**Hexahedral Mesh Quality Improvement via Edge-Angle Optimization**

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**Abstract**

We introduce a simple and practical technique to untangle and improve hexahedral (hex) meshes. We achieve that by enabling the deformation of the boundary surfaces during the untangling process, which provides more space to reach a valid solution. To improve the element quality, an angle optimization strategy is proposed, which has much simpler formulation than the existing method. The deformed volume after optimization is then pulled back to the original one using an inversion-free deformation. In contrast to the current methods, we perform the untangling and quality improvement within a few local regions surrounding elements with undesired quality, which can effectively improve the minimum scaled Jacobian (MSJ) quality of the mesh over the existing method. We demonstrate the effectiveness of our methods by applying it to the hex-meshes generated by a range of methods.

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**1. Introduction**

Hexahedral (or hex-) meshes, are commonly employed by many critical applications that require to solve volumetric partial differential equations. This is mostly due to its naturally embedded tensor product structure, larger tolerance for anisotropy and less numerical stiffness, compared to unstructured meshes (e.g., tetrahedral (or tet-) meshes). These preferred properties enable the convenient imposition of a simulation basis with a higher derivative smoothness between elements of the mesh, and the handling of large deformation during simulations.

However, given any input models, generating hex-meshes with good quality elements while conforming to the surface configuration remains an ongoing challenge. The initially computed hex-meshes, produced by the state-of-the-art methods, such as the polycube mapping or frame-field based methods, often contain inverted elements (i.e., elements with a negative local volume at one or more of its corners), which cannot be

directly applied for finite element calculations [1]. Therefore, there is a need for hex-mesh improvement to eliminate the inverted elements and regulate the element shapes [2] while preserving surface features.

A number of techniques have been proposed to untangle and improve hex-meshes with inverted elements without changing their connectivity [2, 3, 4, 5, 6, 7]. However, none of them is guaranteed to produce inversion-free hex-meshes. Recently, Livesu et al. [8] introduced an untangling method that optimizes the cone-shapes around the individual edges of the hex-mesh to ensure a positive volume for the tetrahedra around the edges. The formulation of their energy function contains several terms that optimize different geometric characteristics of the mesh. However, the optimization is performed globally with varying weights that prefer elements that already have a good shape. While this strategy helps retain the elements with good quality (i.e. by fixing them), it may prevent the improvement of elements with less optimal quality.

In this work, we propose a local untangling and improvement framework so that the optimization is performed only around inverted elements or elements with quality below a user-  
specified minimum value (i.e., minimum scaled Jacobian [9], or MSJ). In our local framework, the focus is on improving those elements with undesired quality (i.e., good quality element may become slightly worse), which relieves the stiffness in the global optimization caused by the elements with good quality, allowing the MSJ quality to be further improved. In the meantime, we introduce a new angle-based distortion energy that characterizes different optimization goals (e.g., orthogonality and straightness) via a unified formulation, largely simplifying the setup and solving of the system. Furthermore, to facilitate the search of a valid solution to our optimization, the boundary surface is relaxed if needed. However, relaxing the surface constraint may lead to a large surface distance between the boundary of the output mesh and the original surface. To address that, we perform an inversion-free deformation that gradually pulls the surface back to its original one while still guaranteeing an inversion-free outcome. Note that this inversion-free deformation is only performed after the untangling process. For the improvement of MSJ, this pull-back process is not applied, as it may worsen the MSJ - against the goal of MSJ improving. Instead, we directly project the surface back to the original one after improving the MSJ of an inversion-free mesh. After improving the MSJ to a user desired level, we perform a Laplacian-like smoothing to improve the average scaled Jacobian (ASJ) of the mesh. Our framework is simple to implement and can handle more challenging inputs than the existing methods. In average, our method takes 2 minutes for a mesh with elements. We have applied our method to over 80 meshes generated by the polycube-based methods, octree-based method, and frame-field based method , respectively, to demonstrate its effectiveness. All our results have been submitted as the supplemental material, and a reference implementation will be released upon acceptance.

**2. Related Work**

In this section, we review the most relevant literature for the creation and optimization of hex-meshes.

Hex-meshing. Considering its importance to finite element simulation [10], a large amount of effort has been dedicated to the generation of valid all-hex meshes. These methods range from the early sweeping and paving [11, 12], gridbased [13, 14, 15, 16] and octree-based methods [17, 18, 19, 20] to the polycube-based and frame-field based approaches [25, 26, 27, 28]. A recent survey [29] provides a detailed look at the advances in this direction. Despite these many existing hex-meshing techniques, most initial hex-meshes generated with these approaches need to undergo a quality optimization process to substantially improve their quality for practical use. Our method can be used to optimize the initial meshes produced by a variety of these methods.

