Simultaneous Localization and Mapping (SLAM)

Lecture 03

Recall

SLAM Fundamentals

Process Model

- The state transition model for the vehicle is given as

$$x_v(k) = f\left(x_v(k-1), u(k-1), w(k-1)\right)$$

where F is

$$F_{[i,j]} = \frac{\partial f_{[i]}}{\partial x_{[j]}}\left(\hat{x}(k-1), u(k-1), 0\right)$$

- The augmented state vector containing both the state of the vehicle and the state of all landmark locations is

$$x(k) = \begin{bmatrix} x_v^T(k) & p_1^T & \dots & p_N^T \end{bmatrix}^T$$

SLAM Fundamentals

Recall

Observation Model

- The observation model is given as

$$z(k) = h\left(x(k), v(k)\right)$$

where

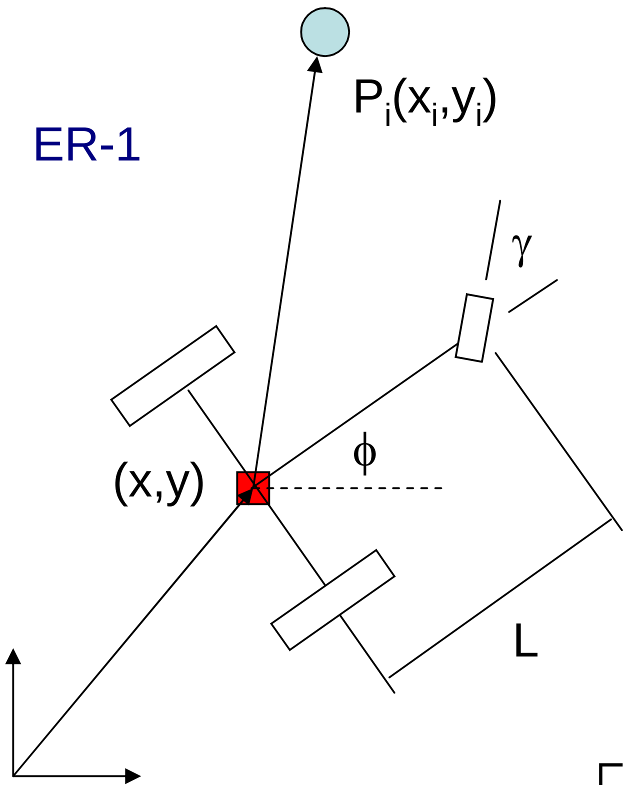
- $v(k)$ is a vector of uncorrelated observation errors with zero mean and variance $R(k)$
- h is a non-linear function that relates the sensor output $z(k)$ to the state vector $x(k)$ when observing a landmark and is written as

$$J_{h_{[i,j]}} = \frac{\partial h_{[i]}}{\partial x_{[j]}}\left(\tilde{x}(k), 0\right)$$

SLAM Example

A Single Landmark

Robot Process Model



Radar Location

Kinematic Equations

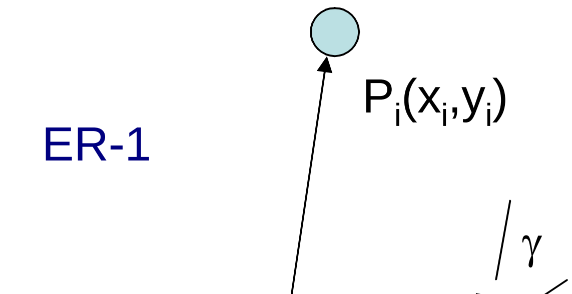
$$\left.\begin{aligned}
\dot{x} &= V\cos\varphi \\
\dot{y} &= V\sin\varphi \\
\dot{\varphi} &= \frac{V\tan\gamma}{L}
\end{aligned}\right\}$$

Non-linear!

f(x,u,w)

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \end{bmatrix} + w(k)$$

Robot Process Model

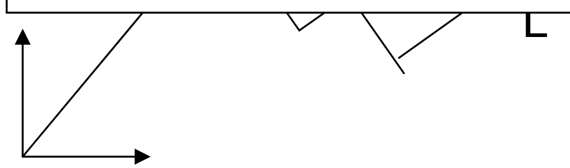


Kinematic Equations

$$\dot{x} = V\cos\varphi$$

$$\dot{y} = V\sin\varphi$$

Objective: Based on system inputs, V and γ (with sensor feedback for regulation, i.e. optical encoders) at time k, estimate the vehicle position at time (k+1)

