Backdoor Criterion

Definition (The Backdoor Criterion)

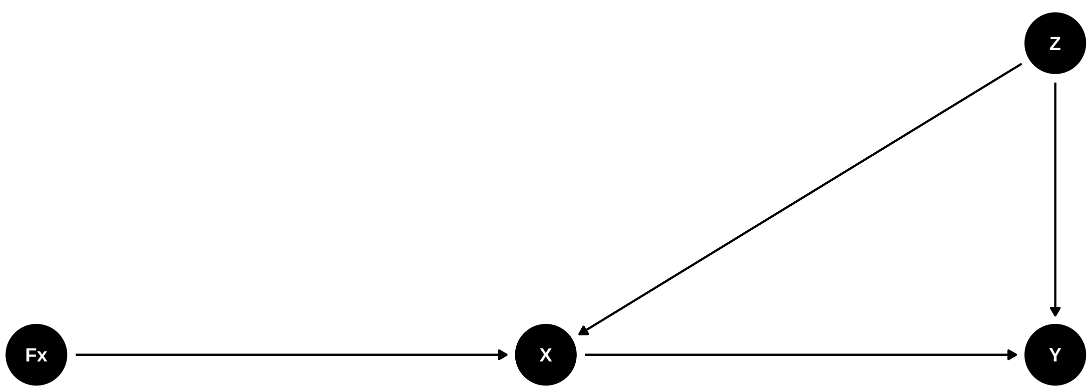
Given an ordered pair of variables (X, Y) in a directed acyclic graph G, a set of variables Z satisfies the backdoor criterion relative to $(X,\ Y)$ if no node in Z is a descendant of X, and Z blocks every path between X and Y that contains an arrow into X.

Causal Effect

If a set of variables Z satisfies the backdoor criterion for X and Y, then the causal effect of X on Y is given by the formula:

$P(Y = y \mid do(X = x)) = \sum_z P(Y = y \mid X = x, Z = z)P(Z = z)$

Introducing regime indicator (F_X) which takes values in {do($x^{'}$), idle (no intervention)} for proof and an example of graphical model, $G^{'}$. Note that a graphical model without F_X is G and $G^{'} = G \bigcup \{F_i \rightarrow X_i\}$.



The conditonal probability, $P(x_i \mid pa(x_i, G^{'}))$, introduces the new parent set of Xi, $PA_i^{'} = PA_i \bigcup \{F_i\}$

$$P(x_i \mid pa(x_i, G^{'})) = \begin{cases} P(x_i \mid pa(x_i, G)) & if\ F_i = idle \\ 1 & if\ F_i = do(x_i^{'})\ and\ x_i = x_i^{'} \\ 0 & if\ F_i = do(x_i^{'})\ and\ x_i \neq x_i^{'} \end{cases}$$

Now, we are ready for proof!

Proof for $P(y \mid do(x)) = \sum_z P(y \mid z, x)P(z)$:

$P(y \mid do(x))$
$= \; P(y \mid F_x = do(x))$
$= \; \sum_z P(y, z \mid F_x = do(x))$
$= \; \sum_z P(y \mid z, F_x = do(x))P(z \mid F_x = do(x))$
$= \; \sum_z P(y \mid z, x, F_x = do(x))P(z \mid F_x = do(x))$
$=^1 \sum_z P(y \mid z, x, F_x = do(x))P(z)$
$=^2 \sum_z P(y \mid z, x)P(z)$

(1) By Parental Markov condition (local Markov condition): A necessary and sufficient condition for a probability distribution P to be Markov relative a DAG G is that every variable be independent of all its nondescendants (in G), conditional on its parents.

