Chapter 3
Trees

Graph theory on September 11, 2023

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Contents

Course outcomes

	Course learning outcomes
CLO.1	Understanding of the basic concepts of graphs
	Special types of graph, computer based graph representation, isomorphism, planar graph, connectivity in graph, graph traversal.
CLO.2	Describe definition of path and circuit
	Identify the existence of Euler path & circuit Identify the existence of Hamilton path & circuit
CLO.3	Compute minimum spanning tree in a (weighted) graph
	Use algorithms: Prim, Kruskal
CLO.4	Determine shortest path in a weighted graph
	Use algorithms: Dijkstra, Bellman-Ford, Floyd-Warshall
CLO.5	Solve maximum flow problem
	Use Ford-Fulkerson's algorithm

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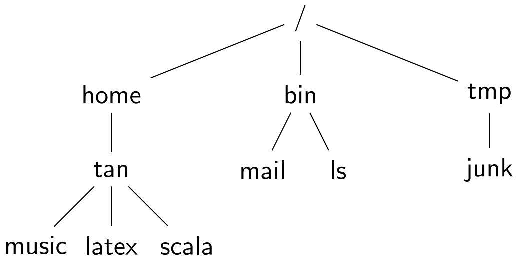
Contents

Introduction

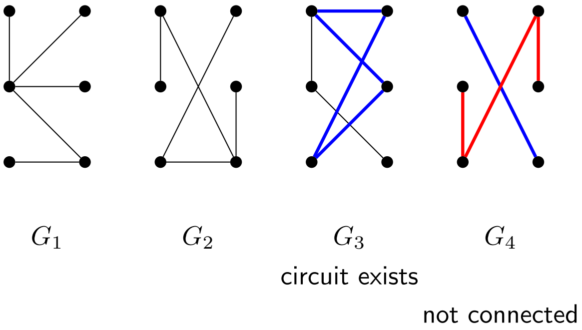
- Very useful in computer science: search algorithm, game winning strategy, decision making, sorting, ...
- Other disciplines: chemical compounds, family trees, organizational tree, ...



Tree

Definition

A tree (*cây*) is a connected undirected graph without any circuits. Consequently, a tree must be a simple graph.



Definition

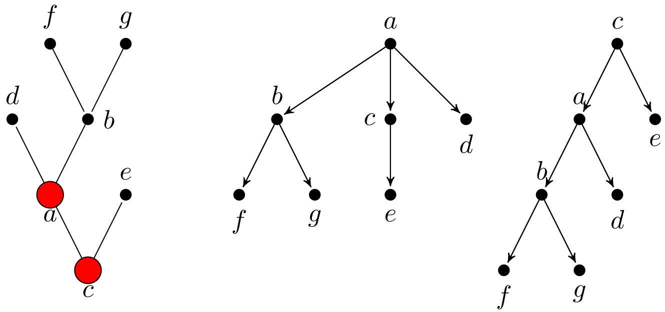
Graphs containing no circuits that are not necessarily connected is forest (*rừng*), in which each connected component is a tree.

Rooted Trees

Definition

A rooted tree (*cây có gốc*) is a tree in which:

- One vertex has been designated as the root and
- Every edge is directed away from the root



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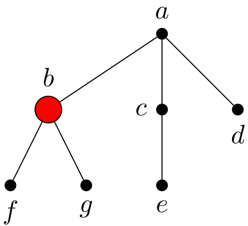
Terminology

Definition

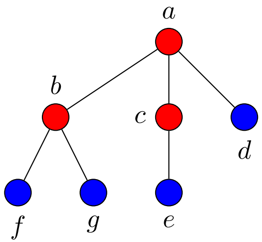
- parent (*cha*) of v is the unique u such that there is a directed edge from u to v
- when u is the parent of v, v is called a child (*con*) of u
- vertices with the same parent are called siblings (*anh em*)
- the ancestors (*tổ tiên*) of a vertex are the vertices in the path from the root to this vertex (excluding the vertex itself)
- descendants (*con cháu*) of a vertex v are those vertices that have v as an ancestor



Terminology

Definition

- a vertex of a tree is called a leaf (*lá*) if it has no children
- vertices that have children are called internal vertices (*đỉnh trong*)



