Geometry Review

Stuyvesant Senior Math Team

Aditya Pahuja

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Contents

0 Conventions

For the most part, diagrams will not be included, both because I'm lazy and because you should get practice drawing diagrams! Drawing a decent diagram can often be one of the hard parts of a problem. (I will probably spare you from drawing all the diagrams in future lessons.)

0.1 Definitions

It's assumed that you know the definitions of the five most common triangle centers, which are listed here for completeness. In $\triangle ABC$,

- the **incenter** is the concurrency point of the (internal) angle bisectors of $\angle A$, $\angle B$, and $\angle C$.
- the **centroid** is the concurrency point of the medians.
- the **circumcenter** is the concurrency point of the perpendicular bisectors of AB, BC, and CA.
- the **orthocenter** is the concurrency point of the altitudes.
- the **A-excenter** is the concurrency point of the internal angle bisector of $\angle A$ and the external angle bisectors of $\angle B$ and $\angle C$. The B-excenter and C-excenter are defined similarly.

The existence of these centers will be assumed for now; methods to prove them will be developed later on. (You should be able to prove the all of the centers' existence if you remember last year's lessons, though.)

0.2 Notation

- When writing $\triangle ABC \cong \triangle XYZ$ or $\triangle ABC \sim \triangle XYZ$, the order of the vertices encodes corresponding vertices. In other words, $\angle A = \angle X$, $\angle B = \angle Y$, and $\angle C = \angle Z$.
- If points P_1, P_2, $\ldots$, P_n are concyclic (i.e. lie on some circle), then $(P_1P_2\ldots P_n)$ is the circle passing through those points.
- Similarly, $[P_1P_2\ldots P_n]$ denotes the area of polygon $P_1P_2\ldots P_n$.
- If not specified otherwise, we denote the side lengths of the sides of $\triangle ABC$ by $BC = a$, $CA = b$, and $AB = c$.

1 Philosophical Rambling

There are two main ways to view a geometry problem: *synthetically* and *analytically*.

Analytic methods involve viewing the configuration using some computational framework: trigonometry and Cartesian coordinates are the standard beginner's tools for this approach.

Synthetic geometry, on the other hand, can be thought of as building up geometry from some set of axioms: A lot of your 9th-grade geometry is the basis of synthetic methods.

A strong geometer is comfortable with both synthetic and analytic techniques: each tool (or perhaps combination of tools) is best suited for different kinds of problems. In my experience, people are generally much more comfortable working with computational techniques, so I'll generally focus more on synthetic techniques here. As such, you should always try to look for synthetic solutions where you can (but bonus points if you can find multiple solutions!).

2 Synthetic Techniques

2.1 Digression: How to approach problems

Generally, solving problems comes in two phases:

1. The "scouting" phase, where you try to get some intuition about why the problem works. This can manifest in multiple ways, such as
   - searching for what "should" be true, such as by working backwards (ex. "$\triangle HBC \sim \triangle ODE$ needs to be true for the problem to be true."),
   - getting a heuristic understanding of what's going on (ex. "$|7^a - 3^b|$ should generally be much larger than $|a^4 + b^2|$, so a and b should be small, whatever *small* means."),
   - thinking about why certain techniques don't work (ex. "I can't show that a randomly selected path in my graph behaves the way I want, so I should try considering the whole set of paths simultaneously."),
   - thinking about what techniques might work (ex. "I have a central right triangle, and my points are easy to define with respect to this right triangle, so Cartesian coordinates could work.")

   and so on. These are the things that people sometimes call "motivation."

2. The "attacking" phase, where you prove things about the problem. This is the part where you actually try computing things, performing induction, etc., ideally solving the problem, but at least getting some sort of intermediate claim. These are much more concrete methods, and are the parts that actually show up in your solution, if you were to write it out.

(This is largely parroting ideas from Evan Chen's blog post on hard and soft techniques.)

Put more simply, you gather information while scouting, and then use that information to mount an attack, hopefully destroying the problem.

Most of the time, scouting is done through a synthetic lens. This can come in the form of redefining points to be more well-behaved (perhaps in hope of finding an analytic approach), angle chasing to look for similar triangles/cyclic quadrilaterals (although that can be thought of as attacking, too), etc. In that vein, scouting strategies will be in **pink, sans-serif font**. Try to come up with some heuristics on your own, too.

2.2 Angles

Angle chasing is a very low-powered technique: you should've already encountered most of the necessary material in 9th-grade geometry. For completeness, here are the two most important theorems that you should know at least as well as the back of your hand (but preferably better).

Theorem 2.1 (Cyclic quadrilaterals and inscribed angles)**.** Let $ABCD$ be a convex quadrilateral. Then, the following are equivalent:

- There exists a circle containing A, B, C, and D.
- $\angle ABC = 180^\circ - \angle ADC$.
- $\angle ACB = \angle ADB$.

Theorem 2.2 (Tangents and chords)**.** There is a triangle $\triangle ABC$ and a line ℓ through A. Let D be a point on ℓ such that B and D are on opposite sides of line AC. Then, ℓ is tangent to (ABC) at A if and only if $\angle ABC$ and $\angle DAC$ are congruent.

2.2.1 Walkthroughs

Example 2.1 (JMO 2011/5)

Points A, B, C, D, E lie on a circle ω and P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$.

Prove that BE bisects AC.

Walkthrough. Let M be the midpoint of AC. We can then phrase the problem as "Show that B, M, and E are collinear."

(a) Assume for a moment that B, M, and E are collinear. What quadruple of points (besides a subset of $\{A, B, C, D, E\}$) must form a cyclic quadrilateral under this assumption?

(b) Get rid of the assumption and prove that the quadrilateral you found in **(a)** really is cyclic.

(c) Using this cyclic quadrilateral, show that M is in fact the midpoint of AC. (One approach is to draw the circumcenter.)

