Chapter 4 Cost Minimization

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4.1 Calculus Analysis of Cost Minimization

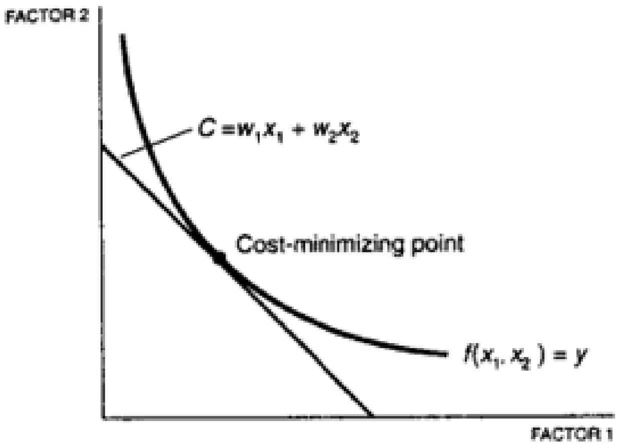
$$\min_{\mathbf{x}} \mathbf{wx}$$
$$\text{such that } f(\mathbf{x}) = y$$
$$\implies \mathcal{L}(\lambda, \mathbf{x}) = \mathbf{wx} - \lambda(f(\mathbf{x}) - y)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x_i} = w_i - \lambda \frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0 \text{ for } i = 1, \cdots, n$$
$$f(\mathbf{x}^*) = y$$
$$\implies \mathbf{w} = \lambda \nabla f(\mathbf{x}^*)$$

where $\nabla f(\mathbf{x}^*)$ is the gradient vector, the vector of partial derivatives of $f(\mathbf{x})$.
We can interpret these FOCs by dividing the
i-th condition by the j-th condition to get

$$
\begin{aligned}
\frac{w_i}{w_j} &= \frac{\partial f(\mathbf{x}^*)/\partial x_i}{\partial f(\mathbf{x}^*)/\partial x_j} \quad i,j = 1,\ldots,n \\
&= -\mathrm{TRS}
\end{aligned}
$$



- Curved line: isoquant

$$f(x_1, x_2) = y$$

When y fixed, the problem of the firm is to find a cost-minimizing point on a given isoquant.

- Straight line: constant cost curve(isocost line)

$$
\begin{aligned}
C &= w_1x_1 + w_2x_2 \\
\implies x_2 &= \frac{C}{w_2} - \frac{w_1}{w_2}x_1
\end{aligned}
$$

1. FOC:

   For fixed w_1 and w_2 the firm wants to find a point on a given isoquant where the associated constant cost curve has minimal vertical intercept. It is clear that such a point will be characterized by the tangency condition that the slope of the constant cost curve must be equal to the slope of the isoquant.

2. SOC:

> The isoquant must lie above the isocost line. Any change in factor inputs that keeps costs constant—that is, a movement along the isocost line—must result in output decreasing or remaining constant.

Let (h_1, h_2) be a small change in factors 1 and 2 and consider the associated change in output. Assuming the necessary differentiability, we can write the second-order Taylor series expansion

$$
\begin{aligned}
f(x_1+h_1, x_2+h_2) \approx f(x_1, x_2) &+ \underbrace{\frac{\partial f}{\partial x_2}}_{f_1} h_1 + \underbrace{\frac{\partial f}{\partial x_2}}_{f_2} h_2 \\
&+ \frac{1}{2}[\underbrace{\frac{\partial^2 f}{\partial x_1^2}}_{f_{11}} h_1^2 + 2 \underbrace{\frac{\partial^2 f}{\partial x_1 \partial x_2}}_{f_{21} f_{12}} h_1 h_2 + \underbrace{\frac{\partial^2 f}{\partial x_2^2}}_{f_{12}} h_2^2].
\end{aligned}
$$

This can be written in matrix form as

$$
\begin{aligned}
f(x_1+h_1, x_2+h_2) \approx f(x_1, x_2) &+ \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\
&+ \frac{1}{2} \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
\end{aligned}
$$

Assume the production cost is holding constant

$$
\begin{gathered}
(x_1, x_2) \to (x_1+h_1, x_2+h_2) \\
w_1 x_1 + w_2 x_2 = w_1(x_1+h_1) + w_2(x_2+h_2) \\
w_1 h_1 + w_2 h_2 = (\lambda f_1) h_1 + (\lambda f_2) h_2 = \lambda[f_1 h_1 + f_2 h_2] = 0
\end{gathered}
$$

