Unstable Modes in Magnetic Nozzle

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1 Equations of Motion

1.1 Linearized Equations of Motion

The dynamics of magnetic nozzle can be characterized by conservation of mass and momentum,

$$\begin{aligned}\frac{\partial n}{\partial t} + B\frac{\partial}{\partial z}\left(\frac{nv}{B}\right) = 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -c_s^2 \frac{1}{n}\frac{\partial n}{\partial z}\end{aligned}$$

Usually, the magnetic field can be described by

$$B(z) = B_0 \left[1 + R\exp\left(-\frac{z^2}{\delta^2}\right)\right]$$

where R and δ are some coefficients.

At equilibrium (stationary solution), we have $\partial n_0/\partial t = 0$ and $\partial v_0/\partial t = 0$, so n_0 and v_0 satisfy

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{n_0 v_0}{B}\right) = 0 \\ v_0 \frac{\partial v_0}{\partial z} = -c_s^2 \frac{1}{n_0}\frac{\partial n_0}{\partial z}\end{aligned}$$

Let $M \equiv v_0/c_s$, then it can be represented by Lambert function,

$$M = \left[-W\left(-M_m^2 \frac{B(z)^2}{B_m^2} e^{-M_m^2}\right)\right]^{1/2}$$

where $B_m \equiv 1 + R$ is the maximum magnetic field (or magnetic field at mid-point), and M_m is the mach number at mid-point. Below shows a few cases of the solution.

- $M_m < 1$, subsonic velocity profile.
- $M_m = 1$, accelerating or decelerating profile (depending on the branch of the Lambert function).
- $M_m > 1$, supersonic velocity profile

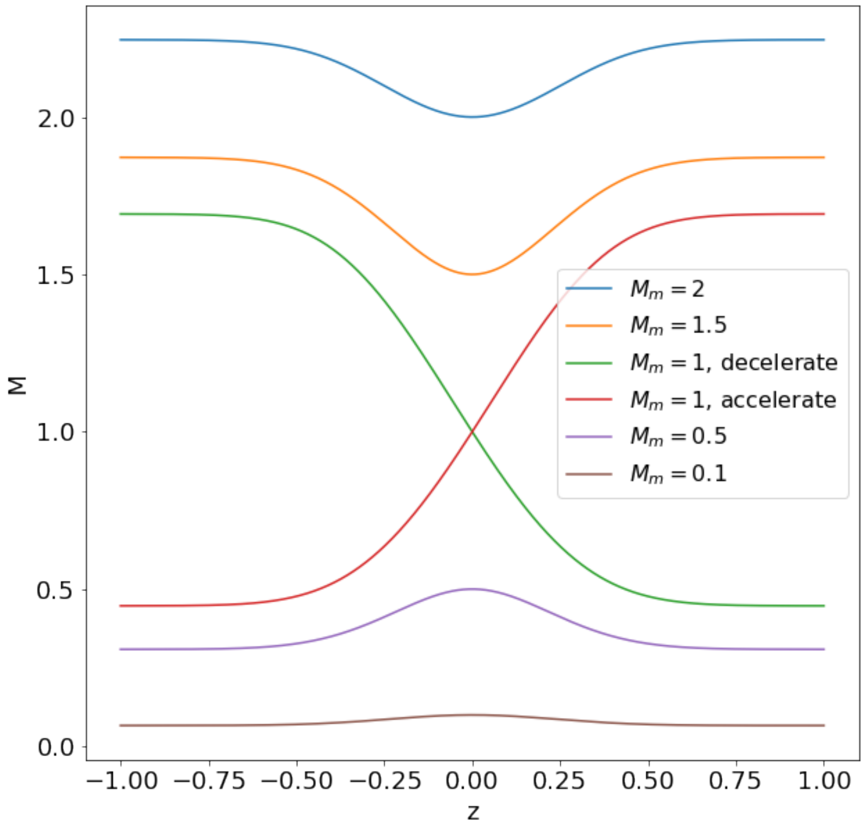


Figure 1: $M_m < 1$, subsonic. $M_m = 1$, accelerating if we select Lambert function branch $k = 0$ for $z < 0$ and brach $k = -1$ for $z >= 0$; decelerating if we choose branch $k = -1$ for $z < 0$ and brach $k = 0$ for $z >= 0$. $M_m > 1$, supersonic.

