New York City Interscholastic Mathematics League

Senior A Division — Fall 2024

Contest 1 Solutions

F24SA01 While reading the last 120 pages of a book, Aditya counted 417 total digits over all of the page numbers of the pages he read. How many pages does the book have? (Assume that the first page of the book is numbered 1, and that pages are numbered sequentially.)

Answer: $\boxed{1056}$

Solution: If all of the last 120 page numbers were 3 digits, then Aditya would count $360 < 417$ total digits, and if they were all 4 digits, then Aditya would count $480 > 417$ total digits. Clearly, some of the page numbers are 3 digits, and some are 4 digits. If we let those amounts be x and y, respectively, we get the following system:

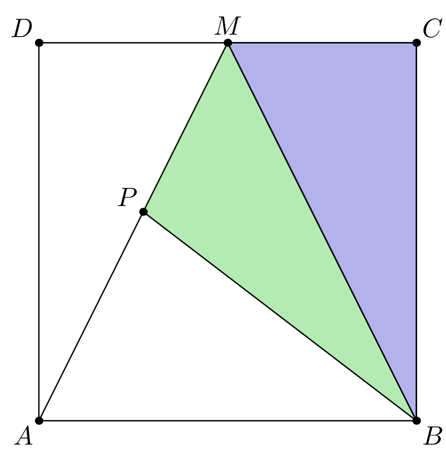
$$3x + 4y = 417$$
$$x + y = 120$$

which, when solved, gives $x = 63$ and $y = 57$. The first 4-digit page number is 1000, so the 57^{th} is $\boxed{1056}$.

F24SA02 Let $ABCD$ be a unit square and let M be the midpoint of side $\overline{CD}$. Point P is positioned on segment $\overline{AM}$ such that $PM = MC$. What is the area of quadrilateral $PBCM$?

Answer: $\boxed{\frac{2\sqrt{5}+5}{20}}$

Solution: Split $[PBCM]$ into $[BCM] + [BPM]$.



The former is simply $\frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$, because $\triangle BCM$ is a right triangle with leg lengths $BC = 1$ and $CM = \frac{1}{2}$. To compute $[BPM]$, note that $\triangle BPM$ and $\triangle BAM$ have the same height from B, so the ratio of their areas is the ratio of their bases: $\frac{[BPM]}{[BAM]} = \frac{PM}{AM}$. We are given $PM = MC = \frac{1}{2}$, and we can compute $AM = \sqrt{AD^2 + DM^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$ using the Pythagorean Theorem on $\triangle ADM$. Finally, $\triangle BAM$ has base 1 and height 1, so its area is $\frac{1}{2}$, and we can find that $[BPM] = \frac{\sqrt{5}}{10}$. The area of $PCBM$ is then $\frac{\sqrt{5}}{10} + \frac{1}{4} = \boxed{\frac{2\sqrt{5}+5}{20}}$.

F24SA03 Positive integers a and b satisfy $a^b + b^a = 2530$. Find the sum of all possible values of a.

Answer: $\boxed{2540}$

Solution: Note that the given equation is symmetric, so WLOG let $a \leq b$. If $a = 1$, then $1^b + b^1 = 1 + b = 2530$, and $b = 2529$. It can be checked that $a = 2$ does not yield integer b, $a = 3$ yields $b = 7$, $a = 4$ does not yield integer b, and $a = 5$ is too large, since $5^5 = 3125 > 2530$. Then, after flipping our ordered pairs to find all four solutions $(1, 2529)$, $(3, 7)$, $(7, 3)$, and $(2529, 1)$, the sum of all possible values of a is $1 + 3 + 7 + 2529 = \boxed{2540}$.

Remark. The checking process can be shortened with the following observations. First, since 2530 is even, a and b must be either both odd or both even. However, if they are both even, then a^b and b^a are both divisible by 4, and since 2530 is not divisible by 4, we can be sure that a and b are both odd.

F24SA04 How many ordered pairs of positive integers (x, y) satisfy $x < y \leq 2024$ and $\frac{\operatorname{lcm}(x,y)}{\gcd(x,y)} = 2024$?

Answer: $\boxed{43}$

Solution: Let $g = \gcd(x, y)$ so that $x = ga$ and $y = gb$ for relatively prime positive integers a and b. Then, $\operatorname{lcm}(x, y) = gab$ and $\frac{\operatorname{lcm}(x,y)}{\gcd(x,y)} = ab = 2024 = 8 \cdot 11 \cdot 23$. Since $a < b$ and a and b are relatively prime, the valid assignments of prime powers are $(a, b) = (1, 2024)$, $(8, 253)$, $(11, 184)$, and $(23, 88)$. Then, to guarantee $y \leq 2024$, we need $g \leq 1, 8, 11$, and 23, respectively, for a total of $1 + 8 + 11 + 23 = \boxed{43}$ pairs.

F24SA05 Three rooks are randomly placed on an 8×8 chessboard such that no two rooks occupy the same square. Compute the probability that no two rooks attack each other. (Two rooks are said to attack each other if they are in the same row or the same column.)

Answer: $\boxed{\frac{14}{31}}$

Solution: After placing the first rook, the second rook cannot be in the same row or column, so there are 7 rows and 7 columns for 7^2 good squares to place it in, out of $8^2 - 1$ possible squares. The third rook cannot be in the same row or column as either of the first two rooks, so there are 6 rows and 6 columns for 6^2 good squares out of $8^2 - 2$ possible squares.

The desired probability is

$$\frac{7^2}{8^2 - 1} \cdot \frac{6^2}{8^2 - 2} = \frac{49}{63} \cdot \frac{36}{62} = \boxed{\frac{14}{31}}.$$

F24SA06 Let $f(x) = x^2 - 20x + 24$. Find all real numbers x satisfying $f(f(x)) = x$.