The first-order terms in this Taylor expansion must vanish for movements along the isocost line. Thus, the requirement that output decreases for any movement along the isocost line can be stated as

$$
\begin{gathered}
\begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \leq 0 \\
\forall\ (h_1, h_2) \text{ such that } \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0.
\end{gathered}
$$

In n-factor case,

$$\mathbf{h}^t \nabla^2 f(\mathbf{x}^*)\mathbf{h} \leq 0$$
$$\forall\ \mathbf{h} \text{ satisfying } \mathbf{wh} = 0.$$

Intuitively, at the cost-minimizing point:

- A first-order movement tangent to the isocost curve implies output remains constant.
- A second-order movement implies output decrease.

4.2 More on Second-order Condition

$$\mathcal{L}(\lambda, x_1, x_2) = w_1 x_2 + w_2 x_2 - \lambda[f(x_1, x_2) - y]$$
$$f_1 = \frac{\partial f}{\partial x_1}, f_2 = \frac{\partial f}{\partial x_2}, f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Hessian matrix of the Lagrangian

$$\nabla^2 \mathcal{L}(\lambda^*, x_1^*, x_2^*) = \begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial^2 x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial^2 x_2} \end{bmatrix} = \begin{bmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{12} \\ -f_2 & -\lambda f_{21} & -\lambda f_{22} \end{bmatrix}$$

The last one is call bordered Hessian matrix.

💡 **Homework:**

If there are three factors of production.

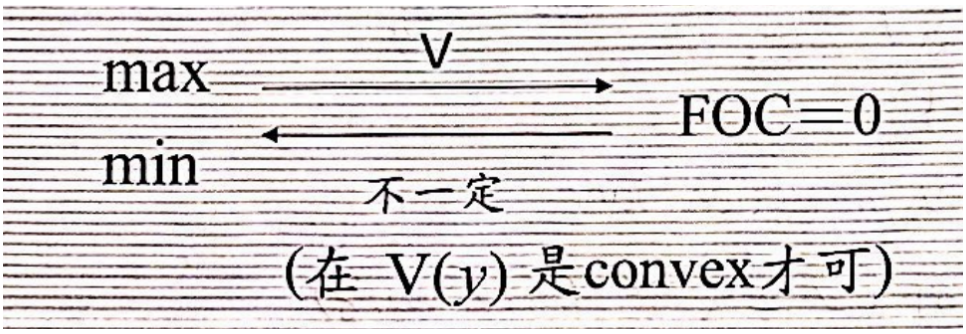
4.3 Difficulties

1. The technology in question may no be representable by a differentiable production function, so the calculus techniques cannot be applied.
2. Conditions are valid only for interior operating positions, they must be modified if a cost minimization point occurs on the boundary.

$$\lambda\frac{\partial f(\mathrm{x}^*)}{\partial x_i} - w_i \le 0 \quad \text{if } x_i^* = 0$$
$$\lambda\frac{\partial f(\mathrm{x}^*)}{\partial x_i} - w_i = 0 \quad \text{if } x_i^* > 0$$

3. bounded subset of $V(y)$. (input requirement set 一定要有界nonempty)

4. The first-order conditions may not determine a unique operating position of the firm. FOC is just a necessary condition, not sufficient for the existence of local optimum. They will uniquely describe a global optimum only under certain convexity conditions. So, $V(y)$ have to be convex for cost minimization problems.



Case 1: Cost function for CD technology

$$C(\mathbf{w}, y) = \min_{x_1, x_2} w_1x_1 + w_2x_2 \quad \text{s.t. } Ax_1^ax_2^b = y$$

- Method 1: (替代法)

  Substitute $Ax_1^ax_2^b = y$ into objective function

  $$C(\mathrm{w}, y) = \min_{x_1} w_1x_1 + w_2(A^{\frac{-1}{b}}y^{\frac{1}{b}}x_1^{\frac{-a}{b}}).$$

