Chapter 1
Introduction to Graph

Graph theory on August 8, 2023

Vo Dang Khoa, Huynh Tuong Nguyen
Faculty of Information Technology
Industrial University of Ho Chi Minh City
{khoavo;htnguyen}@iuh.edu.vn

Contents

Course outcomes

	Course learning outcomes
CLO.1	Understanding of the basic concepts of graphs
	Special types of graph, computer based graph representation, isomorphism, planar graph, connectivity in graph, graph traversal.
CLO.2	Describe definition of path and circuit
	Identify the existence of Euler path & circuit Identify the existence of Hamilton path & circuit
CLO.3	Compute minimum spanning tree in a (weighted) graph
	Use algorithms: Prim, Kruskal
CLO.4	Determine shortest path in a weighted graph
	Use algorithms: Dijkstra, Bellman-Ford, Floyd-Warshall
CLO.5	Solve maximum flow problem
	Use Ford-Fulkerson's algorithm

INDUSTRIAL UNIVERSITY OF HOCHIMINH CITY

Contents

Motivations

The need of the graph

- Representation/Storing
- Searching/sorting
- Optimization

Its applications

- Electric circuit/board
- Chemical structure
- Networking
- Map, geometry, ...

- Graph theory is useful for analysing "things that are connected to other things".
- Some difficult problems become easy when represented using a graph.

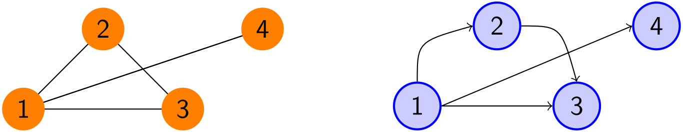
Graph

Definition

A graph (*đồ thị*) G is a pair of (V, E), which are:

- V – nonempty set of vertices (nodes) (*đỉnh*)
- E – set of edges (*cạnh*)

A graph captures abstract relationships between vertices.



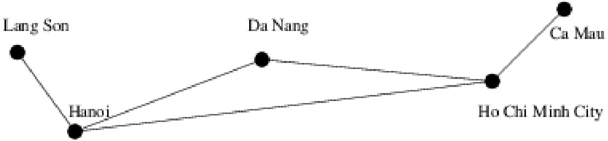
Undirected graph **Directed graph**

Undirected Graph (Đồ thị vô hướng)

Definition (Simple graph (đơn đồ thị))

- Each edge connects two different vertices, and
- No two edges connect the same pair of vertices

An edge between two vertices u and v is denoted as $\{u, v\}$

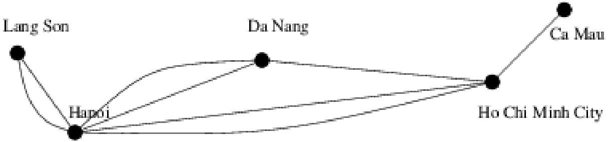


Undirected Graph

Definition (Multigraph (đa đồ thị))

Graphs that may have multiple edges connecting the same vertices.

An unordered pair of vertices $\{u, v\}$ are called multiplicity m (*bội* m) if it has m different edges between.

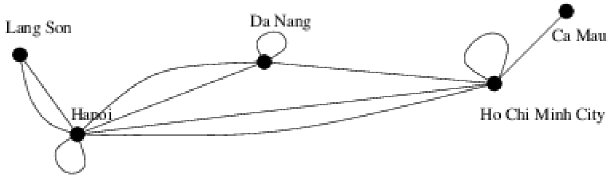


Undirected Graph

Definition (Pseudograph (giả đồ thị))

Are multigraphs that have

- loops (*khuyên*)– edges that connect a vertex to itself



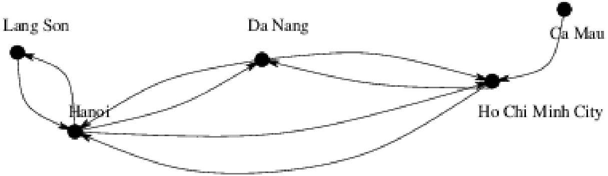
Directed Graph

Definition (Directed Graph (đồ thị có hướng))

A directed graph G is a pair of (V, E), in which:

- V – nonempty set of vertices
- E – set of directed edges (*cạnh có hướng*, arcs)

A directed edge start at u and end at v is denoted as (u, v).



