Chapter 2 - Plasma Physics

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Outline of Presentation

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Tokamak Plasma

The typical plasma in tokamak has the following properties:

Plasma volume	1–100 m^3
Total plasma mass	$10^{-4} - 10^{-2}$ gm
Ion concentration	$10^{19} - 10^{20}\ m^{-3}$
Temperature	1–40 keV
Pressure	0.1–5 atmospheres
Ion thermal velocity	100–1000 km s^{-1}
Electron thermal velocity	$0.01c$–$0.1c$
Magnetic field	1–10 T
Total plasma current	0.1–7 MA

Figure 1: Typical tokamak plasmas

Debye Shielding

Due to the high mobility of electrons, the electric field from heavy ions will be shielded by electron cloud. The electric field is very small at certain distance away from ions. Hence, the plasma is quasineutral.
The number density of ion and electron obey the Boltzmann distribution,

$$n_j = n_0 \exp\left(-\frac{e_j \phi}{T}\right) \tag{1}$$

The electric potential from an ion shielded by electrons is given by,

$$\phi = \frac{e}{4\pi\epsilon_0 r} \exp\left(-\frac{\sqrt{2}r}{\lambda_D}\right) \tag{2}$$

where λ_D is the Debye length and is defined by

$$\lambda_D = \left(\frac{\epsilon_0 T}{ne^2}\right)^2 = 2.35 \times 10^5 \left(\frac{T}{n}\right)^2 \tag{3}$$

Plasma Frequency

If we pin ions as a static and uniform background, and perturb electrons, then electrons will oscillate around the static ions. The oscillation frequency is called electron plasma frequency, and is

$$\omega_{pe} = \left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2} \tag{4}$$

Similarly, the ion plasma has its oscillation frequency as well,

$$\omega_{pi} = \left(\frac{ne_i^2}{\epsilon_0 m_i}\right)^{1/2} \tag{5}$$

where e_i is the charge of ion.

Outline of Presentation

Larmor Orbit

A charged particle gyrates in the plane that is perpendicular to the magnetic field lines. Its gyration radius is called Larmor radius,

$$\rho = \frac{mv_\perp}{eB} \tag{6}$$

where $v_\perp$ is the speed in the direction perpendicular to magnetic field B. The gyration frequency is called Larmor frequency or cyclotron frequency

$$\omega_c = \frac{eB}{m} \tag{7}$$

Particle Motion along B

- If B field is uniform and no E field along **B**, then the particle is moving with constant velocity in the direction of **B**.
- If B field is uniform and there is a non-zero component of E along **B**, then the particle will experience a force $eE_{||}$ along **B**.
- If B field has gradient in the direction of **B** (assuming no E). Then the particle will experience a force $-\frac{\frac{1}{2}mv\perp^2}{B}\boldsymbol{\nabla}_{||}B$.

Particle Drifts - $\mathbf{E} \times \mathbf{B}$ Drift

$$\mathbf{v}_{E\times B} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{8}$$

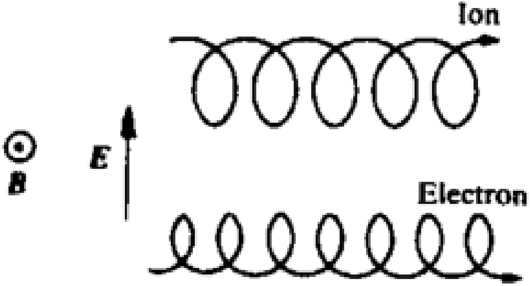


Figure 2: $\mathbf{E} \times \mathbf{B}$ drift of ion and electron. $v_d = E/B$.

Particle Drifts - ∇B Drift

$$\mathbf{v}_{\nabla B} = \mathrm{sign}(e)\frac{1}{2}\rho\frac{\mathbf{B}\times\nabla B}{B^2}v_\perp \tag{9}$$

where sign(e) is the sign of particle charge.

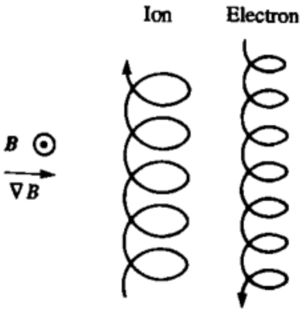


Figure 3: A gradient of B perpendicular to B gives ion and electron drifts in opposite directions.

