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★ Tutorial-4

★ Le-2 Partial Derivatives:-

* Example-1:-

Ex 1 Find limits:-

(a) $\lim_{(x,y)\to(0,1)} \dfrac{x - xy + 3}{x^2y - y^3 + 5xy}$

Putting $x=0$ & $y=1$ we get,

$\Rightarrow \dfrac{(0)-(0)(1)+3}{(0)^2(1) - (1)^3 + 5(0)(1)}$

$\Rightarrow \dfrac{+3}{-1} = \boxed{-3}$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ Having zero in denominator,

By rationalizing we get

$\lim_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \times \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

$\lim_{(x,y)\to(0,0)} \dfrac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$

$\lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y})$

$= 0(\sqrt{0}+\sqrt{0})$

$= \boxed{0}$

(c) $\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y+1}{y-\sin x}$

Putting $x=\pi/2$ & $y=0$ we get,

$\Rightarrow \frac{\cos(0)+1}{0-\sin(\pi/2)} \Rightarrow \frac{1+1}{0-1} = \boxed{-2}$

(d) $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$

$\lim_{(x,y)\to(0,0)} e^y \times \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$

$\lim_{(x,y)\to(0,0)} e^y \times 1$

$\Rightarrow e^{(0)} \times 1 = \boxed{1}$

(e) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

Again by rationalizing we get,

$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} \times \frac{x^2-y^2}{x^2-y^2}$

$\lim_{(x,y)\to(0,0)} \frac{\cancel{(x^4-y^4)}(x^2-y^2)}{\cancel{(x^4-y^4)}}$

$\Rightarrow (0)^2-(0)^2 = \boxed{0}$

Ex 2. By considering diff. paths of approach, show that the function in below e.g. have no limit as $(x,y) \to (0,0)$.

(a) $f(x,y) = \dfrac{xy}{|xy|}$

- along x-axis $\lim_{(x,0) \to (0,0)} \dfrac{xy}{|xy|} = 0$

- along y-axes $\lim_{(0,y) \to (0,0)} \dfrac{xy}{|xy|} = 0$

- Apply diff. path approach $y = mx$

$$\lim_{(x,y) \xrightarrow{mx} (0,0)} \frac{x(mx)}{|x(mx)|}$$

$$\lim_{(x,y) \xrightarrow{mx} (0,0)} \frac{mx^2}{|m|x^2} \Rightarrow \lim_{(x,y) \xrightarrow{mx} (0,0)} \frac{m}{|m|}$$

where $|m| \Rightarrow \begin{cases} 1, & m > 0 \\ -1, & m < 0 \end{cases}$

Thus, limit changes with m.
$\therefore \lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

Hence not continuous.

(b) $g(x,y) = \dfrac{x-y}{x+y}$

- along x-axis

$$\lim_{(x,0)\to(0,0)} \frac{x-y}{x+y} = -1$$

- along y-axis

$$\lim_{(0,y)\to(0,0)} \frac{x-y}{x+y} = 1$$

- Applying diff path approach as $y=mx$

$$\lim_{(x,y)\to(0,0)}^{mx} \frac{x-mx}{x+mx} \Rightarrow \lim_{(x,y)\to(0,0)}^{mx} \frac{x(1-m)}{x(1+m)}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)}^{mx} \frac{1-m}{1+m}$$

Thus, limit changes and depends on m.
$\lim g(x,y)\to(0,0)$ does not exit.
Hence, not continuous.