Answer: $\boxed{\frac{19-\sqrt{341}}{2}, \frac{21-\sqrt{345}}{2}, \frac{19+\sqrt{341}}{2}, \frac{21+\sqrt{345}}{2}}$

Solution: Expanding $f(f(x)) = x$ gives

$$\begin{aligned} (x^2 - 20x + 24)^2 - 20(x^2 - 20x + 24) + 24 &= x \\ x^4 - 40x^2 + 428x^2 - 561x + 120 &= 0. \end{aligned}$$

Rather than attempt to solve this quartic (which has no rational roots), we instead consider the solutions to $f(x) = x$, which must be a subset of those satisfying $f(f(x)) = x$. This means that the roots of $f(x) = x$, which are solutions to $x^2 - 21x + 24 = 0$, are also roots of the quartic, so $f(x) - x$ divides the quartic.

Factoring the quadratic out, we can rewrite the quartic as $(x^2 - 21x + 24)(x^2 - 19x + 5) = 0$, which, by the quadratic formula, has roots $\boxed{\frac{21\pm\sqrt{345}}{2}, \frac{19\pm\sqrt{341}}{2}}$, all of which are real.

Contest 2 Solutions

F24SA07 Let $\tau(k)$ be the number of positive integers dividing k. For example, $\tau(1) = 1$ and $\tau(2025) = 15$. Compute $\tau(1) + \tau(2) + \tau(3) + \cdots + \tau(10)$.

Answer: $\boxed{27}$

Solution: Simply make a list: the first ten values of $\tau(n)$ are

$$1,\ 2,\ 2,\ 3,\ 2,\ 4,\ 2,\ 4,\ 3,\ 4,$$

and the sum of these values is $1 + 2 + 2 + 3 + 2 + 4 + 2 + 4 + 3 + 4 = \boxed{27}$.

F24SA08 How many ways are there to select three letters from the string NYCNYCNYCNYCNYCNYC (NYC repeated six times) so that they spell out "NYC", in that order?

Answer: $\boxed{56}$

Solution: Obviously, the selected letters are N, Y, and C. Let n, y, and c be the integers such that the selected N is in the nth NYC, the selected Y is in the yth NYC, and the selected C is in the cth NYC. Then, it is necessary and sufficient that $n \leq y \leq c$, so we just need to count the number of triples (n, y, c) such that $1 \leq n \leq y \leq c \leq 6$. Pictorially, suppose that there are six stars in a row and three dividers are placed so that each divider has at least one star to its left:

$$* * *|*|**|$$

Then, let n be the number of stars to the left of the leftmost bar, y the number of stars to the left of the middle bar, and c the number of stars to the left of the rightmost bar. This mapping shows that the number of star/divider configurations such that each divider has at least one star to its left is in bijection with the (n, y, c) triples descrribed above, and there are $\binom{8}{3} = \boxed{56}$ such star/bar configurations.

F24SA09 Compute the value of $\sqrt{3^2 + 4^2 + 12^2 + 84^2 - 77^2 + 48^2}$.

Answer: $\boxed{60}$

Solution: The quickest solution follows from the recognition of Pythagorean triples (and difference of squares):

$$\begin{aligned}
&\sqrt{3^2 + 4^2 + 12^2 + 84^2 - 77^2 + 48^2}\\
&= \sqrt{5^2 + 12^2 + 84^2 - 77^2 + 48^2}\\
&= \sqrt{13^2 + 84^2 - 77^2 + 48^2}\\
&= \sqrt{85^2 - 77^2 + 48^2}\\
&= \sqrt{(85 + 77)(85 - 77) + 48^2} = \sqrt{162 \cdot 8 + 48^2}\\
&= \sqrt{36^2 + 48^2}\\
&= \sqrt{60^2} = \boxed{60}.
\end{aligned}$$

Notably, $(13, 84, 85)$ is a Pythagorean triple. In general, given an odd number $2k + 1$, the triple $(2k + 1, 2k^2 + 2k, 2k^2 + 2k + 1)$ is Pythagorean. This fact is easier to remember in light of the observation that

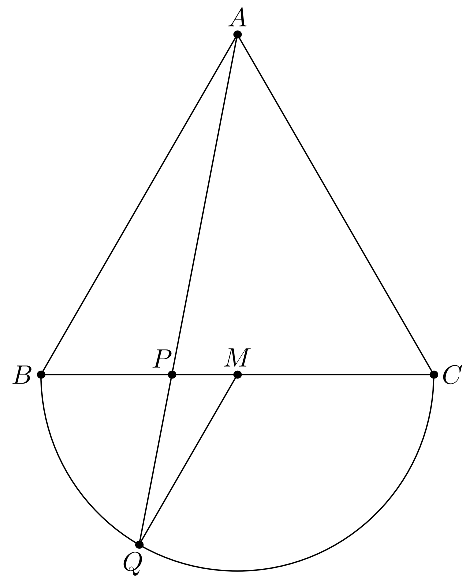
$$2k^2 + 2k = \left\lfloor \frac{(2k+1)^2}{2} \right\rfloor, \qquad 2k^2 + 2k + 1 = \left\lceil \frac{(2k+1)^2}{2} \right\rceil.$$

Alternatively, one can also simply compute all the squares (they are all at most four digits) and find that the radicand evaluates to 3600.

F24SA10 Let $\triangle ABC$ be equilateral. A semicircle ω with diameter $\overline{BC}$ is drawn such that the interiors of ω and $\triangle ABC$ do not overlap. Point P is chosen on segment $\overline{BC}$, and line $\overline{AP}$ intersects ω at Q. Given that $\frac{AP}{PQ} = 2$, compute the sum of all possible values of $\frac{BP}{CP}$.

