Stability

(Dated: 29 March 2023)

$$\frac{\partial}{\partial t}n + B\nabla_{\|}(\frac{nV_{\|}}{B}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}V_{\|} + V_{\|}\nabla_{\|}V_{\|} \;\; = -c_s^2\frac{\nabla_{\|}n}{n}, \tag{2}$$

Equilibrium

$$B\nabla_{\|}(\frac{n_0V_0}{B}) = 0, \tag{3}$$

$$V_0\nabla_{\|}V_0 \;\; = -c_s^2\frac{\nabla_{\|}n_0}{n_0}, \tag{4}$$

Linearized, normalized $\widetilde{V}/c_s \to \widetilde{V}$, $V_0/c_s \to V_0$

$$\frac{\partial}{\partial t}\frac{\widetilde{n}}{n_0} + V_0\frac{\partial}{\partial z}\frac{\widetilde{n}}{n_0} + \frac{\partial}{\partial z}\widetilde{V} - \widetilde{V}\frac{V_0^{'}}{V_0} = 0, \tag{5}$$

$$\frac{\partial}{\partial t}\widetilde{V} + V_0\frac{\partial}{\partial z}\widetilde{V} + V_0^{'}\widetilde{V} = -\frac{\partial}{\partial z}\frac{\widetilde{n}}{n_0} \tag{6}$$

Introducing

$$Y_+ = \frac{\widetilde{n}}{n_0} + \widetilde{V} \tag{7}$$

$$Y_- = \frac{\widetilde{n}}{n_0} - \widetilde{V} \tag{8}$$

$$\frac{\partial}{\partial t}Y_+ + (V_0+1)\,\frac{\partial}{\partial z}Y_+ = \frac{1}{2}(Y_+ - Y_-)\,\frac{V_0^{'}}{V_0}\,(1-V_0) \tag{9}$$

$$\frac{\partial}{\partial t}Y_- + (V_0-1)\,\frac{\partial}{\partial z}Y_- = \frac{1}{2}(Y_+ - Y_-)\,\frac{V_0^{'}}{V_0}\,(1+V_0) \tag{10}$$

I. WKB APPROXIMATION

$kL > 1$, $L^{-1} \simeq V_0^{'}/V_0$

$$(Y_+, Y_-) \sim \exp\left(-i\omega t + ikz\right) \tag{11}$$

$$\omega^2 - 2\omega kV_0 + i\omega V_0^{'} + k^2\left(V_0^2 - 1\right) - ik\frac{V_0^{'}}{V_0}\left(1 + V_0^2\right) = 0 \tag{12}$$

for the instability $V_0 < 2c_s$

II. VARIATIONAL FORM

$$Y_+^* \frac{\partial}{\partial t} Y_+ + (V_0+1)\, Y_+^* \frac{\partial}{\partial z} Y_+ = \frac{1}{2} Y_+^* (Y_+ - Y_-)\ \frac{V_0'}{V_0}\,(1-V_0) \tag{13}$$

$$Y_+ \frac{\partial}{\partial t} Y_+^* + (V_0+1)\, Y_+ \frac{\partial}{\partial z} Y_+^* = \frac{1}{2} Y_+ (Y_+^* - Y_-^*)\ \frac{V_0'}{V_0}\,(1-V_0) \tag{14}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\,|Y_+|^2 + (V_0+1)\,\frac{\partial}{\partial z}\,|Y_+|^2 \\ =\ & \frac{1}{2}(|Y_+|^2 - Y_+^* Y_-)\ \frac{V_0'}{V_0}\,(1-V_0) + \frac{1}{2}(|Y_+|^2 - Y_+ Y_-^*)\ \frac{V_0'}{V_0}\,(1-V_0) \end{aligned} \tag{15}$$

$$Y_-^* \frac{\partial}{\partial t} Y_- + (V_0-1)\, Y_-^* \frac{\partial}{\partial z} Y_- = \frac{1}{2} Y_-^* (Y_+ - Y_-)\ \frac{V_0'}{V_0}\,(1+V_0) \tag{16}$$

$$Y_- \frac{\partial}{\partial t} Y_-^* + (V_0-1)\, Y_- \frac{\partial}{\partial z} Y_-^* = \frac{1}{2} Y_- (Y_+^* - Y_-^*)\ \frac{V_0'}{V_0}\,(1+V_0) \tag{17}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\,|Y_-|^2 + (V_0-1)\,\frac{\partial}{\partial z}\,|Y_-|^2 \\ =\ & \frac{1}{2}(Y_-^* Y_+ - |Y_-|^2)\ \frac{V_0'}{V_0}\,(1+V_0) + \frac{1}{2}(Y_- Y_+^* - |Y_-|^2)\ \frac{V_0'}{V_0}\,(1+V_0) \end{aligned} \tag{18}$$

