Junior Math Team Inversion Notes

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Contents

> **Note —** For typing convenience, the notation Z^* will be used in place of $\hat{Z}$ to denote the image of object Z upon reflecting about some circle. Also, GeoGebra diagrams use Z' instead of Z^*, since I don't know how to write asterisks in GeoGebra (they get interpreted as multiplication).
>
> Additionally, if not specified, reflections are done about an arbitrary circle ω with center O and radius r.

1 Introduction

The geometric transformations that we've encountered thus far have all had many similar properties: they preserve angles, they preserve lengths (either keeping them the same or scaling all lengths by the same amount), and are generally very well-behaved.

Here, however, we will develop a new kind of transformation, which can be thought of as "reflecting about a circle" (in particular, vanilla reflection can be thought of as as special case of this transformation).

> **Definition 1.1** (Reflecting over a circle) — Let ω be a circle with center O and radius r, and let P be a point in the plane. Then, the image of P upon reflection over ω is the point P^* on ray $\overrightarrow{OP}$ such that
> $$OP \cdot OP^* = r^2.$$
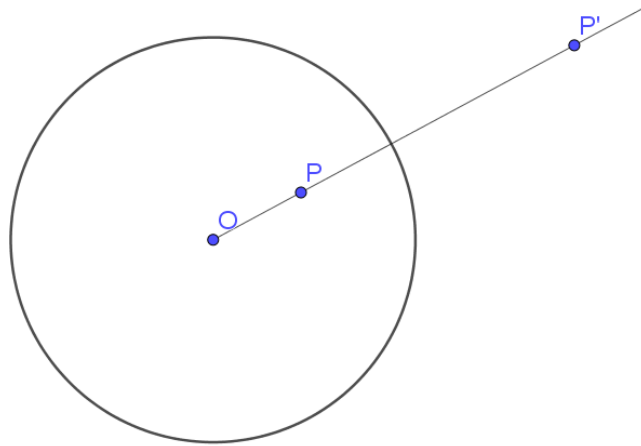
Just from this definition, we can start thinking about a few properties. (Compare these properties to those of reflection over a line!)

- If P is inside the circle, then $OP < r$, so $OP^* > r$, meaning P^* is outside the circle (and vice versa).

- The only points that don't move are the ones on ω, since $OP = r \iff OP^* = r$.

- P and P^* swap with each other, which means that reflecting about the same circle twice is equivalent to doing nothing. As a fun consequence, the converse of pretty much any statement you make will be true (this is a heuristic judgment, not a formal theorem).

At this point, you may have come across the thought, "What happens to O?" since $OO = 0$. To resolve this, we adjoin the "point at infinity" (denoted ∞) to the real plane, which we declare to lie on every single line and be arbitrarily far away from all the other points in the plane; then, we can say that O and ∞ swap.

2 Preliminary properties

The first thing that we should figure out is, "How do I construct P^* given P and ω?"
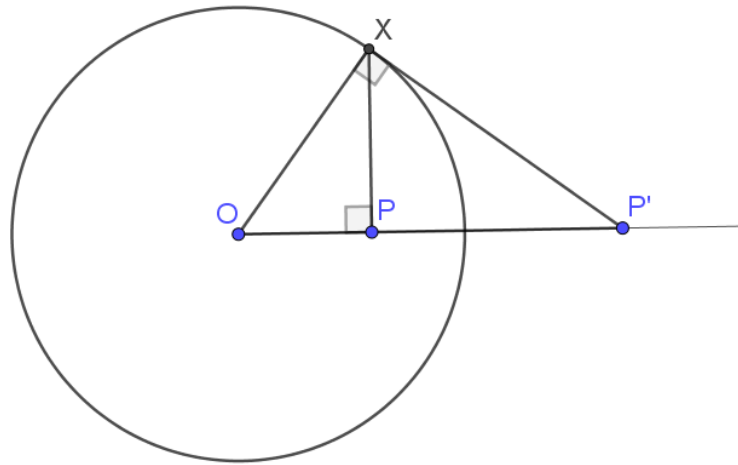
> **Problem 2.1**
>
> Construct P^*.

The only pieces of information we have are that P^* lies on $\overrightarrow{OP}$ and $OP \cdot OP^* = r^2$. The second condition can be rewritten as

$$\frac{OP}{r} = \frac{r}{OP^*},$$

so, really, this is signaling that we should be looking for similar triangles. (In fact, any radius not collinear with OP should generate a pair of similar triangles — prove this.)

If we draw the right radius, this becomes simple:

More precisely, X is one of the intersections of ω and the perpendicular to line OP at P, and P^* is the intersection of $\overrightarrow{OP}$ with the perpendicular to OX at X. Then, $\triangle OPX \sim \triangle OXP^*$ gives the desired ratio relation.

> **Problem 2.2**
>
> If A and B are not collinear with O and do not lie on ω, show that quadrilateral ABB^*A^* is cyclic.