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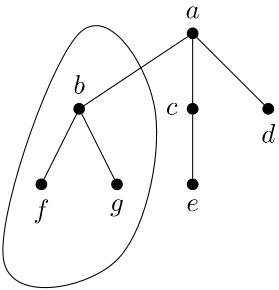
Terminology

Definition

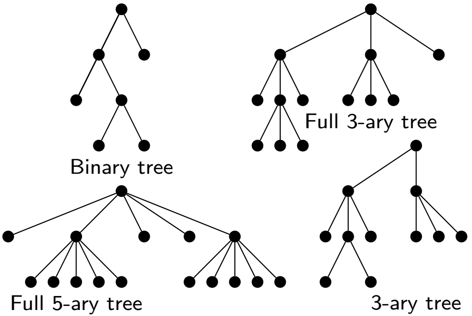
If a is a vertex in a tree, the subtree (*cây con*) with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.



m-ary tree

Definition

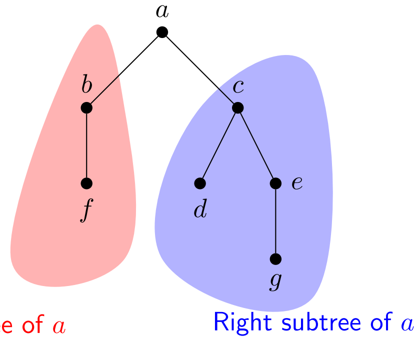
- m-ary tree (*cây m-phân*): at most m children on each internal vertex of a rooted tree.
- **full** m-ary tree (*cây m-phân đầy đủ*): every internal vertex has exactly m children.
- An m-ary tree with $m = 2$ is called a binary tree (*cây nhị phân*).



Ordered Rooted Trees

Definition

- An ordered rooted tree (*cây có gốc có thứ tự*) is a rooted tree where the children of each internal vertex are ordered (e.g. in order from left to right).
- In an ordered binary tree (*cây nhị phân có thứ tự*), if an internal vertex has two children, the first child is called the left child (*con bên trái*) and the second is called the right child (*con bên phải*).



Properties & Theorems

Theorem

A tree with n vertices has $n-1$ edges.

Theorem

A full m-ary tree

1. *n vertices has $(n-1)/m$ internal vertices and $[(m-1)n+1]/m$ leaves*
2. *i internal vertices has $n = mi+1$ vertices and $(m-1)i+1$ leaves*
3. *ℓ leaves has $n = (m\ell-1)/(m-1)$ vertices and $(\ell-1)/(m-1)$ internal vertices*

Example

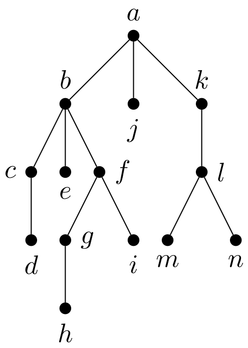
Example (Chain Letter Game)

- Each person who receives the letter is asked to send it on to four other peoples.
- Some peoples do this, but others do not send any letters.
- How many people have seen the letter, including the first person, if no one receives more than one letter and if the chain letter ends after there have been 100 people who read it but did not send it out ?
- How many people sent out the letter?

Level and Height

Definition

- The level (*mức*) of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
- The level of the root is defined to be zero.
- The height (*độ cao*) of a rooted tree is the maximum of the levels of vertices (i.e. the length of the longest path from the root to any vertex).



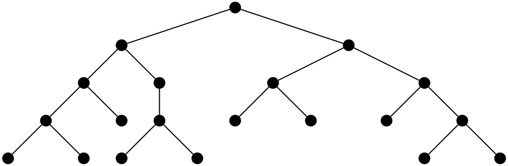
Example

- Level of root $a = 0$, $b, j, k = 1$ and $c, e, f, l = 2 \ldots$
- Because the largest level of any vertex is 4, this tree has height 4.

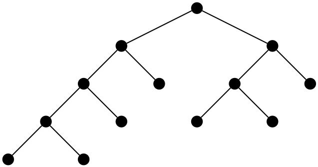
Balanced m-ary Trees

Definition

A rooted m-ary tree of height h is balanced (*cân đối*) if all leaves are at levels h or $h-1$.



T_1



T_2

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Balanced m-ary Tree

Theorem

There are at most m^h leaves in an m-ary tree of height h.

It can be proved by using mathematical induction on the height.

