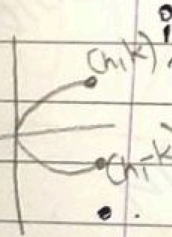
Curve Sketching:

Let $y = f(x)$ or $f(x,y) = 0$ be a rational integral algebraic eq^n.

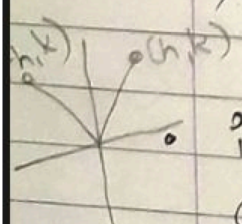
1) Symmetry:

- A curve is symmetric about a line l (or point O) means the shape of a curve on one side of l (or a point O) will be a mirror image on the other side of l (or a point O).
- In other words,
  A curve is said to be symmetric w.r. to a line l (or point O) if, whenever a point A lies on the curve, the point B, which is a mirror image of point A with r. to l (or O) also lies on the curve.



i) Symmetry about the x-axis:
if the eq^n of a curve remains unchanged when y is changed to $-y$.

- if (h,k) lies on curve then $(h,-k)$ lies on curve
  i.e $f(h,k) = 0 \Rightarrow f(h,-k) = 0$



ii) Symmetry about the y-axis:
if the eq^n of a curve remains unchanged when x is changed to $-x$

- if (h,k) lies on curve then $(-h,k)$ lies on curve.
  ie $f(h,k) = 0 \Rightarrow f(-h,k) = 0$

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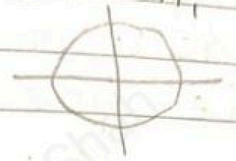
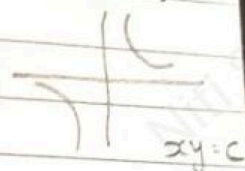
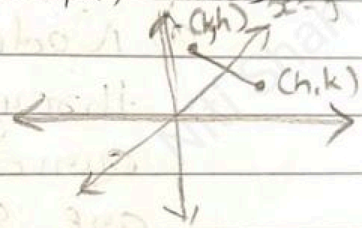
iii) Symmetry about origin : (Symmetry in Opposite quadrants)
if eq^n of a curve remains unchanged when x is changed to $-x$ & y is changed to $-y$
• if (h,k) lies on curve then $(-h,-k)$ lies on curve.
i.e $f(h,k)=0 \Rightarrow f(-h,-k)=0$





iv) Symmetry about $y=x$ line
if eq^n of a curve remains unchanged when x is changed to y & y is changed to $-y$
• if (h,k) lies on curve then (k,h) lies on curve.
i.e $f(h,k)=0 \Rightarrow f(k,h)=0$.



2) Intercepts :
x-intercepts : put $y=0$ in the eq^n of curve.
y-intercept : put $x=0$ in the eq^n of curve.
• Find tangents at these points if necessary by shifting the origin.

3) Origin / tangent at origin :

• If there is no constant term in the eq^n of curve then curve is passing through origin.
• To find tangents at origin :

Theorem : If a curve passing through the origin, the eq^n of the tangents at the origin is obtained by equating to zero the lowest degree terms in the eq^n of curve.

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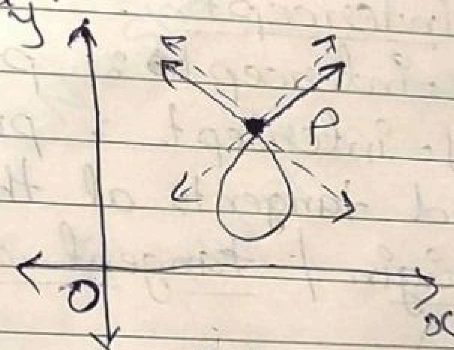
If there are two tangents, we can find their nature.

Def: Double Point — A point traced out twice as a closed is traversed
i.e two braches of the curve pass through it.

