{1}{3}t^{-\frac{2}{3}}dt \cdot \int_0^\infty t^{\frac{4}{3}}e^{-t^2} \cdot \frac{1}{3}t^{-\frac{2}{3}}dt$$

$$= \frac{1}{9}\int_0^\infty e^{-t}t^{-\frac{5}{6}}dt \cdot \int_0^\infty e^{-t^2}t^{\frac{2}{3}}dt$$

$$= \frac{1}{9}\Gamma\left(\frac{1}{6}\right) \cdot \frac{1}{2} \cdot 2\int_0^\infty e^{-t^2}t^{2\left(\frac{5}{6}\right)-1}dt$$

$$= \frac{1}{9}\Gamma\left(\frac{1}{6}\right) \cdot \frac{1}{2}\Gamma\left(\frac{5}{6}\right) \qquad \left[\because 2\int_0^\infty e^{-x^2}x^{2n-1}dx = \Gamma(n)\right]$$

$$= \frac{1}{18}\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right) = \frac{1}{18} \cdot \frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{9}$$

Example 9

Evaluate $\int_0^1 (\log x)^5 \, dx$.

Solution

Let $\log x = -t, x = e^{-t}, dx = -e^{-t}\,dt$

When $x = 0, \quad t \to \infty$

When $x = 1, \quad t = 0$

$$\int_0^1 (\log x)^5 dx = \int_\infty^0 (-t)^5 (-e^{-t})\,dt$$

$$= -\int_0^\infty e^{-t} t^5 dt$$

$$= -\overline{|6} = -120$$

Example 10

Evaluate $\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx.$

Solution

$$\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx = \int_0^1 x^3 \cdot 4 \log\left(\frac{1}{x}\right) dx$$

$$= 4\int_0^1 x^3 \log\left(\frac{1}{x}\right) dx$$

Let $\log\left(\frac{1}{x}\right) = t, \ \frac{1}{x} = e^t, \ x = e^{-t}, \ dx = -e^{-t}dt$

When $x = 0, \quad t \to \infty$

When $x = 1, \quad t = 0$

$$\int_0^1 x^3 \log\left(\frac{1}{x}\right)^4 dx = 4\int_\infty^0 e^{-3t} t(-e^{-t})\,dt$$

$$= 4\int_0^\infty e^{-4t} t^{2-1} dt$$

$$= 4 \cdot \frac{\overline{|2}}{(4)^2} \qquad \left[\because \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\overline{|n}}{k^n}\right]$$

$$= \frac{1}{4}$$