# Adaptive Array Signal Processing

# [5SSC0]

## Assignment Part 1B: Beamformer Design

## REPORT

**Group number: 3**

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# 1.2 Scenario 1: Narrowband beamformers

### a

Given that we have a uniform linear array (ULA) and we assume that the signal is narrowband, we have far field conditions and it is a 2D wavefront, then the steering vector $\underline{\mathbf{a}}[\theta]$ in frequency domain is given by:

$$\begin{aligned}\underline{\mathbf{a}}(\theta) &= \begin{bmatrix} a_1(\theta) & \cdots & a_J(\theta) \end{bmatrix}^\top \\ &= \begin{bmatrix} e^{-j\omega\tau_1} & \cdots & e^{-j\omega\tau_J} \end{bmatrix}^\top,\end{aligned} \tag{1}$$

where $\omega = 2\pi f_d$ is the angular frequency of the wave and $\tau_i$ is the time delay of the signal at sensor $i$ for a total of $J$ sensors. The time delay is dependent on the position of the sensor, direction of arrival (DOA) and propagation speed of the wave. It is described by:

$$\tau_i = \frac{\underline{\mathbf{v}}^\top \cdot \underline{\mathbf{P}}_i}{c}, \tag{2}$$

where $\underline{\mathbf{v}}^\top = \begin{bmatrix} \sin(\theta) & -\cos(\theta) \end{bmatrix}$ is the direction vector of the wave ($\theta$ is the DOA), $\underline{\mathbf{P}}_i^\top = \begin{bmatrix} p_{i,x} & p_{i,y} \end{bmatrix}$ is the position vector of sensor $i$, and finally $c$ is the propagation speed. For the setup in this assignment the sensors are located on the x-axis, thus $p_{i,y} = 0$, and $p_{i,x}$ is described by:

$$p_{i,x} = dx\Big(i - \frac{5}{2}\Big), \tag{3}$$

where $i = \{1\ ,2\ ,3\ ,4\}$ (so 4 sensors) and $dx$ is the distance between the sensors on the x-axis. Thus $\tau_i$ is described by:

$$\begin{aligned}\tau_i &= \frac{1}{c}(\underline{\mathbf{v}}^\top \cdot \underline{\mathbf{P}}_i) = \frac{1}{c}\begin{bmatrix} \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} dx\Big(i - \frac{5}{2}\Big) \\ 0 \end{bmatrix} \\ &= \frac{dx\sin(\theta)}{c}\Big(i - \frac{5}{2}\Big).\end{aligned} \tag{4}$$

The given parameters of the 4 sensor setup are $dx = 3.4{\cdot}10^{-2}$ m, $c = 340$ m/s and $f_d = 2.5{\cdot}10^3$ Hz. Thus $\underline{\mathbf{a}}(\theta)$ is described by:

$$\begin{aligned}\underline{\mathbf{a}}(\theta) &= \begin{bmatrix} e^{-j\omega\tau_1} & e^{-j\omega\tau_2} & e^{-j\omega\tau_3} & e^{-j\omega\tau_4} \end{bmatrix}^\top \\ &= \begin{bmatrix} e^{-j2\pi f_d\tau_1} & e^{-j2\pi f_d\tau_2} & e^{-j2\pi f_d\tau_3} & e^{-j2\pi f_d\tau_4} \end{bmatrix}^\top \\ &= \begin{bmatrix} e^{j\frac{3\pi f_d dx}{c}\sin(\theta)} & e^{j\frac{\pi f_d dx}{c}\sin(\theta)} & e^{-j\frac{\pi f_d dx}{c}\sin(\theta)} & e^{-j\frac{3\pi f_d dx}{c}\sin(\theta)} \end{bmatrix}^\top \\ &\approx \begin{bmatrix} e^{j\frac{3}{4}\pi\sin(\theta)} & e^{j\frac{1}{4}\pi\sin(\theta)} & e^{-j\frac{1}{4}\pi\sin(\theta)} & e^{-j\frac{3}{4}\pi\sin(\theta)} \end{bmatrix}^\top.\end{aligned} \tag{5}$$

The beampattern $B(\theta)$ is described by:

$$B(\theta) = \frac{1}{J^2}|r(\theta)|^2 = \frac{1}{J^2}|\underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta)|^2. \tag{6}$$

When all weights equal one ($\underline{\mathbf{w}}^\top = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}$), the array response $r(\theta)$ is described by:

$$\begin{aligned} r(\theta) = \underline{\mathbf{1}}^h \cdot \underline{\mathbf{a}}(\theta) &= \sum_{i=1}^{J} 1 \cdot a_i(\theta) = \sum_{i=1}^{J} \exp\Big( - j2\pi f_d \frac{dx\sin(\theta)}{c}\Big(i - \frac{5}{2}\Big)\Big) \\ &= \exp\Big(j5\pi f_d \frac{dx\sin(\theta)}{c}\Big) \sum_{i=1}^{J} \exp\Big( - j2\pi i f_d \frac{dx\sin(\theta)}{c}\Big), \end{aligned} \tag{7}$$

where $\exp(x) = e^x$. To get a nice expression for the response vector $r(\theta)$ later, we first multiply it with $\exp\big(j2\pi f_d \frac{dx\sin(\theta)}{c}\big)$:

$$\begin{aligned} r(\theta)\cdot\exp\big(j2\pi f_d \frac{dx\sin(\theta)}{c}\big) &= \exp\Big(j5\pi f_d \frac{dx\sin(\theta)}{c}\Big) \sum_{i=1}^{J} \exp\Big( - j2\pi (i-1) f_d \frac{dx\sin(\theta)}{c}\Big) \\ &= \exp\Big(j5\pi f_d \frac{dx\sin(\theta)}{c}\Big) \sum_{i=0}^{J-1} \exp\Big( - j2\pi i f_d \frac{dx\sin(\theta)}{c}\Big). \end{aligned} \tag{8}$$

