Magnetic Nozzle Eigenvalue Problem Results

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1 Exact Solution

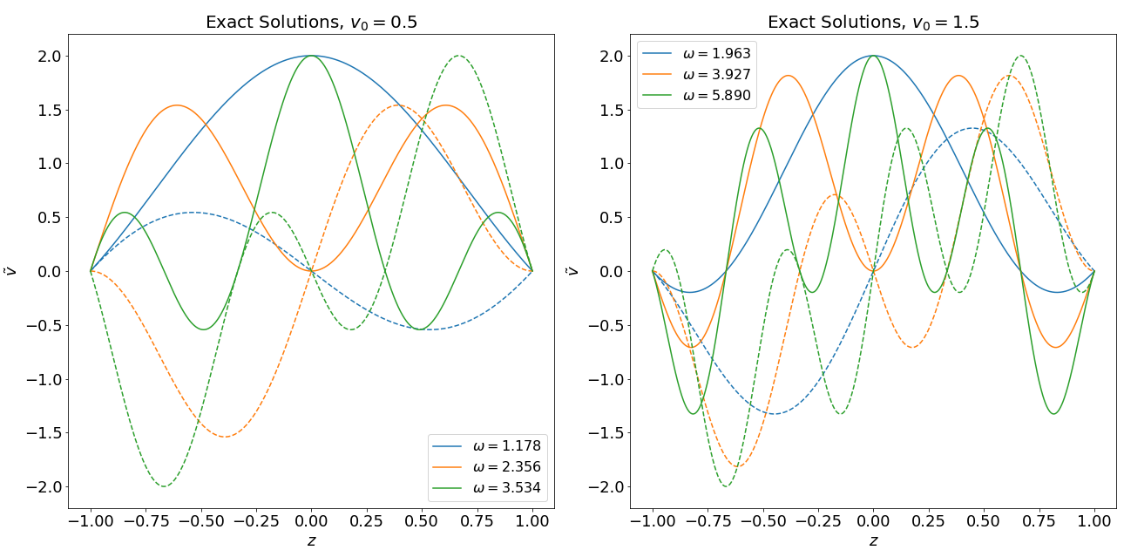
In this document, I will list all the methods I tried and their results for the eigenvalue problem

$$\omega^2 \tilde{v} + 2iv_0\omega \frac{\partial \tilde{v}}{\partial z} + (1 - v_0^2)\frac{\partial^2 \tilde{v}}{\partial z^2} = 0, \tilde{v}(-1) = \tilde{v}(1) = 0 \tag{1}$$

where $v_0 \in \mathbb{R}$ is a constant. In thsi problem, we need to solve for eigenvalues ω and their corresponding eigenfunctions $\tilde{v}$.

$$\tilde{v} = \exp\left(-\frac{i\omega}{v_0 + 1}\right)\left[\exp\left(i\omega\frac{z+1}{v_0+1}\right) - \exp\left(i\omega\frac{z+1}{v_0-1}\right)\right] \text{ where } \omega = \frac{n\pi(1-v_0^2)}{2} \tag{2}$$

All eigenvalues are real, so all modes are stable.



(a) (b)

Figure 1: First few exact eigenfunctions(ground mode, $\omega = 0$, not included).

2 Finite Element

Assume $\tilde{v}(z) = \sum_j c_j u_j(z)$, where $u_j(z)$ are basis functions. Left multiply $u_i(z)$ on both sides of the equation, then take the inner product,

$$\sum_j \left[\omega^2(u_i, u_j) + 2i\omega v_0(u_i, u_j') + (1 - v_0^2)(u_i, u_j'')\right] c_j = 0$$

where $(\cdot, \cdot)$ denotes the inner product.

