1 Introduction

Definition 1. *We say that a function's limit exits at $x = x_0$,i.e*

$$\lim_{x \to x_0} f(x) = L$$

if this statement is true: $\forall \epsilon > 0, \exists \delta > 0$ *such that if* $0 < |x - x_0| < \delta$, *then*

$$|f(x) - L| < \epsilon$$

Example 1. *Let* $f(x) = \frac{(x+1)(x-1)}{(x+1)x}$.

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We see that from the graph

$$\lim_{x \to 1} f(x) = 0$$
$$\lim_{x \to 0} f(x) = DNE$$
$$\lim_{x \to -1} f(x) = 2$$

Notes: The limit of a function at some points x_0 has nothing to do with the function value at that point.

Theorem 1.

$$\begin{aligned} &\lim(f(x) \pm g(x) = \lim f(x) \pm \lim g(x) \\ &\lim(f(x) \times g(x) = \lim f(x) \times \lim g(x) \\ &\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \end{aligned}$$

Example 2.

$$\lim_{x \to 0}(\cos x + \sin x) = \lim_{x \to 0} \cos x + \lim_{x \to 0} \sin x = 1 + 0 = 1$$

Example 3.

$$\lim_{x \to 1} \frac{x^2}{x+1} = \frac{\lim_{x \to 1} x^2}{\lim_{x \to 1}(x+1)} = \frac{1}{2}$$

Example 4.

$$\lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1}$$

If we simply plug in the value $x = -1$, *we will have a 0/0 situation, this is not good. However, we see that the the term causing the problem,* $x + 1$, *can be canceled. Thus*

$$\begin{aligned} \lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1} &= \lim_{x \to -1} \frac{(x+1)(x+2)}{x+1} \\ &= \lim_{x \to -1} x + 2 \\ &= 1 \end{aligned}$$

Although we know that the function value at $x = -1$ is still undefined, but its limit at $x = -1$ has nothing to do with the function.

Example 5.

$$\lim_{x \to 2} \frac{x-2}{\sqrt{x-1}-1}$$

If we simply put the the value $x = 2$, we again encounter the 0/0 situation. However, when we see square root, try to use the square difference formula to simplify the expression first.

$$\begin{aligned}
\lim_{x \to 2} \frac{x-2}{\sqrt{x-1}-1} &= \lim_{x \to 2} \frac{(x-2)(\sqrt{x-1}+1)}{(\sqrt{x-1}-1)(\sqrt{x-1}+1)} \\
&= \lim_{x \to 2} \frac{(x-2)(\sqrt{x-1}+1)}{x-2} \\
&= \lim_{x \to 2} \sqrt{x-1}+1 \\
&= 2
\end{aligned}$$

Note: When dealing with limit, if some situations occur(e.g 0/0), remember to simplify or adjust the expression before you further calculate.

2 One-sided limit

Definition 2. *The **left(right) limit** of a function, $f(x)$, at $x = x_0$ is the value the function is approaching when x is approaching x_0 from the left(right). The notations for them are*

$$\lim_{x \to x_0^-} f(x), \qquad \textit{left limit} \tag{1}$$

$$\lim_{x \to x_0^+} f(x), \qquad \textit{right limit} \tag{2}$$

Example 6. *Let $f(x) = \frac{1}{x}$. Then the one-sided limits at $x = 0$ are*

$$\lim_{x \to 0^-} f(x) = -\infty, \qquad \textit{left limit}$$

$$\lim_{x \to 0^+} f(x) = -\infty, \qquad \textit{right limit}$$

Theorem 2. $\lim_{x \to x_0} f(x)$ *exists if and only if the left and right limit exist and equal, i.e.* $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$.

Example 7.

$$f(x) = \frac{(x+1)}{(x+1)x}$$

The limit of this function at $x = -1$ exists since both left and right limit exist and equal.

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