This is immediate by similar triangles — feel free to work out the details. The key idea is $\triangle OAB \sim \triangle OB^*A^*$. (Often, the similar triangles more relevant than the fact that the points are concyclic.)
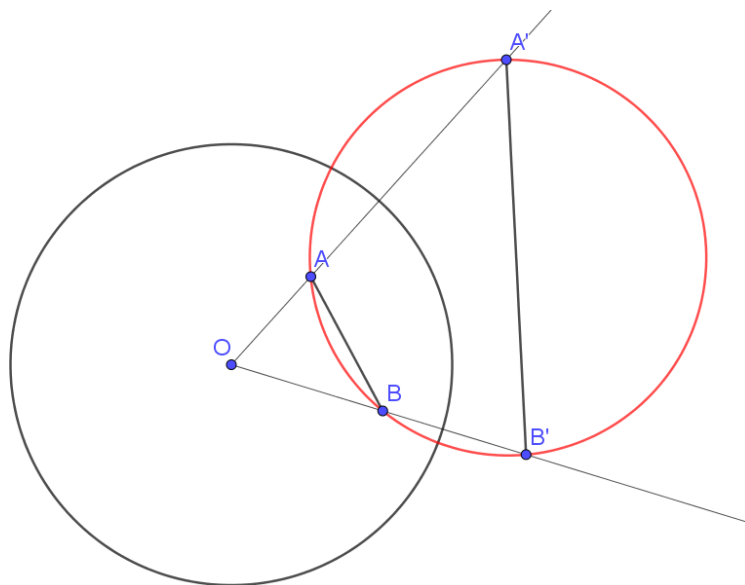

Figure 1: $\triangle OAB \sim \triangle OA^*B^*$.

3 Where do circles and lines go?

In the previous section, we showed that, if A, B, and C are concyclic with O, then A^*, B^*, and C^* must be collinear, which means **that the circumcircle of $ABCO$ actually gets sent to a line**. Conversely, **lines not passing through O map to circles through O**.

However, there are still some cases left: What if a line does go through O? What about circles that don't pass through O?

> **Theorem 3.1** (Images of circles and lines). Here is the full description of where circles and lines get sent under the transformation:
>
> 1. The image of a line passing through O is the same line.
>
> 2. The image of a line not passing through O is a circle passing through O, and vice versa.
>
> 3. The image of a circle not passing through O is another circle not passing through O.

Proof. We show each part separately:

(1) Let ℓ be a line that contains O. Obviously, each point on line ℓ gets sent to another point on ℓ. On the other hand, there are no "holes" in the image of ℓ because the transformation swaps pairs of points A, A^* satisfying $OA \cdot OA^* = r^2$ (it might help to follow along with a discrete analog, say 10 points in a row). This means that the entire line is "preserved," despite the points on the line getting shuffled around.

(2) Here's a second proof of the fact that circles through O get sent to lines (or rather, its converse). Pick some line ℓ not passing through O, and let P be the foot of the altitude from O to ℓ.
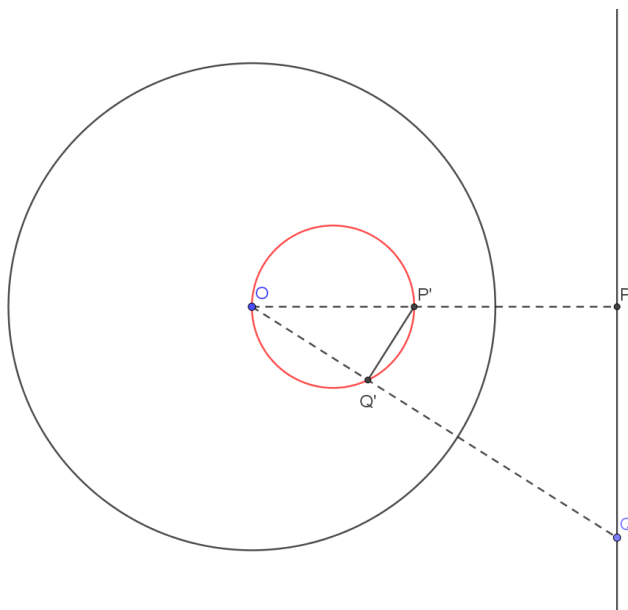

Figure 2: Circles $\leftrightarrow$ lines!

Then, P gets sent to some P^*. Now, arbitrarily choose a point Q on ℓ, and consider its image Q^*. We know that

$$OP \cdot OP^* = OQ \cdot OQ^* = r^2 \implies \frac{OQ}{OP} = \frac{OP^*}{OQ^*},$$

so $\triangle OPQ \sim \triangle OQ^*P^*$. In turn, $\angle OQ^*P^* = 90°$, regardless of Q: that is, Q^* always lies on the circle with diameter OP^*. Varying Q across all points in ℓ in fact shows that every point on the circle is the image of some point on ℓ (with the pre-image of O being ∞).

(3) The proof of this is extremely neat, in my opinion.

