Definition 1. *A function $f(x)$ is continuous at point $x = x_0$ if and only if the following conditions are satisfied:*

$$\lim_{x \to x_0^-} f(x), \ \lim_{x \to x_0^+} f(x), \ \textit{and } f(x_0) \textit{ exist.} \tag{1}$$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0) \tag{2}$$

Example 1. *Show that $f(x)$ is continuous at $x = 1$ if it is defined by*

$$f(x) = \begin{cases} x + 1, & x < 1 \\ x^2 + x, & x \geq 1 \end{cases}$$

Proof. We have to know what left/right limit and also the function value at $x = 2$ are

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + 1 = 2 \\ &\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + x = 2 \\ &f(1) = 1^2 + 1 = 2 \end{aligned}$$

We see that, they all exist, so condition (1) is satisfied. Moreover, since they are all equal, so condition (2) is also satisfied. Thus $f(x)$ is continuous at $x = 1$. □

Example 2. *Find a number k such that the function*

$$f(x) = \begin{cases} \sin(x + k), & x < 0 \\ e^x, & x \geq 0 \end{cases}$$

is continuous at $x = 0$.

Proof. In order to make this function continuous at $x = 1$, we have to make it satisfies the two conditions.

Since

$$\begin{aligned} &\lim_{x \to 0^-} f(x) = \sin k \\ &\lim_{x \to 0^+} f(x) = e^0 = 1 \\ &f(0) = e^0 = 1 \end{aligned}$$

we see that we have to set $\sin k = 1$. Thus $k = \frac{\pi}{2} + 2\pi n, \ n \in \mathbb{Z}$. □