

# Unstable Modes in Magnetic Nozzle

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## 1 Equations of Motion

### 1.1 Linearized Equations of Motion

The dynamics of magnetic nozzle can be characterized by conservation of mass and momentum,

$$\begin{aligned}\frac{\partial n}{\partial t} + B \frac{\partial}{\partial z} \left( \frac{nv}{B} \right) &= 0 \\ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} &= -c_s^2 \frac{1}{n} \frac{\partial n}{\partial z}\end{aligned}$$

Usually, the magnetic field can be described by

$$B(z) = B_0 \left[ 1 + R \exp\left(-\frac{z^2}{\delta^2}\right) \right]$$

where  $R$  and  $\delta$  are some coefficients.

At equilibrium (stationary solution), we have  $\partial n_0 / \partial t = 0$  and  $\partial v_0 / \partial t = 0$ , so  $n_0$  and  $v_0$  satisfy

$$\begin{aligned}\frac{\partial}{\partial z} \left( \frac{n_0 v_0}{B} \right) &= 0 \\ v_0 \frac{\partial v_0}{\partial z} &= -c_s^2 \frac{1}{n_0} \frac{\partial n_0}{\partial z}\end{aligned}$$

Let  $M \equiv v_0 / c_s$ , then it can be represented by Lambert function,

$$M = \left[ -W \left( -M_m^2 \frac{B(z)^2}{B_m^2} e^{-M_m^2} \right) \right]^{1/2}$$

where  $B_m \equiv 1 + R$  is the maximum magnetic field (or magnetic field at mid-point), and  $M_m$  is the mach number at mid-point. Below shows a few cases of the solution.

- $M_m < 1$ , subsonic velocity profile.
- $M_m = 1$ , accelerating or decelerating profile (depending on the branch of the Lambert function).
- $M_m > 1$ , supersonic velocity profile

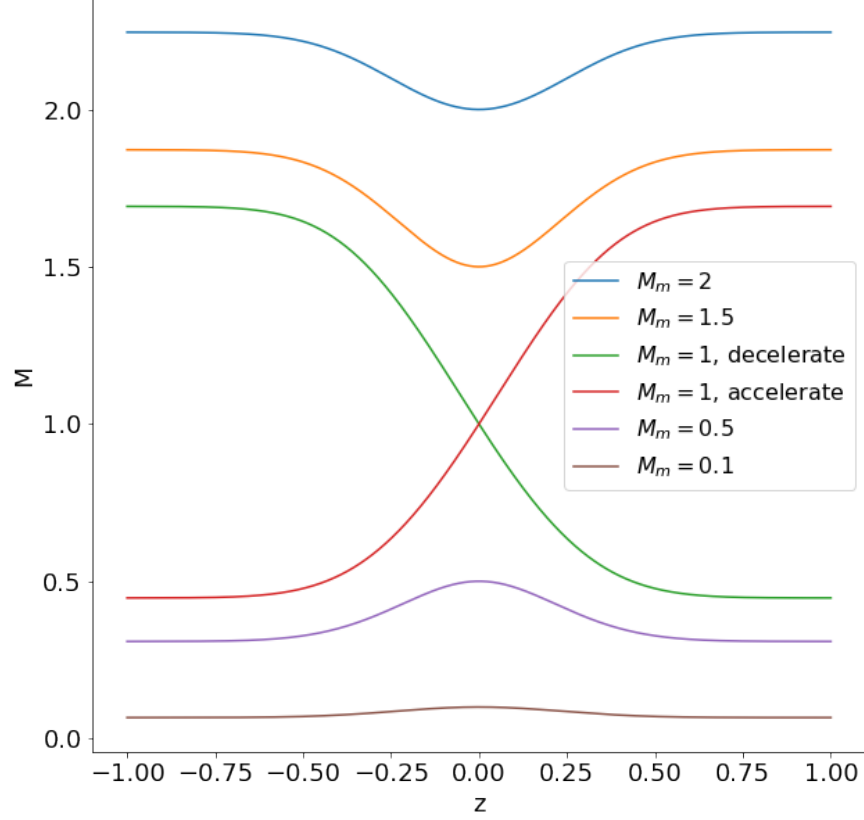


Figure 1:  $M_m < 1$ , subsonic.  $M_m = 1$ , accelerating if we select Lambert function branch  $k = 0$  for  $z < 0$  and brach  $k = -1$  for  $z \geq 0$ ; decelerating if we choose branch  $k = -1$  for  $z < 0$  and brach  $k = 0$  for  $z \geq 0$ .  $M_m > 1$ , supersonic.

For convenience, we nondimensionalize the equations by normalizing the velocity to  $c_s$ ,  $v \mapsto v/c_s$ ,  $z$  to system length  $L$ ,  $z \mapsto z/L$  and time  $t \mapsto c_s t/L$ .

$$\frac{\partial n}{\partial t} + n \frac{\partial v}{\partial z} + v \frac{\partial n}{\partial z} - n v \frac{\partial_z B}{B} = 0 \quad (1)$$

$$n \frac{\partial v}{\partial t} + n v \frac{\partial v}{\partial z} = - \frac{\partial n}{\partial z} \quad (2)$$

and the nondimensionalized equilibrium condition is

$$\frac{\partial}{\partial z} \left( \frac{n_0 v_0}{B} \right) = 0 \quad (3)$$

$$v_0 \frac{\partial v_0}{\partial z} = - \frac{1}{n_0} \frac{\partial n_0}{\partial z} \quad (4)$$