Now we can subtract equation 8 from equation 7, so that we can get rid of the summation:

$$\begin{aligned} r(\theta)\cdot\Big(1 - \exp\big(j2\pi f_d \frac{dx\sin(\theta)}{c}\big)\Big) &= \\ \exp\Big(j5\pi f_d \frac{dx\sin(\theta)}{c}\Big)\Big(\exp\big( - j2\pi J f_d \frac{dx\sin(\theta)}{c}\big) - 1\Big). \end{aligned} \tag{9}$$

Now we can solve for $r(\theta)$ and simplify:

$$\begin{aligned} r(\theta) &= \exp\Big(j(5-2J)\pi f_d \frac{dx\sin(\theta)}{c}\Big)\cdot \frac{1 - \exp\big(j2\pi J f_d \frac{dx\sin(\theta)}{c}\big)}{1 - \exp\big(j2\pi f_d \frac{dx\sin(\theta)}{c}\big)} \\ &= \exp\Big(j(5-2J)\pi f_d \frac{dx\sin(\theta)}{c}\Big)\cdot \frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}. \end{aligned} \tag{10}$$

Now we plug this result into the equation for the beampattern $B(\theta)$:

$$\begin{aligned} B(\theta) &= \frac{1}{J^2}|r(\theta)|^2 \\ &= \frac{1}{J^2}\left|\exp\Big(j(5-2J)\pi f_d \frac{dx\sin(\theta)}{c}\Big)\cdot \frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}\right|^2 \\ &= \frac{1}{J^2}\left(\left|\exp\Big(j(5-2J)\pi f_d \frac{dx\sin(\theta)}{c}\Big)\right|\cdot\left|\frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}\right|\right)^2 \\ &= \frac{1}{J^2}\left(1\cdot\left|\frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}\right|\right)^2 = \frac{1}{J^2}\left|\frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}\right|^2. \end{aligned} \tag{11}$$

Thus the beampattern of the 4 sensor system with the parameters given as before is described by:

$$B(\theta) = \frac{1}{J^2}\left|\frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}\right|^2 = \frac{1}{16}\left|\frac{\sin(\pi\sin(\theta))}{\sin(\frac{\pi}{4}\sin(\theta))}\right|^2. \tag{12}$$

**b**

It can be seen from equation 12, that the position of the sensors is irrelevant, only the spacing between them is. This makes sense since we are considering a far field situation and we assume a flat wavefront. If we have a system with only sensors $p_1$ and $p_2$, of which the distance between them is $dx$, just as before, then the beampattern $B(\theta)$ would be:

$$\begin{aligned} B(\theta) &= \frac{1}{J^2}|\underline{\mathbf{1}}^h \cdot \underline{\mathbf{a}}(\theta)|^2 \\ &= \frac{1}{J^2}\left|\frac{\sin(J\pi f_d \frac{dx\sin(\theta)}{c})}{\sin(\pi f_d \frac{dx\sin(\theta)}{c})}\right|^2 \\ &= \frac{1}{4}\left|\frac{\sin(\frac{2\pi f_d dx}{c}\sin(\theta))}{\sin(\frac{\pi f_d dx}{c}\sin(\theta))}\right|^2 \\ &= \frac{1}{4}\left(\frac{\sin(\frac{2\pi f_d dx}{c}\sin(\theta))}{\sin(\frac{\pi f_d dx}{c}\sin(\theta))}\right)^2. \end{aligned} \tag{13}$$

Since the equation is real-valued, the square of the magnitude of the signal is identical to the square of the signal itself. Thus the absolute value brackets can be ignored. Aliasing occurs when there are grating lobes in the beampattern within the relevant domain for $\theta$. For a 2 sensor system, there are no side lobes, which means that any peak is either the main lobe or grating globes. To find the peaks of the beampattern $B(\theta)$, we take its derivative with respect to $\theta$ and set it to 0. The derivative after simplification is described by:

$$\frac{dB(\theta)}{d\theta} = -\frac{\pi f_d dx}{c}\cos(\theta)\sin\left(\frac{2\pi f_d dx}{c}sin(\theta)\right) = 0. \tag{14}$$

We are interested in the beampattern for $-\pi/2 \leq \theta \leq \pi/2$ and we know that the beampattern is symmetric around the y-axis. The main lobe, which is a peak, is at $\theta = 0$ and can be seen from the equation above ($\sin(0) = 0$). It can also be seen that there are at least 2 other points where the derivative is 0, namely $\theta = \pm\pi/2$, from the term $\cos(\theta)$. Since we have at least 3 points where the derivative is 0 on the given domain. It is desired that there are no other points where the derivative of the beampattern $B(\theta)$ equals 0. If there are 5 points or more, then that means there must be at least 2 peaks besides the main lobe, because between two maxima must be a minimum (there are no saddle points in the square of a sine division function, like the beampattern). Thus to have no grating lobes, we must have exactly 3 points where the derivative of the beampattern is 0. This is achieved when:

$$\sin\left(\frac{2\pi f_d dx}{c}\sin(\theta)\right) \neq 0, \text{ for } \theta \neq 0. \tag{15}$$

We can see that this holds if and only if:

$$-\pi < \frac{2\pi f_d dx}{c} sin(\theta) < \pi. \tag{16}$$

Now we are considering the domain $-\pi/2 < \theta < \pi/2$, as we already discussed the case for $\theta = \pm\pi/2$. For this domain we have that $-1 < \sin(\theta) < 1$. This means that the condition in equation 16 holds if and only if:

$$\frac{2\pi f_d dx}{c} \leq \pi. \tag{17}$$

From this condition we can determine the range of frequencies $f_d$, where no spatial aliasing occurs:

$$f_d \leq \frac{c}{2dx}. \tag{18}$$

To conclude the maximum allowed frequency $f_{\max}$ is:

$$f_{\max} = \frac{c}{2dx} = 5000 \text{ Hz}. \tag{19}$$

**c**

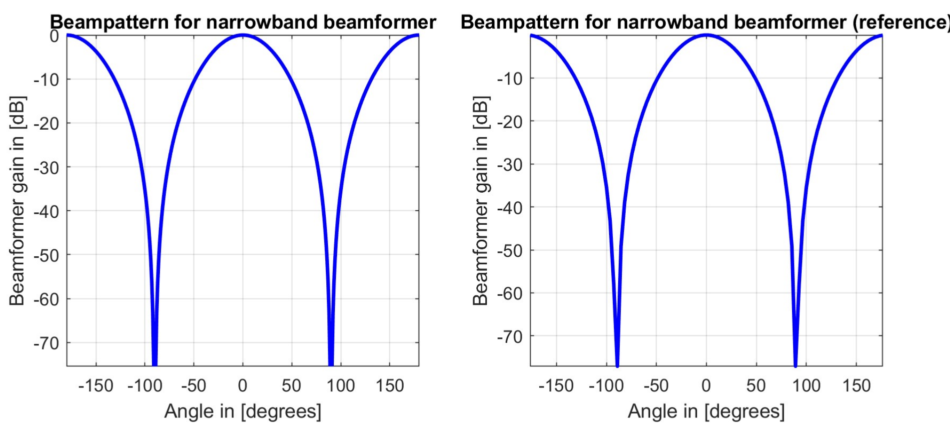



Figure 1: Beampattern of the implemented narrowband beam former using the given parameters (left), (b) Reference beampattern of equation 13
(right). :

Below is the code that calculate elements of the array response vector and below that is the code that subsequently calculates the beampattern:

```
% Line of code in array_response_vector.m:
A = exp((-1j*2*pi*f/340).*(p*wave_vector(theta)));
% beampattern calculation in cal_nb_beampattern.m
w = b.nb_weights;
a_theta = b.array_response_vector(b.angles, b.nb_frequency);
J = b.array.number_of_sensors;
b.nb_beampattern = abs(w'*a_theta).^2/(J^2);
```

The beampattern and the reference beampattern (result of (a.)) are shown in figure 1. The reference beampattern is plotted based on equation12.

It can be seen from the figure that both plots are identical, suggesting that the derivation for the beampattern equation is correct. It can be seen that on the domain $-\pi/2 \leq \theta \leq \pi/2$, there is a single lobe, the main lobe and no side lobes. There are no grating lobes (aliasing), which is to be expected, since $dx \leq \frac{c}{2f_d}$. The amplitude of the main lobe is exactly 1 (0 dB), which is also to be expected, since the beampattern is normalized.

## d

For the desired angle $\theta_d = 30°$, we want the array unit response: $r(\theta_d) = 1$. Thus we have that:

$$r(\theta_d) = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta_d) = 1. \tag{20}$$

First we will determine a general solution to such a problem and then solve the particular case above. Consider a situation with possibly multiple constraints. Then $r(\theta_d)$ becomes a row vector, which we define as $\underline{\mathbf{r}}^h(\theta)$, containing the desired array response for given angles $\theta_d$ and then $\underline{\mathbf{a}}(\theta_d)$ becomes a matrix $\mathbf{A}(\theta_d)$ with $J$ rows and $C$ columns, where $J$ is the amount of sensors and $C$ is the amount of constraints. Every column is the steering vector for a given angle. Then we have:

$$\begin{aligned} \underline{\mathbf{r}}^h(\theta_d) &= \underline{\mathbf{w}}^h \cdot \mathbf{A}(\theta_d) \\ \underline{\mathbf{r}}(\theta_d) &= \mathbf{A}^h(\theta_d) \cdot \underline{\mathbf{w}} \end{aligned} \tag{21}$$

The general solution to such a linear system is described by:

$$\underline{\mathbf{w}} = \left(\mathbf{A}^h(\theta_d)\right)^\dagger \cdot \underline{\mathbf{r}}(\theta_d) + \Big(I - \left(\mathbf{A}^h(\theta_d)\right)^\dagger \cdot \mathbf{A}^h(\theta_d)\Big) \cdot \underline{\mathbf{b}}, \tag{22}$$

where $\left(\mathbf{A}^h(\theta_d)\right)^\dagger$ is the pseudo-inverse of $\mathbf{A}^h(\theta_d)$, described by $\left(\mathbf{A}^h(\theta_d)\right)^\dagger = \mathbf{A}(\theta_d) \cdot \left(\mathbf{A}^h(\theta_d) \cdot \mathbf{A}(\theta_d)\right)^{-1}$ and where $I$ is the identity matrix and $\underline{\mathbf{b}}$ is a vector of free parameters, which is relevant when there is no unique solution. If we now assume that the pseudo-inverse is possible ($\mathbf{A}^h(\theta_d) \cdot \mathbf{A}(\theta_d)$ is invertible) and our system is properly defined, and that there is at least 1 exact solution for the system (so no over-determined system). Then $\underline{\mathbf{b}} = \underline{\mathbf{0}}$ is a valid solution (proof as an exercise for the reader) and for simplicity that is the solution that we will choose. The equation for the weights $\underline{\mathbf{w}}$ then becomes:

$$\begin{aligned} \underline{\mathbf{w}} &= \left(\mathbf{A}^h(\theta_d)\right)^\dagger \cdot \underline{\mathbf{r}}(\theta_d) \\ &= \mathbf{A}(\theta_d) \cdot \left(\mathbf{A}^h(\theta_d) \cdot \mathbf{A}(\theta_d)\right)^{-1} \cdot \underline{\mathbf{r}}(\theta_d). \end{aligned} \tag{23}$$

In the case of the singular constraint $r(\theta_d) = 1$ for $\theta_d = 30°$ described by equation 20, the equation for the weights simply becomes:

$$\begin{aligned} \underline{\mathbf{w}} &= \underline{\mathbf{a}}(\theta_d) \cdot \left(\underline{\mathbf{a}}^h(\theta_d) \cdot \underline{\mathbf{a}}(\theta_d)\right)^{-1} \cdot r(\theta_d) \\ &= \underline{\mathbf{a}}(\theta_d) \cdot \left(\underline{\mathbf{a}}^h(\theta_d) \cdot \underline{\mathbf{a}}(\theta_d)\right)^{-1}. \end{aligned} \tag{24}$$

The resulting beampattern with the weights $\underline{\mathbf{w}}$ calculated as above is shown in figure 2. The beampattern at $\theta_d = 30°$ is equal to $\frac{1}{J^2}|r(\theta_d)|^2 = \frac{1}{16} \approx -12dB$. This can also be seen in the figure.

