Lecture 1

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Course logistics notes: OH Tuesday 3-4 (tentatively)

Side note: see course website for learning goals and some exercises after class. Prof. doesn't expect us to have read textbook before the class.

§1 Linear Vector Spaces

Definition 1.1 (Vector Space)**.** A vector space is a collection of objects $|1\rangle, |2\rangle, \ldots |V\rangle, \ldots$ called vectors for which there is a definite rule for addition and scalar multiplication.

- Vectors are closed under addition
- Scalar multiplication is distributive in the scalars and vectors
- Notion of zero vector such that $\vec{0} + \vec{V_1} = \vec{V_1}$
- Subtraction $V_1 + -V_1 = 0$
- Addition is commutative and associative.

Note that characteristics such as magnitude and direction are not included in the definition. For example, verify that the set of 2x 2 matricies is a vector space, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

However, something like the set of all 2 x 2 matricies where top left is 1 is not a vector space (not closed under addition or scalar multiplication) $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$

Example 1.2 (Vector space example)

Strings f(x) where $0 \leq x \leq L$. Define addition and scalar multiplication pointwise. Some restrictions are possible: i.e. periodic functions where $f(0) = f(L)$, or strings that vanish at the ends.

Definition 1.3 (Linear independence)**.** $V_1 \ldots V_n$ are linearly independent if

$$\alpha_1 V_1 + \alpha_2 V_2 \ldots + \alpha_n V_n = 0 \rightarrow \alpha_i \equiv 0$$

Example 1.4 (Linear independence)

The matricies $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, ... are linearly independent (4 total vectors)

Definition 1.5 (Dimension)**.** The maximal number of linearly independent vectors in a vector space

Suppose there are n linearly independent vectors in an $n-$dimensional vector space, denoted $V_1, \dots, V_n$. This set is known as a **basis**. Any other vector V can be written as a sum of the basis vectors, with $V = \sum_{i=1}^{n} \alpha_i V_i$ where each α_i is the **component** and each V_i is a basis vector, and the v_i are the components.

Theorem 1.6 (Vector expressed uniquely with basis)

For a given basis, each vector V is uniquely defined by its components.

Proof. Suppose there are two representations. $V = \sum \alpha_i V_i$ and $V = \sum \beta_i V_i$. Subtract them to get $0 = \sum(\alpha_i - \beta_i)V_i$. By linear independence of the basis, each component must be unique. $\square$

Vectors will be denoted using braket notation. A vector $\vec{V}$ will be written as: $|V\rangle$. Any vector V is thus expressed as

$$|V\rangle = \sum_{i=1}^{N} v_i|i\rangle$$

where each $|i\rangle$ is a basis vector.

Inner product space: Consider $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_J\vec{k}$, and similarly for $\vec{B}$ with coefficients B_x, etc. Consider the usual definition $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$. This has some properties such as:

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- $A(\alpha B + \beta C) = \alpha A \cdot B + \beta\alpha \cdot C$
- $A \cdot A \geq 0$. This equals zero iff $A = 0$

The properties of an inner product between any two vectors $|V\rangle, |W\rangle$ denoted by $\langle V|W\rangle$ are:

Definition 1.7 (Inner product)**.** A vector space with inner product is called an inner product space. The inner product obeys the following axioms:

- $\langle V|W\rangle = \langle W|V\rangle^*$
- $\langle V|V\rangle \geq 0$, where it is 0 iff $|V\rangle = |0\rangle$
- $\langle Z|(\alpha|V\rangle + \beta|W\rangle)\rangle = \alpha \langle Z|V\rangle + \beta \langle Z|W\rangle$

There is **antilinearity** of the inner product with respect to the first factor:

$$\begin{aligned}
\langle aW + bZ|V\rangle &= \langle V|aW + bZ\rangle^* && (1)\\
&= (a \langle V|W\rangle + b \langle V|Z\rangle)^* && (2)\\
&= a^* \langle V|W\rangle * + b^* \langle V|Z\rangle^* && (3)\\
&= a^* \langle W|V\rangle + b^* \langle Z|V\rangle && (4)
\end{aligned}$$

We will make the definition that vectors are **orthogonal** if their inner product vanishes, and refer to $\sqrt{\langle V|V\rangle} \equiv |V| \equiv \|V\|$ as the norm or **length** of the vector. Finally, if a set of basis vectors all have unit norm and are pairwise orthogonal, they are called an **orthonormal basis** .

Let $|V\rangle = \sum_{i=1}^{N} v_i|i\rangle$, and $|W\rangle = \sum_{j=1}^{N} w_j|j\rangle$. Then, the inner product is calculated via

$$\langle V|W\rangle = \sum_i \sum_j v_i^* w_j \langle i|j\rangle \tag{5}$$

Note that this collapses down to the "usual" definition **assuming** $\langle i|j\rangle = \delta_{ij}$**.** We can usually get to this point by invoking the following:

Theorem 1.8 (Gram-Schmidt)

Given a linearly independent basis we can form linear combinations of the basis vectors to obtain an orthonormal basis.

Thus, the formula reduces to $\langle V|W\rangle = \sum_i v_i^* w_i$. To see why conjugation is important, consider $\langle V|V\rangle = \sum |V_i|^2 \geq 0$.

Since vectors are uniquely specified by its components in a given basis, we may write

$$|V\rangle \to \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{6}$$

and then the inner product of $\langle V|W\rangle$ is given by

$$\langle V|W\rangle = [v_1^*, v_2^*, \ldots, v_n^*] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \tag{7}$$