  The FOC is:

  $$w_1 - \frac{a}{b}w_2(A^{\frac{-1}{b}}y^{\frac{1}{b}}x_1^{-\frac{a+b}{b}}) = 0$$
  $$\implies w_1 = \frac{a}{b}w_2(A^{\frac{-1}{b}}y^{\frac{1}{b}}x_1^{-\frac{a+b}{b}})$$
  $$\implies x_1(w_1, w_2, y) = A^{\frac{-1}{a+b}}[\frac{aw_2}{bw_1}]^{\frac{b}{a+b}}y^{\frac{1}{a+b}}$$
  $$\implies x_2(w_1, w_2, y) = A^{\frac{-1}{a+b}}[\frac{aw_2}{bw_1}]^{-\frac{a}{a+b}}y^{\frac{1}{a+b}}$$

The cost function is

$$
\begin{aligned}
C(w_1, w_2, y) &= w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y) \\
&= A^{\frac{-1}{a+b}} w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}} [(\frac{a}{b})^{\frac{b}{a+b}} + (\frac{a}{b})^{\frac{-a}{a+b}}] \\
&= K w_1^a w_2^{1-a} y
\end{aligned}
$$

CD production function 的 cost function 也為 CD 形式。

> EXAMPLE: Cost function for the Cobb-Douglas technology
>
> Consider the cost minimization problem
>
> $$
> \begin{aligned}
> c(\mathbf{w}, y) = \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\
> \text{such that } A x_1^a x_2^b = y.
> \end{aligned}
> $$
>
> Solving the constraint for x_2, we see that this problem is equivalent to
>
> $$\min_{x_1} w_1 x_1 + w_2 A^{-\frac{1}{b}} y^{\frac{1}{b}} x_1^{-\frac{a}{b}}.$$
>
> The first-order condition is
>
> $$w_1 - \frac{a}{b} w_2 A^{-\frac{1}{b}} y^{\frac{1}{b}} x_1^{-\frac{a+b}{b}} = 0,$$
>
> which gives us the conditional demand function for factor 1:
>
> $$x_1(w_1, w_2, y) = A^{-\frac{1}{a+b}} \left[\frac{a w_2}{b w_1}\right]^{\frac{b}{a+b}} y^{\frac{1}{a+b}}.$$

Case 2: Cost function for CES technology

EXAMPLE: The cost function for the CES technology

Suppose that $f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$. What is the associated cost function? The cost minimization problem is

$$\begin{aligned} \min\ & w_1 x_1 + w_2 x_2 \\ \text{such that } & x_1^\rho + x_2^\rho = y^\rho \end{aligned}$$

The first-order conditions are

$$\begin{aligned} w_1 - \lambda\rho x_1^{\rho-1} &= 0 \\ w_2 - \lambda\rho x_2^{\rho-1} &= 0 \\ x_1^\rho + x_2^\rho &= y^\rho. \end{aligned}$$

Solving the first two equations for x_1^ρ and x_2^ρ, we have

$$\begin{aligned} x_1^\rho &= w_1^{\frac{\rho}{\rho-1}} (\lambda\rho)^{\frac{-\rho}{\rho-1}} \\ x_2^\rho &= w_2^{\frac{\rho}{\rho-1}} (\lambda\rho)^{\frac{-\rho}{\rho-1}}. \end{aligned} \tag{4.5}$$

Substitute this into the production function to find

$$(\lambda\rho)^{\frac{-\rho}{\rho-1}} \left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right] = y^\rho.$$

Solve this for $(\lambda\rho)^{\frac{-\rho}{\rho-1}}$ and substitute into the system (4.5). This gives us the conditional factor demand functions

$$x_1(w_1,w_2,y) = w_1^{\frac{1}{\rho-1}}\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-\frac{1}{\rho}} y$$
$$x_2(w_1,w_2,y) = w_2^{\frac{1}{\rho-1}}\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-\frac{1}{\rho}} y.$$

Substituting these functions into the definition of the cost function yields

$$\begin{aligned} c(w_1,w_2,y) &= w_1x_1(w_1,w_2,y) + w_2x_2(w_1,w_2,y) \\ &= y\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{-\frac{1}{\rho}} \\ &= y\left[w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho-1}{\rho}}. \end{aligned}$$

