Curve Tracing

Generally, a curve is drawn by plotting a number of points and joining them by a smooth line.

If an approximate shape of the curve is sufficient for a given purpose then it is enough to study certain important characteristics. This purpose is served by curve tracing methods.

The points to be observed for tracing of plane algebraic curves are given below.

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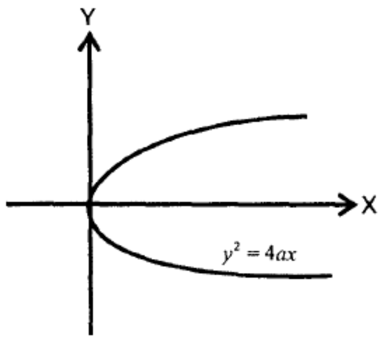
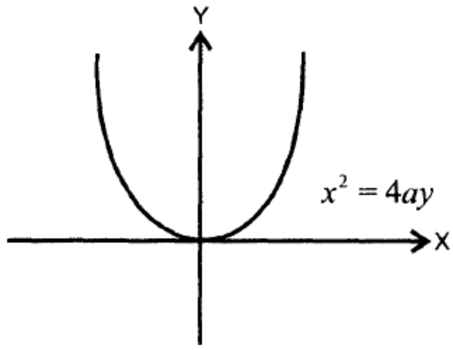
Symmetry

Whether the curve is symmetric about an axis or about other any line . if

1) $F(x,y) = F(x,-y) \Rightarrow curve\ is\ symmetric\ about\ X - axis.$
2) $F(-x,y) = F(x,y) \Rightarrow curve\ is\ symmetric\ about\ Y - axis.$
3) $F(-x,-y) = F(x,y) \Rightarrow curve\ is\ symmetric\ in\ opposite\ quadrants.$
4) $F(y,x) = F(x,y) \Rightarrow curve\ is\ symmetric\ about\ Y = X$
5) $F(-y,-x) = F(x,y) \Rightarrow curve\ is\ symmetric\ about\ Y = -X$



(i)



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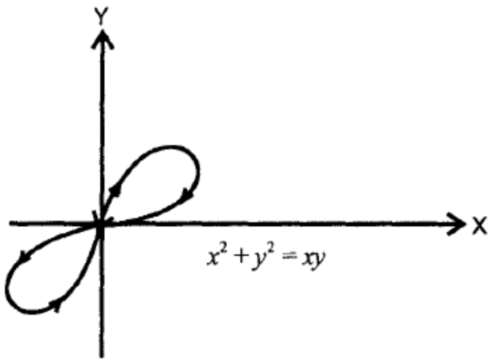

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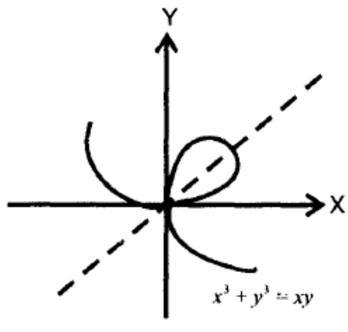

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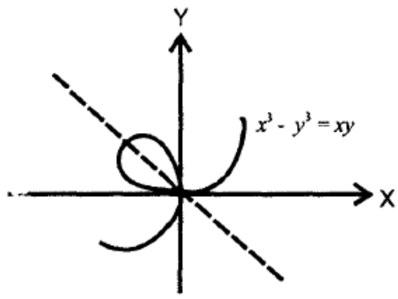


(v)

Origin

Whether the curve is passing through the origin, if so the equations of the tangents to the curve at the origin.

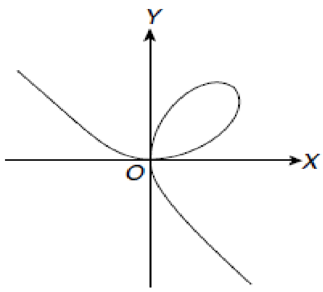
Suppose $F(x, y) = 0$ is the algebraic form of the equation of the curve.

$F(0,0) = 0 \Longrightarrow$The curve is passing through origin(i.e) . If there is no constant term in$F(x, y)$ then origin lies on the curve.