Corollary

- *If an m-ary tree of height h has ℓ leaves, then $h \geq \lceil \log_m \ell \rceil$.*
- *If the m-ary tree is full and balanced, then $h = \lceil \log_m \ell \rceil$.*

Exercise

Exercise (Chess tournament)

Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion. A player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties)

Exercise (Isomorphic)

How many different isomers (*đồng phân*) do the following saturated hydrocarbons have ?

- C_3H_8
- C_5H_{12}
- C_6H_{14}

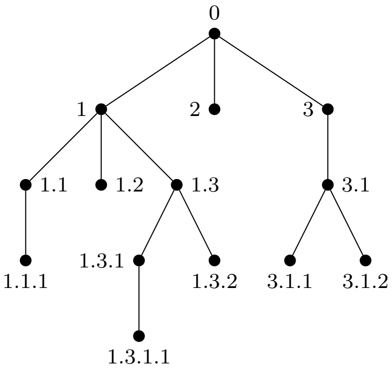
Question

Exercise

- How many vertices and how many leaves does a complete m-ary tree of height h have?
- Show that a full m-ary balanced tree (*cây m-phân hoàn hảo*) of height h has more than m^{h-1} leaves.
- How many edges are there in a forest of t trees containing a total of n vertices?

Labeling Ordered Rooted Trees

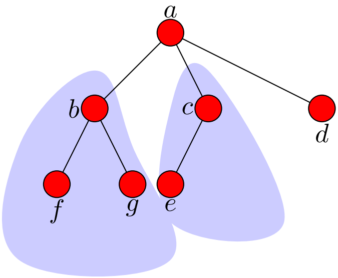
- Ordered rooted trees are often used to store information.
- Need a procedure for visiting each vertex of an ordered rooted tree to access data.
- Ordering and labeling the vertices is important to traverse them in any procedure
- Universal address system (*hệ địa chỉ phổ dụng*)
  $0 < 1 < 1.1 < 1.1.1 < 1.2 < 1.3 < \ldots < 2 < 3 < 3.1 < \ldots$



Traversal Algorithms (Thuật toán duyệt cây)

Preorder Traversal (duyệt tiền thứ tự - NLR)

procedure *preorder*(T: ordered rooted tree)
  r := root of T
  print r
  for each child c of r from left to right
    $T(c)$:= subtree with c as its root
    preorder($T(c)$)

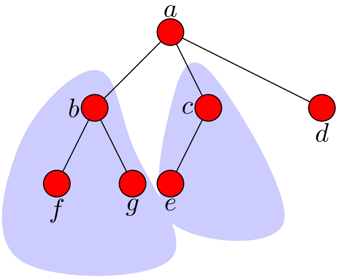


a b f g c e d

Traversal Algorithms

Inorder Traversal (Duyệt trung thứ tự - LNR)

Suppose a tree T with root r. If T consists only of r, then r is inorder traversal of T. Otherwise, suppose r has subtrees T_1, T_2, $\ldots$, T_n from left to right, inorder traversal:
$T_1 \to r \to T_2 \to \ldots \to T_n$.

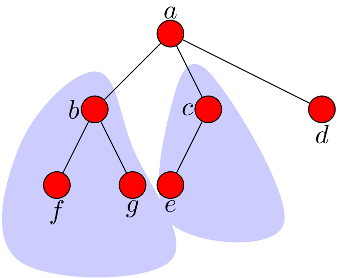


f b g a e c d

Traversal Algorithms

Postorder Traversal (Duyệt hậu thứ tự - LRN)

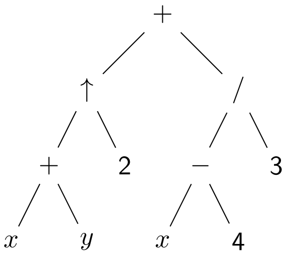
procedure *postorder*(T: ordered rooted tree)
  $r :=$ root of T
  for each child c of r from left to right
    $T(c) :=$ subtree with c as its root
    postorder($T(c)$)
  print r



f g b e c d a

Infix, Prefix and Postfix Notations

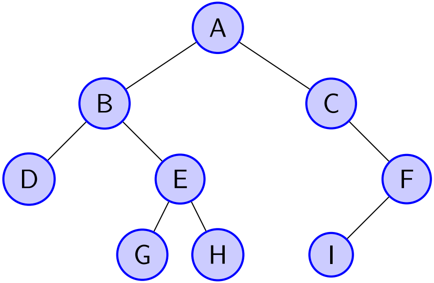
- Infix (*trung tố*):
  $((x + y) \uparrow 2) + ((x - 4)/3)$
- Prefix (*tiền tố*):
  $+ \ \uparrow \ + \ x \ y \ 2 \ / \ - \ x \ 4 \ 3$
- Postfix (*hậu tố*):
  $x \ y \ + \ 2 \ \uparrow \ x \ 4 \ - 3 \ / \ +$



Exercise

Exercise

Implement postorder, inorder and preorder traversal of the following tree.