Let $ABDC$ be a cyclic quadrilateral (sorry, I'm too lazy to change the order of the points in my GeoGebra diagram) with circumcircle Γ, and let T be a tangent from O to Γ.
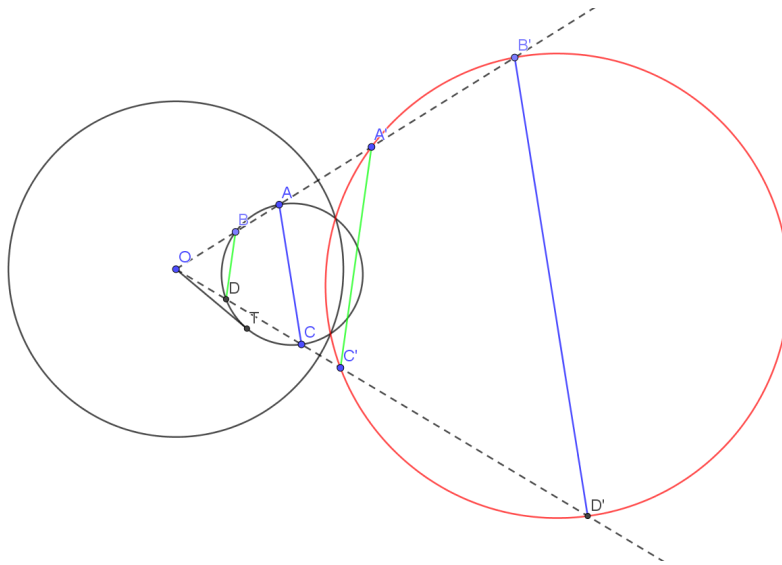
Figure 3: What do you know about segment OT^*?

Power of a point says
$$OA \cdot OB = OC \cdot OD = OT^2,$$
which tells us that
$$OA^* = \frac{r^2}{OA} = \frac{r^2 \cdot OB}{OA \cdot OB} = \frac{r^2 \cdot OB}{OT^2}$$
and the symmetric expressions. In particular, consider $\triangle OBD$. We have

$$OA^* = \frac{r^2}{OT^2} \cdot OB, \quad OC^* = \frac{r^2}{OT^2} \cdot OD,$$

meaning $\triangle OA^*C^*$ is a dilation of $\triangle OBD$ with scale factor $\frac{r^2}{OT^2}$. Similarly, $\triangle OB^*D^*$ is a dilation of $\triangle OAC$ *with the same scale factor*, so we get the incredibly strong result that $A^*B^*D^*C^*$ is a dilation of $BACD$, and thus their circumcircles must map to one another. (Keen-eyed geometers may recognize the key idea of this proof as a variant on Reim's theorem.)

At this point, we have exhausted all the cases, so we are done! $\qquad \square$

**Remark 3.2** (Small detail)**.** When we say "Shape $\mathcal{S}$ gets mapped to shape $\mathcal{T}$ by transformation f," we mean that $f\colon \mathcal{S} \to \mathcal{T}$ is bijective; in particular, f need not send points to any specific location beyond "from $\mathcal{S}$ to $\mathcal{T}$." This becomes pretty important in the study of projective transformations, where things can be very hard to visualize due the generality of the transformations, but it's relevant here, too, as it allows us to formalize the notion of "line ℓ gets sent to itself," for example.

Remark 3.3 (Note for experts)**.** In the second part (where circles get sent to lines), it turns out that ℓ is the radical axis of ω and ℓ^* — thanks to Andrew for pointing this out. Additionally, ω, Γ, and Γ^* are either coaxial or concentric in the third part.

It's worth sketching a few images to get a sense of what they look like.

Problem 8.8. In Figure 8.2C, sketch the inverse of the five solid clines (two lines and three circles) about the dotted circle ω. **Hint:** 279
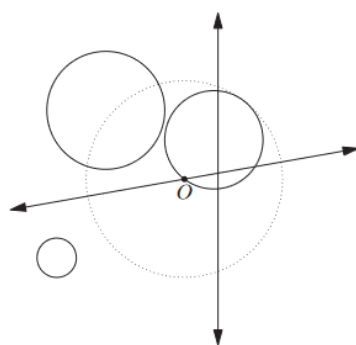
Figure 8.2C. Practice inverting.

Figure 4: From *Euclidean Geometry in Mathematical Olympiads.*

The upshot of all of this is that circles and lines are the same "type of object," in some sense. In particular, this weird reflection allows you to swap concyclicity for collinearity, lines for circles, and circles for lines.

> *It is almost like you are given a choice—which of these two problems looks easier, the inverted one or the original one? Which would you like to solve?*
>
> Evan Chen, *Euclidean Geometry in Mathematical Olympiads*

4 Orthogonal circles

> **Problem 4.1**
>
> Given a circle Γ and a point O, construct ω such that $\Gamma^* = \Gamma$.

As we mentioned in class, an important property of the reflection is the fact that points on ω stay where they are — they are fixed points. This fact is incredibly useful when drawing out the images of shapes (e.g. the exercise above). You might gather, then, that having an entire fixed circle would be extremely nice, which is what makes this a very useful question to answer.

