Indeterminate Forms and L'Hôpital's(L'Hospital's) Rule:

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractionswhose numerators and denominators both approach zero or +q. The rule is knowntoday as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman whowrote the first introductory differential calculus text, where the rule first appeared inprint. Limits
involving transcendental functions often require some use of the rule fortheir calculation.

Indeterminate Form 0/0:

If the continuous functions $f(x)$ and $g\ (x)$ are both zero at $x\ =\ a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x\ =\ a$. The substitution produces **0/0**, a meaningless expression, which we cannot evaluate. We use **0/0** as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such$\frac{\infty}{\infty}$, $\infty\ \cdot \mathbf{0}$, $\infty - \infty$, $\mathbf{0^0}$, $\infty^{\mathbf{0}}$, $\mathbf{1}^{\infty}$which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations.

L'Hospital's rule: If f and g are differentiable functions on an open interval I containing a and suppose that $f(a) = g(a) = 0$, $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example-1: Find $\lim_{x \to 0} \frac{3x - sin\, x}{x}$.

Solution : Here $\lim_{x \to 0} \frac{3x-\sin x}{x} \left({}^{0}/_{0} - form\right)$

So using L'Hospital rule, we get

$\Rightarrow \lim_{x \to 0} \frac{3-\cos x}{1} = \frac{3-1}{1} = 2$.

Example-2: Find $\lim_{x \to 0} \frac{x-\sin x}{x^3}$.

Solution : Here $\lim_{x \to 0} \frac{x-sin\, x}{x^3} \left({}^{0}/_{0} - form\right)$

So using L'Hospital rule, we get

$\Rightarrow \lim_{x \to 0} \frac{1-\cos x}{3x^2} \left({}^{0}/_{0} - form\right)$

Again usingL'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6}\left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right).$$

Example-3: Find $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$.

Solution : Here $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \left({}^{0}/_{0} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}.$$

Example-4: Find $\lim_{x \to 1} \frac{x - x^x}{1+\log x - x}$.

Solution : Here $\lim_{x \to 1} \frac{x - x^x}{1+\log x - x} (0/0 - form)$

So using L'Hospital rule, we get

$$= \lim_{x \to 1} \frac{1 - x^x(1+\log x)}{\frac{1}{x}-1} (0/0 - form)$$

Again using L'Hospital rule, we get

$$= \lim_{x \to 1} \frac{-x^x(1+\log x)^2 - x^{x-1}}{\left(-\frac{1}{x^2}\right)} = 2.$$

Indeterminate Form ∞/∞ :

If f and g are differentiable functions on an open interval I containing a and suppose that $f(a) = g(a) = \infty$, $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example-1: Find $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$.

Solution : Here $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \sin x = 1.$$

Example-2: Find $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Solution : Here $\lim_{x \to \infty} \frac{e^x}{x^2} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{e^x}{2x} \left(\frac{\infty}{\infty} - form\right)$$

Again usingL'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{e^x}{2} = \infty.$$

Example-3: Find $\lim_{x \to \infty} \frac{\log x}{2\sqrt{x}}$.

Solution : Here $\lim_{x \to \infty} \frac{\log x}{2\sqrt{x}} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{1/x}{2/2\sqrt{x}} = \lim_{x \to \infty} \sqrt{x}/x = \lim_{x \to \infty} 1/\sqrt{x} = 0.$$

Example-4: Find $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x}$.

Solution : Here $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{1}{sec^2 x \ (x - \pi/2)} = \lim_{x \to \frac{\pi}{2}} \frac{cos^2 x}{(x - \pi/2)} \left(\frac{0}{0} - form\right)$$

Again usingL'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{-2 \ \sin x \ \cos x}{1} = 0.$$

Indeterminate Form$(0 \cdot \infty)$ or $(\infty - \infty)$:

If f and g are differentiable functions on an open interval I containing a and suppose that $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} f(x)g(x)$ is in $0 \cdot \infty$ form. We write given function as $\lim_{x \to a} \frac{f(x)}{1/g(x)}$ or $\lim_{x \to a} \frac{g(x)}{1/f(x)}$ so it is in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form respectively, which can be solved using L'Hospital's rule.

To evaluate the limits of the type $\lim_{x \to a} [f(x) - g(x)]$, when $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then apply L'Hospital's rule.

