Chapter 1 Technology

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Tags	

1.1 Measurement of inputs and outputs

A firm produces outputs from various combination of inputs, like capital, labor, land, … etc.

1.2 Specification of technology

Assumption: suppose the firm has n possible goods to serve as inputs and/or outputs.

1. Net output

$$\begin{aligned} y_j^i &\implies \text{unit of good } j \text{ serve as an input} \\ y_j^o &\implies \text{unit of good } j \text{ serve as an output} \end{aligned} \quad \forall j = 1, \dots, n$$

The **scalar** $y_j = y_j^o - y_j^i \in \mathbb{R}$ is net output of good j.

$$\begin{cases} y_j > 0 & \implies \text{net output} \\ y_j < 0 & \implies \text{net input} \end{cases}$$

2. Production plan(生產計畫):

the **vector** $\mathrm{y} \in \mathbb{R}^n$ is a list of net output of various goods.

$$\begin{aligned}\mathrm{y} &= (y_1, \ldots, y_n) \\ &= (y,\ -x_1,\ -x_2,\ \ldots,\ -x_n) \\ &= (y, -\mathrm{x})\end{aligned}$$

3. Production probabilities set(生產可能集合):

$$\mathrm{Y} \subset \mathbb{R}^n$$

The set of Y is supposed to describe all patterns of inputs and outputs that are technologically feasible. It gives us a complete description of the technological possibilities facing the firm.

4. The restricted (or short-run) production possibility set:

A vector $\mathrm{z} \in \mathbb{R}^n$ could be a list of the maximum amount of the various inputs and outputs that can be produced in the time period under consideration. The restricted production possibility set $\mathrm{Y}(\mathrm{z})$ is 「在 z 限制水準下所有技術可行生產計畫所形成的集合。」

💡 **Example: short-run production possibilities set**

l units of labor, where labor can be varied immediately;

k units of capital, where capital is fixed at the level $\bar{k}$ in the short run.

The SR production possibility set is written as follows:

$$\mathrm{Y}(\bar{k}) = \{(y, -l, -k) \in \mathrm{Y} : k = \bar{k}\}$$

5. Input requirement set:

The input requirement set $V(\mathrm{y})$ is a set of all input bundles that produce at least y units of output.

$$V(\mathrm{y}) = \{\mathrm{x} \in \mathbb{R}^n_+ : \mathrm{y} = (y, -\mathrm{x}) \in \mathrm{Y}\}$$

6. Isoquant(等產量曲線):

$$Q(y) = \{\mathrm{x} \in \mathbb{R}^n_+ :\ \mathrm{x} \in V(y) \text{ and } \mathrm{x} \notin V(y') \quad \forall\, y' > y\}$$

The isoquant gives all input bundles that produce y units of output.

7. Production function(生產函數):

   If the firm has only one output, then the production function:

$$f(x) = \{y \in \mathbb{R} : y \text{ is the maximum output associate -x in Y}\}$$

8. Transformation function:

   A production plan $\mathrm{y} \in \mathrm{Y}$ is (technologically) efficient if there is no $\mathrm{y}' \in \mathrm{Y}$ such that $\mathrm{y}' \geq \mathrm{y}$; that is a production plan is efficient if there is no way to produce more output with the same inputs or to produce the same output with less inputs.(在既定的投入組合下無法有更高的產出水準。) We often assume that we can describe the set of technologically efficient production plans by a transformation function T.

$$T : R^n \rightarrow R \text{ where } T(\mathrm{y}) = 0 \iff \mathrm{y} \text{ is efficient}$$

   Just as a production function picks out the maximum scalar output as a function of the inputs, the transformation function picks out the maximal vectors of the net outputs.

💡 **Example: Cobb-Douglas technology**

Assume 1 output(y), 2 inputs(x_1, x_2).

