

Stability

(Dated: 29 March 2022)

$$\frac{\partial}{\partial t} n + B \nabla_{\parallel} \left(\frac{n V_{\parallel}}{B} \right) = 0, \quad (1)$$

$$\frac{\partial}{\partial t} V_{\parallel} + V_{\parallel} \nabla_{\parallel} V_{\parallel} = -c_s^2 \frac{\nabla_{\parallel} n}{n}, \quad (2)$$

Equilibrium

$$B \nabla_{\parallel} \left(\frac{n_0 V_0}{B} \right) = 0, \quad (3)$$

$$V_0 \nabla_{\parallel} V_0 = -c_s^2 \frac{\nabla_{\parallel} n_0}{n_0}, \quad (4)$$

Linearized, normalized $\tilde{V}/c_s \rightarrow \tilde{V}$, $V_0/c_s \rightarrow V_0$

$$\frac{\partial}{\partial t} \frac{\tilde{n}}{n_0} + V_0 \frac{\partial}{\partial z} \frac{\tilde{n}}{n_0} + \frac{\partial}{\partial z} \tilde{V} - \tilde{V} \frac{V_0'}{V_0} = 0, \quad (5)$$

$$\frac{\partial}{\partial t} \tilde{V} + V_0 \frac{\partial}{\partial z} \tilde{V} + V_0' \tilde{V} = -\frac{\partial}{\partial z} \frac{\tilde{n}}{n_0} \quad (6)$$

Introducing

$$Y_+ = \frac{\tilde{n}}{n_0} + \tilde{V} \quad (7)$$

$$Y_- = \frac{\tilde{n}}{n_0} - \tilde{V} \quad (8)$$

$$\frac{\partial}{\partial t} Y_+ + (V_0 + 1) \frac{\partial}{\partial z} Y_+ = \frac{1}{2} (Y_+ - Y_-) \frac{V_0'}{V_0} (1 - V_0) \quad (9)$$

$$\frac{\partial}{\partial t} Y_- + (V_0 - 1) \frac{\partial}{\partial z} Y_- = \frac{1}{2} (Y_+ - Y_-) \frac{V_0'}{V_0} (1 + V_0) \quad (10)$$

I. WKB APPROXIMATION

$$kL > 1, \quad L^{-1} \simeq V_0'/V_0$$

$$(Y_+, Y_-) \sim \exp(-i\omega t + ikz) \quad (11)$$

$$\omega^2 - 2\omega k V_0 + i\omega V_0' + k^2 (V_0^2 - 1) - ik \frac{V_0'}{V_0} (1 + V_0^2) = 0 \quad (12)$$

for the instability $V_0 < 2c_s$

II. VARIATIONAL FORM

$$Y_+^* \frac{\partial}{\partial t} Y_+ + (V_0 + 1) Y_+^* \frac{\partial}{\partial z} Y_+ = \frac{1}{2} Y_+^* (Y_+ - Y_-) \frac{V_0'}{V_0} (1 - V_0) \quad (13)$$

$$Y_+ \frac{\partial}{\partial t} Y_+^* + (V_0 + 1) Y_+ \frac{\partial}{\partial z} Y_+^* = \frac{1}{2} Y_+ (Y_+^* - Y_-^*) \frac{V_0'}{V_0} (1 - V_0) \quad (14)$$

$$\begin{aligned} & \frac{\partial}{\partial t} |Y_+|^2 + (V_0 + 1) \frac{\partial}{\partial z} |Y_+|^2 \\ &= \frac{1}{2} (|Y_+|^2 - Y_+^* Y_-) \frac{V_0'}{V_0} (1 - V_0) + \frac{1}{2} (|Y_+|^2 - Y_+ Y_-^*) \frac{V_0'}{V_0} (1 - V_0) \end{aligned} \quad (15)$$

$$Y_-^* \frac{\partial}{\partial t} Y_- + (V_0 - 1) Y_-^* \frac{\partial}{\partial z} Y_- = \frac{1}{2} Y_-^* (Y_+ - Y_-) \frac{V_0'}{V_0} (1 + V_0) \quad (16)$$

$$Y_- \frac{\partial}{\partial t} Y_-^* + (V_0 - 1) Y_- \frac{\partial}{\partial z} Y_-^* = \frac{1}{2} Y_- (Y_+^* - Y_-^*) \frac{V_0'}{V_0} (1 + V_0) \quad (17)$$

$$\begin{aligned} & \frac{\partial}{\partial t} |Y_-|^2 + (V_0 - 1) \frac{\partial}{\partial z} |Y_-|^2 \\ &= \frac{1}{2} (Y_-^* Y_+ - |Y_-|^2) \frac{V_0'}{V_0} (1 + V_0) + \frac{1}{2} (Y_- Y_+^* - |Y_-|^2) \frac{V_0'}{V_0} (1 + V_0) \end{aligned} \quad (18)$$

