Three Wave Interaction

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Abstract

Resonant three-wave interaction is investigated by doing finite-volume simulation. The energy transfer between waves is observed.

1 Introduction

Resonant three-wave interaction is a well known non-dispersive system where nonlinear interactions play an important role in energy transfer. One of interesting applications is the amplification of a short laser pulse in a plasma with a counter-propagating pump laser using the transient Raman backscattering (RBS). With this method, it is possible to achieve not only a highpower laser but also very short laser pulse, which is applicable to the ignition of inertial confinement fusion. [2]

2 Three Wave Interaction

The electromagnetic wave in a plasma is modeled as

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \quad \mathbf{E} = E_{\parallel}\mathbf{e}_x - \frac{\partial \mathbf{A}}{\partial t} \tag{1}$$

where the Coulomb gauge is used ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), and the longitudinal electric field $E_{\parallel}$ is derived from Gauss's law,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\partial E_{\parallel}}{\partial x} = \frac{e}{\epsilon}(n_i - n_e) \tag{2}$$

Here we assume ions are immobile and the ion density $n_i = n$ is constant and is the same as the equilibrium electron density during the considered time scale. Ampere's law gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial E_{\parallel}}{\partial t}\mathbf{e}_x = \mu_0 n_e e \mathbf{v}, \tag{3}$$

and the equation of motion is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} = -\frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4}$$

Here, we consider nonrelativistic motion because the laser intensity is low enough to be in the nonrelativistic region. In the one-dimensional model, the perpendicular and the longitudinal velocities are derived from Eq.(4) as

$$\left(\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}\right)\left(\mathbf{v}_\perp-\frac{e}{m}\mathbf{A}\right)=0 \tag{5}$$

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}=-\frac{e}{m}\left(E_\parallel+\mathbf{v}_\perp\cdot\frac{\partial \mathbf{A}}{\partial x}\right) \tag{6}$$

Combine Eq.(2) and Eq.(3), we get

$$\mathbf{v}_\perp\left(1-\frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial x}\right)=\frac{ec^c}{m\omega_p^2}\left(\frac{\partial^2\mathbf{A}}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right) \tag{7}$$

$$v_x\left(1-\frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial x}\right)=\frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial t} \tag{8}$$

From Eq.(5) we get $\mathbf{v}_\perp = e\mathbf{A}/m$, therefore we have

$$E_\parallel+\frac{1}{\omega_p^2}\frac{\partial^2 E_\parallel}{\partial t^2}\simeq-\frac{e}{m}\mathbf{A}\cdot\frac{\partial \mathbf{A}}{\partial x} \tag{9}$$

$$\mathbf{A}\left(1-\frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial x}\right)=\frac{c^2}{\omega_p^2}\left(\frac{\partial^2\mathbf{A}}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right) \tag{10}$$

Here the second-order nonlinear terms are small enough to be neglected for electrostatic quantities; thus, $v_x\ \partial v_x/\partial x$ and $v_x\ \partial E_\parallel/\partial x$ are approximated by zero. With lineraly polarized waves, the seed and the pump awaves are expressed, respectively, as

$$\mathbf{A}_s(x,t)=\mathbf{e}_y\frac{mc}{e}\,\mathrm{Re}\{\phi_1(x,t)\exp(i\Phi_1)\} \tag{11}$$

$$\mathbf{A}_p(x,t)=\mathbf{e}_y\frac{mc}{e}\,\mathrm{Re}\{\phi_0(x,t)\exp(i\Phi_0)\} \tag{12}$$

where $\Phi_0=-k_0x-\omega_0t, \Phi_1=k_1x-\omega_1t$, and $\mathbf{A}=\mathbf{A}_s+\mathbf{A}_p$. The longitudinal electric field is expressed as

$$E_\parallel=\frac{mc\omega_p}{e}\,\mathrm{Re}\{\phi_2\exp(i\Phi_{PM})\} \tag{13}$$

where the ponderomotive phas, $\Phi_{PM}=\Phi_0-\Phi_1$, has a shorter wavelength and lower frequency than the electromagnetic waves. The exact matching conditions for RBS are

$$\begin{aligned}k_p&=k_0+k_1\\ \omega_p*&=\omega_0-\omega_1\end{aligned} \tag{14}$$

