

Backdoor Criterion

Definition. (The Backdoor Criterion)

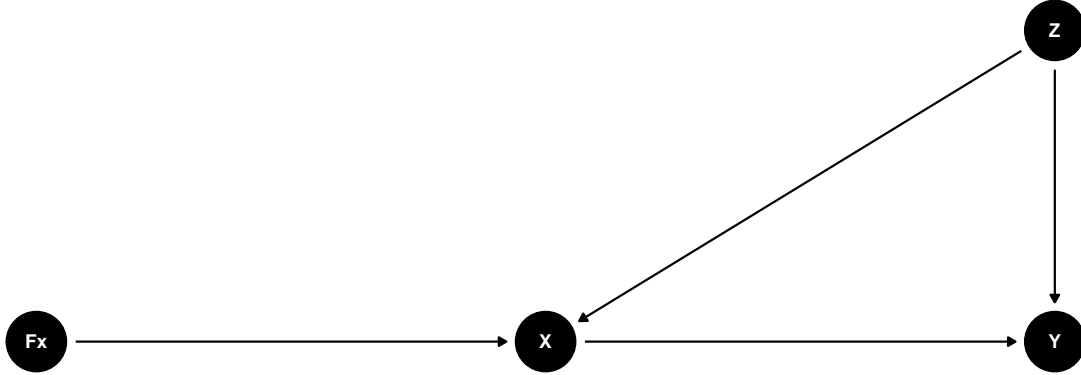
Given an ordered pair of variables (X, Y) in a directed acyclic graph G , a set of variables Z satisfies the backdoor criterion relative to (X, Y) if no node in Z is a descendant of X , and Z blocks every path between X and Y that contains an arrow into X .

Causal Effect.

If a set of variables Z satisfies the backdoor criterion for X and Y , then the causal effect of X on Y is given by the formula:

$$P(Y = y \mid do(X = x)) = \sum_z P(Y = y \mid X = x, Z = z)P(Z = z)$$

Here, I introduce regime indicator (F_X) for proof and an example of graphical model to help understanding



where probability distribution $P(x_i \mid pa(x_i, G'))$ is

$$P(x_i \mid pa(x_i, G')) = \begin{cases} P(x_i \mid pa(x_i, G)) & \text{if } F_i = \text{idle} \\ 1 & \text{if } F_i = do(x'_i) \text{ and } x_i = x'_i \\ 0 & \text{if } F_i = do(x'_i) \text{ and } x_i \neq x'_i \end{cases}$$

Proof for $P(y \mid do(x)) = \sum_z P(y \mid z, x)P(z)$:

$$\begin{aligned}
& P(y \mid do(x)) \\
&= P(y \mid F_x = do(x)) \\
&= \sum_z P(y, z \mid F_x = do(x)) \\
&= \sum_z P(y \mid z, F_x = do(x))P(z \mid F_x = do(x)) \\
&= \sum_z P(y \mid z, x, F_x = do(x))P(z \mid F_x = do(x)) \\
&=^1 \sum_z P(y \mid z, x, F_x = do(x))P(z) \\
&=^2 \sum_z P(y \mid z, x)P(z)
\end{aligned}$$

- (1) By Parental Markov condition (local Markov condition): A necessary and sufficient condition for a probability distribution P to be Markov relative a DAG G is that every variable be independent of all its nondescendants (in G), conditional on its parents.

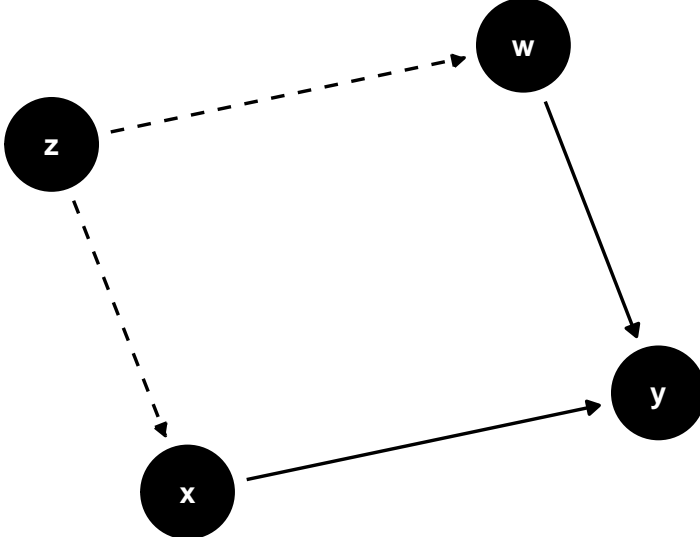
By this theorem, $F_x \perp\!\!\!\perp Z$

