Chapter 2 Profit Maximization

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Outline

Introduction

A firm chooses actions $(a_1, a_2, \ \dots, \ a_n)$ so as to maximize $R(a_1, a_2, \ \dots, \ a_n) - C(a_1, a_2, \ \dots, \ a_n)$.

$$\max_{\mathbf{a}} R(a_1, a_2, \ \dots, \ a_n) - C(a_1, a_2, \ \dots, \ a_n)$$

An optional set of actions, $\mathbf{a}^* = (a_1^*, a_2^*, \ \dots, \ a_n^*)$, is characterized by the conditions that marginal revenue equals to marginal cost

$$\frac{\partial R(\mathbf{a}^*)}{\partial a_i} = \frac{\partial R(\mathbf{a}^*)}{\partial a_i}, \quad i = 1, 2, \dots, n$$

- If marginal revenue > marginal cost: increase activity level
- If marginal revenue > marginal cost: decrease activity level

The production of one more unit of output should produce a marginal revenue equal to its marginal cost of production.

In determining its optimal policy, the firm faces two constraints:

1. technological constraints: feasible production plan
2. market constraints: effect of actions of other agents on the firm

Analyzing profit-maximizing is the simplest kind of market behavior: price-taking behavior and competitive firm assumptions:

1. well-informed consumers
2. homogeneous product
3. a large number of firms

2.1 Profit Maximization

$$\begin{aligned}\pi(\mathbf{p}) &= \max\ \mathbf{p}\cdot\mathbf{y} \quad \text{s.t. } \mathbf{y}\in \mathrm{Y} \\ &= \max\ \begin{pmatrix} p & \mathbf{w}\end{pmatrix}\begin{pmatrix} y \\ -\mathbf{x}\end{pmatrix} \\ &= \max\ py-\mathbf{wx}\end{aligned}$$

The function $\pi(\mathbf{p})$, which gives us the maximum profits as a function of the prices, is called the profit function of the firm. If we are considering a short-run maximization problem, we might define the short-run function, also known as the restricted profit function,

$$\pi(\mathbf{p},\ \mathbf{z}) = \max\ \mathbf{p}\cdot\mathbf{y} \quad \text{s.t. } \mathbf{y}\in \mathbf{Y}(\mathbf{z})$$

If the firm produce only one output, the profit function can be written as

$$\pi(p,\ \mathbf{w}) = \max\ pf(\mathrm{x}) - \mathbf{wx}$$

1 input and 1 output

$$\begin{cases}\pi = py - wx \iff y = \frac{\pi}{p} + \frac{w}{p}x \\ y = f(x)\end{cases}$$

1. FOC:

   The profit-maximizing amount of input occurs where the slope of the isoprofit equals the slope of the production function.

$$
\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx}\left(\frac{\pi}{p} + \frac{w}{p} x\right) \\
\implies \frac{df(x)}{dx} &= \frac{w}{p} \\
\implies p \cdot MPP_x &= w
\end{aligned}
$$

2. SOC:

   At the point of maximal profits the production function must lie below its tangent line at x^*, i.e. must be "locally concave".

$$
\frac{d^2 f(x)}{dx^2} \leq 0
$$

1 output and n inputs

1. FOC:

$$
\begin{aligned}
p \cdot \underbrace{\frac{\partial f(\mathrm{x}^*)}{\partial x_i}}_{MPP_{x_i}} &= w_i, \quad i = 1, \cdots, n \\
\implies p \cdot \begin{bmatrix} \frac{\partial f(\mathrm{x}^*)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathrm{x}^*)}{\partial x_n} \end{bmatrix} &= \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\
\implies p \cdot \nabla f(\mathbf{x}^*) &= \mathbf{w}
\end{aligned}
$$

   The first-order conditions state that "the value of marginal product of each factor must be equal to its price." This is just a special case of the optimization rule we stated earlier: that the marginal revenue of each action be equal to marginal cost.

2. SOC:

   The matrix of second derivatives of the production function must be negative semi-definite at the optimal point.

$$
\nabla^2 f(\mathbf{x}^*) = \left[\frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j}\right]
$$

   that is, the SOC requires that the Hessian matrix must satisfy that the condition that

$$\mathbf{h}\nabla^2 f(\mathbf{x}^*)\mathbf{h}^T \leq 0 \quad \forall \text{ vector } \mathbf{h}$$

2.2 Difficulties

Concept, in 2.1 we assume...

1. for each vector of prices $(p, \mathbf{w})$ there will in general be some optimal choice of factors $\mathbf{x}^*$
2. $\mathbf{x}(p, \mathbf{w})$ factor demand function
3. $y(p, \mathbf{w}) = f(\mathbf{x}(p, \mathbf{w}))$ (output?) supply function

We will often assume that these functions are will-defined and nicely behaved.