Combining

$$\begin{aligned} &\frac{\partial}{\partial t}\,|Y_+|^2 + (V_0+1)\,\frac{\partial}{\partial z}\,|Y_+|^2 \\ =\ & |Y_+|^2 \quad \frac{V_0'}{V_0}\,(1-V_0) - \frac{1}{2}(Y_+ Y_-^* + Y_+^* Y_-)\ \frac{V_0'}{V_0}\,(1-V_0) \end{aligned} \tag{19}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\,|Y_-|^2 + (V_0-1)\,\frac{\partial}{\partial z}\,|Y_-|^2 \\ =\ & -|Y_-|^2\ \ \frac{V_0'}{V_0}\,(1+V_0) + \frac{1}{2}(Y_- Y_+^* \ + Y_-^* Y_+)\ \frac{V_0'}{V_0}\,(1+V_0) \end{aligned} \tag{20}$$

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$$\frac{1}{(1-V_0)}\frac{\partial}{\partial t}|Y_+|^2+\frac{(V_0+1)}{(1-V_0)}\frac{\partial}{\partial z}|Y_+|^2$$
$$=|Y_+|^2\quad\frac{V_0'}{V_0}-\frac{1}{2}(Y_+Y_-^*+Y_+^*Y_-)\ \frac{V_0'}{V_0}\tag{21}$$

$$\frac{1}{(1+V_0)}\frac{\partial}{\partial t}|Y_-|^2+\frac{(V_0-1)}{(1+V_0)}\frac{\partial}{\partial z}|Y_-|^2$$
$$=-|Y_-|^2\ \frac{V_0'}{V_0}\ +\frac{1}{2}(Y_-Y_+^*\ +Y_-^*Y_+)\ \frac{V_0'}{V_0}\tag{22}$$

One form

$$\frac{1}{(1-V_0)}\frac{\partial}{\partial t}|Y_+|^2+\frac{1}{(1+V_0)}\frac{\partial}{\partial t}|Y_-|^2$$
$$+\frac{(V_0+1)}{(1-V_0)}\frac{\partial}{\partial z}|Y_+|^2+\frac{(V_0-1)}{(1+V_0)}\frac{\partial}{\partial z}|Y_-|^2\tag{23}$$
$$=\left(|Y_+|^2\ -|Y_-|^2\right)\quad\frac{V_0'}{V_0}\tag{24}$$

Using

$$\frac{(V_0-1)\,(V_0+1)}{V_0}\left(\frac{V_0+1}{V_0-1}\frac{\partial}{\partial z}|Y_+|^2+|Y_+|^2\quad\frac{V_0'}{V_0}\right)=\frac{\partial}{\partial z}\left(\frac{(V_0+1)^2}{V_0}|Y_+|^2\right)\tag{25}$$
$$\frac{(V_0-1)\,(V_0+1)}{V_0}\left(\frac{(V_0-1)}{(1+V_0)}\frac{\partial}{\partial z}|Y_-|^2+|Y_-|^2\quad\frac{V_0'}{V_0}\right)=\frac{\partial}{\partial z}\left(\frac{(V_0-1)^2}{V_0}|Y_-|^2\right)\tag{26}$$
$$\left(\frac{V_0+1}{V_0-1}\frac{\partial}{\partial z}|Y_+|^2+|Y_+|^2\quad\frac{V_0'}{V_0}\right)=\frac{V_0}{(V_0-1)\,(V_0+1)}\frac{\partial}{\partial z}\left(\frac{(V_0+1)^2}{V_0}|Y_+|^2\right)\tag{27}$$
$$\left(\frac{V_0-1}{1+V_0}\frac{\partial}{\partial z}|Y_-|^2+|Y_-|^2\quad\frac{V_0'}{V_0}\right)=\frac{V_0}{(V_0-1)\,(V_0+1)}\frac{\partial}{\partial z}\left(\frac{(V_0-1)^2}{V_0}|Y_-|^2\right)\tag{28}$$

Finally

$$-\qquad\frac{(V_0+1)}{V_0}\frac{\partial}{\partial t}|Y_+|^2+\frac{V_0-1}{V_0}\frac{\partial}{\partial t}|Y_-|^2=$$
$$=\frac{\partial}{\partial z}\left(\frac{(V_0+1)^2}{V_0}|Y_+|^2\right)\ -\ \frac{\partial}{\partial z}\left(\frac{(V_0-1)^2}{V_0}|Y_-|^2\right)\tag{29}$$

$$\frac{\partial}{\partial t}\left[\frac{V_0-1}{V_0}|Y_-|^2-\frac{(V_0+1)}{V_0}|Y_+|^2\right]=$$
$$=\frac{\partial}{\partial z}\left[\frac{(V_0+1)^2}{V_0}|Y_+|^2-\ \frac{(V_0-1)^2}{V_0}|Y_-|^2\right]\tag{30}$$

