Chapter 2
Graph connectivity

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Acknowledgement

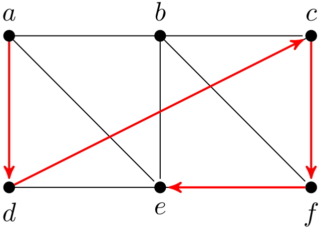
Some slides about Euler and Hamilton circuits are created by Chung Ki-hong and Hur Joon-seok from KAIST.

Course outcomes

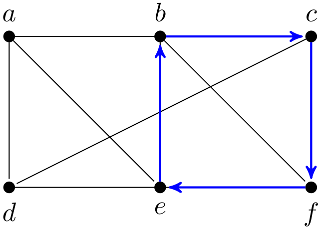
	Course learning outcomes
CLO.1	Understanding of the basic concepts of graphs
	Special types of graph, computer based graph representation, isomorphism, planar graph, connectivity in graph, graph traversal.
CLO.2	Describe definition of path and circuit
	Identify the existence of Euler path & circuit Identify the existence of Hamilton path & circuit
CLO.3	Compute minimum spanning tree in a (weighted) graph
	Use algorithms: Prim, Kruskal
CLO.4	Determine shortest path in a weighted graph
	Use algorithms: Dijkstra, Bellman-Ford, Floyd-Warshall
CLO.5	Solve maximum flow problem
	Use Ford-Fulkerson's algorithm

Contents

Paths and Circuits



Simple path of length 4

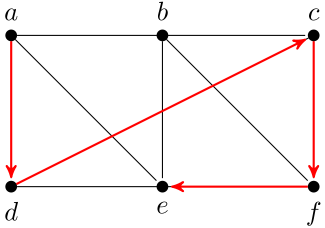


Simple circuit of length 4

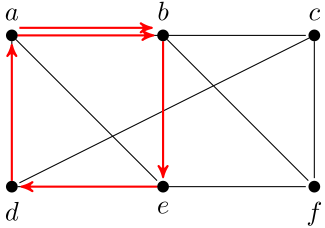
Path and Circuits

Definition (in undirected graph)

- Path (*đường đi*) of length n from u to v: a sequence of n edges $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.
- A path is a circuit (*chu trình*) if it begins and ends at the same vertex, $u = v$.
- A path or circuit is simple (*đơn*) if it does not contain the same edge more than once.



Simple path



Not simple path

Path and Circuits

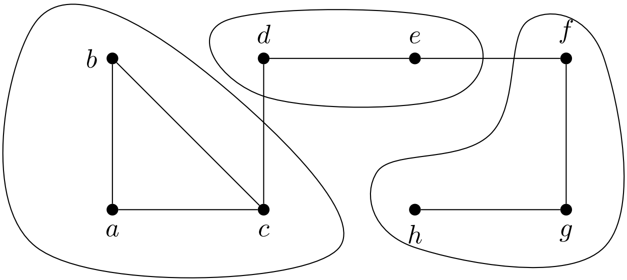
Definition (in directed graphs)

Path is a sequence of $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$, where $x_0 = u$ and $x_n = v$.

Connectedness in Undirected Graphs

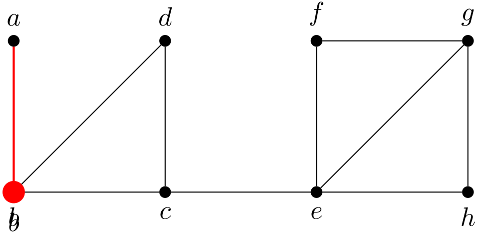
Definition

- An undirected graph is called connected (*liên thông*) if there is a path between every pair of distinct vertices of the graph.
- There is a simple path between every pair of distinct vertices of a connected undirected graph.



Connected graph
Disconnected graph
Connected components (*thành phần liên thông*)

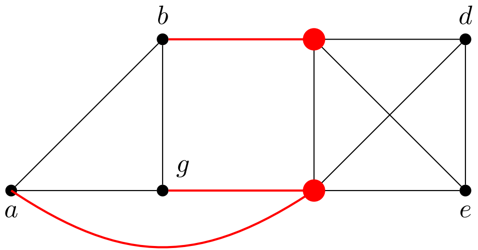
Terminology



Definition

- b is a cut vertex (*đỉnh cắt*) or articulation point (*điểm khớp*). What else?
- $\{a, b\}$ is a cut edge (*cạnh cắt*) or bridge (*cầu*). What else?

How is a graph connected?