INDUSTRIAL UNIVERSITY OF HOCHIMINH CITY

Terminologies For Undirected Graph

Neighborhood

In an undirected graph $G = (V, E)$,

- two vertices u and $v \in V$ are called **adjacent** (*liền kề*) if they are **end-points** (*điểm đầu mút*) of edge $e \in E$, and
- e is **incident with** (*cạnh liên thuộc*) u and v
- e is said to **connect** (*cạnh nối*) u and v;

The degree of a vertex

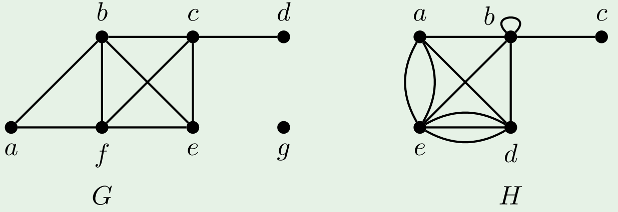
The **degree of a vertex** (*bậc của một đỉnh*), denoted by $\deg(v)$ is the number of edges incident with it, except that a loop contributes twice to the degree of that vertex.

- isolated vertex (*đỉnh cô lập*): vertex of degree 0
- pendant vertex (*đỉnh treo*): vertex of degree 1

Example

Example

What are the degrees and neighborhoods of the vertices in these graphs?



Solution

In G*,* $deg(a) = 2$*,* $deg(b) = deg(c) = deg(f) = 4$*,* $deg(d) = 1, \ldots$
Neiborhoods of these vertices are
$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, \ldots$
In H*,* $deg(a) = 4$*,* $deg(b) = deg(e) = 6$*,* $deg(c) = 1, \ldots$
Neiborhoods of these vertices are
$N(a) = \{b, d, e\}, N(b) = \{a, b, c, d, e\}, \ldots$

Basic Theorems

Theorem (The Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Theorem

An undirected graph has an even number of odd-degree vertices.

Prove that ...

...

If the number of vertices in an undirected graph is an odd number, then there exists an even-degree vertex.

...

If the number of vertices in an undirected graph is an odd number, then the number of vertices with even degree is odd.

...

If the number of vertices in an undirected graph is an even number, then the number of vertices with even degree is even.

Terminologies for Directed Graph

Neighborhood

In an directed graph $G = (V, E)$,

- u is said to be **adjacent to** (*nối tới*) v and v is said to be adjacent from (*được nối từ*) u if (u, v) is an arc of G, and
- u is called **initial vertex** (*đỉnh đầu*) of (u, v)
- v is called **terminal** (*đỉnh cuối*) or **end vertex** of (u, v)
- the initial vertex and terminal vertex of a loop are the same.

The degree of a vertex

In a graph G with directed edges:

- **in-degree** (*bậc vào*) of a vertex v, denoted by $\deg^-(v)$, is the number of arcs with v as their terminal vertex.
- **out-degree** (*bậc ra*) of a vertex v, denoted by $\deg^+(v)$, is the number of arcs with v as their initial vertex.

Note: a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

Basic Theorem

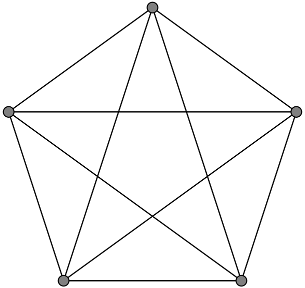
Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

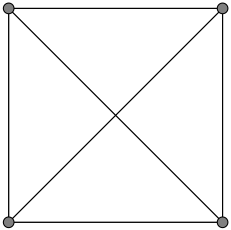
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Complete Graphs

A complete graph (*đồ thị đầy đủ*) on n vertices, K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.



K_5

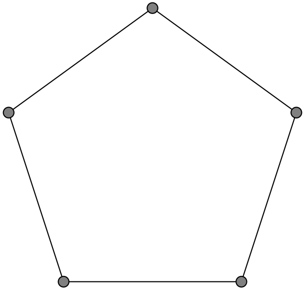


K_4

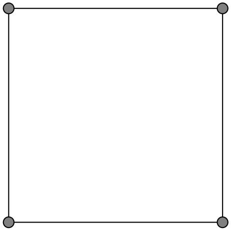
INDUSTRIAL UNIVERSITY OF HOCHIMINH CITY

Cycles

A cycle (*đồ thị vòng*) C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



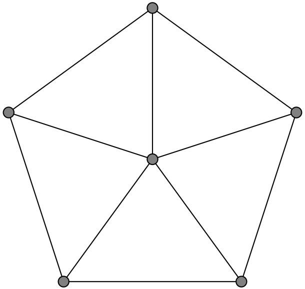
C_5



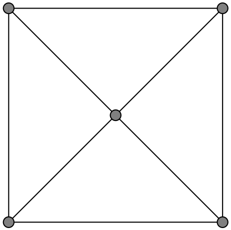
C_4

Wheels

We obtain a wheel (*đồ thị hình bánh xe*) W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n.



