New York City Interscholastic Mathematics League

Senior A Division

Spring 2025

Contest 1 Solutions

S25SA01 Find the smallest positive integer x for which

$$(3x)^{x^2} > (x+28)^{13x+30}$$

holds.

Answer: $\boxed{15}$

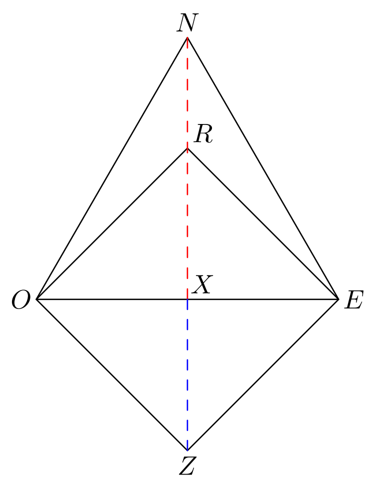
Solution: Setting bases equal, $3x = x + 28$ gives $x = 14$, and the corresponding exponents are 196 and 212. The inequality does not hold. Setting exponents equal, $x^2 = 13x + 30$ gives $x = 15$ as the only positive solution, and the corresponding bases are 45 and 43. The inequality does hold. Now, $1 \le x \le 14$ implies $3x \le x + 28$ and $x^2 \le 13x + 30$. Because of this, we can be sure that $\boxed{15}$ is the smallest positive integer for which the inequality holds.

Remark. The inequalities mentioned are clear algebraically, but we present an intuitive reasoning. The function $3x$ grows faster than $x + 28$, and the function x^2 grows faster than $13x + 30$. If we know that the faster functions do not catch up to the slower ones by $x = 14$, then they clearly don't catch up before then either, when $0 < x < 14$.

S25SA02 Regular polygons $ZERO$ and ONE are oriented in the plane so that R is inside ONE and $NZ = 2$. Find the area of $\triangle ORZ$.

Answer: $\boxed{4 - 2\sqrt{3}}$

Solution: Let the side length of square $ZERO$ be s, so that the side length of equilateral triangle ONE is $s\sqrt{2}$, since $\overline{OE}$ is a diagonal of the square. Let X be the intersection of $\overline{NZ}$ and $\overline{OE}$, noting that it is the center of the square.



Segment $\overline{NX}$ is a height of the equilateral triangle, meaning it has length $s\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \frac{s\sqrt{6}}{2}$. Segment $\overline{XZ}$ is half of a diagonal of the square, meaning it has length $\frac{s\sqrt{2}}{2}$. Then, $NZ = NX + XZ = 2$ implies that $s\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\right) = 2$, and solving gives $s = \frac{4}{\sqrt{6}+\sqrt{2}} = \sqrt{6} - \sqrt{2}$. Finally, since the area of $\triangle ORZ$ is half that of the square, our answer is simply $\frac{1}{2}s^2 = \frac{1}{2} \cdot \left(\sqrt{6} - \sqrt{2}\right)^2 = \boxed{4 - 2\sqrt{3}}$.

S25SA03 Gi-hun has a box containing nine balls, three of which are red, three of which are yellow, and three of which are green. He removes one ball at a time from the box, without replacement. What is the probability that the first and last balls he removes are the same color?

Answer: $\boxed{\frac{1}{4}}$

Solution: No matter what color ball Gi-hun removes first, there will be two of that color and three of the other colors left. Then, the probability that the last ball removed is the same color is simply $\frac{2}{2+3+3} = \boxed{\frac{1}{4}}$.

Remark. The reason why this solution is valid is because Gi-hun is essentially picking an arrangement of the nine balls in a line, uniformly at random, where the ordering along the line determines when each ball is removed. For example, the leftmost ball is removed first and the rightmost ball is removed last. Then, fixing the leftmost ball, we can consider the rightmost ball without fixing any of the balls in between, making the answer $\frac{2}{2+3+3} = \frac{1}{4}$.

We can also consider fixing both the leftmost and rightmost balls at the same time, where there are $\binom{9}{2} = 36$ total ways to choose them and $3 \cdot \binom{3}{2} = 9$ ways where they are the same color. The answer is then $\frac{9}{36} = \frac{1}{4}$.

Most rigorously, out of the $\frac{9!}{3! \cdot 3! \cdot 3!} = 1680$ total arrangements, there are 3 ways to choose the common color and $\frac{7!}{3! \cdot 3! \cdot 1!} = 140$ ways to choose the balls in between for 420 valid arrangements. The answer is then $\frac{420}{1680} = \frac{1}{4}$, as desired.

S25SA04 Find the smallest positive integer n for which the leftmost digits of n, n^2, and n^3 are all equal to the same digit $d \neq 1$.

Answer: $\boxed{97}$

Solution: If n's leftmost digit is d, and has k other digits, then it lies in the interval $\left[d \cdot 10^k, (d+1) \cdot 10^k\right)$. Its square lies in the interval $\left[d^2 \cdot 10^{2k}, (d+1)^2 \cdot 10^{2k}\right)$, and, as a result, has the same leftmost digit as one of the integers in the interval $\left[d^2, (d+1)^2\right)$.

d	d^2	$(d+1)^2$	Possible leftmost digits of n^2
2	4	9	$4, 5, 6, 7, 8$
3	9	16	$9, 1$
4	16	25	$1, 2$
5	25	36	$2, 3$
6	36	49	$3, 4$
7	49	64	$4, 5, 6$
8	64	81	$6, 7, 8$
9	81	100	$8, 9$

The only valid digits are 8 and 9, but 8 doesn't work because $\left[8^3, 9^3\right) = [512, 729)$, meaning n^3 could only have leftmost digit 5, 6, or 7. Indeed, 9 does work, as $\left[9^3, 10^3\right) = [729, 1000)$ allows for a leftmost digit of 9.

Now, $n = 9$ clearly does not satisfy the conditions, so we look to two-digit numbers starting with 9. Through some trial and error, we find that $97^3 > 900000 > 96^3$, so our answer is $\boxed{97}$.

Remark. The final computation aims to find $\left\lceil \sqrt[3]{900000} \right\rceil$, which can be achieved in only three or four cubings using binary search. For example, after noting $95^3 < 900000$, we know the answer must be between 96 and 99 inclusive (or more than two digits, though we now know that the answer is in fact two digits). Then, checking 96, 97, and possibly 98 suffices.

S25SA05 Find the smallest odd, composite, positive integer that is relatively prime to 2025. (Two integers are relatively prime to each other if they do not share any factors other than 1.)

Answer: $\boxed{49}$

Solution: The answer cannot be divisible by 2, 3, or 5 because it is odd and relatively prime to $2025 = 3^4 \cdot 5^2$. And, since it is composite, it must be the product of two or more (not necessarily distinct) primes, the smallest of which is 7. The answer is then clearly $7^2 = \boxed{49}$.

S25SA06 On each edge of a square, five points are drawn to partition the edge into six congruent segments. Let S be the set containing these twenty points and the vertices of the square. How many ways are there to choose four distinct points from S such that they form a parallelogram with positive area?

