Improper Integrals

5.1 INTRODUCTION

The definition of a definite integral $\int_a^b f(x)\,dx$ requires the interval $[a, b]$ be finite. The fundamental theorem of calculus requires that $f(x)$ be continuous on $[a, b]$ or at least bounded. In this chapter, we will study a method of evaluating integrals that fail these requirements—either because their limits of integration are infinite, or because a finite number of discontinuities exist on the interval $[a, b]$. Integrals that fail either of these requirements are known as improper integrals. Improper integrals cannot be computed using a normal Riemann integral.

5.2 IMPROPER INTEGRALS

The integral $\int_a^b f(x)\,dx$ is called an improper integral if

(i) one or both limits of integration are infinite

(ii) function $f(x)$ becomes infinite at a point within or at the end points of the interval of integration.

Improper integrals are classified into three kinds.

5.3 IMPROPER INTEGRALS OF THE FIRST KIND

It is a definite integral in which one or both limits of integration are infinite, e.g. $\int_0^\infty e^{-x}\,dx$ is an improper integral of the first kind since the upper limit of integration is infinite. These integrals are evaluated as follows:

(i) If $f(x)$ is continuous on $[a, \infty)$ then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \ldots (1)$$

(ii) If $f(x)$ is continuous on $(-\infty, b]$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \qquad \ldots (2)$$

(iii) If $f(x)$ is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f(x)\,dx$$

$$= \lim_{a \to -\infty} \int_a^0 f(x)\,dx + \lim_{b \to \infty} \int_0^b f(x)\,dx \quad \ldots(3)$$

The improper integral is said to converge (or exist) when the limit in RHS of (1), (2) and (3) exist (or finite). Otherwise, it is said to diverge.

Example 1

Evaluate $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$.

Solution

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b \to \infty} \left| 2\sqrt{x} \right|_1^b$$

$$= \lim_{b \to \infty} (2\sqrt{b} - 2)$$

$$= \infty$$

Example 2

Evaluate $\int_1^{\infty} \frac{1}{x^2}\,dx$.

Solution

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b \to \infty} \left| -\frac{1}{x} \right|_1^b$$

$$= \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right)$$

$$= 1$$

Example 3

Evaluate $\int_{-\infty}^{0} x \sin x\,dx$.

Solution

$$\int_{-\infty}^{0} x\sin x\, dx = \lim_{a\to-\infty}\int_{a}^{0} x\sin x\, dx$$

$$= \lim_{a\to-\infty}\left|-x\cos x+\sin x\right|_a^0$$

$$= \lim_{a\to-\infty}(a\cos a-\sin a)$$

$$= -\infty \qquad [\because \sin a \text{ and } \cos a \text{ oscillate between } \pm 1]$$

Example 4

Evaluate $\int_{-\infty}^{0} e^{2x}\, dx.$

Solution

$$\int_{-\infty}^{0} e^{2x}\, dx = \lim_{a\to-\infty}\int_{a}^{0} e^{2x}\, dx$$

$$= \lim_{a\to-\infty}\left|\frac{e^{2x}}{2}\right|_a^0$$

$$= \lim_{a\to-\infty}\left(\frac{1}{2}-\frac{1}{2}e^{2a}\right)$$

$$= \frac{1}{2}-0$$

$$= \frac{1}{2}$$

Example 5

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$

Solution

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{1+x^2}\, dx + \lim_{b\to\infty}\int_{0}^{b} \frac{1}{1+x^2}\, dx$$

$$= \lim_{a\to-\infty} |\tan^{-1} x|_a^0 + \lim_{b\to\infty} |\tan^{-1} x|_0^b$$

$$= \lim_{a\to-\infty}(0-\tan^{-1} a) + \lim_{b\to\infty}(\tan^{-1} b-0)$$

$$= \frac{\pi}{2}+\frac{\pi}{2}$$

$$= \pi$$

Example 6

Evaluate $\int_{-\infty}^{\infty} e^x \, dx.$

Solution

$$\int_{-\infty}^{\infty} e^x \, dx = \lim_{a\to-\infty}\int_a^0 e^x \, dx + \lim_{b\to\infty}\int_0^b e^x \, dx$$

$$= \lim_{a\to-\infty} | e^x |_a^0 + \lim_{b\to\infty} | e^x |_0^b$$

$$= \lim_{a\to-\infty} (1-e^a) + \lim_{b\to\infty}(e^b - 1)$$

$$= (1-0) + \lim_{b\to\infty}(e^b - 1)$$

$$= \infty$$

Example 7

Evaluate $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \, dx$.