(c) $f(x,y) = \dfrac{x^4-y^2}{x^4+y^2}$

- along x-axis

$$\lim_{(x,0)\to(0,0)} \frac{x^4-y^2}{x^4+y^2} = -1$$

- along y-axis

$$\lim_{(0,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2} = 1$$

- Applying diff. path approach at $y=mx^2$

$$\lim_{(x,y)\to(0,0)}^{mx^2} \frac{x^4-(mx^2)^2}{x^4+(mx^2)^2} \Rightarrow \lim_{(x,y)\to(0,0)}^{mx^2} \frac{x^4-m^2x^4}{x^4+m^2x^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{1-m^2}{1+m^2}$$

Thus, limit changes & depends on m.
$\lim f(x,y)\to(0,0)$ does not exist
Hence not continuous

(d) $h(x,y) = \dfrac{2xy}{3x^2+y^2}$

- along x-axis
  $\lim_{(x,0)\to(0,0)} \dfrac{2x \cdot y}{3x^2+y^2} = 0$

- along y-axis
  $\lim_{(0,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2} = 0$

- Applying different approach at $y=mx$ we get,

$$\lim_{(x,y)\to(0,mx)} \frac{2x(mx)}{3x^2+(mx)^2}$$

$$\lim_{(x,y)\to(0,mx)} \frac{2x^2m}{3x^2+m^2x^2}$$

$$\lim_{(x,y)\to(0,mx)} \frac{2m}{3+m^2}$$

Thus, it changes & depends on m.
$\lim h(x,y)\to(0,0)$ does not exist.
Hence, not continuous

Ex3 Check whether functions are continuous or not at (0,0).

(a) $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 1 & ; (x,y) \neq (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{\sin x^2+y^2}{x^2+y^2} = 1$$

As both limits are same Thus it is continuous.

(b) $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) \neq (0,0) \end{cases}$

Applying $y=mx$ we get,

$$\lim_{(x,y)\to(0,mx)} \frac{x^2-(mx)^2}{x^2+(mx)^2} \Rightarrow \lim_{(x,y)\to(0,mx)} \frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$\lim_{(x,y)\to(0,mx)} \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)}$$ $\therefore$ It depend on m limit does not exist

Thus, $f(x,y)$ is not continuous at $(0,0)$ because the value $f(0,0)$ is not equal to the limit, with limit does not exist.

(c) $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^3+y^3} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Applying $y=mx$ we get,

$$\lim_{(x,y)\to(0,mx)} \frac{2x^2(mx)}{x^3+(mx)^3} \Rightarrow \lim_{(x,y)\to(0,mx)} \frac{2x^3 \times m}{x^3+m^3x^3}$$

$$\lim_{(x,y)\to(0,mx)} \frac{2\cancel{x^3}\times m}{\cancel{x^3}(1+m^3)} \Rightarrow \lim_{(x,y)\to(0,mx)} \frac{2m}{1+m^3}$$

$$\Rightarrow \frac{2m}{1+m^3}$$ $\therefore$ It depend on m limit does not exit

Thus, $f(x,y)$ is not continuous at $(0,0)$ because value $f(0,0)$ is not equal to limit, with limit does not exist.

* Example - 2

Ex1. Find all first & second order partial derivatives. Hence, verify Mixed Derivative Theorem (Clairaut's Theorem).

(a) $f(x,y) = \ln(2x+3y)$

$f_x = \dfrac{1}{2x+3y} \times 2$ $\qquad f_{xx} = 2(2x+3y)^{-1} \Rightarrow 2[(-1)(2x+3y)^{-2}(2)]$

$\Rightarrow -4/(2x+3y)^2$

$f_y = \dfrac{1}{2x+3y}(3)$ $\qquad f_{yy} = 3(2x+3y)^{-1}$

$\Rightarrow 3[(-1)(2x+3y)^{-2}(3)]$

$\Rightarrow \dfrac{-9}{(2x+3y)^2}$

$f_{xy} = 2(2x+3y)^{-1} \Rightarrow 2[(-1)(2x+3y)^{-2} \times 3]$

$\Rightarrow \dfrac{-6}{(2x+3y)^2}$

$f_{yx} = 3(2x+3y)^{-1} \Rightarrow 3[(-1)(2x+3y)^{-2} \times 2]$

$\Rightarrow \dfrac{-6}{(2x+3y)^2}$

$\therefore$ Mixed Derivative / Clairaut's Theorem :-

$f_{xy} = f_{yx}$

$$\boxed{\dfrac{-6}{(2x+3y)^2} = \dfrac{-6}{(2x+3y)^2}}$$

Hence proved.

