1 Partial derivative

Definition 1. *The partial derivative of function $f(x, y)$ with respect to x is defined by*

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the partial derivative of f with respect to y is defined by

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Note that there are many notations for partial derivative. For example, the following notations are all meaning the same thing

$$\frac{\partial f}{\partial x} \equiv \partial_x f \equiv f_x$$

Example 1. *Let $f(x, y) = x \cos 2y$. Then*

$$\partial_x f = \cos 2y \quad \partial_y f = -2x \sin 2y$$

Proposition 1. *The geometrical meaning of partial derivatives is the instantaneous rate of change of the multi-variable function f in each directions.*

For example f_x means the instantaneous rate of change in x-direction; f_y is the instantaneous rate of change in y-direction.

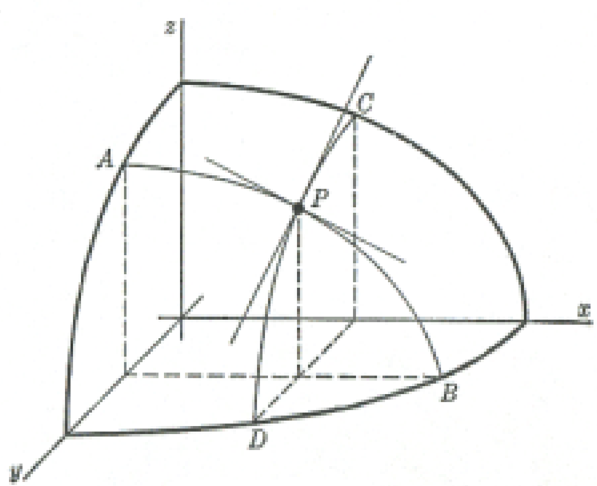

Fig. 1

2 Higher order derivative

We can easily define higher order partial derivatives. They are just partial derivative of the previous partial derivatives. The notation should not be confused

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv \partial_y \partial_x f \equiv f_{xy}$$

And there is a nice result we have to remember.

Theorem 1. ***Clairaut's theorem.***
Suppose a two variable function $f(x, y)$ is defined on some open set U in $\mathbb{R}^2$, and both second-order mixed partial derivatives $f_{xy}(x, y)$ and $f_{yx}(x, y)$ exist and continuous on U, then

$$f_{xy}(x, y) = f_{yx}(x, y) \text{ on } U$$

Note: This theorem can be generalized to n-variable real-valued functions $f(x_1, x_2, \cdots, x_n)$. any second-order mixed derivative has this property. In other words,

$$\partial_{x_i} \partial_{x_j} f = \partial_{x_j} \partial_{x_i} f$$