The key step of this problem is the discovery of the cyclic quadrilateral. Often, the key idea in a geometry problem is locating a particular cyclic quadrilateral, set of collinear points, set of concurrent lines, etc., so it's important to develop some strategies for guessing when these features appear.

One such strategy, which we used in part **(a)**, is **working backwards**. The idea is to assume that the desired result is true, and then formulate equivalent but more tractable statements. In this case, M turns out to be the midpoint *if and only if* this mystery quadrilateral is cyclic, so proving that it's cyclic essentially solves the problem.

(Question: Why should you expect the quadrilateral being cyclic to be *equivalent* to the collinearity?)

Example 2.2 (2022 AIME II/11)

Let $ABCD$ be a convex quadrilateral with $AB = 2$, $AD = 7$, and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of BC. Find the area of $ABCD$.

Walkthrough. The core of this problem is a bunch of angle chasing, followed by a tiny bit of computation.

Let M be the midpoint of BC and let P be the intersection of lines AB and CD; then, M is the incenter of $\triangle APD$.

- **(a)** Solve the following problem first: Let $\triangle ABC$ be a triangle with incenter I. Points X and Y lie on AB and AC such that I is the midpoint of XY. Describe a ruler-and-compass construction for X and Y given A, B, C, and I.
- **(b)** How does the previous part apply in the original problem? Describe $\triangle PBC$.
- **(c)** Show that $\triangle ABM \sim \triangle MCD \sim \triangle AMD$.
- **(d)** Compute the lengths AM, BM, CM, and DM via the similar triangles.
- **(e)** Extract the final answer of $6\sqrt{5}$, either by Heron's formula or by trigonometry.

This is a problem where the original diagram is hard to construct with a ruler and compass without some additional thinking. In such problems, one common approach is to **try phrasing the problem more naturally**: in this case, rewriting the problem with respect to $\triangle APD$ essentially solved the problem, since it allowed us to wrangle the "angle bisectors concur on the midpoint of BC" condition.

Example 2.3 (2020 AIME II/15)

Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16$, $BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$. Find XY.

Walkthrough. There are several viable approaches here, including trigonometry and complex numbers. We will go through a synthetic solution, though.

The most difficult aspect of this problem is that the diagram is very bare: angle chasing will show you a few equal angles, but you won't find anything particularly substantive.

- **(a)** Let M be the foot of the perpendicular from T to BC. What properties does M have? (There are at least three nontrivial ones.)
- **(b)** Show that $TXMY$ is a parallelogram.
- **(c)** The parallelogram law (also called Apollonius' theorem) says that

  $$TX^2 + TY^2 + MX^2 + MY^2 = XY^2 + TM^2.$$

  In other words, the sum of the squares of a parallelogram's side lengths is equal to the sum of the squares of its diagonals. Use this to compute the final answer.
- **(d)** Optionally, prove that M is the orthocenter of $\triangle AXY$.

Actually, this solution is pretty short: there are not many things to do once you add in M; the challenge is realizing that adding M helps.

2.2.2 Extra Problems

In general, these will be problems that show some common configurations or that I simply think are nice/instructive. A couple of problems are marked with a star because I think they're extra cool. They are roughly in difficulty order.

Also, a warning: any AIME problems have been altered to ask for the answer, rather than making you convert it to an integer: it is very possible for the answer to not be an integer (i.e. I ask directly for $\frac{a}{b}$, not $a+b$).

Problem 2.3

Let $\triangle ABC$ have orthocenter H. Lines AH, BH, and CH intersect lines BC, CA, and AB at D, E, and F respectively. Find all six cyclic quadrilaterals with vertices in $\{A, B, C, D, E, F, H\}$, and describe their circumcenters.

(Bonus: prove that the six circumcenters are concyclic, too. This circumcircle is called the **nine-point circle**.)

Problem 2.4 (Incenter-excenter lemma, aka "Fact 5")

Consider a triangle $\triangle ABC$. The angle bisector of $\angle BAC$ intersects its circumcircle ω again at L. Show that L is the circumcenter of quadrilateral $BICI_A$, where I is the incenter and I_A is the A-excenter of the triangle.

Problem 2.5 (Reflecting the orthocenter)

Show that the reflections of the orthocenter of triangle $\triangle ABC$ over BC and the midpoint of BC both lie on the circumcircle of $\triangle ABC$. Moreover, prove that the reflection over the midpoint is the point diametrically opposite from A.

Problem 2.6 (2021 AIME II/14)

Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let X be the intersection of the line tangent to the circumcircle of $\triangle ABC$ at A and the line perpendicular to GO at G. Let Y be the intersection of lines XG and BC. Given that the measures of $\angle ABC, \angle BCA$, and $\angle XOY$ are in the ratio $13 : 2 : 17$, compute the degree measure of $\angle BAC$.

★ Problem 2.7 (USAMO 2021/1)

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

★ Problem 2.8 (BrMO 2013/2, aka the first isogonality lemma)

Let ABC be a triangle and let P be a point inside it satisfying $\angle ABP = \angle PCA$. Let Q be the reflection of P across the midpoint of BC. Prove that $\angle BAP = \angle CAQ$.

2.3 Power of a Point

Power of a point is an extremely useful theorem, and I dedicate a section to it here because it's the most frequently used bridge between length information and angle information (except for maybe trigonometry). This should make sense, as it's essentially similar triangles in a neat package.

Here's the statement as a refresher:

Theorem 2.9 (Power of a point)**.** Let Γ be a circle and P be any point. Then, across all choices of lines through P intersecting Γ at X and Y (it's possible for $X = Y$), the quantity

$$PX \cdot PY$$

is constant.

It's also often phrased in the following equivalent manner:

Corollary 2.10. Let A, B, C, and D be points on a circle. Then, if AB and CD intersect at a point P,

$$PA \cdot PB = PC \cdot PD.$$

As alluded to above, the proof of this follows directly from $\triangle PAC \sim \triangle PDB$.