First, let's gather some observations.

- Circle Γ cannot be entirely inside or entirely outside ω, as reflection swaps inside and outside.

- If P and Q are points on Γ that are collinear with O, then P and Q must be each other's image. In particular, this means $Q = P^*$ and so

$$OP \cdot OQ = OP \cdot OP^* = r^2.$$

- By power of a point, $OP \cdot OQ = OT^2 = r^2$ where OT is a tangent to Γ. This means the tangency points from O to Γ must lie on ω.

Actually, we are now done: there is only one circle centered at O passing through the tangency points from O to Γ; so there's our construction!
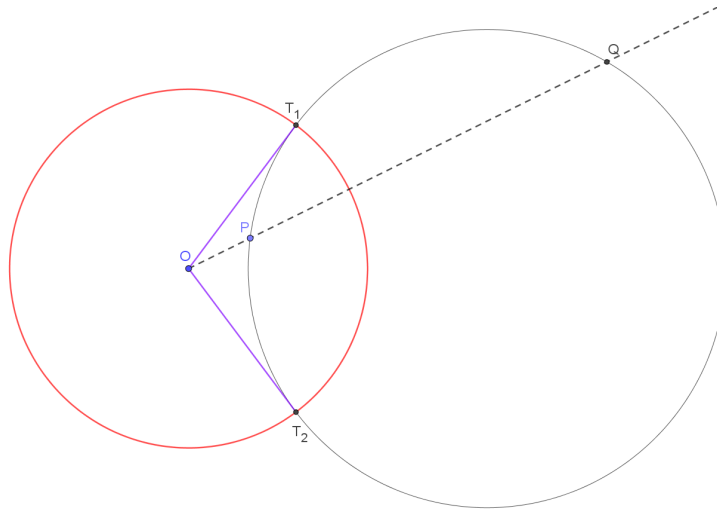
> **Definition 4.2** — Two circles ω_1 and ω_2 are **orthogonal** if the image of ω_1 upon reflection over ω_2 is ω_1 (and vice versa).

> **Warning 4.3** — The circle being preserved does not mean that special points are the same in the image. For example, the center of Γ swaps with the midpoint of T_1T_2.

Exercise 4.4. Suppose two orthogonal circles intersect at T. Show that the tangents to the circles at T are perpendicular.

Exercise 4.5 (Surprisingly useful!)**.** Let two orthogonal circles ω_1 and ω_2 intersect at T, and let Γ be a circle centered at T. Show that the images of ω_1 and ω_2 upon reflection over Γ are perpendicular.

5 Intersections

Of course, normal geometry problems often have more than two lines or circles.

> **Theorem 5.1.** Intersections between circles and lines are preserved by reflection.

The main point is that if, say, a circle and a line are tangent, then they will be tangent after reflection, too — even if the line becomes a circle and the circle becomes a line.

These behave themselves in the nicest way possible: the image of the intersection is the intersection of the images (so, for a concrete example, two circles tangent at O will map to two lines that intersect at $O^* = \infty$, meaning the lines are parallel).

6 Example problems

> **Problem 6.1** (Ptolemy's inequality)
>
> Given four points A, B, C, and D,
>
> $$AB \cdot CD + BC \cdot AD \geq AC \cdot BD,$$
>
> with equality if and only if $ABCD$ is convex and cyclic.

Proof. Let ω be a circle centered at A with radius r, and reflect B, C, and D about ω. From Problem 2.2, we know that $\triangle ABC \sim \triangle AC^*B^*$, $\triangle ACD \sim \triangle AD^*C^*$, and $\triangle ADB \sim \triangle AB^*D^*$. In the former triangle,

$$\frac{BC}{B^*C^*} = \frac{AB}{AC^*} \iff B^*C^* = \frac{AC^*}{AB} \cdot BC = \frac{AC \cdot AC^*}{AB \cdot AC} \cdot BC = \frac{r^2}{AB \cdot AC} \cdot BC.$$

Analogously, we get

$$C^*D^* = \frac{r^2}{AC \cdot AD} \cdot CD, \quad D^*B^* = \frac{r^2}{AD \cdot AB} \cdot DB.$$

Then, the triangle inequality says

$$B^*C^* + C^*D^* \geq B^*D^*$$

$$\iff \frac{r^2}{AB \cdot AC} \cdot BC + \frac{r^2}{AC \cdot AD} \cdot CD \geq \frac{r^2}{AD \cdot AB} \cdot DB$$

$$\iff \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot CD} \geq \frac{DB}{AD \cdot AB}$$

$$\iff BC \cdot AD + CD \cdot AB \geq BD \cdot AC,$$

with equality if and only if B^*, C^*, and D^* are collinear in that order.