Exercise

Exercise

Find the ordered rooted tree representing

$$(\neg(p \wedge q) \vee (\neg q \wedge r)) \to (\neg p \vee \neg r)$$

Then use this rooted tree to find the prefix, postfix and infix forms of this expression

Solution

Exercise

Exercise

Determine postorder of a binary tree with inorder D B H E I A F C J G K and preorder A B D E H I C F G J K.

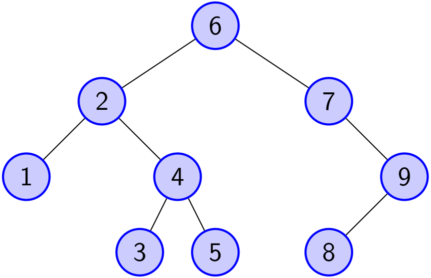
Solution

Binary Search Trees

Definition

Binary search tree (*cây tìm kiếm nhị phân* - BST) is a binary tree in which the assigned key of a vertex is:

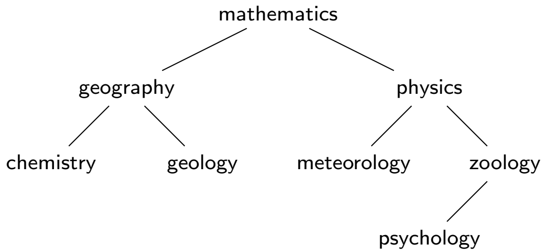
- larger than the keys of all vertices in its left subtree, and
- smaller than the keys of all vertices in its right subtree.



Adding and Locating an Item in BST

Example

Form a BST for the words *mathematics*, *physics*, *geography*, *zoology*, *meteorology*, *geology*, *psychology*, *chemistry* using alphabetical order.



Complexity in searching

$O(\log(n))$ vs. $O(n)$ in linear list

Decision Trees (Cây quyết định)

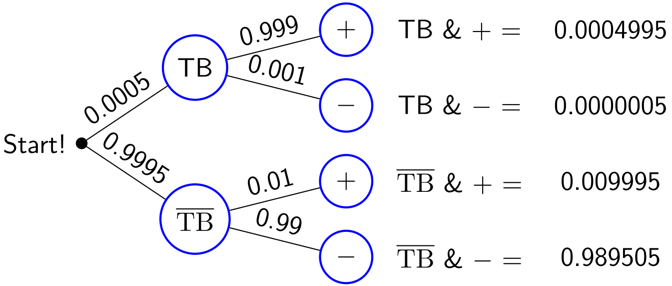
Example

There are seven coins, all with the same weight, and a counterfeit coin that weighs less than the others. How many weighings are necessary using a balance scale to determine which of the eight coins is the counterfeit one? Give an algorithm for finding this counterfeit coin.

Yet Another Application

Example

If we know that the probability that a person has tuberculosis (TB) is $p(\mathrm{TB}) = 0.0005$.
We also know $p(+|\mathrm{TB}) = 0.999$ and $p(-|\overline{\mathrm{TB}}) = 0.99$.
What is $p(\mathrm{TB}|+)$ and $p(\overline{\mathrm{TB}}|-)$?



$$p(\mathrm{TB}|+) = \frac{p(\mathrm{TB}\cap +)}{p(+)} = \frac{0.0004995}{0.0004995+0.009995} \approx 0.0476$$

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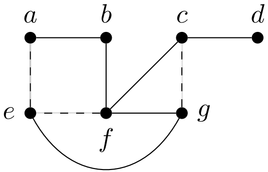
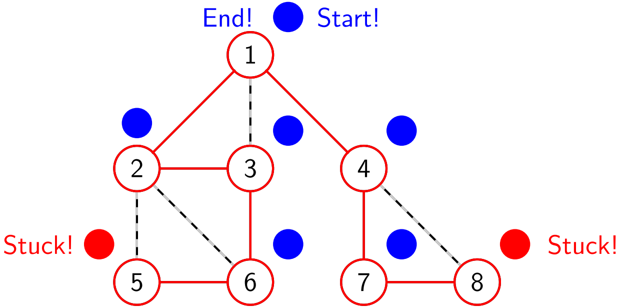
Problem

Definition

- A spanning tree (*cây khung*) in a graph G is a subgraph of G that is a tree which contains all vertices of G.



