MATHEMATICS-II

UNIT-3 : ORDINARY DIFFERENTIAL EQUATIONS

Chapter 4

First Order Differential Equations

Differential equations are of great importance in engineering because many engineering problems when expressed mathematically, appear in the form of differential equations. For instance, the current I in an RLC circuit is governed by the differential equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

where L is the inductance, R is the resistance, C is the capacitance and E is the electromotive force.

In this chapter, we shall introduce the relevant concepts in order to be able to discuss differential equations and we shall discuss some methods to solve first order ordinary differential equations.

4.1 Tutorial : Basic Concepts

Definition

An equation involving the derivatives of a dependent variable with respect to one or more independent variables is called a *differential equation*.

- **Ordinary Differential Equation:** A differential equation that contains derivatives with respect to only one independent variable is called an *ordinary differential equation (ODE)*. For example,

$$\text{(i)}\ \frac{dy}{dx} + y = \sin x, \qquad \text{(ii)}\ \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \qquad \text{(iii)}\ x^3\frac{d^3y}{dx^3} + 2x\frac{dy}{dx} + y = x\log x.$$

- **Partial Differential Equation:** A differential equation that contains derivatives with respect to more than one independent variable is called a *partial differential equation (PDE)*. For example,

$$\text{(i)}\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u, \qquad \text{(ii)}\ \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad \text{(iii)}\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Order and Degree

- The *order* of a differential equation is the order of the highest derivative occurring in the equation.
- The *degree* of a differential equation is the power of the highest ordered derivative occurring in the equation provided all the derivatives are made free from radicals and fractions.

For example,

(1) The differential equation $\frac{dy}{dx} + 4y = e^x$ is of order 1 and degree 1.

(2) The differential equation $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} = \cos x$ is of order 2 and degree 1.

(3) The differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + y\left(\frac{dy}{dx}\right)^3 + y = \cos x$ is of order 2 and degree 2.

(4) The differential equation $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^x$ is of order 3 and degree 1.

(5) The differential equation $\left[1 + \left(\frac{d^3y}{dx^3}\right)^2\right]^{\frac{2}{5}} = \frac{dy}{dx}$ is of order 3 and degree 4.

Concept of Solution

A *solution* of a differential equation is a relation between the variables which does not involve any derivatives and satisfies the given differential equation.

- **General Solution:** The solution of a differential equation is intitutively integrations, hence the solution will contain arbitrary constants. Such a solution is called a *general solution*. The number of arbitrary constants in the general solution is equal to the order of the differential equation.
- **Particular Solution:** A solution of a differential equation obtained from the general solution by giving particular values to the arbitrary constants is called a *particular solution*.
- **Singular Solution:** A solution which cannot be obtained from the general solution is called a *singular solution*. Furthermore, it does not involve any arbitrary constant.

Initial and Boundary Value Problems

A differential equation together with some conditions specified at one value of the independent variable is called an *initial value problem (IVP)*. These conditions are called *initial conditions*. For Example,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = \sin 2x; \quad y(0) = 1,\ y'(0) = 4.$$

A differential equation together with some conditions specified at more than one value of the independent variable is called a *boundary value problem (BVP)*. These conditions are called *boundary conditions*. For Example,

$$\frac{d^2y}{dx^2} + 9y = 0; \quad y(0) = 2,\ y(1) = 3.$$

First Order ODEs

The *standard form* of a first order ordinary differential equation in the unknown function $y(x)$ is

$$\frac{dy}{dx} = f(x, y). \tag{4.1.1}$$

The *differential form* of a first order ordinary differential equation in the unknown function $y(x)$ is

$$M(x, y)dx + N(x, y)dy = 0. \tag{4.1.2}$$

The first order ordinary differential equations can be classified as follows:

(1) Variable Separable Equations;

(2) Homogeneous Differential Equations;

(3) Exact Differential Equations;

(4) Linear Differential Equations.

Variable Separable Equations

A differential equation which can be expressed so that the coefficients of dx is only a function of x and that of dy is only a function of y is called a *variable separable equation*. Thus the general form of such an equation is

$$g(y)dy = f(x)dx \tag{4.1.3}$$

To solve (4.1.3), integrate both sides, we obtain

$$\int g(y)dy = \int f(x)dx + c.$$

The result is an equation involving x and y that determines y as a function of x provided above integrals exist.

