Geometry Review

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Contents

0 Conventions

For the most part, diagrams will not be included, both because I'm lazy and because you should get practice drawing diagrams! Drawing a decent diagram can often be the hardest part of a problem.

0.1 Definitions

It's assumed that you know the definitions of the five most common triangle centers, which are listed here for completeness. In $\triangle ABC$,

- the **incenter** is the concurrency point of the (internal) angle bisectors of $\angle A$, $\angle B$, and $\angle C$.
- the **centroid** is the concurrency point of the medians.
- the **circumcenter** is the concurrency point of the perpendicular bisectors of AB, BC, and CA.
- the **orthocenter** is the concurrency point of the altitudes.
- the **A-excenter** is the concurrency point of the internal angle bisector of $\angle A$ and the external angle bisectors of $\angle B$ and $\angle C$. The B-excenter and C-excenter are defined similarly.

The existence of these centers will be assumed for now; methods to prove them will be developed later on. (You should be able to prove the all of the centers' existence if you remember last year's lessons, though.)

0.2 Notation

- When writing $\triangle ABC \cong \triangle XYZ$ or $\triangle ABC \sim \triangle XYZ$, the order of the vertices encodes corresponding vertices. In other words, $\angle A = \angle X$, $\angle B = \angle Y$, and $\angle C = \angle Z$.
- If points P_1, P_2, $\ldots$, P_n are concyclic (i.e. lie on some circle), then $(P_1P_2 \ldots P_n)$ is the circle passing through those points.
- We sometimes denote the side lengths of the sides of $\triangle ABC$ by $BC = a$, $CA = b$, and $AB = c$.

1 Philosophical Rambling

There are two main ways to view a geometry problem: *synthetically* and *analytically*.

Analytic methods involve viewing the configuration using some computational framework: trigonometry and Cartesian coordinates are the standard beginner's tools for this approach.

Synthetic geometry, on the other hand, can be thought of as building up geometry from some set of axioms: A lot of your 9th-grade geometry is the basis of synthetic methods.

A strong geometer is comfortable with both synthetic and analytic techniques: each tool (or perhaps combination of tools) is best suited for different kinds of problems. In my experience, people are generally much more comfortable working with computational techniques, so I'll generally focus more on synthetic techniques here. As such, you should always try to look for synthetic solutions where you can (but bonus points if you can find multiple solutions!).

2 Synthetic Techniques

2.1 Digression: How to approach problems

Generally, solving problems comes in two phases:

1. The "scouting" phase, where you try to get some intuition about why the problem works. This can manifest in multiple ways, such as
   - searching for what "should" be true, such as by working backwards (ex. "$\triangle HBC \sim \triangle ODE$ needs to be true for the problem to be true."),
   - getting a heuristic understanding of what's going on (ex. "$|7^a - 3^b|$ should generally be much larger than $|a^4 + b^2|$, so a and b should be small, whatever *small* means."),
   - thinking about why certain techniques don't work (ex. "I can't show that a randomly selected path in my graph behaves the way I want, so I should try considering the whole set of paths simultaneously."),
   - thinking about what techniques might work (ex. "I have a central right triangle, and my points are easy to define with respect to this right triangle, so Cartesian coordinates could work.")

   and so on. These are the things that people sometimes call "motivation."

2. The "attacking" phase, where you prove things about the problem. This is the part where you actually try computing things, performing induction, etc., ideally solving the problem, but at least getting some sort of intermediate claim. These are much more concrete methods, and are the parts that actually show up in your solution, if you were to write it out.

(This is largely parroting ideas from Evan Chen's blog post on hard and soft techniques.)

Put more simply, you gather information while scouting, and then use that information to mount an attack, hopefully destroying the problem.

Most of the time, scouting is done through a synthetic lens. This can come in the form of redefining points to be more well-behaved (perhaps in hope of finding an analytic approach), angle chasing to look for similar triangles/cyclic quadrilaterals (although that can be thought of as attacking, too), etc. In that vein, scouting strategies will be in **blue, sans-serif font**. Try to come up with some heuristics on your own, too.

2.2 Angles

Angle chasing is a very low-powered technique: you should've already encountered most of the necessary material in 9th-grade geometry. For completeness, here are the two most important theorems that you should know at least as well as the back of your hand (but preferably better).