It then remains to show that B^*, C^*, and D^* are collinear if and only if A, B, C, and D are concyclic in that order, which is as easy as observing

$$180° = \angle AC^*B^* + \angle AC^*D^* = \angle ABC + \angle ADC$$

meaning the equality case is indeed only when $ABCD$ is convex and cyclic. $\qquad\square$

If anything, two things should be taken away from this proof: one, that circles and lines swap under certain circumstances (elaborated on in the next section), and two, that distances are often easily computable by virtue of similar triangles.

Problem 6.2

Let ω_1, ω_2, ω_3, and ω_4 with consecutive pairs tangent at A, B, C, and D (so ω_1 and ω_2 are tangent at A, for example). Show that A, B, C, D are either collinear or concyclic.
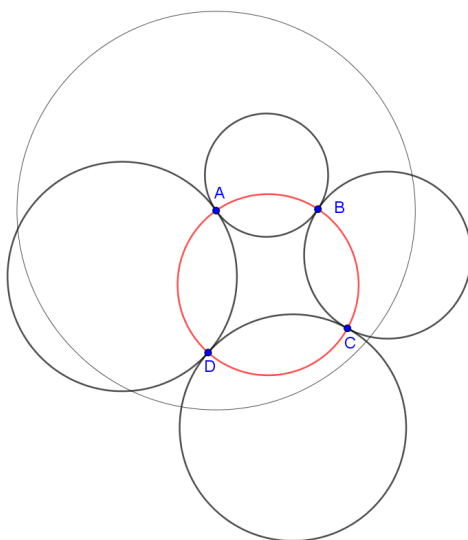
Let's reflect the picture over some circle centered at A (the radius can be any positive real number your heart desires). Try to follow along with a sketch of your own before looking at the diagram on the next page!

- First, ω_1 passes through A, so it gets sent to some line ℓ_1.

- Then, ω_4 also passes through A and is tangent to ω_1, so it gets sent to some line parallel to ℓ_1, say ℓ_4.

- Circles ω_2 and ω_3 become circles ω_2^* and ω_3^*, which are tangent to both ℓ_1 and ℓ_4 at B^* and D^*, as well as tangent to each other at C^*.

Figure 5: The reflected picture, with most of the original diagram hidden away.

Our goal now is to show that B^*, C^*, and D^* are collinear. Let O_2 and O_3 be the centers of ω_2^* and ω_3^*.

Observe that O_2B^* and O_3D^* are parallel since they are perpendicular to the parallel lines ℓ_1 and ℓ_4. This tells us that $\triangle B^*O_2C^*$ and $\triangle D^*O_3C^*$ are similar, so

$$\angle D^*C^*O_3 = \angle B^*C^*O_2 = 180^\circ - \angle B^*C^*O_3$$

which implies the collinearity.

Exercise 6.3 (Inspired by bad diagram). Determine when B^*D^* is perpendicular to ℓ_1 and ℓ_4.

Problem 6.4 (Shortlist 1995 G3)

In $\triangle ABC$, the incircle is tangent to BC, CA, and AB at D, E, and F respectively. Point X is chosen inside $\triangle ABC$ such that the incircle of $\triangle BCX$ is tangent to BC, CX, and BX at D, Y, and Z. Show that E, F, Y, and Z are concyclic.

This is an example of a problem that gets obliterated once you choose the right circle to reflect about and understand where everything goes. Here's the diagram, for reference; try to draw the reflected objects on your own, and make a conjecture about $E^*F^*Z^*Y^*$.
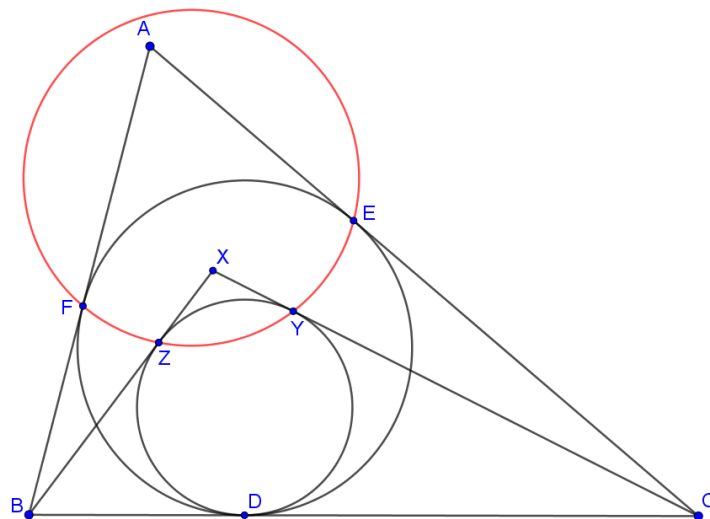
The main point is to center some circle ω at D and reflect everything across ω.