Homogeneous Differential Equations

A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be a *homogeneous differential equation* if all the terms in $M(x, y)$ and $N(x, y)$ have same degree. For example,

(i) $(x^2 - 2xy)dx + (y^2 - x^2)dy = 0,$ (ii) $(x^3 - 3x^2y)dx + (x^2y - y^3)dy = 0.$

This type of differential equations can be solved by substituting $y = vx$. Above two types of differential equations are easy to solve and have been already studied earlier. So, we will start with the exact differential equations.

4.2 Tutorial : Exact Differential Equations

Definition

A first order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{4.2.1}$$

is said to be *exact* if there exists a function $U(x,y)$ such that $Mdx + Ndy = dU$.

Condition For Exactness

The necessary and sufficient condition for the differential equation (4.2.1) to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

where $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ denote partial derivatives of M and N w.r.t. y and x respectively.

Method of Solution

For a given differential equation $M(x,y)dx + N(x,y)dy = 0$,

- Test the condition of exactness, i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- If the condition is satisfied, then the given differential equation is exact and its solution is given by
$$\int\limits_{y \text{ const.}} M dx + \int (\text{Terms in } N \text{ not involving } x)\, dy = c.$$
- If the condition of exactness is not satisfied, i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the differential equation is not exact.

Solved Examples

Example 4.2.1. *Solve* $(x^2 + 2xy)dx + (x^2 - y^2)dy = 0$.

Solution. The given equation is of the form $Mdx + Ndy = 0$ with

$$M = x^2 + 2xy \quad \text{and} \quad N = x^2 - y^2.$$

First we check for exactness. It can be seen that

$$\frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x.$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the equation is exact and its solution is given by

$$\int_{y\ \text{const.}} M\,dx + \int (\text{Terms in } N \text{ not involving } x)\,dy = c$$

$$\Rightarrow \int_{y\ \text{const.}} (x^2 + 2xy)dx - \int y^2 dy = c$$

$$\Rightarrow \frac{x^3}{3} + x^2 y - \frac{y^3}{3} = c. \quad \blacksquare$$

Example 4.2.2. *Find the solution of the differential equation*

$$ye^x dx + (2y + e^x)dy = 0, \text{ where } y(0) = -1.$$

Solution. The given equation is of the form $M dx + N dy = 0$ with

$$M = ye^x \quad \text{and} \quad N = 2y + e^x.$$

First we check for exactness. It can be seen that

$$\frac{\partial M}{\partial y} = e^x \quad \text{and} \quad \frac{\partial N}{\partial x} = e^x.$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the equation is exact and its solution is given by

$$\int_{y\ \text{const.}} M\,dx + \int (\text{Terms in } N \text{ not involving } x)\,dy = c$$

$$\Rightarrow \int_{y\ \text{const.}} (ye^x)dx + \int 2y dy = c$$

$$\Rightarrow ye^x + y^2 = c.$$

Now,

$$y(0) = -1 \quad \Rightarrow \quad (-1)(e^0) + (-1)^2 = c \quad \Rightarrow \quad c = 0.$$

So, the required particular solution is

$$ye^x + y^2 = 0. \quad \blacksquare$$

Example 4.2.3. *Solve the initial value problem*

$$(\cosh x \cos y)dx - (\sinh x \sin y)dy = 0, \qquad y(0) = \pi.$$

Solution. The given equation is of the form $M(x, y)dx + N(x, y)dy = 0$ with

$$M = \cosh x \cos y \quad \text{and} \quad N = -\sinh x \sin y.$$

First we check for exactness. It can be seen that

$$\frac{\partial M}{\partial y} = -\cosh x \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = -\cosh x \sin y.$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, the equation is exact and its solution is given by

$$\int\limits_{y\ \text{const.}} M dx + \int (\text{Terms in } N \text{ not involving } x)\, dy = c$$

$$\Rightarrow \quad \int\limits_{y\ \text{const.}} (\cosh x \cos y) dx = c$$

$$\Rightarrow \quad \cos y \sinh x = c.$$

Now,

$$y(0) = \pi \quad \Rightarrow \quad \cos \pi \sinh 0 = c \quad \Rightarrow \quad c = 0.$$