For convenience, we nondimensionalize the equations by normalizing the velocity to c_s, $v \mapsto v/c_s$, z to system length L, $z \mapsto z/L$ and time $t \mapsto c_s t/L$.

$$\frac{\partial n}{\partial t} + n\frac{\partial v}{\partial z} + v\frac{\partial n}{\partial z} - nv\frac{\partial_z B}{B} = 0 \tag{1}$$

$$n\frac{\partial v}{\partial t} + nv\frac{\partial v}{\partial z} = -\frac{\partial n}{\partial z} \tag{2}$$

and the nondimensionalized equilibrium condition is

$$\frac{\partial}{\partial z}\left(\frac{n_0 v_0}{B}\right) = 0 \tag{3}$$

$$v_0\frac{\partial v_0}{\partial z} = -\frac{1}{n_0}\frac{\partial n_0}{\partial z} \tag{4}$$

Proposition 1. *Let $n = n_0(z) + \tilde{n}(z,t)$ and $v = v_0(z) + \tilde{v}(z,t)$, the linearized equations of motion are*

$$\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0\tilde{Y} + \tilde{v}\frac{\partial_z n_0}{n_0} - \tilde{v}\frac{\partial_z B}{B} = 0 \tag{5}$$

$$\frac{\partial \tilde{v}}{\partial t} + \frac{\partial (v_0\tilde{v})}{\partial z} = -\tilde{Y} \tag{6}$$

where

$$\tilde{Y} \equiv \frac{1}{n_0}\frac{\partial \tilde{n}}{\partial z} - \frac{\partial_z n_0}{n_0^2}\tilde{n} = \frac{\partial}{\partial z}\left(\frac{\tilde{n}}{n_0}\right)$$

Proof. We first derive Eq.(5). We linearize Eq.(3) by setting $n = n_0 + \tilde{n}$ and $v = v_0 + \tilde{v}$. By ignoring the second order perturbations, we obtain

$$
\begin{aligned}
&\frac{\partial(n_0+\tilde{n})}{\partial t} + (n_0+\tilde{n})\frac{\partial(v_0+\tilde{v})}{\partial z} + (v_0+\tilde{v})\frac{\partial(n_0+\tilde{n})}{\partial z} - (n_0+\tilde{n})(v_0+\tilde{v})\frac{\partial_z B}{B} = 0 \\
\Rightarrow &\frac{\partial \tilde{n}}{\partial t} + n_0\frac{\partial v_0}{\partial z} + \tilde{n}\frac{\partial v_0}{\partial z} + n_0\frac{\partial \tilde{v}}{\partial z} + v_0\frac{\partial n_0}{\partial z} + \tilde{v}\frac{\partial n_0}{\partial z} + v_0\frac{\partial \tilde{n}}{\partial z} - (n_0 v_0 + n_0\tilde{v} + \tilde{n}v_0)\frac{\partial_z B}{B} = 0 \\
\Rightarrow &\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial v_0}{\partial z} + \frac{\tilde{n}}{n_0}\frac{\partial v_0}{\partial z} + \frac{\partial \tilde{v}}{\partial z} + \frac{v_0}{n_0}\frac{\partial n_0}{\partial z} + \frac{\tilde{v}}{n_0}\frac{\partial n_0}{\partial z} + \frac{v_0}{n_0}\frac{\partial \tilde{n}}{\partial z} - v_0\frac{\partial_z B}{B} - \tilde{v}\frac{\partial_z B}{B} - \tilde{n}\frac{v_0}{n_0}\frac{\partial_z B}{B} = 0
\end{aligned}
$$

Using the equilibrium condition Eq.(3), some of the terms are canceled and the last term can be written as

$$
\tilde{n}\frac{v_0}{n_0}\frac{\partial_z B}{B} = \frac{\tilde{n}}{n_0}\left(\frac{\partial_z n_0}{n_0}v_0 + \frac{\partial v_0}{\partial z}\right)
$$

Now, we are left with equation

$$
\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0\underbrace{\left(\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial z} - \frac{\tilde{n}}{n_0}\frac{\partial_z n_0}{n_0}\right)}_{\tilde{Y}} + \frac{\tilde{v}}{n_0}\frac{\partial n_0}{\partial z} - \tilde{v}\frac{\partial_z B}{B} = 0
$$