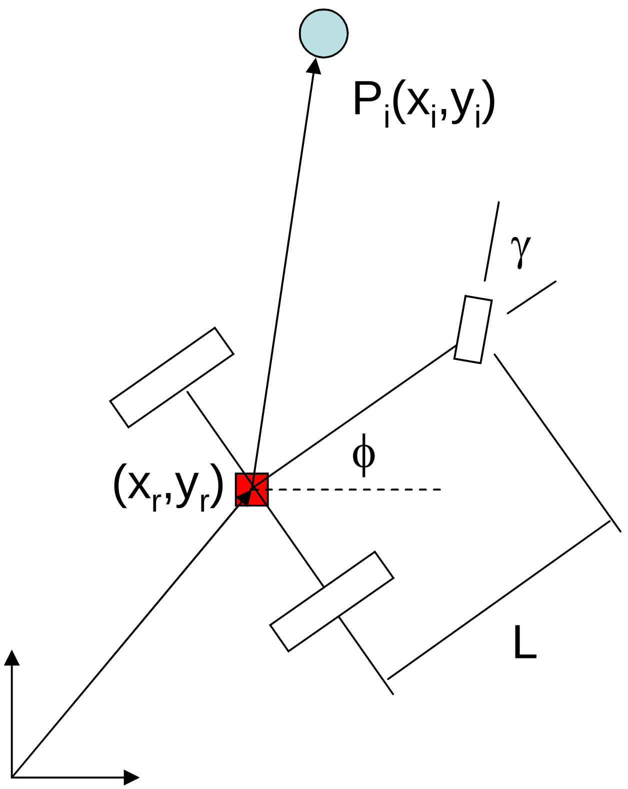


Radar Location

f(x,u,w)

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \end{bmatrix} + w(k)$$

Landmark Process Model



Radar Location

Recall that in the SLAM algorithm, landmarks are assumed to be stationary. Therefore,

$$p_i(k+1) = p_i(k)$$

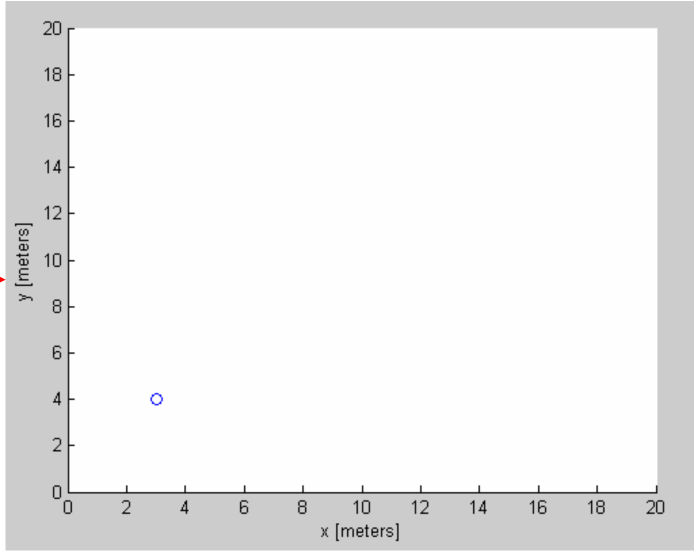
$$\begin{bmatrix} x_i(k+1) \\ y_i(k+1) \end{bmatrix} = \begin{bmatrix} x_i(k) \\ y_i(k) \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ y_1(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ y_1(k) \end{bmatrix}$$