Answer: $\boxed{\frac{5}{2}}$

Solution: Scale the diagram so that the side length of $\triangle ABC$ is 2 and suppose WLOG that P is closer to B than C.



By the given ratio, we get

$$2 = \frac{[ABP]}{[BPQ]} = \frac{[ACP]}{[CPQ]} = \frac{[ABP] + [ACP]}{[BPQ] + [CPQ]} = \frac{[ABC]}{[QBC]}.$$

This means that the distance from Q to $\overline{BC}$ is half the distance from A to $\overline{BC}$, which means the distance from Q to $\overline{BC}$ is $\frac{\sqrt{3}}{2}$. We also know that, since Q is on the circle with diameter $\overline{BC}$, its distance to the midpoint M of the segment is 1. This implies that $\sin(\angle BMQ) = \frac{\sqrt{3}}{2}$, so $\angle BMQ = 60^\circ$ because P being closer to B than C means that $\angle BMQ$ is acute.

One way to finish from here is to note that $\overline{AB} \parallel \overline{MQ}$ and the ratio of the segments' lengths is 2, so

$$\frac{BP}{PM} = \frac{AB}{MQ} = 2$$

which then implies that $PM = \frac{BC}{6}$ and then $\frac{BP}{PC} = \frac{1}{2}$.

Finally, by symmetry, the case of $\frac{BP}{PC} = 2$ is also possible, giving an answer of $2 + \frac{1}{2} = \boxed{\frac{5}{2}}$.

F24SA11 A word bank consists of the four words YOU, BELONG, WITH, and ME. Brandon selects one of the words uniformly at random, and then selects one of the letters in that word uniformly at random. What is the probability that Brandon selects a vowel? (Y is **not** considered a vowel.)

Answer: $\boxed{\frac{7}{16}}$

Solution: Since the probability of choosing each word is $\frac{1}{4}$, the answer is simply the average of the vowels-to-letters ratio in each word, which is

$$\frac{\frac{2}{3} + \frac{2}{6} + \frac{1}{4} + \frac{1}{2}}{4} = \frac{\frac{21}{12}}{4} = \boxed{\frac{7}{16}}.$$

F24SA12 Let $\tau(k)$ be the number of positive integers dividing k. For example, $\tau(1) = 1$ and $\tau(2025) = 15$. Compute $\tau(1) + \tau(2) + \tau(3) + \cdots + \tau(100)$.

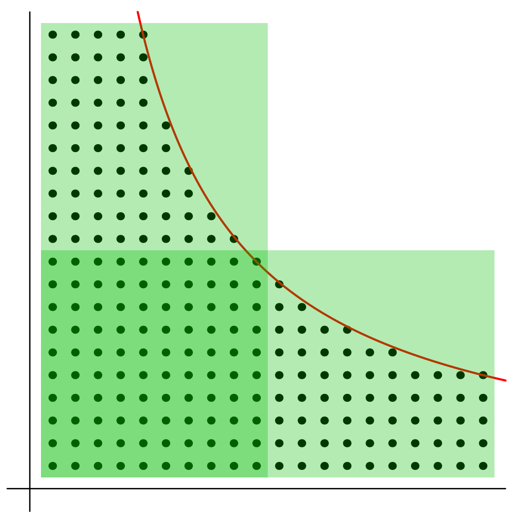
Answer: $\boxed{482}$

Solution: Making a list of the values, like in Problem 7, is no longer reasonable under the time constraint. Instead, we present a faster way to compute this sum. The main idea is the following claim:

$$\sum_{n=1}^{100} \tau(n) = \sum_{ij \le 100} 1,$$

where the sum runs over all ordered pairs (i, j) of positive integers whose product is at most 100. To prove this, we see that $\tau(n)$ counts the number of pairs of positive integers (i, j) such that $ij = n$, and the sum on the left-hand side varies n between 1 and 100, thus each (i, j) with $ij \le 100$ is counted exactly once.

Geometrically, this means that we want to find the number of points with positive integer coordinates within the region in the Cartesian plane bounded by the curves $x = 0$, $y = 0$, and $xy = 100$, since each of these points corresponds to one of the (i, j) pairs that we are counting.



To count the number of these points, we leverage the symmetry of the diagram: the set of points is symmetric with respect to $y = x$, so the count of points is the sum

$$\sum_{\substack{ij \le 100 \\ i \le 10}} 1 + \sum_{\substack{ij \le 100 \\ j \le 10}} 1 - 10^2.$$

That is, if we add the number of points on or to the left of $x = 10$ and the number of points on or below $y = 10$ we are off by exactly 10^2, since the points that are both on or to the left of $x = 10$ and on or below $y = 10$, namely those in the 10×10 square $1 \le x \le 10$, $1 \le y \le 10$, were counted twice. This computation is now not so hard to do: both sums are clearly equal (swapping i and j doesn't change the value), and listing out shows that this common value is

$$\left\lfloor \frac{100}{1} \right\rfloor + \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{6} \right\rfloor + \left\lfloor \frac{100}{7} \right\rfloor + \left\lfloor \frac{100}{8} \right\rfloor + \left\lfloor \frac{100}{9} \right\rfloor + \left\lfloor \frac{100}{10} \right\rfloor$$

$$= 100 + 50 + 33 + 25 + 20 + 16 + 14 + 12 + 11 + 10$$

$$= 291$$

so our final answer is $2 \cdot 291 - 100 = \boxed{482}$.

Alternatively, it's possible to just compute all the distinct values attained by $\left\lfloor \frac{100}{k} \right\rfloor$ as k varies from 1 to 100 as well as how many times each value occurs and sum everything that way.