Definition

- If a graph does not have cut vertices, it is called nonseparable graph (*đồ thị không thể phân tách*).
- The minimum number of vertices in a vertex cut defines the vertex connectivity (*liên thông đỉnh*), e.g., the vertex cut $\{c, f\}$ results in the vertex connectivity $\kappa(G) = 2$.
- The minimum number of edges in an edge cut defines the edge connectivity (*liên thông cạnh*), e.g., the edge cut $\{\{b, c\}, \{a, f\}, \{f, g\}\}$ results in the edge connectivity $\lambda(G) = 3$.

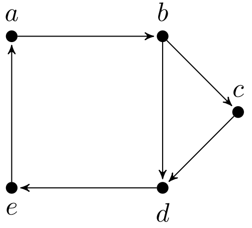
Applications of Vertex and Edge Connectivity

- Reliability of networks
  - Minimum number of routers that disconnect the network
  - Minimum number of fiber optic links that can be down to disconnect the network
- Highway network
  - Minimum number of intersections that can be closed
  - Minimum number of roads that can be closed

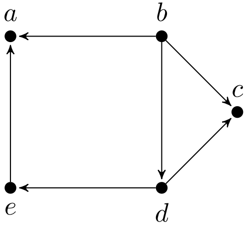
Connectedness in Directed Graphs

Definition

- An directed graph is strongly connected (*liên thông mạnh*) if there is a path between any two vertices in the graph (for both directions).
- An directed graph is weakly connected (*liên thông yếu*) if there is a path between any two vertices in the underlying undirected graph.



Strongly connected

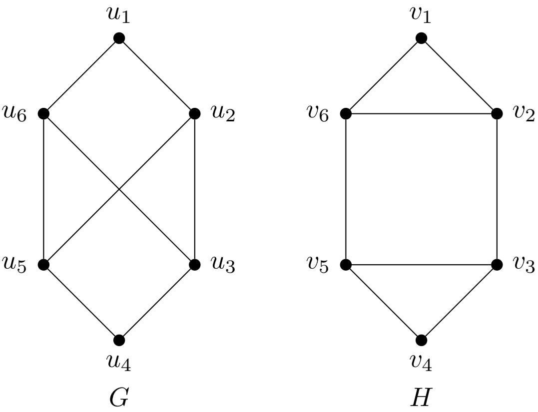


Weakly connected

Applications

Example

Determine whether the graphs below are isomorphic.



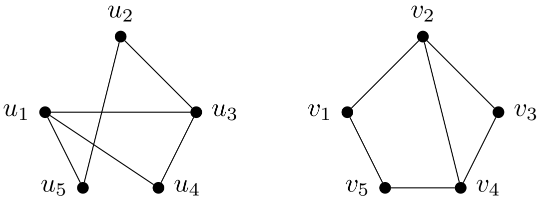
Solution

H has a simple circuit of length three, not G.

Applications

Example

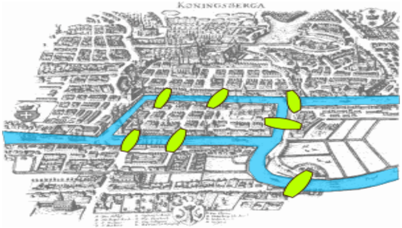
Determine whether the graphs below are isomorphic.



Solution

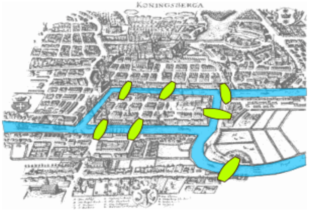
Both graphs have the same vertices, edges, degrees, circuits. They may be isomorphic.
To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degrees.

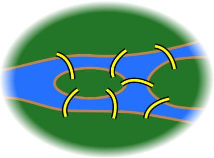
The Famous Problem of Seven Bridges of Königsberg

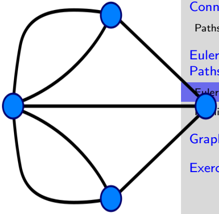


- Is there a route that a person crosses all the seven bridges once?

Euler Solution





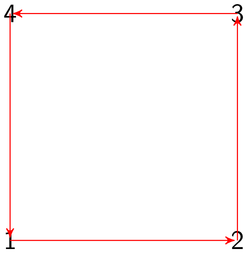


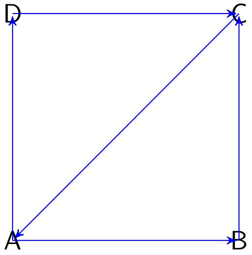
- Euler gave the solution: It is not possible to cross all the bridges exactly once.

What is Euler Path and Circuit?