Hex-Mesh Optimization. Since it is a necessary step in the meshing pipeline, an equally large amount of work for the improvement of the hex-mesh quality has been proposed. There are two different strategies to improve the mesh quality. The first strategy adopts various smoothing (e.g., the Winslow smoothing [30]) and optimization methods (e.g., via the geometric flow [31]) to optimize the mesh without changing its  
connectivity, while the second strategy requires the modification of the mesh connectivity to achieve the desired quality improvement, such as the padding process [18, 32] typically used in the polycube-based methods. Other methods, like the singularity alignment [33] and polycube domain simplification [34, 35] have been proposed to optimize the structure of the hex-meshes. Our method belongs to the first group.

In order to optimize the quality of a hex-mesh, a quality metric has to be identified for the optimizer to improve upon the mesh. According to a Sandia Report by Stimpson et al. [9], there are more than a dozen quality metrics for hex-meshes. Most of these quality metrics measure the difference between a given hexahedron and a canonical cube via either angle distortion, length ratio or tensor distortion. Although there is not a comprehensive study on the effectiveness of these metrics [36], the scaled Jacobian metrics are the most commonly used metrics in the meshing and simulation communities. Intuitively, the Jacobian metric measures the solid angle distortion at the corners of a hexahedron. If the solid angles at the corners are all , the scaled Jacobian achieves the optimal value of 1 . It is well-known that a hexahedron can be decomposed into eight overlapping tetrahedra. It may be natural to use various tetmesh optimization techniques [37, 38] to optimize these individual tetrahedra. It is also worth noting that many simplicial and polygonal map optimization techniques [39, 40, 41] can also be applied to optimize tet-meshes. However, as already shown in the work by Livesu et al. [8], simply optimizing the tetrahedra associated with the corners of a hexahedron may not improve its quality. Fu et al. [42] introduced an advanced MIPS method for computing locally injective mappings, which can be used to substantially improve the quality of a couple hexmeshes. However, only a few simple hex-meshes with no inverted elements were used in their testing. It is unclear how general this can be when applied to other hex-meshes with a substantially lower quality.

Besides that, many other hex-mesh optimization techniques exist. As reviewed by Wilson [43] and Livesu et al. [8], these techniques generally focus on untangling inverted elements (i.e., with negative scaled Jacobian) and improving the average element quality. Knupp introduced techniques to untangle the inverted elements [2] and improve the overall quality of the hex-mesh [3], which later have been integrated into the famous Mesquite library [4]. Specifically, the Mesquite library attempts to simultaneously untangle and improve the hex-mesh by minimizing an function. However, since it optimizes one vertex at a time, the performance of Mesquite is slow when applied to hex-meshes with a large number of inverted elements. Later methods resort to local Gauss-Seidel approaches to iteratively untangle and smooth meshes [5, 6, 7]. Besides the GaussSeidel optimization strategies, non-linear optimization has also been applied to improve the hex-mesh quality [43]. Other optimization techniques for specific types of hex-meshes also exist, such as the quality improvement method for octree-based hexmeshes by Sun et al. [44]. Like many existing approaches, our method can handle hex-meshes generated by different methods (Section 4).

tor that indirectly measures the distortion of the hexahedra via a set of tetrahedra around each mesh edge. Based on this descriptor, a non-linear energy function is defined globally. To solve it, a local-global strategy is applied. As shown by the authors, this approach can untangle meshes that previous methods may fail to untangle. Therefore, we consider this method state-of-the-art and compare our method with it in this paper.

**3. Methodology**

Similar to many mesh optimizers, given an input mesh with a valid all-hex connectivity, our method first corrects the inverted elements, then improves the overall mesh quality. We also allow the boundary vertices to move out of the original volume if a valid solution cannot be found during untangling. This relaxation alleviates the difficulty of untangling elements at the concave areas of the surface. However, different from most methods, we directly measure the distance of the angles between pairs of connected edges from their respective ideal angles, leading to an intuitive and unified distortion energy formulation. In summary, our method consists of the following key steps (Fig. 1 .

Compute target surface. In this step we improve the quality of the surface and associate surface vertices with the features detected from the input mesh (Section 3.1).

Untangling. We detect all inverted elements based on their scaled Jacobians. A local optimizer coupled with a surface relaxation strategy is then used to untangle those inverted elements iteratively until an inversion-free outcome is obtained (Section 3.2).

Inversion-free volume deformation. Due to the relaxation of surface constraint, after the above untangling process, the boundary surface of the output inversion-free hex-mesh may be far away from the original surface. We then perform an inversion-free deformation to pull the current surface back to its original one procedurally (Section 3.3). This step is optional, most models do not need this step.