Answer: $\boxed{206}$

Solution: Recall that a quadrilateral having one pair of congruent, parallel sides is necessary and sufficient criteria for that quadrilateral to be a parallelogram. Thus, if we pick two segments of equal length on opposite sides of the square, those vertices form a parallelogram.

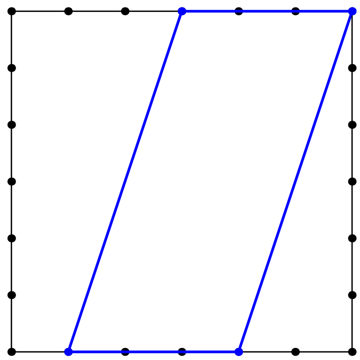


Figure 1: Example of a parallelogram coinciding with some sides of the square

There are 2 choices for the pair of opposite sides (top-bottom or left-right). Then, if the common length is l, there are $6 - l + 1$ ways to choose the segment on each side, for $(7 - l)^2$ ways in total. This gives $6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = \frac{6 \cdot 7 \cdot 13}{6} = 91$ choices of parallelograms. However, the case where the quadrilateral is exactly the square is double-counted, since it coincides with all four sides, so we subtract 1 to get a total of $2 \cdot 91 - 1 = 181$ quadrilaterals in this case.

This covers all quadrilaterals where at least one pair of opposite sides coincides with sides of the square. We now consider parallelograms with no sides parallel to any sides of the square, meaning none of the vertices of the square are chosen, and one point is chosen from each side of the square.

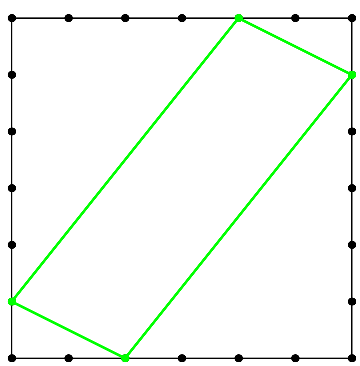


Figure 2: Example of a parallelogram not coinciding with any sides of the square

Again, because we need parallel and congruent sides, fixing the top and left vertices uniquely determines the right and bottom vertices, which are simply rotations of the top and left vertices around the center of the square. This adds an additional $5^2 = 25$ quadrilaterals for a total of $\boxed{206}$.

Contest 2 Solutions

S25SA07 Aditya is flipping a fair coin. After each flip, if the number of heads flipped so far is equal to the number of tails flipped so far, Aditya earns one dollar. Find the probability that Aditya earns exactly one dollar after four flips.

Answer: $\boxed{\frac{3}{8}}$

Solution: We casework on when Aditya earns the dollar.

If Aditya earns the dollar after the second flip, then he must have flipped one head and one tail in total by then, which occurs with probability $\frac{1}{2}$. To avoid earning a second dollar after the fourth flip, both the third and fourth flips must be the same, which occurs with probability $\frac{1}{2}$. The total probability in this case is $\frac{1}{4}$.

If Aditya earns the dollar after the fourth flip, then he needs to have flipped two heads and two tails by the end. To avoid earning a dollar after the second flip, the first two flips must be the same, and the last two flips must be the other outcome. This occurs with $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ probability.

Since it is impossible to earn a dollar after an odd number of flips, the answer is $\frac{1}{4} + \frac{1}{8} + \boxed{\frac{3}{8}}$ as desired.

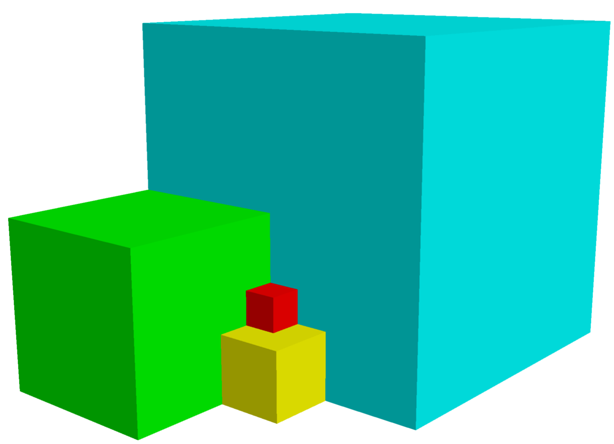
S25SA08 Noam has four cubes, which have side lengths 1, 2, 5, and 10. He wishes to glue them together to form a single solid. What is the minimum possible surface area of the resulting solid?

Answer: $\boxed{708}$

Solution: The minimization of surface area comes from maximizing surface area overlap when gluing together the cubes. For two cubes, the most optimal configuration is to glue one face of the smaller cube entirely onto a face of the larger cube, which "removes" twice the area of the smaller cube's face. If we assume that this happens for every pair of cubes, then the theoretical minimum surface area is

$$6(1^2 + 2^2 + 5^2 + 10^2) - 2(1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 5^2) = \boxed{708}.$$

We now show a construction where every pair of cubes overlaps.

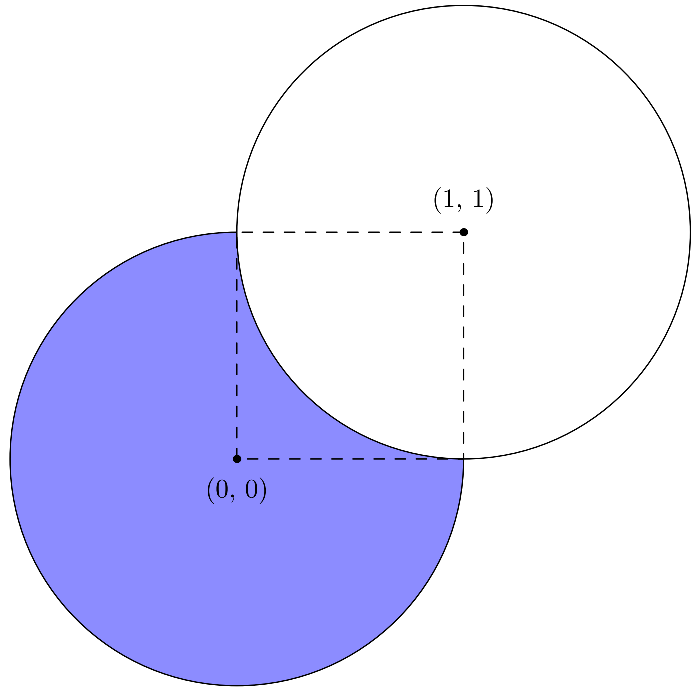


S25SA09 Find the area of the set of all points (x, y) satisfying both of the following equations:

- $x^2 + y^2 \leq 1$
- $(x-1)^2 + (y-1)^2 \geq 1$

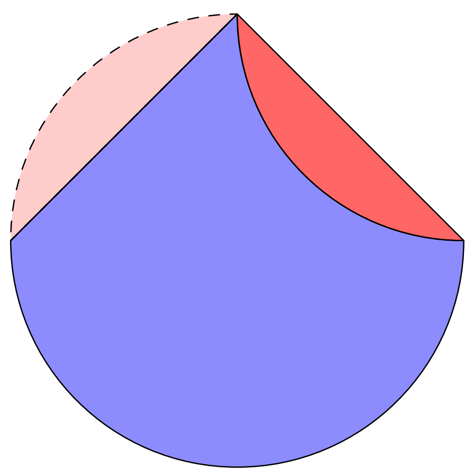
Answer: $\boxed{1 + \frac{\pi}{2}}$

Solution: The first equation represents the disk centered at the origin with radius 1, and the second equation represents the complement of the disk centered at $(1, 1)$ with radius 1. Taking the intersection essentially "chops off" the part of the second disk that overlaps with the first.