To derive Eq.(6), we linearize the LHS of the conservation of momemtum

$$
\begin{aligned}
&(n_0+\tilde{n})\frac{\partial(v_0+\tilde{v})}{\partial t} + (n_0+\tilde{n})(v_0+\tilde{v})\frac{\partial(v_0+\tilde{v})}{\partial z} = -\frac{\partial n}{\partial z} \\
\Rightarrow &\frac{\partial v_0}{\partial t} + \frac{\tilde{n}}{n_0}\frac{\partial v_0}{\partial t} + \frac{\partial \tilde{v}}{\partial t} + \left(v_0 + \tilde{v} + \frac{\tilde{n}}{n_0}v_0\right)\frac{\partial(v_0+\tilde{v})}{\partial z} = -\frac{1}{n_0}\frac{\partial n}{\partial z} \\
\Rightarrow &\frac{\partial v_0}{\partial t} + v_0\frac{\partial v_0}{\partial z} + \tilde{v}\frac{\partial v_0}{\partial z} = -\frac{1}{n_0}\frac{\partial n_0}{\partial z} - \frac{1}{n_0}\frac{\partial \tilde{n}}{\partial z} - v_0\frac{v_0}{z} - \frac{\tilde{n}}{n_0}v_0\frac{\partial v_0}{\partial z}
\end{aligned}
$$

Using the equilibrium condition Eq.(4) on the RHS, we get the desired form. □

1.2 Derivation of Polynomial Eigenvalue Problem

Proposition 2.

$$
\frac{\partial}{\partial z}\ln\left(\frac{n_0}{B}\right) = -\frac{1}{v_0}\frac{\partial v_0}{\partial z} \tag{7}
$$

Proof.

$$
\frac{\partial}{\partial z}\ln\left(\frac{n_0}{B}\right) = \frac{B}{n_0}\frac{\partial}{\partial z}\left(\frac{n_0}{B}\right) = \frac{1}{n_0}\frac{n_0}{z} + B\frac{\partial}{\partial z}\left(\frac{1}{B}\right) = \frac{1}{n_0}\frac{n_0}{z}\underbrace{-\frac{1}{n_0 v_0}\frac{\partial n_0 v_0}{\partial z}}_{Eq.(3)} = -\frac{1}{v_0}\frac{\partial v_0}{\partial z}
$$

□

Proposition 3. *Let $\tilde{n} \sim \exp(-i\omega t)$ and $\tilde{v} \sim \exp(-i\omega t)$, then we have the polynomial eigenvalue problem*

$$
-\Omega^2\tilde{v} + 2\Omega\left(v_0\frac{\partial}{\partial z} + \frac{\partial v_0}{\partial z}\right)\tilde{v} + \left[(1-v_0^2)\frac{\partial^2}{\partial z^2} - \left(3v_0 + \frac{1}{v_0}\right)\frac{\partial v_0}{\partial z}\frac{\partial}{\partial z} - \left(1 - \frac{1}{v_0^2}\right)\left(\frac{\partial v_0}{\partial z}\right)^2 - \left(v_0 + \frac{1}{v_0}\right)\frac{\partial^2 v_0}{\partial z^2}\right]\tilde{v} = 0 \tag{8}
$$

where $\Omega \equiv i\omega$.

Proof.

$$
\begin{aligned}
&\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0\tilde{Y} + \tilde{v}\frac{\partial_z n_0}{n_0} - \tilde{v}\frac{\partial_z B}{B} = 0 \\
&\frac{\partial \tilde{v}}{\partial t} + \frac{\partial(v_0\tilde{v})}{\partial z} = -\tilde{Y}
\end{aligned}
$$

We plug Eq.(6) in to Eq.(5), we have

$$-i\omega\frac{\tilde{n}}{n_0}+\frac{\partial\tilde{v}}{\partial z}-v_0\left(-i\omega\tilde{v}+\frac{\partial(v_0\tilde{v})}{\partial z}\right)+\tilde{v}\frac{\partial_z n_0}{n_0}-\tilde{v}\frac{\partial_z B}{B}=0$$

Using the equilibrium condition Eq.(3), we can eliminate the term $\partial_z B/B$,

$$\begin{aligned}&-i\omega\frac{\tilde{n}}{n_0}+\frac{\partial\tilde{v}}{\partial z}+v_0\left(i\omega\tilde{v}-v_0\frac{\partial\tilde{v}}{\partial z}-\tilde{v}\frac{\partial v_0}{\partial z}\right)-\tilde{v}\frac{\partial_z v_0}{v_0}=0\\ \Rightarrow&-i\omega\frac{\tilde{n}}{n_0}+i\omega v_0\tilde{v}+(1-v_0^2)\frac{\partial\tilde{v}}{\partial z}+\left(v_0+\frac{1}{v_0}\right)\frac{\partial v_0}{\partial z}\tilde{v}=0\end{aligned}$$