Improve MSJ. Even though the mesh is currently inversionfree (i.e., all elements have positive scaled Jacobian), its MSJ may still be too low for practical use. To further improve the MSJ, we adopt the above untangling process but with a larger target MSJ ( ) set by the user and perform the same local optimization (Section 3.4). In other words, the above untangling process can be considered as an optimization with the target MSJ .

After achieving the target MSJ, the obtained hex-mesh may undergo a global optimization to improve its average element quality. However, this step is optional. In the following subsections, we provide more details on the individual steps.

**3.1. Compute Target Surface**

Two different scenarios are considered: 1) the input has a reference triangle mesh of the boundary, and 2) the input does not have a reference triangle mesh of the boundary. For the former, we first smooth and project the surface vertices to the surface of reference mesh, and then take the smoothed and projected mesh as the target surface  
. For the latter, we consider the boundary of the input mesh as the reference mesh to compute the target surface. We first use a simple Laplacian smoothing to improve the surface (e.g., regulate the boundary quad mesh) of the input hex-mesh. Generally, we perform 20 iterations of smoothing. Smoothing the interior vertices in the volume is optional. We then project the smoothed surface to the reference mesh. To do so, we use a perpendicular ray to project a vertex to all planes of triangle facets on the reference mesh. Specifically, a quad facet has 4 overlapping triangle facets. If the intersecting point is inside the triangle (i.e., the parameters of its barycentric coordinates satisfy ), we add it to a set . Finally, we select the intersecting point that is the closest to as the projected point. Via this projection, we obtain the target surface .

For classifying the boundary vertices, we rely on a userspecified angle threshold . If the dihedral angle between two facets sharing a common edge is smaller than , we classify the vertices of as on the sharp feature . If a vertex is adjacent to more than 2 sharp edges, then we consider it as a corner . We mark other surface vertices as regular. During the optimization, a corner could only move within a very small ball, a vertex of sharp edge could move along the feature line, and a regular vertex could move along the tangent plane. See the Eq.(7) for more detailed discussion on how to use this classification.

**3.2. Untangling**

Our untangling process is performed locally. We first detect all the inverted elements based on their scaled Jacobians. We then define a local region surrounding each inverted element. For those inverted elements that form a cluster (i.e., connected with each other), a larger region will be identified. In our implementation, the local region is defined as the union of the two-ring neighborhood surrounding each inverted element. The reason of considering a two-ring neighborhood is that onering neighborhood might not provide sufficient information for the subsequent target edge length computation (i.e., Eq. 8)). If the mesh contains a large portion of inverted elements (e.g., the fandisk model in Figure 6, a larger neighborhood will be constructed to enclose these elements. During the optimization, the boundary vertices of this local region are fixed. To untangle the elements within this local region, an energy function is used to compute the distortion of the individual elements from a canonical cube. In general, any proper distortion energy function can be used here, including the edge cone descriptor [8].

However, we opt for an variant of the edge cone descriptor inspired by the recently introduced local frame description [45] due to the following reasons. First, it is intuitive and easy to implement. Second, it will be shown that all different energy terms can be unified under the same representation. In the next, we describe our distortion energy.

**3.2.1. Distortion Energy**

Given an all-hex mesh that contains the sets of vertices

![](data:application/octet-stream;base64,)

Fig. 1. We optimize a hex-mesh (a) with multiple inverted (red) elements. We first obtain a target surface (b) by aligning boundary vertices to features. We then produce an inversion-free mesh (d) with large surface distance error via optimizing the local regions (c) using a soft constraint on the surface. Next, we use inversion-free deformation to pull the surface of (d) back to the one obtained in (b) and obtain an inversion-free mesh (e). Using a hard constraint on the surface, we further optimize the mesh to improve its MSJ with low surface distance error (f).

Let denote the coordinates of vertex . Our goal is to minimize the following energy.

![](data:application/octet-stream;base64,)

Fig. 2. The relationship of neighboring edges. in (a) is a regular edge, while in (b) is irregular. Different colors indicate different parameterization directions.

Orthogonality term. Consider a set of edges adjacent to vertex , the ideal configuration of two edges that are following two different parameterization directions should be as orthogonal as possible (e.g., edge versus the other edges as shown in Figure 2(a)). This leads to the orthogonality energy.

where indicates that the two edges are on two different parameterization directions that are orthogonal to each other. and are the edge vectors associated with edges and , respectively, pointing outwardly from the center vertex . That is, .

Straightness term. Similarly, we can define the straightness energy among the connected edges that are following the same parameterization direction as follows.

