Chapter 3 Profit Function

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Outline

$$\pi(\mathbf{p}) = \max_{\mathbf{y}} \mathbf{p}\mathbf{y} \quad \text{such that } \mathbf{y} \in Y$$

3.1 Properties of Profit Function

It is important to recognize that these properties follow solely from the assumption of profit maximization. No assumptions about convexity, monotonicity, or other sorts of regularity are necessary.

Theorem: (properties of profit function)

> 1. Non decreasing in output prices, non-increasing in input price. If $p_i' \geq p_i$ and $p_j' \leq p_j$ for all outputs i and inputs j. Then $\pi(\mathbf{p}') \geq \pi(\mathbf{p})$.
> 2. Homogeneous of degree 1 in $\mathbf{p}$. $\pi(t\mathbf{p}) = t\pi(\mathbf{p})\ \forall t \geq 0$.
> 3. Convex in $\mathbf{p}$. Let $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ for $0 \leq t \leq 1$. Then $\pi(\mathbf{p}'') \leq t\pi(\mathbf{p}) + (1-t)\pi(\mathbf{p}')$.
> 4. Continuous in $\mathbf{p}$.
>
> **Proof:**

1.

   a. Let $\mathbf{y}, \mathbf{y'}$ be profit-maximizing net output vectors at $\mathbf{p}, \mathbf{p'}$, respectively. So that

   $$\pi(\mathbf{p}) = \mathbf{py} \quad \pi(\mathbf{p'}) = \mathbf{p'y'}.$$

   b. By definition of profit maximization, we have

   $$\mathbf{p'y'} \geq \mathbf{p'y}.$$

   c. Since $p'_i \geq p_i$ for all i for which $y_i \geq 0$ and $p'_i \leq p_i$ for all i for which $y_i \leq 0$, we also have

   $$\mathbf{p'y} \geq \mathbf{py}.$$

   d. Putting these two inequality together, we have

   $$\pi(\mathbf{p'}) = \mathbf{p'y'} \geq \mathbf{py} = \pi(\mathbf{p})$$

   as required.

2. Let $\mathbf{y}$ be profit-maximizing net output vectors at $\mathbf{p}$.

   $$\begin{aligned} &\implies \mathbf{py} \geq \mathbf{py'} \quad \forall\, \mathbf{y'} \in Y \\ &\implies t\mathbf{py} \geq t\mathbf{py'} \quad (t \geq 0). \end{aligned}$$

   Hence, $\mathbf{y}$ also maximizes profit at price $t\mathbf{p}$

   $$\pi(t\mathbf{p}) = t\mathbf{py} = t\pi(\mathbf{p}).$$

   $\mathbf{y}$ is the profit-maximizing net output vectors at $t\mathbf{p}$.

3.

> a. Let $\mathbf{y}, \mathbf{y'}$ be profit-maximizing net output vectors at $\mathbf{p}, \mathbf{p'}$, respectively. So that
>
> $$\pi(\mathbf{p}) = \mathbf{p}\mathbf{y} \quad \pi(\mathbf{p}') = \mathbf{p'}\mathbf{y'}.$$
>
> b. Let $\mathbf{y}''$ be profit-maximizing net output vectors at $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ where $0 \leq t \leq 1$. Then we have
>
> $$\begin{aligned}\pi(\mathbf{p}'') &= \mathbf{p}''\mathbf{y}'' \\ &= [t\mathbf{p} + (1-t)\mathbf{p}']\mathbf{y}'' \\ &= t\mathbf{p}\mathbf{y}'' + (1-t)\mathbf{p}'\mathbf{y}''.\end{aligned}$$
>
> c. By definition of profit maximization, we known that
>
> $$\begin{aligned}&\Longrightarrow \begin{cases}\mathbf{p}\mathbf{y''} \leq \mathbf{p}\mathbf{y} \\ \mathbf{p'}\mathbf{y''} \leq \mathbf{p'}\mathbf{y'}\end{cases} \\ &\Longrightarrow \begin{cases}t\mathbf{p}\mathbf{y''} \leq t\mathbf{p}\mathbf{y} \\ (1-t)\mathbf{p'}\mathbf{y''} \leq (1-t)\mathbf{p'}\mathbf{y'}\end{cases}\end{aligned}$$
>
> d. Adding these two inequality, we have
>
> $$\begin{aligned}t\mathbf{p}\mathbf{y''} + (1-t)\mathbf{p'}\mathbf{y''} &\leq t\mathbf{p}\mathbf{y} + (1-t)\mathbf{p'}\mathbf{y'} \\ \pi(\mathbf{p}'') &\leq t\pi(\mathbf{p}) + (1-t)\pi(\mathbf{p'})\end{aligned}$$
>
> as required.