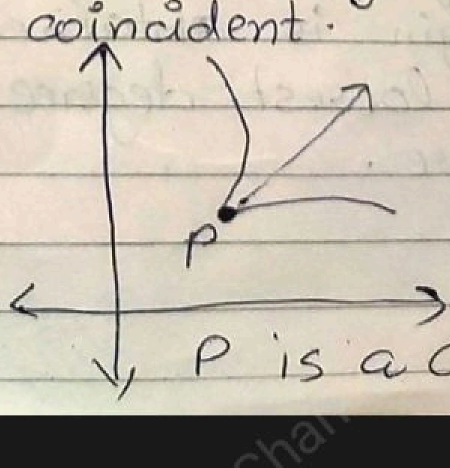
- Classification of Double Points:
- There are three types of double points

a) Node : A node is a point on the curve through which two real braches of the curve pass and two tangents at which are real and distinct.



P is Node in figure

b) Cusp : A cusp is a point on the curve through which two real braches of the curve pass and two tangents at which are real and coincident.



P is a cusp in figure.

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c) Conjugate Point or Isolated Point :
It is a point in the neighbourhood of which there are no other real points of the curve. The tangents at a conjugate point are generally imaginary.

4) Asymptotes :

Defn: A line is said to be an asymptote of a curve, if the distance from a point on the curve to the line tends to zero as the curve extends indefinitely farther away from the origin.

- An asymptote is considered as tangent to the curve at infinity.
- Asymptote gives us the direction and bound of the curve.

i) Asymptote parallel to the x-axis :
To find asymptotes $\parallel^r$ to x-axis, equate to zero the real linear factors in the co-efficient of the highest power of x in the eq^n of the curve.

ii) Asymptote parallel to the y-axis :
To find asymptotes $\parallel^r$ to y-axis, equate to zero the real linear factors in the co-efficient of the highest power of y in the eq^n of the curve.

- Draw Asymptote by dotted lines.

5) Extent / region :

Find those regions of the xy-plane where the curve does not exist.

- It is obtained by solving the eq^n for one variable in terms of the other variable (i.e $y = f(x)$ or $x = f(y)$) and find the set of values of one-variable that make the ~~curve~~ other imaginary.

6) Draw the curve using above 5) points.

Sign of y :

For the curve $y = \dfrac{p(x)}{q(x)}$.

- First take $p(x) = 0$ & $q(x) = 0$ then find values of x.
- Arrange above values in increasing order and make different intervals.
- Check sign of y in above intervals.

Tangents at intercept :

(By Shifting Origin)

- If $x = a$ is a x-intercept, then curve passes through $(a, 0)$.
- For tangent at $(a, 0)$, Shift the origin to the point $(a, 0)$ by putting

$$x = x' + a \quad , \quad y = y' + 0$$

Then transform the given eq^n of curve into new origin (x', y') by substituting above.

- Equating to zero the lowest degree term, we can find the tangent at the new origin at $(a, 0)$.

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Example: Trace the curve $y^2(2a-x) = x^3$ $(a>0)$.

Solⁿ. The eqⁿ of the curve is
$$y^2(2a-x) = x^3 \quad \text{—— (1)}$$

1) Symmetry:
Since eqⁿ remains unchanged when y is replaced by $-y$.
It is symmetric about the x-axis

2) Intercepts:
x-intercepts; put $y=0$ in (1) we get
$$x=0$$
y-intercepts: put $x=0$ in (1) we get
$$y=0$$
∴ The curve meets the axes at origin only

3) Origin/tangent at origin:
Since, curve is passing through origin, we find tangent by equating to zero the lowest degree term in (1),

$$2ay^2 = 0$$
$$\Rightarrow y^2 = 0 \quad \text{as } a>0$$
$$\Rightarrow y=0,\ y=0$$
Since, two tangents are real and coincident, origin is a cusp.

4) Asymptotes:
a) ∥er to x axis: The higest degree term of x is x^2. The coefficient of x^2 is 1
∴ There is no horizontal asymptote.

b) $\parallel^{r}$ to y-axis: The highest degree term of y is y^2. The coefficient of y^2 is $(2a-x)$

By equating to zero, the coefficient of y^2 we get

$$2a-x=0$$
$$\Rightarrow x=2a$$

$\therefore$ The line $x=2a$ is a vertical asymptote of the curve.