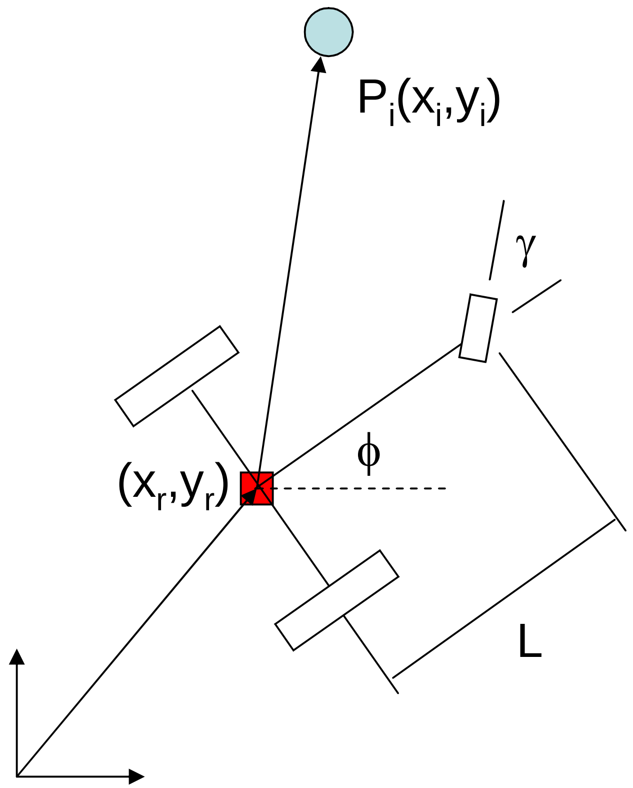
Overall System Process Model

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \\ x_1(k+1) \\ y_1(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

Landmark 1: (3,4)



Observation Model



Radar Location

$$z(k) = h\big(x(k), v(k)\big)$$

The radar used in the experiment returns the range $r_i(k)$ and bearing $\theta_i(k)$ to a landmark i. Thus, the observation model is

$$r_i(k) = \sqrt{\big(x_i - x_r(k)\big)^2 + \big(y_i - y_r(k)\big)^2} + v_r(k)$$

$$\theta_i(k) = \arctan\left(\frac{y_i - y_r(k)}{x_i - x_r(k)}\right) - \varphi(k) + v_\theta(k)$$

The Estimation Process (EKF)

Prediction

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

$$F(k) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \varphi} & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial y_1} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial \varphi} & \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial y_1} \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial y} & \frac{\partial f_5}{\partial \varphi} & \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial y_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta t V(k)\sin\varphi(k) & 0 & 0 \\ 0 & 1 & \Delta t V(k)\cos\varphi(k) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Estimation Process (EKF)

Prediction

$$x(k+1)=\begin{bmatrix} x(k)+\Delta tV(k)\cos\varphi(k) \\ y(k)+\Delta tV(k)\sin\varphi(k) \\ \varphi(k)+\dfrac{\Delta tV(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix}+\begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

$$W(k)=\begin{bmatrix} \frac{\partial f_1}{\partial w_x} & \frac{\partial f_1}{\partial w_y} & \frac{\partial f_1}{\partial w_\varphi} & \frac{\partial f_1}{\partial w_{x_1}} & \frac{\partial f_1}{\partial w_{y_1}} \\ \frac{\partial f_2}{\partial w_x} & \frac{\partial f_2}{\partial w_y} & \frac{\partial f_2}{\partial w_\varphi} & \frac{\partial f_1}{\partial w_{x_1}} & \frac{\partial f_1}{\partial w_{y_1}} \\ \frac{\partial f_3}{\partial w_x} & \frac{\partial f_3}{\partial w_y} & \frac{\partial f_3}{\partial w_\varphi} & \frac{\partial f_1}{\partial w_{x_1}} & \frac{\partial f_1}{\partial w_{y_1}} \\ \frac{\partial f_4}{\partial w_x} & \frac{\partial f_4}{\partial w_y} & \frac{\partial f_4}{\partial w_\varphi} & \frac{\partial f_4}{\partial w_{x_1}} & \frac{\partial f_4}{\partial w_{y_1}} \\ \frac{\partial f_5}{\partial w_x} & \frac{\partial f_5}{\partial w_y} & \frac{\partial f_5}{\partial w_\varphi} & \frac{\partial f_5}{\partial w_{x_1}} & \frac{\partial f_5}{\partial w_{y_1}} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Estimation Process (EKF)