Hence, the required particular solution is

$$\cos y \sinh x = 0. \quad \blacksquare$$

Exercises

Exercise 4.2.1. *Solve* $(2e^y - ye^x)dx + (2xe^y - e^x + y^3)dy = 0.$

Exercise 4.2.2. *Solve* $(1 + 2xy)dx + (1 + x^2 + 2y)dy = 0.$

Exercise 4.2.3. *Test for the exactness and solve* $\left[(x+1)e^x - e^y\right]dx - xe^y dy = 0,\ y(1) = 0.$

Exercise 4.2.4. *Solve the differential equation* $ye^{xy}dx + (xe^{xy} + 2y)dy = 0.$

Exercise 4.2.5. *Find the general solution of the differential equation*

$$\left(-\frac{y}{x^2} + 2\cos 2x\right) dx + \left(\frac{1}{x} - 2\sin 2y\right) dy = 0.$$

Exercise 4.2.6. *Solve* $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

Viva Questions

Question 4.2.7. *When the equation* $Mdx + Ndy = 0$ *is said to be exact ?*

Question 4.2.8. *How to give the general solution if the equation is exact ?*

Question 4.2.9. *Is the differential equation* $(e^{x+y} - y)dx + (xe^{x+y} + 1)dy = 0$ *exact ?*

Question 4.2.10. *Is the differential equation* $(e^x - y)dx + (e^y - x)dy = 0$ *exact ?*

Answers

4.2.1 $2xe^y - ye^x + \frac{y^4}{4} = c$ **4.2.2** $x(1 + xy) + y(1 + y) = c$
4.2.3 $x(e^x - e^y) = e - 1$ **4.2.4** $e^{xy} + y^2 = c$
4.2.5 $\frac{y}{x} + \sin 2x + \cos 2y = c$ **4.2.6** $y\sin x + x\sin y + xy = c$
4.2.9 no **4.2.10** yes

4.3 Tutorial : Integrating Factor

Definition

Sometimes a differential equation which is not exact, can be made so by multiplying a suitable factor called an *integrating factor (I.F.)*.

Rules For Finding an Integrating Factor

Consider a differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0. \tag{4.3.1}$$

Rule-1. If (4.3.1) is a homogeneous equation, then

$$\text{I.F.} = \frac{1}{Mx+Ny}, \text{ provided } Mx+Ny \neq 0.$$

Rule-2. If (4.3.1) is of the form $f(xy)ydx + g(xy)xdy = 0$, then

$$\text{I.F.} = \frac{1}{Mx-Ny}, \text{ provided } Mx-Ny \neq 0.$$

Rule-3. For (4.3.1), if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x only, say $f(x)$, then

$$\text{I.F.} = e^{\int f(x)dx}.$$

Rule-4. For (4.3.1), if $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y only, say $g(y)$, then

$$\text{I.F.} = e^{-\int g(y)dy}.$$

Solved Examples

Example 4.3.1. *Solve* $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0.$

Solution. Here,

$$M = x^2y - 2xy^2, \quad \text{and} \quad N = -x^3 + 3x^2y.$$

Therefore,

$$\frac{\partial M}{\partial y} = x^2 - 4xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -3x^2 + 6xy.$$

Thus

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence, the given equation is not exact. But the equation is homogeneous in x and y. Also,

$$\begin{aligned} Mx + Ny &= (x^2y - 2xy^2)x + (-x^3 + 3x^2y)y \\ &= x^3y - 2x^2y^2 - x^3y + 3x^2y^2 \\ &= x^2y^2 \end{aligned}$$

$$\neq \; 0.$$

Therefore,

$$\text{I.F.} \;=\; \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}.$$

Multiplying the equation throughout by $\frac{1}{x^2y^2}$, we obtain

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0$$

which is exact (verify!). Therefore, the solution is given by

$$\frac{x}{y} - 2\ln x + 3\ln y = c. \quad \blacksquare$$

Example 4.3.2. *Solve* $(1+xy)ydx + (1-xy)xdy = 0$.