We aim to find the area of the blue region. Note that the corresponding circles intersect at $(1, 0)$ and $(0, 1)$, so we draw the unit square. The blue area inside the square is $1^2 - \frac{1}{4}1^2\pi = 1 - \frac{1}{4}\pi$, and the blue area outside the square is $\frac{3}{4}1^2\pi = \frac{3}{4}\pi$, so adding gives $\boxed{1 + \frac{\pi}{2}}$ as desired.

Alternatively, to find the area, we can move a segment of the circle to fill in the curved section on the right:



This is the union of an isosceles right triangle with side length $\sqrt{2}$ and a semicircle of radius 1, which again gives $1 + \frac{\pi}{2}$ as the area.

S25SA10 Find the sum of all primes dividing $12^5 + 13$.

Answer: $\boxed{479}$

Solution: Let $x = 12$. We would like to factor $x^5 + x + 1$. We can rewrite this as

$$\begin{aligned} x^5 + x + 1 &= x^5 + x^4 + x^3 + x^2 + x + 1 - (x^4 + x^3 + x^2) \\ &= \frac{x^6 - 1}{x - 1} - x^2 \cdot \frac{x^3 - 1}{x - 1} \\ &= \left(\frac{x^3 - 1}{x - 1}\right)\left(x^3 + 1 - x^2\right) \\ &= (x^2 + x + 1)(x^3 - x^2 + 1). \end{aligned}$$

Substituting back in, we get $12^5 + 13 = (12^2 + 12 + 1)(12^3 - 12^2 + 1) = 157 \cdot 1585$. Since $1585 = 5 \cdot 317$, and both 157 and 317 are prime, the answer is $5 + 157 + 317 = \boxed{479}$.

Remark. Since the exponents of the expression, 5, 1, and 0, span all remainders mod 3, it follows that the expression is divisible by $x^2 + x + 1$. Generally, a divisor of the form $1 + x + x^2 + \cdots + x^{n-1}$ allows for reduction of exponents mod n, since the divisor is a factor of $x^n - 1$.

S25SA11 Seven students took the AMC 12, and the average of their scores was 103.5. If the average of the lowest four scores was 81 and the average of the highest four scores was 123, find the median score.

Answer: $\boxed{91.5}$

Solution: Let the scores be $a_1, a_2, \ldots a_7$ in nondecreasing order. We are given that

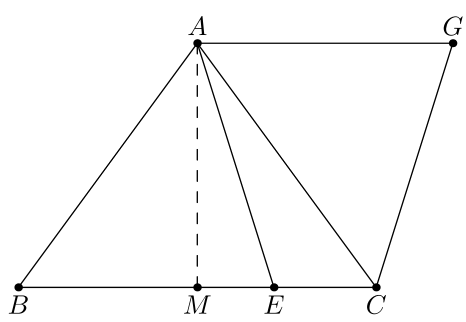
$$\begin{aligned} \frac{a_1 + a_2 + \cdots + a_7}{7} = 103.5 &\implies a_1 + a_2 + \cdots + a_7 = 7 \cdot 103.5 = 724.5 \\ \frac{a_1 + a_2 + a_3 + a_4}{4} = 81 &\implies a_1 + a_2 + a_3 + a_4 = 4 \cdot 81 = 324 \\ \frac{a_4 + a_5 + a_6 + a_7}{4} = 123 &\implies a_4 + a_5 + a_6 + a_7 = 4 \cdot 123 = 492 \end{aligned}$$

so adding the second and third results and subtracting the first tells us that the median a_4 is equal to $324 + 492 - 724.5 = \boxed{91.5}$.

S25SA12 Let $\triangle ABC$ be a triangle with $AB = AC$. Point E lies on $\overline{BC}$ such that $AE = BE$, and segment $\overline{BE}$ is rotated about A such that B coincides with C and E gets sent to point G. Given that the areas of $\triangle ABG$ and $\triangle ABC$ are 5 and 7 respectively, compute $\cos(\angle BAC)$.

Answer: $\boxed{\frac{3}{10}}$

Solution: Below is a diagram for reference.



Observe that $\triangle BEA$ and $\triangle BAC$ are similar because they are isosceles triangles sharing the same base angle $\angle B$ and that $\triangle AGC \cong \angle AEB$ because they are rotations of one another; in particular, $\triangle GAC \sim \triangle ACB$. Then, we can see that $\angle GAC$ and $\angle ACB$, which proves that $\overline{AG} \parallel \overline{BC}$.

Moreover, $GA = AE = BE$, so $\overline{AG}$ and $\overline{BE}$ are parallel and congruent, meaning they determine a parallelogram. This lets us use the information about the area of $\triangle ABG$: its area is half that of parallelogram $ABEG$, so the area is equal to that of $\triangle ABE$ as well. Therefore,

$$\frac{BE}{BC} = \frac{[ABE]}{[ABC]} = \frac{5}{7}.$$

This means there is a real number λ so that $BE = 5\lambda$ and $BC = 7\lambda$. We can make use of $\triangle BEA \sim \triangle BAC$ again to compute

$$\frac{BE}{BA} = \frac{BA}{BC} \iff 35\lambda^2 = BE \cdot BC = BA^2,$$

so $BA = \lambda\sqrt{35}$.

Now, we have all the side lengths of $\triangle ABC$, so the remaining part of the problem is a simple trigonometric calculation. If M is the midpoint of BC, then

$$\sin(\angle BAC/2) = \frac{BM}{BA} = \frac{\frac{7\lambda}{2}}{\lambda\sqrt{35}} = \sqrt{\frac{7}{20}},$$

so $\cos(\angle BAC/2) = \sqrt{\frac{13}{20}}$, and to finish we can apply any of the various double angle formulas for cosine to obtain

$$\cos(\angle BAC) = 2\cos^2(\angle BAC/2) - 1 = \frac{13}{10} - 1 = \boxed{\frac{3}{10}}.$$

Contest 3 Solutions

S25SA13 Aditya sees a bowl with 2025 candies and decides to take x of them. Mark, who has a sweet tooth, takes x^2 candies from the bowl. Aditya is horrified, and steals half of Mark's candies. Finally, Mark takes the rest of the candies in the bowl and says to Aditya, "Nice try, but I have one more candy than you." Find the value of x.

