1 Total Differentiation and Approximation

Definition 1. *The **total derivative** of a function $f(x, y)$ at point (x_0, y_0) is given by*

$$df = \partial_x f(x_0, y_0)dx + \partial_y f(x_0, y_0)dy$$

In general, for functions of n variables $f(\vec{x})$ where $\vec{x} = (x_1, x_2, \cdots, x_n)$, the total differential at point $\vec{x}_0$

$$df = \sum_{i=1}^{n} \partial_{x_i} f(\vec{x}_0)dx_i$$

Note: Just think of this total derivative is just like the differentiation in first year calculus.

Example 1. *Find the Total change of the area of a rectangle with length 10m and width 20m if the change in length is 0.5m and the change in width is 0.1m.*

Solution. *Let the area of the rectangle be $A(l, w) = lw$.*

$$\begin{aligned}\Delta A &\approx \partial_l A \Delta l + \partial_w A \Delta w \\ &= w\Delta l + l\Delta w \\ &= 20 \times 0.5 + 10 \times 0.1 \\ &= 11m^2\end{aligned}$$

2 Tangent Plane

Proposition 1. *Whenever there is a surface $z = f(x, y)$, the tangent plane that at point (x_0, y_0, z_0) can be given by*

$$(z - z_0) = \partial_x f(x_0, y_0)(x - x_0) + \partial_y f(x_0, y_0)(y - y_0)$$

The normal vector of this plane is $(\partial_x f(x_0, y_0), \partial_y f(x_0, y_0), -1)$.

It is easy to see that why this is true. As $(x, y) \to (x_0, y_0)$, the entire equation above comes back to its origin, total differential of the function at point (x_0, y_0).

$$dz = \partial_x f(x_0, y_0)dx + \partial_y f(x_0, y_0)dx$$

Example 2. *Find the tangent plane of the sphere $x^2 + y^2 + z^2 = 1$ at point (0,0,1).*

Solution. *First we convert the surface equation to the form $z = f(x, y)$. Since we only care about the north pole (0,0,1), so we just need to take the upper sphere*

$$z = \sqrt{1 - x^2 - y^2}$$

Thus the tangent plane at (0,0,1) would be

$$
\begin{aligned}
(z - z_0) &= \partial_x f(x_0, y_0)(x - x_0) + \partial_y f(x_0, y_0)(y - y_0) \\
z - 1 &= \left.\frac{-x}{\sqrt{1 - x^2 - y^2}}\right|_{(0,0)} (x - 0) + \left.\frac{-y}{\sqrt{1 - x^2 - y^2}}\right|_{(0,0)} (y - 0) \\
z - 1 &= 0
\end{aligned}
$$

Thus tangent plane is just simply $z = 1$. *Easy to verify this.*