Solution

$$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \, dx = \int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1} \, dx$$

$$= \lim_{a\to-\infty}\int_a^0 \frac{e^x}{e^{2x}+1} \, dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{e^{2x}+1} \, dx$$

Putting $u = e^x$, $du = e^x \, dx$,

$$\int \frac{e^x}{e^{2x}+1} \, dx = \int \frac{du}{u^2+1} = \tan^{-1} u = \tan^{-1} e^x$$

$$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \, dx = \lim_{a\to-\infty} \left|\tan^{-1} e^x\right|_a^0 + \lim_{b\to\infty} \left|\tan^{-1} e^x\right|_0^b$$

$$= \lim_{a\to-\infty}\left(\frac{\pi}{4} - \tan^{-1} e^a\right) + \lim_{b\to\infty}\left(\tan^{-1} e^b - \frac{\pi}{4}\right)$$

$$= \left(\frac{\pi}{4} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

Example 8

Evaluate $\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx$.

Solution

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx = \lim_{a\to-\infty} \int_a^0 \frac{x}{(1+x^2)^2}\, dx + \lim_{b\to\infty} \int_0^b \frac{x}{(1+x^2)^2}\, dx$$

$$= \lim_{a\to-\infty} \int_a^0 (1+x^2)^{-2} \frac{2x}{2}\, dx + \lim_{b\to\infty} \int_0^b (1+x^2)^{-2} \frac{2x}{2}\, dx$$

$$= \lim_{a\to-\infty} \left| -\frac{1}{2(1+x^2)} \right|_a^0 + \lim_{b\to\infty} \left| -\frac{1}{2(1+x^2)} \right|_0^b$$

$$\left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \lim_{a\to-\infty} \left[-\frac{1}{2} + \frac{1}{2(1+a^2)} \right] + \lim_{b\to\infty} \left[-\frac{1}{2(1+b^2)} + \frac{1}{2} \right]$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= 0$$

Example 9

Evaluate $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$.

Solution

$$\int_a^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)} = \lim_{b\to\infty} \int_0^b \frac{\frac{1}{1+v^2}}{1+\tan^{-1} v}\, dv$$

$$= \lim_{b\to\infty} \left| \log |(1+\tan^{-1} v)| \right|_0^b \left[\because \int \frac{f'(v)}{f(v)}\, dv = \log |f(v)| \right]$$

$$= \lim_{b\to\infty} \left[\log |1+\tan^{-1} b| - \log 1 \right]$$

$$= \log(1+\tan^{-1}\infty) - 0$$

$$= \log\left(1+\frac{\pi}{2}\right)$$

5.4 IMPROPER INTEGRALS OF THE SECOND KIND

It is a definite integral in which integrand become infinite (or unbounded or discontinuous) at one or more points within or at the end points of the interval of integration, e.g.

(i) $\int_0^1 \frac{1}{x} dx$ is an improper integral of the second kind as $\frac{1}{x}$ is not continuous at $x = 0$.

(ii) $\int_{-2}^{2} \frac{1}{x^2 - 1} dx$ is an improper integral of the second kind because $\frac{1}{x^2 - 1}$ is not continuous at $x = -1$ and $x = 1$.

There integrals are evaluated as follows:

(i) If $f(x)$ is unbounded at $x = a$ then

$$\int_a^b f(x)dx = \lim_{c \to a} \int_c^b f(x)dx \qquad \dots (1)$$

(ii) If $f(x)$ is unbounded at $x = b$ then

$$\int_a^b f(x)\mathrm{d}x = \lim_{c \to b} \int_a^c f(x)\mathrm{d}x \quad \dots (2)$$

(iii) If $f(x)$ is unbounded at $x = a$ and $x = b$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c_1 \to a} \int_{c_1}^0 f(x)\mathrm{d}x + \lim_{c_2 \to b} \int_0^{c_2} f(x)\mathrm{d}x \quad \dots (3)$$

The improper integral is said to converge (or exist) when the limit in RHS of (1), (2) and (3) exist (or finite). Otherwise, it is said to diverge.

Example 1

Evaluate $\int_0^3 \frac{1}{\sqrt{3-x}}\mathrm{d}x.$

Solution

The integrand is unbounded at $x = 3$.

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\mathrm{d}x &= \lim_{c\to 3}\int_0^c \frac{1}{\sqrt{3-x}}\mathrm{d}x \\ &= \lim_{c\to 3}\left|-2\sqrt{3-x}\right|_0^c \\ &= \lim_{c\to 3}(-2\sqrt{3-c}+2\sqrt{3}) \\ &= 2\sqrt{3}\end{aligned}$$

Example 2

Evaluate $\int_0^{\frac{\pi}{2}} \sec x\mathrm{d}x.$

Solution

The integrand $\sec x$ is not continuous at $x = \frac{\pi}{2}$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sec x\,\mathrm{d}x &= \lim_{c\to\frac{\pi}{2}}\int_0^c \sec x\,\mathrm{d}x \\ &= \lim_{c\to\frac{\pi}{2}}\left|\log|\sec x + \tan x|\right|_0^c\end{aligned}$$

$$= \lim_{c \to \frac{\pi}{2}} \log|\sec c + \tan c|$$

$$= \log\left|\sec\frac{\pi}{2} - \tan\frac{\pi}{2}\right|$$

$$= \infty$$

Example 3

Evaluate $\int_0^1 \frac{1}{x^2} dx$.