Over interval $[a, b]$

$$2\gamma = \left[\frac{(V_0-1)^2}{V_0}|Y_-|^2 - \frac{(V_0+1)^2}{V_0}|Y_+|^2\right]_a^b \left[\int_a^b \left(\frac{1-V_0}{V_0}|Y_-|^2 + \frac{(V_0+1)}{V_0}|Y_+|^2\right) dz\right]^{-1} \tag{31}$$

outgoing boundary condition at $z = b : Y_+ \neq 0$

III. ALTERNATIVE

$$\omega^2 \widetilde{V} + 2i\omega\left(V_0\widetilde{V}\right)' + \left[V_0\left(\frac{\widetilde{V}}{V_0}\right)' - V_0\left(V_0\widetilde{V}\right)'\right]' = 0 \tag{32}$$

IV. PIECE-WISE MODEL

$$B = \begin{array}{ll} B_0 & -1 < z < -\delta \\ B_m & -\delta < z < \delta \\ B_0 & \delta < z < 1 \end{array} \tag{33}$$

$$V_0 = \begin{array}{ll} V_< & -1 < z < -\delta \\ V_m = 1 & -\delta < z < \delta \\ V_> & \delta < z < 1 \end{array} \tag{34}$$

$$\frac{\partial}{\partial t}\frac{\widetilde{n}}{n_0} + V_0\frac{\partial}{\partial z}\frac{\widetilde{n}}{n_0} + \frac{\partial}{\partial z}\widetilde{V} - \widetilde{V}\frac{V_0'}{V_0} = 0, \tag{35}$$

$$\frac{\partial}{\partial t}\widetilde{V} + V_0\frac{\partial}{\partial z}\widetilde{V} + V_0'\widetilde{V} = -\frac{\partial}{\partial z}\frac{\widetilde{n}}{n_0} \tag{36}$$

Introducing

$$Y_+ = \frac{\widetilde{n}}{n_0} + \widetilde{V} \tag{37}$$

$$Y_- = \frac{\widetilde{n}}{n_0} - \widetilde{V} \tag{38}$$

$$\frac{\partial}{\partial t}Y_+ + (V_0+1)\frac{\partial}{\partial z}Y_+ = \frac{1}{2}(Y_+ - Y_-)\frac{V_0'}{V_0}(1 - V_0) \tag{39}$$

$$\frac{\partial}{\partial t}Y_- + (V_0 - 1)\frac{\partial}{\partial z}Y_- = \frac{1}{2}(Y_+ - Y_-)\,\frac{V_0'}{V_0}\,(1 + V_0) \tag{40}$$

$$(Y_+, Y_-) \sim \exp(-i\omega t) \tag{41}$$

Then general solution in the piecewise model are

$$Y_\pm = C_\pm \exp\left(\frac{i\omega z}{V_0 \pm 1}\right) \tag{42}$$

Respectively,

$$\widetilde{N} = C_+ \exp\left(\frac{i\omega z}{V_0 + 1}\right) + C_- \exp\left(\frac{i\omega z}{V_0 - 1}\right) \tag{43}$$

$$\widetilde{V} = C_+ \exp\left(\frac{i\omega z}{V_0 + 1}\right) - C_- \exp\left(\frac{i\omega z}{V_0 - 1}\right) \tag{44}$$

Equilibrium

$$\left(\frac{n_0 V_0}{B_0}\right) = G_0 \tag{45}$$

$$\frac{V_0(z)^2}{2} + \ln\left(n_0(z)/\overline{n}\right) = H_0 \tag{46}$$

Boundary conditions at the interfaces

$$\left[\frac{n_0\widetilde{V}}{B_0} + \frac{\widetilde{n}V_0}{B_0}\right]_{-\delta-\varepsilon}^{-\delta+\varepsilon} = 0 \tag{47}$$

$$\left[V_0\widetilde{V} + \frac{\widetilde{n}}{n_0}\right]_{-\delta-\varepsilon}^{-\delta+\varepsilon} = 0 \tag{48}$$

$$\left[\frac{n_0\widetilde{V}}{B_0} + \frac{\widetilde{n}V_0}{B_0}\right]_{\delta-\varepsilon}^{\delta+\varepsilon} = 0 \tag{49}$$

$$\left[V_0\widetilde{V} + \frac{\widetilde{n}}{n_0}\right]_{\delta-\varepsilon}^{\delta+\varepsilon} = 0 \tag{50}$$

Note that for $-\delta < z < \delta$ when B_m is at maximum, we have $Y_- = 0$ and (chose $C_m = 1$) for convenience

$$\widetilde{N}_m = \exp\left(\frac{i\omega z}{2}\right) \tag{51}$$

$$\widetilde{V}_m = \exp\left(\frac{i\omega z}{2}\right) \tag{52}$$

For $V_0 < 1$, meaningful boundary conditions can be imposed on Y_+ and on Y_- at both ends. For $V_0 > 1$ it seems that conditions are meaningful only at $z = -1$. For accelerating profiles, the conditions are at $z = -1$ and at the critical point $z = z_m$.