Answer: $\boxed{44}$

Solution: Before Aditya steals Mark's candies, Aditya has x candies, and Mark has x^2 candies. Afterwards, Aditya and Mark have $x + \frac{x^2}{2}$ and $\frac{x^2}{2}$ candies respectively. We know that, after Mark takes the rest of the candies, he must have one more than Aditya, or $x + \frac{x^2}{2} + 1$. Now, the bowl is empty, so the sum of Aditya's and Mark's candy counts is

$$\begin{aligned}\left(x + \frac{x^2}{2}\right) + \left(x + \frac{x^2}{2} + 1\right) &= 2025\\ x^2 + 2x + 1 &= 2025\\ (x+1)^2 &= 45^2\\ x + 1 &= 45\end{aligned}$$

So $x = \boxed{44}$.

S25SA14 Thomas the tank engine is traveling at 60 mph down a straight track from point A to point B, starting at A. The distance between A and B is 5 miles, but there are four junctions in between them at 1, 2, 3, and 4 miles along the way. Each junction contains an "in" portal facing towards A, and an "out" portal facing towards B. Each time Thomas enters an "in" portal, he teleports randomly and uniformly to one of the "out" portals. What is the probability that Thomas is able to reach B within five minutes?

Answer: $\boxed{\frac{175}{256}}$

Solution: Thomas can only reach B in time if he is teleported to the fourth "out" portal within the first four minutes (since he also needs one minute to travel between the fourth and fifth mile markers). Since every "in" portal entrance is 1 minute apart, and Thomas comes out of each "out" portal with equal probability, the chance that he does **not** get to the fourth "out" portal within four minutes is $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$, meaning that our desired probability is the complement, or $1 - \frac{81}{256} = \frac{175}{256}$.

S25SA15 A sequence is defined by $a_1 = \frac{37}{71}$ and $a_n = \frac{1}{1-a_{n-1}}$ for all $n \geq 2$. Find the value of a_{2025}.

Answer: $\boxed{-\frac{34}{37}}$

Solution: Inspection reveals that the sequence is periodic with period 3. The computation can be streamlined by letting $a_1 = \frac{p}{q}$, where $p = 37$ and $q = 71$, so that

$$\begin{aligned} a_2 &= \frac{1}{1-a_1} = \frac{1}{1-\frac{p}{q}} = \frac{1}{\frac{q-p}{q}} = \frac{q}{q-p} \\ a_3 &= \frac{1}{1-a_2} = \frac{1}{1-\frac{q}{q-p}} = \frac{1}{\frac{-p}{q-p}} = \frac{p-q}{p} \\ a_4 &= \frac{1}{1-a_3} = \frac{1}{1-\frac{p-q}{p}} = \frac{1}{\frac{q}{p}} = \frac{p}{q} \end{aligned}$$

so $a_4 = a_1$. By the recursive definition, $a_{2025} = a_{2022} = \cdots = a_3 = \frac{p-q}{p} = \frac{37-71}{37} = \boxed{-\frac{34}{37}}$.

Alternatively, since $a_1 < 1$ we can substitute $a_1 = \cos^2\theta$. Then by trigonometric identities, we will quickly get $a_2 = \frac{1}{\sin^2\theta}$ and $a_3 = -\tan^2\theta$. Thus $a_4 = \cos^2\theta$. Note that $\sin^2\theta = \frac{71-37}{71} = \frac{34}{71}$. So $a_{2025} = a_3 = -\frac{34/71}{37/71} = -\frac{34}{71}$.

S25SA16 Jun-ho and 455 other people are playing one round of the Octopus Game. In this game, a positive integer factor of 456 is chosen uniformly at random, and that many people are eliminated uniformly at random. Find the probability that Jun-ho remains.

Answer: $\boxed{\frac{127}{152}}$

Solution: We compute the complement of the answer. If the factor chosen is d, then Jun-ho has a $\frac{d}{456}$ probability of being eliminated. Summing over all d, making sure to divide by the number of possible values of d, we get that the probability that Jun-ho is eliminated is

$$\frac{1}{\sum_{d|456} 1} \cdot \sum_{d|456} \frac{d}{456} = \frac{1}{\tau(456)} \cdot \frac{\sigma(456)}{456}$$

where $\tau(n)$ denotes the number of positive integer factors of n, and $\sigma(n)$ denotes the sum of the positive integer factors of n. Our desired probability is then

$$1 - \frac{\sigma(456)}{456\tau(456)}.$$

Since $456 = 2^3 \cdot 3 \cdot 19$, we have

$$\tau(456) = (3+1)(1+1)(1+1) = 16 = 2^4$$

and

$$\sigma(456) = (1+2+2^2+2^3)(1+3)(1+19) = 15 \cdot 4 \cdot 20 = 1200 = 2^4 \cdot 3 \cdot 5^2$$

for a final answer of

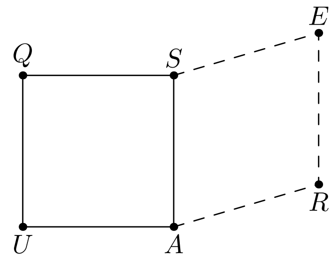
$$1 - \frac{1200}{456 \cdot 16} = 1 - \frac{2^4 \cdot 3 \cdot 5^2}{2^3 \cdot 3 \cdot 19 \cdot 2^4} = 1 - \frac{5^2}{2^3 \cdot 19} = 1 - \frac{25}{152} = \boxed{\frac{127}{152}}.$$

S25SA17 Equilateral hexagon $SQUARE$ with side length 4 has all interior angles measuring less than $180°$. If four of its vertices are the vertices of a square, find its area.

Answer: $\boxed{16+8\sqrt{3}}$

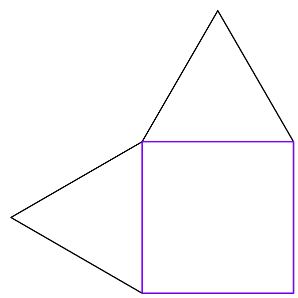
Solution: The two vertices in the hexagon not forming the square must be opposite each other. To see this, we first note that at least two of the edges of the square must coincide with those of the hexagon.

If three edges of $SQUARE$ are part of the square, we can label the vertices so that $SQUA$ is a square.

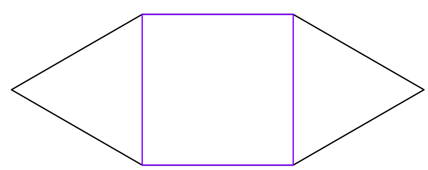


Then, rhombus $ARES$ has either $\angle SAR \geq 90°$ or $\angle ASE \geq 90°$, which implies that either $\angle UAR$ or $\angle QSE$ has measure at least $180°$, contradiction.

Therefore only two edges of the square coincide with those of $SQUARE$. If these edges share an endpoint, then $SQUARE$ is a square with two equilateral triangles coming off two adjacent edges of the square, which we can see is not convex.



This means that the hexagon is made up of a square and two equilateral triangles, each having side length 4.