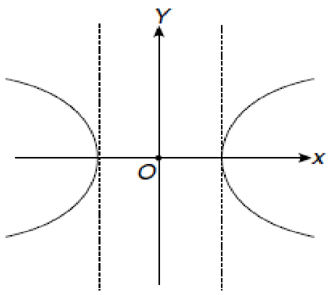
The equation of the tangents to the curve are obtained by equation the lowest degree terms in $F(x, y)$ to zero.

If at $O(0,0)$ the tangent s are:

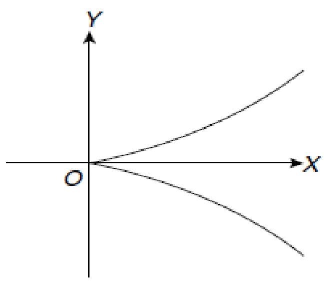
1) Real and coincident then “O” is called **cusp.**
2) Real and different then “O” is called **node.**
3) Imaginary then “O” is called a **conjugate point.**
   a) $y^2 = 4ax$ (Equating to zero, the lowest degree term 4ax we get x=0 (Y-axis) as the tangent at the origin.)
   b) $x^3 + y^3 = 3axy$ (Equating to zero, the lowest degree term x y = 0 we get x = 0, y = 0 as tangents at the origin.)
   c) $x^4 = a^2(x^2 - y^2)$(Equating to zero, the lowest degree term$x^2 = y^2, i.e. y = \pm x$, as the tangent at the origin.)



Node at origin



Conjugate point at origin



Cusp at origin

Point of intersection with coordinate axes:

Find the points of intersection of the curve with the axes. At these points, find equation of tangents either by shifting the origin to the point and equating lowest degree term equal to zero or by finding slope ,i.e, $\frac{dy}{dx}$, at these points.

Region in which the curve lies:

We can determine the region in which the curve liesby solving the equation f (x, y) = 0 for y and knowingthe domain of variation of x and similarly solving theequation for x andknowing the domain of variationof y.If y (or x) is imaginary for any range of values of x(or y) it means that the curve does not lie in that range.

Asymptotes

Finding the asymptotes.

An asymptote is a line that is at a finite distance from (0,0) and is tangential to the curve at infinity (i.e..) the curve approaches the line at infinity.

1) Sum of the coefficients of the highest degree terms in x equated to zero gives the equations of the asymptotes parallel to X- axis.
2) Sum of the coefficients of the highest degree terms in y equated to zero gives the equations of the asymptotes parallel to X- axis.
3) To find the asymptotes that are neither parallel to X-axis nor parallel to Y- axis (i.e.) oblique asymptotes, the following method is suggested.

Substitute $y = mx + c$ in $F(x, y) = 0$ and rewrite the equation as a polynomial equation in "x" as

$$\emptyset_n(m)x^n + \emptyset_{n-1}(m)x^{n-1} + \cdots + \emptyset_n = 0$$

The slopes of the asymptotes are given by $\emptyset_n(m) = 0$. let the slopes be $m_1, m_2, \dots,$

The values of "c" can be obtained from $\emptyset_{n-1}(m) = 0, \emptyset_{n-1}(m) = 0$ (if necessary).

Let the corresponding values of c be $c_1, c_2, \dots,$

Then the asymptotes are $y = m_1x + c_1, y = m_2x + c_2, \dots,$

Note : if$\emptyset_n(m)$ is a constant then there are no oblique asymptotes to the curve.

Increasing (Rising) and Decreasing (Falling) Curves

Calculate$\frac{dy}{dx} from\ f(x, y) = 0$. Then if

1) $\frac{dy}{dx} > 0\ \forall x \in [a, b]$ then the curve is increasing (rising) in$[a, b]$.
2) $\frac{dy}{dx} < 0\ \forall x \in [a, b]$ then the curve is decreasing (falling) in$[a, b]$: and
3) $\frac{dy}{dx} = 0$ at any point P(x_0, y_0), then it is neitherincreasing(rising)nor decreasing(falling): P(x_0, y_0)is a stationarypoint at the top-most or bottom-most point ofthe curve.