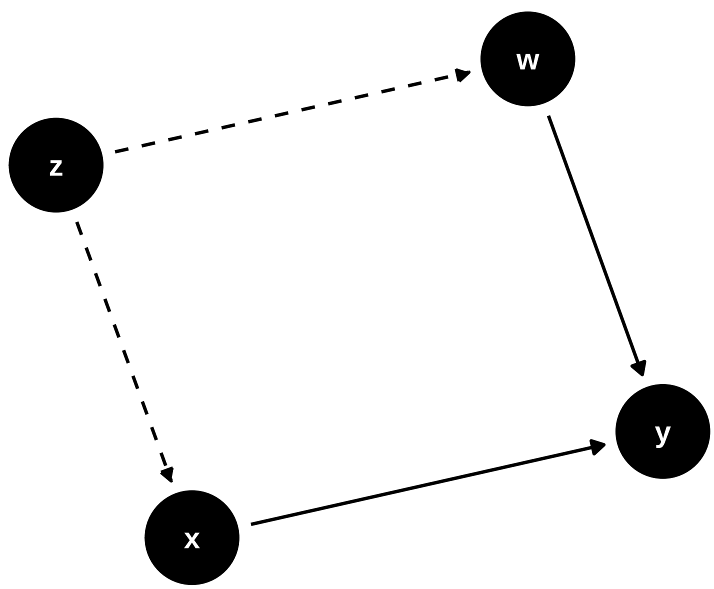
By this theorem, $F_x \perp\!\!\!\perp Z$

(2) By backdoor condition, F_x *and* Y are d-separated, conditional on Z and X => $F_x \perp\!\!\!\perp Y \mid Z, X$

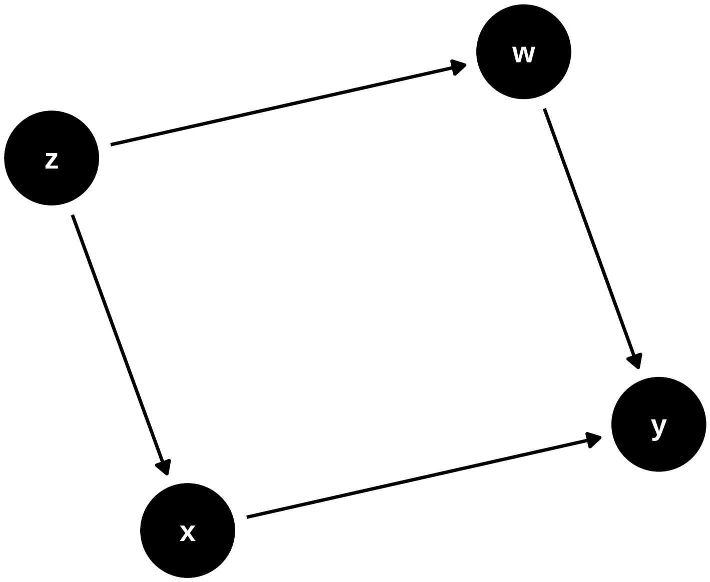
PA(x) always satisfies backdoor criterion:

$P(y \mid pa(x)) = \sum_z P(y \mid pa(x), x)P(pa(x))$

Backdoor Criterion Example1:



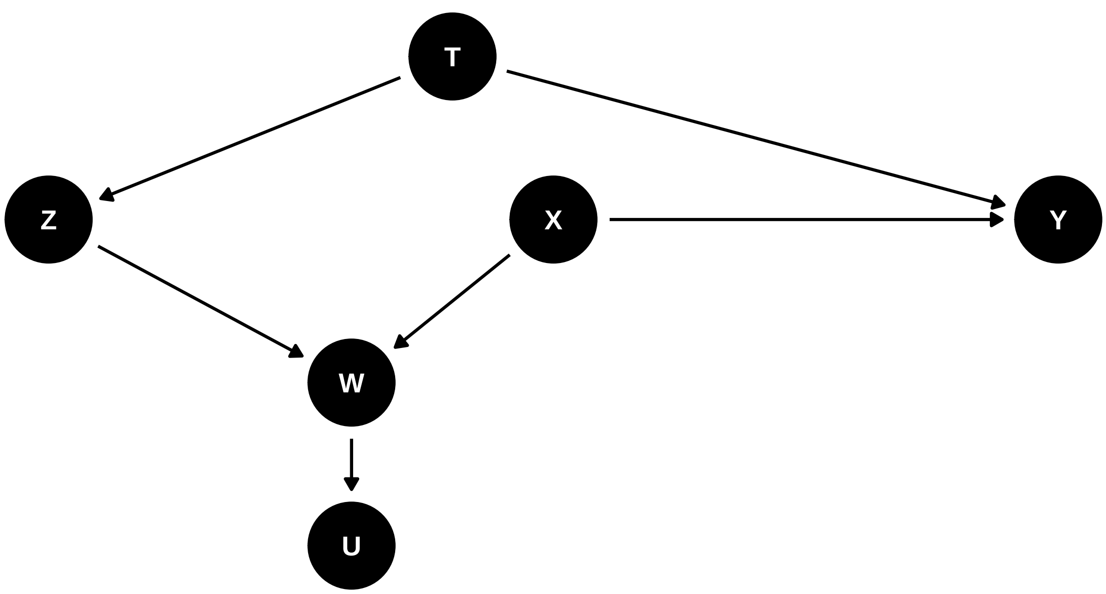
Here, we want to estimate the effect of smoking cigarette (x) on lung function (y). Weight, w, is also measured and we know the type of jobs affect both weight and the choice to smoke cigarette. However, the study did not record the type of jobs. Though, z is not recorded, we can estimate $P(Y = y \mid do(X = x))$ using the backdoor criterion. Here, w is not a descendant of x and also blocks the backdoor path $x \leftarrow z \rightarrow w \rightarrow y$. Thus, $P(Y = y \mid do(X = x)) = \sum_w P(Y = y \mid X = x, W = w)P(W = w)$



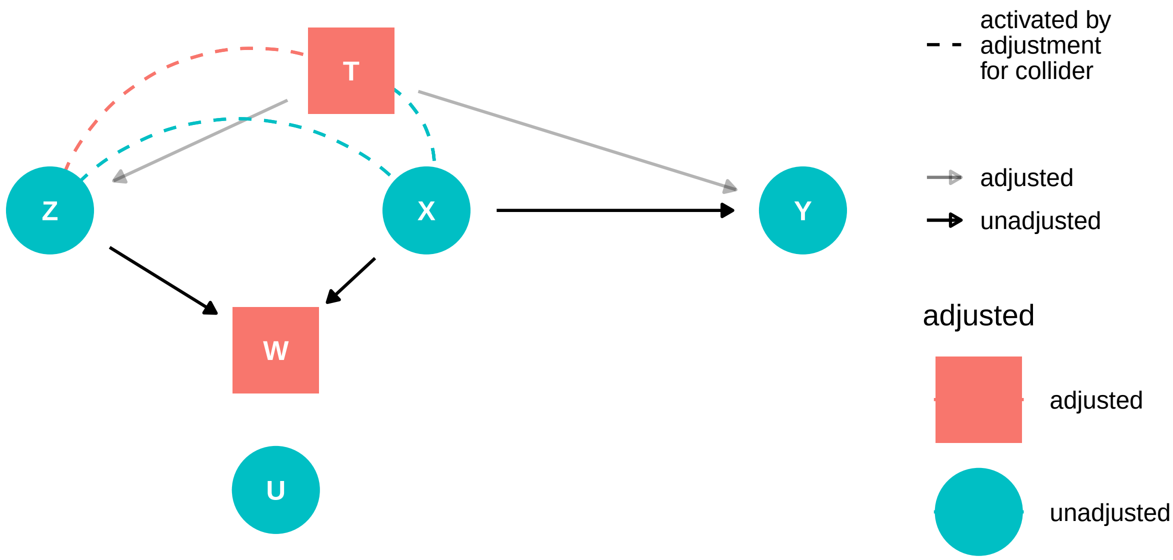
Now assume z is observed as well as w. Then, blocking the backdoor path can be done by adjusting for either z or w. This is useful because the choice can be made depending on which variable is more convenient to measure.