This energy attempts to make the connected edges that are following the same parameterization direction as straight as possible (e.g., the gray edge pair and and the green pair and in Figure 2(a)).

Irregular edge term. The above straightness term cannot handle the irregular edges whose values are not 4 . Consider with valence 5 in Figure 2(b) In this case, the orthogonality between and the rest of the edges around still holds. However, it is impossible to define the straightness among edges due to the irregularity. To address that, we define an energy as the difference between their pairwise angles and their respective ideal angles.

where is the ideal target angle between edge vectors and . For instance, the ideal angle between edges and is , while the ideal angle is between and in Figure 2(b).

Unified energy. In fact, all the above energy terms can be defined as the difference of the angles between pairs of edges from their respective ideal angles. This leads to the following unified expression of all above energy terms

where is the ideal target angle between edge зlength. In our experiments, we use . A detailed  
vectors and . If and are following two orthogonal parameterization directions, their ideal angle is , thus ; if and follow the same parameterization direction, their ideal angle is , thus ; if and are edges around an irregular edge (e.g., Figure 2(b)), their ideal angle is where is the valence of the irregular edge and is the number of edges between and when traversing from to .

In fact, optimizing this angle based distortion energy function is equivalent to optimizing the cone descriptor with the advantage of no need to estimate the valid normal direction for each cone. That is, if all angles around a mesh edge achieve their respective ideal angles, the associated tetrahedra around this edge also have optimal configuration as indicated in Figure 2.

Boundary Handling. To achieve surface conformity, we use the same strategy introduced in [8] that allows the boundary vertices move along the surface. Specifically, the boundary vertices are constrained to stay on their respective tangent planes, feature lines, or corners, based on their classification:

Here is the reference (or closest) surface position for each vertex is the current position of is the surface normal at position is the feature tangent at , and is an auxiliary variable added to the system to enable feature constraints. and are two coefficients that are used to control how strong the boundary constraint is. The larger these two coefficients, the more penalty will be applied to vertices that leave the target surface. In default, we set for all our experiments. During the untangling process, these two coefficients will be updated according to the outcome of the preceding iteration.

Combined energy. By combining the above energy defined in the interior and on boundary of the volume, respectively, we solve for the following optimization problem:

**3.2.2. Numerical Solution**

Equation (7) is not a quadratic function, which means that it is impractical to solve it directly. If we use the nonlinear solver, it will converge at a very slow speed. To address this, we use a local-global like scheme, in which we use the local (or current) values for some variables. Specifically, in the local step, we fix and in Equation (5) (i.e., they are treated as constant with their current values). Also, to determine whether a uniform-size element is enforced or not, we use as the target length for edge if (otherwise, is used). is a user-input parameter and is the average surface edge  
discussion on the effect of is provided in Section 3.5  
Using this method we can construct an over-determined linear system . To minimize the energy (7), we iteratively solve the linear equation . The solver is terminated once it achieves the target MSJ (e.g. for the untangling).

To accelerate the above computation, we use the target length for each edge in the first iteration. The target length can be computed by minimizing the following quadratic energy.

The solution of is an approximate solution. To avoid overshooting, we decrease the step size linearly for each iteration to update the locations of the interior vertices gradually.

**3.2.3. Untangling Pipeline**

We now describe our untangling process. Given an input hexmesh , we first scale its size w.r.t. its center so that its average edge length equals to (we set for all our experiments). This rescaling step is crucial, which enables us to use the same default and values for all different models. Otherwise, different values need to be selected based on the element size of the input mesh. During the scaling, not only needs to be scaled, its target surface has to be scaled to ensure the consistent boundary constraint for the boundary handling. After this normalization, we then identify all inverted elements and construct a local region for each of them. For all these local regions, we perform the following iterative process until is untangled: we first compute target edge lengths in these regions by solving Eq. (8), then set initial step size for updating the vertex positions . Next, we iteratively optimize vertex positions by solving Eq. (7) using the aforementioned local-global strategy until the maximal allowed iterations ( 20 by default) are reached. For each iteration, we check whether the number of inverted elements is reduced within a region. If not (likely due to the overshooting), the solution of this iteration is discarded and is reduced. This process guarantees that the number of inverted elements is monotonically reduced. After locally optimizing the vertices within the region, if the outcome mesh still contains inverted elements, we then decrease so that the uniform-size is not enforced. If is too small (e.g., in our implementation), we decrease and by half and repeat the above process. Algorithm 1 provides the pseudocode of this untangling process. After optimizing the mesh, we scale it back to its original space.

**3.3. Inversion-free Volume Deformation**

After the above untangling process with surface relaxation, the surface of the output untangled mesh may be far away from the target surface (see the inset). Previous methods simply project this deformed surface onto .