Concavity or Convexity

Calculate $\frac{d^2y}{dx^2} from\ f(x, y) = 0$. Then if

1) $\frac{d^2y}{dx^2} > 0$ then the curve is concave upwards (Cup holding coffee)
2) $\frac{d^2y}{dx^2} < 0$ then the curve is concave downwards (cap on the head): and
3) $\frac{d^2y}{dx^2} = 0$ at any point P (x_0, y_0), then P is called an inflexion point where the curve changes direction of concavity/convexity from downward to upward or vice versa.

Summary of Steps:-

POSTAR:-Point of intersection **O**rigin **S**ymmetry **T**angent **A**symptote **R**egion

Example 1

Trace the curve $\boldsymbol{y^2(2a-x)=x^3}$ **(Cissoid of Diocles).**

Solution:

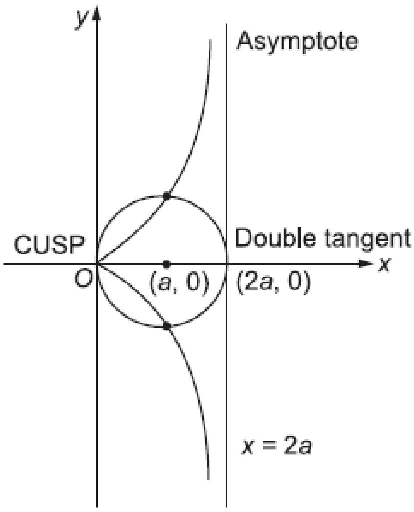
(1) Symmetry Given equation contains even powers of y.So, the curve is symmetric about the x-axis

(2) Passing through the origin Given equationis satisfiedby $x = 0$; $y = 0$. So, the curve passes throughthe origin.

(3) Tangents at the origin Equating to zero thelowest degree terms,$2ay^2 = 0$ we get$y = 0, y = 00$(double root). The two tangents to the curve at the origin are realand coincident so that the origin is a **cusp.**

(4) Region y^2 is negative for negativevalues of x.Thus, y is imaginary for $x < 0$; so, no part of thecurve lies to the left of the $Y - axis$.Again, y is imaginary for $x > 2a$; so, no part of thecurve lies to the right of the straight line $x = 2a$.

(5) Asymptotes Asymptotes parallel to the $Y - axis$are obtained by equating to zero the coefficient of thehighest power of y. Thus$x = 2a$ is an asymptote, andthere is no other asymptote to the curve.



$y^2\,(2a - x) = x^3$

(Cissoid of Diocles)

Example 2

Trace the curve$ay^2 = x^3$**(Semi-cubical parabola).**

Solution

(1) Symmetry The equation contains only evenpowers of y, so the curve is symmetric about thex-axis.

(2) Origin The curve passes through the originsince (0, 0) satisfies the equation.

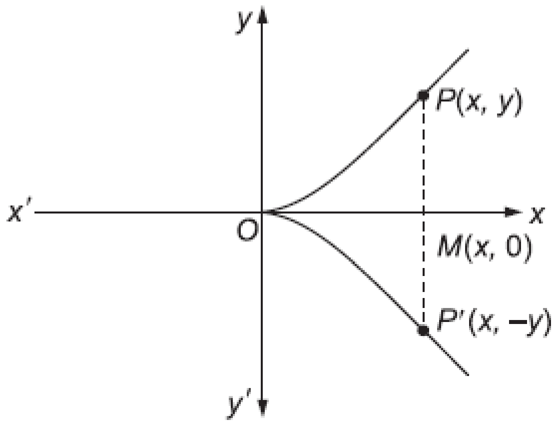
(3) Tangents at (0, 0**)** Equating the lowest degreeterms we get $y^2 = 0 \Rightarrow y = 0, y = 0$. Thus the$X - axis$ is a tangent to the two parts of the curve at theorigin.∴The origin is a **cusp**.

(4) Intersection points with the coordinate axes

$x = 0 \Leftrightarrow y = 0$. So, the curve does not cut thecoordinate axes at any point other than the origin.

(5) Region in which the curve lies $If\ \ x < 0, then\ y$is imaginary. So, no part of the curve lies to the leftof the $Y - axis$.As $x \to \infty, y \to \infty$. The curve extends from theorigin to infinity in the I quadrant and because it issymmetric about the $X - axis$ the reflection of the curve in the $x - axis$ lies in the IV quadrant.

