- **Implicit Function**

Theorem: Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Theorem: Suppose that $F(x, y, z)$ is differentiable and that the equation $F(x, y, z) = 0$ defines z is a differentiable function of x and y. Then at any point where $F_z \neq 0$, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example-1. Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Solution. Take $F(x, y) = y^2 - x^2 - \sin xy$

Here $F_x = \frac{\partial F}{\partial x} = -2x - ycosxy$

$F_y = \frac{\partial F}{\partial y} = 2y - xcosxy$

Therefore, $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x-ycosxy}{2y-xcosxy}$

$= \frac{2x+ycosxy}{2y-xcosxy}$

Example-2. Find $\frac{dy}{dx}$ when $(cosx)^y = (siny)^x$.

Solution. Here $(cosx)^y = (siny)^x$

Taking logarithm function on both sides, we get

$ylog(cosx) = xlog(siny)$

Let $F(x, y) = ylog(cosx) - xlog(siny)$

$F_x = \frac{\partial F}{\partial x} = y\frac{1}{cosx}(-sinx) - \log(siny)$

$= -ytanx - \log(siny)$

$F_y = \frac{\partial F}{\partial y} = log(cosx) - x\frac{1}{siny}(cosy)$

$= log(cosx) - xcoty$

Therefore, $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-ytanx - \log(siny)}{log(cosx) - xcoty}$

$$= \frac{ytanx + \log(siny)}{log(cosx) - xcoty}$$

Example-3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xyz = \cos(x + y + z)$.

Solution. Take $F(x, y, z) = xyz - \cos(x + y + z)$

Here $F_x = \frac{\partial F}{\partial x} = yz + \sin(x + y + z) \cdot 1$

$$F_y = \frac{\partial F}{\partial y} = xz + \sin(x + y + z) \cdot 1$$

$$F_z = \frac{\partial F}{\partial z} = xy + \sin(x + y + z) \cdot 1$$

Therefore, $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

- **Jacobians**

Definition: The Jacobian of the transformation $x = g(u, v),\ y = h(u, v)$ is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Similarly, the Jacobian of the transformation $x = f(u, v, w),\ y = g(u, v, w),\ z = h(u, v, w)$ is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- **Properties of Jacobians:**

1. If $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$ then $JJ' = 1$.
2. If x, y are the function of r, s where r, s are function of u, v then $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(u,v)}$.

Example-1. Find the Jacobian for the transformation $x = rcos\theta, y = rsin\theta$.

Solution. $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= r(cos^2\theta + sin^2\theta)$$

$$= r$$

Example-2. If $u = x + 3y^2 - z^3,\ v = 4x^2yz,\ w = 2z^2 - xy$, evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1,0).

Solution. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$

$$\left.\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|_{(1,-1,0)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20$$

Example-3. If $u = x^2 - y^2, v = 2xy$ and $x = rcos\theta, y = rsin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Solution. We have $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

Since $u = x^2 - y^2, v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

Also $x = rcos\theta, y = rsin\theta,$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= r(cos^2\theta + sin^2\theta)$$

$= r$

Hence $\frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2 + y^2) \cdot r$

$= 4(r^2 cos^2\theta + r^2 sin^2\theta) \cdot r$

$= 4r^3$