Particle Drifts - Curvature Drift

$$\mathbf{v}_{\nabla B} = \frac{v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2}{\omega_c}\frac{\mathbf{B}\times\nabla B}{B^2} \tag{10}$$

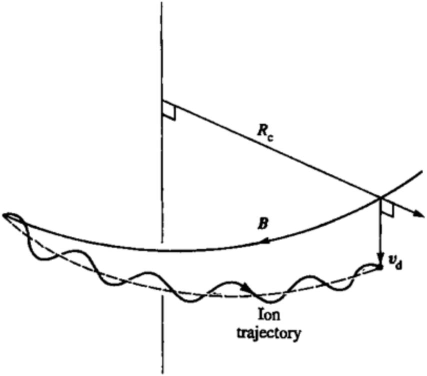


Figure 4: Ion drift due to magnetic field curvature. Electrons drift in the opposite directions.

Particle Drifts - Polarization Drift

$$\mathbf{v}_{\nabla B} = \frac{1}{\omega_c B} \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} \tag{11}$$

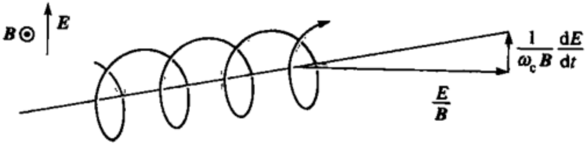


Figure 5: Polarization drift of an ion caused by an increasing electric field perpendicular to the magnetic field. The electron drifts is in the opposite direction.

Adiabatic Invariants

- First adiabatic invariant: The magnetic moment, $\mu = \frac{\frac{1}{2}mv_\perp^2}{B}$, is constant if **B** changes very slowly.
- Second adiabatic invariant: This invariant J is constant when the particle has a larger scale periodic motion. And $J = \oint v_{\|}\,\mathrm{d}l$ where $\mathrm{d}l = \mathbf{b} \cdot \mathrm{d}\mathbf{x}$, **b** is the unit vector along **B**.
- Third adiabatic invariant: If the periodic motion involved J is subject to a drift and results in a larger scale periodic motion, then $J_3 = \int v_d\,\mathrm{d}l$ is invariant.

Outline of Presentation

Fokker-Planck Equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{12}$$

In collisionless plasma, $(\partial_t f)_c = 0$. The collisionless Fokker-Planck equation has another name called Vlasov equation.

Landau Damping

- Landau damping is predicted by kinetic theory.
- Damping exists for waves even in collisionless plasma as well.
- The damping comes from the interaction of wave with particles travelling at the phase velocity, ω/k.

If we perturb the Vlasov equation, with $f = f_0 + f_1$ and $E = E_0 + E_1$, where the subscript 1 indicates that it is a perturbation quantity, we will have dispersion relation

$$\omega = \omega_p(1 + 3k^2\lambda_D^2)^{1/2} - i\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_p}{(k\lambda_D)^3}\exp\left[-\frac{1}{2}\left(\frac{1}{k^2\lambda_D^2} + 3\right)\right] \quad (13)$$

Collision

$$\left(\frac{\partial f}{\partial t}\right)_c = -\sum_{\alpha} \frac{\partial}{\partial v_\alpha}(\langle \Delta v_\alpha \rangle f) + \frac{1}{2}\sum_{\alpha,\beta} \frac{\partial^2}{\partial v_\alpha \partial v_\beta}\left(\langle \Delta v_\alpha \Delta v_\beta \rangle f\right) \tag{14}$$

where $\langle \Delta v_\alpha \rangle$ is coefficient of dynamic friction and $\langle \Delta v_\alpha \Delta v_\beta \rangle$ diffusion tensor, and they are defined as

$$\langle \Delta v_\alpha \rangle = \int \psi \Delta v_\alpha \, \mathrm{d}\Delta \mathbf{v} \, / \Delta t \tag{15}$$

$$\langle \Delta v_\alpha \Delta v_\beta \rangle = \int \psi \Delta v_\alpha \Delta v_\beta \, \mathrm{d}\Delta \mathbf{v} \, / \Delta t \tag{16}$$

The $\psi(\mathbf{v}, \Delta \mathbf{v})$ is the probability of a particle with velocity $\mathbf{v}$ being scattered by $\Delta \mathbf{v}$ in the time Δt.