At the price vector (p^*, w^*) the profit-maximizing production plan (y^*, x^*) yields profits

$$\pi(p^*) = \pi^* = p^*y^* - \mathrm{w}^*\mathrm{x}^* \quad \text{(profit function)}$$

Suppose p increase, but the firm continue using the same production plan (y^*, x^*), then yields profits

$$\Pi(p) = py^* - \mathrm{w}^*\mathrm{x}^* \quad \text{(passive profit function)}$$

The profit from pursuing an optimal policy must be at least as large as the profit from pursuing the passive policy, so the graph of $\pi(p)$ must be above the graph of $\Pi(p)$. The same argument can be repeated for any price p, so the profit function must lie above its tangent line at every point. It follows that $\pi(p)$ must be a convex function.

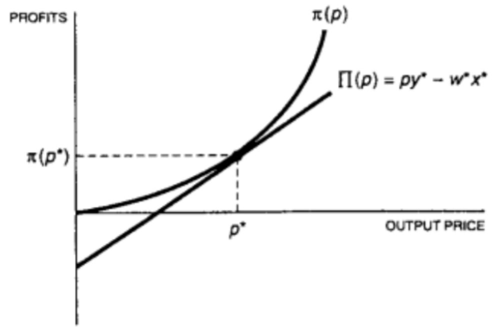

Figure 3.1 **The profit function.** As the output price increases, the profit function increases at an increasing rate.

💡 **Example: The effect of price stabilization**

Output price will be (p_1, p_2) with probability $(q, 1-q)$. It may be desirable to stabilized the price of output at the average price $\bar{p} = p_1 q + p_2(1-q)$. How would this affect profits of a typical firm in the industry?

Since the profit function is convex

$$q\pi(p_1) + (1-q)\pi(p_2) \geq \pi(qp_1 + (1-q)p_2) = \pi(\bar{p})$$
$$\text{average of profit} \geq \text{profit of average price.}$$

Thus, average profit with a fluctuating price are at least as large as with a stabilized price.

3.2 Supply and Demand Function from the Profit Function

Suppose that we are given the profit function and are asked to find the net supply function, which include the output supply function and the input demand function. How can that be done? **Just**

differentiate the profit function!

$$\pi(\mathbf{p}) = \mathbf{p}\mathbf{y}(\mathbf{p})$$

> **Theorem: (Hotelling's Lemma)**
>
> Let $y_i(\mathbf{p})$ be the firm's net supply function for good i. Then
>
> $$y_i(\mathbf{p}) = \frac{\partial \pi(\mathbf{p})}{\partial p_i} \quad \text{for } i = 1, \cdots, n,$$
>
> assuming that the derivative exist and that $p_i > 0$.
>
> **Proof:**
>
> Supply $\mathbf{y}^*$ is a profit-maximizing net output vector at price $\mathbf{p}^*$. Then define the function
>
> $$g(\mathbf{p}) = \pi(\mathbf{p}) - \mathbf{p}\mathbf{y}^*.$$
>
> The plan $\mathbf{y}^*$ will be a profit-maximizing plan at price $\mathbf{p}^*$, so the function g reaches a minimum value of 0 at $\mathbf{p}^*$.
>
> The FOC for a minimum then imply that
>
> $$\begin{aligned} \frac{\partial g(\mathbf{p}^*)}{\partial p_i} = \frac{\partial \pi(\mathbf{p}^*)}{\partial p_i} - y_i^* = 0 \quad \text{for } i = 1, \cdots, n \\ y_i^* = \frac{\partial \pi(\mathbf{p}^*)}{\partial p_i}. \end{aligned}$$

💡 **Example: (1 input, 1 output)**

$$\max \; pf(x) - wx$$

FOC:

$$p\frac{df(x)}{dx} - w = 0$$

Let $x^* = x(p, w)$, the profit function is given by

$$\pi(p,w) \equiv pf(x(p,w)) - wx(p,w) \;\; \text{(indirect profit function)}$$

$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial w} &= \frac{\partial f(x(p,w))}{\partial x}\frac{\partial x}{\partial w} - w\frac{\partial x}{\partial w} - x(p,w) \\ &= \underbrace{[\frac{\partial f(x(p,w))}{\partial x} - w]}_{=0}\frac{\partial x}{\partial w} - x(p,w) \\ &= -x(p,w)\end{aligned}$$

The minus sign come from the fact that we are increasing the price of an input, so profits must decrease.

$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial p} &= f(x) + p\frac{\partial f(x(p,w))}{\partial x}\frac{\partial x}{\partial p} - w\frac{\partial x}{\partial p} \\ &= f(x) + \underbrace{[p\frac{\partial f(x(p,w))}{\partial x} - w]}_{=0}\frac{\partial x}{\partial p} \\ &= f(x) = y\end{aligned}$$

When the price of an output increases by a small amount there will be two effect:

1. Because the price increase the firm will make more profits, even if it continues to produce the same level of output.
2. The increase in output price will induce the firm to change its level of output by a small amount. But it's 0.