Contest 3 Solutions

F24SA13 A pentagon's angle measures form an arithmetic sequence. Its smallest angle measures $5x-58$ degrees, and its second-largest angle measures $155-x$ degrees. Find the measure of its largest angle, in degrees.

Answer: $\boxed{154^\circ}$

Solution: The second-largest angle is three terms after the smallest angle in the arithmetic sequence, so the common difference is

$$\frac{(155-x)-(5x-58)}{3}=\frac{213-6x}{3}=71-2x.$$

Subtracting this from the second-largest angle shows that the middle term of the sequence is

$$155-x-(71-2x)=84+x.$$

However, the middle term is also equal to the average of the five angles, which sum to 540°, so

$$84+x=108 \iff x=24.$$

Finally, we add the common difference $71-2\cdot 24=23$ twice to the third term 108 to get an answer of $108+46=\boxed{154}$.

F24SA14 Find the sum of all prime factors of 8040201.

Answer: $\boxed{281}$

Solution: We factor as follows:

$$\begin{aligned}
8040201 &= 200^3+200^2+200+1\\
&=(200+1)(200^2+1)\\
&=3\cdot 67\cdot(4\cdot 10^4+1)\\
&=3\cdot 67\cdot(4\cdot 10^4+4\cdot 10^2+1-4\cdot 10^2)\\
&=3\cdot 67\cdot(201^2-20^2)\\
&=3\cdot 67\cdot(181\cdot 221)\\
&=3\cdot 67\cdot 181\cdot 13\cdot 17.
\end{aligned}$$

This gives an answer of $3+13+17+67+181=\boxed{281}$.

Remark. the factorization of $4\cdot 10^4+1$ is an instance of the so-called Sophie-Germain identity, which in its general form states that

$$\begin{aligned}
x^4+4y^4 &= x^4+4x^2y^2+4y^4-4x^2y^2\\
&=(x^2+2y^2)^2-(2xy)^2\\
&=(x^2+2xy+2y^2)(x^2-2xy+2y^2).
\end{aligned}$$

F24SA15 The $5!=120$ positive integers whose digits are a permutation of 1, 2, 3, 4, and 5 are written down in increasing order. (So the first three terms are 12345, 12354, and 12435.) What is the 55^{th} number in this list?

Answer: $\boxed{32145}$

Solution: The main idea of this problem is that, if we are given the first k digits of a number the list, then there are $(5-k)!$ possible numbers with those first k digits. For example, there are $3!=6$ numbers whose first two digits are 12. Importantly, these six numbers are all consecutive within the list.

Let N be the 55^{th} number in the list. Based on our observation, we see that the first $4!$ numbers start with 1 and the next $4!$ numbers start with 2. That describes the first 48 numbers, which does not include the 55^{th} one. However, if we skip all the numbers starting with 3, then we are at the $3\cdot 4!+1=73^{\text{rd}}$ number in the list, which is too far. This means that the leftmost digit of N is 3.

More formally, since $55 - 2 \cdot 4!$ is positive while $55 - 3 \cdot 4!$ is not, the first digit must be the third smallest available number, 3. (The coefficient of 2 before $6!$ indicates that we can skip past the first two digits.)

We now want to find the $55 - 2 \cdot 4! = 7^{\text{th}}$ element in the sublist of the numbers starting with 3.

Continuing with the above logic, we get that, since 1 is the largest integer m such that $7 - m \cdot 3!$ is positive, we skip one digit. So, the second digit of N is the 2^{nd} smallest available number, which is 2.

At this point, we want the $7 - 3! = 1^{\text{st}}$ element in the sublist of the numbers starting with 32, which is clearly 145. The value of N is then $\boxed{32145}$, our answer.

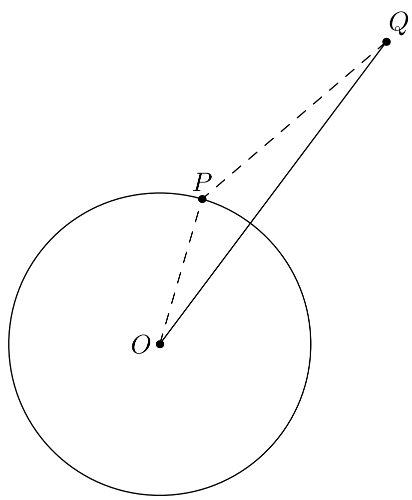
F24SA16 Let P be a point on the graph of $x^2+y^2 = 1$ and let Q be a point on the graph of $y = \sqrt{x^4 + 4x^3 + 5x^2 + 4x + 50}$. Find the smallest possible distance between P and Q.

Answer: $\boxed{6}$

Solution: Let O be the origin. The first thing we notice is that given a point Q, the point P that minimizes PQ is the intersection of segment $\overline{OQ}$ with the graph of $x^2 + y^2 = 1$. This is because

$$OP + PQ \geq OQ \iff PQ \geq OQ - OP = OQ - 1,$$

where equality holds if and only if O, P, and Q are collinear in that order.



Thus, we want to minimize OQ. Letting $Q = (a, b)$, we have

$$OQ^2 = a^2 + b^2 = a^2 + \sqrt{a^4 + 4a^3 + 5a^2 + 4a + 50}^{\,2} = a^4 + 4a^3 + 6a^2 + 4a + 50.$$

We can then recognize the coefficients of 1, 4, 6, and 4 as the coefficients in the expansion of $(a + 1)^4$, which leads us to rewrite the distance OQ^2 as $(a + 1)^4 + 49$. To minimize this quantity, we want to minimize $(a + 1)^4$. In particular, since $(a + 1)^4 \geq 0$ because it's a square and equality holds when $a = -1$,

$$OQ^2 = (a + 1)^4 + 49 \geq 49 \implies OQ \geq 7$$

with equality when $a = -1$. Thus, the smallest possible length of $\overline{PQ}$ is $OQ - 1 = \boxed{6}$, achieved when $Q = (-1, \sqrt{48})$ and $P = (-\frac{1}{7}, \frac{\sqrt{48}}{7})$.