Solution. Here,

$$M = y + xy^2 \quad \text{and} \quad N = x - x^2y.$$

Therefore,

$$\frac{\partial M}{\partial y} = 1 + 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 1 - 2xy.$$

Thus

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence, the given equation is not exact. But the equation is of the form

$$f(xy)ydx + g(xy)xdy = 0.$$

Also,

$$Mx - Ny \;=\; (y+xy^2)x - (x-x^2y)y \;=\; xy + x^2y^2 - xy + x^2y^2 \;=\; 2x^2y^2 \;\neq\; 0.$$

Therefore,

$$\text{I.F.} = \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}.$$

Multiplying the equation throughout by $\frac{1}{2x^2y^2}$, we obtain

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0$$

which is exact (verify!). Therefore, the solution is given by

$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\ln x - \frac{1}{2}\ln y = c' \quad \text{or} \quad \ln\frac{x}{y} - \frac{1}{xy} = c, \quad \text{where } c = 2c'. \quad \blacksquare$$

Example 4.3.3. *Find the general solution of the differential equation*

$$2\sin y^2 dx + xy\cos y^2 dy = 0.$$

Solution. Here,

$$M = 2\sin y^2 \quad \text{and} \quad N = xy\cos y^2.$$

Therefore,

$$\frac{\partial M}{\partial y} = 4y\cos y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = y\cos y^2.$$

Thus

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence, the equation is not exact. But

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy\cos y^2}\left[4y\cos y^2 - y\cos y^2\right] = \frac{3y}{xy} = \frac{3}{x}$$

which is a function of x only. Therefore,

$$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3.$$

Multiplying the given equation by x^3, we obtain

$$2x^3 \sin y^2 dx + x^4 y\cos y^2 dy = 0$$

which is exact (verify!). Therefore, its solution is given by

$$\frac{x^4}{2}\sin y^2 = c. \quad \blacksquare$$

Example 4.3.4. *Solve* $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0.$

Solution. Here,

$$M = xy^3 + y \quad \text{and} \quad N = 2(x^2y^2 + x + y^4)dy = 0.$$

Therefore,

$$\frac{\partial M}{\partial y} = 3xy^2 + 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = 4xy^2 + 2.$$

Thus

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence, the equation is not exact. But

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y(xy^2+1)}(3xy^2 + 1 - 4xy^2 - 2) = \frac{-xy^2 - 1}{y(xy^2+1)} = -\frac{1}{y}$$

which is a function of y only. Therefore,

$$\text{I.F.} = e^{-\int -\frac{1}{y}dy} = e^{\ln y} = y.$$

Multiplying the given equation by y, we obtain

$$(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$$

which is exact (verify!). Therefore, its solution is given by

$$\int_{y\ \text{const.}} (xy^4 + y^2)dx + \int 2y^5 dy = c \quad \Rightarrow \quad \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c. \quad \blacksquare$$

Exercises

Exercise 4.3.1. *Solve* $(x^2y^2+2)ydx+(2-x^2y^2)xdy=0.$

Exercise 4.3.2. *Solve* $(e^{x+y}-y)dx+(xe^{x+y}+1)dy=0.$

Exercise 4.3.3. *Solve* $x^2ydx-(x^3+xy^2)dy=0.$

Exercise 4.3.4. *Solve* $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0.$

Exercise 4.3.5. *Solve* $(xy-2y^2)dx-(x^2-3xy)dy=0.$

Exercise 4.3.6. *Solve* $(1+2x^2y^2)ydx+(1+xy+2x^2y^2)xdy=0.$

Exercise 4.3.7. *Solve* $y(2xy+1)dx+x(1+2xy-x^3y^3)dy=0.$

Exercise 4.3.8. *Solve* $2xydx+3x^2dy=0.$

Exercise 4.3.9. *Solve* $(x^3+y^2)dx-xydy=0.$

Viva Questions

Question 4.3.10. *What is meant by an integrating factor ?*

Question 4.3.11. *Is* $\frac{1}{xy}$ *an integrating factor for the equation* $(2y+xy)dx+2xdy=0$ *?*

Question 4.3.12. *What is the form an integrating factor if the equation is homogeneous ?*