Thus, the total area is

$$4^2 + 2\left(\frac{4^2 \cdot \sqrt{3}}{4}\right) = \boxed{16+8\sqrt{3}}.$$

S25SA18 Compute

$$\frac{23}{24}+\frac{26}{60}+\frac{29}{120}+\frac{32}{210}+\frac{35}{336}+\frac{38}{504}+\frac{41}{720}+\frac{44}{990}$$

Answer: $\boxed{\frac{31}{15}}$

Solution: Note that the numerators form an arithmetic sequence, and the denominators can be expressed as the product of three consecutive integers, so we are trying to find the value of

$$\sum_{n=2}^{9} \frac{3n+17}{n(n+1)(n+2)}.$$

This expression can be decomposed into $\frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$ for real numbers A, B, and C. There are many ways to find the values of A, B, and C; we will multiply through and utilize the resulting polynomial identity.

$$\frac{3n+17}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$
$$3n+17 = A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)$$

Note that, although the first equation does not hold for $n = 0, -1, -2$, it does hold for an infinite number of values, so the second equation must be a polynomial identity and therefore does hold for $n = 0, -1, -2$.

Plugging in $n = 0$, we have $17 = A(1)(2) + 0 + 0$ so $A = \frac{17}{2}$.

Plugging in $n = -1$, we have $14 = 0 + B(-1)(1) + 0$, so $B = -14$.

Plugging in $n = -2$, we have $11 = 0 + 0 + C(-2)(-1)$, so $C = \frac{11}{2}$.

Since $A + B + C = 0$, this summation telescopes.

$$\begin{aligned}
\sum_{n=2}^{9} \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} &= \sum_{n=2}^{9} \frac{A}{n} + \sum_{n=3}^{10} \frac{B}{n} + \sum_{n=4}^{11} \frac{C}{n} \\
&= \left[\frac{A}{2} + \frac{A}{3} + \sum_{n=4}^{9} \frac{A}{n}\right] + \left[\frac{B}{3} + \frac{B}{10} + \sum_{n=4}^{9} \frac{B}{n}\right] + \left[\frac{C}{10} + \frac{C}{11} + \sum_{n=4}^{9} \frac{C}{n}\right] \\
&= \frac{A}{2} + \frac{A+B}{3} + \frac{B+C}{10} + \frac{C}{11} + \sum_{n=4}^{9} \frac{A+B+C}{n} \\
&= \frac{\frac{17}{2}}{2} + \frac{\frac{17}{2} - 14}{3} + \frac{-14 + \frac{11}{2}}{10} + \frac{\frac{11}{2}}{11} + 0 \\
&= \frac{17}{4} - \frac{11}{6} - \frac{17}{20} + \frac{1}{2} = \boxed{\frac{31}{15}}.
\end{aligned}$$

Contest 4 Solutions

S25SA19 In the following addition problem, all letters are digits from 0 to 9, inclusive, and $S \neq 0$.

$$\begin{array}{rcccc}
 & S & T & A & N \\
 & & S & T & A \\
 & & & S & T \\
+ & & & & S \\
\hline
 & 3 & 2 & 3 & 8
\end{array}$$

Find the value of the four-digit number $STAN$.

Answer: $\boxed{2916}$

Solution: We can rewrite the addition problem as $SSSS + TTT + AA + N = 3238$. It then becomes clear that $S \leq 1$ is too small and $S \geq 3$ is too large, so $S = 2$. Subtracting away, we get $TTT + AA + N = 3238 - 2222 = 1016$. It is clear that $T \leq 8$ is too small, so $T = 9$. Subtracting away, we get $AA + N = 1016 - 999 = 17$. Then, $A = 1$ and $N = 6$, so $STAN = \boxed{2916}$.

S25SA20 Find the smallest positive integer that ends in 9999 and is divisible by 97.

Answer: $\boxed{539999}$

Solution: Let the answer be $10000x-1$. We have that

$$\begin{aligned}10000x-1 &\equiv 0 \pmod{97}\\ 9x &\equiv 1 \pmod{97}\\ x &\equiv 9^{-1} \pmod{97}\end{aligned}$$

Since 9 is small, we can find 9^{-1} by checking which k satisfies $9 \mid 97k+1$. In this case, $k=5$ works, so

$$\begin{aligned}9\cdot 9^{-1} &\equiv 1 \pmod{97}\\ 9\cdot 9^{-1} &\equiv 1+5\cdot 97 \pmod{97}\\ 9\cdot 9^{-1} &\equiv 486 \pmod{97}\\ 9^{-1} &\equiv 54 \pmod{97}\end{aligned}$$

This means that the smallest positive integer solution is $x=54$, and our answer is $\boxed{539999}$.

S25SA21 Gary the grasshopper is at 0 on the number line. Gary can travel in either direction along the number line in two ways: short jumps of length 5, and long jumps of length 17. Find the smallest number of jumps Gary must make in order to land on 2025.

Answer: $\boxed{123}$

Solution: Clearly, it is not optimal to jump the same length in both directions, as this achieves no net movement, so all short jumps are in the same direction, and all long jumps are in the same direction. Also, since 2025 is large, all long jumps should be rightward, as we can cover more distance with less jumps. We now only have two cases.

If all short jumps are also rightward, then we have $17x+5y=2025$ for non-negative integers x and y, and we are trying to minimize $x+y$. To do so, we need to maximize the distance we can cover with long jumps, so we want to maximize x. Note that, since $5y$ and 2025 are multiples of 5, so is $17x$. The largest multiple of 85 near 2025 is $1955 = 17\cdot 115$, so $x=115$ and $y=\frac{2025-1955}{5}=14$ gives $x+y=129$ in this case.

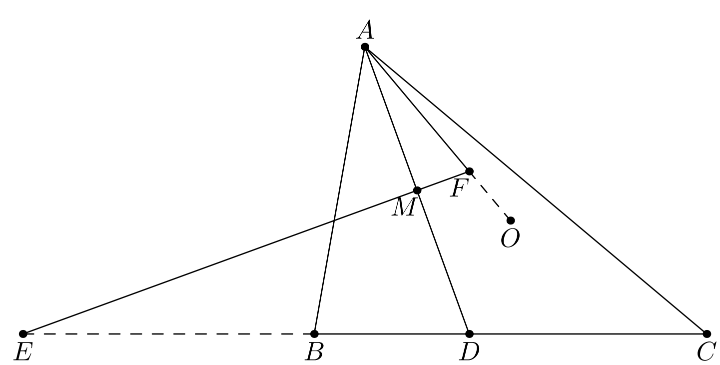
If all short jumps are leftward, then we have $17x-5y=2025$ for non-negative integers x and y. In this case, to minimize $x+y$, we want to minimize the amount of distance we need to backtrack, so we want to minimize y. Rewriting as $17x=5y+2025$, we note that the smallest multiple of 85 at least 2025 is $2040=17\cdot 120$, so $x=120$ and $y=\frac{2040-2025}{5}=3$ gives $x+y=123$ in this case.

The minimum across both cases is $\boxed{123}$ jumps.

