

# Backdoor Criterion

## Definition (The Backdoor Criterion)

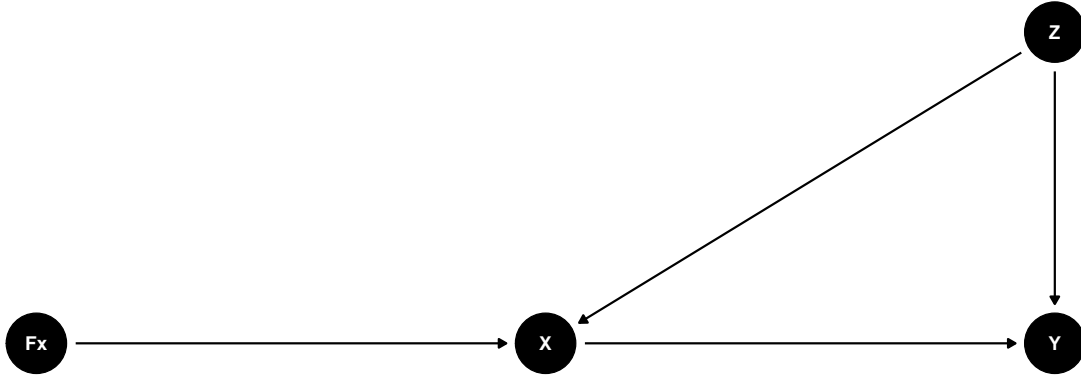
Given an ordered pair of variables  $(X, Y)$  in a directed acyclic graph  $G$ , a set of variables  $Z$  satisfies the backdoor criterion relative to  $(X, Y)$  if no node in  $Z$  is a descendant of  $X$ , and  $Z$  blocks every path between  $X$  and  $Y$  that contains an arrow into  $X$ .

## Causal Effect

If a set of variables  $Z$  satisfies the backdoor criterion for  $X$  and  $Y$ , then the causal effect of  $X$  on  $Y$  is given by the formula:

$$P(Y = y \mid do(X = x)) = \sum_z P(Y = y \mid X = x, Z = z)P(Z = z)$$

Introducing regime indicator ( $F_X$ ) which takes values in  $\{do(x'), \text{idle (no intervention)}\}$  for proof and an example of graphical model,  $G'$ . Note that a graphical model without  $F_X$  is  $G$  and  $G' = G \cup \{F_i \rightarrow X_i\}$ .



The conditional probability,  $P(x_i \mid pa(x_i, G'))$ , introduces the new parent set of  $X_i$ ,  $PA'_i = PA_i \cup \{F_i\}$

$$P(x_i \mid pa(x_i, G')) = \begin{cases} P(x_i \mid pa(x_i, G)) & \text{if } F_i = \text{idle} \\ 1 & \text{if } F_i = do(x'_i) \text{ and } x_i = x'_i \\ 0 & \text{if } F_i = do(x'_i) \text{ and } x_i \neq x'_i \end{cases}$$

Now, we are ready for proof!

**Proof for  $P(y \mid do(x)) = \sum_z P(y \mid z, x)P(z)$ :**

$$\begin{aligned}
& P(y \mid do(x)) \\
&= P(y \mid F_x = do(x)) \\
&= \sum_z P(y, z \mid F_x = do(x)) \\
&= \sum_z P(y \mid z, F_x = do(x))P(z \mid F_x = do(x)) \\
&= \sum_z P(y \mid z, x, F_x = do(x))P(z \mid F_x = do(x)) \\
&=^1 \sum_z P(y \mid z, x, F_x = do(x))P(z) \\
&=^2 \sum_z P(y \mid z, x)P(z)
\end{aligned}$$

- (1) Parental Markov condition (local Markov condition): A necessary and sufficient condition for a probability distribution  $P$  to be Markov relative a DAG  $G$  is that every variable be independent of all its nondescendants (in  $G$ ), conditional on its parents.