Gyro-averaged Kinetic Equations

If the plasma phenomenon involves processes which are slow compared to the Larmor frequency and which vary slowly in space compared to the Larmor radius of the individual particle, then we can average the particle motion over the fast Larmor motion,

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}t} + \mathbf{v}_g \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} + \left(e\mathbf{E} \cdot \mathbf{v}_g + \mu \frac{\partial B}{\partial t} \right) \frac{\partial \bar{f}}{\partial K} = \left(\frac{\partial \bar{f}}{\partial t} \right)_c \tag{17}$$

where $\mu = mv_\perp^2/2B$ is the magnetic moment, $K = \frac{1}{2}mv^2$ is the kinetic energy, and

$$\mathbf{v}_g = v_\parallel \mathbf{b} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{v_\parallel^2 \mathbf{b} \times (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{b} + \mu \mathbf{b} \times \boldsymbol{\nabla} B}{\omega_c} \tag{18}$$

is the guiding center velocity, and $\mathbf{b} = \mathbf{B}/B$ and $\omega_c = eB/m$.

- Gyro-averaged equation reduces 1 dimension compare to the Fokker-Planck equation.
- Not valid if the equilibrium possesses magnetic shear.

Fokker-Planck Equation for a Plasma

We first define the so-called Rosenbluth potentials,

$$\begin{aligned} H_j(\mathbf{v}) &= \left(1 + \frac{m}{m_j}\right) \int \frac{f_j(\mathbf{v})}{|\mathbf{v} - \mathbf{v_j}|} \, \mathrm{d}\mathbf{v_j} \\ G_j(\mathbf{v}) &= \int f_j(\mathbf{v}) |\mathbf{v} - \mathbf{v_j}| \, \mathrm{d}\mathbf{v_j} \end{aligned} \tag{19}$$

The collision term, Eq.(14), can be written using these potentials,

$$\begin{aligned} \left(\frac{\partial f}{\partial t}\right)_c = \sum_j \frac{e^2 Z^2 Z_j^2 \ln \Lambda}{4\pi\epsilon_0^2 m^2} \Bigg[& -\frac{\partial}{\partial v_\alpha} \left(\frac{\partial H_j(\mathbf{v})}{\partial v_\alpha} f(\mathbf{v})\right) \\ & + \frac{1}{2} \frac{\partial^2}{\partial v_\alpha \partial v_\beta} \left(\frac{\partial^2 G_j(\mathbf{v})}{\partial v_\alpha \partial v_\beta} f(\mathbf{v})\right) \Bigg] \end{aligned} \tag{20}$$

Notice that $\langle \Delta v_\alpha \rangle$ and $\langle \Delta v_\alpha \Delta v_\beta \rangle$ can also be expressed using Rosenbluth potentials.

Fokker-Planck Equation for a Plasma - Continued

We can also write the collision term as symmetric Landau integral form,

$$\left(\frac{\partial f}{\partial t}\right)_c = \sum_j \frac{e^2 Z^2 Z_j^2 \ln \Lambda}{4\pi\epsilon_0^2 m^2} \frac{\partial}{\partial v_\alpha} \int \left(\frac{f_j(\mathbf{v_j})}{m} \frac{\partial f(\mathbf{v})}{\partial v_\beta} - \frac{f(\mathbf{v})}{m_j} \frac{\partial f_j(\mathbf{v_j})}{\partial v_{j\beta}} \right) u_{\alpha\beta} \, \mathrm{d}\mathbf{v_j} \tag{21}$$

where $\mathbf{u} = \mathbf{v} - \mathbf{v_j}$, $u_{\alpha\beta} = (u^2 \delta_{\alpha\beta} - u_\alpha u_\beta)/u^3$.
In this way, once we have the initial distributions of each species, we can numerically solve the Fokker-Planck equation, Eq.(12).

Fokker-Planck Coefficient under Maxwellian Distribution

Under Maxwellian distribution, we can calculate Rosenbluth potentials, Eq.(19). The collision term $(\partial_t f)_c$, Eq.(20), becomes

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nabla_v \cdot [\langle \Delta \mathbf{v}_\parallel \rangle f - \nabla_\parallel (D_\parallel f) - D_\perp \nabla_\perp f] \tag{22}$$

where ∇_v means the divergence in velocity space, and the diffusion coefficients are

$$D_\parallel = \frac{1}{2} \langle (\Delta v_\parallel)^2 \rangle \tag{23}$$

$$D_\perp = \frac{1}{2} \langle (\Delta v_{\perp\alpha})^2 \rangle = \frac{1}{2} \langle (\Delta v_{\perp\beta})^2 \rangle \tag{24}$$

