Instability In Magnetic Nozzle and Spectral Pollution

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Abstract

This is the abstract of my thesis.

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I would like to express my deepest appreciation to my professor, Andrei Smolyakov. I am also grateful to the former student of my professor, Ivan Khalzov.

To my wife and my parents.

Contents

1 Introduction

1. What is magnetic nozzle

The magnetic field in a magnetic nozzle is created by a set of coils. The configuration is axisymmetric.

2. History, motivation of creating magnetic nozzle. The configuration of magnetic nozzle is important in many areas. [1]

2. Why care about the stability.

2 Governing Equations

2.1 Equations of Motion

In magnetic nozzle, the magnetic field is along z-axis. The charged particles gyrates about the magnetic field lines, so the velocity of particles can be written as $\mathbf{v} = v\mathbf{B}/B$. Therefore the conservation of density

$$\begin{aligned}\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{v}) &= 0\\ \frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\left(nv\frac{\mathbf{B}}{B}\right) &= 0\\ \frac{\partial n}{\partial t} + B\frac{\partial}{\partial z}\left(\frac{nv}{B}\right) &= 0\end{aligned}$$

In the derivation, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ is used.

Starting from the conservation of momentum,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\boldsymbol{\nabla}p$$

Let $\boldsymbol{\nabla}p = k_B T\ \partial n/\partial z$, we have

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -c_s^2\frac{1}{n}\frac{\partial n}{\partial z}$$

where $c_s^2 = k_B T/m$ is the square of sound speed.

The magnetic field is given by

$$B(z) = B_0\left[1 + R\exp\left(-\left(\frac{x}{\delta}\right)^2\right)\right]$$

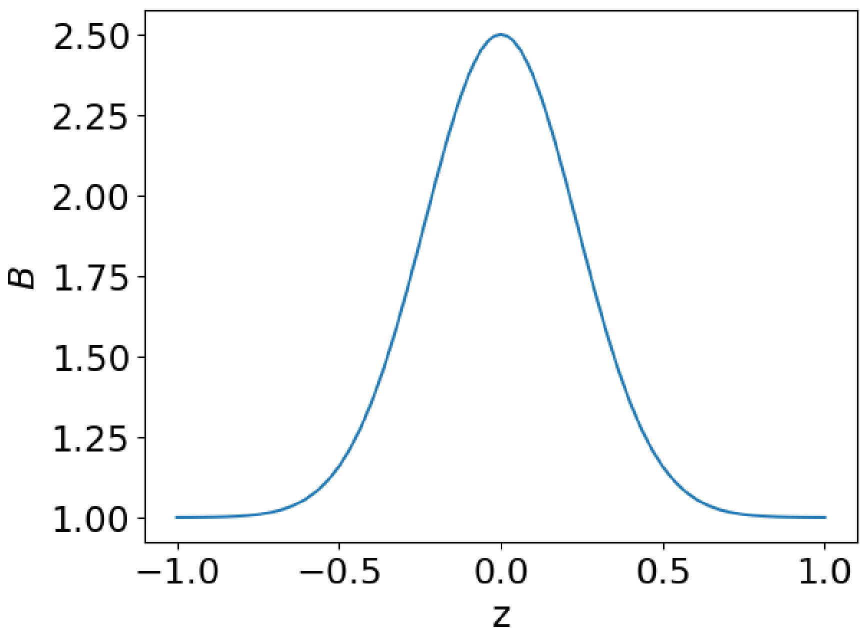

Figure 2.1: This is the magnetic field in nozzle with mirror ratio $1 + R = B_{max}/B_{min} = 2.5$, and the spread of magnetic field, $\delta = 0.1/0.3 = 0.\bar{3}$.

