Instability of Flow In Magnetic Nozzle

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Outline of Presentation

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2. Spectral Method
3. Singular Perturbation
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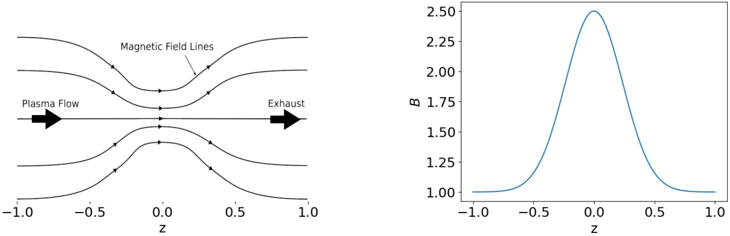
5. Appendix: Numerical Experiments

Linear Instability of Plasma Flow

- The instability of plasma flow refers to the tendency of a plasma system to deviate from a stable, equilibrium state and exhibit perturbations or fluctuations in its behavior. [1]
- To investigate linear instability, we assume oscillating perturbed quantities, $\tilde{n}, \tilde{v} \sim \exp(-i\omega t)$.
  1. If $\mathrm{Im}(\omega) > 0$, then it is unstable flow since the perturbations grow exponential in time, $\exp(\mathrm{Im}(\omega)t)$.
  2. If $\mathrm{Im}(\omega) <= 0$, then it is stable flow since the perturbations decay/unchanged in time.
- In this research we will focus on the linear instability only.

Magnetic Nozzle

- A magnetic nozzle is a device that uses a magnetic field to shape and control the flow of charged particles in a plasma propulsion system.
- Instabilities may affect magnetic nozzle operation and the resulting thrust. [4]



(a) Simplified representation of magnetic nozzle.

(b) A simplified magnetic field of magnetic nozzle.

Figure 1: Simplified representation of magnetic nozzle. Length is normalized.

Governing Equations

The nondimensionalized governing equations for the plasma flow in magnetic nozzle are

$$\text{Cons. of Den.} \qquad \frac{\partial n}{\partial t} + n\frac{\partial v}{\partial z} + v\frac{\partial n}{\partial z} - nv\frac{\partial_z B}{B} = 0 \tag{1}$$

$$\text{Cons. of Mom.} \qquad n\frac{\partial v}{\partial t} + nv\frac{\partial v}{\partial z} = -\frac{\partial n}{\partial z} \tag{2}$$

where n, v are density and velocity, respectively.
The equilibrium quantities n_0, v_0 must satify the condition,

$$\frac{\partial}{\partial z}\left(\frac{n_0 v_0}{B}\right) = 0 \tag{3}$$

$$v_0\frac{\partial v_0}{\partial z} = -\frac{1}{n_0}\frac{\partial n_0}{\partial z} \tag{4}$$

Equilibrium Velocity Profiles

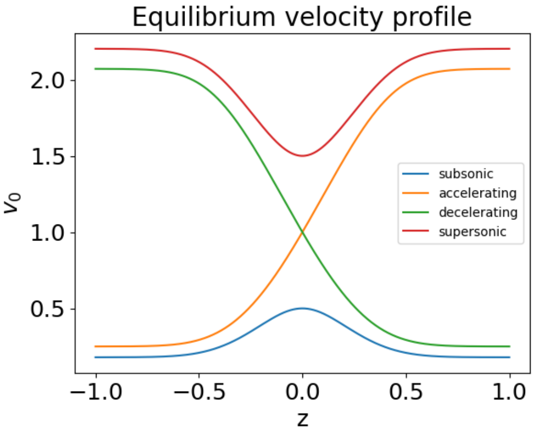


Figure 2: Velocity if normalized to sound speed. There are 4 different cases for velocity profile, subsonic, supersonic, accelerating and decelerating case.