Relaxation Processes

The heat exchange time, τ_{ij}, is defined by

$$\frac{\mathrm{d}T_i}{\mathrm{d}t} = \frac{T_j - T_i}{\tau_{ij}} \tag{25}$$

For electrons and single charged ions the heat exhange time is

$$\tau_{ie} = \tau_{ei} = \frac{m_i}{2m_e}\tau_e \tag{26}$$

where τ_e is the electron collision frequency

$$\tau_e = 3(2\pi)^{3/2}\frac{\epsilon_0^2 m_e^{1/2} T_e^{3/2}}{ne^4 \ln \Lambda} \tag{27}$$

Collision Times

If the temperature is in unit keV, then collision times (in s) are

$$\tau_e = 1.09 \times 10^{16} \frac{T_e^{3/2}}{n_i Z^2 \ln \Lambda} \tag{28}$$

$$\tau_i = 6.60 \times 10^{17} \frac{(m_i/m_p)^{1/2} T_i^{3/2}}{n_i Z^2 \ln \Lambda} \tag{29}$$

Fig.

The Coulomb logarithm is defined as $\ln \Lambda = \int_0^{\lambda_D} \frac{r \mathrm{d} r}{r_0^2 + r^2}$.

- electron-electron collisions

$$\ln \Lambda = 14.9 - \frac{1}{2} \ln\left(n_e/10^20\right) + \ln T_e, \quad T_e \text{ in keV}$$

- electron-ion collisions ($T \gtrsim 10\text{eV}$)

$$\ln \Lambda = 15.2 - \frac{1}{2} \ln\left(n_e/10^20\right) + \ln T_e, \quad T_e \text{ in keV}$$

Collision Times - Figure

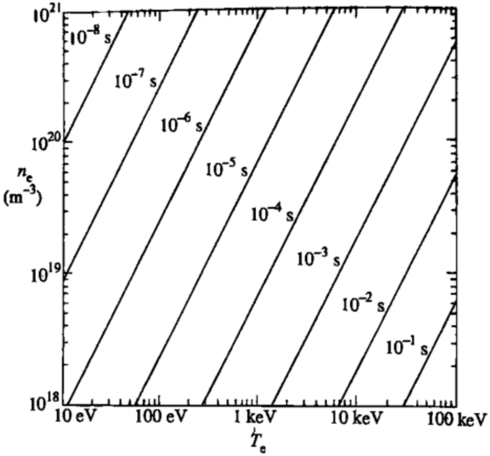


Figure 6: Values of τ_e plotted against electron density and electron temperature.

Resistivity

Resistivity can be defined by Ohm's law, $E = \eta j$.

- Singly charged ions (good for $B = 0$ or in $B_{\|}$ direction):

$$\eta_s = 0.51 \frac{m_e^{1/2} e^2 \ln \lambda}{3\epsilon_0^2 (2\pi T_e)^{3/2}}, \quad T_e \text{ in keV} \tag{30}$$

- Plasma composed of several ion species,

$$\eta = Z_{eff} \eta_s, \quad Z_{eff} = \frac{\sum_j n_j Z_j^2}{\sum_j n_j Z_j}$$

It is obvious that $Z_{eff} = 1$ for hydrogen plasma.

Worth to mention, resistivity in $B_{\perp}$ direction is almost twice of that in $B_{\|}$ direction,

$$\eta_{\perp} = \frac{m_e}{n_e e^2 \tau_e} = 1.96 \eta_{\|}$$

Runaway Electrons - Simple Estimate

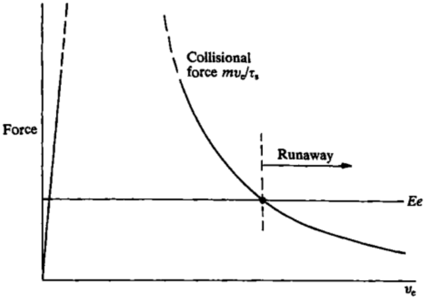


Figure 7: The electric field force, Ee, and the collisional force for an electron with a velocity v_e.