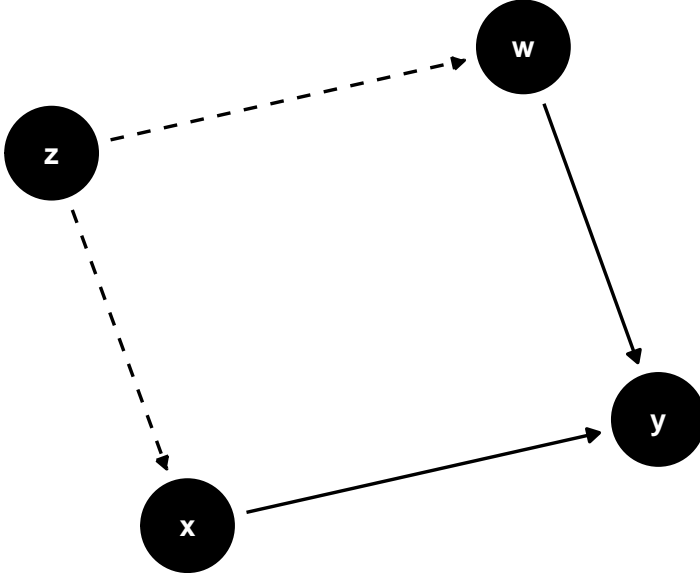
By this theorem,  $F_x \perp\!\!\!\perp Z$

- (2) By backdoor condition,  $F_x$  and  $Y$  are d-separated, conditional on  $Z$  and  $X \Rightarrow F_x \perp\!\!\!\perp Y \mid Z, X$

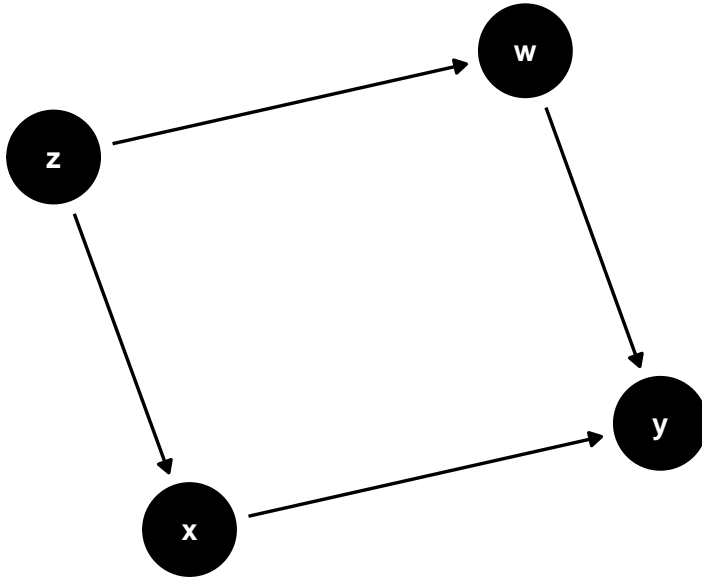
**PA(X) always satisfies backdoor criterion:**

$$P(y \mid pa(x)) = \sum_z P(y \mid pa(x), x)P(pa(x))$$

**Backdoor Criterion Example 1:**



Here, we want to estimate the effect of smoking cigarette ( $x$ ) on lung function ( $y$ ). Weight,  $w$ , is also measured and we know the type of jobs affect both weight and the choice to smoke cigarette. However, the study did not record the type of jobs. Though,  $z$  is not recorded, we can estimate  $P(Y = y \mid do(X = x))$  using the backdoor criterion. Here,  $w$  is not a descendant of  $x$  and also blocks the backdoor path  $x \leftarrow z \rightarrow w \rightarrow y$ . Thus,  $P(Y = y \mid do(X = x)) = \sum_w P(Y = y \mid X = x, W = w)P(W = w)$

