Junior Math Team Inequalities Notes

Based on lessons by Mr. Kats

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1 VB1

The study of inequalities can largely be thought of as manipulations of the following poorly-named inequality:

Theorem 1.1 (Trivial Inequality, or VB1)**.** For all real numbers x,

$$x^2 \geq 0$$

with equality if and only if $x = 0$.

Since the name "trivial inequality" seems a bit disparaging, we instead refer to this as Very Basic 1, or VB1, as Mr. Kats says[1]. The main point to emphasize in this short section is that this simple inequality is the basis for a large portion of the theory of inequalities; it can be thought of as the "machine code" in which the language of inequalities is run. Therefore, when you use inequalities like AM-GM or Cauchy-Schwarz later on, just know that you could in theory (and with a lot of pain) distill these down to applications of good old VB1.

Problem 1.2

Prove the AM-GM inequality for two variables; that is, prove that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

for all positive real numbers a and b, and show that equality occurs if and only if $a = b$.

Problem 1.3 (SophFrosh Practice)

Compute the minimum value of the expression

$$x^4y^2 + x^4 + x^2y^2 - 6x^2y + x^2 + y^2 + 1.$$

Problem 1.4

Let a, b, c, d, and e be real numbers. Show that, if $2a < 5b$, then

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

cannot have all real roots.

2 Inequalities of Averages

Probably the most well-known inequality, besides VB1, is the **arithmetic mean-geometric mean (AM-GM)** inequality, which states that the arithmetic mean of any nonnegative reals is at least as large as their geometric mean, with equality when the numbers are all equal. Symbolically, it says

[1]No, there is no VB2.

Theorem 2.1 (AM-GM)**.** If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The previous section proposed two-variable AM-GM as a practice problem; for completeness, we now prove it here:

$$\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab} \\ \iff \left(\frac{a+b}{2}\right)^2 &\geq ab \\ \iff a^2 + 2ab + b^2 &\geq 4ab \\ \iff a^2 - 2ab + b^2 &= (a-b)^2 \geq 0. \end{aligned}$$

Observe that each manipulation made here is reversible, so we've constructed a chain of equivalent inequalities, the last of which is true; this implies that the original inequality is also true. Equality of course holds only when $a - b = 0$, or equivalently $a = b$.

Using the two-variable version, we can easily prove Theorem 2.1 whenever $n = 2^k$ is a power of two (try it yourself!). One way to do this is by induction on k.

Theorem (AM-GM on 2^k variables)**.** Let $n = 2^k$ for some positive integer k. Then,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. We perform induction on k, with the base case $k = 1$ being proven above.

For the inductive hypothesis, assume that this holds up to some arbitrary k. We then wish to show that the inequality is also true for $k + 1$:

$$\begin{aligned} \frac{a_1 + a_2 + a_3 + \cdots + a_{2^{k+1}}}{2^{k+1}} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_{2_k}}{2^k} + \frac{a_{2^k+1} + a_{2^k+2} + \cdots + a_{2^{k+1}}}{2^k}\right) \\ &\geq \frac{1}{2}\left(\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} + \sqrt[2^k]{a_{2^k+1} a_{2^k+1} \cdots a_{2^{k+1}}}\right) \\ &\geq \sqrt{\sqrt[2^k]{a_{2^k+1} a_{2^k+1} \cdots a_{2^{k+1}}} \cdot \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}} \\ &= \sqrt[2^k]{a_1 a_2 a_3 \cdots a_{2^{k+1}}}. \end{aligned}$$

Intuitively, we're halving the dataset into two smaller powers of two on which we can apply our smaller AM-GMs. □

This extends readily into all positive integers with the following trick:

Problem 2.2

Show that if the AM-GM inequality is true for $n \geq 3$ variables, then it must also be true for $n - 1$ variables. (Be careful not to use circular reasoning here!)

In principle, the idea is that we can find arbitrarily large n for which AM-GM is true, so the ability to prove $n \implies n - 1$ covers all of $\mathbb{N}$.