Definition 1. *A function $f(x)$ is continuous at point $x = x_0$ if and only if the following conditions are satisfied:*

$$\lim_{x \to x_0^-} f(x), \ \lim_{x \to x_0^+} f(x), \text{ and } f(x_0) \text{ exist.} \tag{1}$$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0) \tag{2}$$

Example 1. *Show that $f(x)$ is continuous at $x = 1$ if it is defined by*

$$f(x) = \begin{cases} x+1, & x < 1 \\ x^2 + x, & x \geq 1 \end{cases}$$

Proof. We have to know what left/right limit and also the function value at $x = 2$ are

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + 1 = 2 \\ &\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + x = 2 \\ &f(1) = 1^2 + 1 = 2 \end{aligned}$$

We see that, they all exist, so condition (1) is satisfied. Moreover, since they are all equal, so condition (2) is also satisfied. Thus $f(x)$ is continuous at $x = 1$. □

Example 2. *Find a number k such that the function*

$$f(x) = \begin{cases} \sin(x+k), & x < 0 \\ e^x, & x \geq 0 \end{cases}$$

is continuous at $x = 0$.

Proof. In order to make this function continuous at $x = 1$, we have to make it satisfies the two conditions.

Since

$$\begin{aligned} &\lim_{x \to 0^-} f(x) = \sin k \\ &\lim_{x \to 0^+} f(x) = e^0 = 1 \\ &f(0) = e^0 = 1 \end{aligned}$$

we see that we have to set $\sin k = 1$. Thus $k = \frac{\pi}{2} + 2\pi n, \ n \in \mathbb{Z}$. □

Theorem 1. ***Intermediate Value Theorem.***

If a function $f(x)$ is continuous on [a,b], then for any $\xi \in f([a, b])$, we can find a $a \leq c \leq b$ such that

$$f(x) = \xi$$

Example 3. *Since* $f(x) = x^2$ *is continuous on* $[0,2]$*, and* $f([0,2]) = [0,4]$*. Since* $\pi \in [0,4]$*, thus we can find a* $c \in [0,2]$ *such that* $f(c) = \pi$*.*

Corollary 1. *If a function* $f(x)$ *is continuous on [a,b], and* $f(a)f(b) < 0$*, then there must be at least one* $c \in [a,b]$ *such that* $f(c) = 0$*.*

Proof. Since $f(a)f(b) < 0$, it means one and only one of $f(a)$ and $f(b)$ is negative. Without loss of generality, we assume $f(a) < 0 < f(b)$. Then since $f(x)$ is continuous on [a,b], thus by intermediate value theorem, there must be at least one c such that $f(c) = 0$. □

Example 4. *Show that* $x^20 + 3x^19 - 1 = 0$ *must have at least one solution in [0,1].*

Proof. Since the function $f(x) = x^20 + 3x^19 - 1$ is continuous on [0,1], and $f(0)f(1) = (-1) \times (3) < 0$. Thus it must have one zero in [0,1]. □