**Proposition 1.** Let  $n = n_0(z) + \tilde{n}(z, t)$  and  $v = v_0(z) + \tilde{v}(z, t)$ , the linearized equations of motion are

$$\frac{1}{n_0} \frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0 \tilde{Y} + \tilde{v} \frac{\partial_z n_0}{n_0} - \tilde{v} \frac{\partial_z B}{B} = 0 \quad (5)$$

$$\frac{\partial \tilde{v}}{\partial t} + \frac{\partial (v_0 \tilde{v})}{\partial z} = - \tilde{Y} \quad (6)$$

where

$$\tilde{Y} \equiv \frac{1}{n_0} \frac{\partial \tilde{n}}{\partial z} - \frac{\partial_z n_0}{n_0^2} \tilde{n} = \frac{\partial}{\partial z} \left( \frac{\tilde{n}}{n_0} \right)$$

*Proof.* We first derive Eq.(5). We linearize Eq.(3) by setting  $n = n_0 + \tilde{n}$  and  $v = v_0 + \tilde{v}$ . By ignoring the second order perturbations, we obtain

$$\begin{aligned} & \frac{\partial(n_0 + \tilde{n})}{\partial t} + (n_0 + \tilde{n}) \frac{\partial(v_0 + \tilde{v})}{\partial z} + (v_0 + \tilde{v}) \frac{\partial(n_0 + \tilde{n})}{\partial z} - (n_0 + \tilde{n})(v_0 + \tilde{v}) \frac{\partial_z B}{B} = 0 \\ \Rightarrow & \frac{\partial \tilde{n}}{\partial t} + n_0 \frac{\partial v_0}{\partial z} + \tilde{n} \frac{\partial v_0}{\partial z} + n_0 \frac{\partial \tilde{v}}{\partial z} + v_0 \frac{\partial n_0}{\partial z} + \tilde{v} \frac{\partial n_0}{\partial z} + v_0 \frac{\partial \tilde{n}}{\partial z} - (n_0 v_0 + n_0 \tilde{v} + \tilde{n} v_0) \frac{\partial_z B}{B} = 0 \\ \Rightarrow & \frac{1}{n_0} \frac{\partial \tilde{n}}{\partial t} + \frac{\partial v_0}{\partial z} + \frac{\tilde{n}}{n_0} \frac{\partial v_0}{\partial z} + \frac{\partial \tilde{v}}{\partial z} + \frac{v_0}{n_0} \frac{\partial n_0}{\partial z} + \frac{\tilde{v}}{n_0} \frac{\partial n_0}{\partial z} + \frac{v_0}{n_0} \frac{\partial \tilde{n}}{\partial z} - v_0 \frac{\partial_z B}{B} - \tilde{v} \frac{\partial_z B}{B} - \tilde{n} \frac{v_0}{n_0} \frac{\partial_z B}{B} = 0 \end{aligned}$$

Using the equilibrium condition Eq.(3), some of the terms are canceled and the last term can be written as

$$\tilde{n} \frac{v_0}{n_0} \frac{\partial_z B}{B} = \frac{\tilde{n}}{n_0} \left( \frac{\partial_z n_0}{n_0} v_0 + \frac{\partial v_0}{\partial z} \right)$$

Now, we are left with equation

$$\frac{1}{n_0} \frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0 \underbrace{\left( \frac{1}{n_0} \frac{\partial \tilde{n}}{\partial z} - \frac{\tilde{n}}{n_0} \frac{\partial_z n_0}{n_0} \right)}_{\tilde{Y}} + \frac{\tilde{v}}{n_0} \frac{\partial n_0}{\partial z} - \tilde{v} \frac{\partial_z B}{B} = 0$$

To derive Eq.(6), we linearize the LHS of the conservation of momentum

$$\begin{aligned} & (n_0 + \tilde{n}) \frac{\partial(v_0 + \tilde{v})}{\partial t} + (n_0 + \tilde{n})(v_0 + \tilde{v}) \frac{\partial(v_0 + \tilde{v})}{\partial z} = - \frac{\partial n}{\partial z} \\ \Rightarrow & \frac{\partial v_0}{\partial t} + \frac{\tilde{n}}{n_0} \frac{\partial v_0}{\partial t} + \frac{\partial \tilde{v}}{\partial t} + \left( v_0 + \tilde{v} + \frac{\tilde{n}}{n_0} v_0 \right) \frac{\partial(v_0 + \tilde{v})}{\partial z} = - \frac{1}{n_0} \frac{\partial n}{\partial z} \\ \Rightarrow & \frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial z} + \tilde{v} \frac{\partial v_0}{\partial z} = - \frac{1}{n_0} \frac{\partial n_0}{\partial z} - \frac{1}{n_0} \frac{\partial \tilde{n}}{\partial z} - v_0 \frac{v_0}{z} - \frac{\tilde{n}}{n_0} v_0 \frac{\partial v_0}{\partial z} \end{aligned}$$

Using the equilibrium condition Eq.(4) on the RHS, we get the desired form.  $\square$

## 2 Formulation of the problem

### 2.1 Polynomial Eigenvalue Problem

**Proposition 2.**

$$\frac{\partial}{\partial z} \ln \left( \frac{n_0}{B} \right) = - \frac{1}{v_0} \frac{\partial v_0}{\partial z} \quad (7)$$