Answers

4.3.1 $\frac{1}{2}\log\frac{x}{y}-\frac{1}{2x^2y^2}=c$ **4.3.2** $xe^y+ye^{-x}=c$ **4.3.3** $-\frac{x^2}{2y^2}+\log y=c$
4.3.4 $xy+\frac{2x}{y^2}+y^2=c$ **4.3.5** $\frac{x}{y}-2\log x+3\log y=c$ **4.3.6** $\frac{1}{xy}-2xy-\log y=c$
4.3.7 $\frac{1}{x^2y^2}+\frac{1}{3x^3y^3}+\ln y=c$ **4.3.8** $x^2y^3=c$ **4.3.9** $x-\frac{y^2}{2x^2}=c$ **4.3.11** Yes

4.4 Tutorial : Linear Differential Equations

Definition

A first order differential equation of the form

$$\frac{dy}{dx} + p(x)y = q(x) \qquad \text{(where } p \text{ and } q \text{ are the functions of } x\text{)} \tag{4.4.1}$$

is called a *linear differential equation.*

Method of Solution

- Compare the given equation with the standard form
$$\frac{dy}{dx} + p(x)y = q(x)$$
to determine $p(x)$ and $q(x)$.
- Find *I.F.* using the formula $e^{\int p(x)dx}$.
- Write the general solution as $y(I.F.) = \int q(x)(I.F.)dx + c$.

Solved Examples

Example 4.4.1. *Solve* $\frac{dy}{dx} + y\tan x = \sin 2x,\ y(0) = 1.$

Solution. The given equation is linear with $p(x) = \tan x$ and $q(x) = \sin 2x$. Therefore,

$$\text{I.F.} = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x.$$

Hence, the general solution is

$$\begin{aligned} y\sec x &= \int \sin 2x \sec x x dx + c \\ \Rightarrow \quad y\sec x &= \int 2\sin x\cos x\sec x dx + c \\ \Rightarrow \quad y\sec x &= 2\int \sin x dx + c \\ \Rightarrow \quad y\sec x &= -2\cos x + c \\ \Rightarrow \quad y &= c\cos x - 2\cos^2 x. \end{aligned}$$

Now,

$$y(0) = 1 \quad \Rightarrow \quad c\cos 0 - 2\cos^2 0 = 1 \quad \Rightarrow \quad c(1) - 2(1)^2 = 1 \quad \Rightarrow \quad c = 3$$

Hence, the required particular solution is

$$y = 3\cos x - 2\cos^2 x. \quad \blacksquare$$

Example 4.4.2. *Solve* $\dfrac{dy}{dx} + xy = e^{-\frac{x^2}{2}}$.

Solution. The given equation is linear with $p(x) = x$ and $q(x) = e^{-\frac{x^2}{2}}$. Therefore,

$$\text{I.F.} = e^{\int p(x)dx} = e^{\int x dx} = e^{\frac{x^2}{2}}.$$

Hence, the general solution is

$$\begin{aligned} ye^{\frac{x^2}{2}} &= \int e^{-\frac{x^2}{2}} \cdot e^{\frac{x^2}{2}}\, dx + c \\ \Rightarrow\quad ye^{\frac{x^2}{2}} &= \int 1\, dx + c \\ \Rightarrow\quad ye^{\frac{x^2}{2}} &= x + c \\ \Rightarrow\quad y &= (x+c)e^{-\frac{x^2}{2}}. \quad \blacksquare \end{aligned}$$

Exercises

Exercise 4.4.1. *Solve* $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1}\, y = \dfrac{1}{(x^2+1)^3}$.

Exercise 4.4.2. *Solve the linear equation* $\dfrac{dy}{dx} - y\cos x = e^{\sin x}$.

Exercise 4.4.3. *Solve the initial value problem* $y' + 6x^2 y = \dfrac{e^{-2x^3}}{x^2}$, *where* $y(1) = 0$.

Exercise 4.4.4. *Solve the differential equation* $\dfrac{dy}{dx} + \dfrac{y}{x} = e^x$.

Exercise 4.4.5. *Solve* $(x+1)\dfrac{dy}{dx} - y = e^{3x}(x+1)^2$.

Exercise 4.4.6. *Solve* $(1+y^2)\dfrac{dx}{dy} = \tan^{-1} y - x$.

Exercise 4.4.7. *Solve the initial value problem* $y' - (1+3x^{-1})y = x+2,\ y(1) = e-1$.

