Chapter 3 - Balance

Hunt Feng[1]

[1]Faculty of Physics And Engineering Physics
University of Saskatchewan

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Outline of Presentation

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1. Grad-Shafranov Equation
2. Particle Orbits
3. Current Drive

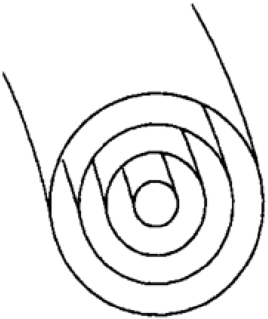
Flux Function

The poloidal magnetic flux function ψ satisfies $\mathbf{B} \cdot \boldsymbol{\nabla}\psi = 0$, that is

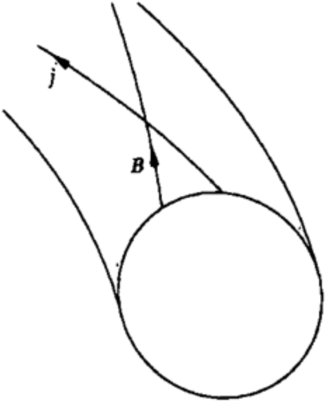
$$B_R = -\frac{1}{R}\frac{\partial \psi}{\partial z}, \quad B_z = \frac{1}{R}\frac{\partial \psi}{\partial z} \tag{1}$$

- ψ can be used a convenient coordinate.
- Pressure is constant on flux surface, so $p = p(\psi)$.
- Define $f = RB_\phi/\mu_0$, $f = f(\psi)$.

Flux Function - Figures



(a) Magnetic flux surfaces forming a set of nested toroids. This is because $\mathbf{B} \cdot \boldsymbol{\nabla} p = 0$, this conclusion comes from $\mathbf{j} \times \mathbf{B} = \boldsymbol{\nabla} p$.



(b) Magnetic field lines and current lines lie in magnetic surfaces. This is because $\mathbf{j} \cdot \boldsymbol{\nabla} p = 0$.

Grad-Shafranov Equation

The balance of tokamak requires $\mathbf{j} \times \mathbf{B} = \nabla p$. Using the flux function ψ, we are able to write it as

$$R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 R^2 p'(\psi) - \mu_0^2 f(\psi) f'(\psi) \tag{2}$$

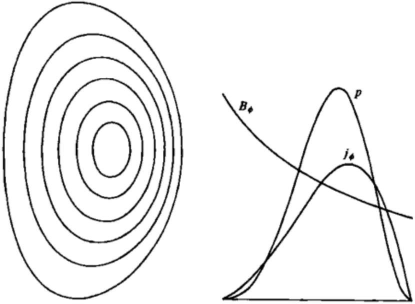


Figure 2: Euilibrium flux surfaces and plots of toroidal current density, plasma pressure, and toroidal magnetic field across the midplane.

Safety Factor, q

$$q = \frac{\phi}{2\pi} \tag{3}$$

- Plays important role in determining stability.
- q characterize the number of winds of the B field line goes around the plasma torus toroidally in one round of θ.

Using B_ϕ and B_p we can express safety factor as

$$q = \frac{1}{2\pi} \oint \frac{1}{R} \frac{B_\phi}{B_p} \, \mathrm{d}s \tag{4}$$

where $\mathrm{d}s$ is the distance moved in θ direction while moving through $\mathrm{d}\phi$. For large aspect-ratio tokamak of circular cross-section, we have approximation,

$$q = \frac{1}{2\pi} \oint \frac{1}{R} \frac{B_\phi}{B_p} \, \mathrm{d}s \tag{5}$$

Safety Factor, q - Figures

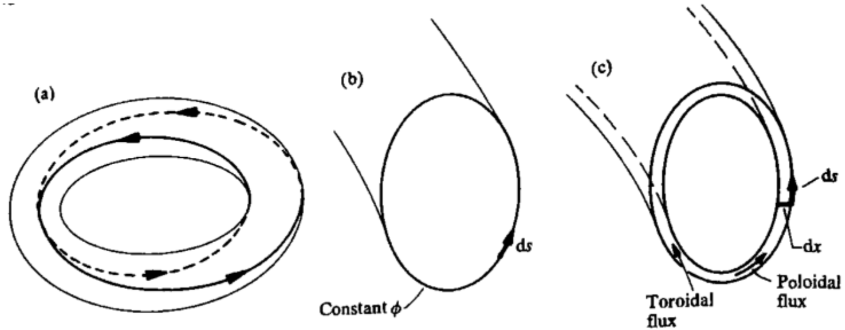


Figure 3: (a) Field line on $q = 2$ surface. (b) Poloidal integration path for Eq.(4). (c) Flux annulus containing toroidal flux $\mathrm{d}\Phi = \oint (B_\phi \, \mathrm{d}x) \, \mathrm{d}s$ and poloidal flux $\mathrm{d}\Psi = 2\pi R B_p dx$