*Proof.*

$$\frac{\partial}{\partial z} \ln \left( \frac{n_0}{B} \right) = \frac{B}{n_0} \frac{\partial}{\partial z} \left( \frac{n_0}{B} \right) = \frac{1}{n_0} \frac{n_0}{z} + B \frac{\partial}{\partial z} \left( \frac{1}{B} \right) = \frac{1}{n_0} \frac{n_0}{z} - \underbrace{\frac{1}{n_0 v_0} \frac{\partial n_0 v_0}{\partial z}}_{Eq.(3)} = - \frac{1}{v_0} \frac{\partial v_0}{\partial z}$$

$\square$

**Proposition 3.** Let  $\tilde{n} \sim \exp(-i\omega t)$  and  $\tilde{v} \sim \exp(-i\omega t)$ , then we have the polynomial eigenvalue problem

$$-\Omega^2 \tilde{v} + 2\Omega \left( v_0 \frac{\partial}{\partial z} + \frac{\partial v_0}{\partial z} \right) \tilde{v} + \left[ (1 - v_0^2) \frac{\partial^2}{\partial z^2} - \left( 3v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \frac{\partial}{\partial z} - \left( 1 - \frac{1}{v_0^2} \right) \left( \frac{\partial v_0}{\partial z} \right)^2 - \left( v_0 + \frac{1}{v_0} \right) \frac{\partial^2 v_0}{\partial z^2} \right] \tilde{v} = 0 \quad (8)$$

where  $\Omega \equiv i\omega$ .

*Proof.*

$$\begin{aligned} \frac{1}{n_0} \frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{v}}{\partial z} + v_0 \tilde{Y} + \tilde{v} \frac{\partial_z n_0}{n_0} - \tilde{v} \frac{\partial_z B}{B} &= 0 \\ \frac{\partial \tilde{v}}{\partial t} + \frac{\partial(v_0 \tilde{v})}{\partial z} &= -\tilde{Y} \end{aligned}$$

We plug Eq.(6) in to Eq.(5), we have

$$-i\omega \frac{\tilde{n}}{n_0} + \frac{\partial \tilde{v}}{\partial z} - v_0 \left( -i\omega \tilde{v} + \frac{\partial(v_0 \tilde{v})}{\partial z} \right) + \tilde{v} \frac{\partial_z n_0}{n_0} - \tilde{v} \frac{\partial_z B}{B} = 0$$

Using the equilibrium condition Eq.(3), we can eliminate the term  $\partial_z B/B$ ,

$$\begin{aligned} -i\omega \frac{\tilde{n}}{n_0} + \frac{\partial \tilde{v}}{\partial z} + v_0 \left( i\omega \tilde{v} - v_0 \frac{\partial \tilde{v}}{\partial z} - \tilde{v} \frac{\partial v_0}{\partial z} \right) - \tilde{v} \frac{\partial_z v_0}{v_0} &= 0 \\ \Rightarrow -i\omega \frac{\tilde{n}}{n_0} + i\omega v_0 \tilde{v} + (1 - v_0^2) \frac{\partial \tilde{v}}{\partial z} + \left( v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \tilde{v} &= 0 \end{aligned}$$

Now we take  $\partial/\partial t$  on Eq.(6). Recall the fact that  $\tilde{Y} = \partial(\tilde{n}/n_0)/\partial z$ , we have

$$\begin{aligned} \omega^2 \tilde{v} + i\omega \left( v_0 \frac{\partial \tilde{v}}{\partial z} + \tilde{v} \frac{\partial v_0}{\partial z} \right) &= \frac{\partial}{\partial t} \frac{\partial}{\partial z} \left( \frac{\tilde{n}}{n_0} \right) \\ \Rightarrow \omega^2 \tilde{v} + i\omega \left( v_0 \frac{\partial \tilde{v}}{\partial z} + \tilde{v} \frac{\partial v_0}{\partial z} \right) &= \frac{\partial}{\partial z} \left( -i\omega v_0 \tilde{v} - (1 - v_0^2) \frac{\partial \tilde{v}}{\partial z} + \left( v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \tilde{v} \right) \end{aligned}$$

Expand the RHS and collect terms, we get

$$\begin{aligned} &\omega^2 \tilde{v} \\ &+ 2i\omega \left( v_0 \frac{\partial}{\partial z} + \frac{\partial v_0}{\partial z} \right) \tilde{v} \\ &+ \left[ (1 - v_0^2) \frac{\partial^2}{\partial z^2} - \left( 3v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \frac{\partial}{\partial z} - \left( 1 - \frac{1}{v_0^2} \right) \left( \frac{\partial v_0}{\partial z} \right)^2 - \left( v_0 + \frac{1}{v_0} \right) \frac{\partial^2 v_0}{\partial z^2} \right] \tilde{v} = 0 \end{aligned}$$

Let  $\Omega \equiv i\omega$ , we obtain Eq.(8). □

## 2.2 Eigenvalue problem

We can decouple the polynomial eigenvalue problem so that it becomes an eigenvalue problem.