Also, the converse is true!

Theorem 2.11 (Converse of POP)**.** If AB and CD are lines intersecting at P, P either lies on both segments AB and CD or on either segment, and

$$PA \cdot PB = PC \cdot PD,$$

then A, B, C, and D form a cyclic quadrilateral.

The nice thing about everything above is that we have a lot of freedom: we can choose AB and CD to be any lines passing through P. In particular, the $PA \cdot PB$ quantity is entirely dependent on the choice of P and the features of the circle. This motivates the following definition:

Definition 2.12 — Let Γ be a circle with center O and radius r, and let P be a point in the plane. We define the **power** of P with respect to Γ by

$$\operatorname{Pow}_\Gamma(P) = OP^2 - r^2.$$

Exercise 2.4. Verify that $|\operatorname{Pow}_\Gamma(P)| = PX \cdot PY$ for any choice of X and Y on Γ such that P, X, and Y are collinear. When is the power positive? Negative? Zero?

While we will discuss this function in more detail later in the year, here's one important result that shows up pretty often.

Theorem 2.13 (Radical axis)**.** Let Γ_1 and Γ_2 be two non-concentric circles. Then, the set of points P satisfying

$$\operatorname{Pow}_{\Gamma_1}(P) = \operatorname{Pow}_{\Gamma_2}(P)$$

is a line that is perpendicular to the line joining the centers of Γ_1 and Γ_2. This line is called the **radical axis** of the two circles.

Proof. Let Γ_i have center O_i and radius r_i. Then, we want to find the set of points P for which

$$O_1P^2 + r_1^2 = O_2P^2 - r_2^2 \iff O_1P^2 - O_2P^2 = r_1^2 - r_2^2.$$

We then want to show that the set of points for which $O_1P^2 - O_2P^2$ is equal to the constant $r_1^2 - r_2^2$ is a line.

Claim 2.14 — Let AB be a segment and let C and D be two points. Then,

$$AC^2 - BC^2 = AD^2 - BD^2$$

if and only if $AB \perp CD$.

Proof. If $AB \perp CD$, then the result follows from Pythagorean theorem.

In the other direction, if the length condition is true, then let H_C and H_D be the feet of C and D onto AB. Then

$$AH_C^2 - BH_C^2 = AC^2 - BC^2 = AD^2 - BD^2 = AH_D^2 - BH_D^2$$

implies that $H_C = H_D$, so $CD \perp AB$. $\square$

This says that the set of points X for which $O_1X^2 - O_2X^2$ is constant is a line perpendicular to O_1O_2, which is what we wanted to prove. $\square$

An alternate (and simpler) proof can also be extracted using coordinates, but the synthetic approach is good to know, since the intermediary claim is pretty useful itself.

Remark 2.15. You may notice the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\operatorname{Pow}_{\omega_1}(X) - \operatorname{Pow}_{\omega_2}(X)$ arising naturally from the proof — the radical axis is the set of points which make the function zero. Interestingly, f is **linear**; that is,

$$f(kA + (1-k)B) = kf(A) + (1-k)f(B)$$

where multiplication (between real numbers and points) and addition (between two points) are done componentwise with the coordinates.

The upshot of this is that knowing the value of f at A and B lets you compute $f(X)$ for all X on line AB, and, if you know f at noncollinear points A, B, and C, then you can compute the value of f at any point in the plane.

2.3.1 Walkthroughs

Example 2.5 (JMO 2012/1)

Given a triangle ABC, let P and Q be on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Walkthrough. This problem is one of my favorites because its solution uses a really unique idea.

(a) Assume for the sake of contradiction that (PRS) and (QRS) are distinct circles. Then, show that AB is tangent to (PRS) at P and AC is tangent to (QRS) at Q.

(b) Convince yourself that RS is the radical axis of the circles (in particular, that the two circles are not concentric).

(c) Find a point not on line RS that has the same power with respect to (PRS) and (QRS), and deduce a contradiction.

I think this solution seems really unnatural at first glance, but it presents itself after noticing that the circles above should be tangent to AB and AC. (Theorem 2.2 paying back in spades!)

Example 2.6 (Euler's theorem)

Let $\triangle ABC$ have circumcenter O and incenter I, as well as circumradius R and inradius r. Then,

$$OI^2 = R^2 - 2Rr.$$

Walkthrough. This problem lends itself very handily to power of a point because of the given equation.

(a) Let Γ be the circumcircle of $\triangle ABC$. Show that $\text{Pow}_\Gamma(I) = -2Rr$. This means we want to show that $XI \cdot YI = 2Rr$ for some choice of $X, Y \in \Gamma$ with X, I, Y collinear.

(b) Draw the angle bisector from A, and let it intersect Γ at D. Then, we want $AI \cdot ID = 2Rr$. If you haven't already done Problem 2.4, show that $DI = DB$.

(c) Imagine that the equation above was the result of setting up a proportion between two similar triangles. Working backwards, what might the original proportion be? Keep in mind which segments would most readily fit in a triangle together.

(d) If you did part **(c)** correctly, then you should be able to draw in one more point to create a pair of similar triangles with the desired proportion.

This is a pretty classical example of power of a point, and it's sometimes used in geometric inequalities, since you can show that

$$R(R - 2r) = OI^2 \geq 0,$$

so $R \geq 2r$ (when does equality happen?). Using some area formulas, you can even extract the inequality

$$4R^2 \geq \frac{abc}{a+b+c},$$

which has no trace of r, surprisingly.

2.3.2 Extra Problems

Problem 2.16

Let Γ_1 and Γ_2 be two intersecting circles. Let a common tangent to Γ_1 and Γ_2 touch Γ_1 at A and Γ_2 at B. Show that the common chord of Γ_1 and Γ_2, when extended, bisects segment AB.