S25SA22 Triangle $\triangle ABC$ has circumcenter O. The bisector of $\angle BAC$ intersects $\overline{BC}$ at D, and the perpendicular bisector of $\overline{AD}$ intersects $\overline{BC}$ at E and $\overline{AO}$ at F. If $AD=20$ and $DE=25$, compute the length of $\overline{AF}$.

Answer: $\boxed{\frac{50\sqrt{21}}{21}}$

Solution: Diagram:



Suppose without loss of generality that $AB \le AC$, as in the picture (otherwise we can swap the labels of B and C without changing the diagram), and let M be the midpoint of $\overline{AD}$. The key observation that we will prove is $\triangle AMF \sim \triangle EMD$. This follows from a fairly straightforward but lengthy angle chase: on one hand,

$$\angle MED = 90^\circ - \angle MDE = 90^\circ - (180^\circ - \angle ABD - \angle BAD) = \angle B + \frac{1}{2}\angle A - 90^\circ,$$

but on the other hand,

$$\begin{aligned}\angle MAF = \angle BAO - \angle BAD &= (90^\circ - \angle C) - \frac{1}{2}\angle A \\ &= (180^\circ - \angle C) - \frac{1}{2}\angle A - 90^\circ \\ &= \angle A + \angle B - \frac{1}{2}\angle A - 90^\circ \\ &= \angle B + \frac{1}{2}\angle A - 90^\circ\end{aligned}$$

where $\angle BAO = 90^\circ - \angle C$ because $\angle AOB = 2\angle C$ by the inscribed angle theorem and $\triangle BAO$ is isosceles. Thus $\angle MAF = \angle MED$ and $\angle AMF = \angle EMD = 90^\circ$, so the aforementioned pair of triangles is similar.

To finish, we can compute

$$EM = \sqrt{DE^2 - DM^2} = \sqrt{25^2 - 10^2} = 5\sqrt{21}$$

and so

$$\frac{AF}{AM} = \frac{ED}{EM} \iff AF \cdot 5\sqrt{21} = 25 \cdot 10 \iff AF = \boxed{\frac{50\sqrt{21}}{21}}.$$

S25SA23 In acute $\triangle ABC$, D is the foot of the altitude from A to $\overline{BC}$. If $AB = 9$, $BD = 1$, and $CD = 8$, find AC.

Answer: $\boxed{12}$

Solution: Since $\triangle ABC$ is acute, D is between B and C, so we have $AD^2 + BD^2 = AB^2$ and $AD^2 + CD^2 = AC^2$ by the Pythagorean Theorem. Plugging in known values, we get $AD^2 = AB^2 - BD^2 = 9^2 - 1^2 = 80$ from the first equation and $AC^2 = 80 + 8^2 = 144$ from the second, so $AC = \boxed{12}$.

S25SA24 Find the sum of all positive integers $x < 10^4$ that have a digit sum of 7.

Answer: $\boxed{233310}$

Solution: Since $7 < 10$, we can model every x as 7 stars and 3 bars, where the bars separate the stars into four (possibly empty) regions. We can have the leftmost region represent the thousands place, the second region represent the hundreds place, and so on. For example, $* \mid\mid ** \mid * * **$ would represent 1024, and $\mid ** \mid * * * * * \mid$ would represent 250. It then follows that there are $\binom{10}{3}$ such integers.

By symmetry, when summing over all x, the digit sum of 7 is equally distributed across all four digits, so each x contributes $\frac{7}{4} \cdot 1111$ to the sum, on average. Then, our answer is $\binom{10}{3} \cdot \frac{7}{4} \cdot 1111 = \boxed{233310}$.

Remark. The aforementioned symmetry may seem slightly unfounded, so we provide both an intuition and a formal proof. Intuitively, the "digit sum" condition does not discriminate between which digits contribute to the sum, so it wouldn't make sense for any digit to be weighted more than another. Formally, consider any $x_1 = 1000a + 100b + 10c + d$ and group it together with its cyclic shifts $x_2 = 1000b + 100c + 10d + a$, $x_3 = 1000c + 100d + 10a + b$, and $x_4 = 1000d + 100a + 10b + c$.

These groups of four are pairwise disjoint, since if $x \sim z$ and $y \sim z$ then $x \sim y$, where $x \sim y$ iff x and y are cyclic shifts of one another (in jargon, the groups of four are equivalence classes under the equivalence relation $\sim$). Clearly, the average within a group is $\frac{1}{4} \cdot 1111(a + b + c + d) = \frac{7}{4} \cdot 1111$.

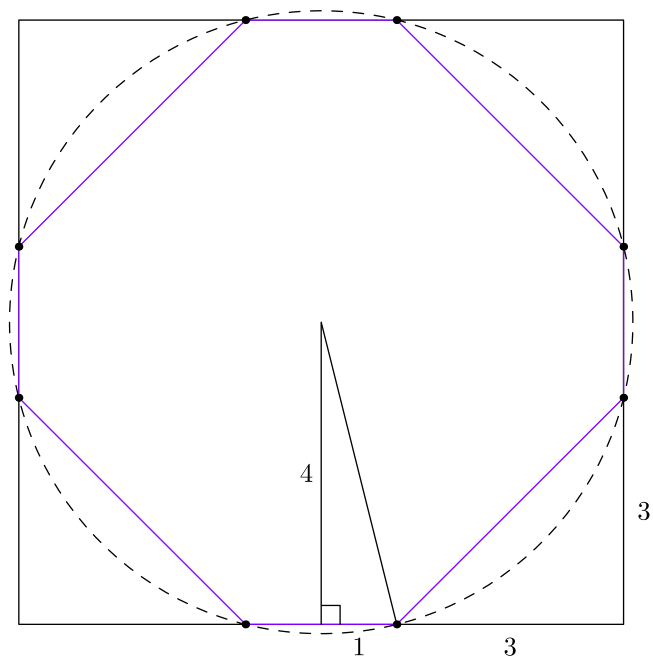
Note. In this specific case, $\gcd(4, 7) = 1$ implies that x_1, x_2, x_3, and x_4 are guaranteed to be pairwise distinct within every group, which means that the average of each group counts every x once. This, however, need not hold. If the cyclic shifts of some general x repeat with period p smaller than the digit sum, we can simply stop before the repeat and only group the p distinct cyclic shifts together. This guarantees counting every distinct x once. It is also true that these smaller groups still have the same average because each digit is counted $1/p$ as many times but there are only $1/p$ times as many numbers being summed, so the factors balance each other out. (One still needs to check that there is no imbalance in how frequently the digits occupy all the possible place values due to cutting off the shifting early, but we leave that as an exercise for the reader.)

Contest 5 Solutions

S25SA25 A square with side length 8 and a circle have the same center. The circle intersects the sides of the square at 8 distinct points, which are the vertices of an octagon with area 46. What is the radius of the circle?