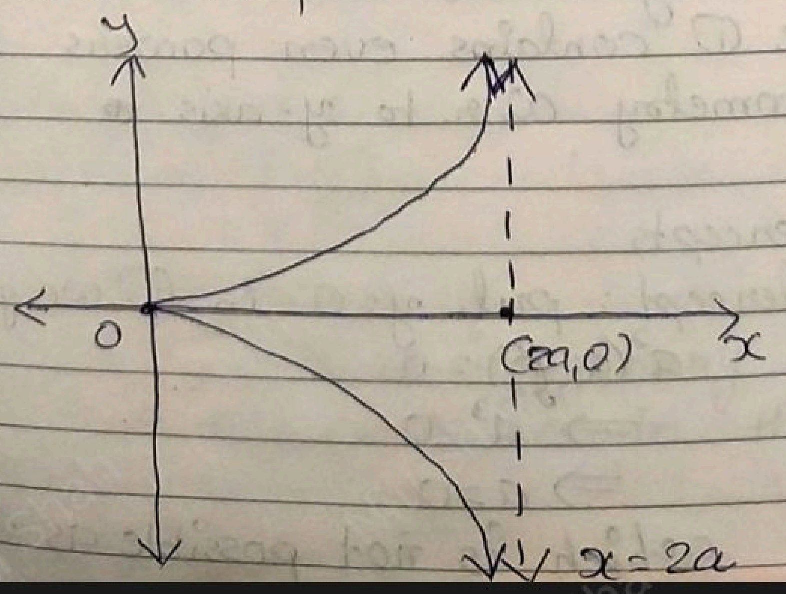
5) Extent / Region:

From ① $y^2 = \dfrac{x^3}{2a-x}$

- When $x<0$, $y^2<0$ $\therefore y$ is imaginary. $\therefore$ No part of the curve lies in the left half plane.

- When $x>2a$, $y^2<0$ $\therefore y$ is imaginary $\therefore$ No part of curve lies to the right of the line $x=2a$.

6) Given Curve is,



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Example: ~~To~~ Trace the curves:

1) $x^2y = a^2(a-y)$; $a>0$
2) $x^2(2a-y) = y^3$
3) $xy^2 = a^2(a-x)$
4) $y(x^2+4a^2) = 8a^3$

5) $x(x^2+y^2) = a(x^2-y^2)$
6) $x^2y\cancel{6} + (y+a)^2(y+2a)=0$; $a>0$
7) $y^2(a-x) = x^2(a+x)$
8) $y^2(a+x) = x^2(3a-x)$

9) $a^2y^2 = x^3(2a-x)$
10) $a^2x^2 = y^3(2a-y)$
11) $ay^2 = x(x-a)^2$

12) $y^2(a^2+x^2) = x^2(a^2-x^2)$
13) $a^4y^2 = x^4(a^2-x^2)$
14) $a^2y^2 = x^2(a^2-x^2)$

Sol^n: 1) $x^2y = a^2(a-y)$ —— ①

i) Symmetry:
Since ① contains even powers of x, it is symmetry w. r. to y-axis.

ii) Intercepts:
x-intercept: put $y=0$ in ① we get

$$a^2(a-\cancel{y})=0$$
$$\Rightarrow a^3=0$$
$$\Rightarrow a=0$$

which is not possible as $a>0$

y-intercept : put $x=0$ in ① we get

$a^2(a-y)=0$

$\Rightarrow \cancel{a^3 - a^2y = 0}$

$\Rightarrow a-y=0$

$\Rightarrow y=a.$

iii) Origin | Tangent at origin :

Since, ① contains constant term (a^3), the curve does not pass through origin.

iv) Asymptotes :

→ ||er to x axis :

Highest degree term in x is x^2.

The coefficient of x^2 is y.

$\therefore$ $y=0$ is horizontal asymptotes.

→ ||er to y-axis :

Highest degree term in y is y

The coefficient of y is x^2+a^2

$\therefore$ $\cancel{x^2-a^2=0}$ $\quad\therefore$ There is no vertical asymptote as no real x is possible.