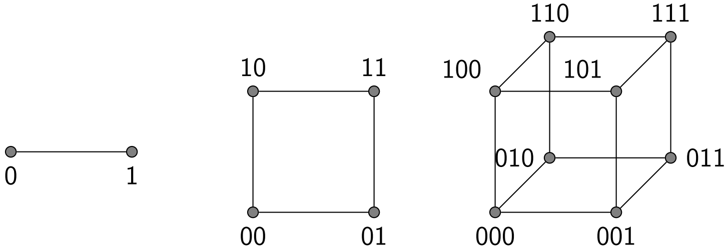
W_5



W_4

n-cube

An n-dimensional hypercube (*khối n chiều*), Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent iff the bit strings that they represent differ in exactly one bit position.



Q_1 Q_2 Q_3

What's about Q_4?

INDUSTRIAL UNIVERSITY OF HOCHIMINH CITY

Contents

Applications of Special Graphs

- Local networks topologies
  - Star, ring, hybrid
- Parallel processing
  - Linear array
  - Mesh network

Exercise

Exercise (5)

Give the number of edges in function of number of vertices in a complete graph K_n.

Exercise (6)

Give an undirected simple graph $G = (V, E)$ with $|V| = n$, show that

- a. $\forall v \in V$, $\deg(v) < n$,
- b. there does not exist simultaneously both a vertex of degree 0 and a vertex of degree $(n-1)$ with $n \geq 2$,
- c. deduce that there are at least two vertices of the same degree.

Exercise (7)

Is it possible that each person has exactly 3 friends in the same group of 9 people ?

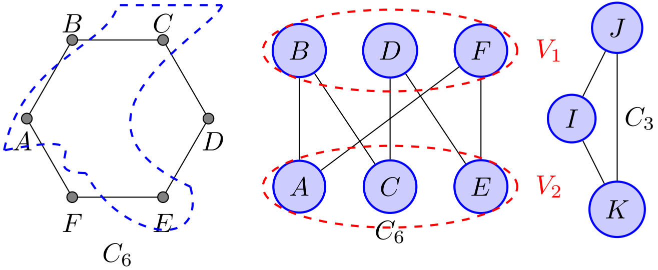
Bipartite Graphs

Definition

A simple graph G is called bipartite (*đồ thị phân đôi*) if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2)

Example

C_6 and C_3 are bipartite ?

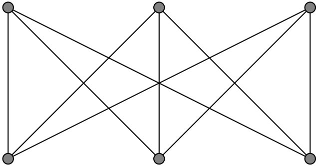


Complete Bipartite Graphs

Definition

A complete bipartite $K_{m,n}$ is a graph that

- has its vertex set partitioned into two subsets of m and n vertices, respectively,
- with an edge between two vertices iff one vertex is in the first subset and the other is in the second one

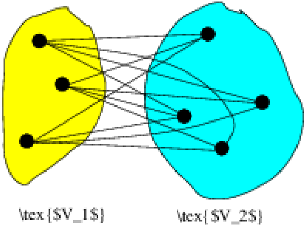


$K_{3,3}$

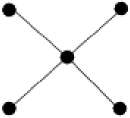
Bipartite graphs

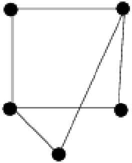
Example (Bipartite graphs?)

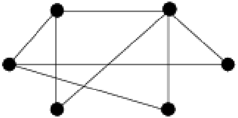
- C_6
- C_n
- K_3
- K_n
- Q_n
- the following graph

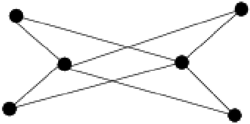


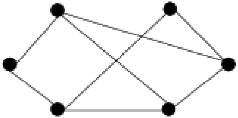
Bipartite graph











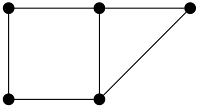
New Graph From Old

Definition

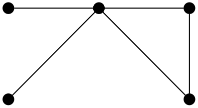
A **subgraph** (*đồ thị con*) of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.