Now we take $\partial/\partial t$ on Eq.(6). Recall the fact that $\tilde{Y}=\partial(\tilde{n}/n_0)/\partial z$, we have

$$\begin{aligned}&\omega^2\tilde{v}+i\omega\left(v_0\frac{\partial\tilde{v}}{\partial z}+\tilde{v}\frac{\partial v_0}{\partial z}\right)=\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{\tilde{n}}{n_0}\right)\\ \Rightarrow&\,\omega^2\tilde{v}+i\omega\left(v_0\frac{\partial\tilde{v}}{\partial z}+\tilde{v}\frac{\partial v_0}{\partial z}\right)=\frac{\partial}{\partial z}\left(-i\omega v_0\tilde{v}-(1-v_0^2)\frac{\partial\tilde{v}}{\partial z}+\left(v_0+\frac{1}{v_0}\right)\frac{\partial v_0}{\partial z}\tilde{v}\right)\end{aligned}$$

Expand the RHS and collect terms, we get

$$\begin{aligned}&\omega^2\tilde{v}\\ &+2i\omega\left(v_0\frac{\partial}{\partial z}+\frac{\partial v_0}{\partial z}\right)\tilde{v}\\ &+\left[(1-v_0^2)\frac{\partial^2}{\partial z^2}-\left(3v_0+\frac{1}{v_0}\right)\frac{\partial v_0}{\partial z}\frac{\partial}{\partial z}-\left(1-\frac{1}{v_0^2}\right)\left(\frac{\partial v_0}{\partial z}\right)^2-\left(v_0+\frac{1}{v_0}\right)\frac{\partial^2 v_0}{\partial z^2}\right]\tilde{v}=0\end{aligned}$$

Let $\Omega\equiv i\omega$, we obtain Eq.(8). □

Definition 1. P and Q functions defined below can be used as an indicator of instabilities.

- P-function:

$$P\equiv-\frac{\partial v_0}{\partial z}\left(3v_0+\frac{1}{v_0}\right)\tag{9}$$

- Q-function:

$$Q\equiv-\left(1-\frac{1}{v_0^2}\right)\left(\frac{\partial v_0}{\partial z}\right)^2-\left(v_0+\frac{1}{v_0}\right)\frac{\partial^2 v_0}{\partial z^2}\tag{10}$$

1.3 Instability condition from Variational Form

Proposition 4.

$$(\omega_r^2-\gamma^2)\left\langle|\tilde{v}|^2\right\rangle-2\omega_r\left\langle v_0\,\mathrm{Im}\left(\tilde{v}^*\frac{\partial\tilde{v}}{\partial z}\right)\right\rangle-\gamma\left\langle\frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle-\left\langle(1-v_0^2)\left|\frac{\partial\tilde{v}}{\partial z}\right|^2\right\rangle+\left\langle\left(Q-\frac{1}{2}\frac{\partial P}{\partial z}\right)|\tilde{v}|^2\right\rangle=0\tag{11}$$

This is the full instability condition, we can determine whether the motion is stable or not by examining the number of solutions γ. If there are two real solutions (two real γ), then it's unstable, otherwise it is stable.

Proof. Starting from Eq.(8), we multiply $\tilde{v}^*$ and take the average over the region, we have

$$\omega^2\left\langle|\tilde{v}|^2\right\rangle+2i\omega\left(\left\langle v_0\tilde{v}^*\frac{\partial\tilde{v}}{\partial z}\right\rangle+\left\langle\frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle\right)+\left[\left\langle(1-v_0^2)\tilde{v}^*\frac{\partial^2\tilde{v}}{\partial z^2}\right\rangle+\left\langle P\tilde{v}^*\frac{\partial\tilde{v}}{\partial z}\right\rangle+\left\langle Q|\tilde{v}|^2\right\rangle\right]=0$$

Let $\omega\equiv\omega_r+i\gamma$, then

$$(\omega_r^2-\gamma^2+2i\omega_r\gamma)\left\langle|\tilde{v}|^2\right\rangle+2(i\omega_r-\gamma)\left(\left\langle v_0\tilde{v}^*\frac{\partial\tilde{v}}{\partial z}\right\rangle+\left\langle\frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle\right)+\left\langle(1-v_0^2)\tilde{v}^*\frac{\partial^2\tilde{v}}{\partial z^2}\right\rangle+\left\langle P\tilde{v}^*\frac{\partial\tilde{v}}{\partial z}\right\rangle+\left\langle Q|\tilde{v}|^2\right\rangle=0$$

We split the equation into real and imaginary part.