Kalman Gain

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

$$z(k) = \begin{bmatrix} r_i(k) \\ \theta_i(k) \end{bmatrix} = \begin{bmatrix} \sqrt{\left(x_i - \hat{x}(k)^{-}\right)^2 + \left(y_i - \hat{y}(k)^{-}\right)^2} \\ \tan^{-1}\left(\dfrac{y_i - \hat{y}(k)^{-}}{x_i - \hat{x}(k)^{-}} \right) - \hat{\varphi}(k)^{-} \end{bmatrix} + v(k)$$

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial \varphi} & \dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial y_1} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} & \dfrac{\partial h_2}{\partial \varphi} & \dfrac{\partial h_2}{\partial x_1} & \dfrac{\partial h_2}{\partial y_1} \end{bmatrix} = \begin{bmatrix} \dfrac{x - x_i}{r} & \dfrac{y - y_i}{r} & 0 & \dfrac{x_i - x}{r} & \dfrac{y_i - y}{r} \\ \dfrac{y_i - y}{r^2} & \dfrac{x - x_i}{r^2} & -1 & \dfrac{y - y_i}{r^2} & \dfrac{x_i - x}{r^2} \end{bmatrix}$$

where $r = \sqrt{(x_i - x)^2 + (y_i - y)^2}$

The Estimation Process (EKF)

Kalman Gain

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

$$z(k) = \begin{bmatrix} r_i(k) \\ \theta_i(k) \end{bmatrix} = \begin{bmatrix} \sqrt{\left(x_i - \hat{x}(k)^{-}\right)^2 + \left(y_i - \hat{y}(k)^{-}\right)^2} \\ \tan^{-1}\left(\dfrac{y_i - \hat{y}(k)^{-}}{x_i - \hat{x}(k)^{-}} \right) - \hat{\varphi}(k)^{-} \end{bmatrix} + v(k)$$

$$V(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial v_r} & \dfrac{\partial h_1}{\partial v_\theta} \\ \dfrac{\partial h_2}{\partial v_r} & \dfrac{\partial h_2}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Estimation Process (EKF)

Measurement Update

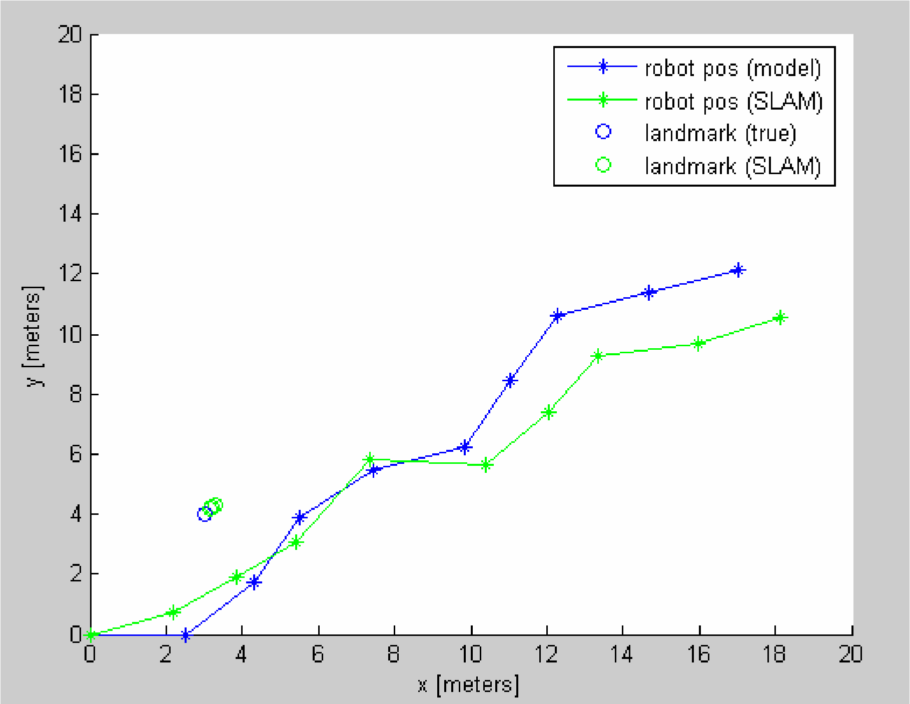
$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\big(\underline{z(k) - H(k)}\big)$$

$$P(k) = \big(I - K(k)J_h(k)\big)P(k)^{-}$$

Innovation

z(k) is 10 fabricated measurements of range and bearing to landmark 1.