(b) $f(x,y) = xy + \frac{e^y}{y^2+1}$

$f_x = y + 0 = y$

$f_y = x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2} \Rightarrow x + \frac{e^y y^2 + e^y - 2ye^y}{(y^2+1)^2}$

$f_{xx} = 0$

$f_{yy} = 0 + \frac{(y^2+1)^2[e^y y^2 + e^y(2y) + e^y - 2ye^y - 2e^y]}{(y^2+1)^4}$

$f_{xy} = 1$

$f_{yx} = 1$

$\therefore$ Mixed Derivative / Clairaut's Theorem:-

$f_{xy} = f_{yx}$

$\boxed{1 = 1}$

Hence proved.

(c) $f(x,y) = \tan^{-1}(y/x)$

$f_x = \frac{1}{1+(y/x)^2} \times \frac{-y}{x^2} \Rightarrow \frac{-y}{x^2+y^2}$

$f_y = \frac{1}{1+(y/x)^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$

$f_{xx} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} \Rightarrow \frac{2xy}{(x^2+y^2)^2}$

$f_{yy} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} \Rightarrow \frac{-2xy}{(x^2+y^2)^2}$

$f_{xy} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} \Rightarrow \frac{-x^2+y^2}{(x^2+y^2)^2}$

$f_{yx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)} = \frac{x^2+y^2-2x^2}{x^2+y^2}$

$= \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\therefore$ Mixed Derivative / Clairaut's Theorem:-

$f_{xy} = f_{yx}$

$$\boxed{\frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}}$$

Hence proved

(d) $g(x,y) = x^2y + \cos y + y\sin x$

$f_x = 2xy + 0 + y\cos x$

$f_y = x^2 + (-\sin y) + \sin x$

$f_{xx} = 2y + y(-\sin x) \Rightarrow 2y - y\sin x$

$f_{yy} = 2x - \cos y$

$f_{xy} = 2x + \cos x$

$f_{yx} = 2x + \cos x$

$\therefore$ Mixed Derivative / Clairaut's Theorem:-

$f_{xy} = f_{yx}$

$$\boxed{2x + \cos x = 2x + \cos x}$$

Hence proved.

Ex2. Evaluate the partial derivatives.

(a) $z = \sqrt{x^2+4y^2}$

$= (x^2+4y^2)^{1/2}$

$z_x = \frac{1}{\cancel{2}}(x^2+4y^2)^{-1/2}(\cancel{2}x)$

$z_x(1,2) = \frac{x}{\sqrt{x^2+4y^2}} = \frac{1}{\sqrt{1+16}} = \boxed{\frac{1}{\sqrt{17}}}$

$z_y = \frac{1}{\cancel{2}}(x^2+4y^2)^{-1/2}(\cancel{8}^{4}y)$

$z_y(1,2) = \frac{4(2)}{\sqrt{1+16}} = \boxed{\frac{8}{\sqrt{17}}}$

(b) $w = x^2 y \cos z$ $(2,1,0)$

$w_x = 2xy\cos z$

$w_y = x^2\cos z$

$w_z = x^2 y(-\sin z)$

$w_x(2,1,0) = 2(2)(1)\cos(0) \Rightarrow 2(2)(1)(1) = \boxed{4}$

$w_y(2,1,0) = (2)^2\cos(0) = (2)^2(1) = \boxed{4}$

$w_z(2,1,0) = (2)^2(1)(-\sin(0)) = (2)^2(1)(0) = \boxed{0}$

Ex3 Find indicated higher order partial derivatives.