Solution 1 (9th grade geometry). This first solution uses nothing but angle chasing, actually.
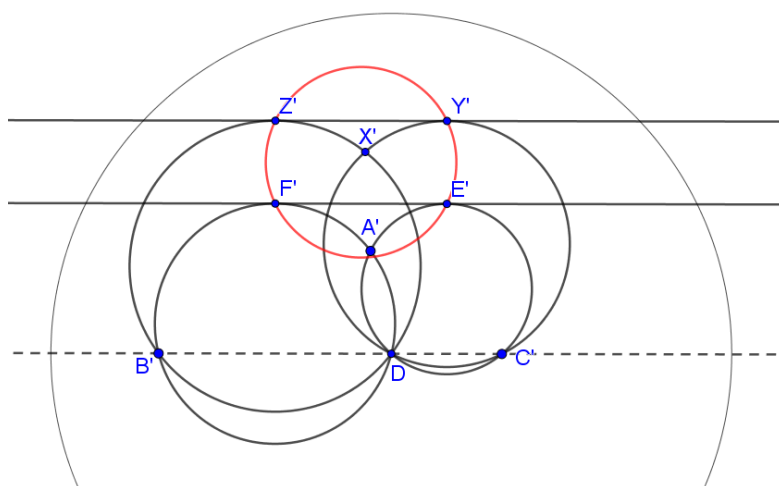
For the sake of comprehensiveness, the objects in the above picture are related in the following ways:

1. Lines B^*C^*, E^*F^*, and Y^*Z^* are parallel.

2. The circumcircles of $\triangle B^*DF^*$ and C^*DE^* are tangent to line E^*F^*.

3. Similarly, the circumcircles of $\triangle B^*DZ^*$ and C^*DY^* are tangent to line Y^*Z^*.

The main mechanism for the solution is then to recognize the following:

> **Claim 6.5** — Triangles $\triangle B^*DF^*$, C^*DE^*, B^*DZ^*, and C^*DY^* are all isosceles, with their bases on line B^*C^*.

Proof. This is just angle chasing. We prove the claim for $\triangle B^*DF^*$; the other three triangles follow the exact same logic. In particular,

$$\angle DF^*E^* = \angle F^*DB^*$$

by parallel lines, and

$$\angle DF^*E^* = \angle F^*B^*D$$

because of the tangency, directly giving the isosceles triangle. $\square$

To finish, F^* and Z^* lie on the perpendicular bisector of B^*D, meaning $\angle Z^*F^*E^* = \angle F^*Z^*Y^* = 90°$. Similarly, $\angle Y^*E^*F^* = \angle E^*Y^*Z^* = 90°$, too, so $E^*F^*Z^*Y^*$ is a rectangle, and thus cyclic!

Solution 2 (orthogonal circles). The main mechanism for this solution is Exercise 4.5.
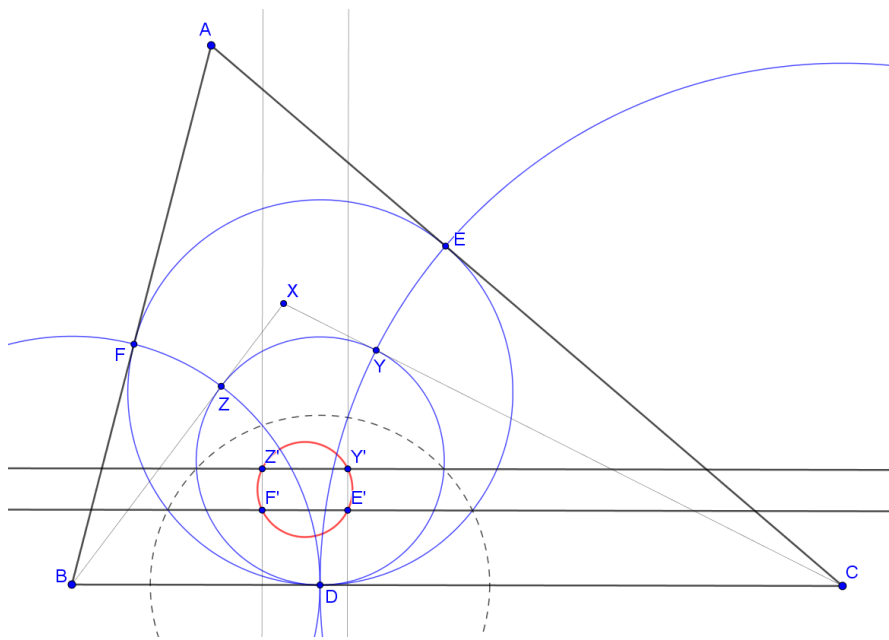
Figure 6: Secret orthogonal circles.

In particular, once you notice that $E^*F^*Z^*Y^*$ seems to be a rectangle, you should realize that there must be a pair of orthogonal circles in the pre-image for each right angle. Moreover, the relevant circles, which are marked blue in the diagram, are precisely the pre-images of the sides of the rectangle.

This means that (DFZ) (meaning "the circumcircle of $\triangle DFZ$") and (DEF) should be orthogonal, for example. But this is obvious by construction, as (DFZ) is the circle centered at B with radius BD (prove it, if it isn't apparent).

Going backwards, the orthogonal circles immediately yield the right angles and therefore the conjectured rectangle. (I did call Exercise 4.5 "surprisingly useful," right?)