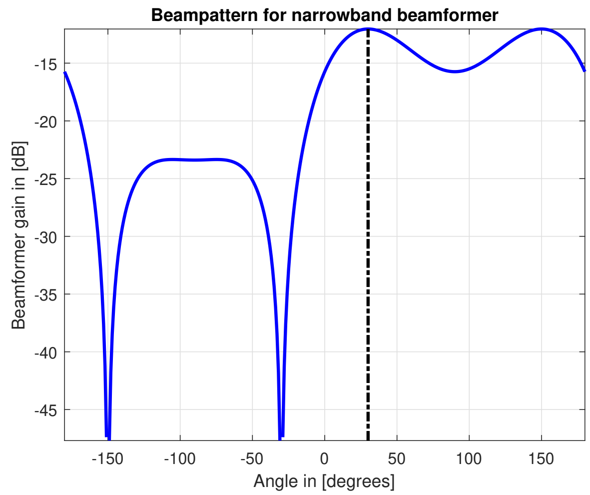



Figure 2: Beampattern of the 4 sensor ULA array with filter weights, that satisfy the constraint of unit array response at $\theta_d = 30°$.

**e**

To rotate the ULA (arrange along x axis originally) by 90°, we can calculate the new position vector $\underline{\mathbf{P}}'$ by multiplying the original position vector $\underline{\mathbf{P}}$ with a rotation matrix $\mathbf{R}(\theta)$:

$$\begin{aligned}
\underline{\mathbf{P}}' &= \begin{bmatrix} \underline{\mathbf{P_1}}' & \underline{\mathbf{P_2}}' & \underline{\mathbf{P_3}}' & \underline{\mathbf{P_4}}' \end{bmatrix} \\
&= \mathbf{R}(\theta) \cdot \begin{bmatrix} \underline{\mathbf{P_1}} & \underline{\mathbf{P_2}} & \underline{\mathbf{P_3}} & \underline{\mathbf{P_4}} \end{bmatrix} \\
&= \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{2}dx & -\frac{1}{2}dx & \frac{1}{2}dx & \frac{3}{2}dx \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{3}{2}dx & -\frac{1}{2}dx & \frac{1}{2}dx & \frac{3}{2}dx \end{bmatrix}
\end{aligned} \tag{25}$$

.

So now the ULA is placed along the y-axis, instead of the x-axis. Since the ULA is rotated by 90°, the beampattern is also rotated by 90°. This can be seen in figure 3, when we have a weight unit vector as in (c.). Before the rotation, the ULA had zero response at $\theta = -90°, 90°$, and unit response at $\theta = 0°$. The rotated ULA has zero response at $\theta = 0°$, and unit response at $\theta = -90°, 90°$. The reference beampattern is then described by:

$$B(\theta) = \frac{1}{16} \left| \frac{\sin(\pi \sin(\theta - 90°))}{\sin(\frac{\pi}{4} \sin(\theta - 90°))} \right|^2 . \tag{26}$$

For the constraint that the array response $r(\theta_d) = 1$ for $\theta_d = 30°$ of the rotated ULA, we have that the beampattern would be identical to $\theta_d = 60°$ for the original ULA, but shifted by $90°$, due to the $90°$ rotation. The beampattern of the rotated ULA is shown in figure 4. When comparing figure 2 and figure 4, we see that they are very similar. The structure of

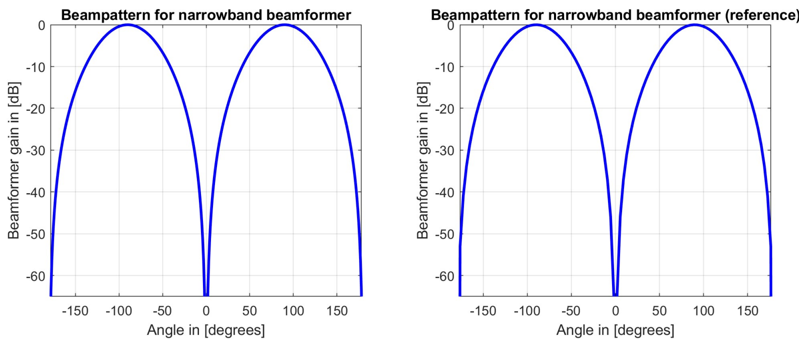



Figure 3: beampattern of the narrowband beam former after rotating the ULA by $90°$ and the reference beampattern according to equation 26.

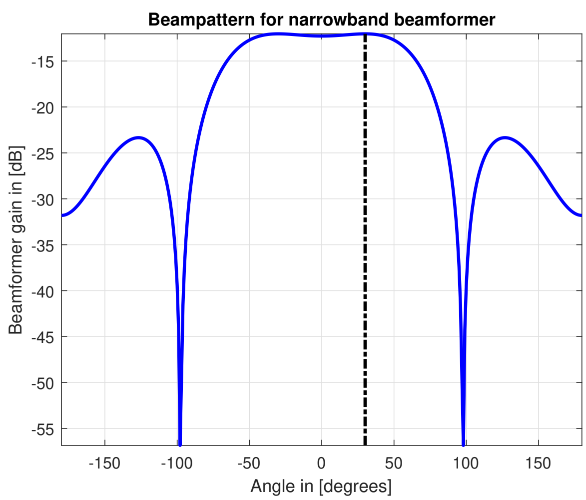



Figure 4: Beampattern of the 4 sensor ULA array rotated by $90°$ with filter weights that satisfy the constraint of unit array response at $\theta_d = 30°$.

the beampattern is very similar. In the first figure the highest peak has the same structure as the lowest peak in the second figure, and vice versa. For the rotated ULA the wide main lobe is close to the edge of the $0 \leq \theta \leq \pi$ domain and thus the main lobe at the back of the array, in the $-\pi \leq \theta \leq 0$ domain, is very close and overlap. As a result there is a broad range of angles with almost identical power. This is also the case for the original ULA, but there it is at approximately -23 dB, which is very small compared to the -12 dB peak.