There are some difficulties:

1. It may happen that the technology cannot be described by a differentiable production function.
2. In many economic problems the variables are naturally nonnegative, and if some variables have value of zero at the optimal choice, the calculus conditions describe above may be inappropriate. (The above conditions are valid only for **interior solutions** where each of the factors are used in a positive amount.)

> **boundary solutions:**
>
> 1. $$\begin{aligned} p \cdot \frac{\partial f(\mathrm{x})}{\partial x_i} - w_i < 0, \text{ if } x_i = 0 \\ \implies \frac{\partial f(\mathrm{x})}{\partial x_i} - \frac{w_i}{p} < 0 \\ \implies MR < MC \end{aligned}$$
>
>   1. Constraint is binding.
>   2. Marginal profit from increasing x_i is **negative**.
>   3. Firm wants to decrease x_i, but $x_i = 0$

2. $$p \cdot \frac{\partial f(\mathrm{x})}{\partial x_i} - w_i = 0, \text{ if } x_i > 0$$
$$\implies \frac{\partial f(\mathrm{x})}{\partial x_i} - \frac{w_i}{p} = 0$$
$$\implies MR = MC$$

   1. Constraint is not binding.
   2. We will have the usual conditions for an interior solution.

> Use Kahn-Tucker Theorem to solve for boundary solution.

3. There may exist no profit-maximizing production plan.

💡 **Example: 1 input, 1 output**

$$y = f(x) = x$$
$$\pi = pf(x) - wx = px - wx$$

- if $p > w$: $x \nearrow \implies f(x) \nearrow \implies \pi \nearrow$
  - For $p > w$ no profit-maximizing plan will exist.
- if $p = w$: $\pi = 0$
  - A maximal profit production plan will exist for the technology only when $p = w$, in which case the maximal level of profit will be zero.
- if $p < w$: $\pi < 0$
  - infeasible.

💡 **Example: CRTS technology, n input, 1 output**

$$\begin{aligned} pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^* &= \pi^* > 0 \\ pf(t\mathbf{x}^*) - \mathbf{w}(t\mathbf{x}^*) &= ptf(\mathbf{x}^*) - t\mathbf{w}(\mathbf{x}^*) \\ &= t(pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^*) \\ &= t\pi^* \end{aligned}$$

- if $t > 1$, $\pi^* > 0$, then $t\pi^* > \pi^*$.
- This means that if profit are ever positive, they could be made larger. Hence, no maximal profit production plan will exist in this case.

4. Even when a profit-maximizing production plan exists, it may not be unique. (In the case of CRTS, if there exists a profit maximizing choice at some $(p, \mathbf{w})$ at all, there will typically be a whole range of production plans that are profit-maximizing.

Example: profit function for CD technology, 1 input, 1 output

$$
\begin{gathered}
y=f(x)=x^a \quad a>0 \\
\pi=pf(x)-wx=px^a-wx
\end{gathered}
$$

1. FOC:

$$
\begin{gathered}
p\frac{df(x)}{dx}-w=0 \\
pax^{a-1}=w
\end{gathered}
$$

2. SOC:

$$
\begin{aligned}
p\frac{d^2f(x)}{dx^2} &\le 0 \\
\implies pa(a-1)x^{a-2} &\le 0 \\
\overset{(\because a>0,\ p>0)}{\implies} (a-1) &\le 0 \\
\implies 0<a &\le 1
\end{aligned}
$$

- if $a=1$,

$$p=w \implies \pi=0$$

When $w=p$ any value of x is a profit-maximizing choice.

- if $a<1$, use FOC to solve for:
  1. The factor demand function:

  2. The output supply function:

  3. Profit function:

$$
\begin{aligned}
\pi(p,w) &= py - wx \\
&= p(\frac{w}{ap})^{\frac{a}{a-1}} - w(\frac{w}{ap})^{\frac{1}{a-1}} \\
&= w(\frac{1-a}{a})(\frac{w}{ap})^{\frac{1}{a-1}}
\end{aligned}
$$

2.3 Properties of demand and supply functions

Factor demand and output supply functions are the solutions for a profit maximization problem. These implies certain restriction on the behavior of the demand and supply functions. In other words, **the factor demand functions must be homogeneous of degree zero.**

$$x_i(tp, t\mathrm{w}) = x_i(p, \mathrm{w}) \quad i = 1, \cdots, n$$

💡 **Homework:**

If all prices are doubled, the levels of goods demanded and supplied by a profit-maximizing firm will not change.