Fig.7 shows that, electrons in the tail of velocity distribution might run away due to the presence of electric field. The critical velocity is

$$v_c^2 = \frac{3ne^3 \ln \Lambda}{4\pi\epsilon_0^2 m_e E} = 2.3 \times 10^{-4} \frac{n}{E} \text{ m}^2 \text{ s}^{-2}, \quad \ln \Lambda = 17 \tag{31}$$

Outline of Presentation

Fluid Equations - Moments of f

From the distribution function $f(\mathbf{x}, \mathbf{v}', t)$, we can get its moments by multiplying it with functions of $\mathbf{v}'$ and integrate them.

$$
\begin{aligned}
n &= \int f(\mathbf{x}, \mathbf{v}', t) \, \mathrm{d}\mathbf{v}' \\
\mathbf{v} &= \frac{1}{n} \int f(\mathbf{x}, \mathbf{v}', t) \mathbf{v}' \, \mathrm{d}\mathbf{v}' \\
\mathbf{P} &= m \int f(\mathbf{x}, \mathbf{v}', t)(\mathbf{v}' - \mathbf{v})(\mathbf{v}' - \mathbf{v}) \, \mathrm{d}\mathbf{v}'
\end{aligned}
\tag{32}
$$

where $\mathbf{P}$ is pressure tensor. For spatial homogeneous distribution function, the scalar pressure is given by

$$
p = \frac{1}{3} m \int f(\mathbf{x}, \mathbf{v}', t)(\mathbf{v}' - \mathbf{v})^2 \, \mathrm{d}\mathbf{v}' \tag{33}
$$

Fluid Equations - Equations of Motion

Apply the same procedure in previous slide to Fokker-Planck Equation, Eq.(12), we are able to get equations of motions,

- Continuity Equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \tag{34}$$

- Momentum Equation:

$$nm\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla \cdot \mathbf{P} + nZe(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{R} \tag{35}$$

  where $\mathbf{R} = \int m\mathbf{v}' \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}\mathbf{v}'$.

We can continue this process and get more equations of higher moments f, e.g. heat equation.
Fluid equations are only valid when the mean free path is small enough compare to the macroscopic scale. So not valid for high temperature plasma.

To close the set of fluid equations, we need an extra equation of state (EOS), the adiabatic equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p\rho^{-\gamma}\right) = 0 \tag{36}$$

Using the vacuum Maxwell's equations, and fluid equations together with EOS, we have the so-called ideal MHD equations,

Table 1: The equations of ideal mhd.

$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v}$	$\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$
$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} - \nabla p$	$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$
$\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma p \nabla \cdot \mathbf{v}$	$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$

To get resistive MHD equations, we only need to replace the adiabatic equation to

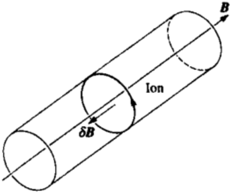
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\gamma - 1}\right) = \frac{\gamma}{\gamma - 1} p \boldsymbol{\nabla} \cdot \mathbf{v} + \eta j^2 \tag{37}$$

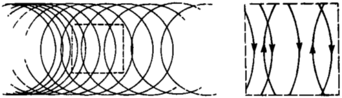
And the $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ to Ohm's law $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$

Plasma Diamagnetism

- Plasma is always diamagnetic due to the Larmor motions of particles weakens the applied magnetic field, see.
- The total diamagnetic current is given by $\mathbf{j} = \mathbf{j_s} + \mathbf{j_d}$, where $\mathbf{j_s} = \mathbf{b} \times \boldsymbol{\nabla}(p/B)$ is the current induced by particle gyration, and $\mathbf{j_d} = \mathbf{b} \times (p/B^2)\boldsymbol{\nabla} B$ is the drift current.

Plasma Diamagnetism - Figures





(a) Ions with a given velocity having Larmor orbits on a cylinder produce a magnetic field δB inside the cylinder, this field being in the opposite direction to the total magnetic field B.

(b) Showing how a density gradient of stationary orbits give rise to a current. The current arises through a local imbalance of upward and downward moving particles as illustrated in the enlarged secion.