Let α be parameter such that $0 < \alpha < 1$.

$$\begin{aligned}
\mathrm{Y} &= \{(y, -x_1, -x_2) \in \mathbb{R}^3 : \ y \leq x_1^{\alpha} x_2^{1-\alpha}\} \\
V(y) &= \{(x_1, x_2) \in \mathbb{R}_+^2 : \ y \leq x_1^{\alpha} x_2^{1-\alpha}\} \\
Q(y) &= \{(x_1, x_2) \in \mathbb{R}_+^2 : \ y = x_1^{\alpha} x_2^{1-\alpha}\} \\
\mathrm{Y(z)} &= \{(y, -x_1, -x_2) \in \mathbb{R}^3 : \ y \leq x_1^{\alpha} x_2^{1-\alpha} \ x_2 = \mathrm{z}\} \\
T(y, x_1, x_2) &= y - x_1^{\alpha} x_2^{1-\alpha} \\
f(x_1, x_2) &= x_1^{\alpha} x_2^{1-\alpha}
\end{aligned}$$

> **Example: Leotief technology**
>
> Assume 1 output(y), 2 inputs(x_1, x_2).
>
> Let α, β be parameters such that $0 < \alpha,\ 0 < \beta$.
>
> Leotief technology is defined as:
>
> $$\begin{aligned} \mathrm{Y} &= \{(y, -x_1, -x_2) \in \mathbb{R}^3 :\ y \le \min(\alpha x_1, \beta x_2)\} \\ V(y) &= \{(x_1, x_2) \in \mathbb{R}^2_+ :\ y \le \min(\alpha x_1, \beta x_2)\} \\ Q(y) &= \{(x_1, x_2) \in \mathbb{R}^2_+ :\ y = \min(\alpha x_1, \beta x_2)\} \\ \mathrm{Y}(\mathrm{z}) &= \{(y, -x_1, -x_2) \in \mathbb{R}^3 :\ y \le \min(\alpha x_1, \beta x_2),\ x_2 = \mathrm{z}\} \\ T(y, x_1, x_2) &= y - \min(\alpha x_1, \beta x_2) \\ f(x_1, x_2) &= \min(\alpha x_1, \beta x_2) \end{aligned}$$

1.3 Activity Analysis

The most straightforward way of describing production sets or input requirement sets is simply to list the feasible production plans. Suppose that we can produce an output good using factor inputs 1 and 2. There are 2 different activities or techniques by which this production can take place.

	Factor 1	Factor 2	Output
Technique A	1	2	1
Technique B	2	1	1

- Production set

$$\mathrm{Y} = \{\mathrm{y}_A,\ \mathrm{y}_B\} = \{(1, -1, -2),\ (1, -2, -1)\}$$

- Input requirement set

$$\begin{aligned} \mathrm{V}(1) &= \{(1, 2),\ (2, 1)\} \\ \mathrm{V}(y) &= \{\underbrace{(y, 2y)}_{\text{tech A}},\ \underbrace{(2y, y)}_{\text{tech B}}\} \end{aligned}$$

However, this set does not include all the relevant possibilities. Why? What about we use a mixture of technique A and B? In this case we have to let y_A be the amount of output produced using technique A and y_B be the amount of output produced using technique B. Assume $y = y_A + y_B$.

$$\mathrm{V}(y) = \{(y_A, 2y_A),\ (2y_B, y_B) : y = y_A + y_B\}$$

💡 **Example:**

$$\mathrm{V}(2) = \{\underbrace{(2,4),}_{\text{tech A}}\ \underbrace{(3,3)}_{\text{tech A, B}},\ \underbrace{(4,2)}_{\text{tech B}}\}$$

1.4 Monotonic technologies

Definition: Monotonicity

> $$\begin{aligned} \mathrm{y} \in \mathrm{Y} \quad \mathrm{y}' \leq \mathrm{y} &\implies \mathrm{y}' \in \mathrm{Y} \\ \mathrm{x} \in \mathrm{V}(y) \quad \mathrm{x}' \geq \mathrm{x} &\implies \mathrm{x}' \in \mathrm{V}(y). \end{aligned}$$

If free disposable is allowed, it is reasonable to argue that if x is a feasible way to produce y units of output and x' is an input vector with at least as much of each input, then x' should be a feasible way to produce y. Thus, the input requirement sets should be monotonic in the following sense:

If $\mathrm{y}' \leq \mathrm{y}$, it means that every component of vector y' is less than or equal to the corresponding component of vector y. This mean that the production plan represented by y' produces an equal or smaller amount of all outputs by using at least as much of all inputs, as compared to y. Hence, it is natural to suppose that if y is feasible, y' is also feasible.

💡 **Example:**

Assume 1 output(y), 3 inputs($x_1,\ x_2,\ x_3$).