At equilibrium, $\frac{\partial}{\partial t} = 0$. Denote n_0 and v_0 as the equilibrium density and velocity profile, they satisfy

$$B \frac{\partial}{\partial z}\left(\frac{n_0 v_0}{B}\right) = 0$$
$$v_0 \frac{\partial v_0}{\partial z} = -c_s^2 \frac{1}{n_0} \frac{\partial n_0}{\partial z}$$

2.2 Velocity Profile

Let $M(z) = v_0(z)/c_s$ be the mach number (nondimensionalized velocity). The equations of motion become

$$B \frac{\partial}{\partial z}\left(\frac{n_0 M}{B}\right) = 0$$
$$M \frac{\partial M}{\partial z} = -\frac{1}{n_0} \frac{\partial n_0}{\partial z}$$

Substitute $\frac{1}{n_0}\, \partial n_0/\partial z$ using first equation, the conservation of momentum becomes

$$(M^2 - 1) \frac{\partial M}{\partial z} = -\frac{M}{B} \frac{\partial B}{\partial z}$$

Notice that there is a singularity at $M = 1$, the sonic speed.

This is a separable equation, integrate it and use the conditions at midpoint $B(0) = B_m, M(0) = M_m$ we get

$$M^2 e^{-M^2} = \frac{B^2}{B_m^2} M_m^2 e^{-M_m^2}$$

We can now express M using the Lambert W function,

$$M(z) = \left[-W_k \left(-\frac{B(z)^2}{B_m^2} M_m^2 e^{-M_m^2} \right) \right]^{1/2}$$

where the subscript k of W stands for branch of Lambert W function. When $k = 0$, it is the subsonic branch; When $k = -1$, it is the supersonic branch.

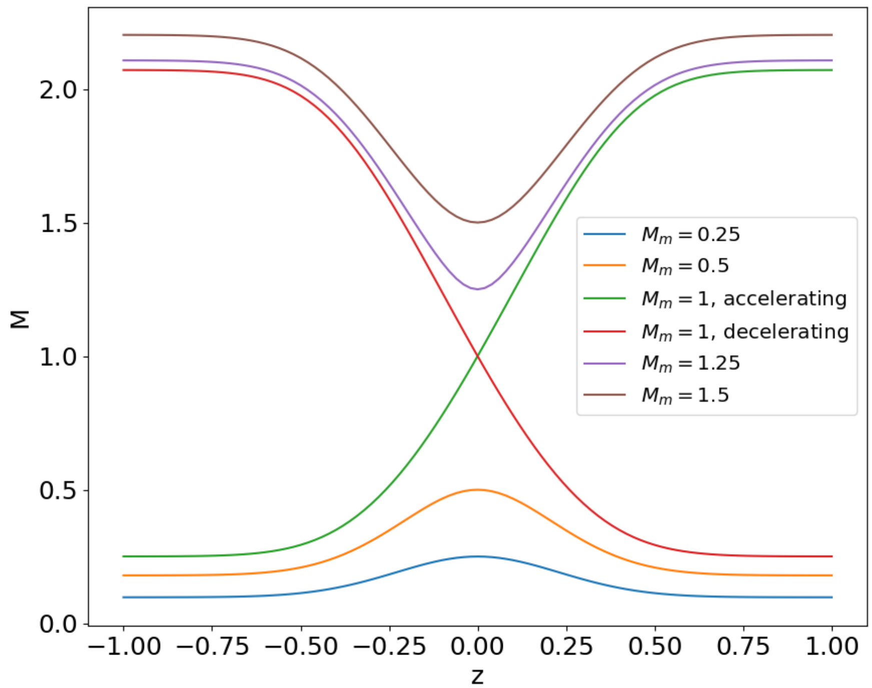

Figure 2.2: The velocity profile in the magnetic nozzle is completely determined by M_m, the velocity at the midpoint, $z = 0$. For the transonic velocity profiles, M_m alone is not enough to determine the profile, we need to specify the branch of Lambert W function to determine whether it is accelerating or decelerating.