Let  $a = \Omega \tilde{v}$ , then Eq.(8) becomes

$$\begin{bmatrix} O & I \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ a \end{bmatrix} = \Omega \begin{bmatrix} \tilde{v} \\ a \end{bmatrix} \quad (9)$$

where  $O$  is zero matrix,  $I$  is identity matrix, and

$$\begin{aligned} A_{21} &= 2 \left( v_0 \frac{\partial}{\partial z} + \frac{\partial v_0}{\partial z} \right) \\ A_{22} &= (1 - v_0^2) \frac{\partial^2}{\partial z^2} - \left( 3v_0 + \frac{1}{v_0} \right) \frac{\partial v_0}{\partial z} \frac{\partial}{\partial z} - \left( 1 - \frac{1}{v_0^2} \right) \left( \frac{\partial v_0}{\partial z} \right)^2 - \left( v_0 + \frac{1}{v_0} \right) \frac{\partial^2 v_0}{\partial z^2} \end{aligned}$$

## 2.3 Variational Form

**Definition 1.** P and Q functions defined below can be used as an indicator of instabilities.

- P-function:

$$P \equiv -\frac{\partial v_0}{\partial z} \left( 3v_0 + \frac{1}{v_0} \right) \quad (10)$$

- Q-function:

$$Q \equiv -\left( 1 - \frac{1}{v_0^2} \right) \left( \frac{\partial v_0}{\partial z} \right)^2 - \left( v_0 + \frac{1}{v_0} \right) \frac{\partial^2 v_0}{\partial z^2} \quad (11)$$

**Proposition 4.**

$$(\omega_r^2 - \gamma^2) \langle |\tilde{v}|^2 \rangle - 2\omega_r \left\langle v_0 \operatorname{Im} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) \right\rangle - \gamma \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle - \left\langle (1 - v_0^2) \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle + \left\langle \left( Q - \frac{1}{2} \frac{\partial P}{\partial z} \right) |\tilde{v}|^2 \right\rangle = 0 \quad (12)$$

*This is the full instability condition, we can determine whether the motion is stable or not by examining the number of solutions  $\gamma$ . If there are two real solutions (two real  $\gamma$ ), then it's unstable, otherwise it is stable.*

*Proof.* Starting from Eq.(8), we multiply  $\tilde{v}^*$  and take the average over the region, we have

$$\omega^2 \langle |\tilde{v}|^2 \rangle + 2i\omega \left( \left\langle v_0 \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right\rangle + \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle \right) + \left[ \left\langle (1 - v_0^2) \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2} \right\rangle + \left\langle P \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right\rangle + \left\langle Q |\tilde{v}|^2 \right\rangle \right] = 0$$

Let  $\omega \equiv \omega_r + i\gamma$ , then

$$(\omega_r^2 - \gamma^2 + 2i\omega_r\gamma) \langle |\tilde{v}|^2 \rangle + 2(i\omega_r - \gamma) \left( \left\langle v_0 \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right\rangle + \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle \right) + \left\langle (1 - v_0^2) \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2} \right\rangle + \left\langle P \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right\rangle + \left\langle Q |\tilde{v}|^2 \right\rangle = 0$$

We split the equation into real and imaginary part.

- Real:

There are two useful simplifications to keep in mind,

$$\begin{aligned} \operatorname{Re} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) &= \frac{1}{2} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} + \tilde{v} \frac{\partial \tilde{v}^*}{\partial z} \right) = \frac{1}{2} \frac{\partial |\tilde{v}|^2}{\partial z} \\ \left\langle \operatorname{Re} \left( \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2} \right) \right\rangle &= \left\langle \frac{1}{2} \left( \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2} + \tilde{v} \frac{\partial^2 \tilde{v}^*}{\partial z^2} \right) \right\rangle = \left\langle \frac{1}{2} \left( \frac{\partial}{\partial z} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) + \frac{\partial}{\partial z} \left( \tilde{v} \frac{\partial \tilde{v}^*}{\partial z} \right) - 2 \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right) \right\rangle = \left\langle \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle \end{aligned}$$

The second formula uses the fact  $\tilde{v} = 0$  on boundaries (Dirichlet condition), so  $\langle \partial \text{product of } \tilde{v} / \partial z \rangle = 0$ .

Now we separate the real part,

$$\begin{aligned} (\omega_r^2 - \gamma^2) \langle |\tilde{v}|^2 \rangle - 2\omega_r \left\langle v_0 \operatorname{Im} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) \right\rangle - \gamma \left\langle v_0 \frac{\partial |\tilde{v}|^2}{\partial z} \right\rangle - 2\gamma \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle \\ - \left\langle (1 - v_0^2) \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle + \left\langle \frac{1}{2} P \frac{\partial |\tilde{v}|^2}{\partial z} \right\rangle + \langle Q |\tilde{v}|^2 \rangle = 0 \end{aligned}$$

The term with  $\gamma$  can combine by swithing the position of  $\partial/\partial z$ ,

$$-\gamma \left\langle v_0 \frac{\partial |\tilde{v}|^2}{\partial z} \right\rangle - 2\gamma \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle = -\gamma \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle$$

The  $P$  and  $Q$  term can combine as well

$$\left\langle \frac{1}{2} P \frac{\partial |\tilde{v}|^2}{\partial z} \right\rangle + \langle Q |\tilde{v}|^2 \rangle = \left\langle \left( Q - \frac{1}{2} \frac{\partial P}{\partial z} \right) |\tilde{v}|^2 \right\rangle$$

The desired instability condition follows.

- Imaginary:

Although the imaginary part is not important, we derive it too.