$\mathrm{y} = (y,\ -x_1,\ -x_2,\ -x_3),\ \mathrm{y}' = (y',\ -x_1',\ -x_2',\ -x_3')$.

$$
\begin{aligned}
\mathrm{y}' \le \mathrm{y} &\implies y' \le y\ \&\ -x_1' \le -x_1\ \&\ -x_2' \le -x_2\ \&\ -x_3' \le -x_3 \\
&\implies y' \le y\ \&\ x_1' \ge x_1\ \&\ x_2' \ge x_2\ \&\ x_3' \ge x_3
\end{aligned}
$$

有一投入組合 $\mathrm{x} = (x_1,\ x_2,\ ,\ldots,\ x_n)$

$$\mathrm{x}' \ge \mathrm{x} \implies x_1' \ge x_1\ \&\ x_2' \ge x_2\ \&\ \ldots\ \&\ x_n' \ge x_n$$

1.5 Convex technologies

Definition: (Convexity)

> If x and x' are in $\mathrm{V}(y)$, then $t\mathrm{x} + (1-t)\mathrm{x}'$ is in $\mathrm{V}(y)$ for all $0 \le t \le 1$. That is, $\mathrm{V}(y)$ is a convex production set.
>
> Convexity of $V(y)$ means that of x and x' can produce y units of output, then any weighted average $t\mathrm{x} + (1-t)\mathrm{x}'$ can also produce y units of output.

Theorem:

Convex production set implies convex input requirement set. If the production set (Y) is a convex set, than the associated input requirement set $\mathrm{V}(y)$ is also a convex set.

Proof:

If Y is a convex set and $\mathrm{y} = (y, -\mathrm{x}),\ \mathrm{y}' = (y, -\mathrm{x}') \in \mathrm{Y}$, we must have

$$
\begin{aligned}
t\mathrm{y} + (1-t)\mathrm{y}' &\in \mathrm{Y} \\
[ty + (1-t)y,\ -t\mathrm{x} - (1-t)\mathrm{x}'] &\in \mathrm{Y} \\
[y,\ -t\mathrm{x} - (1-t)\mathrm{x}'] &\in \mathrm{Y}
\end{aligned}
$$

This is simply requiring that $[y,\ -t\mathrm{x} - (1-t)\mathrm{x}'] \in \mathrm{Y}$.

If follows that if $\mathrm{x},\ \mathrm{x}' \in \mathrm{V}(y)$, $t\mathrm{x} + (1-t)\mathrm{x}' \in \mathrm{V}(y)$ which shows that $\mathrm{V}(y)$ is convex.

Definition: (Quasi-concave)

> The production function $f(\mathrm{x})$ is a quasi-concave function if and only if $\mathrm{V}(y)$ is a convex set.

Theorem:

Convex input requirement set is equivalent to quasi-concave production function.

1.6 Regular technologies

Definition: (Regular)

$V(y)$ is a a closed, nonempty set for all $y \geq 0$.

1.7

Skip.

1.8 The technical rate of substitution

Assume that we have some technology summarized by a smooth production function and we are producing at a particular point $y^* = f(x_1^*, x_2^*)$. Suppose that we want to increase the amount of input 1 and decrease the amount of input 2 so as to maintain a constant level of output. How can we determine this technical rate of substitution between these two factors?

- In the 2-dimensional case, the technical rate of substitution is just the slope of isoquant. Now one have to adjust x_2 to keep output constant when x_1 change by a small amount.
- In the n-dimensional case, the technical rate of substitution is the slope of an isoquant surface measured at a particular direction.

How to express TRS:

- **Method 1: implicit function**

$$f(x_1, x_2(x_1)) = y.$$

Differentiating the above identity, we find:

$$\frac{\partial f(x^*)}{\partial x_1} + \frac{\partial f(x^*)}{\partial x_2} \cdot \frac{\partial x_2(x_1^*)}{\partial x_1} = 0$$
$$\implies \text{TRS} = \frac{\partial x_2(x_1^*)}{\partial x_1} = -\frac{\partial f(x^*)/\partial x_1}{\partial f(x^*)/\partial x_2} = -\frac{\text{MPP}_{x_1}}{\text{MPP}_{x_2}}.$$

