Nonparametric dobuly robust estimation of continuous treatment

This work is to verify the proof of theorems introduced in the paper (Kennedy et al, 2016).

Guide to notation

$Z = (L, A, Y)$ = observed data arising from distribution P with density $p(z) = p(y \mid l, a)p(a \mid l)p(l)$ and support $\mathrm{supp}(Z) = \mathcal{Z} = \mathcal{L} \times \mathcal{A} \times \mathcal{Y}$

$\mathbb{P}_n = \frac{1}{n}\sum_i \delta_{z_i}$ = empirical measure so that $\mathbb{P}_n f(Z) = \frac{1}{n}\sum_i f(z_i)$

$\mathbb{P}(\mho) = \mathbb{P}\{\mho(\mathbb{Z})\} = \int_{\mathcal{Z}} f(z)dP(z)$ = expectation for new Z treating f as fixed (so $\mathbb{P}(\hat{\mho})$ is random if $\hat{f}$ depends on sample, in which case $\mathbb{P}(\hat{\mho}) \neq E(\hat{f})$)

$\pi(a \mid l) = p(a \mid l) = \frac{\partial}{\partial a}P(A \leq a \mid l)$ = conditional density of treatment A

$\hat{\pi}(a \mid l)$ = user-specified estimator of $\pi(a \mid l)$, which converges to limit $\overline{\pi}(a \mid l)$ that may not equal true π

$\omega(a) = p(a) = \frac{\partial}{\partial a}P(A \leq a) = E[\pi(a \mid L)] = \int_{\mathcal{L}} \pi(a \mid l)dP(l)$ = density of A

$\hat{\omega(a)} = \mathbb{P}_{\ltimes}\{\hat{\pi}(a \mid l)\} = \int_{\mathcal{L}} \pi(a \mid l)dP(l)$ = estimator of ω, which converges to limit $\overline{\omega}(a)$ that may not equal true ω

$\mu(l \mid a) = E(Y \mid L = l, A = a) = \int_{\mathcal{Y}} y dP(y \mid l, a)$ = conditional mean outcome

$\hat{\mu}(l \mid a)$ = user-specified estimator of $\mu(l \mid a)$, which converges to limit $\overline{\mu}(l, a)$ that may not equal true μ

$\hat{m}(a) = \mathbb{P}_{\ltimes}\{\hat{\mu}(L, a)\} = \int_{\mathcal{L}} \hat{\mu}(l, a)d\mathbb{P}_{\ltimes}(l) = \frac{1}{n}\sum_i \hat{\mu}(l_i, a)$

$\psi = \int_{\mathcal{A}}\int_{\mathcal{L}} \mu(l, a)\omega(a)dP(l)da$

Theorem 1.

Under a nonparametric model, the efficient influence function for ψ is
$\xi(Z; \pi, u) - \psi + \int_{\mathcal{A}}\{\mu(L, a) - \int_{\mathcal{L}} \mu(l, a)dP(l)\}\omega(a)da$ where

$$\xi(Z; \pi, u) = \frac{Y - \mu(L,A)}{\pi(A|L)}\int_{\mathcal{L}} \pi(A \mid l)dP(l) + \int_{\mathcal{L}} \mu(l, a)dP(l)$$

Importantly, the function $\xi(Z; \pi, u)$ satisfies its desired double robustness property,
i.e., that $E[\xi(Z; \pi, u) \mid A = a] = \theta(a)$ if either $\overline{\pi} = \pi$ or $\overline{\mu} = \mu$ where $\theta(a) = E(Y^a)$

This motivates estimating the effect curve $\theta(a)$ by estimating the nuisance functions (π, μ), and then regressing the estimated psuedo-outcome

$$\hat{\xi}(Z; \hat{\pi}, \hat{\mu}) = \frac{Y - \hat{\mu}(L,A)}{\hat{\pi}(A,L)}\int_{\mathcal{L}} \hat{\pi}(A, L)d\mathbb{P}_{\ltimes}(l) + \int_{\mathcal{L}} \hat{\mu}(l, a)d\mathbb{P}_{\ltimes}(l)$$

on treatment A using off-the-shelf nonparametric regression or machine learning methods.

Proof of Theorem 1

By definition the efficient influence function for ψ is the unique function $\phi(Z)$ that satisfies $\psi_\epsilon'(0) = E[\phi(Z)\ell_\epsilon'(Z;0)]$, where $\psi(\epsilon)$ represents the parameter of interest as a functional on the parametric submodel and $\ell(w \mid \overline{w}; \epsilon = \log p(w \mid \overline{w}; \epsilon)$ for any partition $(W, \overline{W}) \subseteq Z$. Therefore

$$\ell_\epsilon'(z;\epsilon) = \ell_\epsilon'(y \mid l, a; \epsilon) + \ell_\epsilon'(a \mid l; \epsilon) + \ell_\epsilon'(l;\epsilon)$$

The authors give two important properties of such score functions of such score functions $\ell_\epsilon'(w \mid \overline{w}; \epsilon)$ that will be used throughout this proof. First note that since $\ell_\epsilon(w \mid \overline{w}; \epsilon)$ is a log transformation of $p(w \mid \overline{w}; \epsilon)$, it follows that $\ell_\epsilon'(w \mid \overline{w}; \epsilon) = p'(w \mid \overline{w}; \epsilon)/p(w \mid \overline{w}; \epsilon)/$. Similarly, as with any score function, note that $E[\ell_\epsilon'(W \mid \overline{W}; \epsilon) \mid \overline{W}] = 0$ since

$$\int_{\mathcal{W}} \ell_\epsilon'(w \mid \overline{w}; 0) dP(w \mid \overline{w}) = \int_{\mathcal{W}} dP'(w \mid \overline{w}) = \frac{\partial}{\partial \epsilon} \int_{\mathcal{W}} P(w \mid \overline{w}) = 0$$

The goal in this proof is to show that $\psi_\epsilon'(0) = E[\phi(Z)]\ell_\epsilon'(Z;0)$ for the proposed influence function $\phi(Z) = \xi(Z;\pi,u) - \psi + \int_{\mathcal{A}}\{\mu(L,a) - \int_{\mathcal{L}}\{\mu(l,a)dP(l)\}\omega(a)da$ given in the main text. First we will give an expression for $\psi_\epsilon'(0)$. By definition $\psi(\epsilon)$