

# Backdoor Criterion

## Definition. (The Backdoor Criterion)

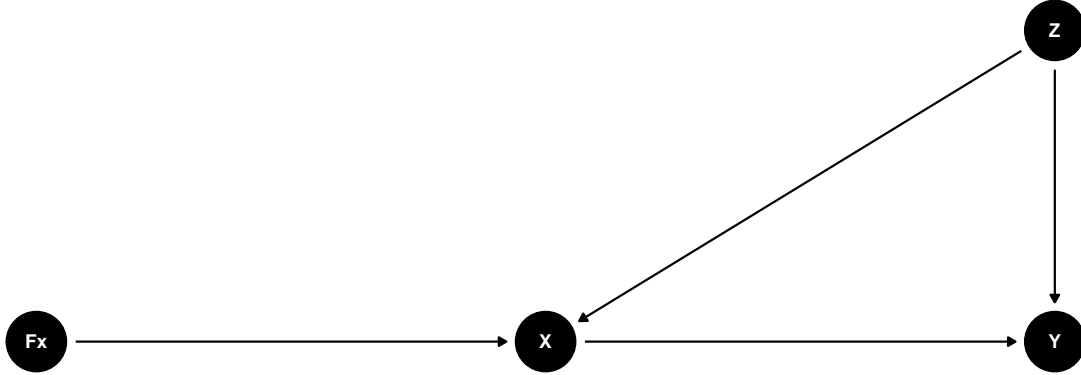
Given an ordered pair of variables  $(X, Y)$  in a directed acyclic graph  $G$ , a set of variables  $Z$  satisfies the backdoor criterion relative to  $(X, Y)$  if no node in  $Z$  is a descendant of  $X$ , and  $Z$  blocks every path between  $X$  and  $Y$  that contains an arrow into  $X$ .

## Causal Effect.

If a set of variables  $Z$  satisfies the backdoor criterion for  $X$  and  $Y$ , then the causal effect of  $X$  on  $Y$  is given by the formula:

$$P(Y = y \mid do(X = x)) = \sum_z P(Y = y \mid X = x, Z = z)P(Z = z)$$

Here, I introduce regime indicator ( $F_X$ ) for proof and an example of graphical model to help understanding



where probability distribution  $P(x_i \mid pa(x_i, G'))$  is

$$P(x_i \mid pa(x_i, G')) = \begin{cases} P(x_i \mid pa(x_i, G)) & \text{if } F_i = \text{idle} \\ 1 & \text{if } F_i = do(x'_i) \text{ and } x_i = x'_i \\ 0 & \text{if } F_i = do(x'_i) \text{ and } x_i \neq x'_i \end{cases}$$

**Proof for  $P(y \mid do(x)) = \sum_z P(y \mid z, x)P(z)$ :**

$$\begin{aligned}
& P(y \mid do(x)) \\
&= P(y \mid F_x = do(x)) \\
&= \sum_z P(y, z \mid F_x = do(x)) \\
&= \sum_z P(y \mid z, F_x = do(x))P(z \mid F_x = do(x)) \\
&= \sum_z P(y \mid z, x, F_x = do(x))P(z \mid F_x = do(x)) \\
&\stackrel{1}{=} \sum_z P(y \mid z, x, F_x = do(x))P(z) \\
&\stackrel{2}{=} \sum_z P(y \mid z, x)P(z)
\end{aligned}$$

- (1) By Parental Markov condition (local Markov condition): A necessary and sufficient condition for a probability distribution  $P$  to be Markov relative a DAG  $G$  is that every variable be independent of all its nondescendants (in  $G$ ), conditional on its parents.

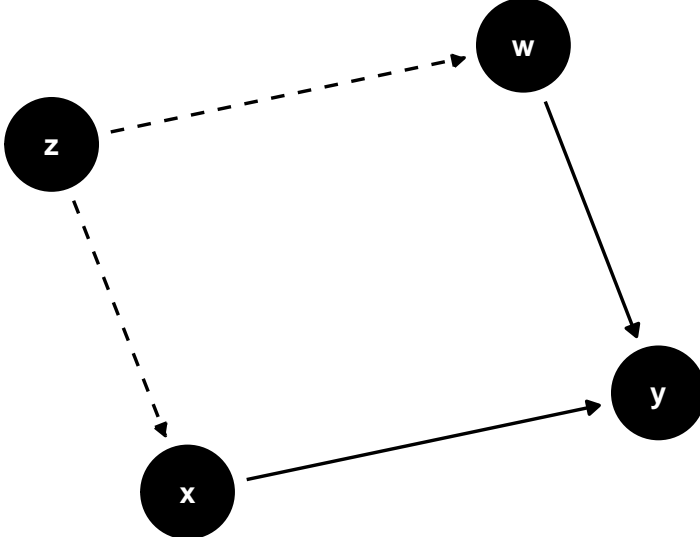
By this theorem,  $F_x \perp\!\!\!\perp Z$