Definition

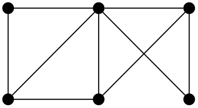
The **union** (*hợp*) of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



G_1

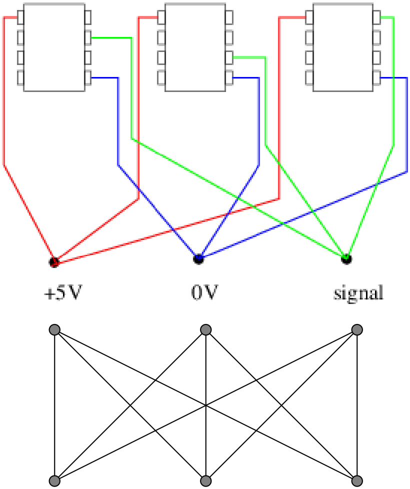


G_2



$G_1 \cup G_2$

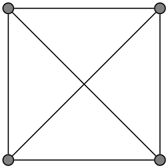
Planar Graphs



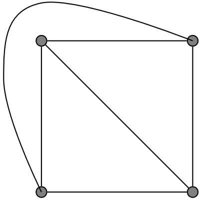
Planar Graphs

Definition

- A graph is called planar (*phẳng*) if it can be drawn in the plane without any edges crossing.
- Such a drawing is called planar representation (*biểu diễn phẳng*) of the graph.



K_4

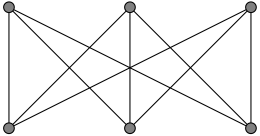


K_4 with no crossing

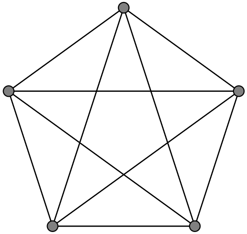
Important Corollaries

Corollary

- *If G is a* **connected planar simple graph** *with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.*
- *If G is a* **connected planar simple graph** *with e edges and v vertices where $v \geq 3$, and no circuits of length 3, then $e \leq 2v - 4$.*



$K_{3,3}$
Non-planar

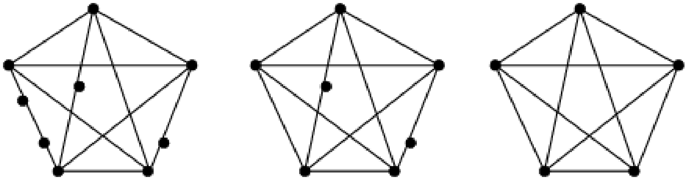


K_5
Non-planar

Elementary Subdivision

Definition

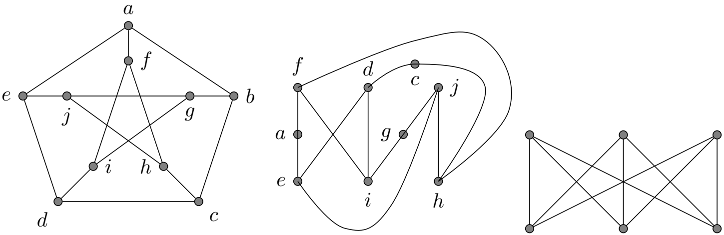
- Given a planar graph G, an elementary subdivision (*phân chia sơ cấp*) is removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic (*đồng phôi*) if they can obtained from the same graph by a sequence of elementary subdivisions.



Kuratowski's Theorem

Theorem

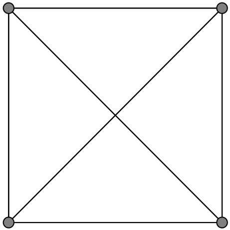
A graph is nonplanar iff it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.



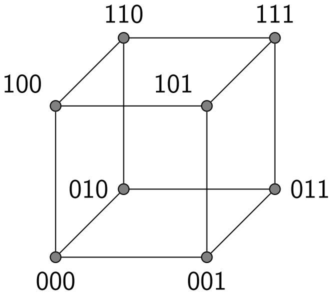
Exercise

Exercise

- Is K_4 planar?
- Is Q_3 planar?