7 Overlapping diagrams

So far, our general strategy has been "Let's reflect the entire diagram, and, in the process, generate an entirely new problem that we can solve separately." As a consequence, all of the problems in the previous section only cared about the center of the circle of reflection. This, however, loses a key piece of information about the transformation, namely the radius associated with it. Moreover, it completely ignores the original structure of the problem.

This section is thus dedicated to using this information via choosing your circles to make pre-images and images interact cleanly.

> **Problem 7.1** (Shoemaker's knife)
>
> Let Γ be a semicircle with diameter AB and let C be a point on segment AB that lies closer to B than A. Define Ω to be the semicircle contained in Γ with diameter AC. Then, define the sequence of circles (ω_k) such that ω_0 has diameter BC and, for $k > 0$, ω_k is tangent to ω_{k-1}, Ω, and Γ.
>
> Prove that the distance from the center of ω_n to AB is n times the diameter of ω_n.

There are so many circles here that it would be a crime not to invert.

> **Exercise 7.2.** Guess the center of the reflection, and reflect the diagram about an arbitrary circle centered at this point. What happens?

Figure 7: This configuration is called a Pappus chain embedded in a shoemaker's knife.

Reflecting about a circle centered at A sends Γ and Ω to some parallel lines, and then, since each of the ω_i is tangent to both circles and the previous circle, they transform into a chain of congruent circles, with ω_0 becoming a semicircle centered on line AB.
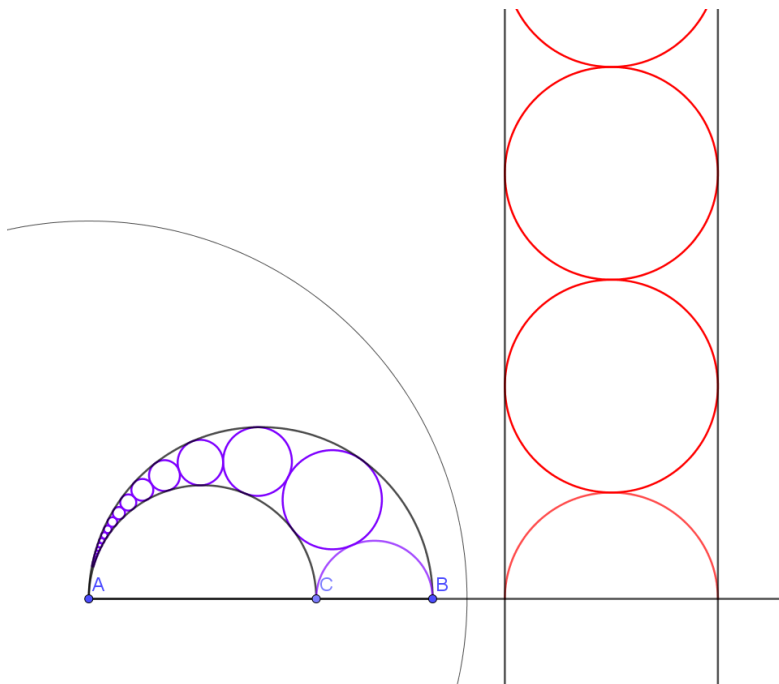
Figure 8: All the circles in the chain become congruent!

However, the issue is that makes the information about the radius of ω_k a lot less tractable, since any relevant calculations would feature the arbitrary radius of inversion heavily. While this can eventually lead to a solution, we can do better.

The key trick is to preserve the radius of ω_k. We have a natural mechanism for preserving radii under inversion using orthogonal circles, in fact. Thus, the circle about which we are reflecting is the circle centered at A orthogonal to ω_k.
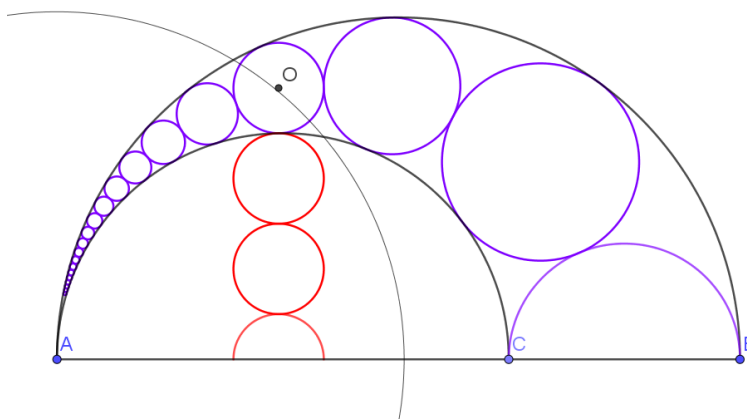

Figure 9: We have forced ω_3 to map to itself here.

This in fact gives us the result immediately — just look at the diagram!

Often, a picture is centered about a triangle ABC, in which case we don't want to offhandedly discard information about the vertices. Here, we can use a composition of an inversion and a line reflection cleverly to swap B and C.