- Real:
  There are two useful simplifications to keep in mind,

$$\mathrm{Re}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z}\right) = \frac{1}{2}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z} + \tilde{v}\frac{\partial \tilde{v}^*}{\partial z}\right) = \frac{1}{2}\frac{\partial |\tilde{v}|^2}{\partial z}$$

$$\left\langle \mathrm{Re}\left(\tilde{v}^*\frac{\partial^2 \tilde{v}}{\partial z^2}\right)\right\rangle = \left\langle \frac{1}{2}\left(\tilde{v}^*\frac{\partial^2 \tilde{v}}{\partial z^2} + \tilde{v}\frac{\partial^2 \tilde{v}^*}{\partial z^2}\right)\right\rangle = \left\langle \frac{1}{2}\left(\frac{\partial}{\partial z}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\tilde{v}\frac{\partial \tilde{v^*}}{\partial z}\right) - 2\left|\frac{\partial \tilde{v}}{\partial z}\right|^2\right)\right\rangle = \left\langle \left|\frac{\partial \tilde{v}}{\partial z}\right|^2\right\rangle$$

  The second formula uses the fact $\tilde{v} = 0$ on boundaries (Dirichlet condition), so $\langle \partial\text{product of } \tilde{v}/\partial z\rangle = 0$.

  Now we separate the real part,

$$(\omega_r^2 - \gamma^2)\left\langle |\tilde{v}|^2\right\rangle - 2\omega_r\left\langle v_0\,\mathrm{Im}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z}\right)\right\rangle - \gamma\left\langle v_0\frac{\partial |\tilde{v}|^2}{\partial z}\right\rangle - 2\gamma\left\langle \frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle - \left\langle (1-v_0^2)\left|\frac{\partial \tilde{v}}{\partial z}\right|\right\rangle + \left\langle \frac{1}{2}P\frac{\partial |\tilde{v}|^2}{\partial z}\right\rangle + \left\langle Q|\tilde{v}|^2\right\rangle = 0$$

  The term with γ can combine by swithing the position of $\partial/\partial z$,

$$-\gamma\left\langle v_0\frac{\partial |\tilde{v}|^2}{\partial z}\right\rangle - 2\gamma\left\langle \frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle = -\gamma\left\langle \frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle$$

  The P and Q term can combine as well

$$\left\langle \frac{1}{2}P\frac{\partial |\tilde{v}|^2}{\partial z}\right\rangle + \left\langle Q|\tilde{v}|^2\right\rangle = \left\langle \left(Q - \frac{1}{2}\frac{\partial P}{\partial z}\right)|\tilde{v}|^2\right\rangle$$

  The desired instability condition follows.

- Imaginary:
  Although the imaginary part is not important, we derive it too.

  This time, we notice that

$$\left\langle \mathrm{Im}\left(\tilde{v}^*\frac{\partial^2 \tilde{v}}{\partial z^2}\right)\right\rangle = \left\langle \frac{1}{2}\left(\tilde{v}^*\frac{\partial^2 \tilde{v}}{\partial z^2} - \tilde{v}\frac{\partial^2 \tilde{v}^*}{\partial z^2}\right)\right\rangle = \left\langle \frac{1}{2}\left(\frac{\partial}{\partial z}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z}\right) - \frac{\partial}{\partial z}\left(\tilde{v}\frac{\partial \tilde{v^*}}{\partial z}\right)\right)\right\rangle = 0$$

$$2\omega_r\gamma\left\langle |\tilde{v}|^2\right\rangle + \omega_r\left\langle v_0\frac{\partial |\tilde{v}|^2}{\partial z}\right\rangle + 2\omega_r\left\langle \frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle - 2\gamma\left\langle v_0\,\mathrm{Im}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z}\right)\right\rangle + \left\langle P\,\mathrm{Im}\left(\tilde{v}^*\frac{\partial \tilde{v}}{\partial z}\right)\right\rangle = 0$$

□

Proposition 5. *For oscillations where $\omega_r \to 0$, the instability condition becomes*

$$-\gamma^2\left\langle |\tilde{v}|^2\right\rangle + \gamma\left\langle \frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle - \left\langle (1-v_0^2)\left|\frac{\partial \tilde{v}}{\partial z}\right|^2\right\rangle + \left\langle \left(Q - \frac{1}{2}\frac{\partial P}{\partial z}\right)|\tilde{v}|^2\right\rangle = 0$$

