G-estimation

Suppose conditional exchangeability holds and then the outcome distribution in the treated and the untreated would be the same if both groups had received the same treatment level, namely $A \perp\!\!\!\perp Y^a \mid L$. Here, let Y is the outcome, A is the treatment, and L is the set of all measured covariates.

Then, $Pr[A = 1 \mid Y^a, L] = Pr[A = 1 \mid L]$

Now, suppose we propose the following parametric logistic model for the probability of treatment:

$logitPr[A = 1 \mid Y^{a=0}, L] = \alpha_0 + \alpha_1 Y^{a=0} + \alpha_2 L$ where α_2 is a vector of parameters, one for each component of L.

If L has P components, $\alpha_2 L = \sum_{j=1}^{p} \alpha_{2j} L_j$

Structural nested mean models

Suppose we are interested in estimating the average causal effect of treatment A within levels of L, that is, $E[Y^{a=1} \mid L] - E[Y^{a=0} \mid L]$

The equation is the same as $E[Y^{a=1} - Y^{a=0} \mid L]$ because the difference of the means is equal to the mean of the differences.

If there is no effect measure modification by L, these differences would be constant across strata:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1$

And the structural model for the conditional causal effect would be:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1 a$

More generally, there may be effect modification by L and the structural model would be:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1 a + \beta_2 aL$ which is referred to as a *structural nested mean model*.

The parameters, β_1 and β_2 (a vector), are estimated by g-estimation and they quantify the average causal effect of treatment A on Y within levels of A and L.

Rank Preservation

Suppose we rank every individual according to $Y^{a=1}$ and according to $Y^{a=0}$ as well. If those individuals in two lists are ordered identically, there is *rank preservation*. For example, if treatment A gives the same effect on the outcome Y to everyone, then the ranking of those individuals according to $Y^{a=0}$ would be equal to that of those individuals according to $Y^{a=1}$

The conditioanl additive rank preservation holds if the effect of treatment A on the outcome Y is exactly the same for all individuals with the same values of L.

An example of an (additive conditional) rank-preserving structural model is:

$Y_i^a - Y_i^{a=0} = \psi_1 a + \psi_2 a L_i$ for all individuals i

where $\psi_1 + \psi_2 l$ s the constant causal effect for all individuals with covariate values L = l.

Although rank preservation is implausible, it is introduced to introduce g-estimation and for easier understanding of it.

G-estimation

Suppose we want to estimate the parameters of structural nested mean model $E[Y^a - Y^{a=0} \mid A = a, L] = \beta_1 a$

Also assume that the additive rank-preserving model is $Y_i^a - Y_i^{a=0} = \psi_1 a$ for all individuals i.

Then the individual causal effect ψ_1 is equal to the average causal effect β_1.

The rank-preserving model can be also written as:

$Y^{a=0} = Y^a - \psi_{1a}$

If the model were correct and the value of ψ_1 were known, then it would be possible to compute the counterfactual outcome under no treatment $Y^{a=0}$ for each individual in the study population. The challenge is to estimate ψ_1

Let us notate the possible values of ψ_1 as $\psi_1^\dagger$ and define function:

$H(\psi^\dagger) = Y - \psi^\dagger A$

Among possible values of $\psi^\dagger$, only one of them is the true ψ and its corresponding candidate $H(\psi^\dagger)$ is the counterfactual outcome $Y^{a=0}$.

In order to find the right candidate for ψ among possible $\psi^\dagger$, we fit separate logistic models for every candidate $H(\psi^\dagger)$:

$logit Pr[A = 1 \mid H(\psi^\dagger), L] = \alpha_0 + \alpha_1 H(\psi^\dagger) + \alpha_2 L$

The trick is to use conditional exchangeability, $A \perp\!\!\!\perp Y^a \mid L$.

Namely, we find the $H(\psi^\dagger)$ with $\alpha_1 = 0$ in its logistic model and that $H(\psi^\dagger)$ is the counterfactual $Y^{a=0}$

The example for the possible range of $\psi^\dagger$ is from -20 to 20 and test each value of $\psi^\dagger$ by increments of 0.01.

Structural nested models with effect modification

Suppose we have effect modification by some components V of L.

Then, the structural nested mean model is:

$E[Y^{a=1} - Y^{a=0} \mid L] = \beta_1 a + \beta_2 aV$

The corresponding rank-preserving model is:

$Y_i^a - Y_i^{a=0} = \psi_1 a + \psi_2 a V_i$

Because the structural model has two parameters, ψ_1 and ψ_2, we also need to include two parameters in the IP Weighted logistic model for $Pr[A = 1 \mid H(\psi^\dagger), L]$:

$logit Pr[A = 1 \mid H(\psi^\dagger), L] = \alpha_0 + \alpha_1 H(\psi^\dagger) + \alpha_2 H(\psi^\dagger)V + \alpha_3 L$

And find the combination of $\psi_1^\dagger$ and $\psi_2^\dagger$ that gives $H(\psi^\dagger)$ that makes both α_1 and α_2 equal to zero.

For the search must be conducted over a two-dimensional space, this grid search is computationally demanding. Less computationally intensive approaches, known as directed search methods, for approximate searching are available in statistical software. It is for linear mean models and the estimate can be directly calculated using a formula:

$\sum_{i=1}^n [C_i = 0] W_i^C H_i(\psi^\dagger)(A_i - E[A \mid L_i]) = 0$

The solution to this equation has a closed form and hence can be calculated directly by using the fact $H_i(\psi^\dagger) = Y_i - \psi^\dagger A_i$:

$\sum_{i=1}^n [C_i = 0] W_i^C (A_i - E[A \mid L_i])(Y_i - \psi^\dagger A_i) = 0 \ \Leftrightarrow \ \hat{\psi}_1 = \frac{\sum_{i=1}^n [C_i=0] W_i^C Y_i (A_i - E[A|L_i])}{\sum_{i=1}^n [C_i=0] W_i^C A_i (A_i - E[A|L_i])}$

G-estimation for time-varying treatments