Definition 1. *Suppose a multi-variable function $f : \mathbb{R}^2 \to \mathbb{R}$, then if the limit of a function $f(\vec{x})$ at point $\vec{x}_0$ is L, it can be denoted by*

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = L$$

Theorem 1. $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = L$ *if and only if no matter which path $\vec{x}$ is chosen to approach $\vec{x}_0$, the limit is L.*

Example 1.

$$\lim_{(x,y)\to(0,0)} xy + e^x = 1$$

When the function is continuous at that point, we can easily compute the limit by calculating the function value at that point.

Example 2.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \textit{ does not exist}$$

Proof. We will show it by using polar coordinate system.
Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} &= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} \\ &= \lim_{r\to 0} \cos\theta\sin\theta\end{aligned}$$

The result depends on the choice of angle (path dependent). Thus the limit at point (0,0) does not exist. □

Example 3.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \textit{ does not exist}$$

Proof. Using polar coordinate will be somewhat troublesome, we will use the rectangular coordinate instead.

When (x,y) approaches (0,0) by the path $y = mx$.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} &= \lim_{x\to 0} \frac{x^3m^2}{x^2+m^4x^4} \\ &= \lim_{x\to 0} \frac{xm^2}{1+m^4x^2} \\ &= 0\end{aligned}$$

When (x,y) approaches (0,0) by the path $x = y^2$.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} &= \lim_{y\to 0} \frac{y^4}{y^4+y^4} \\ &= \frac{1}{2}\end{aligned}$$

□