F24SA17 A triangle with positive area has two sides whose lengths differ by 19. Find the minimum possible integer value of the perimeter of this triangle.

Answer: $\boxed{39}$

Solution: Of the two side lengths mentioned in the problem, let the shorter one be $\varepsilon > 0$, so the longer one is $\varepsilon + 19 > 19$. By the triangle inequality, the third side length s must satisfy $\varepsilon + s > \varepsilon + 19$, or $s > 19$.

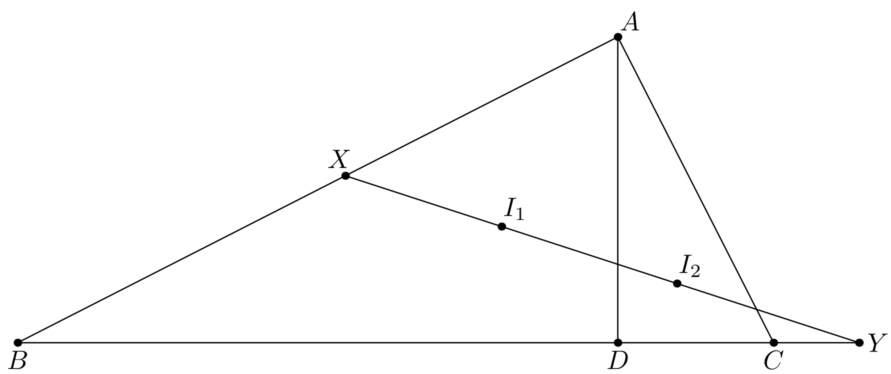
Then, the perimeter is strictly greater than 38, and the next smallest integer $\boxed{39}$ is achievable with $\varepsilon = \frac{1}{3}$ and $s = \frac{58}{3}$.

F24SA18 Let $\triangle ABC$ have angles $\angle B = 18°$ and $\angle C = 72°$. The altitude from A to $\overline{BC}$ intersects $\overline{BC}$ at D. What is the degree measure of the acute angle formed by line $\overline{BC}$ and the line through the incenters of $\triangle ABD$ and $\triangle ACD$?

Answer: $\boxed{27°}$

Solution: We present two solutions.

Solution 1. Let I_1 and I_2 be the incenters of $\triangle BAD$ and $\triangle CAD$ respectively.



We see that $\triangle DBA \sim \triangle DAC$ and $\triangle DI_1B \sim \triangle DI_2A$ by simple angle chasing arguments, so we get $\triangle DI_1I_2 \sim \triangle DBA \sim \triangle ABC$.

Now, let line $\overline{I_1I_2}$ intersect $\overline{AB}$ at X. Since

$$\angle ABD = \angle XBD = \angle DI_1I_2 = 180° - \angle DI_1X,$$

we find that

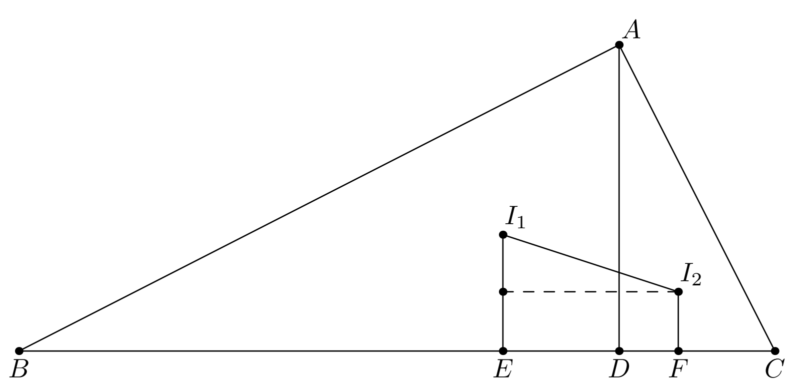
$$\angle BXI_1 = 360° - \angle XBD - \angle XI_1D - \angle BDI_1 = 360° - 180° - \angle 45° = 135°,$$

meaning $\overline{I_1I_2}$ makes a $45°$ angle with $\overline{AB}$, and by symmetry, it also makes a $45°$ angle with $\overline{AC}$.

To finish, if $\overline{I_1I_2}$ intersects $\overline{BC}$ at Y, then

$$\angle XYB = 180° - \angle XBY - \angle BXY = 180° - \angle 135° - \angle XBY = 45° - 18° = \boxed{27°}.$$

Solution 2. Let E and F be the feet of the perpendiculars from I_1 and I_2 to $\overline{BC}$ respectively.



Then, since $\triangle I_1ED$ and $\triangle I_2FD$ are isosceles right triangles, $I_1E = DE$ and $I_2F = DF$, so $I_1E + I_2F = EF$. Now, let $r_1 = I_1E$ and $r_2 = I_2F$. Note that $r_1 > r_2$ and that the tangent of the angle θ between $\overline{I_1I_2}$ and $\overline{BC}$ is

$$\tan\theta = \frac{I_1E - I_2F}{EF} = \frac{r_1 - r_2}{r_1 + r_2}.$$

Moreover, because $\triangle DBA \sim \triangle DAC$, the ratio of their inradii is $\frac{r_1}{r_2} = \frac{AB}{AC} = \tan\angle C$, which means

$$\tan\theta = \frac{r_2\tan\angle C - r_2}{r_2\tan\angle C + r_2} = \frac{\tan\angle C - 1}{\tan\angle C + 1} = \frac{\tan\angle C - \tan 45^\circ}{1 + \tan\angle C \cdot \tan 45^\circ} = \tan(\angle C - 45^\circ).$$

Therefore, $\theta = \angle C - 45^\circ = \boxed{27^\circ}$.