Braginskii Equations

- Coulomb collisions do not change the number of particles.
- Friction force: $\mathbf{R_u} = -(m_e n/\tau_e)(0.51 u_\parallel + u_\perp) = ne(\eta_\parallel j_\parallel + \eta_\perp j_\perp)$, where $\mathbf{u} = \mathbf{v_e} - \mathbf{v_i}$
- Thermal force: $\mathbf{R_T} = -0.71 n \nabla_\parallel T_e - \frac{3}{2}\frac{n}{|\omega_{ce}\tau_e|\mathbf{b}\times\nabla T_e}$
- Ion heating: $Q_i = \frac{3m_e}{m_i}\frac{n}{\tau_e}(T_e - T_i)$.
- Electron heating:
  $Q_e = -\mathbf{R}\cdot\mathbf{u} - Q_i = \eta_\parallel j_\parallel^2 + \eta_\perp j_\perp^2 + \frac{1}{ne}\mathbf{j}\cdot\mathbf{R_T} + \frac{3m_e}{m_i}\frac{n}{\tau_e}(T_i - T_e)$

Some interesting things,

- The ratio of parallel to perpendicular thermal conductivity is $(\omega_c\tau)^2$. It is 10^{13} for electrons and 10^{16} for ions.
- In parallel direction, electron thermal conductivity is larger than that of ions by a factor $\sim (m_i/m_e)^2$ because of their long collision time.
- In perpendicular direction, the relationship is reversed because of the larger ion Larmor radius.
- The ohm heating ηj^2 appears only in electrons because they only transfer $\sim m_e/m_i$ of energy to ions.

Plasma Waves - Dispersion Relations

- Transverse EM wave:
$$\omega^2 = k^2c^2 + \omega_p^2 \tag{38}$$
- Sound wave:
$$\omega^2 = k^2c_s^2, \quad c_s^2 = \gamma\frac{p_i + p_e}{nm_i} \tag{39}$$
- Shear Alfven wave, see Fig:
$$\omega = k_x v_A, \quad v_A = B_0/\sqrt{\mu_0\rho} \tag{40}$$
- Magnetosonic waves:
$$\frac{\omega^2}{k^2} = \frac{1}{2}\{c_s^2 + v_A^2 \pm [(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2\theta]^{1/2}\} \tag{41}$$

The fast magnetosonic wave given by choosing the $+$ sign, and the slow magnetosonic wave given by choosing the $-$ sign.

Plasma Wave - Alfven Wave

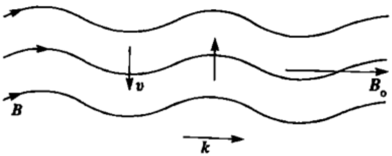


Figure 9: Simple Alfven wave with $k \parallel B_0$. The fluid velocity oscillates in the plane of the figure.

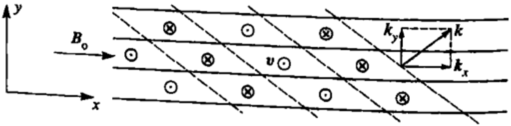


Figure 10: General shear Alfven wave. B_0 and k lie in the plane of the figure and the velocity oscillation is perpendicular to this plane. The wave propagates along x with the Alfven velocity v_A.

Plasma Wave - Magnetosonic Wave

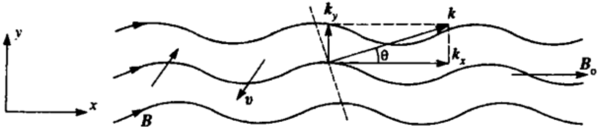


Figure 11: The magnetosonic wave has velocity oscillations in the plane containing B_0 and k.

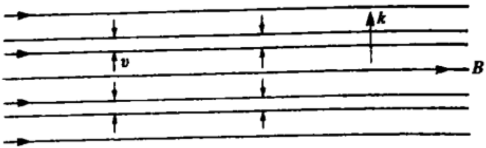


Figure 12: The fast magnetosonic with $k \perp B_0$. The oscillations involve compression of both the fluid and the magnetic field.

F. Chen.
Introduction to Plasma Physics and Controlled Fusion.
Springer, dec 29 2015.

M. Delage, A. Froese, D. Blondal, and D. Richardson.
Progress towards acoustic magnetized target fusion: An overview of the r&d program at general fusion.
In *Canadian Nuclear Society - 33rd Annual Conference of the Canadian Nuclear Society and 36th CNS/CNA Student Conference 2012: Building on Our Past... Building for the Future*, volume 1, pages 285–297, 2012.

M. Laberge.
Experimental results for an acoustic driver for mtf.
Journal of fusion energy, 28(2):179–182, 2009.

J. Wesson and D. J. Campbell.
Tokamaks.
International Monographs on Ph, oct 13 2011.