Gamma Functions:

The definite integral $\int_0^\infty e^{-x}x^{n-1}dx, \quad n>0$ is a function of n and is called gamma function. It is denoted by $\overline{)n} = \int_0^\infty e^{-x}x^{n-1}dx, \quad n>0$

Properties of Gamma function:

1. $\overline{)n+1} = \begin{cases} n\overline{)n} \\ n! \quad n \text{ is a positive integer} \end{cases}$
2. $\int_0^\infty e^{-ax}x^{n-1}dx = \dfrac{\overline{)n}}{a^n}$
3. $2\int_0^\infty e^{-x^2}x^{2n-1}dx = \overline{)n}$
4. $\overline{)\frac{1}{2}} = \sqrt{\pi}$

Examples:

1. Evaluate $\int_{-\infty}^{\infty} e^{-k^2x^2}dx$

Solution: Let

$$I = \int_{-\infty}^{\infty} e^{-k^2x^2}dx$$

$$= 2\int_0^\infty e^{-k^2x^2}dx$$

Let, $k^2x^2 = u \quad \therefore 2k^2xdx = du$

$$dx = \frac{1}{2k^2x}du = \frac{1}{2k}u^{-\frac{1}{2}}du$$

$$x: 0 \to \infty \Rightarrow u: 0 \to \infty$$

$$\therefore I = 2\int_0^\infty e^{-u}\ \frac{1}{2k} u^{-\frac{1}{2}} du$$

$$= \frac{1}{k}\int_0^\infty e^{-u} u^{\frac{1}{2}-1} du$$

$$= \frac{1}{k}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{k}\sqrt{\pi}$$

2. Evaluate $\int_0^\infty e^{-x^2} x^5 dx$

Solution: Take $x^2 = u$

$$\therefore 2xdx = du$$

$$\therefore x : 0 \to \infty \Rightarrow u : 0 \to \infty$$

$$\therefore I = \int_0^\infty e^{-x^2} x^4 dx$$

$$\therefore I = \int_0^\infty e^{-u} u^2 du$$

$$= \frac{1}{2}\int^\infty e^{-u} u^{3-1} du$$

$$= \frac{1}{2}\Gamma(3)$$

$$= 1$$

3. Evaluate $\int_0^1 x^4 e^{-x^4} dx$

Solution: Take $x^4 = u$

$$\therefore 4x^3 dx = du$$

$$\therefore x^3 dx = \frac{1}{4} du$$

Also $x : 0 \to \infty \Rightarrow u : 0 \to \infty$

$$\therefore I = \int_0^\infty x\, e^{-x^4}\, x^3 dx$$

$$= \int_0^\infty u^{\frac{1}{4}} e^{-u}\, \frac{1}{4} du$$

$$= \frac{1}{4}\int_0^\infty e^{-u} u^{\frac{5}{4}-1} du$$

$$= \frac{1}{4}\Gamma\left(\frac{5}{4}\right)$$

$$= \frac{1}{4}\,\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

Practice Examples:

1. Evaluate $\int_0^1 x^m (\log x)^n\, dx$
2. Evaluate $\int_0^\infty \frac{x^5}{5^x}\, dx$
3. Evaluate $\int_0^1 \frac{1}{\sqrt{x \log\left(\frac{1}{x}\right)}}\, dx$
4. Evaluate $\int_0^\infty 5^{-3x^2}\, dx$
5. Evaluate $\int_0^\infty 5^{-4x^2}\, dx$

Beta Functions:

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m>0, n>0$ is a function of m and n is called Beta function. It is denoted by $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m>0, n>0$

Properties of Beta function:

1. $B(m,n) = B(n,m)$
2. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$
3. $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$
4. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\,dx$
5. Relation between Beta and Gamma function: $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$
6. Duplication Formula or Legendre's formula:
   $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$

Examples:

1. Prove that $\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\,d\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

   Solution: By definition

   $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m>0, n>0$$

   Take

   $x = \sin^2\theta \quad \therefore dx = 2\sin\theta\cos\theta d\theta$

$$x : 0 \to 1 \Rightarrow \theta : 0 \to \frac{\pi}{2}$$

$$\therefore B(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} \; 2 \sin\theta \cos\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \; d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \; d\theta = \frac{1}{2} B(m,n)$$

Taking $2m - 1 = p \quad and \quad 2n - 1 = q$ we get

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad \textbf{Or} \quad \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

2. Prove that $n \; B(m+1, n) = n \; B(m, n+1)$

Solution:

$$n \; B(m+1, n) = n \times \frac{\Gamma(m+1) \; \Gamma(n)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma(m) \times n\Gamma(n)}{\Gamma(m+n+1)}$$

$$= m \times \frac{\Gamma(m) \; \Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= mB \; (m, n+1)$$