Therefore, unstable modes occur if

$$\left\langle \frac{\partial v_0}{\partial z}|\tilde{v}|^2\right\rangle^2 + 4\left\langle |\tilde{v}|^2\right\rangle\left(-\left\langle (1-v_0^2)\left|\frac{\partial \tilde{v}}{\partial z}\right|^2\right\rangle + \left\langle \left(Q - \frac{1}{2}\frac{\partial P}{\partial z}\right)|\tilde{v}|^2\right\rangle\right) > 0$$

2 Numerical Experiments

The stability condition Eq.(11 is a quadratic equation about γ. Let

$$
\begin{aligned}
a &= -\left\langle |\tilde{v}|^{2} \right\rangle \\
b &= -\left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^{2} \right\rangle \\
c &= \omega_r^2 \left\langle |\tilde{v}|^{2} \right\rangle - 2\omega_r \left\langle v_0 \operatorname{Im}\left(\tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) \right\rangle - \left\langle (1 - v_0^2) \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle + \left\langle \left(Q - \frac{1}{2} \frac{\partial P}{\partial z} \right) |\tilde{v}|^{2} \right\rangle = 0
\end{aligned}
$$

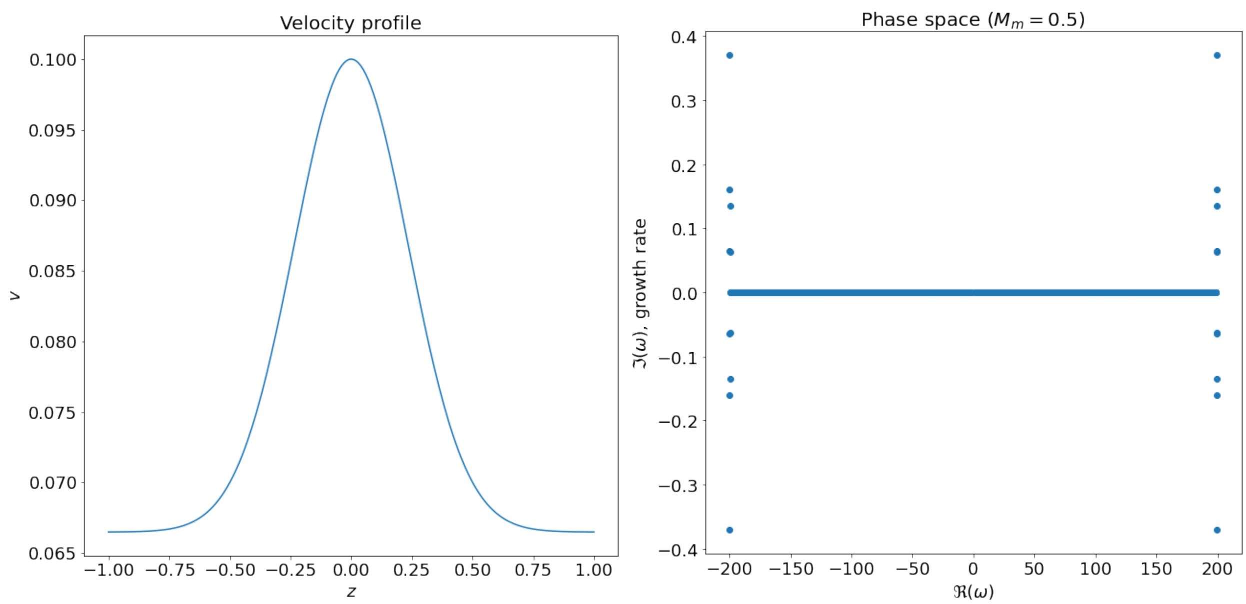
then the full instability condition becomes

$$D \equiv b^2 - 4ac$$

If the discriminant $D > 0$, it is unstable. If $D < 0$, it is stable.

2.1 Subsonic case

Subsonic modes are stable. The stability condition $D < 0$. Moreover, the eigenvalues are real.



(a) Subsonic velocity profile.