Example-1: Find $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution : Here $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $(\infty - \infty \ form)$

$\therefore \lim\limits_{x \to 0} \frac{x - \sin x}{x \sin x} \left(\frac{0}{0} - form\right)$

So using L'Hospital rule, we get

$\Longrightarrow \lim\limits_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \left(\frac{0}{0} - form\right)$

Again usingL'Hospital rule, we get

$\Longrightarrow \lim\limits_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = 0.$

Example-2: Find $\lim\limits_{x \to \infty} \left(a^{1/x} - 1\right) x.$

Solution : Here $\lim\limits_{x \to \infty} \left(a^{1/x} - 1\right) x \;(0 \cdot \infty - form)$

$\therefore \lim\limits_{x \to \infty} \frac{\left(a^{1/x} - 1\right)}{\frac{1}{x}} \left(\frac{0}{0} - form\right)$

So using L'Hospital rule, we get

$\Longrightarrow \lim\limits_{x \to \infty} \frac{a^{1/x}\left(-\frac{1}{x^2}\right) \log a}{\left(-\frac{1}{x^2}\right)} = \log a \; .$

Example-3: Find $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right].$

Solution : Here $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right] (\infty - \infty \; form)$

$\therefore \lim\limits_{x \to 2} \left[\frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)}\right] \left(\frac{0}{0} - form\right)$

So using L'Hospital rule, we get

$$\Longrightarrow \lim\limits_{x \to 2} \left[\frac{\frac{1}{x-1} - 1}{log(x-1) + \frac{(x-2)}{x-1}}\right] = \lim\limits_{x \to 2} \frac{1 - (x-1)}{(x-2) + (x-1)\, log(x-1)} \left(\frac{0}{0} - form\right)$$

Again usingL'Hospital rule, we get

$$= \lim\limits_{x \to 2} \frac{-1}{1 + \frac{x-1}{x-1} + log(x-1)} = \frac{1}{2}.$$

Example-4: Find $\lim\limits_{x \to 1} (x^2 - 1)\, tan\left(\frac{\pi x}{2}\right).$

Solution : Here $\lim\limits_{x \to 1} (x^2 - 1)\, tan\left(\frac{\pi x}{2}\right) \; (0 \cdot \infty - form)$

$\therefore \lim\limits_{x \to 1} \frac{(x^2 - 1)}{\cot\left(\frac{\pi x}{2}\right)} (0/0 - form)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 1} \frac{2x}{-cosec^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)} = \frac{2}{-\left(\frac{\pi}{2}\right)} = -\frac{4}{\pi}.$$

Indeterminate Forms $1^\infty, \infty^0, 0^0$:

To evaluate the limits of the type $\lim_{x \to a}[f(x)]^{g(x)}$, which takes any one of the indeterminate forms 1^∞, ∞^0, 0^0 for $f(x) > 0$, we proceed as follows:

Let $\quad l = \lim_{x \to a}[f(x)]^{g(x)}$, $f(x) > 0$

Applying log function on both the sides we get, $L = \log l = \lim_{x \to a}[g(x)\log f(x)]$ which takes the form $0 \cdot \infty$ which can be solved using L'Hospital rule.

So $\lim_{x \to a}[f(x)]^{g(x)} = \lim_{x \to a} e^{g(x)\log f(x)} = e^L$.

Example-1: Find $\lim_{x \to o^+}(1+x)^{1/x}$.

Solution : Here $\lim_{x \to o^+}(1+x)^{1/x} (1^\infty - form)$

$$\therefore \quad L = \lim_{x \to o^+}\frac{1}{x}\log(1+x)\ (0 \cdot \infty - form)$$

$$= \lim_{x \to o^+}\frac{\log(1+x)}{x}(0/0 - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to o^+}\frac{1}{1+x} = 1$$

Therefore, $\lim_{x \to o^+}(1+x)^{1/x} = \lim_{x \to o^+} e^{\frac{1}{x}\log(1+x)} = e^1 = e.$

Example-2: Find $\lim_{x \to \infty} x^{1/x}$.

Solution : Here $\lim_{x \to \infty} x^{1/x}\ (\infty^0 - form)$

$$\therefore \quad L = \lim_{x \to \infty}\frac{1}{x}\log x \quad (0 \cdot \infty - form)$$

$$= \lim_{x \to \infty}\frac{\log x}{x}(\infty/\infty - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to \infty}\frac{1}{x} = 0$$

Therefore, $\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\frac{1}{x}\log x} = e^0 = 1.$

Example-3: Find $\lim_{x \to o^+} x^x$.

Solution : Here $\lim_{x \to o^+} x^x (0^0 - form)$

$$\therefore \quad L = \lim_{x \to o^+} x \log x \ (0 \cdot \infty - form)$$

$$= \lim_{x \to o^+} \frac{\log x}{\frac{1}{x}} (\infty/\infty - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to o^+} \frac{1/x}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to o^+} -x = 0$$

Therefore, $\lim_{x \to o^+} x^x = \lim_{x \to o^+} e^{x \log x} = e^0 = 1.$