Spectral Pollution

Hunt Feng

July 13, 2022

1 Background

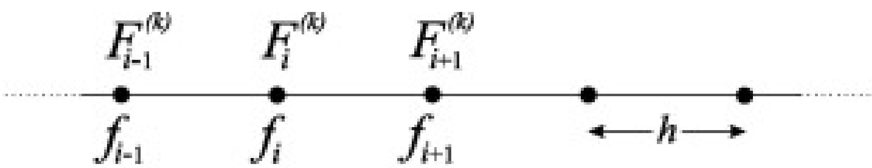
In this document, we are going to investigate the spectral pollution in the problem

$$\omega^2 v + 2iv_0 \frac{\mathrm{d}v}{\mathrm{d}x} + (1 - v_0^2)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0, \quad v(\pm 1) = 0 \tag{1}$$

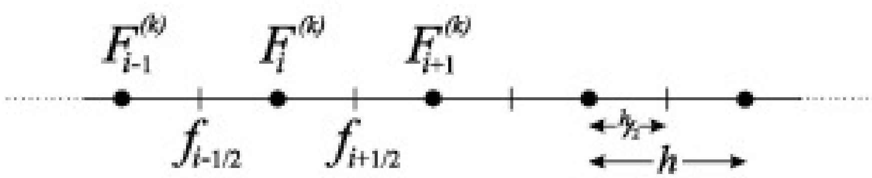
where the dispersion relation is known,

$$\omega = k(v_0 \pm 1) \tag{2}$$

If we introduce staggered grid, then there are 2 possible ways to discretize the Eq.(1).



(a) Collocated grid.



(b) Staggered grid.

1. Discretize the equation on the OPPOSITE grid where the function v is defined on.
2. Discretize the equation on the SAME grid where the function v is defined on.

If we assume $v \sim \exp(-ikx)$, and let $\beta \equiv kh/2$. Then the differential operators $\mathrm{d}^n/\mathrm{d}x^n$ are equivalent to the following factors [1],

- Evaluate equation on the OPPOSITE grid align

$$\begin{aligned}
H_0 &= [\exp(i\beta) + \exp(-i\beta)]/2 = \cos(2\beta) \\
H_1 &= [\exp(i\beta) - \exp(-i\beta)]/h = (2i/h)\sin(\beta) \\
H_2 &= [\exp(3i\beta) - \exp(i\beta) - \exp(i\beta) + \exp(-3i\beta)]/2h^2 = H_1^2 H_0
\end{aligned} \tag{3}$$

- Evaluate equation on the SAME grid

$$\begin{aligned} G_0 &= 1 \\ G_1 &= [\exp(2i\beta) - \exp(-2i\beta)]/2h = (i/h)\sin(2\beta) = H_1 H_0 \\ G_2 &= [\exp(2i\beta) - 2 - \exp(-2i\beta)]/h^2 = (2/h^2)(\cos(2\beta) - 1) = H_1^2 \end{aligned} \tag{4}$$

2 Analysis of Numerical Spectrum

2.1 Discretize on the Same Grid

Using the G-operator, Eq.(4), the discretized equation of Eq.(1) is

$$\omega^2 + \omega(2iv_0 H_1 H_0) + (1 - v_0^2)H_1^2 = 0$$

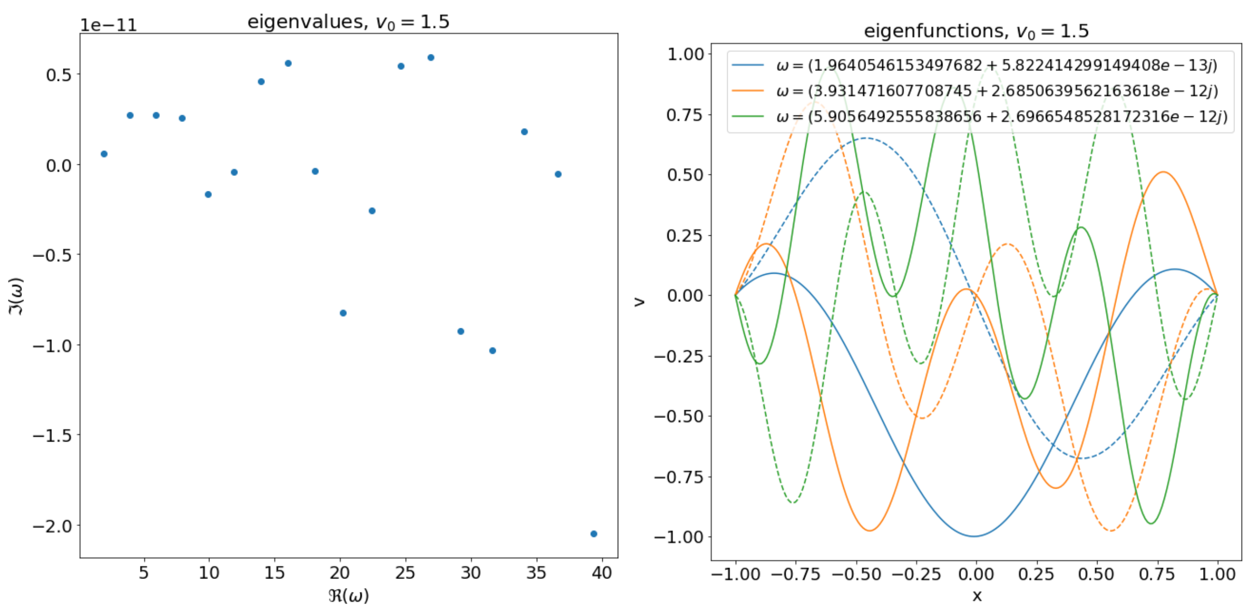
Thus the numerical dispersion relation is