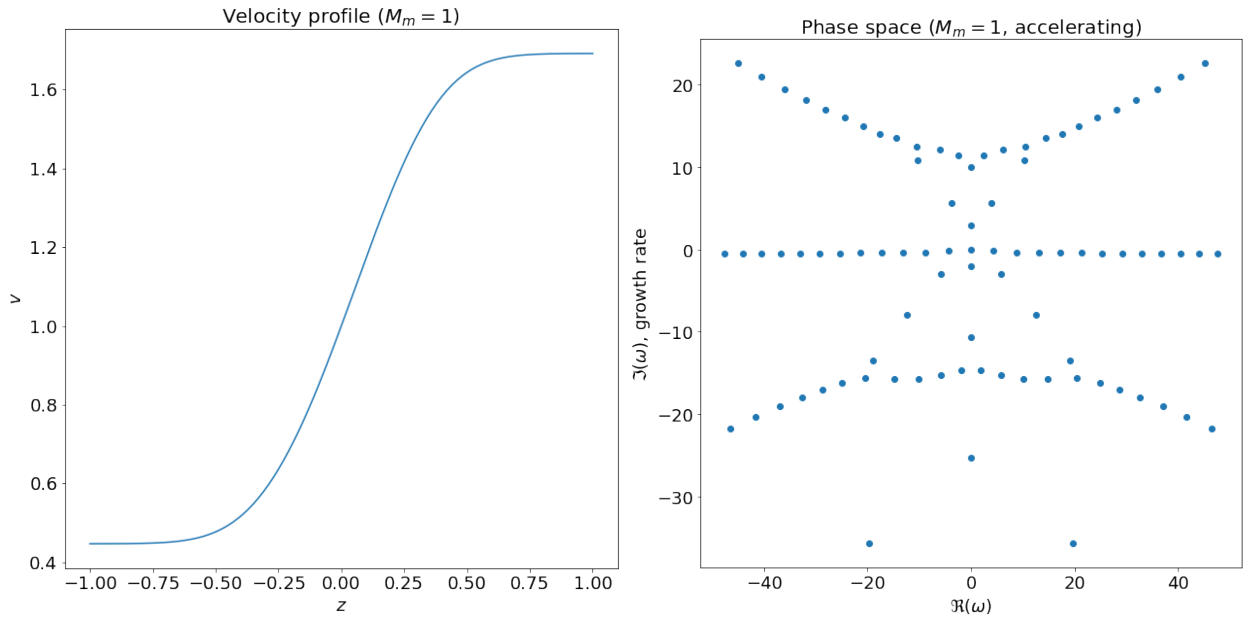
(b) ω's are real, meaning that it is stable.

Figure 2: Solution to the polynomial eigenvalue problem Eq.(8) with subsonic velocity profile. ω's are pure imaginary numbers, meaning that subsonic mode should be stable for most cases.

2.2 Accelerating case

Accelerating modes are unstable.

The stability condition for ground state is $D = 0.00062 > 0$.



(a) Accelerating velocity profile.

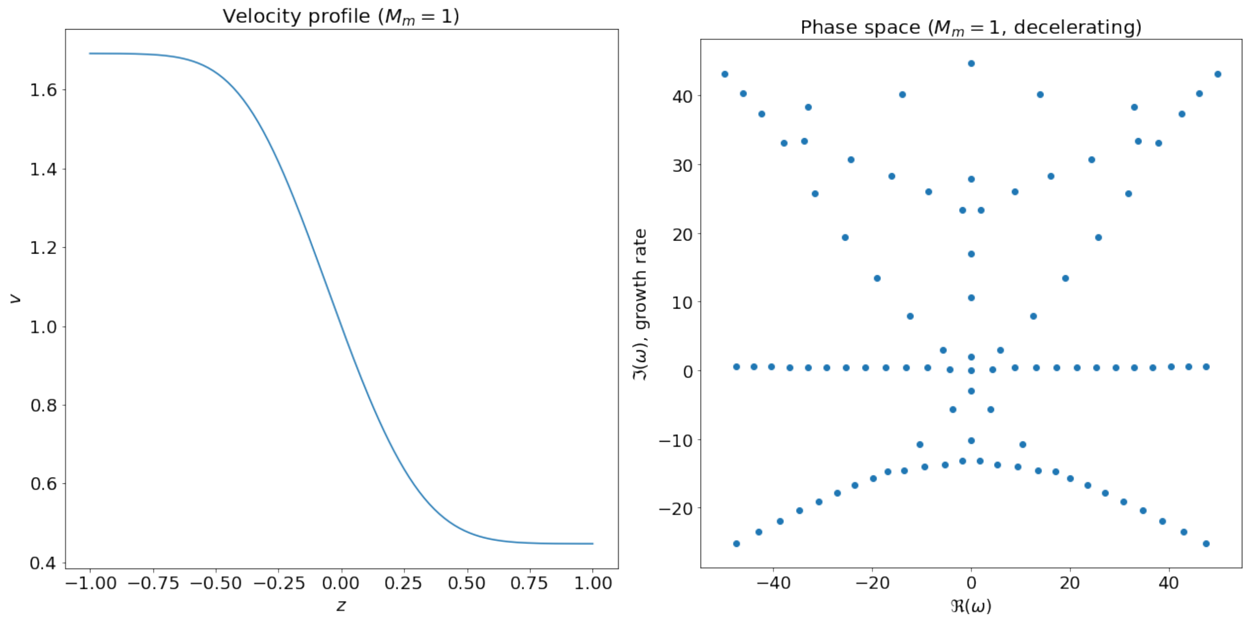
(b) Some modes are unstable.