Depth-First Search (Tìm kiếm ưu tiên chiều sâu)



Property

- Go deeper as you can
- Backtrack (*quay lui*) to possible branch when you are stuck.
- $O(e)$ or $O(n^2)$

Depth-First Search

Algorithm

procedure *DFS* (G)
  T := tree consisting only vertex v_1
  visit(v_1)

procedure *visit*$(v$: vertex of $G)$ /* recursive */
  for each vertex w adjacent to v and not in T
    add w and edge $\{v, w\}$ to T
    visit(w)

A pseudocode of DFS

void DFS(G)

1.loop (more vertex v in G)
  1.color[v]= **White**
  2.father[v]=*null*
2.time=0
3.loop (more vertex v in G)
  1.if (color[v]== **White**)
    1.DFSVisit(G, v)

void DFSVisit (G, v)

1.color[v]=**Gray**1.color[v]=Gray
  2.time= time+1**2.time= time+1**
  3.d[v]= time**3.d[v]= time**
  4.loop(more u adjacent to v)
    1.if(color[u]==**White**)
      1. father[u]=v**1. father[u]=v**

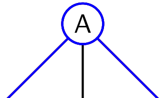
2.DFSVisit(G, u)**2.DFSVisit(G, u)**

5.color[v]=**Black**5.color[v]=Black

6.time=time+1**6.time=time+1**
  7.f[v]=time**7.f[v]=time**

vertex	father	color	c
A	-	WGB	•
B	-A	WGB	•
C	-B	WGB	•
D	-A	WGB	•1
E	-F	WGB	•
F	-C	WGB	•
G	-D	WGB	•1
H	-G	WGB	•1

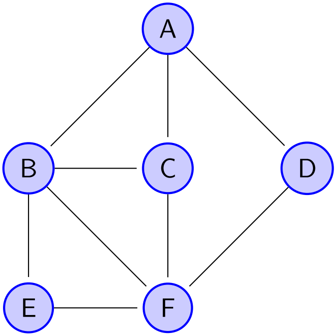
time = 0 1 2 3 4 5 6 7 8 9 10 11 12 1



Exercise

Exercise

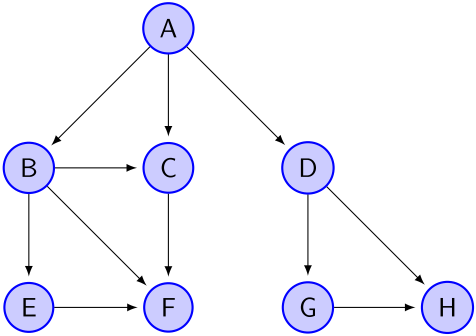
Apply DFS into the following graph.



Exercise

Exercise

Apply DFS into the following graph.



Exercise

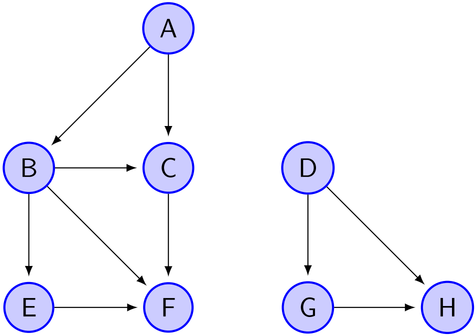
How to apply & modify DFS algorithm to ...

a. determine whether an undirected graph is connected,
b. determine whether there exists a cycle in an undirected graph,
c. determine whether there exists a cycle in an digraph,
d. determine whether a graph is bipartite,
e. determine topological order,
f. calculate number of connected components,
g. identify articulation points,
h. determine a longest path in a given digraph...

