Expansions of functions :

In this section we shall discuss expansions of functions into infinite series with the help of Maclaurin's theorem and Taylor's theorem.

Power Series:

An infinite series of the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots\cdots + a_n(x-a)^n + \cdots\cdots$$

is called a power series in $(x-a)$, where the a_i's are constants.
If $a = 0$ then a power series can be written as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots + a_nx^n + \cdots\cdots$$

Taylor's and Maclaurin's Series:

Let f be a function with derivatives of all orders throughout some interval containing $a$as an interior point. Then the **Taylor series generated by** f at $x = a$is ,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

If $a = 0$ then the **Maclaurin'sseries**of$\boldsymbol{f}$ is the Taylor's series generated by f at $x = 0$ is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots$$

Remark:

Let $f(x)$ be a differentiable function of order n at a point $x = a$ then its Taylor's series expansion is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

In above expansion if we substitute $x - a = h$ then

$$f(a+h) = f(a) + f'(a)\,h + \frac{f''(a)}{2!}h^2 + \cdots\cdots (*)$$

If $h = x$ in above formula we get,

$$f(a+x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots\cdots$$

If $a = x$ in (*) formula we get,

$$f(x+h) = f(x) + f'(x)\,h + \frac{f''(x)}{2!}h^2 + \cdots\cdots$$

Which is the **alternative form**(another form) of the **Taylor's series.**

Maclaurin's expansion of some standard functions :

1. $f(x) = e^x$.
   Here $f(x) = e^x$, $a = 0$ so $f(0) = 1$

$f^{'}(x) = e^x, f'(0) = 1$

$f^{''}(x) = e^x$, $f''(0) = 1$

$f^{'''}(x) = e^x$, $f^{'''}(0) = 1$

$f^{IV}(x) = e^x$, $f^{IV}(0) = 1$

$\vdots\vdots$

$\vdots\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf^{'}(0) + \frac{x^2}{2!}f^{''}(0) + \frac{x^3}{3!}f^{'''}(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$\therefore e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots + \frac{x^n}{n!} + \cdots\cdots$

If we replace by x by $-x$ in above series expansion , we get

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots + (-1)^n\frac{x^n}{n!} + \cdots\cdots$$

2. $f(x) = \sin x$, $\therefore f(0) = 0$
   $f'(x) = \cos x$, $\therefore f'(0) = 1$

$f''(x) = -\sin x \therefore f''(0) = 0$

$f'''(x) = -\cos x \therefore f'''(0) = -1$

$f^{IV}(x) = \sin x \therefore f^{IV}(0) = 0$

$\vdots\vdots$

$\vdots\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf^{'}(0) + \frac{x^2}{2!}f^{''}(0) + \frac{x^3}{3!}f^{'''}(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$\therefore sin\, x = x - \frac{x^3}{3!} + \cdots\cdots\cdots\cdots$

3. $f(x) = \cos x$, $\therefore f(0) = 1$
   $f^{'}(x) = -\sin x$, $\therefore f^{'}(0) = 0$

$f''(x) = -\cos x, \therefore f''(0) = -1$

$f'''(x) = \sin x \therefore f'''(0) = 0$

$f^{IV}(x) = \cos x \therefore f^{IV}(0) = 1$

$\vdots\vdots$

$\vdots\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots\cdots\cdots$$

4. $f(x) = \log(1+x)$, $\therefore f(0) = 0$

$f'(x) = \frac{1}{1+x}$, $\therefore f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$, $\therefore f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$, $\therefore f'''(0) = 2 = 2!$

$f^{IV}(x) = -\frac{6}{(1+x)^4}$, $\therefore f^{IV}(0) = -6 = -3!$

$\vdots\vdots$

$\vdots\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$$\therefore\ log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\cdots\cdots$$

If we replace x by $-x$ in above series expansion, we get

$$f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots\cdots\cdots$$

5. $f(x) = sinhx = \frac{e^x - e^{-x}}{2}$

So using standard expansions of e^x and e^{-x}, we get

$f(x) = sinhx$

$$= \frac{1}{2}\left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right]\right\}$$

$$= \frac{1}{2}\left\{2x + 2\frac{x^3}{3!} + \cdots\cdots\cdots\right\} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots + \frac{x^{(2n+1)}}{(2n+1)!} + \cdots\cdots$$