Theorem 2.2.1 (Cyclic quadrilaterals and inscribed angles)**.** Let $ABCD$ be a convex quadrilateral. Then, the following are equivalent:

- There exists a circle containing A, B, C, and D.
- $\angle ABC = 180^\circ - \angle ADC$.
- $\angle ACB = \angle ADB$.

Theorem 2.2.2 (Tangents and chords)**.** There is a triangle $\triangle ABC$ and a line ℓ through A. Let D be a point on ℓ such that B and D are on opposite sides of line AC. Then, ℓ is tangent to (ABC) at A if and only if $\angle ABC$ and $\angle DAC$ are congruent.

2.2.1 Walkthroughs

Example 2.2.3 (JMO 2011/5)

Points A, B, C, D, E lie on a circle ω and P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$.

Prove that BE bisects AC.

Walkthrough. Let M be the midpoint of AC. We can then phrase the problem as "Show that B, M, and E are collinear."

(a) Assume for a moment that B, M, and E are collinear. What quadruple of points (besides a subset of $\{A, B, C, D, E\}$) must form a cyclic quadrilateral under this assumption?

(b) Get rid of the assumption and prove that the quadrilateral you found in **(a)** really is cyclic.

(c) Using this cyclic quadrilateral, show that B, M, and E are collinear. One approach is to show that $\angle BMC = \angle BED$ (assuming that A, B, C, D, E are on the circle in that order).

The key step of this problem is the discovery of the cyclic quadrilateral. Often, the key idea in a geometry problem is locating a particular cyclic quadrilateral, set of collinear points, set of concurrent lines, etc., so it's important to develop some strategies for guessing when these features appear.

One such strategy, which we used in part **(a)**, is **working backwards**. The idea is to assume that the desired result is true, and then formulate equivalent but more tractable statements. In this case, M turns out to be the midpoint *if and only if* this mystery quadrilateral is cyclic, so proving that it's cyclic essentially solves the problem.

(Question: Why should you expect the quadrilateral being cyclic to be *equivalent* to the collinearity?)

Example 2.2.4 (2022 AIME II/11)
Let $ABCD$ be a convex quadrilateral with $AB = 2$, $AD = 7$, and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of BC. Find the area of $ABCD$.

Walkthrough. The core of this problem is a bunch of angle chasing, followed by a tiny bit of computation.

Let M be the midpoint of BC and let P be the intersection of lines AB and CD; then, M is the incenter of $\triangle APD$.

- **(a)** Solve the following problem first: Let $\triangle ABC$ be a triangle with incenter I. Points X and Y lie on AB and AC such that I is the midpoint of XY. Describe a ruler-and-compass construction for X and Y given A, B, C, and I.
- **(b)** How does the previous part apply in the original problem? Describe $\triangle PBC$.
- **(c)** Show that $\triangle ABM \sim \triangle MCD \sim \triangle AMD$.
- **(d)** Compute the lengths AM, BM, CM, and DM via the similar triangles.
- **(e)** Extract the final answer of $6\sqrt{5}$, either by Heron's formula or by trigonometry.

This is a problem where the original diagram is hard to construct with a ruler and compass without some additional thinking. In such problems, one common approach is to **try phrasing the problem more naturally**: in this case, rewriting the problem with respect to $\triangle APD$ essentially solved the problem, since it allowed us to wrangle the "angle bisectors concur on the midpoint of BC" condition.

Example 2.2.5 (2020 AIME II/15)
Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16$, $BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$. Find XY.

Walkthrough. There are several viable approaches here, including trigonometry and complex numbers. We will go through a synthetic solution, though.

The most difficult aspect of this problem is that the diagram is very bare: angle chasing will show you a few equal angles, but you won't find anything particularly substantive.

- **(a)** Let M be the foot of the perpendicular from T to BC. What properties does M have? (There are at least three nontrivial ones.)
- **(b)** Show that $TXMY$ is a parallelogram.
- **(c)** The parallelogram law (also called Apollonius' theorem) says that
$$TX^2 + TY^2 + MX^2 + MY^2 = XY^2 + TM^2.$$
In other words, the sum of the squares of a parallelogram's side lengths is equal to the sum of the squares of its diagonals. Use this to compute the final answer.
- **(d)** Optionally, prove that M is the orthocenter of $\triangle AXY$.

Actually, this solution is pretty short: there are not many things to do once you add in M; the challenge is realizing that adding M helps.

2.2.2 Extra Problems

In general, these will be problems that show some common configurations or that I simply think are nice/instructive. A couple of problems are marked with a star because I think they're extra cool. They are roughly in difficulty order.