Problem 2.17 (Radical center)

Let ω_1, ω_2, and ω_3 be three circles whose centers are not collinear. Show that the radical axes of each pair of circles concur at some point P, which is called the **radical center** of the three centers.

Problem 2.18

Describe a ruler-and-compass construction for the radical axis of *any* two (possibly disjoint) circles.

Problem 2.19 (2019 AIME II/11)

Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Compute AK.

★ **Problem 2.20** (Shortlist 2022 G2)

In the acute-angled triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E, respectively. Points $X \neq A$ and $Y \neq A$ lie on the circles ABD and ACE, respectively, such that $DA = DX$ and $EA = EY$. Prove that B, C, X, and Y are concyclic.

Problem 2.21 (IMO 2000/1)

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

★ **Problem 2.22** (Japan 2014/4)

Let Γ be the circumcircle of triangle ABC, and let ℓ be the tangent line of Γ passing through A. Let D, E be points on sides AB and AC such that $BD : DA = AE : EC$. Line DE meets Γ at points F and G. The line parallel to AC through D meets ℓ at H, the line parallel to AB through E meets ℓ at I. Prove that there exists a circle passing through four points F, G, H, I, and tangent to line BC.

3 Analytic Techniques

3.1 Theorems

The following are things that we saw last year. If you are new to math team, or simply don't remember how to prove one of these, their proofs are reproduced in the appendix.

- (Extended) law of sines: In triangle $\triangle ABC$,
$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
- Law of cosines: $a^2 + b^2 - 2ab\cos C = c^2$.
- Ptolemy's inequality: for any non-collinear points A, B, C, D,
$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD,$$
with equality if and only if $ABCD$ is a convex cyclic quadrilateral. Ptolemy's theorem is the fact that equality occurs iff $ABCD$ is cyclic.
- Triangle inequality: $AB + BC \geq AC$, with equality if and only if A, B, C are collinear in that order.
- Stewart's theorem: $dad + man = bmb + cnc$.
- Area formulas: $K = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{1}{2}ab\sin C$

You should also already know the trig angle addition formulas from Algebra II or Precalc:
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$
If you ever forget these, you can equate the real and imaginary parts of
$$e^{i(x+y)} = e^{ix} \cdot e^{iy}$$
by using $e^{i\theta} = \cos\theta + i\sin\theta$.

In addition, there are the product-to-sum and sum-to-product formulas below, but I can never remember them. You should at least know how to derive them; I rederive them every single time.
$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$
$$\sin x \cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$
$$\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$
$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
The latter two equations are, of course, just the second and third equations in the other direction (i.e. $(x,y) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$).

If you're not so familiar with any of these formulas, then go through and verify them.

3.2 Walkthroughs

Example 3.1 (2015 AIME II/15)

Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ. Given that the areas of $\triangle DBA$ and $\triangle ACE$ are equal, determine their common value.

Walkthrough. The key observation is that these circles are dilations of one another about A; we can leverage this fact to get a lot of length relations.

(a) Let any line through A intersect $\mathcal{P}$ at X and $\mathcal{Q}$ at Y. Show that $\frac{AX}{AY} = \frac{1}{4}$.

(b) Show that $\angle BAC = 90°$, and then compute AB and AC. One approach is to extend AB to hit $\mathcal{Q}$ at B' and find similar triangles.

(c) Let $\angle BAD = \alpha$. What are $\sin\alpha$ and $\cos\alpha$? (Hint: What do the equal areas tell you?)

(d) What are BD and AD?

(e) If you did the previous parts correctly, you now know AB, BD, and AD; pick an area formula and finish.

Example 3.2 (Fermat point)

Let $\triangle ABC$ be a triangle whose angles are all less than $120°$ and let P be a point in the plane. Show that $PA + PB + PC$ achieves its minimum value if and only if $\angle APB = \angle BPC = \angle CPA = 120°$.

Walkthrough. The trick with this problem is to draw X, the point on the other side of BC from A such that $\triangle BXC$ is equilateral.

(a) Let F be the point such that $\angle AFB = \angle BFC = \angle CFA$. Show that $FBXC$ is cyclic and A, F, X are collinear. Which two inequalities does F therefore optimize?

(b) Use the inequalities from the previous part to show that

$$PA + PB + PC \geq AX,$$

with equality if and only if $P = F$.

(c) Where was the stipulation that $\angle A$, $\angle B$, and $\angle C$ are less than $120°$ used?

(d) Some extra things: If Y and Z are constructed similarly to X so that $\triangle CYA$ and $\triangle AZB$ are equilateral, show that $AX = BY = CZ$ and that those three lines are concurrent (at F).

This is an example where **trying to understand the weird condition** (namely, the equality case of $\angle AFB = \angle BFC = \angle CFA$) will get you pretty close to a solution; I found it by trying to construct F.

Example 3.3 (2023 AIME I/12)
Let $\triangle ABC$ be an equilateral triangle with side length 55. Points D, E, and F lie on sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, with $BD = 7$, $CE = 30$, and $AF = 40$. A unique point P inside $\triangle ABC$ has the property that

$$\measuredangle AEP = \measuredangle BFP = \measuredangle CDP.$$

Find $\tan(\measuredangle AEP)$.

Walkthrough. I didn't solve this in contest, unfortunately, but here's a synthetic solution I found afterwards using the Fermat point.

(a) Show that the angle condition implies that $AEPF$, $BFPD$, and $CDPE$ are cyclic. What does the equilateral condition say about $\angle DPE$, $\angle EPF$, and $\angle FPD$?

(b) Extend DP to meet (FPE) at X, so that it suffices to compute $\tan(\angle XDC)$. What do you know about $\triangle EFX$? What about quadrilateral $AXDC$?

(c) Compute the length of AX using Ptolemy's theorem.

(d) Redraw quadrilateral $AXDC$ separately, and eradicate the problem.

