Summary

$$\omega^2 \hat{v} + 2i\omega\left(V_0 \frac{\partial \hat{v}}{\partial z} + \hat{v}\frac{\partial V_0}{\partial z}\right) + \frac{\partial^2 \hat{v}}{\partial z^2}(1-V_0^2)$$
$$+ P\frac{\partial \hat{v}}{\partial z} + Q\hat{v} = 0$$

$$P \equiv -V_0'\left(\frac{1}{V_0} + 3V_0\right) \qquad Q = V_0'^2\left(\frac{1}{V_0^2} - 1\right) - V_0''\left(V_0 + \frac{1}{V_0}\right)$$

$$\omega = \omega_r + i\gamma \qquad |\hat{v}|^2 = \hat{v}\hat{v}^*$$

Variational forms

$$(\omega_r^2 - \gamma^2)\langle|\hat{v}|^2\rangle - 2\omega_r\left\langle V_0 \operatorname{Im}\left(\hat{v}\frac{\partial \hat{v}^*}{\partial z}\right)\right\rangle$$
$$- \gamma\langle V_0' |\hat{v}|^2\rangle - \left\langle (1-V_0^2)\left|\frac{\partial \hat{v}}{\partial z}\right|^2\right\rangle$$
$$+ \left\langle (Q - P')\, |\hat{v}|^2\right\rangle = 0$$

$$2\omega_r\gamma\langle|\hat{v}|^2\rangle + \omega_\gamma\langle|\hat{v}|^2 V_0'\rangle$$
$$+ 2\gamma\left\langle V_0 \operatorname{Im}\left(\hat{v}\frac{\partial \hat{v}^*}{\partial z}\right)\right\rangle$$
$$+ \left\langle P \operatorname{Im}\left(\hat{v}^*\frac{\partial \hat{v}}{\partial z}\right)\right\rangle = 0$$

Assume $\operatorname{Im}\left(\hat{v}\frac{\partial \hat{v}^*}{\partial z}\right) \to K|\hat{v}|^2$, $K \to 0$

$\omega_r \to 0$

$$-\gamma^2 \langle |\hat{v}|^2 \rangle - \gamma \langle v_s' |\hat{v}|^2 \rangle$$
$$+ \langle (Q - P') |\hat{v}|^2 \rangle = 0$$

$|\hat{v}|^2 \simeq$ const.

$$\boxed{\gamma^2 \simeq \langle (Q - P') \rangle}$$

$$Q - P' = 2(v_s'^2 - v_s'')$$

For the instability

$\langle Q - P' \rangle > 0$ is required

More accurate would be to use a test function $\hat{v} = (z+1)(z-1)$

and

$$\gamma^2 \simeq \frac{2\int_{-1}^{1} (v_s'^2 - v_s'') |\hat{v}|^2 \, dz}{\int_{-1}^{1} |\hat{v}|^2 \, dz}.$$