There is only one landmark and it is incorporated into the model from the start.



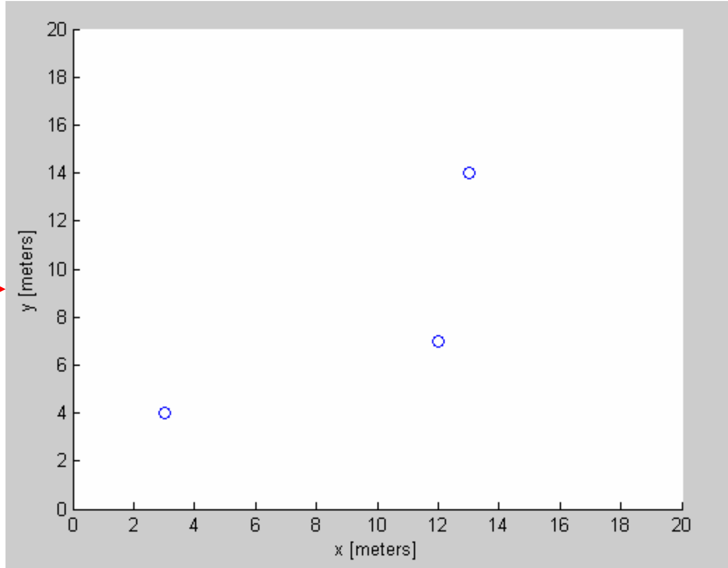
SLAM Example

Multiple Landmarks

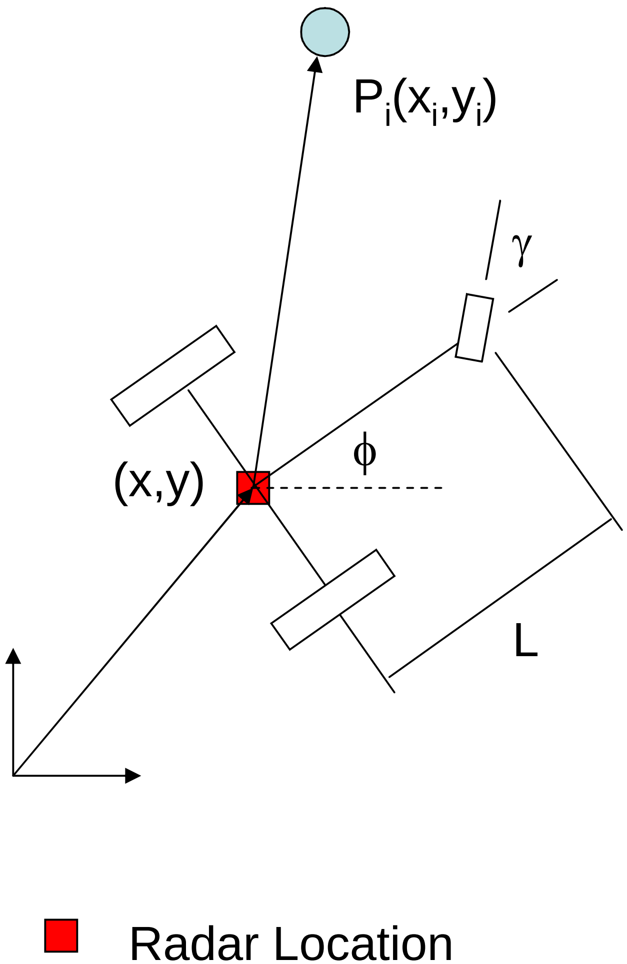
Overall System Process Model

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \\ p_1(k+1) \\ \vdots \\ p_N(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k) \cos\varphi(k) \\ y(k) + \Delta t V(k) \sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k) \tan\gamma(k)}{L} \\ p_1(k) \\ \vdots \\ p_N(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Landmark 1: (3,4)
Landmark 2: (12,7)
Landmark 3: (13,14)