Answer: $\boxed{\sqrt{17}}$

Solution: By symmetry, the octagon is formed by taking the square and chopping off congruent isosceles right triangles off all four corners. The total area of these four triangles is $64 - 46 = 18$, so the area of one is $\frac{18}{4} = \frac{9}{2}$, which means that if s is the leg length of one of the triangles, then $\frac{s^2}{2} = \frac{9}{2}$ implies $s = 3$.



We now note that the circle's radius is the hypotenuse of a right triangle with legs of length 4 and $4 - 3 = 1$, so its length is $\sqrt{4^2 + 1^2} = \boxed{\sqrt{17}}$.

S25SA26 The outer surface of a white $3 \times 3 \times 3$ cube is painted red. Then, the cube is split into 27 unit cubes, one of which is chosen uniformly at random and rolled. Given that the chosen unit cube's top face is white, what is the probability that it has exactly one red face?

Answer: $\boxed{\frac{5}{18}}$

Solution: We aim to find

$$\frac{P(\text{top face is white AND has exactly 1 red face})}{P(\text{top face is white})}.$$

The only unit cubes with exactly one red face are the ones in the center of each face of the $3 \times 3 \times 3$ cube, so there are 6 out of 27 total. The probability of rolling a white face on any cube with exactly 1 red face is $\frac{5}{6}$, so the desired numerator is $\frac{6}{27} \cdot \frac{5}{6} = \frac{5}{27}$. Also note that the act of choosing a unit cube uniformly at random and rolling it is equivalent to choosing a face out of all unit cubes uniformly at random. Out of the $6 \cdot 3^3 = 162$

faces of the individual unit cubes, $6 \cdot 3^2 = 54$ of them were painted red, so the probability of choosing a white face is $1 - \frac{54}{162} = \frac{2}{3}$, our desired denominator. The answer is then $\frac{\frac{5}{27}}{\frac{2}{3}} = \boxed{\frac{5}{18}}$.

S25SA27 Let r and s be the zeroes of $x^2 + 5x + 1$. Find $\left(r + \frac{1}{r}\right)^3 + \left(r - \frac{1}{r}\right)^3 + \left(s + \frac{1}{s}\right)^3 + \left(s - \frac{1}{s}\right)^3$.

Answer: $\boxed{-250}$

Solution: By Vieta's formulas, $rs = 1$, so $\frac{1}{r} = s$ and $\frac{1}{s} = r$, and $r + s = -5$. This allows us to compute

$$\begin{aligned}
&\left(r - \frac{1}{r}\right)^3 + \left(s - \frac{1}{s}\right)^3 + \left(r + \frac{1}{r}\right)^3 + \left(s + \frac{1}{s}\right)^3 \\
&= (r-s)^3 + (s-r)^3 + (r+s)^3 + (s+r)^3 \\
&= (r-s)^3 - (r-s)^3 + (-5)^3 + (-5)^3 \\
&= 0 + 2(-125) = \boxed{-250}.
\end{aligned}$$

Alternatively, one could expand the desired expression to get $2r^3 + 2s^3 + \frac{6}{r} + \frac{6}{s} = 2(r^3 + s^3) + 6\left(\frac{1}{r} + \frac{1}{s}\right)$. The sum $\frac{1}{r} + \frac{1}{s}$ evaluates to $\frac{r+s}{rs} = \frac{-5}{1} = -5$, and to find the sum of the cubes of the roots, we factor to get

$$\begin{aligned}
r^3 + s^3 &= (r+s)(r^2 - rs + s^2) \\
&= (-5)((r+s)^2 - 3rs) \\
&= (-5)((-5)^2 - 3) = -140.
\end{aligned}$$

Thus, the answer is $2(-140) + 6(-5) = -250$.

S25SA28 Find the largest integer that is equal to twice the sum of the squares of its digits.

Answer: $\boxed{298}$

Solution: The answer cannot have more than three digits, since

$$10^{n-1} > 2 \cdot 9^2 \cdot n$$

holds when $n \geq 4$, where the left-hand side is the smallest positive n-digit number and the right-hand side is twice the largest possible sum of squares of digits for an n-digit number (achieved when all digits are equal to 9). Therefore, we only need to consider integers of the form $100a + 10b + c$, where a, b, c are the hundreds, tens, and ones digits respectively. The condition translates to

$$\begin{aligned}
100a + 10b + c &= 2(a^2 + b^2 + c^2) \\
100a - 2a^2 &= 2b^2 - 10b + 2c^2 - c
\end{aligned}$$

In order to make $100a + 10b + c$ as large as possible, we have to maximize a. Since $100a - 2a^2$ is increasing on $[0, 9]$, maximizing a is equivalent to making the left-hand side as large as possible.

We can find an upper bound on the left-hand side by computing the largest possible value of the right-hand side. Since $2b^2 - 10b$ and $2c^2 - c$ are quadratic in b and c with positive leading coefficients, they are each maximized when b and c are as far away from the respective polynomials' vertices as possible. The vertex of $2b^2 - 10b$ is at $b = \frac{5}{2}$, so we want to take $b = 9$. The vertex of $2c^2 - c$ is at $c = \frac{1}{4}$, so we'd like to take $c = 9$ — however, we can make a small optimization by noting that c must be even because all the other terms in the equation are even, so in fact $c = 8$ is the highest we can go. This gives us that

$$100a - 2a^2 = (2b^2 - 10b) + (2c^2 - c) \leq (2 \cdot 9^2 - 10 \cdot 9) + (2 \cdot 8^2 - 8) = 192.$$

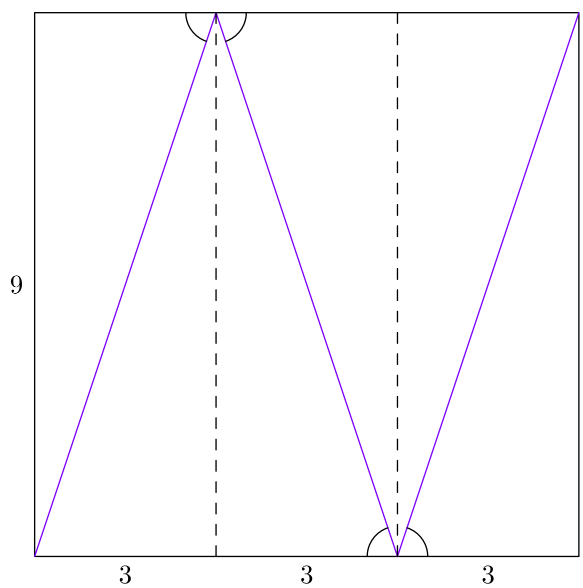
Luckily, this bound is sharp when $a = 2$, as $100 \cdot 2 - 2 \cdot 2^2 = 192$ as well, so $(a, b, c) = (2, 9, 8)$ for an answer of $\boxed{298}$ is largest number equaling twice the sum of the squares of its digits.

Remark. While the common value of 192 seems like divine intervention, more classical bounding techniques quickly give $a < 3$, and yield this solution in a similar fashion.

S25SA29 A square pool table with side length 9 has a pool ball at one corner. Noam strikes the ball, which bounces off the rails twice before reaching the diagonally opposite corner. Find the total length of the ball's path.