Viva Questions

Question 4.4.8. *What is the standard form of a linear equation ?*

Question 4.4.9. *What is the integrating factor for a linear equation ?*

Question 4.4.10. *Which of the following equations are linear ?*

(i) $\dfrac{dy}{dx} + x^3 y = \sin x$ (ii) $\dfrac{dy}{dx} + xy^2 = e^x$ (iii) $\dfrac{dy}{dx} = \dfrac{1}{6x+e^y}$ (iv) $\dfrac{dy}{dx} + y = \dfrac{x}{y}$

Answers

4.4.1 $y(x^2+1)^2 = \tan^{-1} x + c$ **4.4.2** $y = (x+c)e^{\sin x}$ **4.4.3** $ye^{2x^3} = 1 - \frac{1}{x}$
4.4.4 $y = \left(1 - \frac{1}{x}\right)e^x + \frac{c}{x}$ **4.4.5** $y = (x+1)\left(\frac{e^{3x}}{3} + c\right)$ **4.4.6** $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$
4.4.7 $y = x(e^x x^2 - 1)$ **4.4.10** (i)-linear in y, (iii)-linear in x

4.5 Tutorial : Bernoulli Equation

Definition

An equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad \text{(where } p \text{ and } q \text{ are functions of } x\text{)} \tag{4.5.1}$$

is called the *Bernoulli equation.*

Method of Solution

- Divide equation (4.5.1) by y^n to get
$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x). \tag{4.5.2}$$
- Put $y^{1-n} = z$, so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$. Thus equation (4.5.2) becomes
$$\frac{1}{1-n}\frac{dz}{dx} + p(x)z = q(x) \quad \text{or} \quad \frac{dz}{dx} + p(x)(1-n)z = q(x)(1-n) \tag{4.5.3}$$
which is a linear equation in z.
- Solve equation (4.5.3) by the usual method of linear differential equation.
- Replace z by y^{1-n} in the solution of equation (4.5.3) which will give the general solution of equation (4.5.1).

Solved Examples

Example 4.5.1. *Find the general solution of the differential equation*

$$\frac{dy}{dx} + 2y = y^2.$$

Solution. The given equation is a Bernoulli equation. Dividing the equation throughout by y^2, we get

$$y^{-2}\frac{dy}{dx} + 2y^{-1} = 1. \tag{4.5.4}$$

Put $y^{-1} = z$, so that $-y^{-2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$. Therefore, equation (4.5.4) becomes

$$-\frac{dz}{dx} + 2z = 1 \quad \text{or} \quad \frac{dz}{dx} - 2z = -1. \tag{4.5.5}$$

which is a linear equation in z with $p(x) = -2$ and $q(x) = -1$. So, we have

$$\text{I.F.} = e^{\int p(x)dx} = e^{-\int 2dx} = e^{-2x}.$$

Thus the general solution of equation (4.5.5) is given by

$$ze^{-2x} = -\int e^{-2x}dx + c \quad \Rightarrow \quad ze^{-2x} = \frac{e^{-2x}}{2} + c.$$

Replacing z by y^{-1}, we get the general solution of the given equation as

$$y^{-1}e^{-2x} = \frac{e^{-2x}}{2} + c \quad \Rightarrow \quad \frac{1}{y} = \frac{1}{2} + ce^{2x}. \quad \blacksquare$$

Example 4.5.2. *Find the solution of the differential equation*

$$\frac{dy}{dx} + y = -\frac{x}{y}.$$

Solution. The given equation is a Bernoulli equation. Multiplying the equation throughout by y, we get

$$y\frac{dy}{dx} + y^2 = -x. \tag{4.5.6}$$

Let $y^2 = z$. Then $2y\dfrac{dy}{dx} = \dfrac{dz}{dx}$. Therefore, equation (4.5.6) becomes

$$\frac{1}{2}\frac{dz}{dx} + z = -x \quad \text{or} \quad \frac{dz}{dx} + 2z = -2x. \tag{4.5.7}$$

which is a linear equation in z with $p(x) = 2$ and $q(x) = -2$. So, we have

$$\text{I.F.} \ = e^{\int p(x)dx} = e^{\int 2dx} = e^{2x}.$$

Thus the general solution of equation (4.5.7) is given by

$$\begin{aligned}
ze^{2x} &= \int -2xe^{2x}dx + c \\
\Rightarrow \quad ze^{2x} &= -2\left[(x)\left(\frac{e^{2x}}{2}\right) - (1)\left(\frac{e^{2x}}{4}\right)\right] + c \\
\Rightarrow \quad ze^{2x} &= -\frac{1}{2}(2xe^{2x} - e^{2x}) + c \\
\Rightarrow \quad z &= -\frac{1}{2}(2x - 1) + ce^{-2x}.
\end{aligned}$$