$P(Y = y \mid do(X = x)) = \sum_w P(Y = y \mid X = x, W = w)P(W = w) = \sum_w P(Y = y \mid X = x, Z = z)P(Z = z)$

Backdoor Criterion Example2:



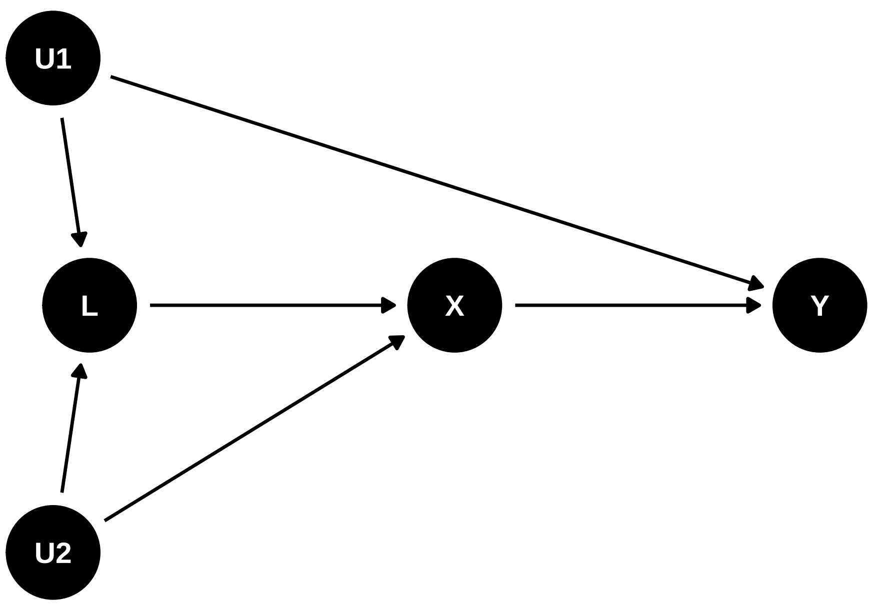
Here, we want to estimate the effect of X on Y for a specific value w of W. Though W is a collider and conditioning on W opens a path, T (not a descendant of X) can be used to block the spurious path $X \rightarrow W \leftarrow Z \leftrightarrow T \rightarrow Y$.



We can see the backdoor path is blocked when adjusting for W and T.

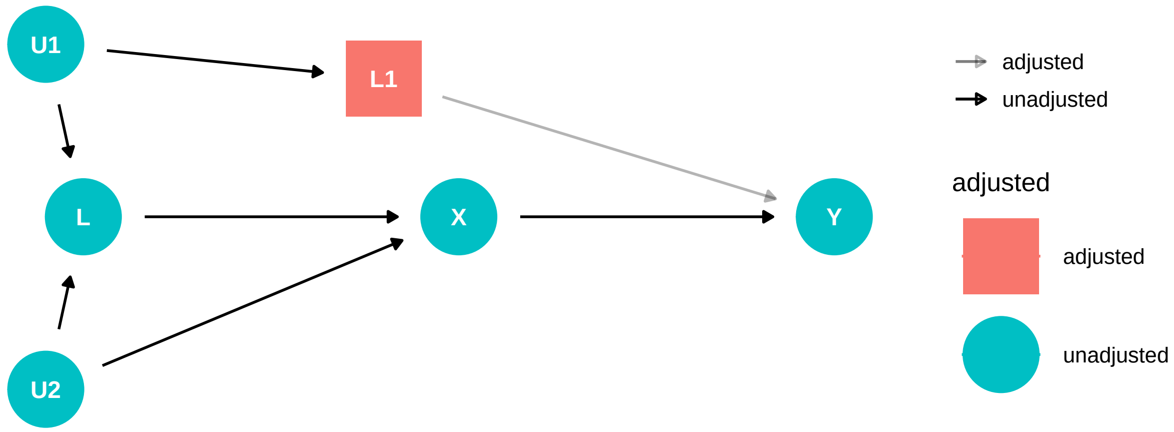
Thus, $P(Y = y \mid do(X = x), W = w) = \sum_t P(Y = y \mid X = x, W = w, T = t)P(T = t \mid W = w)$