(6) Asymptotes Since the coefficients of the highestand the next highest degree terms (3rd and 2nd degree terms) are 1 and 0, respectively, there are noasymptotes to the curve.The curve is called a semi-cubical parabola whosegraph is shown in Fig



$ay^2 = x^3$

(Semi-cubical parabola)

Example 3

Trace the curve$y^2 = x^2 \frac{a+x}{a-x}$**(Strophoid).**

Solution

(1) Symmetry Given equation contains only even powersof y. So, the curve is symmetrical about the $X - axis$.

(2) Origin The curve passes through the originsince $x = 0, y = 0$ satisfy given equation.

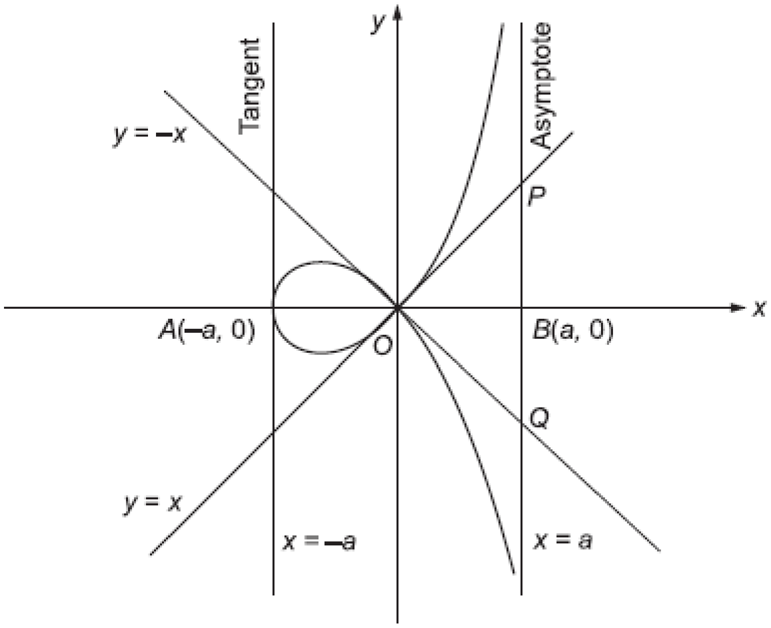
(3) Tangents at the origin Equating to zero thelowest degree terms, we get $y = \pm x$ as the tangentsat the origin. These are real and different. So, theorigin is a node.

(4) Intersection with the coordinate axes $y = 0\ when\ x = 0\ or\ -\ a$. The curve cuts the $x - axis\ at\ x = -a\ and\ x = 0$.

(5) Region Given equation can be written as$y = \pm x\sqrt{\frac{a+x}{a-x}}$When $x < -a\ or\ x > a, y$ is imaginary. So no partof the curve lies outside the lines$x = -a, x = a$.As x increases from $-a\ to\ \frac{-a}{2}$, y decreases from0 to – $\frac{a}{2}$and as $x\ increases\ from -\frac{a}{2} to\ 0, y\ increases\ from\ \frac{-a}{2}\ \ to\ 0$.

(6) Asymptotes Asymptotes parallel to the $y - axis$are obtained by equating to zero thecoefficient of thehighest power of y; thus $x = a$ is an asymptote.The coefficient of the highest power of $x\ is$ 1. So,there is no asymptote parallel to the $X - axis$.

The graph for the curve is given in Fig.



$$y^2 = x^2 \cdot \left(\frac{a+x}{a-x}\right)$$

(Strophoid)

Exercise:

Trace the following curves:

1) $x^2y^2 = a^2(y^2 - x^2)$

2) $x^3 + y^3 = 3axy$ (Folium of Descartes)

3) $xy^2 = 4a^2(2a - x)$ (witch of Agnesior versiera)

4) $xy^2 = a^2(x - a)$

5) $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1 \quad or$

$$x = acos^3t, \quad y = bsin^3t \quad (Hypocycloid)$$

6) $y^2(a - x) = (a + x)$

7) $y = \frac{x^2+1}{x^2-1}$