Let $A_2 = (u_i, u_j)$, $A_1 = 2iv_0(u_i, u_j')$, and $A_0 = (1 - v_0^2)(u_i, u_j'')$. Denote $\mathbf{c} = [c_j]^T$. We have a polynomial eigenvalue problem

$$(\omega^2 A_2 + \omega A_1 + A_0)\mathbf{c} = \mathbf{0}$$

If $u_j(z)$ are orthonormal functions on $[-1, 1]$, then the polynomial eigenvalue problem can be written as an algebraic eigenvalue problem

$$\begin{bmatrix} O & I \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \omega\mathbf{c} \end{bmatrix} = \omega \begin{bmatrix} \mathbf{c} \\ \omega\mathbf{c} \end{bmatrix}$$

where O is $N \times N$ zero matrix, I is $N \times N$ identity matrix, and

$$A_{21} = (1 - v_0^2)(u_i', u_j'), \quad A_{22} = -2iv_0(u_i, u_j')$$

2.1 DVR methods

Let $u_j(z)$ be basis functions that satisfy the boundary condition $u_j(-1) = u_j(1) = 0$, and the Kronecker delta property

$$u_j(z_k) = \delta_{jk}$$

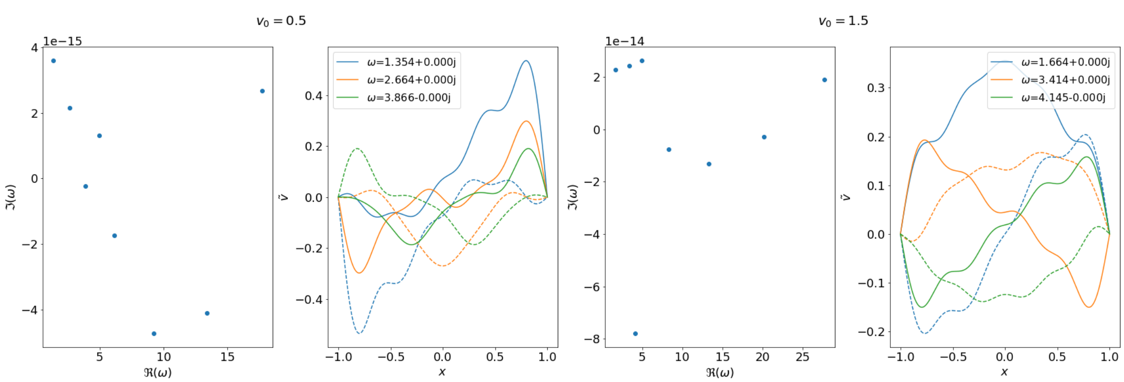
By using Gaussian quadrature with quadrature nodes $z_k \in [-1, 1]$, we see that $u_i(z)$ are othornormal on $[-1, 1]$ if we scale them correctly,

$$\int_{-1}^{1} u_i(z)u_j(z) = \sum_k w_k u_i(z_k)u_j(z_k) = \delta_{ij}$$

2.1.1 Sine DVR

Let $\psi_n(z) = \sin\left(\frac{n\pi}{2}(z+1)\right), n = 1, \cdots, N$ be the orthorgonal functions on $[-1, 1]$, and define the DVR basis functions by

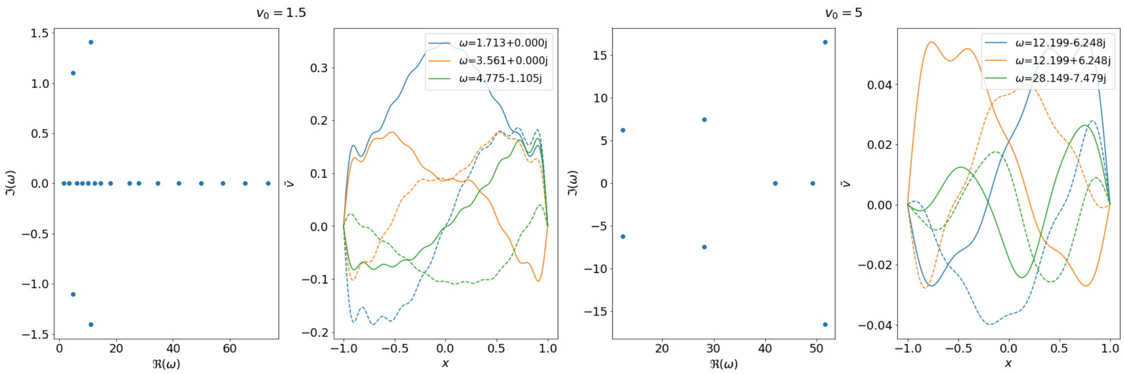
$$u_j(z) = w_j \sum_{n=1}^{N} \psi_n(z)\psi_n^*(z_j)$$



(a) (b)

Figure 2: Using $N = 9$ basis functions. All modes are stable.

