1 Instantaneous Rate of Change

Definition 1. ***Instantaneous rate of change*** *of a function $f(x)$ at $x = x_0$ is defined by*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

or

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example 1. *Suppose a car is in motion, and its position can be described by the function $x(t) = 5t + 1$ (t in second and x in meter), then the instantaneous rate of change(in this case, instantaneous velocity) at $t = 3$ can be calculated by*

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{t \to 3} \frac{f(t) - f(3)}{t - 3} = \lim_{t \to 3} \frac{(5t + 1) - (16)}{t - 3} = \lim_{t \to 3} \frac{5t - 15}{t - 3} = 5$$

That means the instantaneous velocity at $t = 3s$ is 5m/s.

Example 2. *Let $f(x) = x^2 + 1$, show that the instantaneous rate of change at point $x = x_0$ is $2x_0$.*

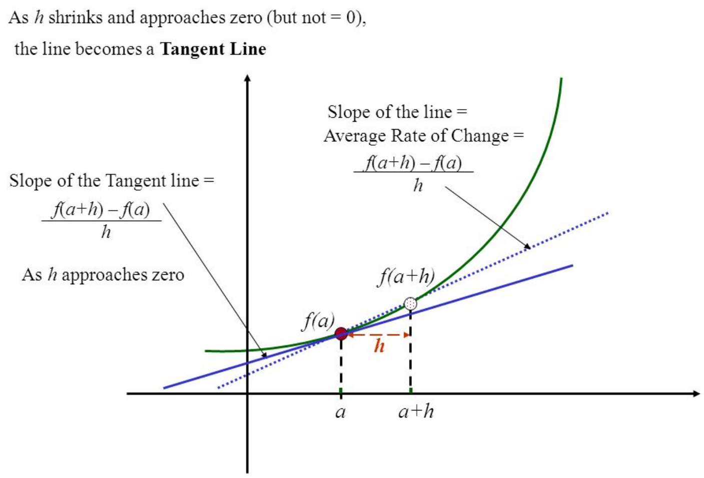
Proof.

$$\begin{aligned}
\lim_{h \to 0} \frac{\Delta y}{\Delta x} &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{[(x_0 + h)^2 + 1] - (x_0^2 + 1)}{h} \\
&= \lim_{h \to 0} \frac{x_0^2 + 2x_0 h + h^2 - x_0^2}{h} \\
&= \lim_{h \to 0} 2x_0 + h \\
&= 2x_0
\end{aligned}$$

□

2 Geometrical Interpretation

The instantaneous rate of change of a function $f(x)$ at a point $x = a$ is the slope of the tangent line at the point $(a, f(a))$.



Example 3. *Show that the slope of the tangent line of the function $f(x) = \sqrt{x}$ at the point $x = 4$ is 1/4.*

Proof.

$$\begin{aligned}
\lim_{h\to 0} \frac{\Delta y}{\Delta x} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Thus when $x = 4$, the instantaneous rate of change is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$ $\square$