G-estimation

Suppose conditional exchangeability holds and then the outcome distribution in the treated and the untreated would be the same if both groups had received the same treatment level, namely $A \perp\!\!\!\perp Y^a \mid L$. Here, let Y is the outcome, A is the treatment, and L is the set of all measured covariates.

Then, $Pr[A = 1 \mid Y^a, L] = Pr[A = 1 \mid L]$

Now, suppose we propose the following parametric logistic model for the probability of treatment:

$logitPr[A = 1 \mid Y^{a=0}, L] = \alpha_0 + \alpha_1 Y^{a=0} + \alpha_2 L$ where α_2 is a vector of parameters, one for each component of L.

If L has P components, $\alpha_2 L = \sum_{j=1}^{p} \alpha_{2j} L_j$

Structural nested mean models

Suppose we are interested in estimating the average causal effect of treatment A within levels of L, that is, $E[Y^{a=1} \mid L] - E[Y^{a=0} \mid L]$

The equation is the same as $E[Y^{a=1} - Y^{a=0} \mid L]$ because the difference of the means is equal to the mean of the differences.

If there is no effect measure modification by L, these differences would be constant across strata:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1$

And the structural model for the conditional causal effect would be:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1 a$

More generally, there may be effect modification by L and the structural model would be:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1 a + \beta_2 aL$ which is referred to as a *structural nested mean model.*

The parameters, β_1 and β_2 (a vector), are estimated by g-estimation and they quantify the average causal effect of treatment A on Y within levels of A and L.

Rank Preservation

Suppose we rank every individual according to $Y^{a=1}$ and according to $Y^{a=0}$ as well. If those individuals in two lists are ordered identically, there is *rank preservation.* For example, if treatment A gives the same effect on the outcome Y to everyone, then the ranking of those individuals according to $Y^{a=0}$ would be equal to that of those individuals according to $Y^{a=1}$

The conditioanl additive rank preservation holds if the effect of treatment A on the outcome Y is exactly the same for all individuals with the same values of L.

An example of an (additive conditional) rank-preserving structural model is:

$Y_i^a - Y_i^{a=0} = \psi_1 a + \psi_2 a L_i$ for all individuals i

where $\psi_1 + \psi_2 l$ s the constant causal effect for all individuals with covariate values L = l.

Although rank preservation is implausible, it is introduced to introduce g-estimation and for easier understanding of it.

G-estimation

Suppose we want to estimate the parameters of structural nested mean model $E[Y^a - Y^{a=0} \mid A = a, L] = \beta_1 a$

Also assume that the additive rank-preserving model is $Y_i^a - Y_i^{a=0} = \psi_1 a$ for all individuals i.

Then the individual causal effect ψ_1 is equal to the average causal effect β_1.

The rank-preserving model can be also written as:

$Y^{a=0} = Y^a - \psi_{1a}$

If the model were correct and the value of ψ_1 were known, then it would be possible to compute the counterfactual outcome under no treatment $Y^{a=0}$ for each individual in the study population. The challenge is to estimate ψ_1

Let us notate the possible values of ψ_1 as $\psi_1^\dagger$ and define function:

$H(\psi^\dagger) = Y - \psi^\dagger A$

Among possible values of $\psi^\dagger$, only one of them is the true ψ and its corresponding candidate $H(\psi^\dagger)$ is the counterfactual outcome $Y^{a=0}$.

In order to find the right candidate for ψ among possible $\psi^\dagger$, we fit separate logistic models for every candidate $H(\psi^\dagger)$:

$logit Pr[A = 1 \mid H(\psi^\dagger), L] = \alpha_0 + \alpha_1 H(\psi^\dagger) + \alpha_2 L$

The trick is to use conditional exchangeability, $A \perp\!\!\!\perp Y^a \mid L$.

Namely, we find the $H(\psi^\dagger)$ with $\alpha_1 = 0$ in its logistic model and that $H(\psi^\dagger)$ is the counterfactual $Y^{a=0}$

The example for the possible range of $\psi^\dagger$ is from -20 to 20 and test each value of $\psi^\dagger$ by increments of 0.01.

Structural nested models with effect modification

Suppose we have effect modification by some components V of L.

Then, the structural nested mean model is:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1 a + \beta_2 a V$

The corresponding rank-preserving model is:

$Y_i^a - Y_i^{a=0} = \psi_1 a + \psi_2 a V_i$

Because the structural model has two parameters, ψ_1 and ψ_2, we also need to include two parameters in the IP Weighted logistic model for $Pr[A = 1 \mid H(\psi^\dagger), L]$:

$logit Pr[A = 1 \mid H(\psi^\dagger), L] = \alpha_0 + \alpha_1 H(\psi^\dagger) + \alpha_2 H(\psi^\dagger) V + \alpha_3 L$

And find the combination of $\psi_1^\dagger$ and $\psi_2^\dagger$ that gives $H(\psi^\dagger)$ that makes both α_1 and α_2 equal to zero.

For the search must be conducted over a two-dimensional space, this grid search is computationally demanding. Less computationally intensive approaches, known as directed search methods, for approximate searching are available in statistical software.