You can also approach this with complex numbers, coordinates, trigonometry, or other synthetic approaches (one I saw on AoPS was to drop the altitudes from P); I encourage looking for an alternate solution.

Finally, here's a number theory problem from IMO 2001 with a strange geometric solution, just as a silly example.

Example 3.4 (IMO 2001/6)
Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Walkthrough. This problem is. . . weird. Let's first reveal where the geometry really is:

(a) Expand the equation (yes, really). You should end up with

$$a^2 - ac + c^2 = b^2 + bd + d^2.$$

What do the left- and right-hand sides of this equation look like?

In light of this, we can construct a quadrilateral $WXYZ$ with sides $WX = a$, $XY = c$, $YZ = b$, $ZX = d$, and diagonal WY having length

$$\sqrt{a^2 - ac + c^2} = \sqrt{b^2 + bd + d^2},$$

so that, by law of cosines, $\angle WXY = 60^\circ$ and $\angle WZY = 120^\circ$. This means $WXYZ$ is cyclic!

On the other hand, the diagonals of a cyclic quadrilateral can *always* be expressed in terms of their sides.

(b) Show that there are exactly three quadrilaterals with side lengths a, b, c, d in some order. Moreover, show that, among these quadrilaterals, there are three possible lengths t, u, v for the diagonals; that is, one quadrilateral has diagonals (t, u), one has diagonals (u, v), and one has diagonals (v, t).

(c) Apply Ptolemy's theorem in all three quadrilaterals to get the system of equations

$$\begin{aligned} ab + cd &= tu \\ ac + bd &= uv \\ ad + bc &= vt \end{aligned}$$

where $t = WY$ and $u = XZ$. Then, compute t^2.

(d) Finally, we can use the inequality condition: show (ex. by rearrangement) that

$$ab + cd > ac + bd > ad + bc,$$

so, if $ab + cd$ is prime, then t^2 can never be an integer. However, from earlier, $WY^2 = b^2 + bd + d^2$ is an integer — contradiction!

The takeaway from here is primarily that quadratic expressions in two variables can sometimes be thought of as applications of the law of cosines. The additional geometric structure then lets you, well, do geometry to get nontrivial, non-geometric information (such as the fact that $(ac + bd) \mid (ab + cd)(ad + bc)$ in this case).

3.3 Extra Problems

Problem 3.1 (NIMO, Evan Chen)

Let $AXYZB$ be a convem pentagon inscribed in a semicircle with diameter AB. Suppose that $AZ - AX = 6$, $BX - BZ = 9$, and $BY = 5$. Find the perimeter of quadrilateral $OXYZ$, where O is the midpoint of AB.

Problem 3.2 (2014 AIME I/15)

In $\triangle ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, compute DE.

Problem 3.3 (2012 AIME I/13)

Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. Compute the largest possible area of the triangle.

Problem 3.4 (Pompeiu's theorem)

Let P be a point in the plane of equilateral triangle $\triangle ABC$. Show that PA, PB, and PC form the sides of a (maybe degenerate) triangle. When is the triangle degenerate?

Problem 3.5 (AMC 12A 2021/24)

Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, find the area of $\triangle PQR$.

Problem 3.6 (ARML 2023 T10)

Parallelogram $ABCD$ is rotated about A in the plane, resulting in $AB'C'D'$, with D on $\overline{AB'}$. Suppose that $[B'CD] = [ABD'] = [BCC']$. Compute $\tan\angle ABD$.

★ **Problem 3.7** (USAMO 1998/6)

Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2\ldots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle, where indices are taken modulo n.

3.3.1 More trig practice, suggested by Ashley Zhu (from Hunter)

I don't use trig very often (read: ever), so I asked Ashley to provide some problems.

Problem 3.8 (Feb HMMT 2022 G3)

Let $ABCD$ and $AEFG$ be unit squares such that the area of their intersection is $\frac{20}{21}$. Given that $\angle BAE < 45$, $\tan \angle BAE$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Problem 3.9 (2023 AIME I/5)

Let P be a point on the circle circumscribing square $ABCD$ that satisfies $PA \cdot PC = 56$ and $PB \cdot PD = 90$. Find the area of $ABCD$.

Problem 3.10 (Feb HMMT 2023 G6)

Convex quadrilateral $ABCD$ satisfies $\angle CAB = \angle ADB = 30°$, $\angle ABD = 77°$, $BC = CD$, and $\angle BCD = n°$ for some positive integer n. Compute n.

Problem 3.11 (2014 AIME II/12)

Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Problem 3.12 (2018 AIME I/13)

Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

Problem 3.13 (2020 AIME I/15)

Let $\triangle ABC$ be an acute triangle with circumcircle ω, and let H be the intersection of the altitudes of $\triangle ABC$. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3, HX = 2$, and $HY = 6$. The area of $\triangle ABC$ can be written in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Remark 3.14 (If you want even more practice)**.** Most problems from AIMEs, HMMTs, PUMaCs, ... are pretty nice in general. If you ever find yourself wanting more problems to do (particularly for AMC/AIME practice), these contests provide a plethora of good ones.

4 Appendix: Proofs to theorems in Section 3

Here are the proofs for most of the items in the list of theorems.

Theorem 4.1 (Law of sines). In $\triangle ABC$ with circumradius R,

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Proof. We prove that $2R = \frac{a}{\sin A}$; the rest follows analogously. The main idea is to introduce a right angle by drawing B', the point diametrically opposite to B in (ABC); then, $\triangle BB'C$ has $\angle C = 90°$.

Now, since A, B, B', and C are concyclic, angle $\angle BB'C$ is equal to either $\angle BAC = \angle A$ or $180° - \angle BAC = 180° - \angle A$. In either case,

$$\sin(\angle A) = \sin(\angle BB'C) = \frac{BC}{BB'} = \frac{a}{2R}$$

because $\sin A = \sin(180° - A)$. $\square$

Theorem 4.2 (Law of cosines). In $\triangle ABC$,

$$a^2 + b^2 - 2ab\cos C = c^2.$$

Proof. We will prove the theorem only in the case where the triangle isn't obtuse; the obtuse case follows similarly. (Alternatively, both cases can be handled simultaneously with directed lengths.)