### f

For the undesired source active at $\theta_u = -60°$, we set the constraint $r(\theta_u) = 0$. Now we have to consider two constraints: unit response at $\theta_d = 30°$ and zero response at $\theta_u = -60°$:

$$\begin{cases} \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta_d) = 1 \\ \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta_u) = 0. \end{cases} \tag{27}$$

As described in question d, we put these constraints in matrix notation:

$$\begin{aligned} \mathbf{A}(\theta) &\equiv \begin{bmatrix} \underline{\mathbf{a}}(\theta_d) & \underline{\mathbf{a}}(\theta_u) \end{bmatrix} \\ \underline{\mathbf{r}}(\theta) &\equiv \begin{bmatrix} \underline{\mathbf{r}}(\theta_d) & \underline{\mathbf{r}}(\theta_u) \end{bmatrix}^h. \end{aligned} \tag{28}$$

Now we simply solve the linear system of equations using equation 23. The solution of $\underline{\mathbf{w}}$ is thus described as:

$$\underline{\mathbf{w}} = \left(\mathbf{A}^h(\theta)\right)^\dagger \cdot \underline{\mathbf{r}}(\theta). \tag{29}$$

The steered response of the ULA can be seen in figure 5. It can be seen that the constraints are met, i.e. zero response at $\theta_u = -60°$, and unit response at $\theta_d = 30°$. The maximum is at $\theta_d = 30°$, which is ideal, since that is the desired direction and thus the signal-to-noise (SNR) is maximized. However, when the noise source $\theta_u$ moves closer to $\theta_d$, the slope between the two angles must become steeper. This fast change in array response over a short range of angles causes that $\theta_d$ is no longer the maximum, as the function keeps increasing when it shifts away from $\theta_d$. This means that the SNR is not maximized. It is still unit response, but there are other angles where the response is larger, which means the array system may receiving signal from undesired direction. This can be seen in figure 6. When $\theta_d$ and $\theta_u$ come very close, the slope becomes so steep that the signal gets amplified for most angles. This can be seen in graph (c) and (d) in figure 6. For example, the response at 50° and 90° changes from -3dB in (c) to 15 dB in (d).

### g

The code with which the array is generated is shown below. If the amount of sensors does not fit a square exactly, the code will compute the closest square number that smaller than assigned amount.

```
% amount of sensors is floor(sqrt(J))^2
% this avoids problems if sqrt(J) is not an integer
length=floor(sqrt(J)); p = zeros(length^2,2);
% fill p with coordinates:
for i=1:length
    for j=1:length
        p((i-1)*length+(j),:)= [dx*((1-length)/2+(i-1)), dy*((1-length)/2+(j-1))];
    end
end
```

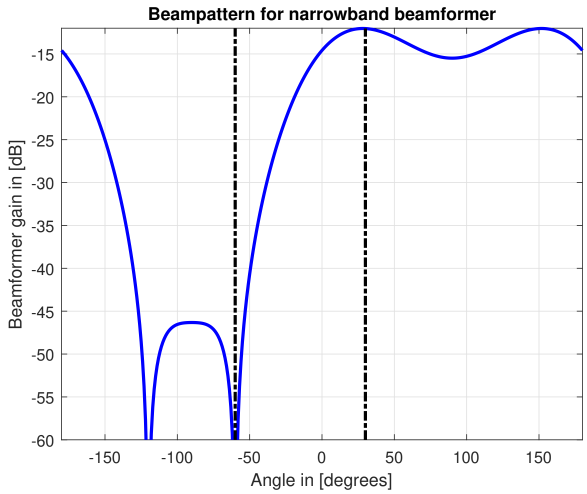



Figure 5: Beampattern of the 4 sensor ULA array with filter weights that satisfy the constraint of unit array response at $\theta_d = 30°$ and zero array response at $\theta_u = -60°$.

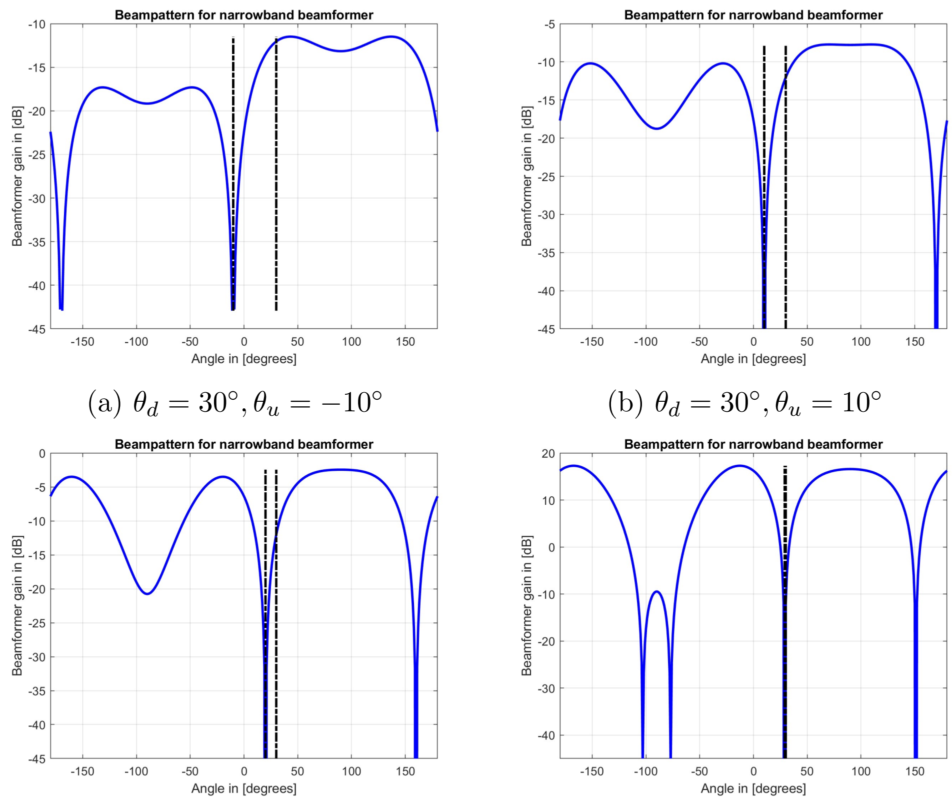
The beampattern with weights calculated such that $r(30°) = 1$ and $r(-60°) = 0$ is shown in figure 7. It can be seen that for this array configuration the requirements are met, but that again the maximum is not at 30°. If $\theta_d$ and $\theta_u$ are moved closer to each other, the same behavior can be observed as before. The slope between the 2 angles increases and as a result the array response maximum shifts further away from $\theta_d$. Again the SNR is not maximized.