This looks hopeful! However, when we increase velocity, v_0, or the number of basis function, N. Unstable modes occur in supersonic case,



(a) Using $N = 21$ basis function.

(b) Using $N = 9$ basis function.

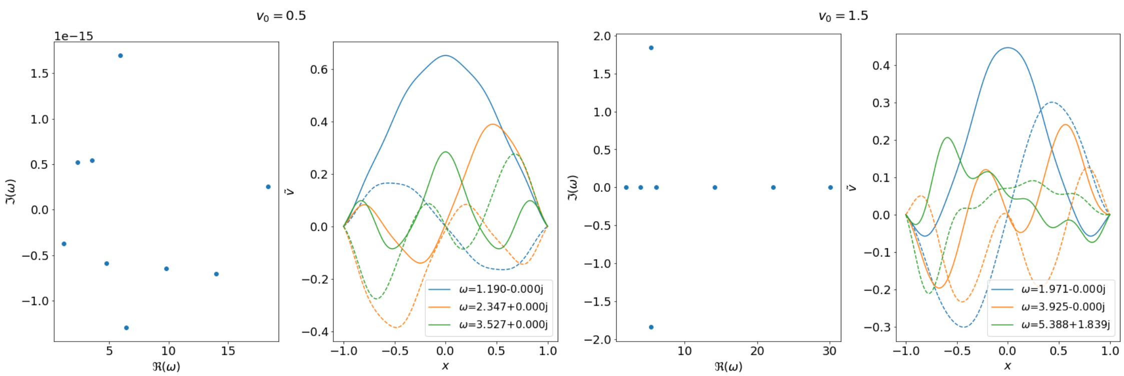
Figure 3: Increasing N or v_0, things become more unstable in supersonic case.

2.1.2 Sinc DVR

Using the normalized $\text{sinc}(x) = \sin(\pi x)/(\pi x)$, the basis functions are

$$u_j(z) = \frac{\text{sinc}((z - z_j)/\Delta z)}{\sqrt{\Delta z}}$$

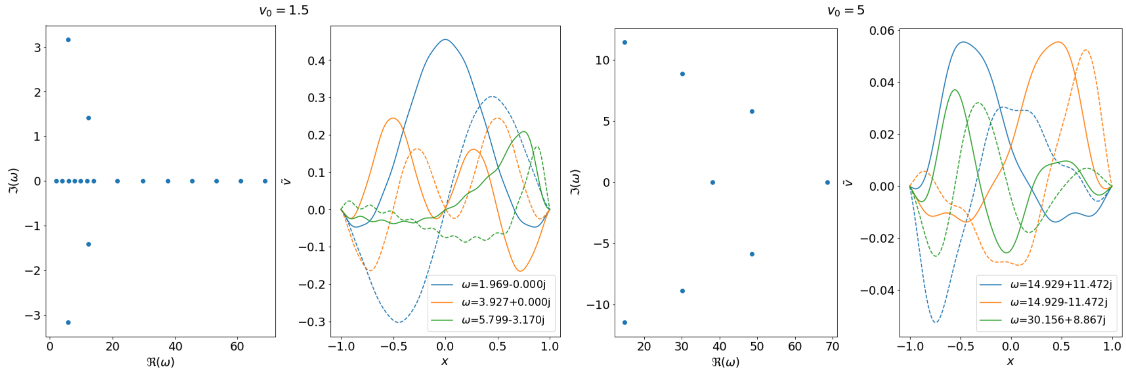
where $z_j \in [-1, 1]$ are the quadrature nodes.



(a) Using $\Delta x = 0.2$.

(b) Using $\Delta x = 0.2$.

Figure 4: No matter what Δx I use, there will be unstable modes in supersonic case.



(a) Using $\Delta x = 0.1$

(b) Using $\Delta x = 0.2$.