Figure 3: Solution to the polynomial eigenvalue problem Eq.(8) with accelerating velocity profile. ω's are not on the real line, meaning that accelerating mode should be unstable for most cases.

2.3 Decelerating case

Decelerating modes are unstable.

The stability condition for ground state is $D = 0.00062 > 0$.



(a) Decelerating velocity profile.

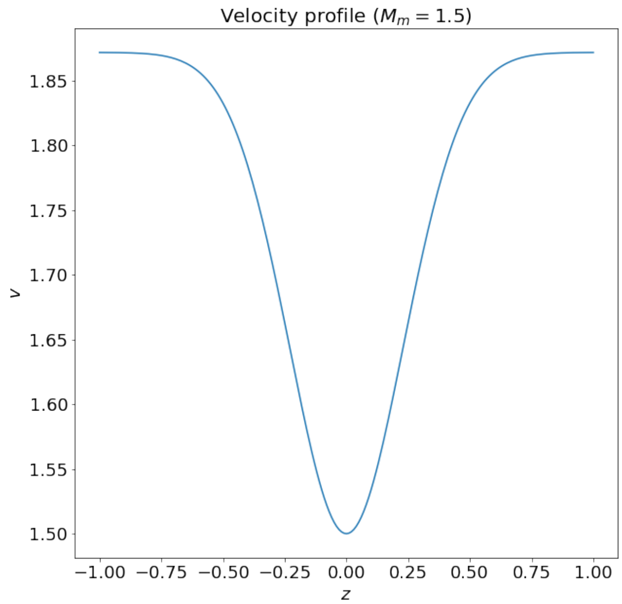
(b) Some modes are unstable.

Figure 4: Solution to the polynomial eigenvalue problem Eq.(8) with decelerating velocity profile. ω's are not on the real line, meaning that decelerating mode should be unstable for most cases.

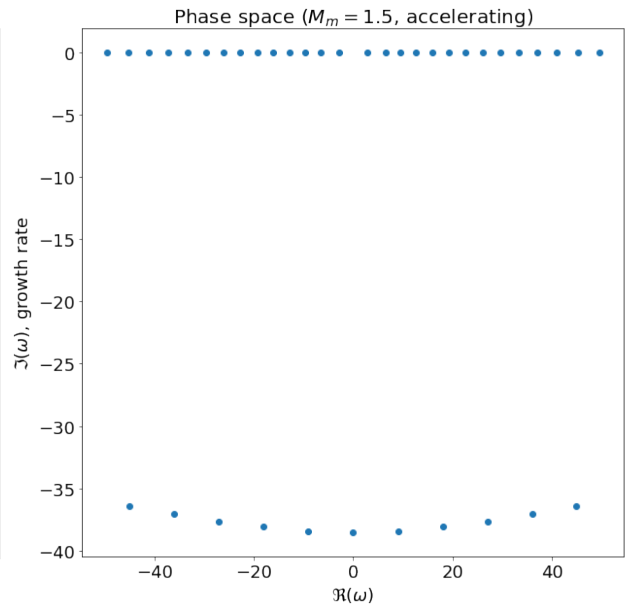
2.4 Supersonic case

Supersonic modes are unstable.

The stability condition for ground state is $D = 1.21025 > 0$.



(a) Supersonic velocity profile.



(b) Some modes are unstable.

Figure 5: Solution to the polynomial eigenvalue problem Eq.(8) with supersonic velocity profile. ω's are not on the real line, meaning that supersonic mode should be unstable for most cases.