Polynomial Eigenvalue Problem

By linearizing the governing equations, and assume oscillating perturbed quantities, $\tilde{n}, \tilde{v} \sim \exp(-i\omega t)$. We can derive the following equation,

$$
\begin{aligned}
&\omega^2 \tilde{v} \\
&+2i\omega \left(v_0 \frac{\partial}{\partial z} + \frac{\partial v_0}{\partial z} \right) \tilde{v} \\
&+ \left[(1 - v_0^2) \frac{\partial^2}{\partial z^2} - \left(3v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \frac{\partial}{\partial z} \right. \\
&\quad \left. - \left(1 - \frac{1}{v_0^2} \right) \left(\frac{\partial v_0}{\partial z} \right)^2 - \left(v_0 + \frac{1}{v_0} \right) \frac{\partial^2 v_0}{\partial z^2} \right] \tilde{v} = 0
\end{aligned}
\tag{5}
$$

It is a polynomial eigenvalue problem.

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Spectral Method

Eq.(5) can be reformulated as

$$\begin{bmatrix} 0 & 1 \\ \hat{M} & \hat{N} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \omega \tilde{v} \end{bmatrix} = \omega \begin{bmatrix} \tilde{v} \\ \omega \tilde{v} \end{bmatrix} \tag{6}$$

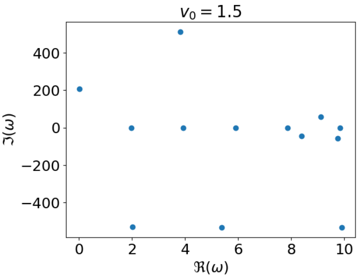
where the operators $\hat{M}$ and $\hat{N}$ are defined as

$$\begin{aligned} \hat{M} = & - \left[(1 - v_0^2) \frac{\partial^2}{\partial z^2} - \left(3v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \frac{\partial}{\partial z} \right. \\ & \left. - \left(1 - \frac{1}{v_0^2} \right) \left(\frac{\partial v_0}{\partial z} \right)^2 - \left(v_0 + \frac{1}{v_0} \right) \frac{\partial^2 v_0}{\partial z^2} \right] \\ \hat{N} = & - 2i \left(v_0 \frac{\partial}{\partial z} + \frac{\partial v_0}{\partial z} \right) \end{aligned}$$

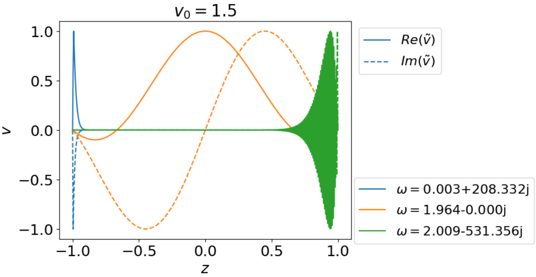
Then by discretizing operators $\hat{M}, \hat{N}$, this becomes an algebraic eigenvalue problem.

Spectral Pollution

- Analytical result shows all modes of Eq.(5) with $v_0 = \text{const}$ are stable.
- Finite-difference, finite-element, and spectral element discretization all show spurious unstable modes.



(a) Unfiltered eigenvalues.



(b) First few unfiltered eigenfunctions.

Figure 3: Finite difference discretization was used. Spurious modes occurs regardless of the resolution.