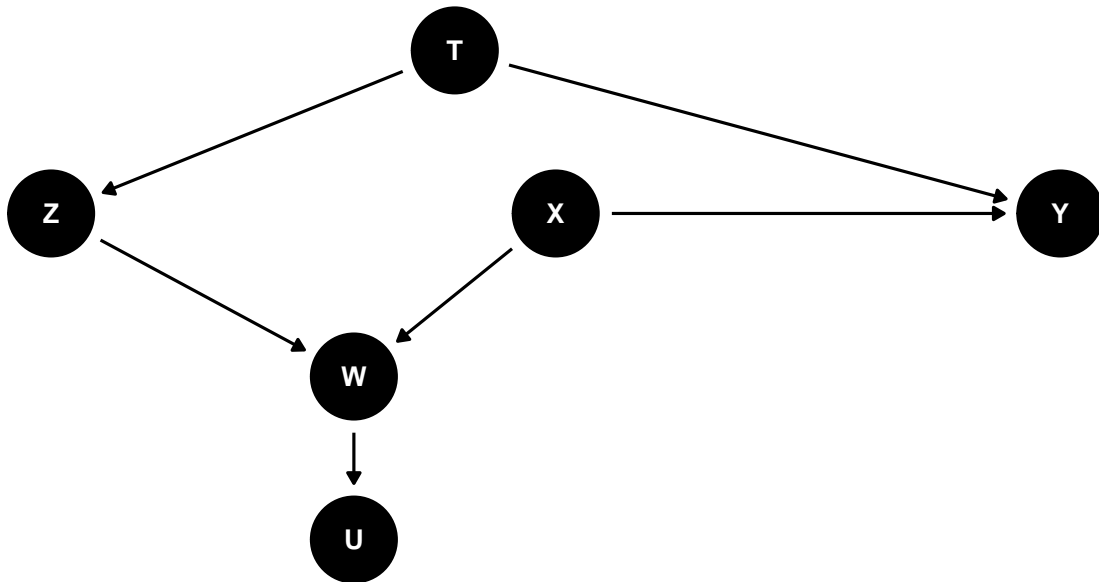


Now assume  $z$  is observed as well as  $w$ . Then, blocking the backdoor path can be done by adjusting for either  $z$  or  $w$ . This is useful because the choice can be made depending on which variable is more convenient to measure.

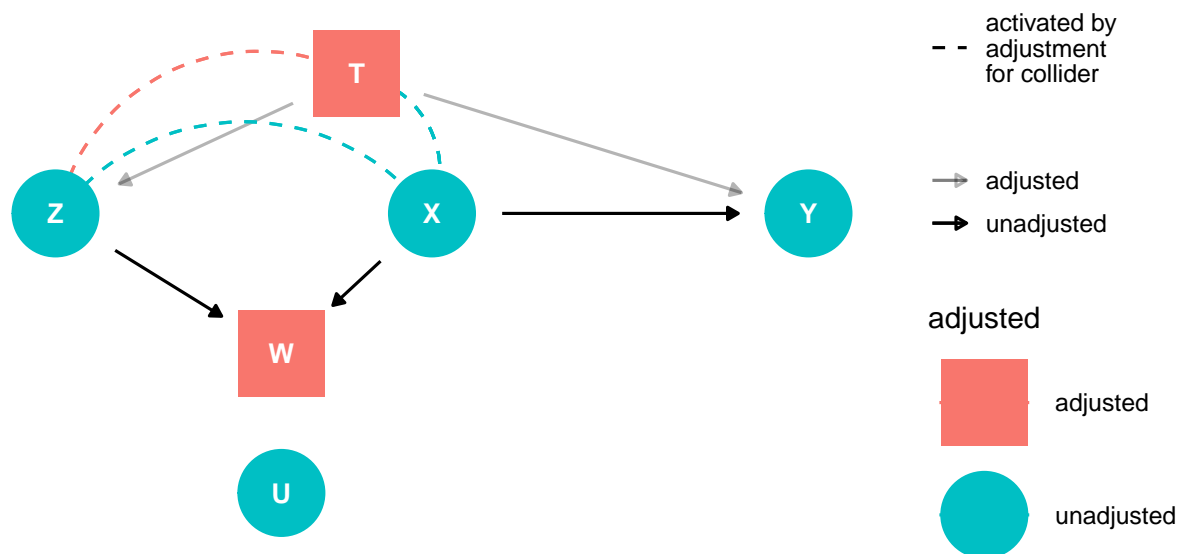
$$P(Y = y \mid do(X = x)) =$$

$$\sum_w P(Y = y \mid X = x, W = w)P(W = w) = \sum_w P(Y = y \mid X = x, Z = z)P(Z = z)$$