- **Euler Path** (*đường đi Euler*) is a path in the graph that passes each edge only once.
  The problem of Seven Bridges of Königsberg can be also stated: Does Euler Path exist in the graph?
- **Euler Circuit** (*chu trình Euler*) is a path in the graph that passes each edge only once and return back to its original position.
  From Definition, Euler Circuit is a subset of Euler Path.

Examples of Euler Path and Circuit





Euler Circuit

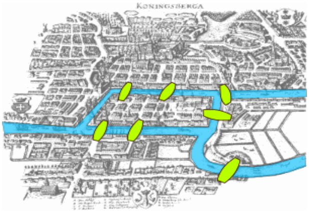
Euler Path

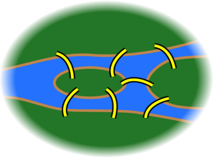
Conditions for Existence

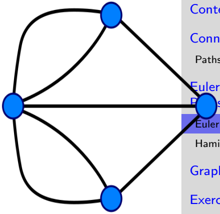
In a connected multigraph,

- Euler Circuit existence: **no odd-degree nodes exist** in the graph.
- Euler Path existence: **2 or no odd-degree nodes exist** in the graph.

Back to the Seven Bridges Problem







- Four vertices of odd degree
- No Euler circuit $\rightarrow$ cannot cross each bridge exactly once, and return to starting point
- No Euler path, either

Searching Euler Circuits and Paths – Fleury's Algorithm

- Choose a random vertex (if circuit) or an odd degree vertex (if path)
- Pick an edge joined to another vertex so that it is not a cut edge unless there is no alternative
- Remove the chosen edge. The above procedure is repeated until all edges are covered.

Searching Euler Circuits and Paths – Hierholzer's Algorithm

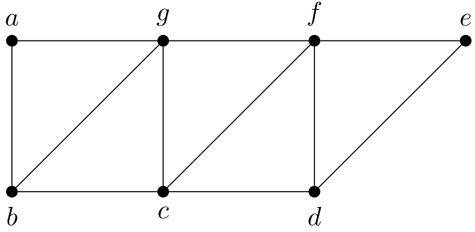
- Choose a starting vertex and find a circuit
- As long as there exists a vertex v that belongs to the current tour but that has adjacent edges not part of the tour, start another circuit from v

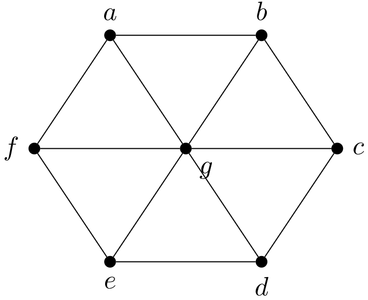
More efficient algorithm, $O(n)$.

Exercise

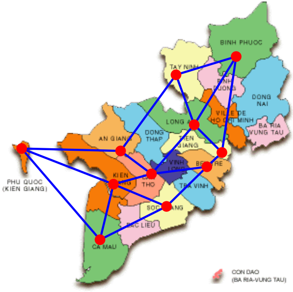
Example

Are these following graph Euler path (circuit)? If yes, find one.





Traveling Salesman Problem

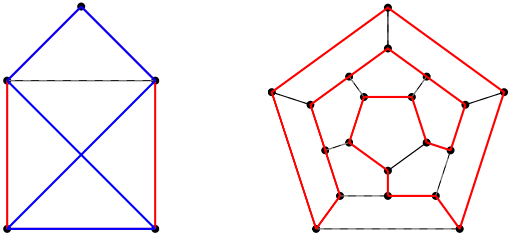


Is there the possible tour that visits each city exactly once?

What Is A Hamilton Circuit?

Definition

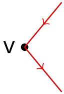
The circuit that visit each vertex in a graph once



Rules of Hamilton Circuits

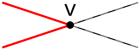
$\deg(v) = 2$ for $\forall v$ in Hamilton circuit!

Rule 1 if $\deg(v) = 2$, both edge must be used.

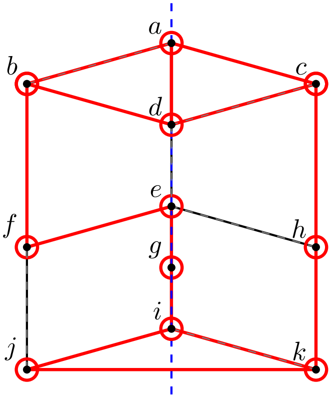


Rule 2 No subcircuit (*chu trình con*) can be formed.

Rule 3 Once two edges at a vertex v is determined, all other edges incident at v must be removed.



Finding Hamilton Circuits



We get Hamilton circuit!