Convergence Test

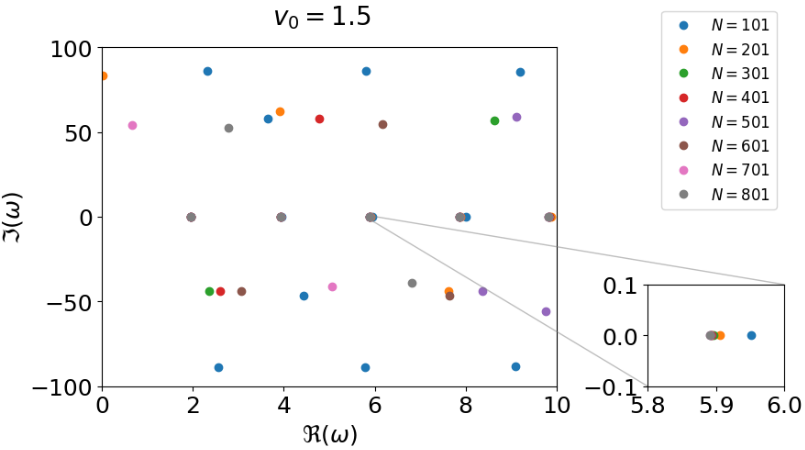
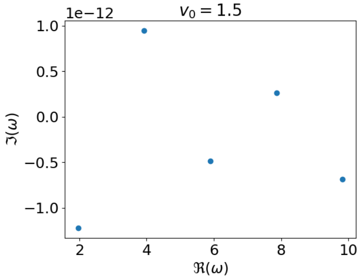
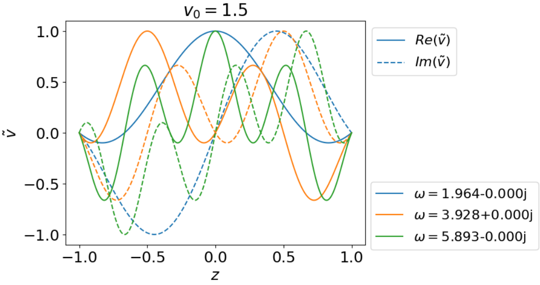


Figure 4: Pickup the convergent eigenvalues using convergence test.

Filtering Spurious Modes



(a) Filtered eigenvalues.



(b) First few filtered eigenfunctions.

Figure 5: The spurious modes are changing under different resolution. We can filter them by convergence test.

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Existence of Singularity

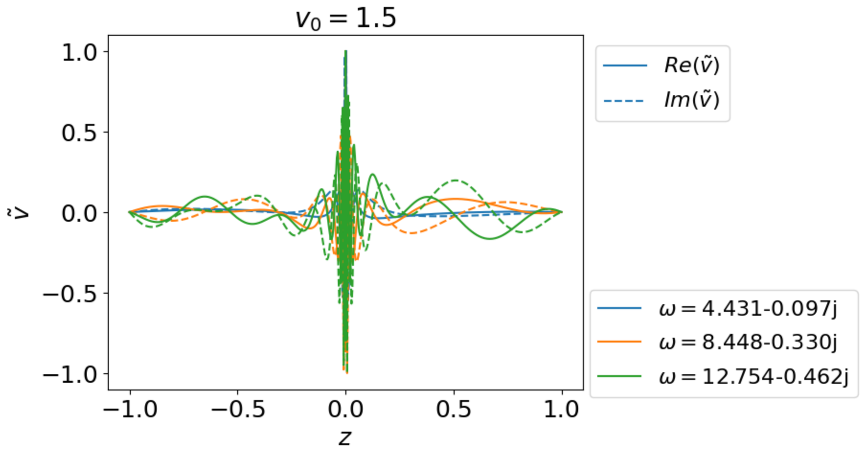

Figure 6: Dirichlet boundary conditions are set at the two ends, all eigenfunctions are squeezed to the singular point.

Interesting Connection to Black Hole

- The sonic horizon is an exact sonic analogue of black hole horizon. [5]
- A quasi-1D fluid flow is ruled by [2]

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A v) = 0 \tag{7}$$

$$\frac{\partial}{\partial t}(\rho A v) + \frac{\partial}{\partial x}[(\rho v^2 + p)A] = 0 \tag{8}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} - \frac{p}{1-\gamma}A\right) + \frac{\partial}{\partial x}\left[\left(\frac{\rho v^2}{2} - \frac{\gamma}{1-\gamma}A\right)Av\right] = 0 \tag{9}$$

In [2, 3], the acoustic analogue of tortoise coordinate is used to transform the above to Schrödinger-type equation,

$$x^* = c_{s0}\int [c_s(x)(1 - M(x)^2)]^{-1} dx$$

where c_{s0} denotes the stagnation speed of sound, $c_s = \mathrm{d}p/\mathrm{d}\rho$ is the local speed of sound and $M(x) = v(x)/c_s(x)$ is the Mach number.