Contest 4 Solutions

F24SA19 Circles ω_1 and ω_2, both with radius 1, are externally tangent to each other and are both internally tangent to another circle ω_3. If the centers of ω_1, ω_2, and ω_3 make an equilateral triangle, what is the area of ω_3?

Answer: $\boxed{9\pi}$

Solution: The distance between the centers of ω_1 and ω_2 is 2, so the side length of the equilateral triangle is 2 and thus the radius of ω_3 is $1 + 2 = 3$, giving an answer of $3^2 \cdot \pi = \boxed{9\pi}$.

F24SA20 A hunter and an invisible rabbit are playing a series of ping pong games, none of which end in a tie. The first player to win 3 games wins the series. They are equally skilled, so normally they both have a 50% chance of winning each game. However, due to self confidence issues, the rabbit's chances of winning a game drop to 25% if and only if the hunter has won strictly more games than the rabbit so far. Compute the probability that the rabbit wins the series.

Answer: $\boxed{\frac{101}{256}}$

Solution: We make a table with all ten possible cases, where W stands for a win for the rabbit and L a loss for the rabbit.

WWW	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$
LWWW	$\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$
WLWW	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$
WWLW	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$
LLWWW	$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{256}$
LWLWW	$\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{128}$
LWWLW	$\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{64}$
WLLWW	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{64}$
WLWLW	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$
WWLLW	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$

The sum of all the probabilities is then

$$\frac{32 + 8 + 16 + 16 + 3 + 2 + 4 + 4 + 8 + 8}{256} = \boxed{\frac{101}{256}}.$$

F24SA21 Let $f(x) = x^2 - 10x + 28$ and $g(x) = f(f(f(x)))$. Compute $g(1) + g(4) - g(9)$.

Answer: $\boxed{4}$

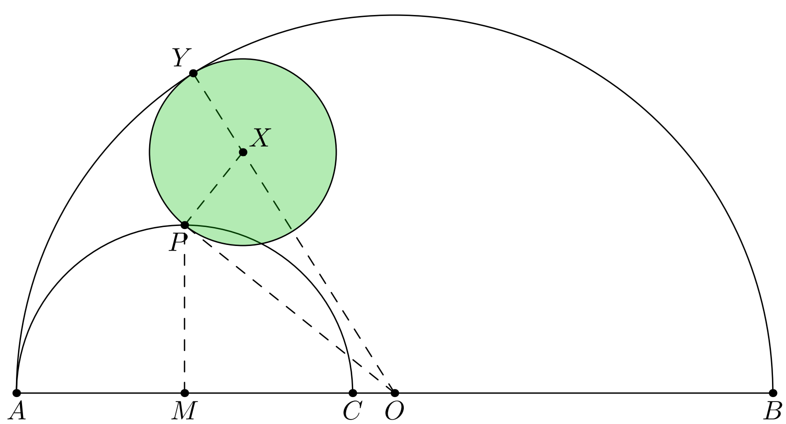
Solution: Since $f(x) = (x-5)^2 + 3$, we see that $f(1) = f(9) = 4^2 + 3$, so $g(1) = f(f(f(1))) = f(f(f(9))) = g(9)$, meaning $g(1) - g(9) = 0$. And, since $f(4) = (4-5)^2 + 3 = 4$, we have

$$g(4) = f(f(f(4))) = f(f(4)) = f(4) = \boxed{4}.$$

F24SA22 Let $\mathcal{S}_1$ be a semicircle with diameter $\overline{AB}$ and center O. Point C lies on segment $\overline{AB}$ such that $AC = 20$ and $BC = 25$. Semicircle $\mathcal{S}_2$ is drawn with diameter $\overline{AC}$ in the region bounded by arc $\mathcal{S}_1$ and line $\overline{AB}$. Point P on $\mathcal{S}_2$ is equidistant from A and C, and circle ω is tangent to $\overline{OP}$ at P and internally tangent to $\mathcal{S}_1$. What is the radius of ω?

Answer: $\boxed{\frac{50}{9}}$

Solution: Let the center of $\mathcal{S}_2$ be M. Let $a = AC$, $b = BC$, and let r be the radius of ω. Finally, let X be the center of ω and let Y be the point at which ω touches $\mathcal{S}_1$.



We will compute the length of OX in two different ways, which result in an equation that we can solve for r. First, due to the tangency of ω and $\mathcal{S}_1$,

$$OX = OY - XY = \frac{AB}{2} - r = \frac{a+b}{2} - r.$$

Second, since $\overline{OP}$ is tangent to ω, $\angle OPX$ is right, which lets us use the Pythagorean theorem in the form

$$OX^2 = OP^2 + PX^2 = OP^2 + r^2.$$

We then need to compute OP^2. To do this, we note that $\angle OMP = 90°$; this is because $AP = CP$ implies that P lies on the perpendicular bisector of $\overline{AC}$, which of course contains M, the midpoint of $\overline{AC}$. Thus, applying the Pythagorean theorem again tells us that

$$\begin{aligned} OP^2 &= OM^2 + MP^2 \\ &= (OA - MA)^2 + MA^2 \\ &= \left(\frac{AB - AC}{2}\right)^2 + \left(\frac{AC}{2}\right)^2 \\ &= \left(\frac{BC}{2}\right)^2 + \left(\frac{AC}{2}\right)^2 \\ &= \frac{a^2 + b^2}{4}. \end{aligned}$$