2.3 Linearized Equations

The dynamics of magnetic nozzle can be characterized by conservation of mass and momentum,

$$\frac{\partial n}{\partial t} + B \frac{\partial}{\partial z}\left(\frac{nv}{B}\right) = 0$$
$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = -c_s^2 \frac{1}{n} \frac{\partial n}{\partial z}$$

Usually, the magnetic field can be described by

$$B(z) = B_0 \left[1 + R \exp\left(-\frac{z^2}{\delta^2} \right) \right]$$

where R and δ are some coefficients.

At equilibrium (stationary solution), we have $\partial n_0/\partial t = 0$ and $\partial v_0/\partial t = 0$, so n_0 and v_0 satisfy

$$\frac{\partial}{\partial z}\left(\frac{n_0 v_0}{B}\right) = 0$$
$$v_0\frac{\partial v_0}{\partial z} = -c_s^2\frac{1}{n_0}\frac{\partial n_0}{\partial z}$$

Let $M \equiv v_0/c_s$, then it can be represented by Lambert function,

$$M = \left[-W\left(-M_m^2\frac{B(z)^2}{B_m^2}e^{-M_m^2}\right)\right]^{1/2}$$

where $B_m \equiv 1 + R$ is the maximum magnetic field (or magnetic field at mid-point), and M_m is the mach number at mid-point. Below shows a few cases of the solution.

- $M_m < 1$, subsonic velocity profile.
- $M_m = 1$, accelerating or decelerating profile (depending on the branch of the Lambert function).
- $M_m > 1$, supersonic velocity profile

For convenience, we nondimensionalize the equations by normalizing the velocity to c_s, $v \mapsto v/c_s$, z to system length L, $z \mapsto z/L$ and time $t \mapsto c_s t/L$.

$$\frac{\partial n}{\partial t} + n\frac{\partial v}{\partial z} + v\frac{\partial n}{\partial z} - nv\frac{\partial_z B}{B} = 0 \tag{2.1}$$
$$n\frac{\partial v}{\partial t} + nv\frac{\partial v}{\partial z} = -\frac{\partial n}{\partial z} \tag{2.2}$$

and the nondimensionalized equilibrium condition is

$$\frac{\partial}{\partial z}\left(\frac{n_0 v_0}{B}\right) = 0 \tag{2.3}$$
$$v_0\frac{\partial v_0}{\partial z} = -\frac{1}{n_0}\frac{\partial n_0}{\partial z} \tag{2.4}$$

Proposition 1. *Let $n = n_0(z) + \tilde{n}(z,t)$ and $v = v_0(z) + \tilde{v}(z,t)$, the linearized equations of motion are*

$$\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0\tilde{Y} + \tilde{v}\frac{\partial_z n_0}{n_0} - \tilde{v}\frac{\partial_z B}{B} = 0 \tag{2.5}$$
$$\frac{\partial \tilde{v}}{\partial t} + \frac{\partial(v_0\tilde{v})}{\partial z} = -\tilde{Y} \tag{2.6}$$

where

$$\tilde{Y} \equiv \frac{1}{n_0}\frac{\partial \tilde{n}}{\partial z} - \frac{\partial_z n_0}{n_0^2}\tilde{n} = \frac{\partial}{\partial z}\left(\frac{\tilde{n}}{n_0}\right)$$

Proof. We first derive Eq.(2.5). We linearize Eq.(2.3) by setting $n = n_0 + \tilde{n}$ and $v = v_0 + \tilde{v}$. By ignoring the second order perturbations, we obtain