Lemma 7.3 (Force-Overlaid Inversion)**.** Let $\triangle ABC$ be a triangle. Then, inverting about A with radius $\sqrt{AB \cdot AC}$ followed by reflecting over the bisector of $\angle BAC$ swaps B and C.

Problem 7.4 (Russian Olympiad 2009)

In triangle ABC with circumcircle Ω, the internal angle bisector of $\angle A$ meets BC at D and Ω again at E. The circle with diameter DE meets Ω again at F. Prove that AF is a symmedian of triangle ABC.

First, we define a symmedian:

Definition 7.5 — The A-symmedian of $\triangle ABC$ is the reflection of the A-median over the angle bisector of $\triangle ABC$.

This means we should of course introduce the midpoint M of BC. Moreover, it turns out that M lies on the circle Γ with diameter DE (why?), so the force-overlaid inversion should actually swap M and F.
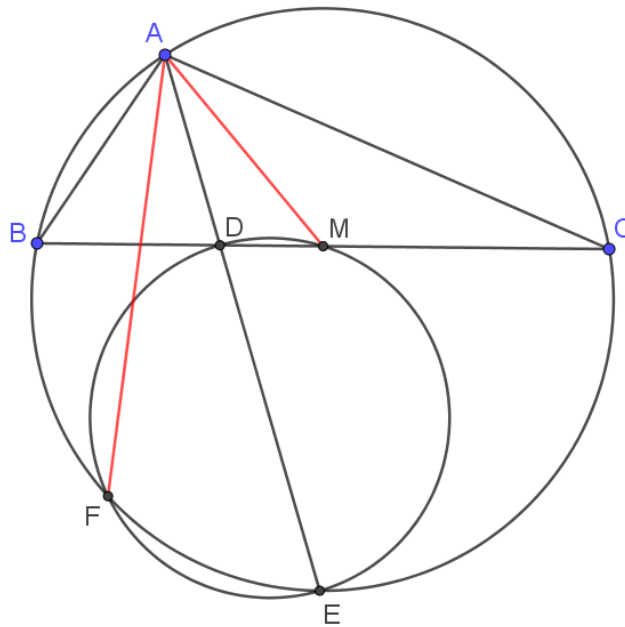

Figure 10: Lines AM and AF should be symmetric about AE.

From $\triangle ABD \sim \triangle AEC$, we get

$$\frac{AB}{AD} = \frac{AE}{AC} \iff AB \cdot AC = AD \cdot AE,$$

so D and E get swapped, meaning the circle with diameter DE is fixed. Also, line BC gets sent to the circumcircle of AC^*B^*, which is just the circumcircle of ABC since $(B, C) = (C^*, B^*)$.

Putting all this together means that the intersection of BC and the circle with diameter DE gets sent to the second intersection of Ω and the circle with diameter DE — that is, M gets sent to F as desired.

Exercise 7.6. Obliterate AIME II 2012/15.

8 Practice problems

Problem 8.1 (By me!)

Circles ω_1, ω_2, ω_3, and ω_4 all pass through point P, with ω_1 and ω_3 being tangent as well as ω_2 and ω_4. Moreover, the tangents to ω_1 and ω_2 at P are perpendicular, and ω_i and ω_{i+1} intersect at X_i, where $i = 1, 2, 3, 4$ and $\omega_5 = \omega_1$. If $PX_1 = 2$, $PX_2 = 6$, and $PX_3 = 3$, what is PX_4?

Problem 8.2 (AMC 12A 2017/16)

Semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter JK. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P?

Problem 8.3 (USAMO 1993/2)

Let $ABCD$ be a quadrilateral whose diagonals AC and BD are perpendicular and intersect at E. Prove that the reflections of E across AB, BC, CD, and DA are concyclic.

Problem 8.4 (*Euclidean Geometry in Mathematical Olympiads*, Problem 8.23)

Let ABC be a right triangle with $\angle C = 90°$ and let X and Y be points in the interiors of CA and CB, respectively. Construct four circles passing through C, centered at A, B, X, Y. Prove that the four points lying on exactly two of these four circles are concyclic.

Problem 8.5 (Brocard's theorem)

Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Define P to be the intersection of AB and CD, Q to be the intersection of AC and BD, and R to be intersection of AD and BC. Show that O is the orthocenter of $\triangle PQR$.

Definition — Let $P \neq O$ be a point in the plane. Then, the **polar of P with respect to** ω is the line through P^* perpendicular to line OP.

Problem 8.6

Show that, if P lies on the polar of Q, then the circle with diameter PQ is orthogonal to ω.

Problem 8.7 (Circles inscribed in segments)

Let BC be a chord of circle Ω. Let ω be a circle tangent to BC at K and internally tangent to ω at T. Show that

- ray TK passes through M, and

- the power of M with respect to ω is MC^2 (that is, $MT \cdot MK = MC^2$).

Problem 8.8 (Shortlist 2003 G4)

Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P and Γ_1, Γ_1 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2, Γ_2 and Γ_3, Γ_3 and Γ_4, Γ_4 and Γ_1 meet at A, B, C, D, respectively. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}$$