Advanced Strategy in Nim: The Impact of the Freeze Move

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1 Introduction

This report introduces an advanced variation with the inclusion of "freeze moves", adding a predictive and strategic layer to the Nim strategy game .

2 Definition

2.1 Game: Nim (Impartial)

Position: A collection of stacks of counters, with each stack containing a non-negative number of counters. The collection of no stacks is considered a stack.

Moves: A player selects a stack and removes some or all of the counters in that stack.

End of game: Player with last move win

2.2 Freeze Move

The advanced variation retains the basic rules of Nim but introduces a "freeze move" mechanic:

- **Execution**: A player may declare a freeze move before their opponent's turn, selecting a stack to immobilize temporarily. The opponent must be notified of this action.
- **Usage**: Every player gets one freeze move in each game to start with. An additional move is granted if a player successfully predicts their opponent's next move. Freeze moves can't be used as the last stack of the game. To prevent an infinite game, players can't freeze a single stack more than two times in a row.
- **Effect**: If the opponent's chosen move includes a frozen stack, the move will be discarded.

3 Sample Game

Moves in the game are represented by ordered tuples. The blue number represents the stack we freeze. For example, if a game position changes from having stacks of 2, 2, and 1 counters to having stacks of 2 and 2 counters, this can be represented as:

Move Sequence:

$$
\begin{aligned}
(2,2,1) &\rightarrow_{\text{B use the freeze move}} (2,2,\textcolor{blue}{1}) \rightarrow_{\text{A}} (2,2) \rightarrow_{\text{+1 freeze move for B}} \\
&\rightarrow (2,2,1) \rightarrow_{\text{A use the freeze move}} (2,2,\textcolor{blue}{1}) \rightarrow_{\text{B}} (1,2,1) \rightarrow_{\text{B use the freeze move}} (1,\textcolor{blue}{2},1) \\
&\rightarrow_{\text{A}} (1,2) \rightarrow_{\text{B}} (1,1) \rightarrow_{\text{A}} (1) \rightarrow_{\text{B}} (0)
\end{aligned}
$$

Text description:

1. Player A goes first, so Player B uses the freeze move to freeze the third stack.
2. Player A removes the third stack; the move is discarded because Player B froze that stack on their last move, and the game comes back to default. Player B also receives one additional freeze move.
3. Next move is Player B turn so Player A uses the freeze move to freeze the third stack.
4. Player B removes one counter from the first stack
5. Next move is Player A turn so Player B uses the freeze move to freeze the second stack.
6. Player A removes 1 counter from the third stack
7. Player B removes 1 counter from the second stack
8. Player A removes 1 counter from the first stack
9. Player B removes 1 counter from the second stack. B win since that the last move

This sequence demonstrates the strategic depth added by the freeze move, forcing players to adapt and predict beyond the traditional gameplay.

4 The Analysis

4.1 Option Notation

Standard Moves

For a position (x, y, z), the options for the next move can be represented as:

$$\mathcal{O}(x,y,z) = \{(x',y,z),(x,y',z),(x,y,z') \mid 0 \le x' < x, 0 \le y' < y, 0 \le z' < z\}$$

where $\mathcal{O}(x, y, z)$ denotes the set of all positions reachable from (x, y, z) by making a standard move.

Freeze Moves

For freeze moves, we extend this notation. If a freeze move is applied to a stack, we denote this as a barred number. For instance, $(\overline{x}, y, z)$ indicates that stack x is frozen for the next turn. The option set for a position with a potential freeze move is:

$$\mathcal{F}(x, y, z) = \{(\overline{x}, y, z), (x, \overline{y}, z), (x, y, \overline{z})\}$$

This set includes the positions reachable by applying a freeze move.

Combining Standard and Freeze Moves

The combined set of options for a player in any given position is the union of $\mathcal{O}$ and $\mathcal{F}$ sets.

Game Value

It would encapsulate both the number of objects in each stack and the strategic implications of freeze moves.

4.2 Outcome Classes

In optimal play by both players, the standard Nim game often easily decides which player wins by determining whether the Nim sum at the start of the game is equal to 0 or not. The addition of freeze moves can alter the Nim game's outcome classes. For instance, a stack with a single counter, typically in a disadvantageous position, can be turned favorable by a well-timed freeze move. Because of that, the outcome classes are harder to determine since there are more possible moves and ways to play. However, we can still determine some outcome classes for games with small stacks:

- In a one-stack game, since players can't use freeze move as their last stack of the game, it's a first-player win like the standard Nim

- In a two-stacks game, the winner can often be predicted based on the initial position with optimal play. However, using the freeze move significantly diversifies potential outcomes. Below are some outcome classifications for the Nim game incorporating the freeze move:

  1. If either player, in their first move, eliminates all counters from a single stack and no freeze move is used (either because the opponent did not use it or their prediction was incorrect), then the player who makes the subsequent move will win the game.
  2. If the initial position of the game is (1,1), the previous player wins if no freeze move is used or they predicted the next player's first move wrong. The next player can win if the previous player freezes the next player's first move.

3. For $x \neq 1$, after the position is set to (x,1) or (1,x), the optimal freeze move set for both players is:
$$\mathcal{F}(x, y) = \{(\overline{x}, 1), (1, \overline{x}), \}$$
Since players can't freeze a single stack more than two times in a row, the player who sets the position to (x,1) or (1,x) wins the game.

4.3 Predictive Strategy

A crucial aspect of this variation is the predictive use of the freeze move. Players will use the Nim sum strategy to predict the opponent's next move, freeze their winning move, and force them to make another move in order not to lose the move. Players must not only consider the immediate game state but also anticipate future moves. Specifically, in a two-stacks game, players can utilize the outcome classes from part 4.2 to maneuver into a more advantageous position and strategically employ the freeze move to secure a win. This predictive element adds complexity to the standard combinatorial game theory applied to Nim.

5 Conclusion

The addition of the freeze move fundamentally transforms the strategic dynamics of Nim. This report emphasizes the importance of foresight and flexibility in gameplay, thereby enriching the strategic complexity of Nim. Furthermore, the concept of 'freeze moves' can be applied to various other games. The analysis shows that even a simple rule change can profoundly impact game theory dynamics, opening new avenues for strategic exploration.