This expression looks a bit nicer if we set $r = \rho/(\rho-1)$ and write

$$c(w_1,w_2,y) = y\left[w_1^r + w_2^r\right]^{\frac{1}{r}}.$$

Note that this cost function has the same form as the original CES production function with r replacing ρ. In the general case where

$$f(x_1,x_2) = \left[(a_1x_1)^\rho + (a_2x_2)^\rho\right]^{\frac{1}{\rho}},$$

similar computations can be done to show that

$$c(w_1,w_2,y) = \left[(w_1/a_1)^r + (w_2/a_2)^r\right]^{\frac{1}{r}} y.$$

$$y = ax + b$$

💡 **Example:**

qweqwe

$$y = ax + b$$

Theorem: (name)

qweqwe

$$y = ax + b$$

Proof:

qweqwe

$$y = ax + b$$

Case 3: Cost function for Leontif technology

EXAMPLE: The cost function for the Leontief technology

Suppose $f(x_1, x_2) = \min\{ax_1, bx_2\}$. What is the associated cost function? Since we know that the firm will not waste any input with a positive price, the firm must operate at a point where $y = ax_1 = bx_2$. Hence, if the firm wants to produce y units of output, it must use y/a units of good 1 and y/b units of good 2 no matter what the input prices are. Hence, the cost function is given by

$$c(w_1, w_2, y) = \frac{w_1 y}{a} + \frac{w_2 y}{b} = y\left(\frac{w_1}{a} + \frac{w_2}{b}\right).$$

$$y = ax + b$$

Example:

qweqwe

$$y = ax + b$$

Theorem: (name)

qweqwe

$$y = ax + b$$

Proof:

qweqwe

$$y = ax + b$$

Case 4: Cost function for linear technology

EXAMPLE: The cost function for the linear technology

Suppose that $f(x_1, x_2) = ax_1 + bx_2$, so that factors 1 and 2 are perfect substitutes. What will the cost function look like? Since the two goods are perfect substitutes, the firm will use whichever is cheaper. Hence, the cost function will have the form $c(w_1, w_2, y) = \min\{w_1/a, w_2/b\}y$.

In this case the answer to the cost-minimization problem typically involves a boundary solution: one of the two factors will be used in a zero amount. Although it is easy to see the answer to this particular problem, it is worthwhile presenting a more formal solution since it serves as a nice example of the Kuhn-Tucker theorem in action. The Kuhn-Tucker theorem is the appropriate tool to use here, since we will almost never have an interior solution. See Chapter 27, page 503, for a statement of this theorem.

For notational convenience we consider the special case where $a = b = 1$. We pose the minimization problem as

$$\begin{aligned} \min\ & w_1x_1 + w_2x_2 \\ \text{s.t. } & x_1 + x_2 = y \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

The Lagrangian for this problem can be written as

$$\mathcal{L}(\lambda, \mu_1, \mu_2, x_1, x_2) = w_1x_1 + w_2x_2 - \lambda(x_1 + x_2 - y) - \mu_1x_1 - \mu_2x_2.$$

The Kuhn-Tucker first-order conditions are

$$\begin{aligned} w_1 - \lambda - \mu_1 &= 0 \\ w_2 - \lambda - \mu_2 &= 0 \\ x_1 + x_2 &= y \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

and the complementary slackness conditions are

$$\begin{aligned} \mu_1 \geq 0,\ \mu_1 = 0 \text{ if } x_1 > 0 \\ \mu_2 \geq 0,\ \mu_2 = 0 \text{ if } x_2 > 0. \end{aligned}$$

In order to determine the solution to this minimization problem, we have to examine each of the possible cases where the inequality constraints are binding or not binding. Since there are two constraints and each can be binding or not binding, we have four cases to consider.

$$y = ax + b$$

💡 **Example:**

qweqwe

$$y = ax + b$$

Theorem: (name)

qweqwe

$$y = ax + b$$

Proof:

qweqwe

$$y = ax + b$$

4.4 Conditional Factor Demand Function

The conditional factor demand functions $\mathbf{x}(\mathbf{w}, y)$ must satisfy the FOC:

$$\begin{aligned} f(\mathbf{x}(\mathbf{w}, y)) &\equiv y \\ \mathbf{w} - \lambda \nabla f(\mathbf{x}(\mathbf{w}, y)) &\equiv \mathbf{0} \end{aligned}$$