Let H be the foot of the altitude from A to BC, and, for conciseness, let $AH = h$ and $CH = x$. Then, by the Pythagorean theorem, we get a system

$$x^2 + h^2 = b^2$$
$$(a - x)^2 + h^2 = c^2.$$

Subtracting the first equation from the second,

$$a^2 - 2ax = c^2 - b^2.$$

By definition, $x = b\cos C$, so

$$a^2 + b^2 - 2ab\cos C = c^2$$

as desired. $\square$

Theorem 4.3 (Ptolemy's inequality). Let A, B, C, D be four non-collinear points in the plane. Then,

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD,$$

with equality if and only if $ABCD$ is a convex cyclic quadrilateral.

We'll present two proofs.

Proof 1 (Similar triangles). Let P be the point such that $\triangle APB \sim \triangle ADC$ with the same orientation (i.e. A, P, B and A, D, C are both clockwise/both counterclockwise along the respective triangles). Then, $\triangle APD \sim \triangle ABC$ as well, because $\frac{AP}{AD} = \frac{AB}{AC}$ and $\angle PAD = \angle BAC$. Thus

$$\frac{AB}{AC} = \frac{PB}{DC}, \quad \frac{AC}{AD} = \frac{BC}{PD},$$

so

$$AD \cdot BC = AC \cdot PD, \quad AB \cdot CD = AC \cdot PB$$

which implies that

$$AB \cdot CD + AD \cdot BC = AC \cdot (PB + PD) \geq AC \cdot BD$$

by triangle inequality.

Moreover, equality occurs when B, P, and D are collinear, with P between B and D. Since line AP is the reflection of line AC over the bisector of $\angle BAD$ by the angle condition, the latter constraint is satisfied by requiring $ABCD$ to be convex. As for the former constraint, we need

$$\angle ABD = \angle ABP = \angle ACD,$$

which is equivalent to $ABCD$ being cyclic, the end. □

Proof 2 (Inversion). Invert about A with radius r. Then

$$B^*C^* = \frac{BC \cdot r^2}{AB \cdot AC}, \quad C^*D^* = \frac{CD \cdot r^2}{AC \cdot AD}, \quad B^*D^* = \frac{BD \cdot r^2}{AB \cdot AD}$$

by the inversion distance formula. Triangle inequality then says

$$\begin{aligned} B^*C^* + C^*D^* &\geq B^*D^* \\ \iff \frac{BC \cdot r^2}{AB \cdot AC} + \frac{CD \cdot r^2}{AC \cdot AD} &\geq \frac{BD \cdot r^2}{AB \cdot AD} \\ \iff BC \cdot AD + CD \cdot AB &\geq BD \cdot AC. \end{aligned}$$

Equality occurs when B^*, C^*, D^* are collinear in that order, which is equivalent to $ABCD$ being convex and cyclic. □

> **Remark.** If this seemed like black magic to you, feel free to check out my notes on inversion from Mr. Kats's class!

Theorem 4.4 (Stewart's theorem)**.** Point D lies on side BC of $\triangle ABC$. Then, if $BD = m$, $CD = n$, and $AD = d$,

$$man + dad = bmb + cnc.$$

Mnemonic: A man and his dad put a bomb in the cinc (sink).

Proof. You can do this with trigonometry or three applications of the Pythagorean theorem, but we present a neat proof using Ptolemy's theorem.

Let D' be the second intersection of line AD with (ABC). By power of a point,

$$mn = BD \cdot CD = AD \cdot D'D = d \cdot DD',$$

so $DD' = \frac{mn}{d}$. Then, since $\triangle ABD \sim \triangle CD'D$ and $\triangle ACD \sim \triangle BD'D$, we can also compute

$$BD' = \frac{AC \cdot BD}{AD} = \frac{bm}{d}, \quad CD' = \frac{AB \cdot CD}{AD} = \frac{cn}{d}.$$

Therefore, by Ptolemy's theorem,

$$c \cdot \frac{cn}{d} + b \cdot \frac{bm}{d} = (m+n)\left(d + \frac{mn}{d}\right) = ad + \frac{amn}{d}$$

which gives what we want upon multiplying by d. $\square$

Theorem 4.5 (Miscellaneous area formulas)**.** In $\triangle ABC$,

$$K = rs = \frac{1}{2}ab\sin C = \frac{abc}{4R}\sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. For the first formula, let I be the incenter of $\triangle ABC$. Then,

$$[ABC] = [IBC] + [ICA] + [IAB] = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = r \cdot \frac{1}{2}(a+b+c) = rs.$$

For the second formula, let H be the foot of the altitude from A, so

$$\frac{1}{2} \cdot AH \cdot BC = \frac{1}{2} \cdot b\sin C \cdot a.$$

For the third formula, we simply take the previous one and write

$$\sin C = \frac{c}{2R}.$$

For the fourth formula (Heron's formula), we start by using the law of cosines to compute $\cos C$:

$$a^2 + b^2 - 2ab\cos C = c^2 \implies \frac{a^2+b^2-c^2}{2ab} = \cos C.$$

Since $\sin^2 C + \cos^2 C = 1$,

$$\sin C = \sqrt{1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2}.$$

Then the area is

$$\begin{aligned}
\frac{1}{2}ab\sin C &= \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \\
&= \frac{1}{2} \cdot \frac{ab}{2ab} \cdot \sqrt{4a^2b^2 - (a^2+b^2-c^2)^2} \\
&= \frac{1}{4}\sqrt{(2ab - (a^2+b^2-c^2))(2ab + (a^2+b^2-c^2))} \\
&= \frac{1}{4}\sqrt{(c^2 - (a^2 - 2ab + b^2))((a^2+2ab+b^2) - c^2)} \\
&= \frac{1}{4}\sqrt{(c-a+b)(c+a-b)(a+b+c)(a+b-c)} \\
&= \sqrt{\frac{a+b+c}{2} \cdot \frac{-a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}} \\
&= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$

Yay! $\square$

5 Solutions to Walkthroughs

5.1 Solution 2.1, JMO 2011/5

Let M be the midpoint of AC, so we want to show that B, M, and E are collinear.