Shooting Method

- Employed shooting method.
- Expanded $\tilde{v}$ near singularity.
- Picked up regular solution.
- Shoot regular solution to the left and match boundary condition.

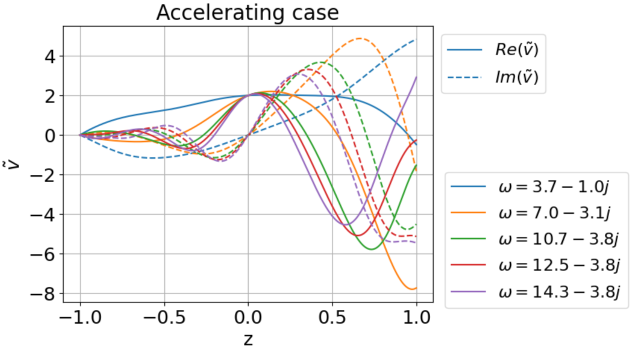

Figure 7: The solutions crosses the singular point smoothly. All modes are stable.

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Future Work

- Investigate and interpret the instability of an accelerating flow with non-zero left boundary. See Fig.8
- Compare results to analytically solvable problems with similar configuration.

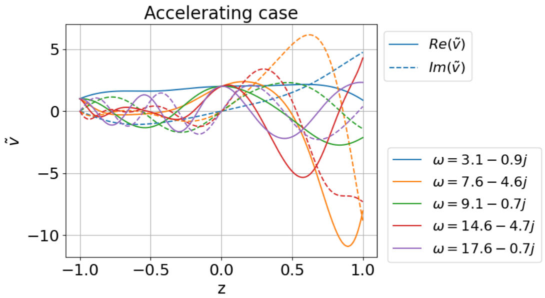

Figure 8: What is the physical interpretation of "non-zero" boundary value? How do we interpret these eigenvalues?

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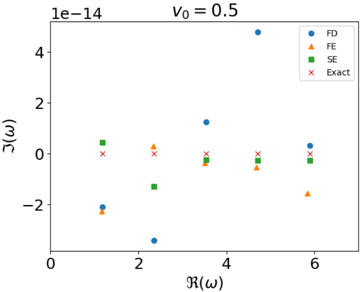
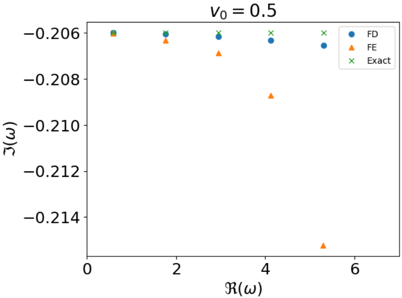
Constant Velocity Case - Subsonic



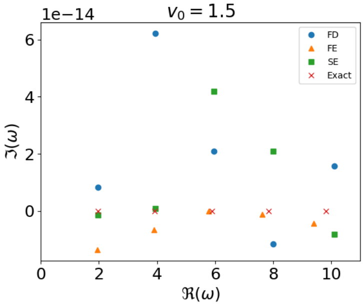
(a) Dirichlet boundary, all modes are stable.



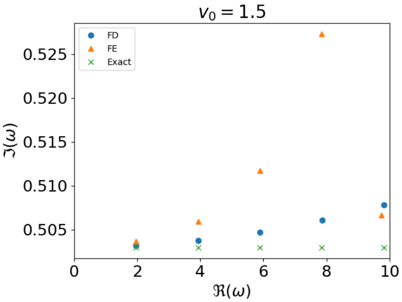
(b) Fixed-open boundary, all modes are stable.