![](data:application/octet-stream;base64,)

periments, and 20 iterations are performed.

This simple projection does not guarantee that the obtained mesh is still inversion-free. To address that, we formulate the above problem as a volumetric mapping problem , where is the boundary of . A number of inversion-free local injective mapping techniques [41, 42] can be used to achieve the above deformation. In this work, we select the recently introduced SLIM solver [46]. To utilize the SLIM solver, we decompose each hexahedron of into eight tetrahedra (i.e., one tet for each corner). The AMIPS exponential energy [42] is used in our ex-

Algorithm 1: Local untangle  
 Input: $\mathcal{H}, \Omega\_{t}$  
 Output: $\mathcal{H}^{\prime}$  
 Scale $\mathcal{H}$ and $\Omega\_{t}$;  
 Set $\alpha=1000, \beta=1000, \xi=0.6$;  
 while current $M S J \leq 0$ do  
 while not reach maximum global iteration (default 20)  
 do  
 Identify inverted elements $\mathcal{I}$;  
 Extract local regions $\mathcal{R}$ (a copy from $\mathcal{H}$ );  
 Classify surface vertices for $\mathcal{R}$;  
 Compute target edge length by solving Eq. (8) for  
 $\mathcal{R}$;  
 $\tau=1 ;$  
 while not reach maximum local iteration (default  
 20) do  
 $\tau=0.9 \tau ;$  
 Solve Eq. (7) for $\mathcal{R}$;  
 Save $\mathcal{R}$;  
 $\mathcal{R} \leftarrow$ Update vertices. using Eq (9);  
 Project surface vertices of $\mathcal{R}$ to its original  
 surface;  
 if \#invertedElements increased then  
 Recover the saved $\mathcal{R}$;  
 end  
 Update the vertices of $\mathcal{R}$;  
 if current $M S J>0$ then  
 output $\mathcal{H}^{\prime}$;  
 end  
 $\xi=\xi-0.1$;  
 if $\xi<0.2$ then  
 $\alpha=0.5 \times \alpha, \beta=0.5 \times \beta, \xi=0.6 ;$  
 end

**3.4. Improving MSJ**

Similar to the above untangling process. The improvement of the MSJ can be performed locally. In fact, the same process  
sto the above untangling can be employed with only the modification of the target MSJ , which is specified by the user. Given this target MSJ, the optimizer will first identify the elements whose scaled Jacobian is smaller than . A local region is then constructed for each identified element, which will be used to perform the local improvement. In all our experiments of improving MSJ, we avoid using to control distance error. In fact, most of the time we can achieve the target MSJ using . The bigger are, the stronger the surface constraint is. In practice, if a larger is set (e.g., > 0.5), the optimizer will take longer time to converge. Sometime, it may not even find a solution. Therefore, we suggest to achieve this procedurally. That is, we optimize the mesh so that its MSJ is positive, then , until it reaches a value above or close to . This procedural strategy is shown very effective in practice (Figure 10.

**3.5. Discussion on User Parameters**

Our approach allows four user-input parameters: (1) target minimum scaled Jacobian , (2) surface constraint , (3) angle threshold for sharp feature and corner identification, and (4) edge length constraint that controls whether a uniform-size hex-mesh is preferred.  
Effects of different and Figure 3 shows the untangling results with different values of and . Figure 3(a) shows the results with . The output mesh has small surface distance from the original surface. However, the untangler fails to correct all inverted elements (see the red elements). Figure 3(b) is the result of the same input mesh with . Note that the untangler successfully corrects all inverted elements. However, the surface distance from the original surface is larger than the one shown in Figure 3(a). Generally, larger result in smaller distance error but lower MSJ; in the opposite, smaller lead to larger distance error but higher MSJ. In our untangling process and MSJ improvement, the values of and are automatically adjusted to find a desired solution. For a large user-specified target MSJ , due to the configurations of the individual surfaces, smaller and may be used to achieve , which may lead to large surface error. Although an inversion-free deformation can be applied to reduce the surface error, it may worsen the MSJ at the same time. Therefore, in our experiments, we do not allow the values of and to be smaller than 500 during the improvement of MSJ, which also ensures a small surface distance error. However, the user may choose to lower the values of and to achieve even better MSJ with the possible larger surface error.  
Effects of different . Parameter is used to control the extraction of surface features. Figure 3(e) and 3(f) show the effect of different . In general, the larger is, the more surface features will be detected, thus more constraints will be applied to the surface vertices. In practice, we set . Nonetheless, the accurate detection of surface sharp features is non-trivial and tends to be very sensitive to noise. Addressing this is beyond the scope of this work.  
Effects of different . As briefly mentioned earlier, parameter is used to control whether a mesh with uniform-size elements (i.e., with constant edge length) is desired or not. In particular,  
the larger is the stronger the constraint on uniform-size elements. For instance, in Figure 3(b) and 3(c) both and are set as 100 , while is 0.4 in 3 (c) and 0.2 in 3 (b). For the , the untangler fails to correct all inverted elements. This shows that enabling some variation in the element size will in fact help enhance the success rate of untangling. Similarly, in the MSJ improvement, a larger will constraint the optimizer from find- ing a good solution (Figure 4(b) and 4(c)).