Problem 2.2.1

Let $\triangle ABC$ have orthocenter H. Lines AH, BH, and CH intersect lines BC, CA, and AB at D, E, and F respectively. Find all six cyclic quadrilaterals with vertices in $\{A, B, C, D, E, F, H\}$, and describe their circumcenters.

(Bonus: prove that the six circumcenters are concyclic, too.)

Problem 2.2.2 (Incenter-Excenter Lemma)

Consider a triangle $\triangle ABC$. The angle bisector of $\angle BAC$ intersects its circumcircle ω again at L. Show that L is the circumcenter of quadrilateral $BICI_A$, where I is the incenter and I_A is the A-excenter of the triangle.

Problem 2.2.3 (Reflecting the orthocenter)

Show that the reflections of the orthocenter of triangle $\triangle ABC$ over BC and the midpoint of BC both lie on the circumcircle of $\triangle ABC$. Moreover, prove that the reflection over the midpoint is the point diametrically opposite from A.

Problem 2.2.4 (2021 AIME II/14)

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let X be the intersection of the line tangent to the circumcircle of $\triangle ABC$ at A and the line perpendicular to GO at G. Let Y be the intersection of lines XG and BC. Given that the measures of $\angle ABC, \angle BCA$, and $\angle XOY$ are in the ratio $13 : 2 : 17$, compute the degree measure of $\angle BAC$.

★ Problem 2.2.5 (USAMO 2021/1)

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

★ Problem 2.2.6 (BrMO 2013/2, aka the first isogonality lemma)

Let ABC be a triangle and let P be a point inside it satisfying $\angle ABP = \angle PCA$. Let Q be the reflection of P across the midpoint of BC. Prove that $\angle BAP = \angle CAQ$.

2.3 Power of a Point

Power of a point is an extremely useful theorem, and I dedicate a section to it here because it's the most frequently used bridge between length information and angle information (except for maybe trigonometry). This should make sense, as it's essentially similar triangles in a neat package.

Here's the statement as a refresher:

Theorem 2.3.1 (Power of a point)**.** Let Γ be a circle and P be any point. Then, across all choices of lines through P intersecting Γ at X and Y (it's possible for $X = Y$), the quantity

$$PX \cdot PY$$

is constant.

It's also often phrased in the following equivalent manner:

Corollary 2.3.2. Let A, B, C, and D be points on a circle. Then, if AB and CD intersect at a point P,

$$PA \cdot PB = PC \cdot PD.$$

As alluded to above, the proof of this follows directly from $\triangle PAC \sim \triangle PDB$.

Also, the converse is true!

Theorem 2.3.3 (Converse of POP)**.** If AB and CD are lines intersecting at P and

$$PA \cdot PB = PC \cdot PD,$$

then A, B, C, and D form a cyclic quadrilateral.

The nice thing about everything above is that we have a lot of freedom: we can choose AB and CD to be any lines passing through P. In particular this quantity is entirely dependent on the choice of P and the features of the circle. This motivates the following definition:

Definition 2.3.4 — Let Γ be a circle with center O and radius r, and let P be a point in the plane. We define the **power** of P with respect to Γ by

$$\mathrm{Pow}_\Gamma(P) = OP^2 - r^2.$$

Exercise. Verify that $|\mathrm{Pow}_\Gamma(P)| = PX \cdot PY$ for any choice of X and Y on Γ such that P, X, and Y are collinear. When is the power positive? Negative? Zero?

While we will discuss this function in more detail later in the year, here's one important result that shows up pretty often.

Theorem 2.3.5 (Radical axis). Let Γ_1 and Γ_2 be two non-concentric circles. Then, the set of points P satisfying

$$\mathrm{Pow}_{\Gamma_1}(P) = \mathrm{Pow}_{\Gamma_2}(P)$$

is a line that is perpendicular to the line joining the centers of Γ_1 and Γ_2. This line is called the **radical axis** of the two circles.

Proof. Let Γ_i have center O_i and radius r_i. Then, we want to find the set of points P for which

$$O_1P^2 + r_1^2 = O_2P^2 - r_2^2 \iff O_1P^2 - O_2P^2 = r_1^2 - r_2^2.$$

We then want to show that the set of points for which $O_1P^2 - O_2P^2$ is equal to the constant $r_1^2 - r_2^2$ is a line.