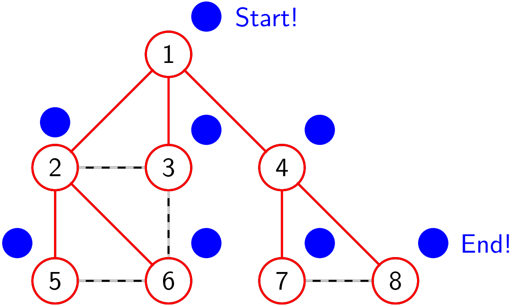
Exercise

Exercise

Apply DFS into the following graph.



Breadth-First Search (Tìm kiếm ưu tiên chiều rộng)



vertex	L
	$\emptyset$
1	2, 3, 4
2	3, 4, 5, 6
3	4, 5, 6
4	5, 6, 7, 8
5	6, 7, 8
6	7, 8
7	8
8	$\emptyset$

Property

- $O(e)$ or $O(n^2)$

Breadth-First Search

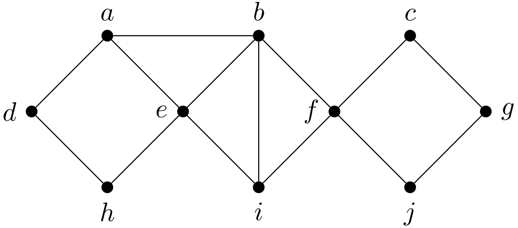
Algorithm

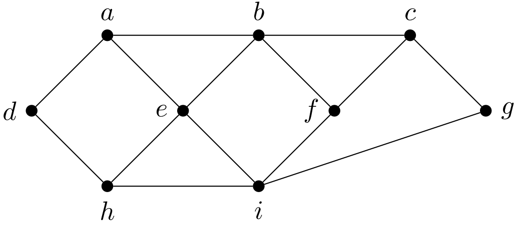
procedure *BFS* (G)
  $T :=$ tree consisting only vertex v_1
  $L :=$ empty list
  put v_1 in the list L of unprocessed vertices
  while L is not empty
    remove the first vertex, v, from L
    for each neighbor w of v
      if w is not in L and not in T **then**
        add w to the end of the list L
        add w and edge $\{v, w\}$ to T

Exercise

Exercise

Find spanning tree in the following graphs.

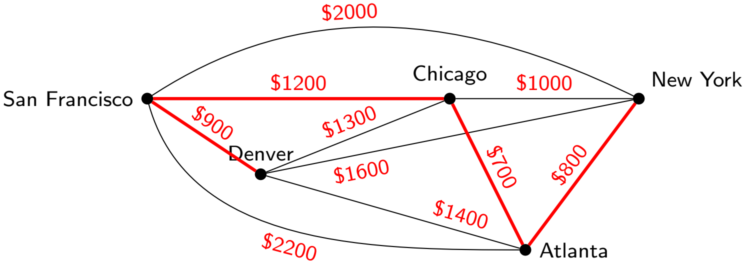




Minimum Spanning Trees

Definition

- A minimum spanning tree (*cây khung nhỏ nhất*) in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.



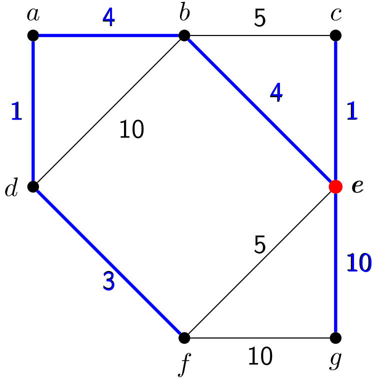
Prim's Algorithm (Nearest-Neighbor)

Prim's Algorithm (1957)

procedure *Prim*(G)
  $T :=$ a minimum-weight edge incident to a **given vertex**
  for $i := 1$ to $n - 2$
    $e :=$ an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T
    $T := T$ with e added
  return T

Prim's Algorithm (Nearest-Neighbor)

- Pick a vertex to start from
- Iteratively absorb smallest edge possible



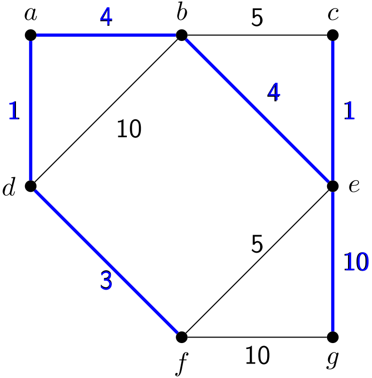
Kruskal's Algorithm (Lightest-Edge)