- (2)  $F_x$  and  $Y$  are d-separated, conditional on  $Z$  and  $X \Rightarrow F_x \perp\!\!\!\perp Y \mid Z, X$

**PA(x) always satisfies backdoor criterion:**

$$P(y \mid pa(x)) = \sum_z P(y \mid pa(x), x)P(pa(x))$$

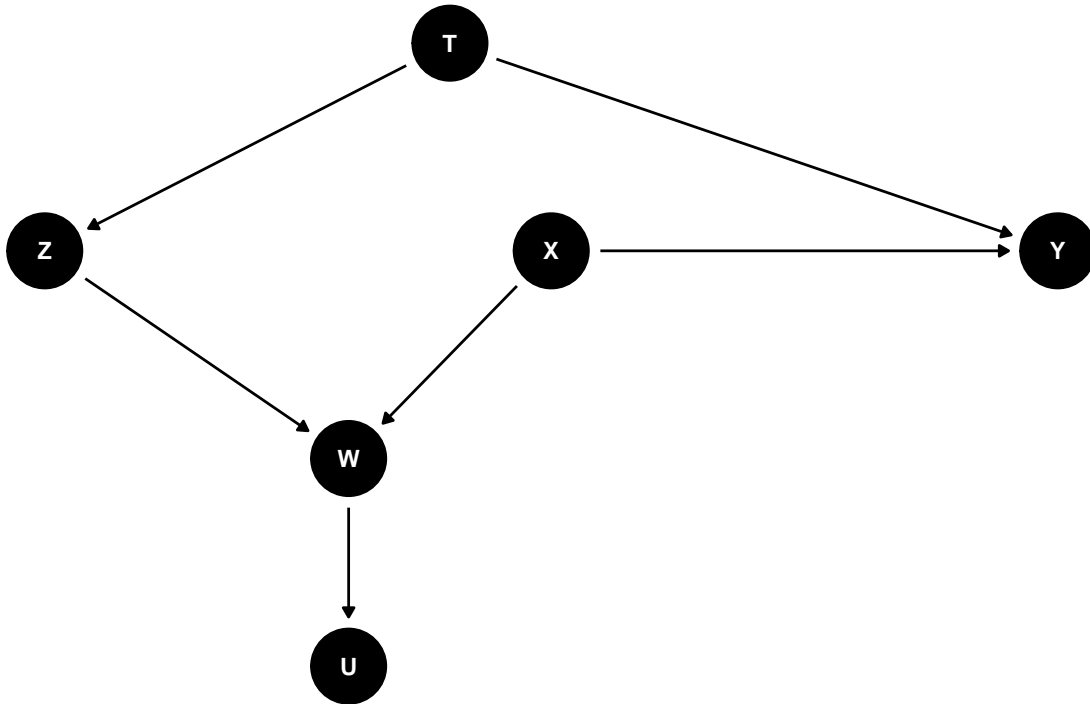
**Backdoor Criterion Example.1:**



Here, we want to estimate the effect of smoking cigarette ( $x$ ) on lung function ( $y$ ). Weight,  $w$ , is also measured and we also know the type of jobs affect both weight and the choice to smoke cigarette. However, the study did not record the type of jobs. Though,  $z$  is not recorded, we can estimate  $P(Y = y \mid do(X = x))$  using the backdoor criterion. Here,  $w$  is not a descendant of  $x$  and also blocks the backdoor path  $x \leftarrow z \rightarrow w \rightarrow y$ .

Thus,  $P(Y = y \mid do(X = x)) = \sum_w P(Y = y \mid X = x, W = w)P(W = w)$

## Backdoor Criterion Example.2:



Here, we want to estimate the effect of  $X$  on  $Y$  for a specific value  $w$  of  $W$ . Though  $W$  is a collider and conditioning on  $W$  opens a path,  $T$  (not a descendant of  $X$ ) can be used to block the spurious path  $X \rightarrow W \leftarrow Z \leftrightarrow T \rightarrow Y$ .

Thus,  $P(Y = y \mid do(X = x), W = w) = \sum_t P(Y = y \mid X = x, W = w, T = t)P(W = w, T = t)$