## Backdoor Criterion Example 2:



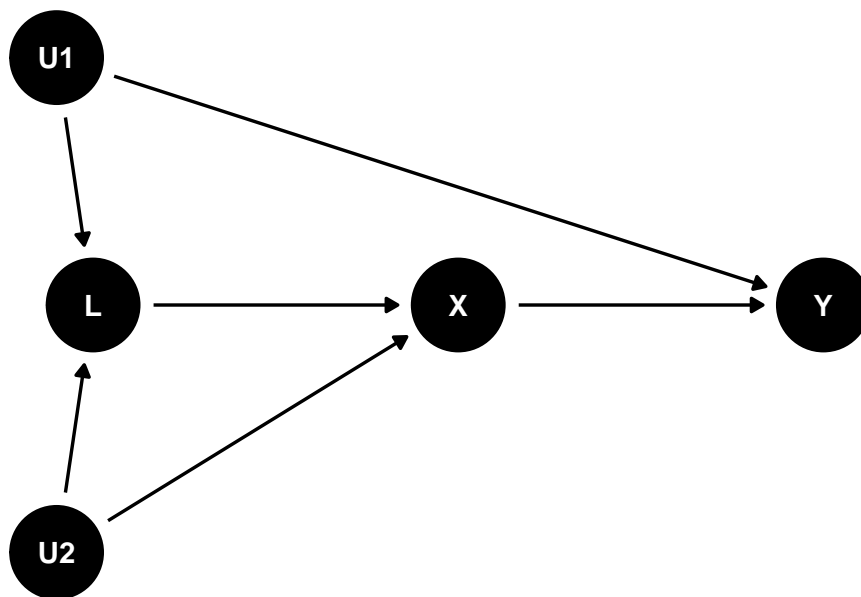
Here, we want to estimate the effect of  $X$  on  $Y$  for a specific value  $w$  of  $W$ . Though  $W$  is a collider and conditioning on  $W$  opens a path,  $T$  (not a descendant of  $X$ ) can be used to block the spurious path  $X \rightarrow W \leftarrow Z \leftrightarrow T \rightarrow Y$ .



We can see the backdoor path is blocked when adjusting for W and T.

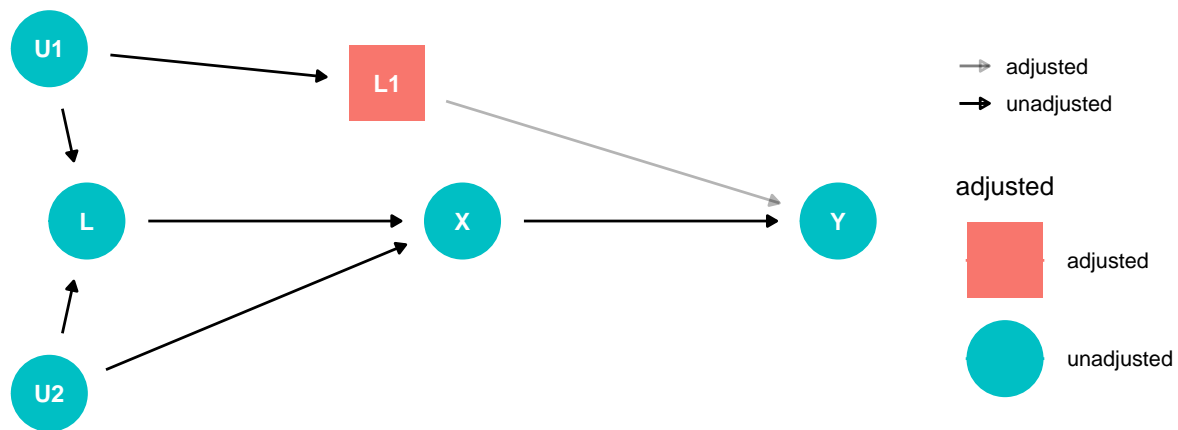
Thus,  $P(Y = y \mid do(X = x), W = w) = \sum_t P(Y = y \mid X = x, W = w, T = t)P(T = t \mid W = w)$