Claim 5.1 — Quadrilateral $BMDP$ is cyclic.

Proof. Let O be the center of ω. Then,

$$\angle OBP = \angle ODP = \angle OMP = 90^\circ,$$

so B, M, D, P, and O all lie on the circle with diameter OP. □

Now, we can see that

$$\angle BMP = \angle BDP = \angle BED$$

which implies that B, M, and E are collinear because $MP \parallel ED$.

5.2 Solution 2.2, 2022 AIME II/11

Let P be the intersection of AB and CD. By construction, M is the incenter of $\triangle APD$. Since PM is both an angle bisector and a median in $\triangle PBC$, the triangle must be isosceles. In particular, $\angle PAD = \angle PDA = 90^\circ - \frac{1}{2}\angle BPC$.

Moreover, $\angle AMD = 180^\circ - \frac{1}{2}\angle PAD - \frac{1}{2}\angle PDA = 90^\circ + \frac{1}{2}\angle APD$. Therefore

$$\angle ABM = \angle MCD = \angle AMD = 90^\circ + \frac{1}{2}\angle APD.$$

From $\angle BAM = \angle MAD$ we then get $\triangle ABM \sim \triangle AMD$, and $\angle CDM = \angle MDA$ gives $\triangle MCD \sim \triangle AMD$.

This finally lets us compute

$$\frac{AB}{AM} = \frac{AM}{AD} \iff AM^2 = AB \cdot AD = 14$$

and

$$\frac{DC}{DM} = \frac{DM}{DA} \iff DM^2 = DC \cdot DA = 21,$$

so $AM = \sqrt{14}$ and $DM = \sqrt{21}$. We also have

$$\frac{BM}{CD} = \frac{AB}{CM},$$

and, since M is the midpoint of BC, $BM = CM = \sqrt{6}$. Finally, applying Heron's formula, we extract the answer of $\boxed{6\sqrt{5}}$.

5.3 Solution 2.3, 2020 AIME II/15

Let M be the midpoint of BC. Then

$$\angle TMB = \angle TBC = \angle TXB = \angle TYC = 90^\circ,$$

so $TMBX$ and $TMCY$ are cyclic.

Now, consider the following angle relations:

$$\angle C = 180^\circ - \angle A - \angle B = 180^\circ - \angle B - \angle CBT = \angle TBX = \angle TMX$$
$$\angle C = 180^\circ - \angle MCY = \angle MTY$$

These tell us $\angle TMX = \angle MTY$, so $MX \parallel TY$. Analogously, we can find $MY \parallel TX$, i.e. $MXTY$ is a parallelogram.

Using Apollonius's theorem (aka the parallelogram law), we then get

$$XY^2 + MT^2 = XY^2 + (16^2 - 11^2) = 2TX^2 + 2TY^2 = 2(1143 - XY^2)$$

which spits out $XY^2 = 717$ and thus $\boxed{XY = \sqrt{717}}$.

5.4 Solution 2.5, JMO 2012/1

Suppose for the sake of contradiction that (PRS) and (QRS) are distinct. Then, the angle conditions imply that AB is tangent to (PRS) at P and AC is tangent to (QRS) at Q (recall Theorem 2.2).

Now, observe that, since the circles intersect at R and S, their radical axis is line RS: that is, no point outside line RS has the same power with respect to the two circles. However, since $AP = AQ$,

$$\operatorname{Pow}_{(PRS)}(A) = AP^2 = AQ^2 = \operatorname{Pow}_{(QRS)}(A),$$

and A obviously doesn't lie on line RS — contradiction!

5.5 Solution 2.6, Euler's theorem

The equation is asking us to show that the power of I with respect to (ABC) is $-2Rr$. Now, we define a couple of points:

- Let D be the point where the bisector of $\angle A$ hits (ABC).
- Let F be the foot of the perpendicular from I on AB.
- Let D' be the point diametrically opposite from D.

First, we prove an important claim:

Claim 5.2 (Problem 2.4, incenter-excenter lemma) — Point D is the circumcenter of $\triangle IBC$.

Proof. Simple angle chasing: observe that

$$\angle IBD = \frac{1}{2}\angle B + \frac{1}{2}\angle A, \quad \angle IDB = \angle C$$

implies

$$\angle BID = 180^\circ - \angle C - \frac{1}{2}\angle B - \frac{1}{2}\angle A = \frac{1}{2}\angle A + \frac{1}{2}\angle B,$$

so $\triangle BID$ is isosceles and $BD = ID$.

Similar logic (or just using $BD = CD$) shows that $BD = CD = ID$ too. □

Now, we have $\angle BD'D = \angle BAD = \frac{1}{2}\angle A$ and $\angle AFI = \angle D'BD = 90^\circ$, so $\triangle IAF \sim \triangle DD'B$. This means

$$\frac{AI}{FI} = \frac{D'D}{BD} \iff AI \cdot ID = AI \cdot BD = 2R \cdot r.$$

Since I is inside (ABC),

$$\text{Pow}_{(ABC)}(I) = -AI \cdot ID = -2Rr$$

as desired.

5.6 Solution 3.1, 2015 AIME II/15

Extend AB to meet $\mathcal{Q}$ at B'. Angle chasing gives $\angle BAC = 90^\circ$, so $\angle BCB' = 90^\circ$ as well with AC as the altitude from C; this means $AB \cdot AB' = AC^2$ implying $AC = 2AB$. This forces $AB = \frac{4}{\sqrt{5}}$.