Kruskal's Algorithm (1958)

procedure *Kruskal*(G)
    $T :=$ empty graph
    for $i := 1$ **to** $n - 1$
        $e :=$ any edge in G with smallest weight that does not form a simple circuit when added to T
        $T := T$ with e added
    return T

Kruskal's Algorithm (Lightest-Edge)

- Iteratively add smallest edge possible

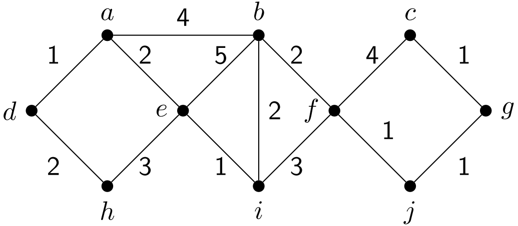


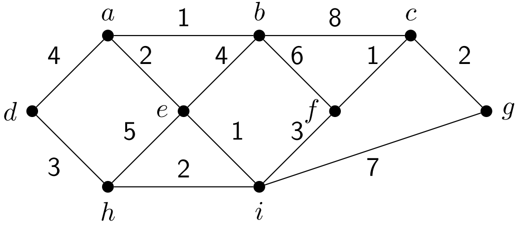
ad	OK
ce	OK
df	OK
ab	OK
be	OK
bc	KO
ef	KO
eg	OK & Stop
fg	

Exercise

Exercise

By using Prim's and Kruskal's algorithm, determine minimum spanning tree in the following graphs. (and maximum spanning tree (*cây khung cực đại*).





Exercise

Cho một cây có gốc với n đỉnh. Giả thiết một đỉnh trong tập đỉnh có bậc là $n-1$. Chiều cao của cây là

- (A) 1
- (B) $n-1$
- (C) n
- (D) 2

Exercise

Xác định tiền tố (prefix) của cây nhị phân có gốc và có thứ tự (ordered rooted tree) dùng để biểu diễn

$$(\neg(p \wedge q) \vee (\neg q \wedge r)) \rightarrow (\neg p \vee \neg r)$$

A $\rightarrow \vee \neg \wedge p\ q \vee \neg q\ r \vee \neg p\ r$

B $p\ q \wedge \neg \vee q \neg r \wedge p \neg r \vee \rightarrow$

C $p\ q \neg \vee q \neg \wedge r \rightarrow p \neg \vee r$

D $p\ q \neg \vee q \neg \wedge r \rightarrow p \neg \vee r$

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Exercise

Có bao nhiêu cây nhị phân có tiền tố (pre-order traversal) là ABC?

- A 1
- B 3
- C 5
- D 7

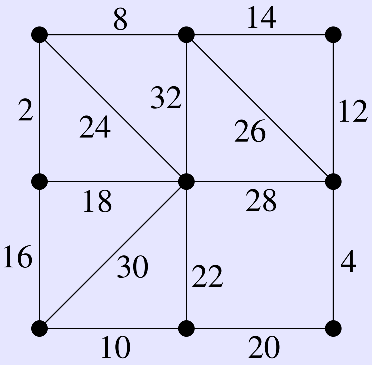
Exercise

Hãy cho biết hậu tố (post-order traversal) của một cây nhị phân biết rằng tiền tố (pre-order traversal) là $HBGFDECIA$ và trung tố (in-order traversal) là $GBFHCEIDA$.

- A $GFBCIEADH$
- B $BGFDECIAH$
- C $GFBCIEJADH$
- D $GFBHCIEADH$

Exercise

Cho đồ thị như trong hình vẽ dưới.



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Cây phủ tối thiểu có tổng trọng số là

- (A) 40
- (B) 60
- (C) 84
- (D) 100

Exercise

Cho trước số tự nhiên $a > 1$, và xét đồ thị đầy đủ K_{2a+3}. Số lượng cạnh ta phải xóa khỏi đồ thị K_{2a+3} để thu được một cây phủ (cây khung hay bao trùm, *spanning tree*) của K_{2a+3} là bao nhiêu?

- A $2a + 2$
- B $2a^2 + 3a - 1$
- C $4a^2 + 3a + 1$
- D $2a^2 + 3a + 1$