This is the resonance conditions for three-wave interaction, where k_p is the wave number of the longitudinal wave. The equation for the longitudinal field is obtained from Eq.(9). When we consider only the terms having the ponderomotive phase $\exp(i\Phi_{PM})$ in order to treat the coupling with the longitudinal field,

$$\mathbf{A}\cdot\frac{\partial\mathbf{A}}{\partial x}=-\,\mathrm{Re}\left\{\frac{i}{2}\left(\frac{mc}{2}\right)^2k_p\phi_1^*\phi_0\exp(i\Phi_{PM})\right\} \tag{15}$$

Also, the left-hand side of Eq.(9) can be converted to

$$\left(1+\frac{1}{\omega_p^2}\frac{\partial^2}{\partial t^2}\right)E_\parallel = \mathrm{Re}\left\{\frac{mc}{e}\left(\frac{1}{\omega_p}\frac{\partial^2\phi_2}{\partial t^2}-2i\frac{\partial\phi_2}{\partial t}\right)\exp(i\Phi_{PM})\right\} \tag{16}$$

Therefore, Eq.(9) becomes

$$\frac{\partial\phi_2}{\partial t}+\frac{i}{2\omega_p}\frac{\partial^2\phi_2}{\partial t^2} = -\frac{c}{4}k_p\phi_1^*\phi_0 \tag{17}$$

The group velocity of the plasma wave is zero in a cold plasma, and the second term on the left-hand side can be neglected because ϕ_2 varies very slowly in time. In the same way, Eq.(10) gives the equations for the pump and the seed pulses, respectively, with phase terms of $\exp(i\Phi_0)$ and $\exp(i\Phi_1)$. For the pump wave,

$$-\omega_p\frac{c}{4}k_p\phi_1\phi_2 \simeq k_0c^2\frac{\partial\phi_0}{\partial x}-\omega_0\frac{\partial\phi_0}{\partial t} \tag{18}$$

and for the seed wave,

$$\omega_p\frac{c}{4}k_p\phi_0\phi_2 \simeq -k_1c^2\frac{\partial\phi_1}{\partial x}-\omega_0\frac{\partial\phi_1}{\partial t} \tag{19}$$

With the definitions of the group velocities, $v_{g,0}=k_0c^2/\omega_0$ and $v_{g,1}=k_1c^2/\omega_1$, we finally get the governing equations of the nonlinear three-wave interaction as [3]

$$\begin{aligned}\frac{\partial\phi_0}{\partial t}-v_{g,0}\frac{\partial\phi_0}{\partial x} &= \beta_0\phi_1\phi_2\\ \frac{\partial\phi_1}{\partial t}+v_{g,1}\frac{\partial\phi_1}{\partial x} &= -\beta_1\phi_0\phi_2^*\\ \frac{\partial\phi_2}{\partial t} &= -\beta_2\phi_0\phi_1*\end{aligned} \tag{20}$$

where

$$\beta_0=\beta_2\frac{\omega_p}{\omega_0},\quad \beta_1=\beta_2\frac{\omega_p}{\omega_1},\quad \beta_2=\frac{c}{4}k_p \tag{21}$$

Here, $\phi_0=eE_0/mc\omega_0$ and $\phi_1=eE_1/mc\omega_1$ are the normalized electric fields for the pump and the seed pulses, respectively, and $\phi_2=eE_\parallel/mc\omega_p$ is the normalized longitudinal electric field. The seed ($j=1$) and the pump ($j=0$) laser satisfy the dispersio relation of an electromagnetic wave in a plasma:

$$\omega_j^2=\omega_p^2+c^2k_j^2 \tag{22}$$

3 Methodology

We will use finite-volume simulation to verify the amplification of seed laser.