![](data:application/octet-stream;base64,)

Fig. 3. (a) The output mesh has small surface distance from the original surface with . However, the untangler fails to correct all inverted elements (red). (b) Untangler successfully corrects all inverted elements. However, the surface distance from the original surface is larger than the one in (a). (c) Larger greatly impacts the untangler for this model (cap) even using very small and . However, using a more relaxed helps untangling (d). (e) and (f) show the detected surface features (in black) with different values. Small results in less sharp feature lines and corners

Note that among the above four parameters, the default values for and (i.e. 1000, 1000, and 0.6 ) usually work well for the majority of the models, thus need not be adjusted. However, in some cases, and still need careful selection in order to produce an ideal result, which we will show next.

**4. Results**

We have applied our untangling and MSJ improvement technique to a number of hex-meshes produced by a variety of

![](data:application/octet-stream;base64,)

Fig. 4. This experiment fixes the parameters , and make varying. Large step of target MSJ or larger will make improvement fail. (b) fails with 315 inverted elements while (c) fails with 275 elements. The volume of each element is mapped to red-togreen to show the effect of using different . The more constant the color, the more uniform sized the elements are.

ing the MSJ of these meshes. We also achieve the best ASJ for the RockerArm mesh.

Table 1. Comparison with the edge cone technique. denotes results of edge-cones [8], \* denotes ours. We use metro tools to compute Hausdorff distance w.r.t. bounding box diagonal [47].

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Model | #hexes | #flip | Error | MSJ | ASJ |
| Armadillo | 29935 | 323 | 0.011262 | 0.14 | 0.9 |
| Armadillo\* | 29935 | 323 | 0.019349 | 0.163 | 0.834 |
| Block | 2520 | 31 | 0.004555 | 0.250 | 0.870 |
| Block\* | 2520 | 31 | 0.002737 | 0.252 | 0.857 |
| Bunny | 37734 | 1 | 0.007955 | 0.606 | 0.972 |
| Bunny\* | 37734 | 1 | 0.006477 | 0.651 | 0.953 |
| Bust | 5258 | 30 | 0.007404 | 0.114 | 0.922 |
| Bust(Fig1)\* | 5258 | 30 | 0.008843 | 0.201 | 0.869 |
| Bust\* | 5258 | 30 | 0.006795 | 0.183 | 0.865 |
| Cap | 4420 | 50 | 0.009933 | 0.106 | 0.870 |
| Cap\* | 4420 | 50 | 0.005320 | 0.114 | 0.775 |
| Dancing | 35293 | 5 | 0.008480 | 0.354 | 0.942 |
| Children\* | 35293 | 5 | 0.006905 | 0.582 | 0.931 |
| Hanger | 4539 | 3930 | 0.002709 | 0.716 | 0.987 |
| Hanger\* | 4539 | 3930 | 0.001486 | 0.723 | 0.974 |
| Impeller | 11174 | 8857 | 0.000935 | 0.184 | 0.942 |
| Impeller\* | 11174 | 8857 | 0.000493 | 0.192 | 0.934 |
| KingKong | 159488 | 11 | 0.010483 | 0.268 | 0.967 |
| KingKong\* | 159488 | 11 | 0.016948 | 0.500 | 0.954 |

Table 2. Comparison with AMIPS and edge cone. The first row of each model shows the result by AMIPS [42]. denotes results of edge-cone [8], \* denotes ours.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Model | #hexes | input MSJ | MSJ | ASJ |
| Fertility | 10600 | 0.209 | 0.46 | 0.937 |
| Fertility | 10600 | 0.209 | 0.478 | 0.951 |
| Fertility\* | 10600 | 0.209 |  | 0.933 |
| RockerArm | 19870 | 0.196 | 0.550 | 0.923 |
| RockerArm | 19870 | 0.196 | 0.556 | 0.937 |
| RockerArm\* | 19870 | 0.196 |  |  |

Improve hex-meshes from polycube map. Next, we apply our technique to improve tens of hex-meshes generated from the polycube map database by Fu et al. [49]. The initial hex-meshes consist of varying numbers of inverted elements. Our technique successfully untangled all of these meshes and managed to improve their MSJ substantially. As a comparison, we also show the results of the improved elephant and bottle1 meshes produced by the authors of the edge-cone technique. The comparison shows that our method produces much higher-quality meshes for these two cases in all quality metrics. Note that the surface error of the elephant mesh produced by the edge-cone method is not measurable as the resulting mesh does not have the same scale as the input mesh. We also note that a padding process is applied during the generation of the initial polycube hex-meshes to push the surface singularities into volume. This ensures that degenerate cases (i.e., corner is located on the flat region of the surface) do not occur; otherwise, the meshes may not be able to untangle as already shown by Livesu et al. [8].