Also $f(x) = coshx = \frac{e^x + e^{-x}}{2}$

$$= \frac{1}{2}\left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots + \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right]\right\}$$

$=\frac{1}{2}\left\{2+2\frac{x^2}{2!}+\cdots\cdots\right\}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\cdots+\frac{x^{2n}}{(2n)!}+\cdots\cdots$

Summary :

Function	Maclaurin's Series expansion
$y=e^x$	$1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$
$y=e^{-x}$	$1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+(-1)^n\frac{x^n}{n!}+\cdots$
$y=\sin x$	$x-\frac{x^3}{3!}+\cdots\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$
$y=\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$
$y=\log(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots\cdots\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots$
$y=\log(1-x)$	$-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\cdots\cdots\cdots-\frac{x^n}{n}+\cdots$
$y=sinhx$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\cdots+\frac{x^{(2n+1)}}{(2n+1)!}+\cdots\cdots$
$y=coshx$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\cdots+\frac{x^{2n}}{(2n)!}+\cdots\cdots$

Example:1 Find the Maclaurin's series expansion of $\tan x$.

Solution: Let $y=f(x)=tanx \therefore y(0)=f(0)=0$

$$y_1=f^{'}(x)=sec^2x=1+tan^2x \therefore y_1(0)=f'(0)=1$$

$$=1+y^2$$

$$y_2=f^{''}(x)=2yy_1 \therefore y_2(0)=f^{''}(0)=2(0)(1)=0$$

$$y_3=f^{'''}(x)=2{y_1}^2+2yy_2 \therefore y_3(0)=f^{'''}(0)=2(1)+2\cdot 0=2$$

$y_4=f^{IV}(x)=6y_1y_2+2yy_3 \therefore y_4(0)=f^{IV}(0)=0$

$y_5=f^{V}(x)=6{y_2}^2+8\,y_1y_3+2yy_4 \quad \therefore\ y_5(0)=f^{V}(0)=16$

$$\vdots\vdots$$

$$\vdots\vdots$$

Now, by Maclaurin's series expansion, we have

$$f(x)=f(0)+xf^{'}(0)+\frac{x^2}{2!}f^{''}(0)+\frac{x^3}{3!}f^{'''}(0)+\frac{x^4}{4!}f^{IV}(0)+\cdots\cdots$$

$\therefore \tan x=x+\frac{x^3}{3!}(2)+\frac{x^5}{5!}(16)+\cdots\cdots=\ x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\cdots$

Example:2 Expand $\sec x$ in powers of x up to x^4 by Maclaurin's series.

Solution: Let $y = f(x) = \sec x \therefore y(0) = f(0) = 1$

$$y_1 = \sec x \tan x = y \tan x \therefore y_1(0) = f'(0) = 0$$

$y_2 = y_1 \tan x + y\, sec^2 x = y_1 \tan x + y^3 ,\therefore y_2(0) = f''(0) \; = 1$

$$y_3 = y_2 \tan x + 2y_1 sec^2 x + 2y sec^2 x \; \tan x$$

$= y_2 \tan x + 2y_1 y^2 + 2y^3 \tan x, \qquad \therefore y_3(0) = f'''(0)=0$

$y_4 = 3y_2 y^2 + y_3 \; \tan x + 4y\, {y_1}^2 \; + 6y^2 y_1 \tan x + 2\, y^5 \,, \; \therefore y_4(0) = \; 5$

$$\vdots\vdots$$

$$\vdots\vdots$$

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf^{'}(0) + \frac{x^2}{2!} f^{''}(0) + \frac{x^3}{3!} f^{'''}(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$\therefore \sec x = 1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + \cdots\cdots\cdots$

Example:3 Find the Maclaurin's series expansion of the function $y = \log(1 + \sin x)$.