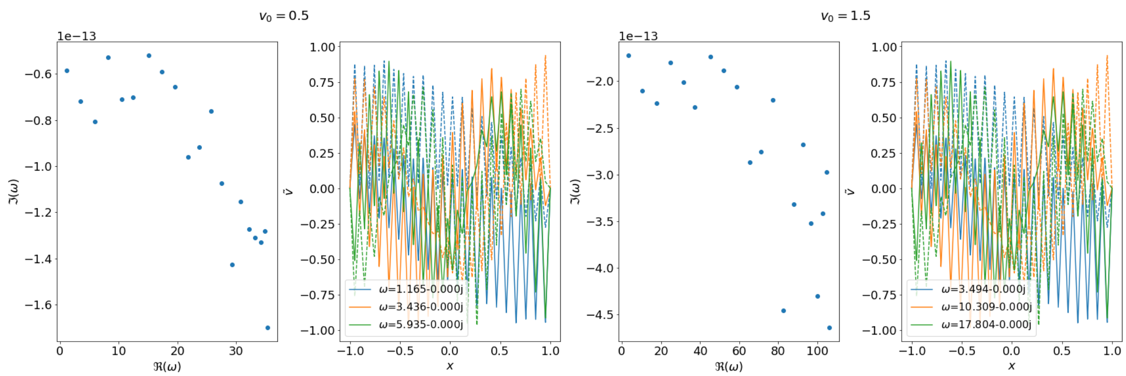
Figure 5: Decreasing Δx (equivalent to increasing number of basis functions) or increasing v_0, things become more unstable in supersonic case.

Similar to Sine DVR, if we decrease the Δz (equivalent to increasing number of basis functions), or increase v_0. Modes become unstable in supersonic case.

2.2 Non-DVR methods

2.2.1 Linear Element

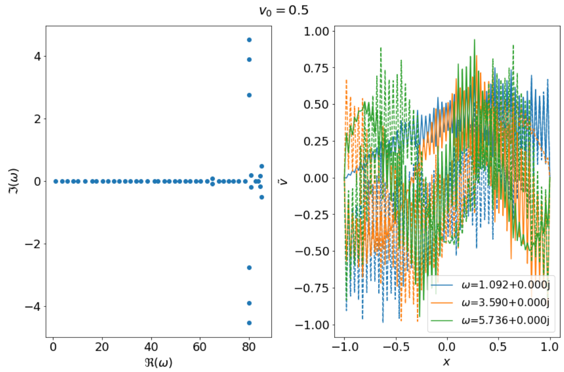
Let $\tilde{v}(z) = \sum_{j=1}^{N} c_j u_j(z)$ where $u_j(z)$ are tent functions that peaks at $x_j \in (-1, 1)$.

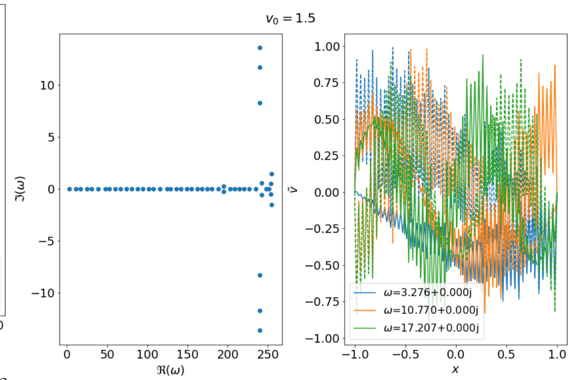


(a) All modes are stable.

(b) Some modes are unstable.

Figure 6: Using $N = 40$, all modes are stable.





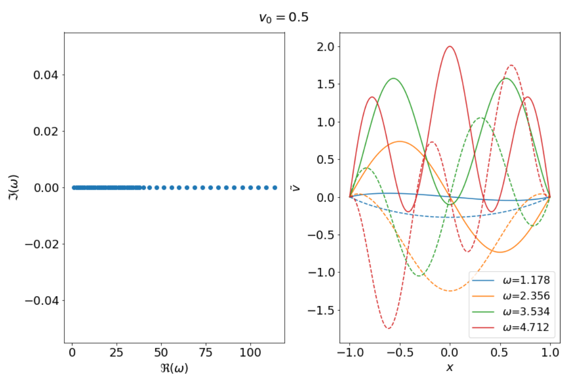
(a) Using $N = 100$. This time unstable modes even occur in subsonic case.