This time, we notice that

$$\begin{aligned} \left\langle \operatorname{Im} \left( \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2} \right) \right\rangle &= \left\langle \frac{1}{2} \left( \tilde{v}^* \frac{\partial^2 \tilde{v}}{\partial z^2} - \tilde{v} \frac{\partial^2 \tilde{v}^*}{\partial z^2} \right) \right\rangle = \left\langle \frac{1}{2} \left( \frac{\partial}{\partial z} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) - \frac{\partial}{\partial z} \left( \tilde{v} \frac{\partial \tilde{v}^*}{\partial z} \right) \right) \right\rangle = 0 \\ 2\omega_r \gamma \left\langle |\tilde{v}|^2 \right\rangle + \omega_r \left\langle v_0 \frac{\partial |\tilde{v}|^2}{\partial z} \right\rangle + 2\omega_r \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle - 2\gamma \left\langle v_0 \operatorname{Im} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) \right\rangle + \left\langle P \operatorname{Im} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) \right\rangle &= 0 \end{aligned}$$

□

**Proposition 5.** *For oscillations where  $\omega_r \rightarrow 0$ , the instability condition becomes*

$$-\gamma^2 \left\langle |\tilde{v}|^2 \right\rangle + \gamma \left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle - \left\langle (1 - v_0^2) \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle + \left\langle \left( Q - \frac{1}{2} \frac{\partial P}{\partial z} \right) |\tilde{v}|^2 \right\rangle = 0$$

*Therefore, unstable modes occur if*

$$\left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle^2 + 4 \left\langle |\tilde{v}|^2 \right\rangle \left( - \left\langle (1 - v_0^2) \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle + \left\langle \left( Q - \frac{1}{2} \frac{\partial P}{\partial z} \right) |\tilde{v}|^2 \right\rangle \right) > 0$$

### 3 Numerical Experiments

#### 3.1 Constant velocity profile

If we set  $v_0$  to Constant, then the eigenvalue

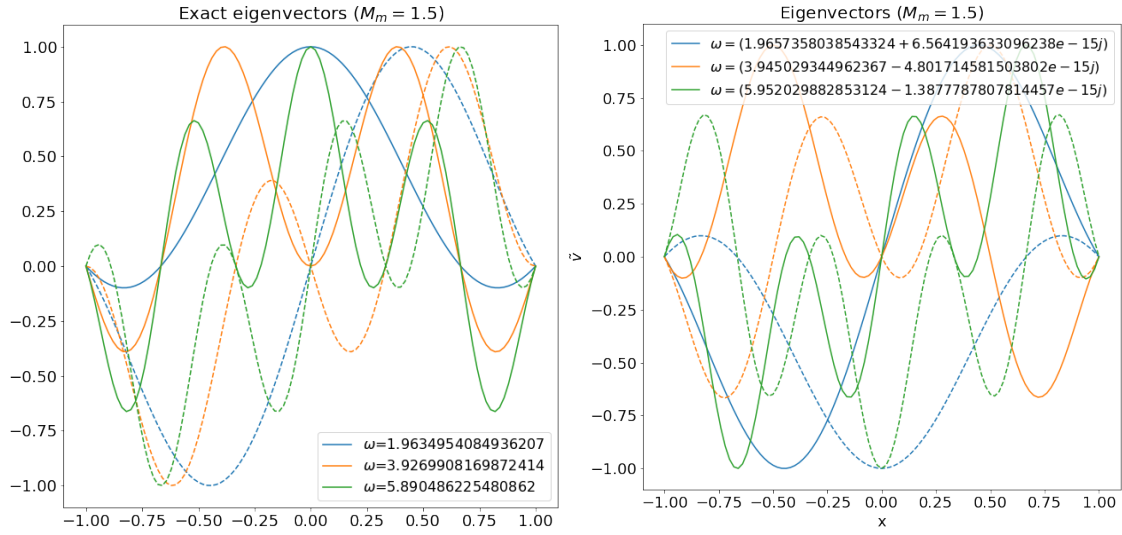
$$-\Omega^2 \tilde{v} + 2\Omega v_0 \frac{\partial}{\partial z} \tilde{v} + (1 - v_0^2) \frac{\partial^2}{\partial z^2} \tilde{v} = 0 \quad (13)$$

where  $\Omega \equiv i\omega$ .

Then we have non-zero analytical solution

$$\tilde{v} = \exp\left(-\frac{\Omega}{v_0 + 1}\right) \left[ \exp\left(\Omega \frac{z+1}{v_0 + 1}\right) - \exp\left(\Omega \frac{z+1}{v_0 - 1}\right) \right] \text{ where } \Omega = \frac{in\pi(1 - v_0^2)}{2} \quad (14)$$

Therefore,  $\omega = -i\Omega = n\pi(1 - v_0^2)/2$  are real. All modes are stable.



(a) First few exact eigenfunctions (ground mode not included). (b) First few numerical eigenfunctions by solving Eq.(9).

Figure 2: We see that the numerical eigenfunctions match the theoretical results. The numerical results are obtained by solving Eq.(9). Same conclusion can be drawn by solving Eq.(8).