Now, we can equate our two expressions for OX, which gives us

$$OX^2 = \left(\frac{a+b}{2} - r\right)^2 = \frac{a^2+b^2}{4} + r^2.$$

Simplifying the left-hand side, we find that

$$\begin{aligned} \frac{(a+b)^2}{4} - r(a+b) + r^2 &= \frac{a^2+b^2}{4} + r^2 \\ \iff \frac{a^2+b^2+2ab}{4} - r(a+b) + r^2 &= \frac{a^2+b^2}{4} + r^2 \\ \iff \frac{2ab}{4} - r(a+b) &= 0 \\ \iff r(a+b) &= \frac{ab}{2} \\ \iff r &= \frac{ab}{2(a+b)}. \end{aligned}$$

Substituting in $a = 20$ and $b = 25$, we get

$$r = \frac{20 \cdot 25}{2(20+25)} = \frac{500}{90} = \boxed{\frac{50}{9}}.$$

F24SA23 A right triangle has area 7 and hypotenuse 6. Find its perimeter.

Answer: $\boxed{14}$

Solution: Let a and b be the unknown leg lengths. Because the triangle is right, we know that the area is $\frac{ab}{2} = 7$ and the square of the hypotenuse is $a^2 + b^2 = 6^2$ by the Pythagorean Theorem. Then,

$$(a+b)^2 = a^2 + b^2 + 2ab = 36 + 4 \cdot 7 = 64,$$

so, since $a + b$ is positive, $a + b = 8$ and the perimeter is $a + b + 6 = \boxed{14}$.

F24SA24 For a positive integer n, define

$$f_n(x) = \sum_{k=1}^{n} \lfloor kx \rfloor.$$

What is the smallest integer n such that $f_n(x)$ achieves at least 50 values as x varies across the interval $(0, 1)$?

Answer: $\boxed{13}$

Solution: We first note that f_n is non-decreasing, and its smallest output is 0, achievable by any $x < \frac{1}{n}$. Now, since $\lfloor kx \rfloor = m$ if and only if $x \in \left[\frac{m}{k}, \frac{m+1}{k}\right)$ for integer m, the value of f_n is determined by where x is relative to integer multiples of $\frac{1}{k}$ for all positive integers $k \leq n$, inclusive.

For example, when $n = 4$, the relevant multiples are $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2} = \frac{2}{4}$, $\frac{2}{3}$, and $\frac{3}{4}$, “splitting” the interval $(0, 1)$ into 6 pieces, which corresponds to 6 distinct values.

We want the smallest n such that the distinct integer multiples of $\frac{1}{k}$ in $(0, 1)$ over all positive integers $k \leq n$ split the interval into at least 50 pieces. In general, increasing $n - 1$ to n introduces $\varphi(n)$ new, distinct integer multiples of $\frac{1}{n}$, and $f_1(x)$ achieves $\varphi(1) = 1$ value, so we want

$$\sum_{k=1}^{n} \varphi(k) \geq 50$$

for which the smallest n is $\boxed{13}$.

Contest 5 Solutions

F24SA25 Aditya randomly and uniformly chooses two distinct integers from the set $\{1, 2, 3, \ldots, 2025\}$. Find the probability that the sum of these two integers is even.

Answer: $\boxed{\frac{1012}{2025}}$

Solution: The two integers Aditya chooses must be of the same parity. We then have two cases: out of the $\binom{2025}{2}$ total ways to choose two distinct integers from the set, there are $\binom{1012}{2}$ ways to choose two even integers and $\binom{1013}{2}$ ways to choose two odd integers, for an answer of $\frac{\binom{1012}{2}+\binom{1013}{2}}{\binom{2025}{2}} = \frac{1012\cdot1011+1013\cdot1012}{2025\cdot2024} = \boxed{\dfrac{1012}{2025}}$.

F24SA26 Andrew's K-pop playlist has an average song length of 2 minutes and 59 seconds. His friend recommends the song "Fatal Trouble" by ENHYPEN, which is 2 minutes and 50 seconds long. Andrew adds it to his playlist, and the average song length becomes 2 minutes and 58 seconds. Andrew likes the song so much that he also adds five similar songs, which total 16 minutes and 56 seconds in length. Now, what is his playlist's average song length, in seconds?

Answer: $\boxed{187}$

Solution: Let n be the number of songs in Andrew's playlist before any additions. Converting everything into seconds, we have that

$$179n + 170 = 178(n + 1)$$

where the common value is the length of the playlist after adding Fatal Trouble. Solving, we get $n = 8$. Now, since the other five songs total $16 \cdot 60 + 56 = 1016$ seconds in length, we want to compute the value of

$$\frac{179(8) + 170 + 1016}{8 + 1 + 5} = \frac{2618}{14} = \boxed{187}.$$

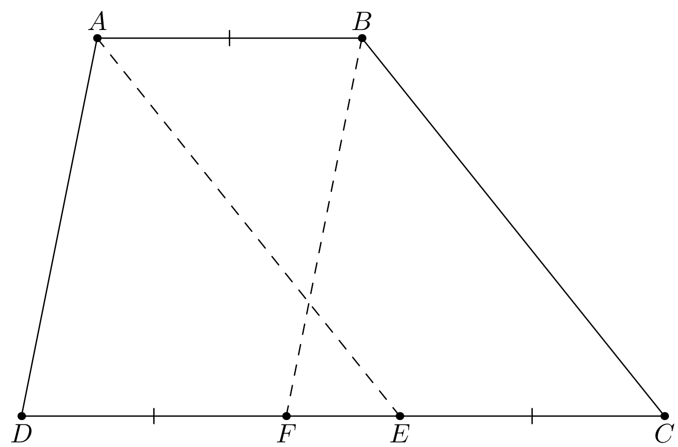
F24SA27 Let $ABCD$ be a trapezoid with area 588 and parallel bases $AB = 7$ and $CD = 17$. The line through A parallel to $\overline{BC}$ intersects line $\overline{CD}$ at E, and the line through B parallel to $\overline{AD}$ intersects $\overline{CD}$ at F. What is the area of the quadrilateral formed by A, B, E, and F?