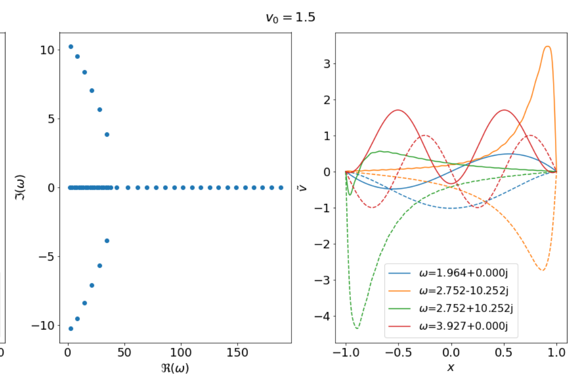
(b) Using $N = 100$.

Figure 7: Increasing N, things become more unstable in supersonic case. Increasing v_0 will increase the magnitude of growth rate of the modes, but the effect is not significant.

2.2.2 Sine

Let $\tilde{v}(z) = \sum_{j=1}^{N} c_j u_j(z)$ where $u_j(z) = \sin\left(\frac{n\pi}{2}(z+1)\right)$.

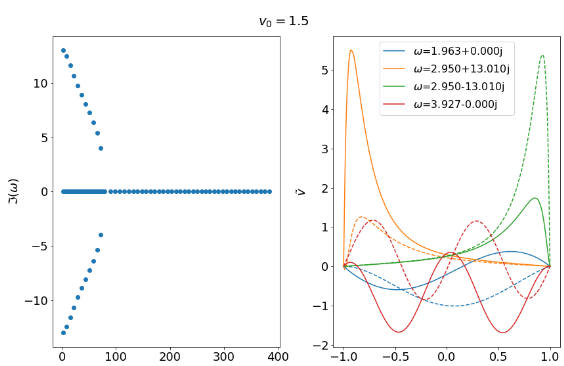


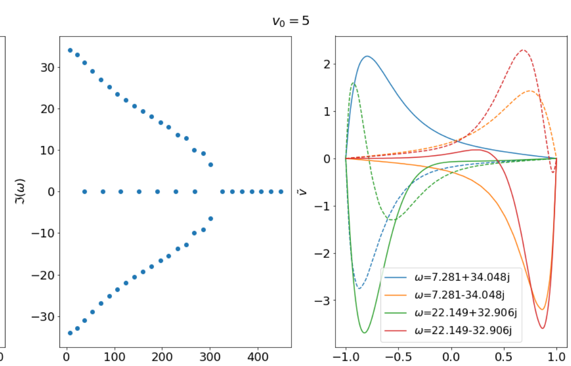


(a) Using $N = 50$.

(b) Using $N = 50$.

Figure 8: Unstable modes in supersonic case.





(a) Using $N = 50$.

(b) Using $N = 50$.

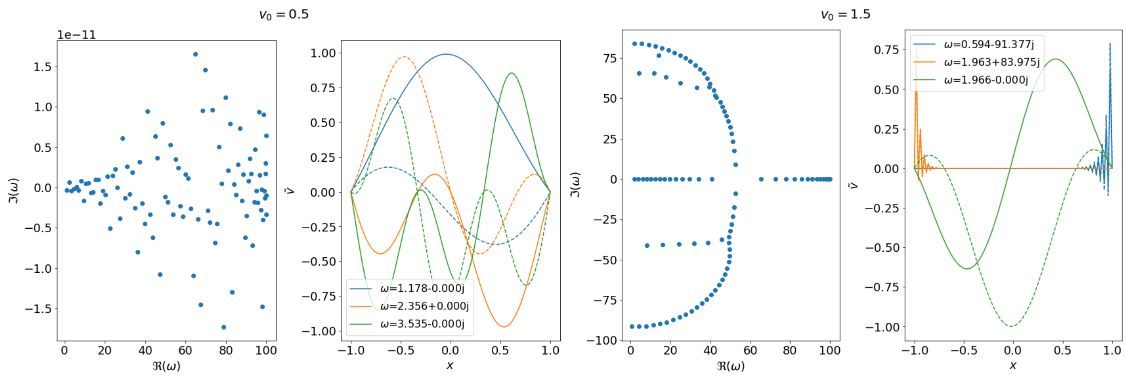
Figure 9: Increasing N or v_0 makes things more unstable in supersonic case.