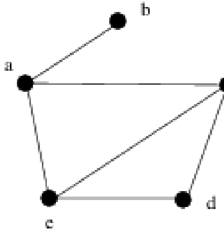
K_4



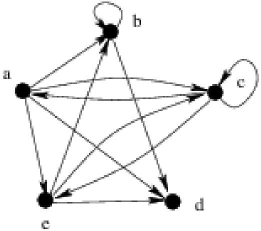
Q_3

Adjacency Lists (Danh sách kề)

Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d



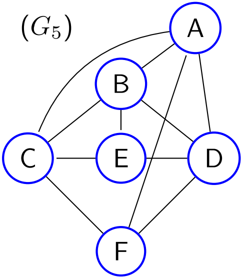
Initial vertex	Terminal vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	c, e
e	b, c, d



Examples

Example

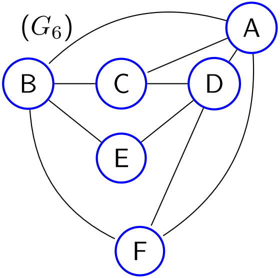
Determine adjacency list of vertices A & B from the following graph (Write the elements in the alphabetical order)



Examples

Example

Determine adjacency list of vertices D & F from the following graph (Write the elements in the alphabetical order)

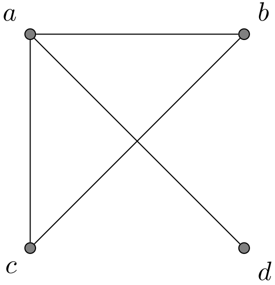


Adjacency Matrices

Definition

Adjacency matrix (*Ma trận kề*) A_G of $G = (V, E)$

- Dimension $|V| \times |V|$
- Matrix elements
  $a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$



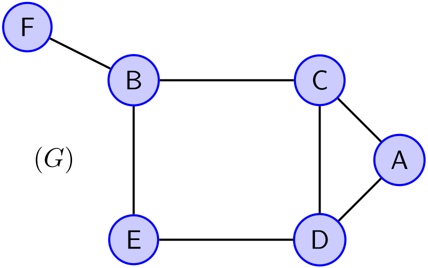
$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array}$$

INDUSTRIAL UNIVERSITY OF HOCHIMINH CITY

Examples

Example

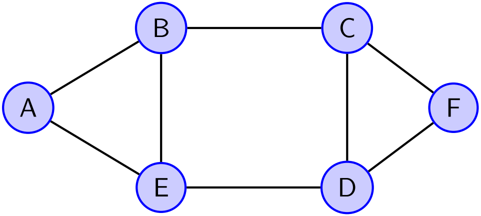
Give adjacency matrix according to the following graph



Examples

Example

Give adjacency matrix according to the following graph



(G)

Examples

Example

Give the graph defined by the following adjacency matrix

$$
\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} &
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

Examples

Example

Give the directed graph defined by the following adjacency matrix

$$
\begin{array}{c}
 \\ A \\ B \\ C \\ D \\ E
\end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

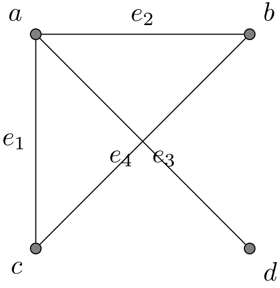
Incidence Matrices

Definition

Incidence matrix (*ma trận liên thuộc*) M_G of $G = (V, E)$

- Dimension $|V| \times |E|$
- Matrix elements

$$m_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

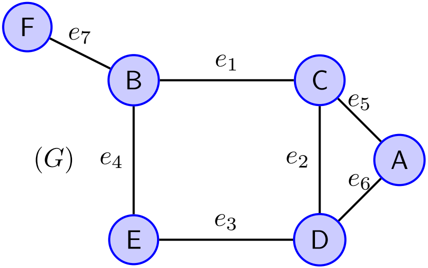


$$\begin{array}{c c} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Examples

Example

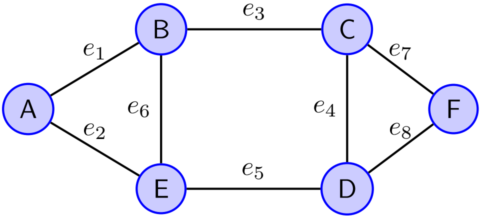
Give incidence matrix according to the following graph



