Binomial distribution is one of most common seen distributions, it is powerful for many situations.

Definition 1. *Binomial distribution is a family of distributions, it has two parameters, namely n and p. If a discrete random variable X has distribution*

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0, 1, 2, ..., n \\ 0 & , otherwise \end{cases}$$

we say that X is binomially distributed, i.e. $X \sim Binomial(n, p)$.

Of course, the pmf of binomial distribution has the following two properties.

Property 1. $p(x) \geq 0, \ \ \forall x$

Property 2. $\sum_x p(x) = 1$

Proof. The first property is obvious. We are going to prove the second property.

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (1 + (1-p))^n = 1$$

The first equality holds because of the binomial theorem. □

Definition 2. *If $X \sim Binomial(n, p)$, then the cumulative distribution function(cdf) of X is defined by*

$$F_X(x) = P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

We should talk about the expectations now.

Proposition 1.

$$E(X) = \sum_x x \binom{n}{x} p^x (1-p)^{n-x} = np \tag{1}$$

$$E(X^2) = \sum_x x^2 \binom{n}{x} p^x (1-p)^{n-x} = np((n-1)p + 1) \tag{2}$$

$$Var(X) = E(X^2) - (E(X))^2 = np(1-p) \tag{3}$$

Proof. We first prove (1),

$$\begin{aligned} \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} &= 0 + \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \\ &= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \end{aligned}$$

We now make two substitutions, $x-1 \to k$, and $n-1 \to m$, then

$$E(X) = np \sum_{k=0}^{n} \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} = np$$

The equation (2) can be derived in the similar manner, but with a little trick. Notice that $x^2 = x(x-1) + x$, then

$$\begin{aligned}
E(X^2) &= E(X(X-1)) + E(X) \\
&= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} + np \\
&= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x} + np \\
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-1)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x} + np \\
&= n(n-1)p^2 \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} p^k (1-p)^{n-x} + np \\
&= n(n-1)p^2 + np \\
&= np((n-1)p + 1)
\end{aligned}$$

Equation (3) follows from the previous two.

$$Var(X) = E(X^2) - (E(X))^2 = np((n-1)p+1) - (np)^2 = np(1-p)$$

$\square$