Solution: Here $y = f(x) = \log(1 + \sin x) \therefore f(0) = 0$

$y_1 = f^{'}(x) = \frac{\cos x}{1+\sin x} \qquad\qquad \therefore f'(0) = 1$

$$y_2 = f^{''}(x) = \frac{(1 + \sin x)(-\sin x) - cos^2 x}{(1 + \sin x)^2}$$

$y_2 = \; -\frac{1}{1+\sin x} \qquad\qquad \therefore f''(0) =$ -1

$$\vdots\vdots$$

$$\vdots\vdots$$

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf^{'}(0) + \frac{x^2}{2!} f^{''}(0) + \frac{x^3}{3!} f^{'''}(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\log(1 + \sin x) = x - \frac{x^2}{2!} + \cdots\cdots\cdots$$

Example:4 Arrange the following Polynomial in powers of x using Maclaurin's series
$f(x) = 5 + (x + 3) + 7(x + 3)^2$.

Solution: Here $f(x) = 5 + (x + 3) + 7(x + 3)^2 \; , \therefore f(0) = 71$

$f^{'}(x) = 1 + 14(x + 3), \qquad\qquad \therefore f^{'}(0) = 43$

$f^{''}(x) = 14 \,, \qquad\qquad \therefore f^{''}(0) = 14$

$f^{'''}(x) = 0 \,, \qquad\qquad \therefore f^{'''}(0) = 0$

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf^{'}(0) + \frac{x^2}{2!}f^{''}(0) + \frac{x^3}{3!}f^{'''}(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$\therefore\ 5 + (x+3) + 7(x+3)^2 = 71 + 43x + 7x^2$.

Example:5 Expand $\log x$ in powers of $(x-1)$ up to three power and hence evaluate $\log 1.1$ correct to four decimal places.

Solution: Here $(x) = \log x$, $\quad a = 1 \therefore f(1) = 0$

$f^{'}(x) = \frac{1}{x}$, $\quad \therefore f^{'}(1) = 1$

$f^{''}(x) = -\frac{1}{x^2}$, $\quad \therefore\ f^{''}(1) = -1$

$f^{'''}(x) = \frac{2}{x^3}$, $\quad \therefore f^{'''}(1) = 2$

$\vdots\vdots$

$\vdots\vdots$

Now, by Taylor's series expansion, we have

$$f(x) = f(1) + (x-1)f^{'}(1) + \frac{(x-1)^2}{2!}f^{''}(1) + \cdots\cdots$$

$\therefore \log x = (x-1)\ 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}2 + \cdots\cdots$

$\therefore \log x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3} + \cdots\cdots$

Now taking $x = 1.1$, we get

$\therefore \log 1.1 = 0.09533.$

Example:6 Expand $2x^3 + 7x^2 + 1$ in powers of $(x-3)$ by using Taylor's series expansion.

Solution: Here $(x) = 2x^3 + 7x^2 + 1$, $\quad a = 3$, $\quad f(3) = 118$

$f^{'}(x) = 6x^2 + 14x$, $\quad f^{'}(3) = 96$

$f^{''}(x) = 12x + 14$, $\quad f^{''}(3) = 50$

$f^{'''}(x) = 12$, $\quad f^{'''}(3) = 12$

$f^{IV}(x) = 0\ , f^{IV}(3) = 0$

Now, by Taylor's series expansion, we have

$$f(x) = f(3) + (x-3)f^{'}(3) + \frac{(x-3)^2}{2!}f^{''}(3) + \cdots\cdots$$

$\therefore 2x^3 + 7x^2 + 1 = 118 + (x-3)\ 96 + \frac{(x-3)^2}{2!}\ 50 + \frac{(x-3)^3}{3!}12 + 0$

$= 118 + 96(x-3) + 25(x-3)^2 + 2\ (x-3)^3.$

Example:7 Find the Taylor's series expansion of $\sin\left(x+\frac{\pi}{4}\right)$ in powers of x and hence find the value of $\sin 46°$.

Solution: Here let $f(x) = \sin x$ and $h = \frac{\pi}{4}$

So using another form of Taylor's series we have

$f(x+h) = f(h) + f'(h)x + \frac{f''(h)}{2!}x^2 + \cdots\cdots$ ....(1)

Now $f(x) = \sin x$, $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x$, $f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x$, $f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and so on

Hence from equation (1), we have

$$\sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + x\left(\frac{1}{\sqrt{2}}\right) + \frac{x^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots\cdots$$

Consider $x = 1° = 0.01745$,

$\therefore \sin\left(1° + \frac{\pi}{4}\right) = \sin 46° = 0.71934.$