Claim 2.3.5.1 — Let AB be a segment and let C and D be two points. Then,

$$AC^2 - BC^2 = AD^2 - BD^2$$

if and only if $AB \perp CD$.

Proof. If $AB \perp CD$, then the result follows from Pythagorean theorem.

In the other direction, if the length condition is true, then let H_C and H_D be the feet of C and D onto AB. Then

$$AH_C^2 - BH_C^2 = AC^2 - BC^2 = AD^2 - BD^2 = AH_D^2 - BH_D^2$$

implies that $H_C = H_D$, so $CD \perp AB$. $\square$

This says that the set of points X for which $O_1X^2 - O_2X^2$ is constant is a line perpendicular to O_1O_2, which is what we wanted to prove. $\square$

An alternate (and simpler) proof can also be extracted using coordinates, but the synthetic approach is good to know, since the intermediary claim is pretty useful itself.

Remark 2.3.6. You may notice the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\mathrm{Pow}_{\omega_1}(X) - \mathrm{Pow}_{\omega_2}(X)$ arising naturally from the proof — the radical axis is the set of points which make the function zero. Interestingly, f is **linear**; that is,

$$f(kA + (1-k)B) = kf(A) + (1-k)f(B)$$

where multiplication (between real numbers and points) and addition (between two points) are done componentwise with the coordinates.

The upshot of this is that knowing the value of f at A and B lets you compute $f(X)$ for all X on line AB, and, if you know f at A, B, and C, then you can compute the value of f at any point in the plane.

2.3.1 Walkthroughs

Example 2.3.7 (JMO 2012/1)

Given a triangle ABC, let P and Q be on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Walkthrough. This problem is one of my favorites because its solution uses a really unique idea.

(a) Assume for the sake of contradiction that (PRS) and (QRS) are distinct circles. Then, show that AB is tangent to (PRS) at P and AC is tangent to (QRS) at Q.

(b) Convince yourself that RS is the radical axis of the circles (in particular, that the two circles are not concentric).

(c) Find a point not on line RS that has the same power with respect to (PRS) and (QRS), and deduce a contradiction.

I think this solution seems really unnatural at first glance, but it presents itself after noticing that the circles above should be tangent to AB and AC. (Theorem 2.2 paying back in spades!)

Example 2.3.8 (Euler's theorem)

Let $\triangle ABC$ have circumcenter O and incenter I, as well as circumradius R and inradius r. Then,

$$OI^2 = R^2 - 2Rr.$$

Walkthrough. This problem lends itself very handily to power of a point because of the given equation.

(a) Let Γ be the circumcircle of $\triangle ABC$. Show that $\mathrm{Pow}_\Gamma(I) = -2Rr$. This means we want to show that $XI \cdot YI = 2Rr$ for some choice of $X, Y \in \Gamma$ with X, I, Y collinear.

(b) Draw the angle bisector from A, and let it intersect Γ at D. Then, we want $AI \cdot ID = 2Rr$. If you haven't already done Problem 1.2, show that $DI = DB$.

(c) Imagine that the equation above was the result of setting up a proportion between two similar triangles. Working backwards, what might the original proportion be? Keep in mind which segments would most readily fit in a triangle together.

(d) If you did part **(c)** correctly, then you should be able to draw in one more point to create a pair of similar triangles with the desired proportion.

This is a pretty classical example of power of a point, and it's sometimes used in geometric inequalities, since you can show that

$$R(R - 2r) = OI^2 \geq 0,$$

so $R \geq 2r$ (when does equality happen?). Using some area formulas, you can even extract the inequality

$$4R^2 \geq \frac{abc}{a+b+c},$$

which has no trace of r, surprisingly.

2.3.2 Extra Problems

Problem 2.3.1

Let Γ_1 and Γ_2 be two intersecting circles. Let a common tangent to Γ_1 and Γ_2 touch Γ_1 at A and Γ_2 at B. Show that the common chord of Γ_1 and Γ_2, when extended, bisects segment AB.

Problem 2.3.2 (Radical center)

Let ω_1, ω_2, and ω_3 be three circles whose centers are not collinear. Show that the radical axes of each pair of circles concur at some point P, which is called the **radical center** of the three centers.

Problem 2.3.3

Describe a ruler-and-compass construction for the radical axis of *any* two (possibly disjoint) circles.

Problem 2.3.4 (2019 AIME II/11)

Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Compute AK.