q Profiles

The radial profile of q is given by

$$q(r) = \frac{2B_\phi}{\mu_0 \langle j \rangle_r R} \tag{6}$$

where $\langle j \rangle_r = \int_0^r j(r')r' \,\mathrm{d}r' \,/(r^2/2)$ is the average current density inside the radius r.

- An approximation: $q_{cyl} = 2\pi a b B_\phi / \mu_0 I R$.
- With $j = j_0(1 - r^2/a^2)^\nu$,

$$q = \frac{2(\nu+1)}{\mu_0 j_0} \frac{B_\phi}{R} \frac{r^2/a^2}{1 - (1 - r^2/a^2)^{\nu+1}}$$

When near the X-point, the limiting form of q as $d \to 0$ is

$$q \to \frac{B_\phi}{\pi R |\nabla B_p|} \ln \frac{\lambda}{d} \quad (d \to 0)$$

Beta

The efficiency of confinement of plasma is represented by the ratio

$$\beta = \frac{p}{B^2/2\mu_0}$$

There are other slightly different definitions for different purposes,

- For a reactor, the important quantity is the thermonuclear power obtained for a given B field,

$$\beta^* = \frac{\left(\int p^2 \mathrm{d}\tau / \int \mathrm{d}\tau\right)^{1/2}}{B_0^2/2\mu_0}$$

- A commonly used averaged beta

$$\langle\beta\rangle = \frac{\int p \mathrm{d}\tau / \int \mathrm{d}\tau}{B_0^2/2\mu_0}$$

- Poloidal β

$$\beta_p = \frac{\int p \, \mathrm{d}S \,/ \int \mathrm{d}S}{B_a^2/2\mu_0}$$

where the ingrals are surface integrals over the poloidal cross-section

Large Aspect-Ratio

For large aspect-ratio plasmas of circular cross-section with low β, the tokamak equilibria take a simple form.
If the flux surface ψ is displaced a distance $\Delta(\psi_0(r))$, ψ may be written as

$$\psi = \psi_0 - \Delta(r)\cos\theta \frac{\mathrm{d}\psi_0}{\mathrm{d}r} \tag{7}$$

The Grad-Shafranov equation, Eq.(2), takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_{\theta 0}^2 \frac{\mathrm{d}\Delta}{\mathrm{d}r}\right) = \frac{r}{R_0}\left(2\mu_0 r \frac{\mathrm{d}p_0}{\mathrm{d}r} - B_{\theta 0}^2\right) \tag{8}$$

Eq.(7) and Eq.(8) together provide the solution $\psi(r,\theta)$.

Large Aspect-Ratio - Figure

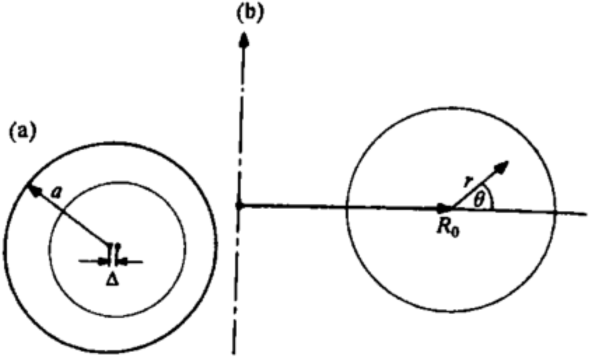


Figure 4: (a) Showing circular flux surface displaced by a distance Δ with respect to outer flux surface whose centre is at a distance R_0 from the major axis. (b) Coordinate system (r, θ) with centre at major radius R_0.

Shafranov Shift

The displacement Δ appears in Eq.(7) is called the Shafranov shift.
If the profiles of pressure and current are given by

$$p = \hat{p}\left(1 - \frac{r^2}{a^2}\right), \quad j = \hat{j}\left(1 - \frac{r^2}{a^2}\right)^{\nu}$$

we can numerically solve the Shafranov shift Δ_s, see Fig.5.