### 3.1.1 "Unstable modes" for $v_0 > 1$

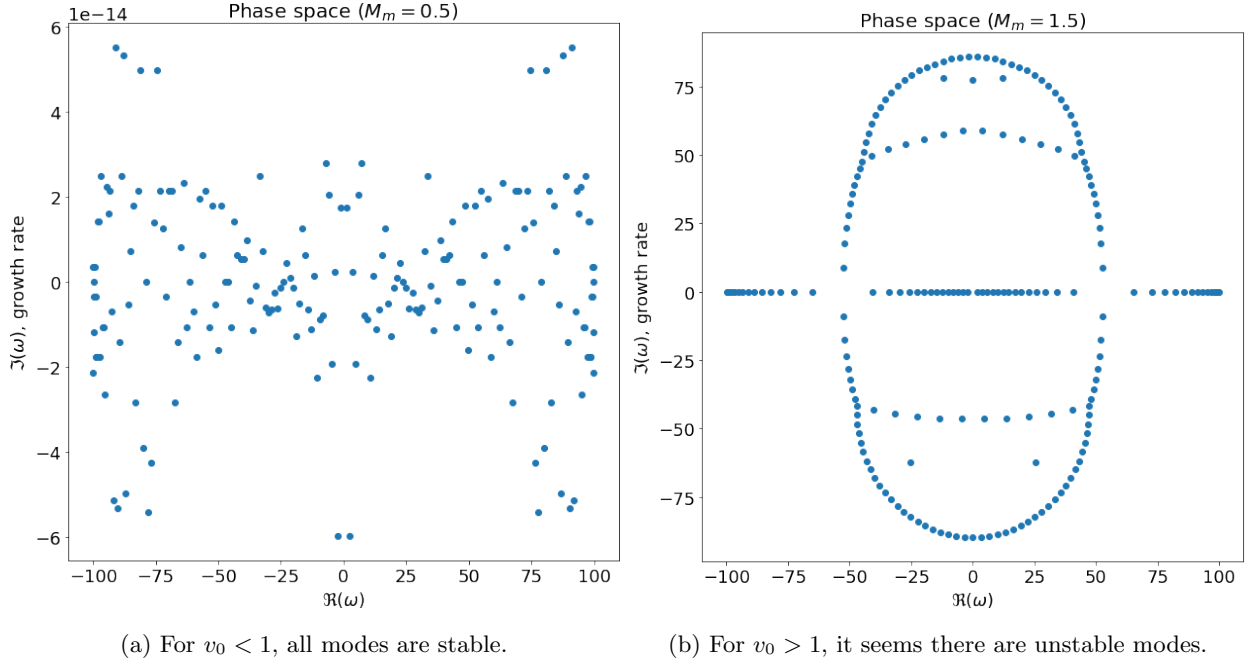


Figure 3: It seems there are unstable modes for  $v_0 > 1$ .

If we plot the eigenfunctions for the "unstable modes", we see that

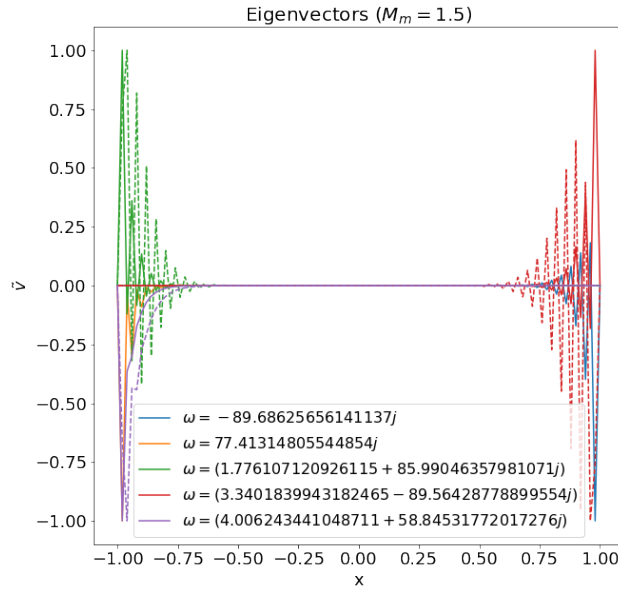


Figure 4: The eigenfunctions do not look right. For both equally-spaced finite difference differentiation matrix and Chebyshev differentiation matrix, these eigenfunctions will occur.



### 3.2 Subsonic case

The stability condition Eq.(12 is a quadratic equation about  $\gamma$ . Let

$$\begin{aligned} a &= -\langle |\tilde{v}|^2 \rangle \\ b &= -\left\langle \frac{\partial v_0}{\partial z} |\tilde{v}|^2 \right\rangle \\ c &= \omega_r^2 \langle |\tilde{v}|^2 \rangle - 2\omega_r \left\langle v_0 \operatorname{Im} \left( \tilde{v}^* \frac{\partial \tilde{v}}{\partial z} \right) \right\rangle - \left\langle (1 - v_0^2) \left| \frac{\partial \tilde{v}}{\partial z} \right|^2 \right\rangle + \left\langle \left( Q - \frac{1}{2} \frac{\partial P}{\partial z} \right) |\tilde{v}|^2 \right\rangle = 0 \end{aligned}$$

then the full instability condition becomes

$$D \equiv b^2 - 4ac$$

If the discriminant  $D > 0$ , it is unstable. If  $D < 0$ , it is stable.

Subsonic modes are stable. The stability condition  $D < 0$ . Moreover, the eigenvalues are real.