Figure 9: Showing the first 5 eigenvalues. In the Dirichlet boundary case, all methods are close to the exact eigenvalues. Meanwhile, finite-difference method has higher accuracy than finite-element method in fixed-open case.

Constant Velocity Case - Supersonic



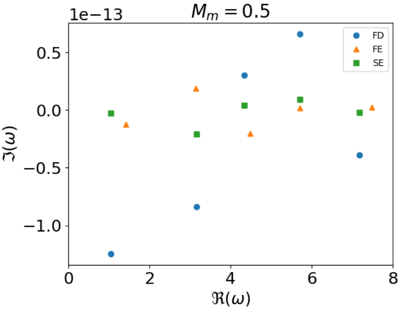
(a) Dirichlet boundary, filtered modes are stable.



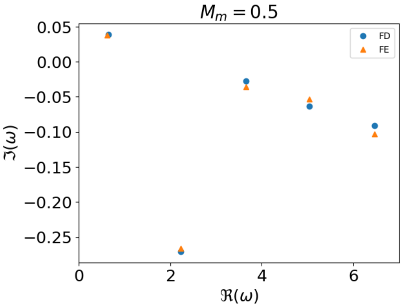
(b) Fixed-open boundary, all modes are unstable.

Figure 10: Showing the first 5 eigenvalues. In the Dirichlet boundary case, all methods are close to the exact eigenvalues. Meanwhile, finite-difference method has higher accuracy than finite-element method in fixed-open case.

Subsonic Case



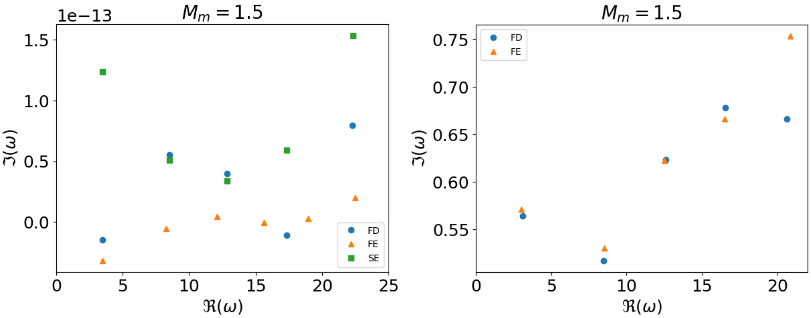
(a) Dirichlet boundary, all modes are stable.



(b) The ground mode is unstable, other modes are stable.

Figure 11: Showing the first 5 modes. It suggests that the subsonic flow in magnetic nozzle is stable.

Supersonic Case



(a) Dirichlet boundary, filtered modes are stable.

(b) Fixed-open boundary, all modes are unstable.

Figure 12: This suggests that the supersonic flow is stable if the boundary is Dirichlet and unstable if the boundary is left-fixed-right-open.

Accelerating Case

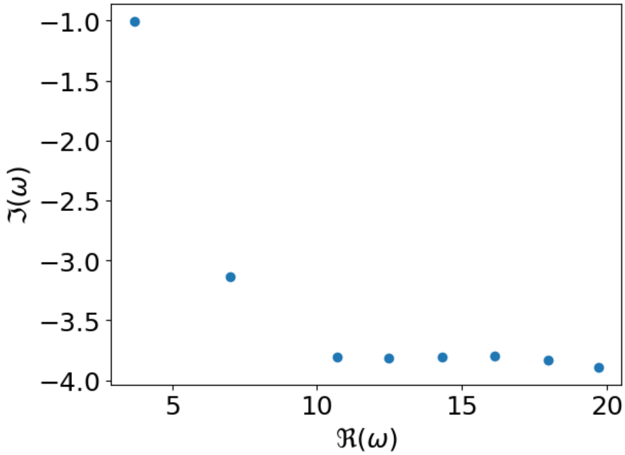


Figure 13: All modes are stable.