3. Evaluate $\int_0^m x^m (\mathrm{m} - \mathrm{x})^n \, dx$

Solution: take

$x = mu \quad \therefore dx = mdu$

$x : 0 \to 1 \Rightarrow u : 0 \to 1$

$$\therefore I = \int_0^m (mu)^m (m - mu)^n \, m \, du$$

$$= m^{m+n+1} \int_0^1 u^m \; (1-u)^n \, du$$

$$= m^{m+n+1} \int_0^1 u^{(m+1)-1} \; (1-u)^{(n+1)-1} \, du$$

$$= m^{m+n+1} B(m+1, n+1)$$

Practice Examples:

1. Evaluate $\int_0^1 x^5 (1-x^3)^{10} \; dx$ in terms of beta function.
2. Evaluate $\int_0^2 x^4 (8-x^3)^{-\frac{1}{3}} \; dx$
3. Evaluate $\int_0^\infty \frac{x^8 (1-x^6)}{(1+x)^{24}} \; dx$
4. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \; d\theta \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}} \; d\theta$
5. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 \theta (1+\cos\theta)^4 \; d\theta$ by using gamma function.

Reduction Formulae:

Useful Properties:

1. $\int_a^b f(x) \; dx = \int_a^b f(\text{t}) \; dt$
2. $\int_a^b f(x) \; dx = -\int_b^a f(\text{t}) \; dt$
3. For $a < c < b$, $\int_a^b f(x) \; dx = \int_a^c f(x) \; dx + \int_c^b f(x) \; dx$
4. $\int_0^a f(x) \; dx = \int_0^a f(a-x) \; dx$

5. $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2\mathrm{a}-x)\,dx$

6. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx; & \text{if } f(x) \text{ is an even function} \\ 0; & \text{if } f(x) \text{ is an odd function} \end{cases}$

Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ **and** $\int_0^{\frac{\pi}{2}} \cos^n x\,dx \quad (n \in N,\ n > 1)$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)(n-3)\ldots\ \ 2 \text{ or } 1}{n(n-2)\ldots\ \ 2 \text{ or } 1} \times K \ \text{ where } K = \begin{cases} \frac{\pi}{2}; & n \text{ is even} \\ 1; & n \text{ is odd} \end{cases}$$

Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^m x \cos^{\mathrm{n}} x\,dx \quad (m, n \in N, \quad m, n > 1)$

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^{\mathrm{n}} x\,dx = \frac{[(m-1)(m-3)\ldots\ \ 2 \text{ or } 1][(n-1)(n-3)\ldots\ \ 2 \text{ or } 1]}{(m+n)(m+n-2)\ldots\ \ 2 \text{ or } 1} \times K$$

$$\text{Where } K = \begin{cases} \frac{\pi}{2}; & m \text{ and } n \text{ both are even} \\ 1; & \text{otherwise} \end{cases}$$

Examples:

1. Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 \theta\,d\theta$

Solution: $I = \int_0^{\frac{\pi}{2}} \sin^6 \theta\,d\theta$

$$= \frac{5\times3\times1}{6\times4\times2}\times\frac{\pi}{2}$$
$$= \frac{5\pi}{32}$$

2. Evaluate $\int_0^{\frac{\pi}{8}} \cos^3 4\theta \, d\theta$

Solution: Let $4\theta = x \quad \therefore d\theta = \frac{1}{4}dx$

Also, $\theta : 0 \to \frac{\pi}{8} \Rightarrow x : 0 \to \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \cos^3 x \times \frac{1}{4}dx$$
$$= \frac{1}{4}\times\frac{2}{3\times1}\times1$$
$$= \frac{1}{6}$$

3. Evaluate $\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^5\theta d\theta$

Solution: let $I = \int_0^{\frac{\pi}{2}} \sin^7\theta\cos^5\theta d\theta$

$$= \frac{[6\times4\times2][4\times2]}{[12\times10\times8\times6\times4\times2]}\times1$$

$$= \frac{1}{120}$$

Practice Examples:

1. Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 6\theta\cos^6 3\theta \, d\theta$
2. Evaluate $\int_0^{\pi} \theta\sin^8\theta\cos^6\theta \, d\theta$

3. Evaluate $\int_0^{\pi} \sin^2\theta\,(1+\cos\theta)^4\; d\theta$

4. $\int_0^{\infty} \frac{x^2}{(1+x^2)^8}\, dx$

5. Evaluate $\int_0^{4} x^3\sqrt{4x-x^2}\; dx$

6. Evaluate $\int_0^{2a} x^3(2ax-x^2)^{\frac{3}{2}}\; dx$