### Backdoor Criterion Example 3:



Here, we want to estimate the effect of X on Y. Conditioning on L would block the backdoor path,  $X \leftarrow L \leftarrow U_1 \rightarrow Y$ , but would simultaneously open a new backdoor path,  $X \rightarrow U_2 \rightarrow L \leftarrow U_1 \rightarrow Y$ . Namely, the attempt to block the confounding path brings about a selection bias. Here, A solution would be to measure either (i) a variable  $L_1$  between  $U_1$  and either A or Y, or (ii) a variable  $L_2$  between  $u_2$  and either A or L.

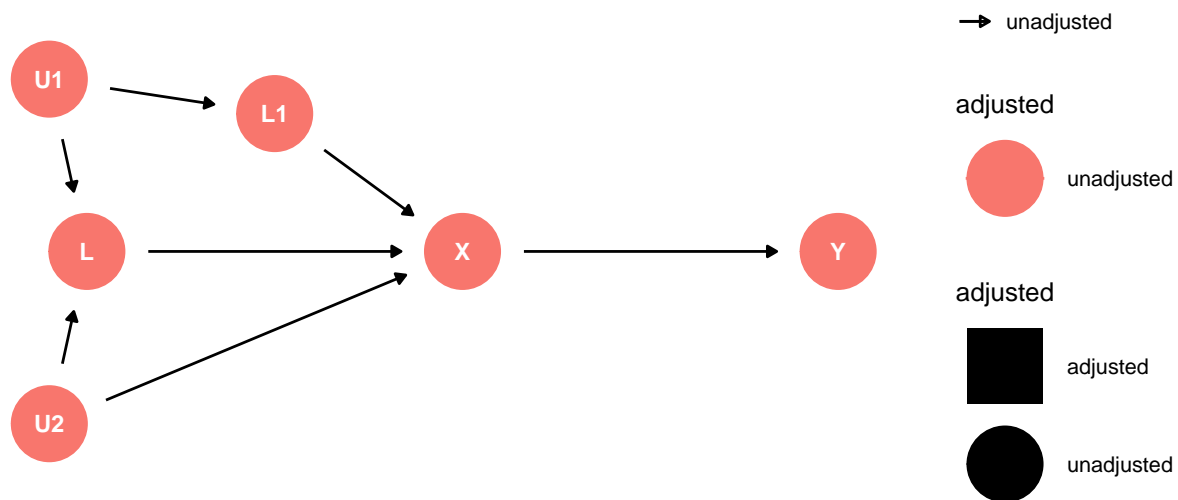
### Example 3 Solution 1 ( $L_1$ between $U_1$ and $Y$ ):



Adjusting for  $L_1$  blocks the backdoor path.

Thus,  $P(Y = y \mid do(X = x)) = \sum_{l_1} P(Y = y \mid X = x, L_1 = l_1)P(L_1 = l_1)$

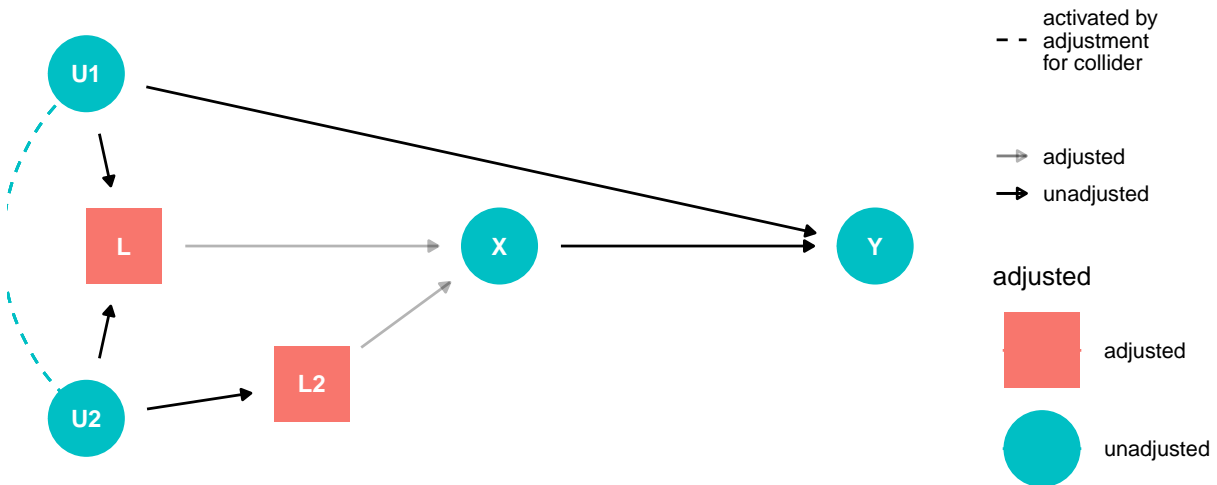
### Example 3 Solution 1 ( $L_1$ between $U_1$ and $X$ ):



