Variational forms

Original equation

1. $$\omega^2 \tilde{v} + 2i\omega\left(v_s \frac{\partial \tilde{v}}{\partial z} + v_s' \tilde{v}\right) + \frac{\partial^2 \tilde{v}}{\partial z^2}(1-v_s^2)$$
$$- v_s'\left(\frac{1}{v_s} + 3v_s\right)\frac{\partial \tilde{v}}{\partial z} +$$
$$+ \tilde{v}\left(v_s'^2\left(\frac{1}{v_s^2}-1\right) - v_s''\left(\frac{1}{v_s} + v_s\right)\right) = 0$$

$$\omega = \omega_r + i\gamma$$

$$(\omega_r^2 + 2i\omega_r\gamma - \gamma^2)\tilde{v} + 2i\omega_r\left(v_s \frac{\partial \tilde{v}}{\partial z} + v_s'\tilde{v}\right)$$
$$- 2\gamma\left(v_s \frac{\partial \tilde{v}}{\partial z} + v_s'\tilde{v}\right) + \frac{\partial^2 \tilde{v}}{\partial z^2}(1-v_s^2)$$
$$- v_s'\left(\frac{1}{v_s} + 3v_s\right)\frac{\partial \tilde{v}}{\partial z} + \tilde{v}\, Q = 0$$

$$P = - v_s'\left(\frac{1}{v_s} + 3v_s\right)$$

$$(\omega_r^2 + 2i\omega_r\gamma - \gamma^2)|\tilde{v}|^2 + 2i\omega_r\left(v_s \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} + v_s'|\tilde{v}|^2\right)$$
$$- 2\gamma\left(v_s \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} + v_s'|\tilde{v}|^2\right) + \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2}(1-v_s^2)$$
$$+ P\, \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} + Q|\tilde{v}|^2 = 0$$

Taking real and imaginary parts ②

$$(\omega_r^2-\gamma^2)\,|\hat{v}|^2 - 2\omega_r V_0 \,\mathrm{Im}\left(\hat{v}^*\frac{\partial \hat{v}}{\partial z}\right) - 2\gamma V_0 \,\mathrm{Re}\left(\hat{v}^*\frac{\partial \hat{v}}{\partial z}\right) - 2\gamma V_0'|\hat{v}|^2 + (1-V_0^2)\,\mathrm{Re}\left(\hat{v}^*\frac{\partial^2 \hat{v}}{\partial z^2}\right) + P\,\mathrm{Re}\left(\hat{v}^*\frac{\partial \hat{v}}{\partial z}\right) + Q|\hat{v}|^2 = 0$$

$$(\omega_r^2-\gamma^2)\langle|\hat{v}|^2\rangle - 2\omega_r\left\langle V_0 \,\mathrm{Im}\left(\hat{v}^*\frac{\partial \hat{v}}{\partial z}\right)\right\rangle - \gamma\left\langle V_0\frac{\partial}{\partial z}|\hat{v}|^2\right\rangle - 2\gamma V_0'|\hat{v}|^2 - \left\langle(1-V_0^2)\left|\frac{\partial \hat{v}}{\partial z}\right|^2\right\rangle + \frac{1}{2}\left\langle P\frac{\partial}{\partial z}|\hat{v}|^2\right\rangle + \langle Q|\hat{v}|^2\rangle = 0$$

$$(\omega_r^2-\gamma^2)\langle|\hat{v}|^2\rangle - 2\omega_r\left\langle V_0 \,\mathrm{Im}\left(\hat{v}^*\frac{\partial \hat{v}}{\partial z}\right)\right\rangle - \gamma\langle V_0'|\hat{v}|^2\rangle - \left\langle(1-V_0^2)\left|\frac{\partial \hat{v}}{\partial z}\right|^2\right\rangle + \left\langle\left(Q-\frac{1}{2}P'\right)|\hat{v}|^2\right\rangle = 0$$

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$$2\omega_r \gamma |\hat{v}|^2 + 2\omega_r V_0 \, \mathrm{Re}\left(\hat{v}^* \frac{\partial \hat{v}}{\partial z}\right) + 2\omega_r V_0' |\hat{v}|^2 - 2\gamma V_0 \, \mathrm{Im}\left(\hat{v}^* \frac{\partial \hat{v}}{\partial z}\right) + P\left(\mathrm{Im}\, \hat{v}^* \frac{\partial \hat{v}}{\partial z}\right) = 0$$

$$Q - \frac{1}{2}P' = V_0'^2\left(\frac{1}{V_0^2} - 1\right) - V_0''\left(\frac{1}{V_0} + V_0\right) + \frac{1}{2}\left[V_0'\left(\frac{1}{V_0} + 3V_0\right)\right]'$$

$$= V_0'^2\left(\frac{1}{V_0^2} - 1\right) - V_0''\left(\frac{1}{V_0} + V_0\right) + \frac{V_0''}{2}\left(\frac{1}{V_0} + 3V_0\right) + \frac{1}{2}V_0'^2\left(-\frac{1}{V_0^2} + 3\right)$$

$$= V_0'^2\left(\frac{1}{2}\frac{\cancel{1}}{V_0^2} + \frac{1}{2}\right) - V_0''\left(\frac{1}{2V_0} + \frac{1}{2}V_0\right)$$

$$= \frac{1}{2}V_0'^2\left(\frac{1}{V_0^2} + 1\right) - \frac{1}{2}V_0''\left(\frac{1}{V_0} + V_0\right)$$

$$= \frac{1}{2}\left(\frac{1}{V_0^2} + 1\right)\left(V_0'^2 - V_0'' V_0\right)$$