Shafranov Shift

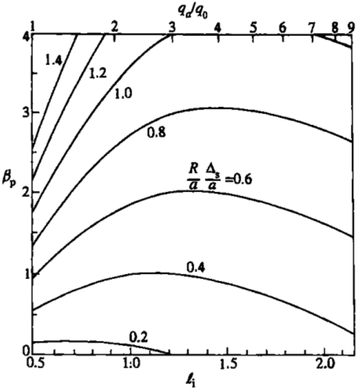


Figure 5: Graphs giving the Shafranov shift Δ_s in the form of contours of equal values of $(R/a)\Delta_s/a$ in the (β_p, l_i) plane for a parabolic pressure profile and current profiles given by above, where $l_i = \overline{B_\theta^2}/B_{\theta a}^2$ is the internal inductance.

Vacuume Magnetic Field

The Eq.(8) enables us to determine the magnetic required to maintain the plasma equilibrium,

$$B_\theta(a) = B_{\theta 0}(a)\left(1 + \frac{a}{R_0}\Lambda\cos\theta\right) \tag{9}$$

where $\Lambda = \beta_p + \frac{l_i}{2} - 1$.
The vaccume vertical magnetic field necessary to maintain the plasma in equilibrium is given by

$$B_v = -\frac{\mu_0 I}{4\pi R_0}\left(\ln\frac{8R_0}{a} + \Lambda - \frac{1}{2}\right)$$

Its effect is to provide inward force to balance the outward hoop force on the plasma.

Electric Field

This is just $\mathbf{j} \times \mathbf{B}$

Outline of Presentation

Particle Orbits

- Particles with high velocity can go around the torus.
- Particles with low velocity will be trapped due to mirror effect.

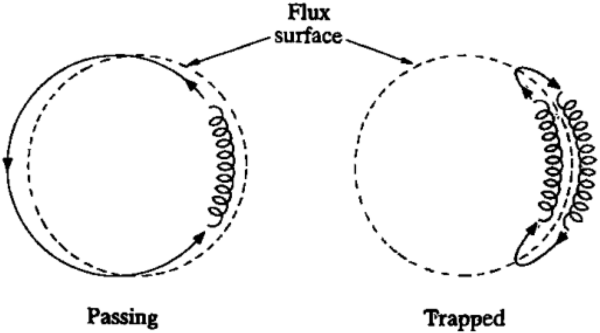


Figure 6: Diagram illustrating drift surfaces for the orbit of a passing particle and the (banana) orbit of a trapped particle. The major axis of the torus is on the left.

Particle Trapping

- The vacuume toroidal magnetic field is propotional to $1/R$, the field is smaller on the outer side of the torus.
- Particles with a small $v_{||}$ undergo a magnetic mirror reflection as they move into the region of higher field.

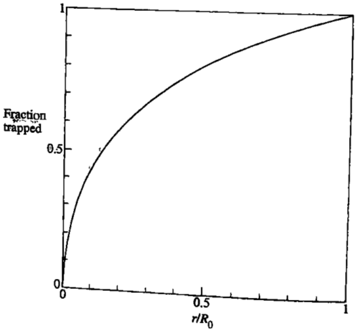
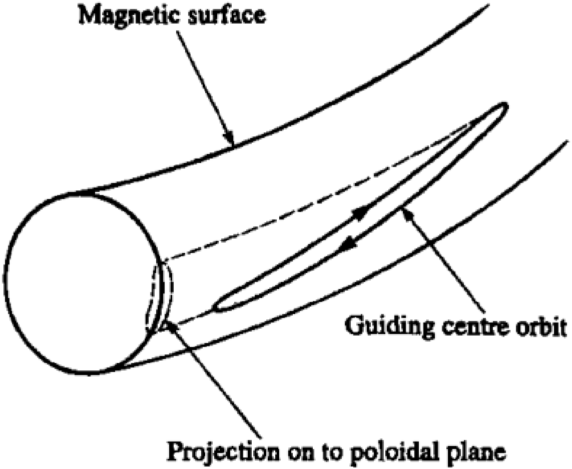


Figure 7: Graph of the fraction of the particles which are trapped as a function of the inverse aspect-ratio of the magnetic surface, r/R_0.

Particle Orbits - Banana Orbit

- For deuterons, $r^{3/2} \gg q\rho R^{1/2} \sim 10^0$cm, so their orbits usually have banana shape.