- (2) F_x and Y are d-separated, conditional on Z and $X \Rightarrow F_x \perp\!\!\!\perp Y \mid Z, X$

PA(x) always satisfies backdoor criterion:

$$P(y \mid pa(x)) = \sum_z P(y \mid pa(x), x)P(pa(x))$$

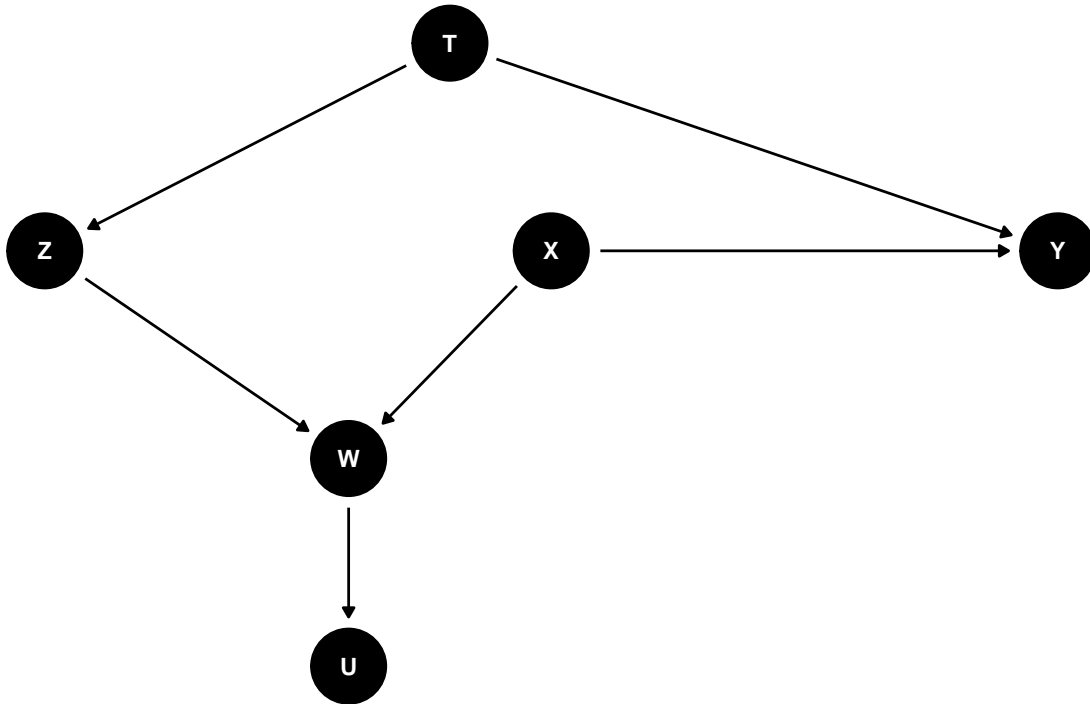
Backdoor Criterion Example.1:



Here, we want to estimate the effect of smoking cigarette (x) on lung function (y). Weight, w , is also measured and we also know the type of jobs affect both weight and the choice to smoke cigarette. However, the study did not record the type of jobs. Though, z is not recorded, we can estimate $P(Y = y \mid do(X = x))$ using the backdoor criterion. Here, w is not a descendant of x and also blocks the backdoor path $x \leftarrow z \rightarrow w \rightarrow y$.

Thus, $P(Y = y \mid do(X = x)) = \sum_w P(Y = y \mid X = x, W = w)P(W = w)$

Backdoor Criterion Example.2:



Here, we want to estimate the effect of X on Y for a specific value w of W . Though W is a collider and conditioning on W opens a path, T (not a descendant of X) can be used to block the spurious path $X \rightarrow W \leftarrow Z \leftrightarrow T \rightarrow Y$.

Thus, $P(Y = y \mid do(X = x), W = w) = \sum_t P(Y = y \mid X = x, W = w, T = t)P(W = w, T = t)$