### h

For frequencies other than $f_d = 2500$ Hz the constraints are not met. This can be seen in figure 8. At the locations of the dashed lines, indicating $\theta_u$ and $\theta_d$, the gain is not approximately $-\infty$ dB and approximately $-12$ dB respectively, and thus the constraint is not met. For 1000 Hz the gain difference between the two angles is approximately 5 dB and for 4000 Hz approximately 7.5 dB. However if the source locations shift slightly in angle or frequencies shift a small amount, then this difference can become smaller. For 2500 Hz the gain around $\theta_u$ is at maximum approximately -30 dB, which is still 18 dB lower than the gain at $\theta_d$ even when the source locations shift slightly or the frequency shifts slightly. The SNR is much better for 2500 Hz than for the other frequencies and shows that the array performs much better for frequencies close to 2500 Hz.

### i

A solution for processing a broadband signal would be to split the whole frequency domain of all potential signals into many narrowband ranges. The size of the bandwidth of the narrow bands should be such that the gain difference of the beam pattern between $\theta_d$ and




(a) $\theta_d = 30°, \theta_u = -10°$

(b) $\theta_d = 30°, \theta_u = 10°$

(c) $\theta_d = 30°, \theta_u = 20°$

(d) $\theta_d = 30°, \theta_u = 29°$

Figure 6: Beampattern of the 4 sensor ULA array with filter weights that satisfy the constraint of unit array response at $\theta_d = 30°$ and zero array response at different $\theta_u$ that move closer to $\theta_d$.

$\theta_u$ for all frequencies in that band is above a desired threshold and that the gain at $\theta_d$ for all frequencies in the band is roughly the same to avoid amplification/muting of certain frequencies of the desired source signal. For every narrow-band different weights should be computed according to the constraint of $\theta_u$ and $\theta_d$.

The signal measured at the array should be converted to frequency domain by using the discrete Fourier transform (DFT), split up into the many bands. These narrowband signal should be processed using the corresponding narrowband weights. Then these output should be combined again and brought back to time domain by using the inverse DFT.

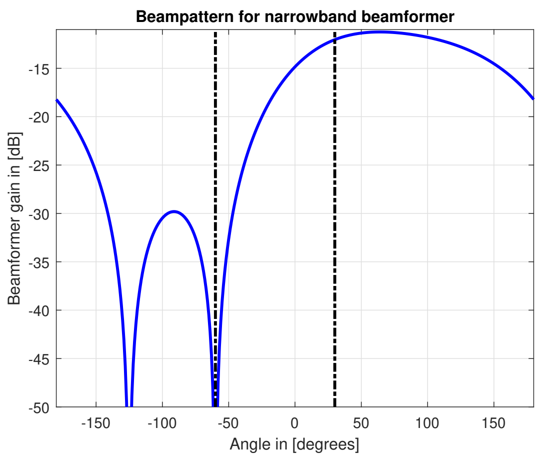



Figure 7: Beampattern of the 4 sensor rectangular array with filter weights that satisfy the constraint of unit array response at $\theta_d = 30°$ and zero array response at $\theta_u = -60°$

## 1.3 Scenario 2: Delay and Sum Beamformer

### a

The inverse Fourier transform of $D(e^{j\theta})$ can be expressed by:

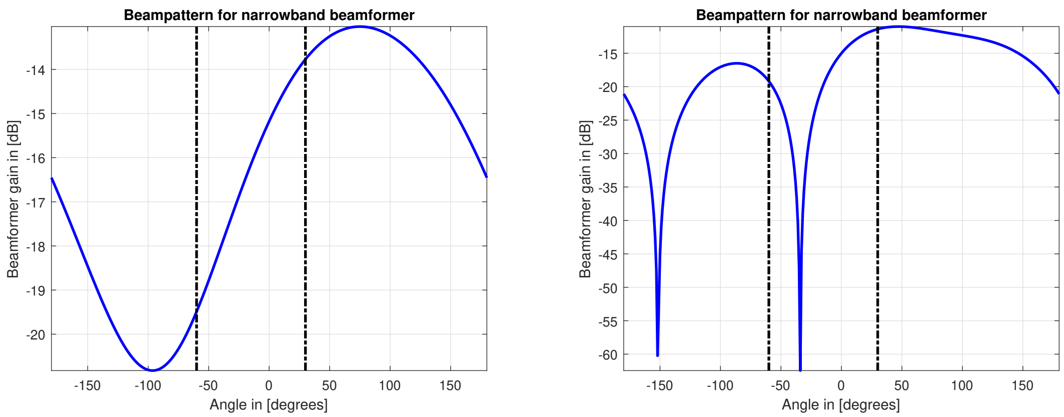
$$\begin{aligned} d(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D(e^{j\theta})\cdot e^{j\theta t}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(\tau-t)\theta}d\theta \\ &= \frac{1}{2\pi}\cdot\frac{e^{-j(\tau-t)\theta}}{-j(\tau-t)}\Bigg|_{\theta=-\pi}^{\pi} \\ &= \frac{1}{2\pi}\frac{e^{-j(\tau-t)\pi}-e^{j(\tau-t)\pi}}{-j(\tau-t)} \end{aligned} \tag{30}$$

