PARTIAL DERIVATIVES

- **Limit of a Function of Two Variables**
  The function $f(x, y)$ has the *limit* L as $(x, y) \to (x_0, \ y_0)$ provided that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
  $$|f(x, y) - L| < \varepsilon \text{ when } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$
  We say that
  $$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \mathrm{L}.$$
- **Remark.** When we use the definition of a limit to show that a particular limit exists, we usually employ certain basic inequalities such as
  $$|x| \le \sqrt{x^2 + y^2}, \qquad |y| \le \sqrt{x^2 + y^2}, \qquad \frac{x}{x+1} < 1, \qquad \frac{x^2}{x^2+y^2} \le 1$$
  $$|x - a| = \sqrt{(x - a)^2} \le \sqrt{(x - a)^2 + (y - a)^2}$$

- **Continuity of a Function of Two Variables**
  A function $f(x, y)$ is said to be *continuous* at the point $(x_0, \ y_0)$ if
  1. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists;
  2. $f(x_0, \ y_0)$ is defined;
  3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, \ y_0)$

Example-1. Find the limit
$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2 + y^2} .$$

Solution. Observe that the point (1, 2) does not cause division by zero or other domain issues. So,
$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2 + y^2} = \frac{5(1)^2(2)}{(1)^2 + (2)^2} = 2.$$

Example-2. Show that
$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

is not continuous at the origin.

Solution. Let us apply different path approach. We check the limit along different paths $y = mx, x \neq 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2m}{1+m^2}.$$

This limit changes with m. Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence the function $f(x,y)$ is not continuous at the origin.

- **Partial Derivative**

  When the function involves two or more independent variables, like $u = f(x,y)$ or $u = f(x,y,z)$, then the derivative of u with respect to any one of the independent variables, treating all other variables as constant is referred as *partial derivative* of u with respect to that variable.

- **Mathematical Form**

  The partial derivative of $u = f(x,y)$ w. r. t. x at a point $(x_0,\ y_0)$ is denoted by $\frac{\partial f}{\partial x}(x_0,\ y_0)$ or $f_x(x_0,\ y_0)$ or $\frac{\partial u}{\partial x}(x_0,\ y_0)$ or $u_x(x_0,\ y_0)$ and is defined as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0,\ y_0)}{h}$$

  provided the limit exists.

  The partial derivative of $u = f(x,y)$ w. r. t. y at a point $(x_0,\ y_0)$ is denoted by $\frac{\partial f}{\partial y}(x_0,\ y_0)$ or $f_y(x_0,\ y_0)$ or $\frac{\partial u}{\partial y}(x_0,\ y_0)$ or $u_y(x_0,\ y_0)$ and is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0,\ y_0)}{h}$$

  provided the limit exists.

- **Higher order Partial Derivatives**

  For a function $f(x,y)$ the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are themselves are functions of x and y, so we can take partial derivatives of them as

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \qquad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

  Higher order partial derivatives (e.g. f_{xxy}) can also be calculated. Using the subscript notation, the order of differentiation is from left to right.

Example-1. Let $f(x,y) = 3x^2 + e^{-xy^2}$. Find f_x, f_y.

Solution. $f_x(x,y) = 6x - y^2e^{-xy^2}$ and $f_y(x,y) = -2xye^{-xy^2}$.

Example-2. Let $f(x,y) = ycos(xy)$. Find f_x, f_y.

Solution. $f_x(x,y) = -y^2 \sin(xy)$ and $f_y(x,y) = cos(xy) - xysin(xy)$.

Example-3. Let $f(x,y) = x^2 - 4xy^3$. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

Solution.

$$f_x(x,y) = 2x - 4y^3 \qquad f_y(x,y) = -12xy^2$$
$$f_{xx}(x,y) = 2 \quad f_{xy}(x,y) = f_{yx}(x,y) = -12y^2 \quad f_{yy}(x,y) = -24xy.$$

Example-4. If $z = x + y^x$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

Solution. Here

$$z = x + y^x$$

Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = 1 + y^x \log y.$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = xy^{x-1}.$$

Differentiating $\frac{\partial z}{\partial y}$ partially w. r. t. x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = y^{x-1} \cdot 1 + xy^{x-1} \log y = y^{x-1}(1 + x \log y) \dots \dots \dots \dots (1)$$

Differentiating $\frac{\partial z}{\partial x}$ partially w. r. t. y, we get

$$\frac{\partial^2 z}{\partial y \partial x} = y^x \cdot \frac{1}{y} + \log y \cdot xy^{x-1} = y^{x-1}(1 + x \log y) \dots \dots \dots \dots (2)$$

By (1) and (2),

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Example-5. If $u = \frac{x^2+y^2}{x+y}$, then prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Solution. Here,

$$u = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)2x - (x^2+y^2)1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)2y - (x^2+y^2)1}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now,

$$LHS = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left(\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right)^2$$

$$= \left(\frac{2(x^2-y^2)}{(x+y)^2}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2$$

$$RHS. = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 4\left(1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy - x^2}{(x+y)^2}\right) = 4\left(\frac{x-y}{x+y}\right)^2$$

Thus

$$LHS = RHS$$

Example-6. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Solution. Here

$$u = \log(x^3 + y^3 - x^2y - xy^2)$$

$$= \log(x^3 - x^2y + y^3 - xy^2)$$

$$= \log[x^2(x-y) - y^2(x-y)]$$

$$= \log[(x-y)(x^2-y^2)]$$

$$= \log[(x+y)(x-y)^2]$$

$$= \log(x+y) + 2\log(x-y).$$

Differentiating u w. r. t. x partially,

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}.$$

Differentiating $\frac{\partial u}{\partial x}$ w. r. t. x partially,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2} \ldots\ldots\ldots (1)$$

Differentiating u w. r. t. y partially,

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}.$$

Differentiating $\frac{\partial u}{\partial y}$ w. r. t. y partially,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2} \ldots\ldots\ldots (2)$$

Differentiating $\frac{\partial u}{\partial y}$ w. r. t. x partially,

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{(x+y)^2} + \frac{2}{(x-y)^2} \ldots\ldots\ldots (3)$$

By (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Example-7. If $x = r\cos\theta, y = r\sin\theta$, show that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$.

Solution. Here

$$x = r\cos\theta, y = r\sin\theta$$

$$\Rightarrow\ x^2 + y^2 = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating (1) w.r.t. x partially

$$2x = 2r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating (1) w.r.t. y partially

$$2y = 2r\frac{\partial r}{\partial y} \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

Hence

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1.$$