$$
\begin{aligned}
x(p,w) &= (\frac{w}{ap})^{\frac{a}{a-1}} \\
x(tp,tw) &= (\frac{tw}{atp})^{\frac{a}{a-1}} = t(\frac{w}{ap})^{\frac{a}{a-1}} \quad (\mathrm{HD0})
\end{aligned}
$$

2.4 Comparative statics using the first-order conditions

Case 1: 1 inputs, 1 output

$$\max_x pf(x) - wx$$

Under the assumption of 1 input and 1 output, $f(x)$ is differentiable. If $f(x)$ is differentiable, the demand function $x(p, w)$ must satisfy:

- FOC: $pf'(x(p, w)) - w = 0$
- SOC: $pf''(x(p, w)) \leq 0$

Under any (p, w) optimal choice x has to satisfy the following conditions:

$$\frac{\partial FOC}{\partial w_i} \implies pf''(x(p, w))\frac{dx(p, w)}{dw} - 1 = 0$$

If we have a strict maximum $\implies f''(x) \neq 0$

$$\frac{dx(p, w)}{dw} = \frac{1}{pf''(x(p, w))}$$

how factor demand x responds to w express $\frac{dx}{dw}$ in terms of the production function $f(x)$.

Implication:

1. If the production function is very curved in a neighborhood of the optimum so that the second derivative is large in magnitude then the change in factor demand as the factor price changes will be small.
2. $f''(x(p, w)) < 0 \implies \frac{dx(p,w)}{dw} = \frac{1}{pf''(x(p,w))} < 0$

Case 2: 2 inputs, 1 output

- FOC:

$$f_1 = p\frac{\partial f[x_1(w_1, w_2), x_1(w_1, w_2)]}{\partial x_1} = w_1$$
$$f_2 = p\frac{\partial f[x_1(w_1, w_2), x_1(w_1, w_2)]}{\partial x_2} = w_2$$

- SOC:

$$\frac{\partial f_1}{\partial w_1} = f_{11}\frac{\partial x_1}{\partial w_1} + f_{12}\frac{\partial x_2}{\partial w_1} = 1 \quad \frac{\partial f_2}{\partial w_1} = f_{21}\frac{\partial x_1}{\partial w_1} + f_{22}\frac{\partial x_2}{\partial w_1} = 0$$
$$\frac{\partial f_1}{\partial w_2} = f_{11}\frac{\partial x_1}{\partial w_2} + f_{12}\frac{\partial x_2}{\partial w_2} = 0 \quad \frac{\partial f_2}{\partial w_2} = f_{21}\frac{\partial x_1}{\partial w_2} + f_{22}\frac{\partial x_2}{\partial w_2} = 1$$
$$\Longrightarrow \underbrace{\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}}_{\text{Hessian matrix}} \overbrace{\begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{bmatrix}}^{\text{Substitution matrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let us assume that we have a **strict maximum**. This implies the Hessian matrix is strictly negative definite, and therefore nonsingular. Solving for the matrix of first derivative, we have:

$$\begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1}$$

Describes how the firm substitutes 1 input for another as the factor prices change.

Theorem:

> - Hessian matrix is a symmetric negative definite matrix.
> - The inverse of a symmetric negative definite matrix is a symmetric negative definite matrix.
>   - $\frac{\partial x_i}{\partial w_i} < 0 \quad \forall\, i = 1, 2, \ldots, n$
>   - $\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i} \quad \forall\, i \neq j$

Case 3: more than 3 inputs, 1 output, let $p = 1$

1. FOC:

$$\nabla f[\mathbf{x}(\mathbf{w})] - \mathbf{w} \equiv 0$$

2. SOC:

$$\underbrace{\nabla^2 f[\mathbf{x}(\mathbf{w})]}_{\text{Hessian matrix}} \quad \cdot \quad \underbrace{\nabla[\mathbf{x}(\mathbf{w})]}_{\text{Substitution matrix}} \quad - I \equiv 0$$

3. From SOC, we have

$$\underbrace{\nabla[\mathbf{x}(\mathbf{w})]}_{\text{Substitution matrix}} \quad \equiv \quad \underbrace{[\nabla^2 f[\mathbf{x}(\mathbf{w})]]^{-1}}_{\text{Inverse of Hessian matrix}}$$

Suppose that the vector of factor prices change from $\mathrm{w} = (w_1, w_2)$ to $\mathbf{w} + d\mathbf{w}$. Then the associated change in the factor demands is

$$\begin{gathered} d\mathrm{x} = \nabla \mathrm{x}(\mathrm{w}) d\mathrm{w}^T \\ d(x_1(w_1, w_2), x_2(w_1, w_2)) = \frac{\partial x_1(w_1, w_2)}{\partial w} dw + \frac{\partial x_2(w_1, w_2)}{\partial w} dw \\ d\mathrm{w} d\mathrm{x} = d\mathrm{w} \nabla \mathrm{x}(\mathrm{w}) d\mathrm{w}^T \end{gathered}$$