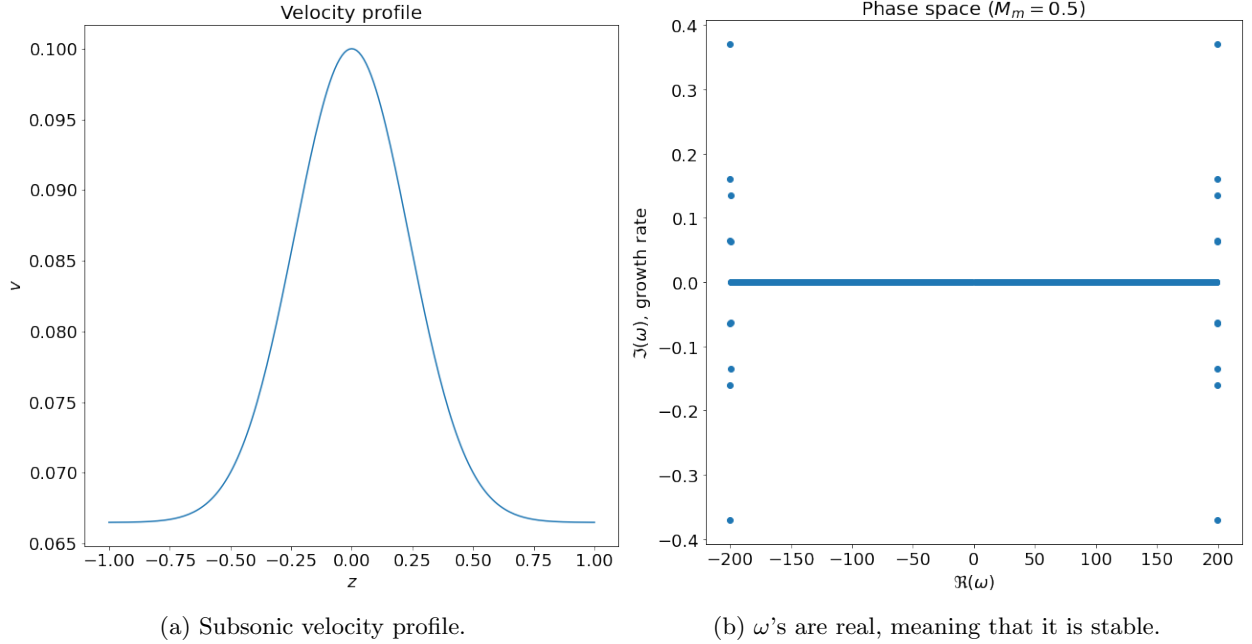


Figure 5: Solution to the polynomial eigenvalue problem Eq.(8) with subsonic velocity profile.  $\omega$ 's are pure imaginary numbers, meaning that subsonic mode should be stable for most cases.

### 3.3 Accelerating case

Accelerating modes are unstable.

The stability condition for ground state is  $D = 0.00062 > 0$ .

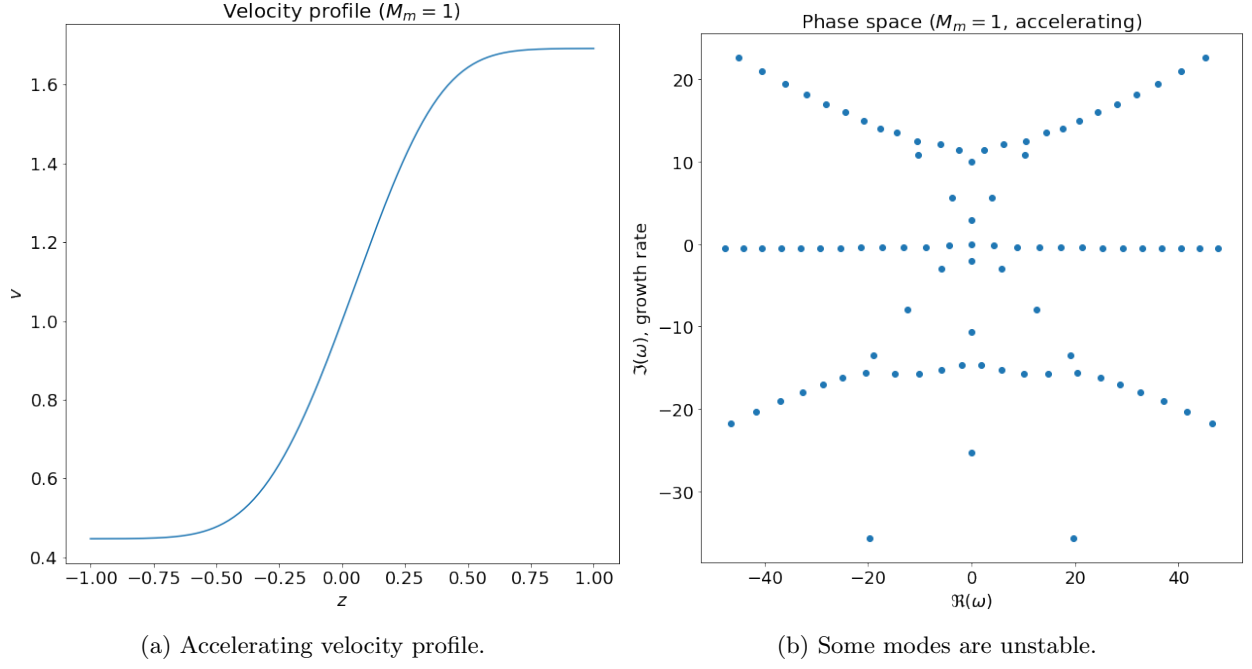


Figure 6: Solution to the polynomial eigenvalue problem Eq.(8) with accelerating velocity profile.  $\omega$ 's are not on the real line, meaning that accelerating mode should be unstable for most cases.

### 3.4 Decelerating case

Decelerating modes are unstable.

The stability condition for ground state is  $D = 0.00062 > 0$ .