1 output, 2 inputs caseb

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &\implies f(x_1(w_1, w_2, y), x_2(w_1, w_2, y)) && \equiv y \\ \frac{\partial \mathcal{L}}{\partial x_1} &\implies w_1 - \lambda \frac{\partial f(x_1(w_1, w_2, y), x_2(w_1, w_2, y))}{\partial x_1} && \equiv 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &\implies w_2 - \lambda \frac{\partial f(x_1(w_1, w_2, y), x_2(w_1, w_2, y))}{\partial x_2} && \equiv 0 \end{aligned}$$

$$
\begin{aligned}
\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial w_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial w_1} &\equiv 0 \\
1-\lambda[\frac{\partial^2 f}{\partial x_1^2}\frac{\partial x_1}{\partial w_1}+\frac{\partial^2 f}{\partial x_1 x_2}\frac{\partial x_2}{\partial w_1}]-\frac{\partial f}{\partial x_1}\frac{\partial \lambda}{\partial w_1} &\equiv 0 \\
0-\lambda[\frac{\partial^2 f}{\partial x_1 x_2}\frac{\partial x_1}{\partial w_1}+\frac{\partial f}{\partial x_2^2}\frac{\partial x_2}{\partial w_1}]-\frac{\partial f}{\partial x_2}\frac{\partial \lambda}{\partial w_1} &\equiv 0
\end{aligned}
$$

$$
\begin{bmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{bmatrix}
\begin{bmatrix} \frac{\partial \lambda}{\partial w_1} \\ \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{bmatrix}
=
\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}
$$

Use **Carmer's Rule,**

$$
\frac{\partial x_1}{\partial w_1}=\frac{\begin{vmatrix} 0 & 0 & -f_2 \\ -f_1 & -1 & -\lambda f_{21} \\ -f_2 & 0 & -\lambda f_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{vmatrix}}=\frac{f_2^2}{H}<0
$$

$$
\frac{\partial x_1}{\partial w_2}=\frac{\begin{vmatrix} 0 & -f_1 & 0 \\ -f_1 & -\lambda f_{11} & -1 \\ -f_2 & -\lambda f_{12} & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{vmatrix}}=-\frac{f_2 f_1}{H}>0
$$

Use matrix expression,

$$
\begin{aligned}
f(\mathbf{x}(\mathbf{w})) &\equiv y \\
\mathbf{w}-\lambda\nabla f(\mathbf{x}(\mathbf{w})) &\equiv \mathbf{0}
\end{aligned}
$$

Differentiating these identities

$$
\begin{aligned}
\nabla f(\mathbf{x}(\mathbf{w}))\nabla\mathbf{x}(\mathbf{w}) &\equiv \mathbf{0} \\
\mathbf{I}-\lambda\nabla^2 f(\mathbf{x}(\mathbf{w}))\nabla\mathbf{x}(\mathbf{w})-\nabla f(\mathbf{x}(\mathbf{w}))\nabla\mathbf{x}(\mathbf{w}) &\equiv \mathbf{0}
\end{aligned}
$$

Rearrange give us

$$\begin{pmatrix} 0 & -\mathbf{D}f(\mathbf{x}) \\ -\mathbf{D}f(\mathbf{x})^t & -\lambda\mathbf{D}^2 f(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{D}\lambda(\mathbf{w}) \\ \mathbf{D}\mathbf{x}(\mathbf{w}) \end{pmatrix} = -\begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix}.$$

4.5 Algebraic approach to cost minimization

Theorem: Weak Axiom of Cost Minimization(WACM)

> From output level y^t for factor price $\mathbf{w}^t$ and factor levels $\mathbf{x}^t$ for $t = 1, \ldots, T$
>
> $$\mathbf{w}^t\mathbf{x}^t \leq \mathbf{w}^t\mathbf{x}^s \quad \forall s, t \quad \text{s.t. } y^s \geq y^t.$$
>
> **Proof:**
>
> $$\mathbf{w}^t\mathbf{x}^t \leq \mathbf{w}^t\mathbf{x}^s$$
> $$\mathbf{w}^s\mathbf{x}^s \leq \mathbf{w}^s\mathbf{x}^t$$
> $$\implies \begin{aligned} \mathbf{w}^t\mathbf{x}^t - \mathbf{w}^t\mathbf{x}^s &\leq 0 \\ -\mathbf{w}^s\mathbf{x}^t + \mathbf{w}^s\mathbf{x}^s &\leq 0 \end{aligned}$$
> $$\implies \mathbf{w}^t\mathbf{x}^t - \mathbf{w}^t\mathbf{x}^s - \mathbf{w}^s\mathbf{x}^t + \mathbf{w}^s\mathbf{x}^s \leq 0$$
> $$\implies (\mathbf{w}^t - \mathbf{w}^s)(\mathbf{x}^t - \mathbf{x}^s) \leq 0$$
> $$\implies \Delta\mathbf{w}\Delta\mathbf{x} \leq 0$$