Answer: $\boxed{9\sqrt{10}}$

Solution: Let the starting corner be the bottom-left one. The two bounces separate the ball's path into three congruent diagonal segments, each of which traverse a third of the length and the entire width of the table; the segments must be arranged this way in order to obey the rule that the angle of reflection equals the angle of incidence.



This means each segment has a length of $\sqrt{3^2+9^2} = 3\sqrt{10}$ by the Pythagorean Theorem, so the total length of the ball's path is $3 \cdot 3\sqrt{10} = \boxed{9\sqrt{10}}$.

S25SA30 Compute the number of triples of integers (a, b, c) such that $0 \le a, b, c < 288$ and $a^n + bn + c$ is divisible by 288 for all positive integers n.

Answer: $\boxed{20}$

Solution: Let $m = 288 = 2^5 \cdot 3^2$ be the divisor. We will first characterize the valid triples (a, b, c). Plugging in $n = 1$ implies that

$$a + b + c \equiv 0 \pmod{m} \implies c \equiv -a - b \pmod{m}$$

and subtracting the case of $n = 1$ from that of $n = 2$ implies

$$0 \equiv 0 - 0 \equiv (a^2 + 2b + c) - (a + b + c) \equiv a^2 - a + b \pmod{m} \implies b \equiv a - a^2 \pmod{m}$$

This demonstrates that b and c are dependent on a. Finally, subtracting the case of $n = 2$ from that of $n = 3$ gives

$$0 \equiv 0 - 0 \equiv (a^3 + 3b + c) - (a^2 + 2b + c) \equiv a^3 - a^2 + a - a^2 \equiv a(a-1)^2 \pmod{m}$$

so $a(a-1)^2$ is divisible by m.

Now suppose (a, b, c) does satisfy these three properties:

i. $c \equiv -a - b \pmod{m}$

ii. $b \equiv a - a^2 \pmod{m}$

iii. $a(a-1)^2 \equiv 0 \pmod{m}$

Then for all $n > 1$,

$$(a^{n+1} - a^n) - (a^n - a^{n-1}) \equiv a^{n+1} - 2a^n + a^{n-1} \equiv a^{n-1}(a-1)^2 \equiv a^{n-2}a(a-1)^2 \equiv 0 \pmod{m}$$

meaning that a^n is an arithmetic sequence for $n \geq 1$. We can then write a^n as $a + (n-1)(a^2 - a)$, so

$$\begin{aligned} a^n + bn + c &\equiv (a + (n-1)(a^2 - a)) + n(a - a^2) - (a + (a - a^2)) \\ &\equiv a + (a - a^2) - (a + (a - a^2)) \equiv 0 \pmod{m} \end{aligned}$$

for all $n \geq 1$.

This means that the three conditions above are necessary and sufficient for (a, b, c) to be a valid triple. Since b and c are uniquely determined by a, the number of valid triples is simply the number of a in $[0, m)$ such that m divides $a(a-1)^2$.

Note that a and $(a-1)^2$ are relatively prime, so we can casework on which prime powers divide a. If $P \mid a$ for some P dividing m, we have that $\frac{m}{P}$ divides $(a-1)^2$, and we can find the minimal Q that must divide $a-1$ to satisfy that condition. Importantly, P and Q must be relatively prime, since a and $a-1$ are relatively prime. Once we have $a \equiv 0 \pmod{P}$ and $a \equiv 1 \pmod{Q}$, the Chinese Remainder Theorem states that there is a unique solution for a mod PQ, so the number of solutions is $\frac{m}{PQ}$.

P that divides a	$\frac{m}{P}$ that divides $(a-1)^2$	Q that divides $a-1$	Number of solutions
1	$2^5 \cdot 3^2$	$2^3 \cdot 3$	$2^2 \cdot 3 = 12$
2^5	3^2	3	3
3^2	2^5	2^3	$2^2 = 4$
$2^5 \cdot 3^2$	1	1	1

The total among all cases is $12 + 3 + 4 + 1 = \boxed{20}$.

Remark. To generalize, we will separately consider each prime power p^k in the prime factorization of m to compute generally the number of a satisfying $0 \leq a < p^k$ and $p^k \mid a(a-1)^2$. Since $\gcd(a, (a-1)^2) \leq \gcd(a, a-1) = 1$, we have two cases: Either $p^k \mid a$, which has one solution (namely $a = 0$), or

$$p^k \mid (a-1)^2 \iff p^{\lceil \frac{k}{2} \rceil} \mid a - 1 \iff a \equiv 1 \pmod{p^{\lceil \frac{k}{2} \rceil}},$$

which has $\frac{p^k}{p^{\lceil \frac{k}{2} \rceil}} = p^{\lfloor \frac{k}{2} \rfloor}$ nonnegative integer solutions less than p^k.

Let the prime factorization of m be $p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. Given a tuple $(a_1, a_2, \ldots, a_r)$ such that for each k between 1 and r, $0 \leq a_k < p_k^{e_k}$ and $a_k(a_k - 1)^2 \equiv 0 \pmod{p_k^{e_k}}$, the Chinese Remainder Theorem says that there exists a unique $a \in [0, m)$ satisfying

$$a \equiv a_k \pmod{p^k}$$

for all k and thus

$$m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} \mid a(a-1)^2$$

Furthermore, two distinct triples $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_r)$ correspond to different numbers a and b upon application of the Chinese Remainder Theorem, so the number of solutions to the equation $a(a-1)^2 \equiv 0 \pmod{m}$ is equal to the product of the number of solutions for each $p_k^{e_k}$. This gives us an answer of

$$\prod_{k=1}^{r} \left(1 + p_k^{\lfloor e_k/2 \rfloor}\right)$$

and plugging in $m = 288 = 2^5 \cdot 3^2$ gives

$$\left(1 + 2^{\lfloor 5/2 \rfloor}\right)\left(1 + 3^{\lfloor 2/2 \rfloor}\right) = \left(1 + 2^2\right)\left(1 + 3^1\right) = 5 \cdot 4 = \boxed{20}.$$

Remark. Alternatively, since $bn + c$ is an arithmetic sequence mod m, we must also have that a^n is an arithmetic sequence mod m. Every such a generates one triple, as we can see by just plugging in $n = 1$ and $n = 2$ and solving for b and c modulo m. Thus, it is sufficient to count the number of values of a for which a^n is an arithmetic sequence mod m.

The first three terms are a, a^2, and a^3, so we need

$$a^2 - a \equiv a^3 - a^2 \pmod{m}.$$

We can also see that all a satisfying this relation generate an arithmetic sequence, since

$$a^n - a^{n-1} \equiv a^{n+1} - a^n \pmod{m} \implies a^{n+1} - a^n \equiv a^{n+2} - a^{n+1} \pmod{m}$$

by multiplying by a, so the rest of the sequence is also arithmetic. Now, we have

$$a^3 - 2a^2 + a \equiv 0 \pmod{m},$$

meaning $m \mid a^3 - 2a^2 + a = a(a-1)^2$. We can then finish like the above solutions.