Vertices : cities

Edges : possible routes

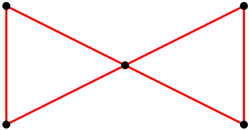
Rule 1
$\deg(v) = 2$

Rule 3
Once two edges are determined, other edges must be removed

Existence of Hamilton Circuit

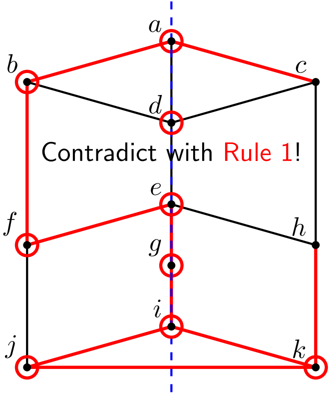
Hamilton circuit does not exist for all graph. But, there is no specific way to find whether Hamilton circuit exists or not.

Simple check by rules of Hamilton circuit



Violates Rule 2! (No subcircuit)

We can verify nonexistence of the graph during find Hamilton circuit.



Hamilton circuit doesn't exist!

Application – Gray Code

Definition

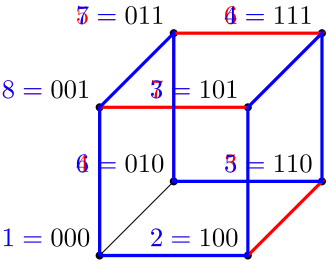
The binary sequence that express consecutive numbers by differing just one position of sequence.

Decimal number		Binary number	Gray code
1	=	001	000
2	=	010	100
3	=	011	110
4	=	100	010
5	=	101	011
$\vdots$		$\vdots$	$\vdots$

Used at digital communication for reduce the effect of noise; it prevents serious changes of information by noise.

Gray Code

n-digit gray code can be generated by finding Hamilton circuits of n-dimensional hypercube! Consider the case $n = 3$.

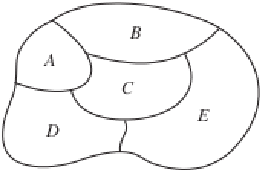


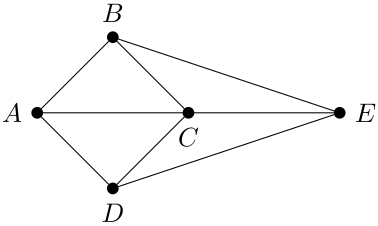
Coordinate of each vertex is 3-digit binary sequences. Coordinates of adjacent vertices differ in just on place. Hamilton circuits of a cubic graph makes the order of binary sequences!

Maps and Graphs

Definition

- Every map can be represented by a graph. We call it dual graph.
- Problem of coloring the regions of a map $\rightarrow$ coloring the vertices of the dual graph so that no two adjacent vertices have the same color.

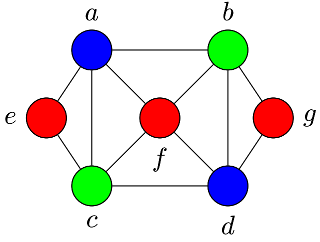


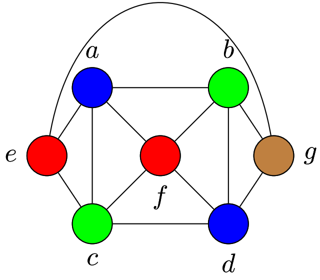


Graph coloring

Definition

- A coloring (*tô màu*) of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- The chromatic number (*số màu*) of a graph, denoted by $\chi(G)$, is the least number of colors needed for a coloring of this graph.





Four color theorem

Theorem (Four color theorem)

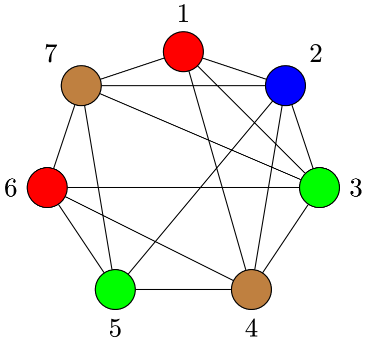
The chromatic number of a planar graph is no greater than four.

- Was a conjecture in the 1850s
- Was not proved completely until 1976 by Kenneth Appel and Wolfgang Haken, using computer
- No proof not relying on a computer has yet been found

Applications of Graph coloring

Scheduling Final Exam

- How can the final exams at a university be scheduled so that no student has two exams at the same time?
- Suppose we have 7 finals, numbered 1 through 7.
- The pairs of courses have common students are depicted in the following graph



Applications of Graph Coloring

Other Applications

- Frequency Assignments: Television channels 2 through 12 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?
- Index Registers: In an execution of loop, the frequently used variables should be stored in index registers to speed up. How many index registers are needed?

Exercise

Determine a shortest path from a to other vertices in the following graph.