Table 3. Stress Test. Fandisk is created by the frame field method, Kitty is generated by the PolyCube [23], and airplane is obtained using MeshGems [48]. #flip shows the number of inverted elements after the artificial perturbation. \* denotes our results. We use metro tools to compute Hausdorff distance w.r.t. bounding box diagonal [47]. '-' means the mesh has no reference surface to compute the Hausdorff distance error.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Model | #hexes | #flip | Error | MSJ | ASJ |
| Fandisk | 357 | 0 | - | 0.609 | 0.936 |
| Fandisk | 357 | 286 | 0.004769 |  |  |
| Kitty | 7083 | 0 | - | 0.424 | 0.910 |
| Kitty | 7083 | 3232 | 0.012656 |  |  |
| airplane1 | 4972 | 0 | - | 0.030 | 0.838 |
| airplane1 | 4972 | 3510 | 0.004369 |  |  |

Table 4. The results on a set of meshes produced from the polycube map database [49]. denotes results of edge-cone [8], \* denotes ours. We use metro tools to compute Hausdorff distance wrt. bounding box diagonal [47]. '-' means the mesh's surface is not in the same scale with the input for computing the Hausdorff distance error.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Model | #hexes | #flip | Error | MSJ | ASJ |
| airplane1\* | 17913 | 467 | 0.006257 | 0.731 | 0.959 |
| bird\* | 16934 | 288 | 0.005774 | 0.732 | 0.961 |
| cup1\* | 16862 | 40 | 0.006944 | 0.723 | 0.960 |
| chair1\* | 20344 | 709 | 0.004720 | 0.690 | 0.941 |
| horse\* | 44145 | 304 | 0.017018 | 0.600 | 0.944 |
| blade\* | 14792 | 141 | 0.008885 | 0.650 | 0.946 |
| kiss\* | 19976 | 247 | 0.014755 | 0.500 | 0.913 |
| bottle1 | 15478 | 127 | 0.008066 | 0.132 | 0.925 |
| bottle1\* | 15478 | 127 | 0.009675 | 0.604 | 0.948 |
| elephant | 46525 | 421 | - | 0.012 | 0.881 |
| elephant\* | 46525 | 421 | 0.009899 | 0.500 | 0.915 |

![](data:application/octet-stream;base64,)

Fig. 5. Blade example(created by the Octree-based method)

![](data:application/octet-stream;base64,)

Fig. 6. Fandisk example(created by the framed field method)

![](data:application/octet-stream;base64,)

Fig. 7. Kitty example(created by the -PolyCube method)

**Improve hex-meshes generated by the octree-based method.**

We also apply our technique to improve hex-meshes generated using the MeshGems [48]-an octree-based method [19, 50]. The initial hex-meshes consist of elements with very low scaled Jacobian (< 0.1). For most models, our method can improve their MSJ to be greater than 0.2 . Figure 5 and Table 5 provide examples of the improvement of octree-based meshes. More details are in the supplementary document.

Fig. 8. airplane1 example(created by the octree-based method)

![](data:application/octet-stream;base64,)

Stress test. In the stress test experiments, we untangle hexahedral meshes perturbed from the initial hex-meshes generated by the frame-field based, polycube based and octree-based methods, respectively. Our method successfully untangles the perturbed meshes and produces meshes with much better quality than the original ones (Figures 6, 7) and 8). Despite the meshes

Table 5. The results on a set of meshes produced from MeshGems [48]. We use metro tools to compute Hausdorff distance w.r.t. bounding box diagonal [47].'-' means the there is no reference surface to compute the Hausdorff distance error. and show the input and output, respectively