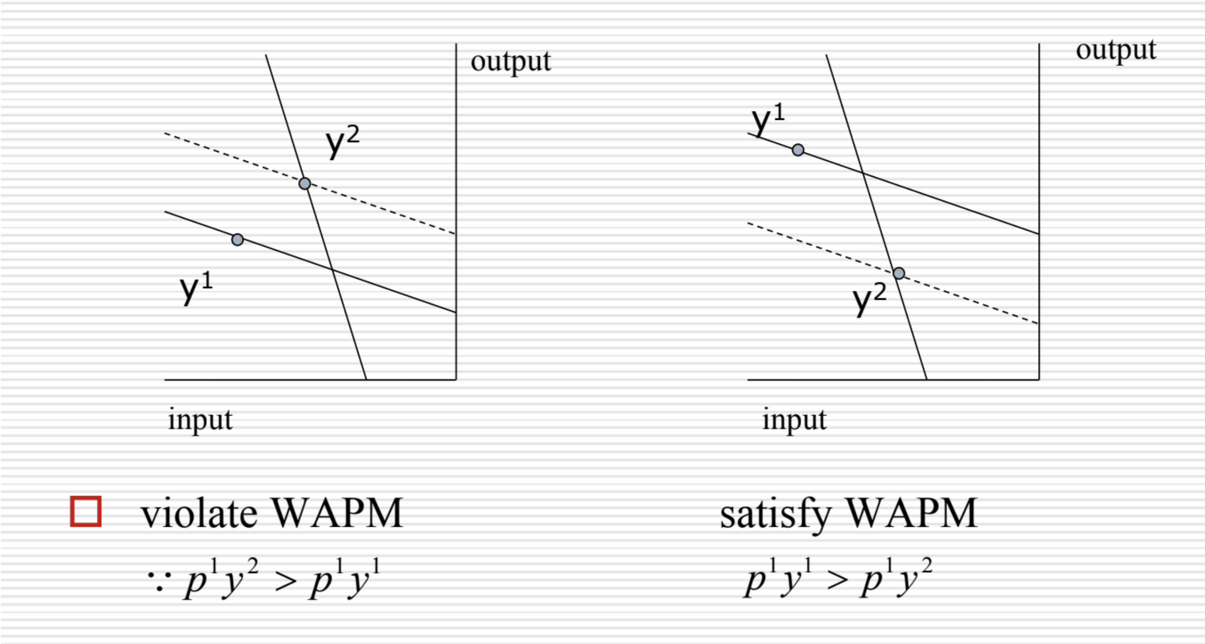
2.5 Comparative statics using algebra

Suppose that we are given a list of observed price vector $\mathbf{p}^t$, and the associated net output vectors $\mathbf{y}^t$, for $t = 1, \ldots, T$. In terms of the supply functions $(\mathrm{p}^t, \mathrm{y}(\mathrm{p}^t))$ for some observations $t = 1, \ldots, T$.

If the firm is maximizing profits, then the observed net output choice $\mathbf{y}^t$ at price $\mathbf{p}^t$ must have a level of profit at least as great as the profit at any other net output firm could have chosen, say $\mathbf{y}^s$ for $s = 1, \ldots, T$.

Theorem: (Weak Axiom of Profit Maximization, WAPM)

> A necessary condition for profit maximization is that
>
> $$\mathbf{p}^t \mathbf{y}^t \geq \mathbf{p}^t \mathbf{y}^s \quad \forall\, t, s = 1, 2, \ldots, T.$$
>
> We refer to this conditions the **Weak Axiom of Profit Maximization(WAPM)**.



Theorem:

> $$\begin{cases} \mathbf{p}^t\mathbf{y}^t \geq \mathbf{p}^t\mathbf{y}^s \\ \mathbf{p}^s\mathbf{y}^s \geq \mathbf{p}^s\mathbf{y}^t \end{cases} \implies \begin{cases} \mathbf{p}^t(\mathbf{y}^t - \mathbf{y}^s) \geq 0 \\ -\mathbf{p}^s(\mathbf{y}^t - \mathbf{y}^s) \geq 0 \end{cases}$$
> $$\implies (\mathbf{p}^t - \mathbf{p}^s)(\mathbf{y}^t - \mathbf{y}^s) \geq 0 \implies \Delta\mathbf{p}\Delta\mathbf{y} \geq 0$$
>
> The inner product of a vector of price changes with the associated vector of changes in the output must be **non-negative**.