(a) $f(x,t) = x^2 e^{-ct}$ $\quad f_{ttt} = ?$ $\quad f_{txx} = ?$

$f_t = x^2 e^{-ct}(-c)$

$f_{tt} = -c[x^2 e^{-ct}(-c)] = +c^2 x^2 e^{-ct}$

$f_{ttt} = +c[x^2 e^{-ct}(-c)] = -c^3 x^2 e^{-ct}$

$f_{tx} = -c[2x e^{-ct}] = -2cx e^{-ct}$

$f_{txx} = -2c[e^{-ct}] = -2ce^{-ct}$

$$\boxed{\begin{aligned} f_{ttt} &= -c^3 x^2 e^{-ct} \\ f_{txx} &= -2ce^{-ct} \end{aligned}}$$

(b) $f(x,y,z) = \cos(4x+3y+2z)$ $\quad f_{xyz} = ?$ $\quad f_{yzz} = ?$

$f_x = -\sin(4x+3y+2z) \times 4$

$f_{xy} = -\cos(4x+3y+2z) \times 12$

$f_{xyz} = +\sin(4x+3y+2z) \times 24$

$f_y = -\sin(4x+3y+2z) \times 3$

$f_{yz} = -\cos(x+3y+2z) \times 6$

$f_{yzz} = \sin(x+3y+2z) \times 12$

$$\boxed{\begin{aligned} f_{xyz} &= 24\sin(4x+3y+2z) \\ f_{yzz} &= 12\sin(x+3y+2z) \end{aligned}}$$

(c) $f(x,y,z) = 1 - 2xy^2z + x^2yz + 3z \qquad f_{yxyz} = ?$

$f_y = 0 - \cancel{2}4xyz + x^2z + 0$
$f_{yx} = -4yz + 2xz$
$f_{yxy} = -4z + 20$
$f_{yxyz} = \underline{\underline{-4}}$

$$\boxed{f_{yxyz} = -4}$$

Ex4 Use limit definition of partial derivation to compute at specified points.

(a) $f(x,y) = 1 - x + y - 3x^2y \qquad$ at $(1,2)$

$\frac{\partial f}{\partial x} = 0 - 1 + 0 - 6xy = -1 - 6xy$

$\frac{\partial f}{\partial x}$ at $(1,2) = -1 - 6(1)(2) = -1 - 12 = \boxed{-13}$

$\frac{\partial f}{\partial y} = 0 - 0 + 1 - 3x^2 = +1 - 3x^2$

$\frac{\partial f}{\partial y}$ at $(1,2) = 1 - 3(1)^2 = 1 - 3 = \boxed{-2}$

(b) $w(x,y,z) = x^2yz^2 \qquad$ at $(1,2,3)$
$\frac{\partial w}{\partial z} = x^2y \times 2z = 2x^2yz$

$\frac{\partial w}{\partial z}$ at $(1,2,3) = 2(1)^2(2)(3) = 4 \times 3 = \boxed{12}$

Ex5 Show each statisfies the Laplace eq.

(a) $u = \ln\sqrt{x^2+y^2}$

$u = \frac{1}{2}\log(x^2+y^2)$

$u_x = \frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2} \times \cancel{2}x\right]$

$= \frac{x}{x^2+y^2}$

$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$

$= \frac{-x^2+y^2}{(x^2+y^2)^2}$

$u_y = \frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2} \times \cancel{2}y\right]$

$= \frac{y}{x^2+y^2}$

$u_y = \frac{(x^2+y^2)(1) - y(2y)}{x^2+y^2}$

$= \frac{x^2-y^2}{x^2+y^2}$

Laplace eq:-

$u_{xx} + u_{yy} = 0$

$\frac{-\cancel{x^2}+\cancel{y^2}+\cancel{x^2}-\cancel{y^2}}{x^2+y^2} = \boxed{0}$ Hence proved

(b) $u = e^{-2y} \cos 2x$

$u_x = e^{-2y} \times -\sin 2x \times 2$

$= -2e^{-2y} \times \sin 2x$

$u_{xx} = -2e^{-2y} \times \cos 2x \times 2$

$= -4e^{-2y} \times \cos 2x$

$u_y = \cos 2x \, e^{-2y} \times (-2)$

$u_{yy} = \cos 2x \, e^{-2y} \times 4$

Laplace eq :-

$u_{xx} + u_{yy} = 0$

$-4e^{-2y} \times \cancel{\cos 2x} + 4\cancel{\cos 2x}\, e^{-2y}$

$= \boxed{0}$ Hence proved.