3 Finite Difference

Expressing the differentiation operator as matrices, we have

$$(\omega^2 A_2 + \omega A_1 + A_0)\tilde{\mathbf{v}} = \mathbf{0}$$

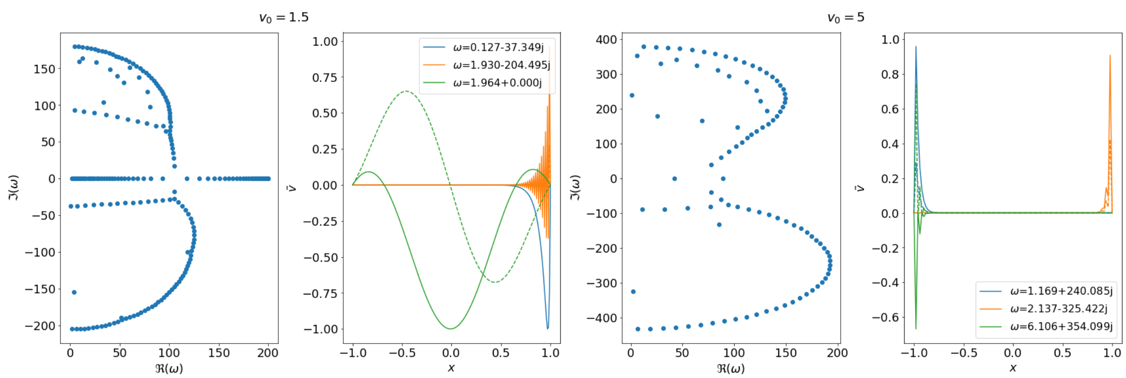
where $A_2 = I$, $A_1 = 2iv_0\, \partial/\partial z$, and $(1-v_0^2)\, \partial^2/\partial z^2$. Lastly, $\tilde{\mathbf{v}} = [\tilde{v}_j]^T$ is the value of $\tilde{v}$ at each grid point $x_j \in [-1, 1]$.



(a) All modes are stable.

(b) Some modes are unstable.

Figure 10: Using $\Delta x = 0.1$, some modes are unstable in supersonic case. In fact, no matter what Δx I use, there will be unstable modes in supersonic case.



(a)

(b)

Figure 11: Increasing N or v_0 makes the unstable modes more unstable in supersonic case.

3.1 Reduction to Normal Form

The equation

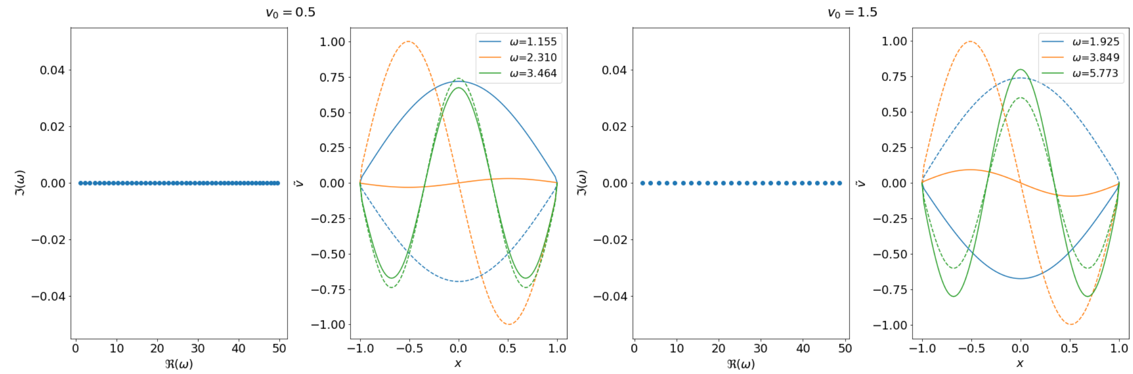
$$\omega^2 \tilde{v}(z) + 2i\omega\tilde{v}'(z) + (1-v_0^2)\tilde{v}''(z) = 0, \tilde{v}(-1) = \tilde{v}(1) = 0$$

can be reduced to normal form

$$u''(z) + r(z)u(z) = 0, u(-1) = u(1) = 0$$

by doing the following variable change

$$u(z) = \exp\left(\frac{iv_0\omega}{1-v_0^2}\right)\tilde{v}(z); \quad r(z) = \frac{\omega^2}{(1-v_0^2)^2}$$



(a) (b)

Figure 12: Finally, all modes are stable.