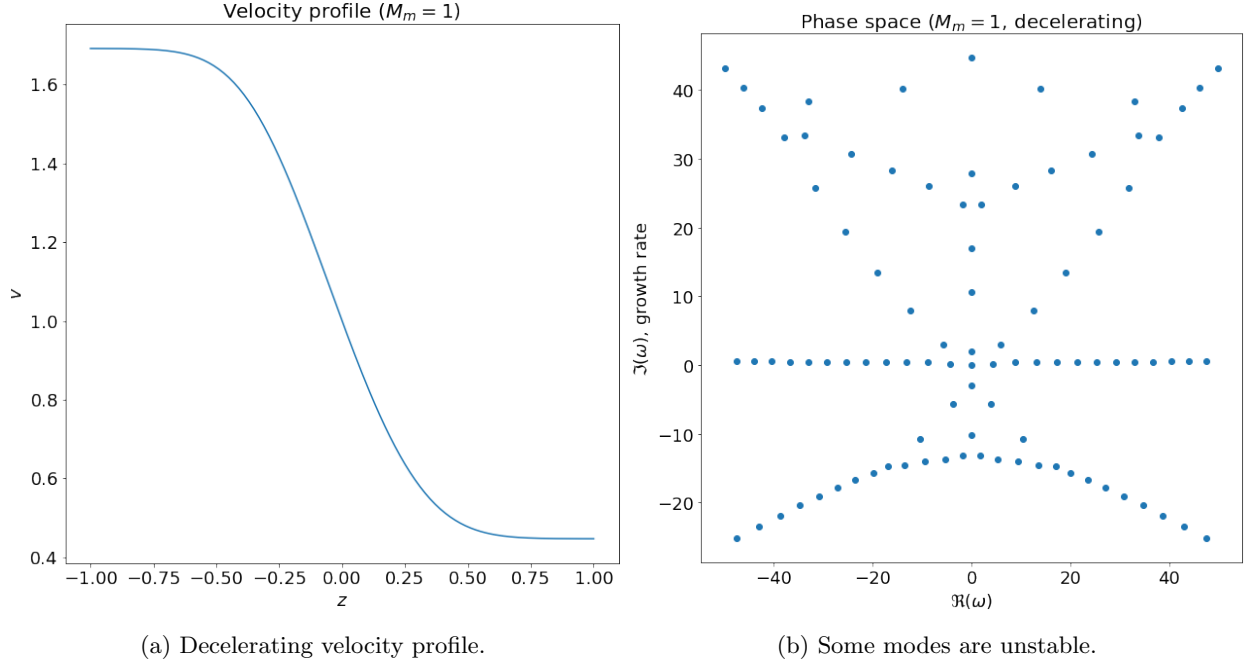


Figure 7: Solution to the polynomial eigenvalue problem Eq.(8) with decelerating velocity profile.  $\omega$ 's are not on the real line, meaning that decelerating mode should be unstable for most cases.

### 3.5 Supersonic case

Supersonic modes are unstable.

The stability condition for ground state is  $D = 1.21025 > 0$ .

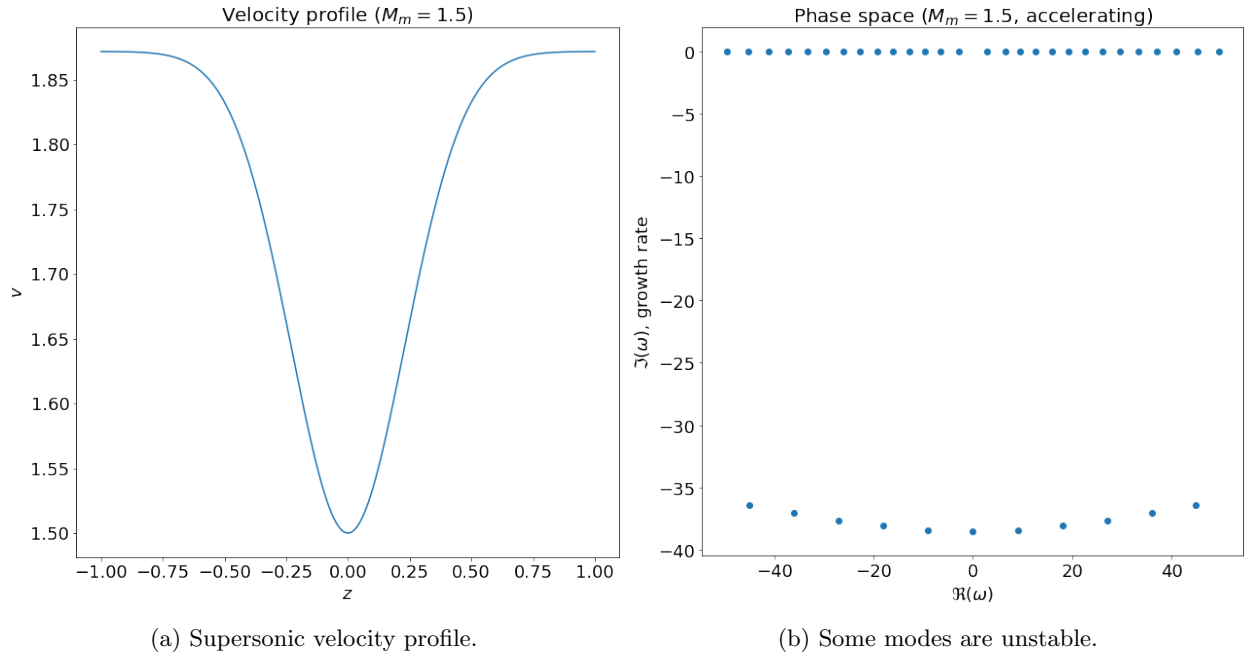


Figure 8: Solution to the polynomial eigenvalue problem Eq.(8) with supersonic velocity profile.  $\omega$ 's are not on the real line, meaning that supersonic mode should be unstable for most cases.