(c) $u = (x^2+y^2+z^2)^{-1/2}$

$u_x = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}$

$u_{xx} = -\left[\dfrac{(x^2+y^2+z^2)^{3/2}(1) - (x)\left(\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2} \times \cancel{2}x\right)}{\left[(x^2+y^2+z^2)^{3/\cancel{2}}\right]^{\cancel{2}}}\right]$

$= -\left[(x^2+y^2+z^2)^{1/2}\left\{\dfrac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^3}\right\}\right]$

$= -\left[\dfrac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\right]$

$$u_y = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$u_{yy} = -\left[\frac{(x^2+y^2+z^2)^{3/2}(1) - (y)\left(\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2} \times \cancel{2}y\right)}{\left[(x^2+y^2+z^2)^{3/2}\right]^2}\right]$$

$$= -\left[(x^2+y^2+z^2)^{1/2}\left\{\frac{x^2+y^2+z^2-3y^2}{(x^2+y^2+z^2)^3}\right\}\right]$$

$$= -\left[\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$u_z = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$u_{zz} = -\left[\frac{(x^2+y^2+z^2)^{3/2}(1) - (z)\left(3/2\,(x^2+y^2+z^2)^{1/2} \times 2z\right)}{\left[(x^2+y^2+z^2)^{3/2}\right]^2}\right]$$

$$= -\left[(x^2+y^2+z^2)^{1/2}\left\{\frac{x^2+y^2+z^2-3z^2}{(x^2+y^2+z^2)^3}\right\}\right]$$

$$= -\left[\frac{+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$\therefore$ Laplace eq:-

$$u_{xx}+u_{yy}+u_{zz}=0$$

$$\Rightarrow \frac{2x^2-y^2-z^2-x^2+2y^2-z^2-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow \frac{2\cancel{x^2}-2\cancel{x^2}-\cancel{2y^2}+\cancel{2y^2}-\cancel{2z^2}+\cancel{2z^2}}{(x^2+y^2+z^2)^{5/2}} = \boxed{0}$$

Hence proved.

(d) $u = e^{-x}\cos y - e^{-y}\cos x$

$u_x = e^{-x}\cos y\,(-1) + e^{-y}\sin x$
$= -e^{-x}\cos y + e^{-y}\sin x$

$u_{xx} = -e^{-x}\cos y(-1) + e^{-y}\cos x$
$= e^{-x}\cos y + e^{-y}\cos x$

$u_y = -e^{-x}\sin y + e^{-y}\cos x$

$u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$

$\therefore$ Laplace eq:-

$\therefore \Rightarrow +e^{-x}\cancel{\cos y} + e^{-y}\cancel{\cos x} - e^{-x}\cancel{\cos y} - e^{-y}\cancel{\cos x}$

$= \boxed{0}$ Hence proved

Ex6 $\Rightarrow$ $R_1 = 30 \quad R_2 = 45 \quad R_3 = 90$

Eq. $\Rightarrow \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

By Solving we get,

$$R = \frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}$$

Thus, by differentiating with respect to R_2 we get,

$$\frac{dR}{\partial R_2} = \frac{(R_2R_3+R_1R_3+R_1R_2)(R_1R_3)-(R_1R_2R_3)[R_3+R_1]}{(R_2R_3+R_1R_3+R_1R_2)^2}$$

