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LINEAR ALGEBRA

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DETERMINANTS & THEIR PROPERTIES

- **DETERMINANTS:**

Determinant is a scalar value which is the function of entries of a square matrix. It characterizes the properties of a matrix. Following are the formulas for calculating determinants of 2x2 and 3x3 matrices.

DETERMINANT OF 2X2 MATRIX:

If a, b, c, d are scalar values than,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \boxed{ad - bc}$$

DETERMINANT OF 3X3 MATRIX:

If a, b, c, d, e, f, g, h, i are scalar values then,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= \boxed{a(ei-hf) - b(di-gf) + c(dh-ge)}$$

- **PROPERTIES OF MATRIX'S DETERMINANT:**

1. **Reflection Property:**

The determinant remains unchanged if rows are changed into columns and columns into rows i.e. transpose of a matrix is taken.

Eg:

$$\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \rightarrow \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = 10-12 = \boxed{-2}$$

Taking transpose of matrix.

$$\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} = 10-12 = \boxed{-2}$$

Hence, proved.

2. **All-zero Property:**

If every element of a row or column of a matrix is zero, then the determinant is zero.

Eg:

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 0(18-20)-1(0-0)+2(0-0) = \boxed{0} \text{ (proved)}$$

3. **Proportionality Property:**

If the row or column's elements are identical to all elements of some other row or column, then determinant is zero.

Eg:

$$\begin{vmatrix} 1 & 5 & 1 \\ 2 & 5 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 1(15-10)-5(6-6)+1(10-15) = 5-0+(-5)$$

$$= \boxed{0} \text{ proved.}$$

4. **Switching Property:**

If we switch/interchange any two rows or columns of a matrix, then the determinant of new matrix changes its sign.

eg:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-6)-2(0-6)+3(4-5) = \boxed{3}$$

Interchanging C_1 and C_2:

$$\begin{vmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 1 & 1 & 0 \end{vmatrix} = 2(0-6)-1(0-6)+3(5-4) = \boxed{-3}$$

Hence, proved.

5. Scalar Multiple Property:

If every element of a row or column is multiplied by any non-zero constant, then the determinant gets multiplied by the same constant.

eg:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{vmatrix} = \boxed{3} \Rightarrow \text{let it be } det_1$$

Multiply C_1 by 2:

$$\begin{vmatrix} 2 & 2 & 3 \\ 8 & 5 & 6 \\ 2 & 1 & 0 \end{vmatrix} = 2(0-6)-2(0-12)+3(8-10) = 6 = 2\times3$$

$$= \boxed{2(det_1)}$$

Hence, proved.

6. Sum Property:

If elements of a row or column are expressed as sum of two or more terms, then the determinant can be expressed as sum of two or more determinants.

$$\begin{vmatrix} a_1+x_1 & b_1 \\ a_2+x_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 \\ x_2 & b_2 \end{vmatrix}$$

Eg:

$$\begin{vmatrix} 1+2 & 3 \\ 4+5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$LHS = \begin{vmatrix} 3 & 3 \\ 9 & 6 \end{vmatrix} = 18 - 27 = \boxed{-9}$$

$$RHS = (6-12) + (12-15) = \boxed{-9}$$

Hence; proved LHS = RHS.

7. Property of invariance:

If each element of a row and column of a determinant is added with equimultiples of elements of another row or column of a determinant, then the value of determinant remains unchanged.

$$A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad B = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 \\ a_2 & b_2 \end{vmatrix}$$

then $\det(A) = \det(B)$

Eg:

$$A = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix}, \quad B = \begin{vmatrix} 2+2(6) & 4+2(8) \\ 6 & 8 \end{vmatrix} = \begin{vmatrix} 14 & 20 \\ 6 & 8 \end{vmatrix}$$

$$\det(A) = 16 - 24 = \boxed{-8}.$$

$$\det(B) = 112 - 120 = \boxed{-8}$$

Hence, proved.

8. Triangular Property:

If elements above or below main diagonal are equal to zero, then the value of determinant is equal to product of elements of diagonal matrix.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot b_2 \cdot b_3$$

eg:

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{vmatrix} = 1(15-0)-2(0-0)+4(0-0) = \boxed{15}$$

and $(1)(3)(5) = \boxed{15}$

— Hence, proved.

9. Factor Property:

If a determinant becomes 0 while considering the value of $x = \alpha$, then $(x-\alpha)$ is considered as a factor of determinant.

10. Co-factor of Matrix Property:

The cofactor of an element a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} \times M_{ij}$ where M_{ij} is minor of a_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor of a_{12} : $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of a_{12} : $A_{12} = (-1)^{1+2} M_{12}$

$$= (-1)^{1+3} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$