Examples

Example

Give incidence matrix according to the following graph

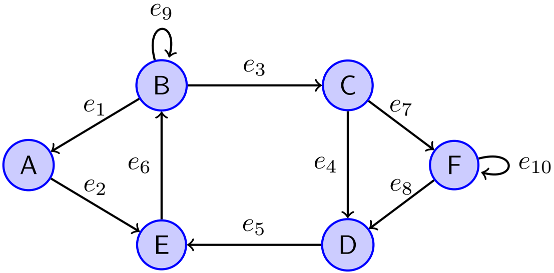


(G)

Examples

Example

Give incidence matrix according to the following graph

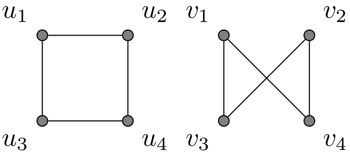


(G)

Graph Isomorphism

Definition

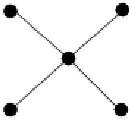
$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** (*đẳng cấu*) if there is a **one-to-one function** f from V_1 to V_2 with the property that a and b are adjacent in G_1 iif $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an **isomorphism** (*một đẳng cấu*).
(i.e. there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.)

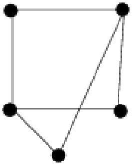


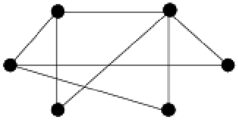
Isomorphism function $f : U \longrightarrow V$ with

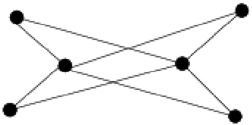
$$f(u_1) = v_1 \qquad f(u_2) = v_4$$
$$f(u_3) = v_3 \qquad f(u_4) = v_2$$

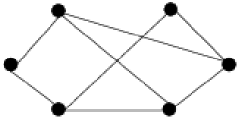
Bipartite graph



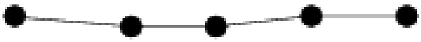


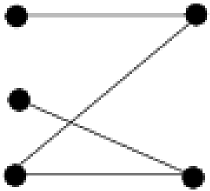


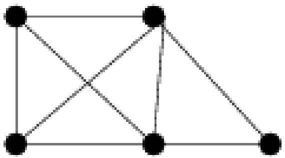


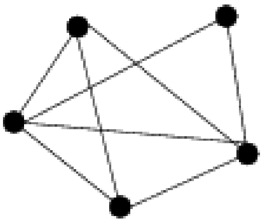


Isomorphism ?

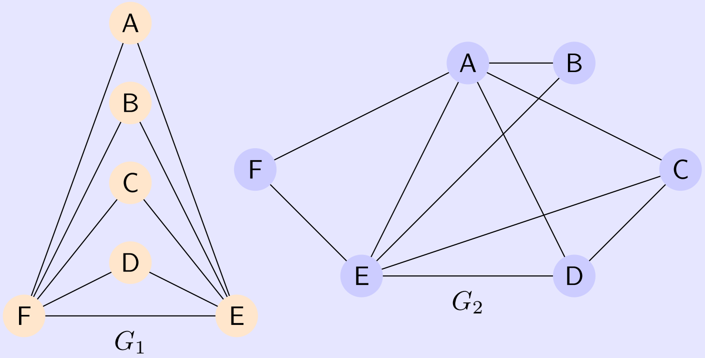




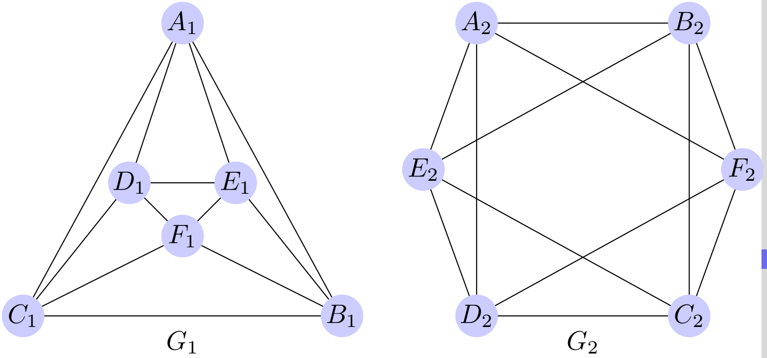




Isomorphism?



Isomorphism?



Isomorphism

Are the simple graphs with the following adjacency matrices isomorphic ?

1. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Isomorphism

Determine whether the graphs (without loops) with the incidence matrices are isomorphic.

- $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$
- Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.
- Define isomorphism of directed graphs