- **Method 2: Total differential** $f(x_1, x_2) = y$.

$$dy = \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2.$$

Since output remains constant, we have

$$\frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 = 0$$

which we can solve for

$$\begin{aligned}
\text{TRS} = \frac{dx_2}{dx_1} &= -\frac{\partial f(x)/\partial x_1}{\partial f(x)/\partial x_2} \\
&= -\frac{\text{mpp}_{x_1}}{\text{mpp}_{x_2}} \\
&= -\frac{\text{mpp}_{x_1}}{\text{mpp}_{x_2}} \cdot \frac{p}{p} \\
&= -\frac{w_1}{w_2}
\end{aligned}$$

where w_i is factor price.

💡 **Example: TRS for CD technology**

$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, $\mathrm{x} = (x_1, x_2)$

$$\frac{\partial f(\mathrm{x})}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha}$$
$$\frac{\partial f(\mathrm{x})}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha}$$
$$\mathrm{TRS} = \frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial f(x)/\partial x_1}{\partial f(x)/\partial x_2} = -\frac{a}{1-a} \cdot \frac{x_2}{x_1}$$

1.9 The elasticity rate of substitution

- The TRS measures the slope of an isoquant.
- The elasticity of substitution measure the curvature of an isoquant.

Definition: (Elasticity of substitution)

> Measure the percentage change in the factor ratio divided by the percentage change in TRS, with output being held fixed.
>
> $$\sigma = \frac{\frac{\Delta x_2/x_1}{x_2/x_1}}{\frac{\Delta \mathrm{TRS}}{\mathrm{TRS}}}$$

It expresses how the ratio of factor inputs changes as the slope of the isoquant changes. If we assume the percents change is small and take the limit of this expression as Δ goes to zero.

$$\sigma = \frac{\mathrm{TRS}}{x_2/x_1} \cdot \frac{d(x_2/x_1)}{d(\mathrm{TRS})} = \frac{d\ln(x_2/x_1)}{d\ln|\mathrm{TRS}|}$$
$$\left(\because \begin{aligned} \frac{d\ln x}{dx} &= \frac{1}{x} \\ \implies d\ln x &= \frac{1}{x} \cdot dx \end{aligned} \right)$$

💡 **Example: CD production function**

$f(x_1, x_2) = x_1^a x_2^{1-a}$

$$
\begin{aligned}
\text{TRS} &= \frac{-a}{1-a}\frac{x_2}{x_1} \\
\frac{x_2}{x_1} &= -\frac{a}{1-a} \cdot \text{TRS} \\
\ln \frac{x_2}{x_1} &= \ln \frac{a}{1-a} + \ln |\text{TRS}| \\
\sigma &= \frac{d \ln \frac{x_2}{x_1}}{d \ln |\text{TRS}|} = 1
\end{aligned}
$$

1.10 Returns to scale

Suppose that we are using some vector of inputs x to produce some output y and we decide to scale all inputs up or down by some amount $t \geq 0$. What will happen to the level of output?

Definition:

> - A technology exhibits constant return to scale if any of the following are satisfied:
>
>   Constant Returns to Scale(CRS, CRTS):
>
>   1. $\mathrm{y} \in \mathrm{Y} \implies t\mathrm{y} \in \mathrm{Y} \quad \forall t \geq 0$
>   2. $\mathrm{x} \in V(y) \implies t\mathrm{x} \in V(ty) \quad \forall t \geq 0$
>   3. $f(t\mathrm{x}) = tf(\mathrm{x}) \quad \forall t \geq 0 \implies$ production function $f(\mathrm{x})$ is in homogeneous of degree 1
>
> - Increasing Returns to Scale(IRTS, IRS):
>
>   A technology exhibits increasing returns to scale if $f(t\mathrm{x}) > tf(\mathrm{x}) \quad \forall t > 1$

> - Decreasing Returns to Scale
>
>   A technology exhibits decreasing returns to scale if $f(t\mathrm{x}) < tf(\mathrm{x}) \quad \forall t > 1$

Definition: (The elasticity of scale)

> The elasticity of scale measures the percentage increase in the output due to one percentage increase in all inputs.
>
> $$e(\mathrm{x}) = \frac{dy(t)/y}{dt/t}$$

Let $y = f(\mathrm{x})$ be the production function. Let t be a positive scalar, and consider the function $y(t) = f(t\mathrm{x})$. Evaluated at $t = 1$, we have

$$e(\mathrm{x}) = \frac{dy(t)}{dt} \cdot \frac{t}{y} \ |_{t=1} = \frac{df(t\mathrm{x})}{dt} \cdot \frac{t}{f(t\mathrm{x})} \ |_{t=1}$$

💡 **Example: Returns to scale and the CD technology**

Suppose $y = x_1^a x_2^b$.