There is no backdoor path by adding  $L_1$  variable.

Thus,  $P(Y = y \mid do(X = x)) = P(Y = y \mid X = x)$

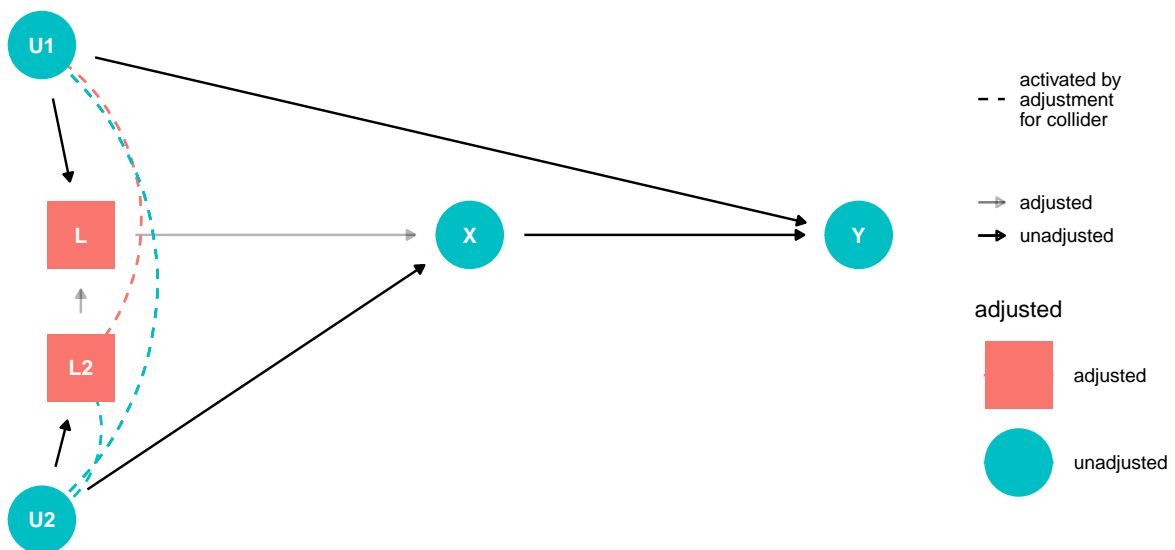
### Example 3 Solution 2 ( $L_2$ between $U_2$ and X):



Adjusting for  $L$  and  $L_2$  blocks the backdoor path.

Thus,  $P(Y = y \mid do(X = x)) = \sum_{l, l_2} P(Y = y \mid X = x, L = l, L_2 = l_2)P(L = l, L_2 = l_2)$

### Example 3 Solution 2 ( $L_2$ between $U_2$ and $L$ ):



Adjusting for  $L$  and  $L_2$  blocks the backdoor path.

Thus,  $P(Y = y \mid do(X = x)) = \sum_{l, l_2} P(Y = y \mid X = x, L = l, L_2 = l_2)P(L = l, L_2 = l_2)$

## Reference

Pearl, J., Glymour, M., & Jewell, N. P. (2016). Causal inference in statistics a primer. Wiley.

Hernán MA, Robins JM (2020). Causal Inference: What If. Boca Raton: Chapman & Hall/CRC.