Answer: $\boxed{245}$

Solution: Let h be the distance between lines $\overline{AB}$ and $\overline{CD}$. Then, we see that $ABEF$ is a trapezoid with bases $\overline{AB}$ and $\overline{EF}$, which means that

$$[ABEF] = \frac{AB + EF}{2} \cdot h = \frac{AB + EF}{AB + CD} \cdot \frac{AB + CD}{2} \cdot h = \frac{AB + EF}{AB + CD} \cdot [ABCD].$$

In words, this highlights the idea that, since $ABEF$ and $ABCD$ share the same height, their areas are related by the ratios of the sums of their bases.



Now, we know the values of AB, CD, and $[ABCD]$, so it remains to compute EF. This can be done from the observation that $ABCE$ and $BADF$ are both parallelograms, so $DF = EC = AB = 7$, which implies that

$$EF = CD - DF - CE = 17 - 7 - 7 = 3.$$

Therefore, the answer is

$$\frac{7+3}{7+17} \cdot 588 = \frac{5}{12} \cdot 588 = \boxed{245}.$$

Remark. In general, if the base lengths are a and b with $a < b$, then the answer is

$$\frac{b-a}{b+a} \cdot [ABCD].$$

F24SA28 How many positive integers $n \leq 2024$ are there such that $n^{22} - 1$ is divisible by 2024?

Answer: $\boxed{176}$

Solution: Note that divisibility by 2024 is equivalent to divisibility by its prime powers, namely $2^3 = 8$, 11, and 23.

It is clear that n must be odd (if n were even, then $n^{22} - 1$ would be odd and cannot be divisible by 8). It can be checked that $1^2 - 1 \equiv 3^2 - 1 \equiv 5^2 - 1 \equiv 7^2 - 1 \equiv 0 \pmod 8$, so n can be any of 4 remainders mod 8.

We have $n^{22} - 1 \equiv n^2 - 1 \pmod{11}$ by Fermat's Little Theorem, and $n^2 - 1 = (n+1)(n-1) \equiv 0 \pmod{11}$ means that n is $\pm 1 \pmod{11}$ for 2 possible remainders mod 11.

Also by Fermat's Little Theorem, we have $n^{22} - 1 \equiv 0 \pmod{23}$ unless $23 \mid n$, so n can be any of 22 nonzero remainders mod 23.

By the Chinese Remainder Theorem, given remainders mod 8, mod 11, and mod 23, there exists a unique positive integer at most 2024 whose remainders upon division by 8, 11, and 23 are the given numbers. Therefore, we can multiply the number of valid remainders to get $4 \cdot 2 \cdot 22 = \boxed{176}$ positive integers.

F24SA29 Real numbers x and y satisfy $x + y = xy = \frac{x}{y}$. Find the value of $x - y$.

Answer: $\boxed{\frac{3}{2}}$

Solution: From $xy = \frac{x}{y}$, we have $y \neq 0$. Multiplying by y and moving everything to one side gives $xy^2 - x = x(y^2 - 1) = x(y-1)(y+1) = 0$, meaning either $x = 0$, $y = 1$, or $y = -1$.

In the first case, we have $0 + y = 0 \cdot y \implies y = 0$, a contradiction.

In the second case, we have $x + 1 = x \cdot 1$, which has no solution.

In the third case, we have $x + (-1) = x \cdot -1 \implies x = \frac{1}{2}$, which can be verified to be a valid solution.

The only solution is $(x, y) = \left(\frac{1}{2}, -1\right)$, so $x - y = \frac{1}{2} - (-1) = \boxed{\dfrac{3}{2}}$.

F24SA30 Let S be the set of lattice points with $1 \leq x \leq 4$ and $1 \leq y \leq 6$. How many ways are there to color each of the points in S either red or blue such that, for every four distinct points in S that form a rectangle with sides parallel to the axes, not all of them are the same color?

Answer: $\boxed{720}$

Solution: *Lemma:* If any row contains three or more points of the same color, it is impossible to satisfy the condition.

Assume WLOG that the first three points of the bottom-most row, e.g. $(1,1)$, $(2,1)$, and $(3,1)$, are all colored red. We proceed by contradiction. Clearly, none of the other five rows may have two or more of their first three points colored red; otherwise, it is possible to choose a rectangle with sides in that row and the first row whose vertices are all red.

Additionally, the colorings of the first three points of rows must be pairwise distinct; if two rows are identical with respect to their first three points, it is possible to choose a rectangle with sides in those rows whose vertices are all the same color, since coloring three points with two colors guarantees at least two points being of the same color (by the Pigeonhole principle).

This means that there are only four other possible colorings, being RBB?, BRB?, BBR?, and BBB?, so the sixth row cannot be colored without failing the condition (not even considering the fact that BBB? cannot exist with any of the other colorings). $\square$

With our lemma proved, every row must have two red and two blue points. As in the lemma, two rows may not be identical, and there are exactly $\binom{4}{2} = 6$ different colorings with two red and two blue points; namely, RRBB, RBRB, RBBR, BRRB, BRBR, and BBRR, so each must appear once. It turns out that the coloring

RRBB
RBRB
RBBR
BRRB
BRBR
BBRR

does satisfy the condition. Since swapping rows and columns does not affect the condition, there are $6! = \boxed{720}$ different ways to arrange the rows, with each arrangement corresponding to a valid coloring.