Observation Model



$$z(k) = h\big(x(k), v(k)\big)$$

The radar used in the experiment returns the range $r_i(k)$ and bearing $\theta_i(k)$ to a landmark i. Thus, the observation model is

$$r_i(k) = \sqrt{\big(x_i - x_r(k)\big)^2 + \big(y_i - y_r(k)\big)^2} + v_r(k)$$

$$\theta_i(k) = \arctan\left(\frac{y_i - y_r(k)}{x_i - x_r(k)}\right) - \varphi(k) + v_\theta(k)$$

The Estimation Process (EKF)

Prediction

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \end{bmatrix} + w(k)$$

$$\hat{x}(k)^{-} = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$P(k)^{-} = F(k)P(k-1)F(k)^{T} + W(k)Q(k-1)W(k)^{T}$$

Initially, before landmarks are added

$$F(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial \varphi} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial \varphi} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta t V(k)\sin\varphi(k) \\ 0 & 1 & \Delta t V(k)\cos\varphi(k) \\ 0 & 0 & 1 \end{bmatrix} \qquad W(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial w_x} & \dfrac{\partial f_1}{\partial w_y} & \dfrac{\partial f_1}{\partial w_\varphi} \\ \dfrac{\partial f_2}{\partial w_x} & \dfrac{\partial f_2}{\partial w_y} & \dfrac{\partial f_2}{\partial w_\varphi} \\ \dfrac{\partial f_3}{\partial w_x} & \dfrac{\partial f_3}{\partial w_y} & \dfrac{\partial f_3}{\partial w_\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Estimation Process (EKF)

Kalman Gain

$$K(k) = P(k)^- J_h(k)^T \left(J_h(k) P(k)^- J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

$$z(k) = \begin{bmatrix} r_i(k) \\ \theta_i(k) \end{bmatrix} = \begin{bmatrix} \sqrt{\left(x_i - \hat{x}(k)^-\right)^2 + \left(y_i - \hat{y}(k)^-\right)^2} \\ \tan^{-1}\left(\dfrac{y_i - \hat{y}(k)^-}{x_i - \hat{x}(k)^-} \right) - \hat{\varphi}(k)^- \end{bmatrix} + v(k)$$

Initially, before landmarks are added

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial \varphi} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} & \dfrac{\partial h_2}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \dfrac{x - x_i}{r} & \dfrac{y - y_i}{r} & 0 \\ \dfrac{y_i - y}{r^2} & \dfrac{x - x_i}{r^2} & -1 \end{bmatrix} \qquad V(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial v_r} & \dfrac{\partial h_1}{\partial v_\theta} \\ \dfrac{\partial h_2}{\partial v_r} & \dfrac{\partial h_2}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $r = \sqrt{(x_i - x)^2 + (y_i - y)^2}$

The Estimation Process (EKF)

Measurement Update

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\big(z(k) - H(k)\big)$$

$$P(k) = \big(I - K(k)J_h(k)\big)P(k)^{-}$$

Now, if a landmark is observed at t(k+1),
the state model is updated

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \\ x_1(k+1) \\ y_1(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$x_1(k+1) = x(k) + r\cos\theta \qquad\qquad y_1(k+1) = y(k) + r\sin\theta$$

The Estimation Process (EKF)

Prediction (2)

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k) \cos\varphi(k) \\ y(k) + \Delta t V(k) \sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k) \tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

$$F(k) = \begin{bmatrix} \dfrac{\partial f}{\partial(x, y, \varphi)} & 0 \\ 0 & I^{2N \ x \ 2N} \end{bmatrix}$$

where N is the number of landmarks

The Estimation Process (EKF)

Kalman Gain (2)

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

If observing the 1st landmark

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h}{\partial(x, y, \varphi)} & \dfrac{\partial h}{\partial(x_i, y_i)} & 0 & \dots & 0 \end{bmatrix}$$

If observing the 2nd landmark

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h}{\partial(x, y, \varphi)} & 0 & \dfrac{\partial h}{\partial(x_i, y_i)} & 0 & \dots & 0 \end{bmatrix}$$

Must repeat for each landmark!!

The Estimation Process (EKF)

Measurement Update (2)

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\big(z(k) - H(k)\big)$$

$$P(k) = \big(I - K(k)J_{h}(k)\big)P(k)^{-}$$