Solution

The integrand $\frac{1}{x^2}$ is discontinuous at $x = 0$.

$$\int_0^1 \frac{1}{x^2} dx = \lim_{c \to 0} \int_c^1 \frac{1}{x^2} dx$$

$$= \lim_{c \to 0} \left| -\frac{1}{x} \right|_c^1$$

$$= \lim_{c \to 0} \left[-1 - \left(-\frac{1}{c} \right) \right]$$

$$= \infty$$

Example 4

Evaluate $\int_{-1}^1 \frac{1}{x^{\frac{2}{3}}} dx.$

Solution

The integrand $\frac{1}{x^{\frac{2}{3}}}$ is unbounded at $x = 0$.

$$\int_{-1}^1 \frac{1}{x^{\frac{2}{3}}} dx = \lim_{c_1 \to 0} \int_{-1}^{c_1} \frac{1}{x^{\frac{2}{3}}} dx + \lim_{c_2 \to 0} \int_{c_2}^1 \frac{1}{x^{\frac{2}{3}}} dx$$

$$= \lim_{c_1 \to 0} \left| 3x^{\frac{1}{3}} \right|_{-1}^{c_1} + \lim_{c_2 \to 0} \left| 3x^{\frac{1}{3}} \right|_{c_2}^1$$

$$= \lim_{c_1 \to 0} \left[3c_1^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} \right] + \lim_{c_2 \to 0} \left[3 - 3c_2^{\frac{1}{3}} \right]$$

$$= [0 - 3(-1)^{\frac{1}{3}}] + [3 - 0]$$

$$= 6$$

Example 5

Evaluate $\int_0^5 \frac{1}{(x-2)^2}\,dx$.

Solution

The integrand $\frac{1}{(x-2)^2}$ is unbounded at $x = 2$.

$$\int_0^5 \frac{1}{(x-2)^2}\,dx = \lim_{c_1 \to 2} \int_0^{c_1} \frac{1}{(x-2)^2}\,dx + \lim_{c_2 \to 2} \int_{c_2}^5 \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{c_1 \to 2} \left| -\frac{1}{x-2} \right|_0^{c_1} + \lim_{c_2 \to 2} \left| -\frac{1}{x-2} \right|_{c_2}^5$$

$$= \lim_{c_1 \to 2} \left(-\frac{1}{c_1 - 2} - \frac{1}{2} \right) + \lim_{c_2 \to 2} \left(-\frac{1}{3} + \frac{1}{c_2 - 2} \right)$$

$$= -\infty + \infty, \quad \text{indeterminate form}$$

Hence, no conclusion can be made about the value of the integral.

Example 6

Evaluate $\int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\,dx$.

Solution

The integrand $\frac{1}{\sqrt{a^2 - x^2}}$ is unbounded at $x = \pm a$.

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\,dx = \lim_{c_1 \to -a} \int_{c_1}^{0} \frac{1}{\sqrt{a^2 - x^2}}\,dx + \lim_{c_2 \to a} \int_0^{c_2} \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$$= \lim_{c_1 \to -a} \left| \sin^{-1} \frac{x}{a} \right|_{c_1}^{0} + \lim_{c_2 \to a} \left| \sin^{-1} \frac{x}{a} \right|_0^{c_2}$$

$$= \lim_{c_1 \to -a} \left[\sin^{-1} 0 - \sin^{-1} \frac{c_1}{a} \right] + \lim_{c_2 \to a} \left[\sin^{-1} \frac{c_2}{a} - \sin^{-1} 0 \right]$$

$$= -\sin^{-1}\left(-\frac{a}{a} \right) + \sin^{-1}\left(\frac{a}{a} \right)$$

$$= \sin^{-1} 1 + \sin^{-1} 1$$

$$= 2 \sin^{-1} 1$$

$$= 2 \cdot \frac{\pi}{2}$$

$$= \pi$$

Example 7

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}} dx.$

Solution

The integrand is unbounded at $x = 0$.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}} dx = \lim_{c\to 0} \int_c^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-\cos x}} dx$$

$$= \lim_{c\to 0} \int_c^{\frac{\pi}{2}} (1-\cos x)^{-\frac{1}{2}} \sin x \, dx$$

$$= \lim_{c\to 0} \left| \frac{(1-\cos x)^{\frac{1}{2}}}{\frac{1}{2}} \right|_c^{\frac{\pi}{2}} \qquad \left[\because \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \lim_{c\to 0} \left| 2(1-\cos x)^{\frac{1}{2}} \right|_c^{\frac{\pi}{2}}$$

$$= \lim_{c\to 0} 2\left[1-(1-\cos c)^{\frac{1}{2}}\right]$$

$$= 2$$