★ **Problem 2.3.5** (Shortlist 2022 G2)

In the acute-angled triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X, and Y are concyclic.

Problem 2.3.6 (IMO 2000/1)

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

★ **Problem 2.3.7** (Japan 2014/4)

Let Γ be the circumcircle of triangle ABC, and let ℓ be the tangent line of Γ passing through A. Let D, E be points on sides AB and AC such that $BD : DA = AE : EC$. Line DE meets Γ at points F and G. The line parallel to AC through D meets ℓ at H, the line parallel to AB through E meets ℓ at I. Prove that there exists a circle passing through four points F, G, H, I, and tangent to line BC.

3 Analytic Techniques

3.1 Theorems

The following are things that we saw last year:

- (Extended) law of sines: In triangle $\triangle ABC$,
$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
- Law of cosines: $a^2 + b^2 - 2ab\cos C = c^2$.
- Ptolemy's inequality:
$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD,$$
with equality when $ABCD$ is cyclic, or when A, B, C, D are collinear, with the order of the points on the line being one of (A, B, C, D), (B, C, D, A), (C, D, A, B), (D, A, B, C). (Ptolemy's theorem is just the part where $ABCD$ is cyclic.)
- Triangle inequality: $AB + BC \geq AC$, with equality if and only if A, B, C are collinear in that order.
- Stewart's theorem: $dad + man = bmb + cnc$.
- Area formulas: $K = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)} = rs = ab\sin C$

You should also already know the trig angle addition formulas from Algebra II or Precalc:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

If you ever forget these, you can equate the real and imaginary parts of

$$e^{i(x+y)} = e^{ix} \cdot e^{iy}$$

by using $e^{i\theta} = \cos\theta + i\sin\theta$.

In addition, there are the product-to-sum and sum-to-product formulas below, but I can never remember them. You should at least know how to derive them; I rederive them every single time.

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$
$$\sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$
$$\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$
$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

The latter two equations are, of course, just the second and third equations in the other direction (i.e. $(x, y) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$).

If you're not so familiar with any of these formulas, then go through and verify them.

3.2 Walkthroughs

Example 3.2.1 (2015 AIME II/15)

Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ. Given that the areas of $\triangle DBA$ and $\triangle ACE$ are equal, determine their common value.

Walkthrough. The key observation is that these circles are dilations of one another about A; we can leverage this fact to get a lot of length relations.

(a) Let any line through A intersect $\mathcal{P}$ at X and $\mathcal{Q}$ at Y. Show that $\frac{AX}{AY} = \frac{1}{4}$.

(b) Show that $\angle BAC = 90^\circ$, and then compute AB and AC. One approach is to extend AB to hit $\mathcal{Q}$ at B' and find similar triangles.

(c) Let $\angle BAD = \alpha$. What are $\sin\alpha$ and $\cos\alpha$? (Hint: What do the equal areas tell you?)

(d) What are BD and AD?

(e) If you did the previous parts correctly, you now know AB, BD, and AD; pick an area formula and finish.

Example 3.2.2 (Fermat point)

Let $\triangle ABC$ be a triangle whose angles are all less than 120° and let P be a point in the plane. Show that $PA + PB + PC$ achieves its minimum value if and only if $\angle APB = \angle BPC = \angle CPA = 120^\circ$.

Walkthrough. The trick with this problem is to draw X, the point on the other side of BC from A such that $\triangle BXC$ is equilateral.

(a) Let F be the point such that $\angle AFB = \angle BFC = \angle CFA$. Show that $FBXC$ is cyclic and A, F, X are collinear. Which two inequalities does F therefore optimize?

(b) Use the inequalities from the previous part to show that

$$PA + PB + PC \geq AX,$$

with equality if and only if $P = F$.

(c) Where was the stipulation that $\angle A$, $\angle B$, and $\angle C$ are less than 120° used?

(d) Some extra things: If Y and Z are constructed similarly to X so that $\triangle CYA$ and $\triangle AZB$ are equilateral, show that $AX = BY = CZ$ and that those three lines are concurrent (at F).

This is another instance where **trying to understand the weird condition** (namely, the equality case of $\angle AFB = \angle BFC = \angle CFA$) will get you pretty close to a solution; I found it by trying to construct F.

Example 3.2.3 (2023 AIME I/12)

Let $\triangle ABC$ be an equilateral triangle with side length 55. Points D, E, and F lie on sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, with $BD = 7$, $CE = 30$, and $AF = 40$. A unique point P inside $\triangle ABC$ has the property that

$$\measuredangle AEP = \measuredangle BFP = \measuredangle CDP.$$

Find $\tan(\measuredangle AEP)$.