By putting values of R_1, R_2, R_3 we get,

$$= \frac{[4050+2700+1350][2700]-[121500][120]}{8100 \times 8100}$$

$$= \frac{21870000 - 14580000}{8100 \times 8100}$$

$$= \frac{729\cancel{0}\cancel{0}\cancel{0}\cancel{0}}{81\cancel{0}\cancel{0} \times 81\cancel{0}\cancel{0}} = \frac{\overset{81}{\cancel{729}}}{\underset{9}{\cancel{81}} \times \cancel{81}} = \frac{1}{9}$$

Thus Ans:- $\boxed{\frac{\partial R}{\partial R_2} = \frac{1}{9}}$

Ex 7 Find partial derivatives with constrained variables:

(a) $x = r\cos\theta \quad y = r\sin\theta$

(i) $\frac{dr}{\partial x} = \frac{\partial x}{\partial r}, \quad r\frac{\partial\theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial\theta}$

$r^2 = x^2+y^2 \Rightarrow \sqrt{x^2+y^2} \Rightarrow r$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \times \cancel{2}x \Rightarrow \frac{x}{\sqrt{x^2+y^2}} \Rightarrow \frac{\cancel{r}\cos\theta}{\cancel{r}} \Rightarrow \cos\theta$$

$\frac{\partial x}{\partial r} = \cos\theta$ Thus L.H.S = R.H.S
Hence proved.

$$r\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial \theta}$$

$$\frac{y}{x} = \frac{\cancel{r}\cos\theta}{\cancel{r}\sin\theta} = \tan\theta$$

$\theta = \tan^{-1}(y/x)$

By differentiating with respect to x we get,

$$\frac{\partial \theta}{\partial x} = \frac{1}{(1 + y^2/x^2)} \times \left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \frac{-y}{x^2+y^2} \Rightarrow \frac{-\cancel{r}\sin\theta}{r^{\cancel{2}}} = \frac{-\sin\theta}{r} \quad (1)$$

$$\frac{1}{r} \times \frac{\partial x}{\partial \theta} \Rightarrow \frac{1}{\cancel{r}}(-\cancel{r}\sin\theta) \Rightarrow -\sin\theta \quad (2)$$

By putting eq. (1) in L.H.S

$$r \times \frac{\partial \theta}{\partial x} \Rightarrow \cancel{r} \times \frac{-\sin\theta}{\cancel{r}} = -\sin\theta \quad (3)$$

Thus eq. (2) & (3) are same,

$$\boxed{\text{Hence L.H.S = R.H.S} \quad \text{Hence proved.}}$$

(ii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

$\theta = \tan^{-1}(y/x)$

By differentiating w.r.t x & then w.r.t to y we get,

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 \theta}{\partial y x^2} = \frac{+2xy}{x^2+y^2} \quad \text{[By differentiating again w.r.t } x\text{]} \quad (1)$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{(x^2+y^2)(0) - x(2y)}{x^2+y^2}$$

$$= -\frac{2xy}{x^2+y^2} \quad \text{—— (2)}$$

Adding eq (1) & (2) we get,

$$\boxed{\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = +\frac{2\cancel{xy}}{\cancel{x^2+y^2}} - \frac{2\cancel{xy}}{\cancel{x^2+y^2}} \Rightarrow 0}$$

Hence proved.

(iii) $$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial x}{\partial y}\right)^2\right]$$

Differentiating w.r.t x

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

Again, Differentiating it we get,

$$\frac{\partial^2 r}{\partial x^2} = \frac{(\sqrt{x^2+y^2})(1) - (x)\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{(\sqrt{x^2+y^2})^2}$$

$$\Rightarrow \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} \Rightarrow \frac{\cancel{x^2}+y^2-\cancel{x^2}}{(x^2+y^2)\sqrt{x^2+y^2}} \Rightarrow \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Differentiating it w.r.t to y

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{(\sqrt{x^2+y^2})(1) - (y)\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{x^2+y^2} \Rightarrow \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Adding both we get,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}\left[\frac{x^2}{(x^2+y^2)} + \frac{y^2}{(x^2+y^2)}\right]$$

$$\boxed{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]}$$

Hence proved.