Now we simplify:

$$\begin{aligned} d(t) &= \frac{1}{(\tau-t)\pi}\frac{e^{j(\tau-t)\pi}-e^{-j(\tau-t)\pi}}{2j} \\ &= \frac{\sin\big((\tau-t)\pi\big)}{(\tau-t)\pi} \\ &= \text{sinc}\,\big((\tau-t)\pi\big). \end{aligned} \tag{31}$$

### b

The theoretical impulse response of $d_0 = \text{sinc}\,\big((\tau_0 - t)\pi\big)$ and $d_1 = \text{sinc}\,\big((\tau_1 - t)\pi\big)$ is shown in figure 9. It can be seen that the maximum of the sinc function corresponds to the delay

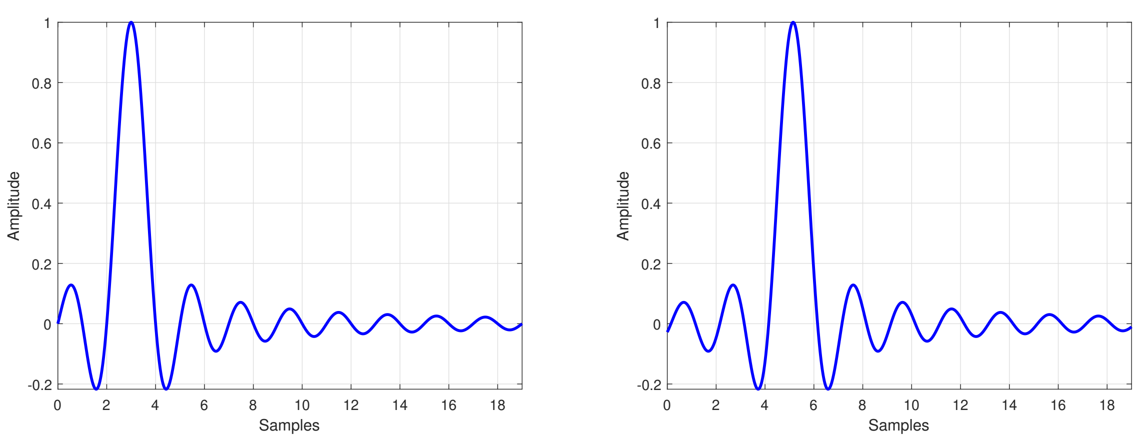



(a) $f_d = 1000$ Hz (b) $f_d = 4000$ Hz

Figure 8: Beampattern for 2 different frequencies of the 4 sensor rectangular array with filter weights that satisfy the constraint of unit array response at $\theta_d = 30°$ and zero array response at $\theta_u = -60°$ and $f_d = 2500$ Hz.

$\tau$, where $\tau$ is measured in numer of samples. The code that generates the figures is shown below:

```
NSample = 20; samples = 0:0.01:NSample-1;
d0 = sinc((3-samples)); d1 = sinc((5.15-samples));
figure
plot(samples,d0,'b-','LineWidth',2)
grid on; axis tight; ylabel("Amplitude");xlabel("Samples")
figure
plot(samples,d1,'b-','LineWidth',2)
grid on; axis tight; ylabel("Amplitude");xlabel("Samples")
```



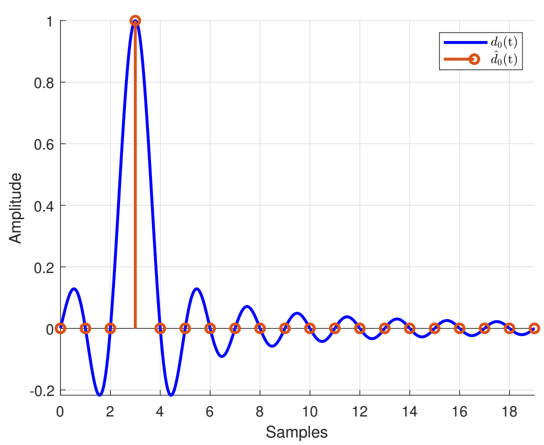

(a) $d_0(t)$, delay $\tau_0 = 3$ (b) $d_1(t)$, delay $\tau_1 = 5.15$

Figure 9: Theoretical impulse response of $d_0(t)$ (a) and $d_1(t)$ (b) for a period of 20 samples.

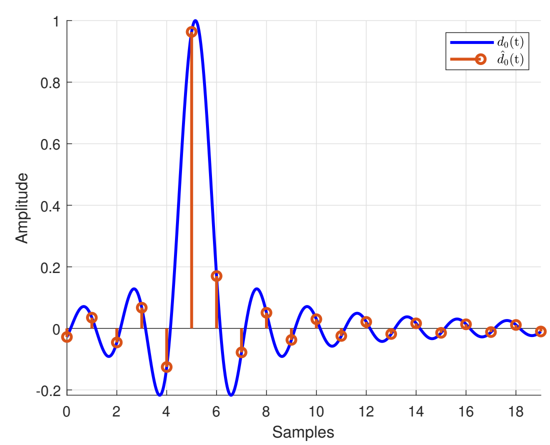
**c**

The impulse response $\hat{d}_0$ and $\hat{d}_1$ are plotted in figure 10. It can be seen that the sampled response corresponds to the result from the previous question. It can be seen that for an integer delay all sampled values are 0, except for the value corresponding to the time delay. For a fractional delay, the samples are taken slightly shifted from the zero crossings. Causality is dealt with by overlaying the sampled signal with the theoretical one and then removing all values corresponding to negative time domain, since those time values violate causality. The code that generates the figures is shown below:

```
% Set the number of delay sample
tau = 5.15; FracDelay = tau;
% Largest integer that is less than the fractional delay
FracDelayInt = floor(FracDelay);
%Fractional part
FracDelayRem = FracDelay - FracDelayInt;
%Design the delay filter
h = delay(FracDelayInt, round(FracDelayRem * 100), 100, ceil(FracDelay));
%The processing delay is determined by the length of the filter and dmax
ProcessingDelay = length(h) - ceil(FracDelay);
d0_hat = h(floor(ProcessingDelay / 2) + 1: floor(ProcessingDelay / 2) + NSample);
figure; hold on; grid on; axis tight; xlabel('Samples'),ylabel('Amplitude');
plot(samples,d0,'b-','LineWidth',2)
stem(0:NSample-1,d0_hat,'LineWidth',2); hold off
legend('$d_0$(t)','$\hat{d}_0$(t)','Interpreter','latex')
```