Combining

$$\begin{aligned} & \frac{\partial}{\partial t} |Y_+|^2 + (V_0 + 1) \frac{\partial}{\partial z} |Y_+|^2 \\ &= |Y_+|^2 \frac{V_0'}{V_0} (1 - V_0) - \frac{1}{2} (Y_+ Y_-^* + Y_+^* Y_-) \frac{V_0'}{V_0} (1 - V_0) \end{aligned} \quad (19)$$

$$\begin{aligned} & \frac{\partial}{\partial t} |Y_-|^2 + (V_0 - 1) \frac{\partial}{\partial z} |Y_-|^2 \\ &= -|Y_-|^2 \frac{V_0'}{V_0} (1 + V_0) + \frac{1}{2} (Y_- Y_+^* + Y_-^* Y_+) \frac{V_0'}{V_0} (1 + V_0) \end{aligned} \quad (20)$$

$$\begin{aligned} & \frac{1}{(1-V_0)} \frac{\partial}{\partial t} |Y_+|^2 + \frac{(V_0+1)}{(1-V_0)} \frac{\partial}{\partial z} |Y_+|^2 \\ = & |Y_+|^2 \frac{V_0'}{V_0} - \frac{1}{2} (Y_+ Y_-^* + Y_+^* Y_-) \frac{V_0'}{V_0} \end{aligned} \quad (21)$$

$$\begin{aligned} & \frac{1}{(1+V_0)} \frac{\partial}{\partial t} |Y_-|^2 + \frac{(V_0-1)}{(1+V_0)} \frac{\partial}{\partial z} |Y_-|^2 \\ = & -|Y_-|^2 \frac{V_0'}{V_0} + \frac{1}{2} (Y_- Y_+^* + Y_-^* Y_+) \frac{V_0'}{V_0} \end{aligned} \quad (22)$$

One form

$$\begin{aligned} & \frac{1}{(1-V_0)} \frac{\partial}{\partial t} |Y_+|^2 + \frac{1}{(1+V_0)} \frac{\partial}{\partial t} |Y_-|^2 \\ & + \frac{(V_0+1)}{(1-V_0)} \frac{\partial}{\partial z} |Y_+|^2 + \frac{(V_0-1)}{(1+V_0)} \frac{\partial}{\partial z} |Y_-|^2 \end{aligned} \quad (23)$$

$$= \left(|Y_+|^2 - |Y_-|^2 \right) \frac{V_0'}{V_0} \quad (24)$$

Using

$$\frac{(V_0-1)(V_0+1)}{V_0} \left(\frac{V_0+1}{V_0-1} \frac{\partial}{\partial z} |Y_+|^2 + |Y_+|^2 \frac{V_0'}{V_0} \right) = \frac{\partial}{\partial z} \left(\frac{(V_0+1)^2}{V_0} |Y_+|^2 \right) \quad (25)$$

$$\frac{(V_0-1)(V_0+1)}{V_0} \left(\frac{(V_0-1)}{(1+V_0)} \frac{\partial}{\partial z} |Y_-|^2 + |Y_-|^2 \frac{V_0'}{V_0} \right) = \frac{\partial}{\partial z} \left(\frac{(V_0-1)^2}{V_0} |Y_-|^2 \right) \quad (26)$$

$$\left(\frac{V_0+1}{V_0-1} \frac{\partial}{\partial z} |Y_+|^2 + |Y_+|^2 \frac{V_0'}{V_0} \right) = \frac{V_0}{(V_0-1)(V_0+1)} \frac{\partial}{\partial z} \left(\frac{(V_0+1)^2}{V_0} |Y_+|^2 \right) \quad (27)$$

$$\left(\frac{V_0-1}{1+V_0} \frac{\partial}{\partial z} |Y_-|^2 + |Y_-|^2 \frac{V_0'}{V_0} \right) = \frac{V_0}{(V_0-1)(V_0+1)} \frac{\partial}{\partial z} \left(\frac{(V_0-1)^2}{V_0} |Y_-|^2 \right) \quad (28)$$

Finally

$$\begin{aligned} - & \frac{(V_0+1)}{V_0} \frac{\partial}{\partial t} |Y_+|^2 + \frac{V_0-1}{V_0} \frac{\partial}{\partial t} |Y_-|^2 = \\ = & \frac{\partial}{\partial z} \left(\frac{(V_0+1)^2}{V_0} |Y_+|^2 \right) - \frac{\partial}{\partial z} \left(\frac{(V_0-1)^2}{V_0} |Y_-|^2 \right) \end{aligned} \quad (29)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{V_0-1}{V_0} |Y_-|^2 - \frac{(V_0+1)}{V_0} |Y_+|^2 \right] = \\ = & \frac{\partial}{\partial z} \left[\frac{(V_0+1)^2}{V_0} |Y_+|^2 - \frac{(V_0-1)^2}{V_0} |Y_-|^2 \right] \end{aligned} \quad (30)$$

Over interval $[a, b]$

$$2\gamma = \left[\frac{(V_0 - 1)^2}{V_0} |Y_-|^2 - \frac{(V_0 + 1)^2}{V_0} |Y_+|^2 \right]_a^b \left[\int_a^b \left(\frac{1 - V_0}{V_0} |Y_-|^2 + \frac{(V_0 + 1)}{V_0} |Y_+|^2 \right) dz \right]^{-1} \quad (31)$$

outgoing boundary condition at $z = b : Y_+ \neq 0$

III. ALTERNATIVE

$$\omega^2 \tilde{V} + 2i\omega (V_0 \tilde{V})' + \left[V_0 \left(\frac{\tilde{V}}{V_0} \right)' - V_0 (V_0 \tilde{V})' \right]' = 0 \quad (32)$$