$$\frac{\partial(n_0+\tilde{n})}{\partial t} + (n_0+\tilde{n})\frac{\partial(v_0+\tilde{v})}{\partial z} + (v_0+\tilde{v})\frac{\partial(n_0+\tilde{n})}{\partial z} - (n_0+\tilde{n})(v_0+\tilde{v})\frac{\partial_z B}{B} = 0$$
$$\Rightarrow \frac{\partial \tilde{n}}{\partial t} + n_0\frac{\partial v_0}{\partial z} + \tilde{n}\frac{\partial v_0}{\partial z} + n_0\frac{\partial \tilde{v}}{\partial z} + v_0\frac{\partial n_0}{\partial z} + \tilde{v}\frac{\partial n_0}{\partial z} + v_0\frac{\partial \tilde{n}}{\partial z} - (n_0v_0 + n_0\tilde{v} + \tilde{n}v_0)\frac{\partial_z B}{B} = 0$$
$$\Rightarrow \frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial v_0}{\partial z} + \frac{\tilde{n}}{n_0}\frac{\partial v_0}{\partial z} + \frac{\partial \tilde{v}}{\partial z} + \frac{v_0}{n_0}\frac{\partial n_0}{\partial z} + \frac{\tilde{v}}{n_0}\frac{\partial n_0}{\partial z} + \frac{v_0}{n_0}\frac{\partial \tilde{n}}{\partial z} - v_0\frac{\partial_z B}{B} - \tilde{v}\frac{\partial_z B}{B} - \tilde{n}\frac{v_0}{n_0}\frac{\partial_z B}{B} = 0$$

Using the equilibrium condition Eq.(2.3), some of the terms are canceled and the last term can be written as

$$\tilde{n}\frac{v_0}{n_0}\frac{\partial_z B}{B} = \frac{\tilde{n}}{n_0}\left(\frac{\partial_z n_0}{n_0}v_0 + \frac{\partial v_0}{\partial z}\right)$$

Now, we are left with equation

$$\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0 \underbrace{\left(\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial z} - \frac{\tilde{n}}{n_0}\frac{\partial_z n_0}{n_0}\right)}_{\tilde{Y}} + \frac{\tilde{v}}{n_0}\frac{\partial n_0}{\partial z} - \tilde{v}\frac{\partial_z B}{B} = 0$$

To derive Eq.(2.6), we linearize the LHS of the conservation of momemtum

$$\begin{aligned}
&(n_0 + \tilde{n})\frac{\partial(v_0 + \tilde{v})}{\partial t} + (n_0 + \tilde{n})(v_0 + \tilde{v})\frac{\partial(v_0 + \tilde{v})}{\partial z} = -\frac{\partial n}{\partial z} \\
\Rightarrow &\frac{\partial v_0}{\partial t} + \frac{\tilde{n}}{n_0}\frac{\partial v_0}{\partial t} + \frac{\partial \tilde{v}}{\partial t} + \left(v_0 + \tilde{v} + \frac{\tilde{n}}{n_0}v_0\right)\frac{\partial(v_0 + \tilde{v})}{\partial z} = -\frac{1}{n_0}\frac{\partial n}{\partial z} \\
\Rightarrow &\frac{\partial v_0}{\partial t} + v_0\frac{\partial v_0}{\partial z} + \tilde{v}\frac{\partial v_0}{\partial z} = -\frac{1}{n_0}\frac{\partial n_0}{\partial z} - \frac{1}{n_0}\frac{\partial \tilde{n}}{\partial z} - v_0\frac{v_0}{z} - \frac{\tilde{n}}{n_0}v_0\frac{\partial v_0}{\partial z}
\end{aligned}$$

Using the equilibrium condition Eq.(2.4) on the RHS, we get the desired form. □

Bibliography

[1] A. I. Smolyakov, A. Sabo, P. Yushmanov, and S. Putvinskii. On quasineutral plasma flow in the magnetic nozzle. *Physics of Plasmas*, 28(6):060701, June 2021.

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[1] A. I. Smolyakov, A. Sabo, P. Yushmanov, and S. Putvinskii. On quasineutral plasma flow in the magnetic nozzle. *Physics of Plasmas*, 28(6):060701, June 2021.

Appendix A

Sample Appendix

Stuff for this appendix goes here.