Walkthrough. I didn't solve this in contest, unfortunately, but here's a synthetic solution I found afterwards using the Fermat point.

(a) Show that the angle condition implies that $AEPF$, $BFPD$, and $CDPE$ are cyclic. What does the equilateral condition say about $\angle DPE$, $\angle EPF$, and $\angle FPD$?

(b) Extend DP to meet (FPE) at X, so that it suffices to compute $\tan(\angle XDC)$. What do you know about $\triangle EFX$? What about quadrilateral $AXDC$?

(c) Compute the length of AX using Ptolemy's theorem.

(d) Redraw quadrilateral $AXDC$ separately, and eradicate the problem.

You can also approach this with complex numbers, coordinates, trigonometry, or other synthetic approaches (one I saw on AoPS was to drop the altitudes from P); I encourage looking for an alternate solution.

Finally, here's a number theory problem from IMO 2001 with a strange geometric solution.

Example 3.2.4 (IMO 2001/6)

Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Walkthrough. This problem is... weird. Let's first reveal where the geometry really is:

(a) Expand the equation (yes, really). You should end up with

$$a^2 - ac + c^2 = b^2 + bd + d^2.$$

What do the left- and right-hand sides of this equation look like?

In light of this, we can construct a quadrilateral $WXYZ$ with sides $WX = a$, $XY = c$, $YZ = b$, $ZX = d$, and diagonal WY having length

$$\sqrt{a^2 - ac + c^2} = \sqrt{b^2 + bd + d^2},$$

so that, by law of cosines, $\angle WXY = 60^\circ$ and $\angle WZY = 120^\circ$. This means $WXYZ$ is cyclic!

On the other hand, the diagonals of a cyclic quadrilateral can *always* be expressed in terms of their sides.

(b) Show that there are exactly three quadrilaterals with side lengths a, b, c, d in some order. Moreover, show that, among these quadrilaterals, there are three possible lengths t, u, v for the diagonals; that is, one quadrilateral has diagonals (t, u), one has diagonals (u, v), and one has diagonals (v, t).

(c) Apply Ptolemy's theorem in all three quadrilaterals to get the system of equations

$$\begin{aligned} ab + cd &= tu \\ ac + bd &= uv \\ ad + bc &= vt \end{aligned}$$

where $t = WY$ and $u = XZ$. Then, compute t^2.

(d) Finally, we can use the inequality condition: show (ex. by rearrangement) that

$$ab + cd > ac + bd > ad + bc,$$

so, if $ab + cd$ is prime, then t^2 can never be an integer. However, from earlier, $WY^2 = b^2 + bd + d^2$ is an integer — contradiction!

The takeaway from here is primarily that quadratic expressions in two variables can sometimes be thought of as applications of the law of cosines. The additional geometric structure then lets you, well, do geometry to get nontrivial, non-geometric information (such as the fact that $(ac + bd) \mid (ab + cd)(ad + bc)$ in this case).

3.3 Extra Problems

Problem 3.3.1 (NIMO, Evan Chen)

Let $AXYZB$ be a convem pentagon inscribed in a semicircle with diameter AB. Suppose that $AZ - AX = 6$, $BX - BZ = 9$, and $BY = 5$. Find the perimeter of quadrilateral $OXYZ$, where O is the midpoint of AB.

Problem 3.3.2 (2014 AIME I/15)

In $\triangle ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, compute DE.

Problem 3.3.3 (2012 AIME I/13)

Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. Compute the largest possible area of the triangle.

★ **Problem 3.3.4** (Pompeiu's theorem)

Let P be a point in the plane of equilateral triangle $\triangle ABC$. Show that PA, PB, and PC form the sides of a (maybe degenerate) triangle. When is the triangle degenerate?

Problem 3.3.5 (AMC 12A 2021/24)

Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, find the area of $\triangle PQR$.

Problem 3.3.6 (ARML 2023 T10)

Parallelogram $ABCD$ is rotated about A in the plane, resulting in $AB'C'D'$, with D on $\overline{AB'}$. Suppose that $[B'CD] = [ABD'] = [BCC']$. Compute $\tan \angle ABD$.

★ **Problem 3.3.7** (USAMO 1998/6)

Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2 \dots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle, where indices are taken modulo n.