Backdoor Criterion Example3:



Here, we want to estimate the effect of X on Y. Conditioning on L would block the backdoor path, $X \leftarrow L \leftarrow U_1 \rightarrow Y$, but would simultaneously open a new backdoor path, $X \rightarrow U_2 \rightarrow L \leftarrow U_1 \rightarrow Y$. Namely, the attempt to block the confounding path brings about a selection bias. Here, A solution would be to measure either (i) a variable L_1 between U_1 and either A or Y, or (ii) a variable L_2 between u_2 and either A or L.

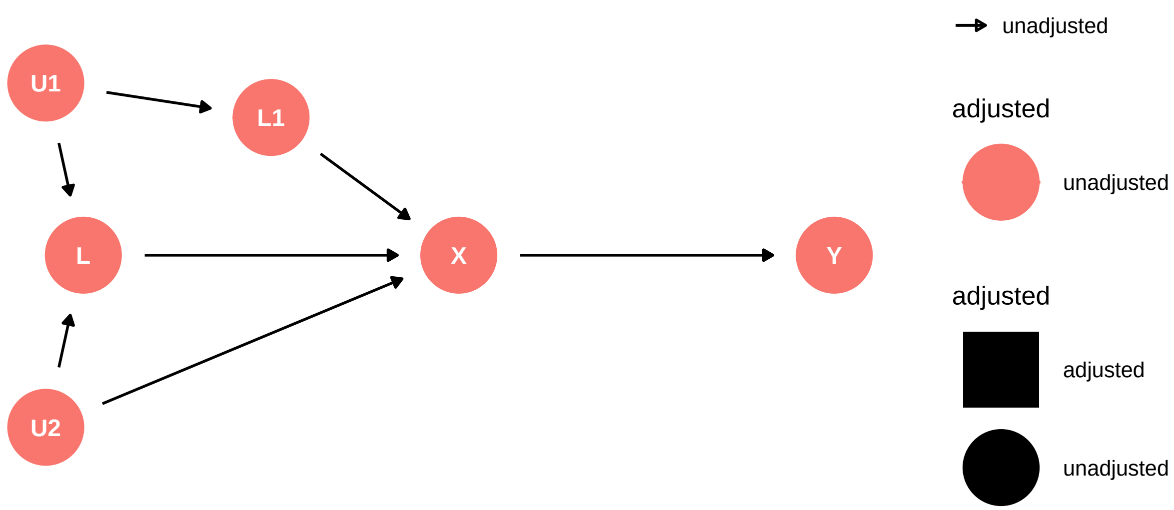
Example3 Solution1 (L_1 between U_1 and Y):



Adjusting for L_1 blocks the backdoor path.

Thus, $P(Y = y \mid do(X = x)) = \sum_{l_1} P(Y = y \mid X = x, L_1 = l_1)P(L_1 = l_1)$

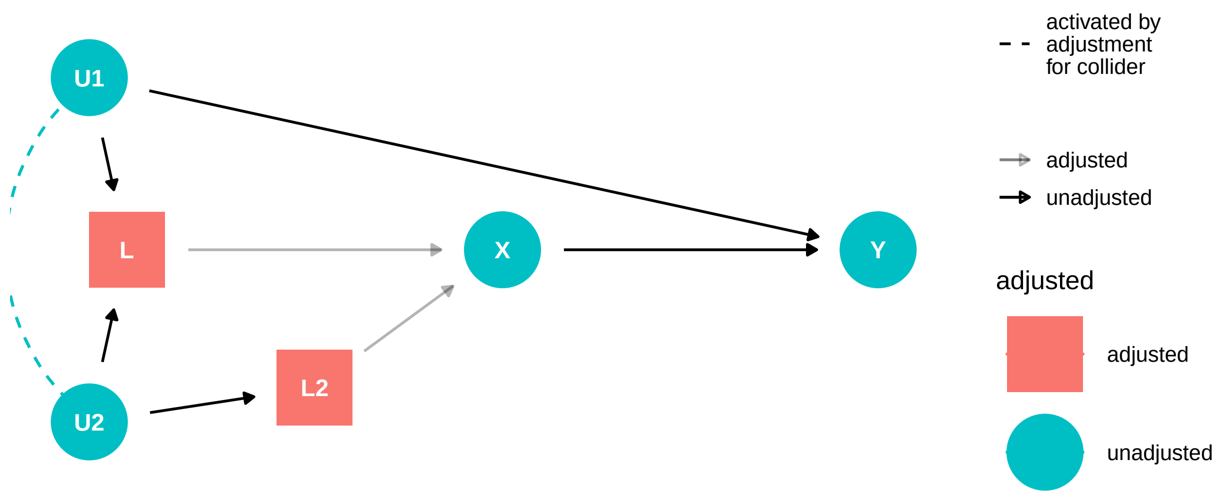
Example3 Solution1 (L_1 between U_1 and X):