3.3 The Envelope Theorem

Consider an arbitrary maximization problem where the objective function depends on some parameter a.

$$M(a) = \max_x g(x, a).$$

In the case of profit function,

- a would be some price,
- x would be some factor demand,
- $M(a)$ would be the maximized value of profits as function of the price.

Let $x(a)$ be the value of x that solves the maximization problem. Then we can also write

$$M(a) = g(x(a), a).$$

It is often of interest to know how $M(a)$ changes as a change. The envelope theorem tells us the answer:

$$\begin{aligned}\frac{dM(a)}{da} &= \underbrace{\frac{\partial g(x,a)}{\partial x}}_{=0}\frac{\partial x}{\partial a} + \frac{\partial g(x,a)}{\partial a}\\ &= \frac{\partial g(x,a)}{\partial a}\\ &= \frac{\partial g(x,a)}{\partial a}\Big|_{x=x(a)}\end{aligned}$$

The expression says that the derivative of M with respect of a is given by the partial derivative of g with respect to a, holding x fixed at optimal choice.

💡 **Example: (1 input, 1 output, profit maximization)**

At the application of profit function $M(a) \to \pi(p,w)$, $g(x) \to pf(x) - wx$
$g(x) = f(x)x(a) \to x(p,w)$.

$$\pi(p,w) = \max \ pf(x) - wx$$
$$p\frac{df(x)}{dx} - w = 0$$

1.
$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial p} &= f(x) + p\frac{\partial f(x)}{\partial x}\frac{\partial x}{\partial p} - w\frac{\partial x}{\partial p} \\ &= f(x) + (p\frac{\partial f(x)}{\partial x} - w)\frac{\partial x}{\partial p} \\ &= f(x) = y \\ &= f(x)|_{x=x(p,w)} \\ &= f(x(p,w))\end{aligned}$$

2. Profit-maximizing net supply of the factor

$$\frac{\partial \pi(p,w)}{\partial w} = -x|_{x=x(p,w)} = -x(p,w)$$

3.4 Comparative Statics using the Profit Function

1. (Profit function 1st Property) Profit function is a monotonic function of the prices.

$$\frac{\partial \pi(\mathbf{p})}{\partial p_i}\begin{cases} > 0 & \text{when } i \text{ is an output} \\ < 0 & \text{when } i \text{ is an input}\end{cases}$$

2. (Profit function 2nd Property) Profit function is HD-1 in prices. This implies that the partial derivatives of the profit function must be HD-0. Scaling all prices by a positive factor t won't change the optimal choice of the firm, and therefore profits will scale by the same factor t.

3. Profit function is a convex function of p

   - Profit function 對 Price 做二階微分，可得 Hessian Matrix。
   - 由於 Profit function 具 convexity， Hessian Matrix is symmetric and positive semidefinite.

$$
\mathrm{H} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial p_1^2} & \frac{\partial^2 \pi}{\partial p_1 p_2} \\ \frac{\partial^2 \pi}{\partial p_2 p_1} & \frac{\partial^2 \pi}{\partial p_2^2} \end{bmatrix} \overset{\text{Hotelling's Lemma}}{=} \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{bmatrix}
$$

- 第二個是substitution matrix。

> **Theorem: (Le Chatelier principle)**
>
> The firm will response more to price change in long run, since it has more factor to adjust in the long run than in the short run.
>
> **Proof:**
>
> Assume only 1 output and input prices are all fixed. → Profit function only depends on output price p.
>
> - Short run profit function: $\pi_S(p, z)$ (z is vector fixed in short run)
> - Long run profit function: $\pi_L(p) = \pi_S(p, z(p))$
> - Let p^* be some given output price and $z^* = z(p^*)$ be the optimal LR demand for the z-factor at p^*.
>
> The LR profits $\geq$ SR profits because the set of factors can be adjusted in the LR includes the subset of factors that can be adjusted in the short run.
>
> $$
> \begin{aligned}
> \pi(p, z(p)) &\geq \pi(p, z) \\
> h(p) = \pi_L(p) &- \pi_S(p, z^*) \\
> = \pi_S(p, z(p)) &- \pi_S(p, z^*) \geq 0
> \end{aligned}
> $$
>
> for all price p because $h(p)$ reaches a minimum at $p = p^*$.
>
> - FOC $= 0$ at p^*
>
>   By Hotelling's Lemma, we see that the LR and SR supplies for each good must be equal at p^*.
>
> - SOC ≥ 0 ($\because p^*$ is a minimum choice of $h(p)$)

$$\frac{\partial^2 h(p)}{\partial p^2} = \frac{\partial^2 \pi_L(p^*)}{\partial p^2} - \frac{\partial^2 \pi_S(p^*, z^*)}{\partial p^2} \geq 0$$

$$\underset{\text{(By Hotelling's Lemma)}}{=} \frac{dy_L(p^*)}{dp} - \frac{\partial y_S(p^*, z^*)}{\partial p} \geq 0$$

This expression implies that LR supply response to a price change in prices at least as large as the SR supply response at $z^* = z(p^*)$