Next, noting that $AE = 4AD$, we get

$$[BAD] = \frac{1}{2}(AB \cdot AD \cdot \sin\alpha) = \frac{1}{2}(AC \cdot AE \cdot \cos\alpha) = \frac{1}{2}(2AB \cdot 4AD \cdot \cos\alpha)$$

where $\angle BAD = \alpha$, so $\tan\alpha = 8$. This means $\sin\alpha = \frac{8}{\sqrt{65}}$ and $\cos\alpha = \frac{1}{\sqrt{65}}$.

By the law of sines, we now have

$$\frac{BD}{\sin\alpha} = 2 \implies BD = \frac{16}{\sqrt{65}}.$$

This means we can compute $AD = x$:

$$x^2 + \frac{16}{5} - \frac{8}{5\sqrt{13}}x = \frac{256}{65}$$

by the law of cosines, which yields $x = \frac{4}{\sqrt{13}}$. To finish, Heron's formula shows

$$[ABD] = \frac{1}{2} \cdot \frac{4}{\sqrt{13}} \cdot \frac{4}{\sqrt{5}} \cdot \frac{8}{\sqrt{65}} = \boxed{\frac{64}{65}}.$$

5.7 Solution 3.2, Fermat point

Let X be the point on the opposite side as A from BC such that $\triangle BXC$ is equilateral. Then, by Ptolemy,

$$PX \cdot BC \leq PB \cdot CX + PC \cdot BX.$$

Since $BC = BX = CX$, this means $PX \leq PB + PC$. Then, by triangle inequality,

$$PA + PB + PC \geq PA + PX \geq AX.$$

Equality occurs when P lies on (BCX) and AX, in which case

$$\angle BPX = \angle CPX = 60^\circ$$

because $BX = CX$ and $\angle BPC = 120^\circ$, allowing us to conclude $\angle APB = \angle APC = 120^\circ$ as well.

5.8 Solution 3.3, 2023 AIME I/12

The angle condition obviously shows that $AEPF$, $BFPD$, and $CDPE$ are cyclic. Moreover, the equilateral condition shows that $\angle DPE = \angle EPF = \angle FPD$ (seem familiar?).

Now, let X be the point on the opposite side of EF from D so that $\triangle EFX$ is equilateral. Since $\angle EAF = 60^\circ$, A, E, F, P, X are concyclic. We then see that

$$\angle XAF = \angle XEF = 60^\circ = \angle ABD,$$

so $AX \parallel CD$.

By Ptolemy, $AX + AE = AF$ and thus $AX = 15$. This means $AXDC$ is a trapezoid, and, since D, P, X are collinear due to P being the Fermat point of $\triangle DEF$, we want to compute $\tan(\angle XDC)$.

Letting M be the midpoint of BC, we have $AX = 15$, $MD = 48 - \frac{55}{2} = \frac{41}{2}$, and $AD = 55\sqrt{3}$, as AD is perpendicular to both bases of trapezoid $AXDM$. This means the "slope" of DX (pretending BC is the x-axis) is

$$\frac{\frac{55\sqrt{3}}{2}}{\frac{41}{2} - 15} = \frac{55\sqrt{3}}{11} = \boxed{5\sqrt{3}}.$$

(The actual AIME problem asked for the square of the tangent, so even if you got $-5\sqrt{3}$, you would've been fine; however, drawing the diagram accurately shows that $\angle AEP = \angle CDP$ is acute.)

5.9 Solution 3.4, IMO 2001/6

Expanding the given equation gives

$$a^2 - ac + c^2 = b^2 + bd + d^2.$$

Construct a quadrilateral $WXYZ$ with sides $WX = a$, $XY = c$, $YZ = b$, $ZX = d$, and

$$WY^2 = a^2 - ac + c^2 = b^2 + bd + d^2.$$

(You can easily verify that this does not violate the triangle inequality.) Then, by law of cosines, $\angle WXY = 60^\circ$ and $\angle WZY = 120^\circ$, so $WXYZ$ is cyclic.

Now, we compute the diagonal lengths directly in terms of a, b, c, d.

Theorem 5.3 (Diagonals of a cyclic quadrilateral)**.** The length of diagonal WY is

$$WY = \sqrt{\frac{(ab+cd)(ad+bc)}{ac+bd}}.$$

Proof. There are exactly three cyclic quadrilaterals whose side lengths are a permutation of (a, b, c, d); you can directly construct these by reflecting vertices of $WXYZ$ over the perpendicular bisector of the diagonal not containing the vertex.

These three quadrilaterals have congruent circumcircles, too, and thus, because congruent chords subtend congruent arcs, we can conclude that any two of the quadrilaterals share a diagonal length, so there are only three unique diagonal lengths among the quadrilaterals.

Therefore, let $t = WY$, $u = XZ$, and v the remaining diagonal length. By Ptolemy's theorem, we can set up the system

$$ab + cd = tu \tag{1}$$
$$ac + bd = uv \tag{2}$$
$$ad + bc = vt, \tag{3}$$

and then the theorem is obtained from multiplying (1) and (3) and dividing by (2). $\square$

This means
$$WY^2 = \frac{(ab + cd)(ad + bc)}{ac + bd} = a^2 - ac + c^2$$
is an integer, hence $ac + bd \mid (ab + cd)(ad + bc)$. By the rearrangement inequality,

$$ab + cd > ac + bd > ad + bc.$$

Assuming $ab + cd$ is prime for the sake of contradiction, we find that $ac + bd$ cannot be divisible by $ab + cd$ due to size, so $\gcd(ac + bd, ab + cd) = 1$, which means

$$ac + bd \mid ad + bc.$$

This, however, is impossible because $ac + bd > ad + bc$, so we have found a contradiction, meaning $ab + cd$ is indeed not prime.