1. Method 1:

   Hence,

$$\begin{aligned} f(tx_1,\, tx_2) &= t^{a+b} \cdot y \\ a+b>1 &\implies \text{IRTS} \\ a+b=1 &\implies \text{CRTS} \\ a+b<1 &\implies \text{DRTS} \end{aligned}$$

2. Method 2: elasticity of scale

$$\frac{d}{dt} f(t\mathrm{x}) = \frac{d(tx_1)^a (tx_2)^b}{dt} = (a+b)t^{a+b-1} x_1^a x_2^b$$

   Evaluating derivative at $t=1$ and dividing by $f(x_1,\, x_2) = x_1^a x_2^b$ gives us the result.

$$\frac{df(t\mathrm{x})}{dt} \frac{t}{f(t\mathrm{x})} |_{t=1} = \frac{d(tx_1)^a (tx_2)^b}{dt} \frac{t}{t^{a+b} x_1^a x_2^b} |_{t=1} = (a+b)$$

1.11 Homogeneous and homothetic technologies

Definition: (HD-k)

> A function is homogeneous of degree k if
>
> $$f(t\mathrm{x}) = t^k f(\mathrm{x}) \quad \forall t > 0$$

- HD-0(homogeneous of degree 0):

$$f(t\mathrm{x}) = t^0 f(\mathrm{x}) = f(\mathrm{x})$$

- HD-1(homogeneous of degree 1):

$$f(t\mathrm{x}) = t^1 f(\mathrm{x}) = t f(\mathrm{x})$$

Comparing this definition to the definition of constant returns to scale, we see that a technology has constant returns to scale if and only if its production function is homogeneous of degree one.

Definition: (Positive monotonic transformation)

> A function $g : \mathbb{R} \to \mathbb{R}$ is said to be a positive monotonic transformation if g is a strictly increasing function; that is a function for which $x > y \implies g(x) > g(y)$.

Definition: (Homothetic function)

A homothetic function is a (positive) monotonic transformation of a function that is homogeneous of degree 1. $f(x)$ is homothetic if and only if it can be written as $f(x) = g(h(x))$ where $g(\cdot)$ is monotonic function and $h(\cdot)$ is a homogeneous of degree one.

Example: The CES production function

The constant elasticity of substitution(CES) production function has the form

$$y=[a_1x_1^\rho+a_2x_2^\rho]^{\frac{1}{\rho}}$$

$$\begin{aligned}\text{TRS}&=-(\frac{x_1}{x_2})^{\rho-1}\\ \implies|-\text{TRS}|&=(\frac{x_2}{x_1})^{1-\rho}\\ \implies|\text{TRS}|^{\rho-1}&=(\frac{x_2}{x_1})\\ \implies\ln|\text{TRS}|^{\rho-1}&=\ln(\frac{x_2}{x_1})\\ \implies\sigma&=\frac{d\ln(\frac{x_2}{x_1})}{d\ln|\text{TRS}|^{\rho-1}}=\frac{1}{\rho-1}\end{aligned}$$

The CES production function takes on a variety of shapes depending on the value of the parameter ρ.

- $\rho=1$ linear production function

  $y=x_1+x_2$ 完全替代

- $\rho=0$ CD production function

$$\text{TRS}=-(\frac{x_1}{x_2})^{\rho-1}=-(\frac{x_1}{x_2})^{-1}=-(\frac{x_2}{x_1})$$

- $\rho=-\infty$ Leontief production function

$$\text{TRS}=-(\frac{x_1}{x_2})^{\rho-1}=-(\frac{x_1}{x_2})^{-1-\infty}=-(\frac{x_2}{x_1})^{1+\infty}$$

  - $x_1<x_2\implies|\text{TRS}|=-\infty$
  - $x_1>x_2\implies|\text{TRS}|=0$