There is no backdoor path by adding L_1 variable.

Thus, $P(Y = y \mid do(X = x)) = P(Y = y \mid X = x)$

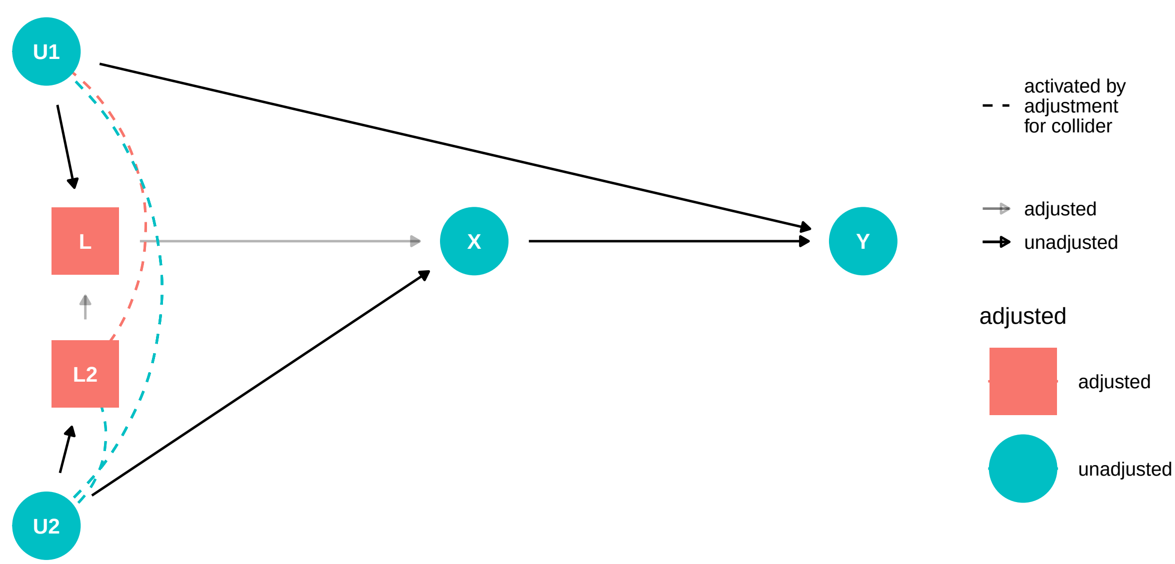
Example3 Solution2 (L_2 between U_2 and X):



Adjusting for L and L2 blocks the backdoor path.

Thus, $P(Y = y \mid do(X = x)) = \sum_{l,l_2} P(Y = y \mid X = x, L = l, L_2 = l_2)P(L = l, L_2 = l_2)$

Example3 Solution2 (L_2 between U_2 and L):



Adjusting for L and L2 blocks the backdoor path.

Thus, $P(Y = y \mid do(X = x)) = \sum_{l,l_2} P(Y = y \mid X = x, L = l, L_2 = l_2)P(L = l, L_2 = l_2)$