Replacing z by y^2, we get the general solution of the given equation as

$$y^2 = -\frac{1}{2}(2x - 1) + ce^{-2x} \quad \Rightarrow \quad 2y^2 = 1 - 2x + 2ce^{-2x}. \quad \blacksquare$$

Exercises

Exercise 4.5.1. *Find the general solution of the differential equation*

$$\frac{dy}{dx} + 3y = e^{2x}y^3.$$

Exercise 4.5.2. *Solve* $xy' = y^2 + y$.

Exercise 4.5.3. *Solve* $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$.

Exercise 4.5.4. *Find the general solution of the differential equation*

$$\frac{dy}{dx} + (x + 1)y = e^{x^2}y^3.$$

Exercise 4.5.5. *Solve the Bernoulli equation* $y' + \frac{1}{3}y = \frac{1}{3}(1 - 2x)y^4$.

Exercise 4.5.6. *Solve the differential equation* $y' + xy = xy^{-1}$.

Viva Questions

Question 4.5.7. *What is the standard form of a Bernoulli equation ?*

Question 4.5.8. *How to reduce a Bernoulli equation into a linear equation ?*

Question 4.5.9. *Which of the following are Bernoulli equations ?*

(i) $\dfrac{dy}{dx} + 2xy = y^4$ (ii) $\dfrac{dy}{dx} + y\sin x = xy^{-1}$ (iii) $\dfrac{dy}{dx} + x^2y^3 = x^2$ (iv) $\dfrac{dy}{dx} + x^2y^3 = x^2y$

Answers

4.5.1 $\frac{2}{y^2} = e^{2x}(1 + ce^{4x})$ **4.5.2** $x(1 + y) = cy$ **4.5.3** $e^{-y} = \frac{1}{2x} + cx$ **4.5.5** $\frac{1}{y^3} = ce^x - 2x - 1$
4.5.4 $\frac{1}{y^2} = e^{x^2}(1 + ce^{2x})$ **4.5.6** $y = \sqrt{1 + ce^{-x^2}}$ **4.5.9** (i), (ii), (iv)

4.6 Tutorial : Clairaut's Equation

An equation of the form
$$y = px + f(p) \tag{4.6.1}$$
is known as *Clairaut's equation*.

Differentiating equation (4.6.1) w. r. t. x, we get
$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0 \quad \Rightarrow \quad x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0 \quad \Rightarrow \quad \left[x + f'(p)\right]\frac{dp}{dx} = 0.$$
Thus
$$\frac{dp}{dx} = 0 \quad \text{or} \quad x + f'(p) = 0. \tag{4.6.2}$$
The solution of first equation in (4.6.2) is $p = c$, where c is an arbitrary constant. Thus the general solution of equation (4.6.1) is given by
$$y = cx + f(c). \tag{4.6.3}$$
If we eliminate p form equation (4.6.1) and the second equation in (4.6.2), we get a singular solution of equation (4.6.1).

Solved Examples

Example 4.6.1. *Solve* $y = xp + \frac{1}{p}$*. Also, find the singular solution.*

Solution. The given equation is of the form
$$y = px + f(p) \quad \text{(Clairaut's equation)} \tag{4.6.4}$$
Therefore, the general solution is given by
$$y = cx + f(c)$$
Also, from the second equation in (4.6.2), we have
$$x - f'(p) = 0 \quad \Rightarrow \quad x - \frac{1}{p^2} = 0.$$
Eliminating p from the above equation and given equation, we get
$$y^2 = 4x,$$
which is a singular solution of the given equation. ∎

Exercises

Exercise 4.6.1. *Find the general and singular solution of* $y = xp + \frac{a}{p}$.

Exercise 4.6.2. *Solve* $p^2(y - px) = 1 - p + \sin p, \quad p > 0$.

Answers

4.6.1 $y = cx + a/c,\ y^2 = 4ax$

4.6.2 $y = cx + 1 - c + \sin c/c^2$