$\cancel{\Rightarrow x=\pm a}$

~~Thus, $x=\pm a$ are vertical asymptotes.~~

v) Region | Extent :

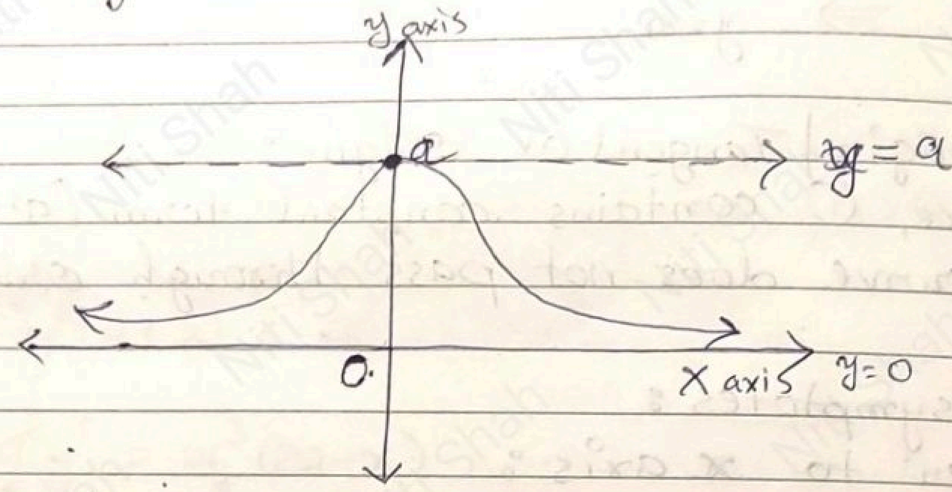
$x^2 = \dfrac{a^2(a-y)}{y}$

When $y<0$, $x^2<0$

$\therefore$ x is imaginary.

i.e No part of curve lies in the lower half plane.

when $y>a$, $x^2<0$
$\therefore$ x is imaginary
i.e. No part of curve lies ~~be~~ above the line $y=a$.



5 $x(x^2+y^2) = a(x^2-y^2)$ —— ①

Sol^n: i) Symmetry:
Since, ① contains even powers of y, it is symmetry w.r.to x-axis.

ii) Intercepts:
~~y~~-intercept: put $x=0$ in ①
then $0 = a(0-y^2)$
$\Rightarrow -ay^2=0$
$\Rightarrow y=0$
x~~y~~-intercept: put $y=0$ in ①
$x(x^2+0) = ax^2$
$\Rightarrow x^2 = ax$
$\Rightarrow x(x-a)=0$
$\therefore x=0, x=a$
$\therefore$ Curve passes through $(0,0)$ & $(a,0)$
Tangent at $(a,0)$: Let us shift the origin to the $(a,0)$ by putting $x=x'+a$ & $y=y'+0$.

$\therefore$ (1) transforms to $(x'+2a)y'^2 = -x'(x'+a)^2$

lowest degree term, $-a^2x'$ equaling to zero we get $x'=0$

$\Rightarrow x = a$ is tangent at $(a,0)$

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iii) Origin / Tangent at origin:

Since, there is no constant term in (1), it passes through origin.

$$x^2 - y^2 = 0$$
$$\Rightarrow x^2 = y^2$$
$$\Rightarrow x = \pm y$$

Since, two tangents are real and different, origin is node.

iv) Asymptote:

$\parallel^{er}$ to x-axis:

Highest ~~power~~ degree term in x is x^3, its co-efficient is 1

$\therefore$ there is no horizontal asymptote.

$\parallel^{er}$ to y-axis:

Highest degree term in y is y^2; its coefficient is $x+a$

$$\therefore x + a = 0$$
$$\Rightarrow x = -a$$

$x=a$ is vertical asymptote.

v) Region / Extent:

From (1) $x^3 + xy^2 - ax^2 + ay^2 = 0$

$$\Rightarrow x^3 + (x+a)y^2 - ax^2 = 0$$
$$\Rightarrow (x+a)y^2 = ax^2 - x^3$$
$$\therefore y^2 = \frac{x^2(a-x)}{a+x}$$

when $x > a$, $y^2 < 0$,

$\therefore$ y is imaginary

$\therefore$ No part of curve lies in to the right of line $x=a$.

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when $x < -a$, $y^2 < 0$

$\therefore$ y is imaginary

$\therefore$ No part of curve lies to the left of the curve $x = -a$