(a) $\tau_0 = 3$



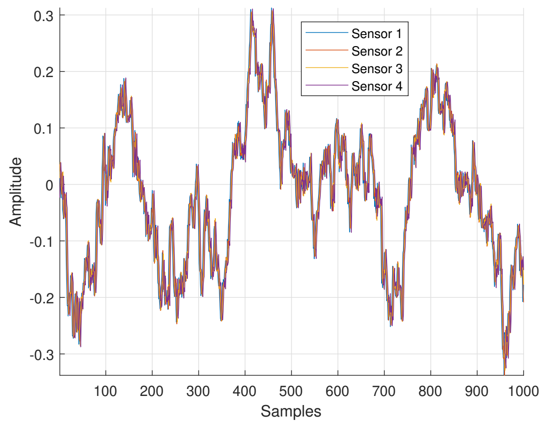

(b) $\tau_1 = 5.15$

Figure 10: Theoretical and estimated impulse response of $d_0(t)$ (a) and $d_1(t)$ (b) for a period of 20 samples.

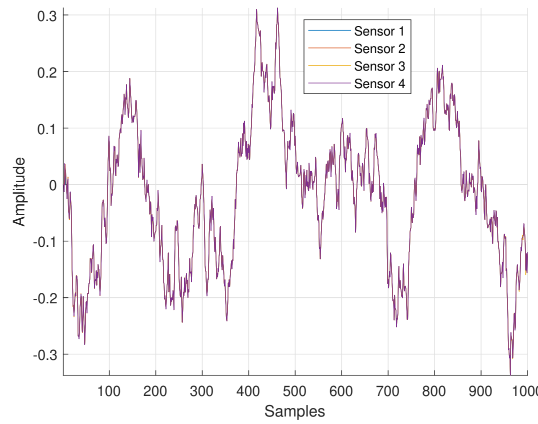
### d

The original signal received by the sensors and the signals after time-alignment is show in figure 11. Also a zoomed in version is shown so that it can clearly be seen that the signal received at the receiver is indeed time-aligned properly. Below the code is shown that time-aligns the signals received at the sensors and plots the result:

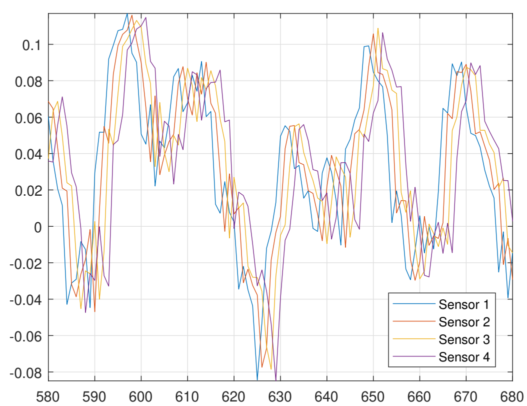
```
FilteredSignal = zeros(J,length(x));
for i = 1:J
    % The sensor that first receive signal -> delay to align with the farest sensor
    % The farest sensor-> no delay
    FracDelay = (dx*(J-i+1)*sind(DOA)/340)*fs;
    FracDelayInt = floor(FracDelay);
    FracDelayRem = FracDelay - FracDelayInt;
    DelayFilter = delay(FracDelayInt, round(FracDelayRem * 100),100, ceil(FracDelay));
    % Apply the delay filter to the corresponding sensor
    SignalFracDelay = conv(x(i,:),DelayFilter);
    ProcessingDelay = length(DelayFilter) - ceil(FracDelay);
    ResultSignal = SignalFracDelay(...
    floor(ProcessingDelay / 2) + 1:floor(ProcessingDelay / 2) + length(x));
    % To cope with casuality
    CasualFactor =  ceil(FracDelay);
    FilteredSignal(i,:) = ResultSignal.*...
        [zeros(1,CasualFactor), ones(1, length(x)-CasualFactor)];
end

figure; hold on; grid on; axis tight; xlabel('Samples'),ylabel('Amplitude');
plot(x')
legend('Sensor 1','Sensor 2','Sensor 3','Sensor 4')
figure; hold on; grid on; axis tight; xlabel('Samples'),ylabel('Amplitude');
plot(FilteredSignal')
legend('Sensor 1','Sensor 2','Sensor 3','Sensor 4')
```



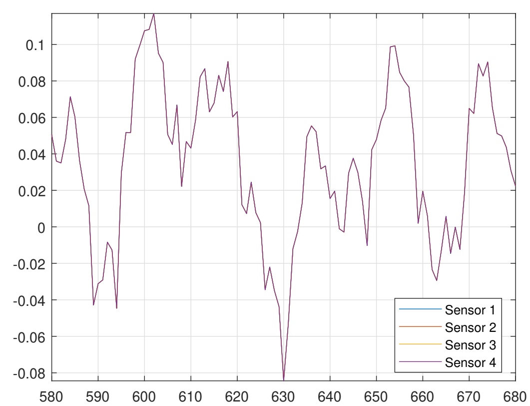

(a) Before time aligning




(b) After time aligning




(c) Before time aligning (Detailed)




(d) After time aligning (Detailed)

Figure 11: Signal received at each sensor before (left) and after (right) time aligning.