Trapped Particle Orbit - Potato Orbits

- α-particles have large Larmor radius and are predominantly produced in the core of plasma, their orbits are called potato orbit.
- The potato orbit is given by

$$\left(\frac{r}{R_0}\right)^3 = \left(\frac{2\rho q}{R_0}\right)^2 \cos\theta$$

where $q = B/R_0 B'_\theta$.

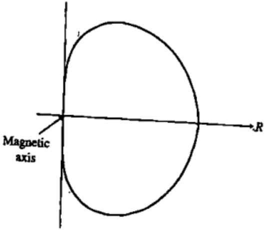
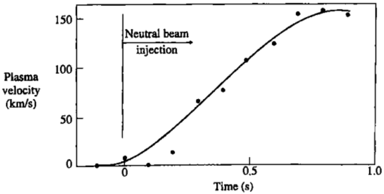


Figure 9: Potato orbit for particle passing through the magnetic axis.

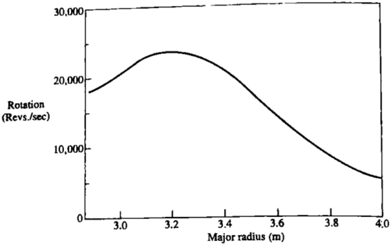
Outline of Presentation

Plasma Rotation

- Neutral beams are injected into the plasma with toroidal velocity causes plasma to spin.
- The spin of plasma introduces a centrifugal force, hence breaking the Grad-Shafranov equation.



(a) Plasma velocity during neutral beam injection as measured by the Doppler shift of the impurity spectral lines (JET).



(b) Typical radial profile of toroidal rotation of a plasma spun by neotral beam injection (JET).

Current Drive

- **Neutral beam injection**: ions in the plasma center will be accelerated, hence creating a toroidal current.
- **Lower hybrid current drive**: the energy of low waves with high $v_{||}$ is deposited into the plasma by Landau damping. Creating a toroidal current. High efficiency in the outer half of the plasma.
- **Fast wave electron current drive**: uses fast magnetosonic waves in the ion cyclotron frequency range. Similar to lower hybrid waves, the energy is deposited into plasma by Landau damping. The force on the electrons is due to the $E_{||}$, $\nabla_{||} B$ of the waves and the magnetic moment of the electrons. Transit time magnetic pumping (TTMP).
- **Electron cyclotron current drive**: heating only those electrons circulating in a particular toroidal direction. It is to accelerate electrons primarily in the perpendicular direction. Good for control of mhd instabilities.
- **Fast wave minority ion current drive**: Similar to electron cyclotron current drive, but this time we heat up ion.

Current Drive - Figure

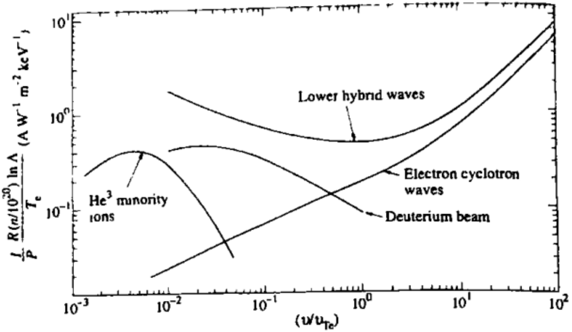


Figure 11: Comparison of theoretical current drive effecicieny for (i) deuterium beams injected into a D-T plasma; (ii) Landau damping of lower hybrid waves. (iii) electron cyclotron waves, and (iv) He^3 minority ions in a deuterium plasma. The scale gives the ratio of the total current I(A) to the otal power injected, P(W), into a tokamak plasma of major radius R(m), density $n(10^{20})\mathrm{m}^{-3}$, and temperature T_e(keV). v is the wave phase velocity, or beam ion velocity, and v_{T_e} is the electron thermal velocity.

All figures are adopted from [4].

F. Chen.
Introduction to Plasma Physics and Controlled Fusion.
Springer, dec 29 2015.

M. Delage, A. Froese, D. Blondal, and D. Richardson.
Progress towards acoustic magnetized target fusion: An overview of the r&d program at general fusion.
In *Canadian Nuclear Society - 33rd Annual Conference of the Canadian Nuclear Society and 36th CNS/CNA Student Conference 2012: Building on Our Past... Building for the Future*, volume 1, pages 285–297, 2012.

M. Laberge.
Experimental results for an acoustic driver for mtf.
Journal of fusion energy, 28(2):179–182, 2009.

J. Wesson and D. J. Campbell.
Tokamaks.
International Monographs on Ph, oct 13 2011.