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Model | #hexes | Error | MSJ | ASJ |
| bird | 4247 | - | 0.0313 | 0.820 |
| bird | 4247 | 0.011580 | 0.553 | 0.868 |
| blade | 10996 | - | 0.025 | 0.845 |
| blade | 10996 | 0.007911 | 0.312 | 0.868 |
| block | 1624 | - | 0.179 | 0.661 |
| block | 1624 | 0.016877 | 0.550 | 0.815 |
| bone | 2751 | - | 0.154 | 0.781 |
| bone | 2751 | 0.006475 | 0.207 | 0.794 |
| dragonstand2 | 23917 | - | 0.013 | 0.837 |
| dragonstand | 23917 | 0.004598 | 0.304 | 0.857 |
| fish1 | 9537 | - | 0.015 | 0.815 |
| fish1 | 9537 | 0.008377 | 0.308 | 0.845 |
| gargoyle | 41610 | - | 0.024 | 0.834 |
| gargoyle | 41610 | 0.005247 | 0.200 | 0.849 |
| kiss | 18418 | - | 0.027 | 0.844 |
| kiss | 18418 | 0.005075 | 0.224 | 0.857 |
| rocker | 16608 | - | 0.108 | 0.865 |
| rocker | 16608 | 0.007742 | 0.241 | 0.874 |

in our stress test contain large portions of inverted elements, they are generated with artificial perturbation. In the future, we plan to further assess our optimization technique with meshes containing large numbers of inverted elements in practice.  
Performance study. As mentioned in Section 3.4, our improvement of MSJ is performed in several stages. The benefit of this divide-and-conquer strategy is that the number of elements that have quality lower than the current target MSJ remains small each step, which facilitates our optimizer to quickly find a solution. Figure 10 shows a timing plot of this gradual improvement process. The times spent on the individual stages are shown as the histogram, and the orange curve shows the accumulated time. As expected, more time will be needed to achieve a higher MSJ as more elements will have quality lower than the target MSJ. In general, our method takes about 2 minutes on average to process a mesh with elements. The smaller the MSJ, the faster the computation will be as already shown in Figure 10.

![](data:application/octet-stream;base64,)

Fig. 10. Performance plot of our technique.

![](data:application/octet-stream;base64,)

Fig. 9. Result gallery. Elements with scaled Jacobian are transparent in the input (first figure for each model). Output meshes are colored using element volume info (last figure for each model). Elements with are also showed in the output (e.g., the middle image for each model). We hope to further improve these elements in the future work. If all elements of a mesh have , we do not show its middle image (e.g., the airplane and cup models).

![](data:application/octet-stream;base64,)

Fig. 11. SLIM fails to map back the surface of dragon.

Failure case. Figure 11 shows a failure case where the SLIM solver fails to map the deformed surface after MSJ improvement back to the target surface. This may be an issue of the SLIM solver that typically requires an accurate correspondence between the boundary vertices of the current mesh with those on the target surface, while our current projection based method may not be sufficiently accurate. In the future, we plan to experiment with other more robust inversion-free mapping technique and improve our surface correspondence calculation.

**5. Conclusion**

In this paper, we introduce a simple yet effective hex-mesh improvement technique. This technique is based on a new and intuitive angle based optimization strategy. To enable our optimizer to find a valid solution, we allow the boundary surface to move out from the original volume, which will be mapped to the original surface with the inversion-free guarantee. To accelerate the computation, we perform the optimization within a local region surrounding the inverted elements or elements with quality lower than the user-specified threshold. Our method is easy to implement. We also discuss the effects of the different values of a number of parameters used in our framework to help users choose proper values for their needs. We have applied our method to a large number of hex-meshes generated with a variety of methods to demonstrate its effectiveness.

Limitations.. First, although our method produces meshes with higher MSJ for all the test meshes and better Hausdorff distance for the majority of meshes when compared to the state-of-theart techniques, our method may not improve the average scaled Jacobian substantially. Again, this is due to the relaxation of the constraint on uniform element sizes. Also, our sub-optimal ASJ may also attribute to the selection of the parameter . For most models, we find that can produce ideal results. But for some models (e.g. Hanger), the result using is better than the one obtained with other values of . Nonetheless, the fixed value of throughout the entire mesh may constrain the improvement of ASJ. Should the of an edge be a function with respect to its neighboring configuration, ASJ might be improved further. Second, to ensure an inversion-free outcome, the meshes generated with our method may have a surface distance larger than the user-specified error. Third, our current surface feature detection is sensitive to the user-specified angle threshold . A robust feature detection technique may be required to resolve this issue. Fourth, in the extreme case (i.e., a complete inverted mesh), our angle based energy will vanish. However, since we enforce the boundary constraint of  
non-inverted elements, such an extreme case will not occur. Finally, our method for solving the non-linear energy minimization problem is not a typical local-global scheme, which may not converge to meet the required mesh quality. However, it enables us to effectively minimize our angle distortion energy. We plan to address these limitations in the future.

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1. 56 Recently, Livesu et al. [8] introduced the edge cone descrip-