$$\omega = -iH_1\left(v_0 \pm \sqrt{v_0^2 H_0^2 + (1 - v_0^2)}\right) = \frac{2\sin(\beta)}{h}\left(v_0 \pm \sqrt{1 - v_0^2 \sin^2(\beta)}\right) \tag{5}$$

We see that

- ω is real for all k if $v_0 < 1$.
- ω is complex for large k, more specifically $k > h/2 \arcsin(1/v_0)$, if $v_0 > 1$.
- For small k, meaning $k \to 0$, Eq.(5) is a good representation for the analytical dispersion relation, Eq.(2).

This explains why the spurious unstable modes occur when $v_0 > 1$.



(a) Filtered eigenvalues

(b) Filtered eigenfunctions

Figure 1: Filter out the spurious modes with $k > h/2 \arcsin(1/v_0)$.

2.2 Discretize on the Opposite Grid

Using the H-operator, Eq.(3), Eq.(1) becomes

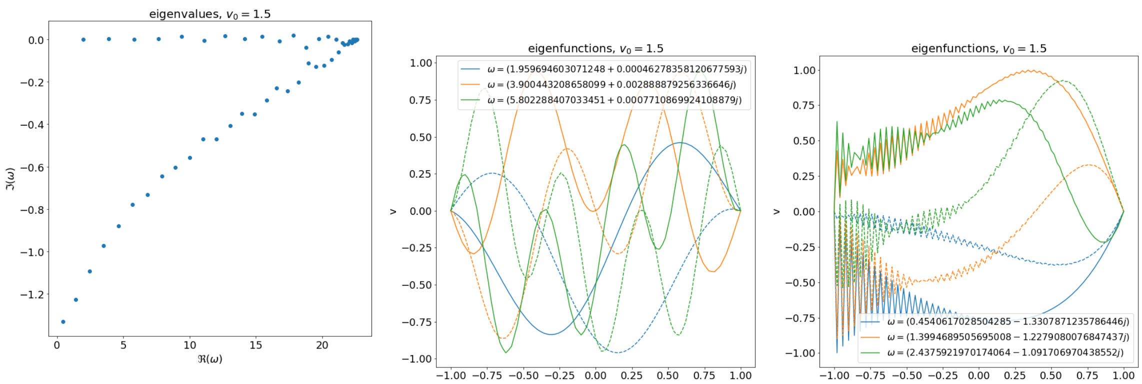
$$\omega^2 H_0 + \omega(2iv_0 H_1) + (1 - v_0^2) H_1^2 H_0 = 0$$

So the numerical dispersion relation is

$$\omega = -iH_1 \left(v_0 \pm \sqrt{v_0^2 + (1 - v_0^2) H_0^2} \right) = \frac{2\sin(\beta)}{h} \left(v_0 \pm \sqrt{\cos^2(\beta) + v_0^2 \sin^2(\beta)} \right) \tag{6}$$

We see that

- ω is real for all k for all v_0.
- For small k, Eq.(6) dramatically deviates from the analytical dispersion relation Eq.(2).



(a) Deformed eigenvalues, accumulation point occurs.

(b) Good eigenfunctions

(c) Spurious modes.

Figure 2: Although the spurious unstable modes are much smaller, but the eigenvalues deviate from the analytical dispersion relation.

3 Conclusion

1. If we discretize Eq.(1) on the same grid as the function v is defined on, then the eigenvalues are good for small k modes. To get the good modes, we can filter out the modes with wave number $k > h/2 \arcsin(1/v_0)$.
2. While we get less spurious unstable modes if we discretize Eq.(1) on the opposite grid as the v is defined on, we suffer the inaccurate eigenvalues.

References

[1] X. Llobet, K. Appert, A. Bondeson, and J. Vaclavik. On spectral pollution. *Computer Physics Communications*, 59(2):199–216, 1990.