3.1 Finite Volume

The three-wave interaction is governed by the PDE system Eq.(20). The PDE system can be written as the non-conservative form [1]

$$\frac{\partial \mathbf{U}}{\partial t} + A\frac{\partial \mathbf{U}}{\partial x} = \mathbf{S}(\mathbf{U}), \tag{23}$$

or in the conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}(\mathbf{U}) \tag{24}$$

where $\mathbf{U} = [\phi_0, \phi_1, \phi_2]^T$ is the state variable, $A = \mathrm{diag}\{-v_{g,0}, v_{g,1}, 0\}$ is the Jacobian matrix $\partial\mathbf{F}/\partial\mathbf{U}$, and $\mathbf{S}(\mathbf{U}) = [\beta_0\phi_0\phi_2, -\beta_1\phi_0\phi_2^*, -\beta_2\phi_0\phi_1^*]^T$ is the source term.

Then we discretize the conservative form Eq.(24) by applying integrals on a control volume (a cell) $[x_{i-1/2}, x_{i+1/2}]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}dx + \mathbf{F}(\mathbf{U})\Big|_{x_{i-1/2}}^{x_{i+1/2}} = \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{S}(\mathbf{U})dx \tag{25}$$

Define the mean value of $\mathbf{U}$ in control volume i as

$$\mathbf{U}_i = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}dx, \quad \Delta x_i = x_{i+1/2} - x_{i-1/2} \tag{26}$$

Then Eq.(25) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_i \approx -\frac{1}{\Delta x_i}\,\mathbf{F}(\mathbf{U})\Big|_{x_{i-1/2}}^{x_{i+1/2}} + \mathbf{S}(\mathbf{U}_i) \tag{27}$$

The mean value of source term $\mathbf{S}(\mathbf{U})_i$ is approximated by $\mathbf{S}(\mathbf{U}_i)$. This is the semidiscrete form of Eq.(24).

3.2 Evaluate flux difference

3.2.1 Spatial reconstruction and flux limiter

The first task we need to do is to evaluate $\mathbf{F}(\mathbf{U})\big|_{x_{i-1/2}}^{x_{i+1/2}}$. We see that this requires the value at the interface of the cell i. We need to reconstruct the value of $\mathbf{U}(x,t)$ within the control volume so that we can extrapolate the value of $\mathbf{U}(x_{i\pm1/2}, t)$ at the interface.

In this experiment, we do linear reconstruction of $\mathbf{U}$. Together with the use of flux limiter, we can achieve second-order spatial accuracy. In this experiment, minmod limiter is chosen. It is defined as

$$\phi^{(m)}(r_i) = \max[0, \min(1, r)], \quad r_i = \frac{u_i^{(m)} - u_{i-1}^{(m)}}{u_{i+1}^{(m)} - u_i^{(m)}} \tag{28}$$

Here $u_i^{(m)}$ means the m-th component of $\mathbf{U}_i$. The linear extrapolation of $\mathbf{U}$ at interface $x_{i+1/2}$ is therefore,

$$\begin{aligned} u_{i+1/2}^{(m)L} &= u_i^{(m)} + \phi^{(m)}(r_i)\frac{u_{i+1}^{(m)} - u_i^{(m)}}{2} \\ u_{i-1/2}^{(m)R} &= u_i^{(m)} + \phi^{(m)}(r_i)\frac{u_i^{(m)} - u_{i-1}^{(m)}}{2} \end{aligned} \tag{29}$$

where L and R stands for the left limit and right limit of the linear reconstruction.

3.2.2 Lax-Friedrichs flux

The Lax-Friedrichs method is an alternative to Godunov's scheme, where one avoids solving a Riemann problem at each cell interface, at the expense of adding artificial viscosity.

$$\hat{\mathbf{F}}_{i+1/2} = \frac{1}{2}(\mathbf{F}_{i+1/2}^L + \mathbf{F}_{i+1/2}^R) - \frac{\Delta x}{2\Delta t}(\mathbf{U}_{i+1/2}^R - \mathbf{U}_{i+1/2}^L) \tag{30}$$

Once we compute the Lax-Friedrichs flux, the flux difference can be estimated,

$$\mathbf{F}(\mathbf{U})\Big|_{x_{i-1/2}}^{x_{i+1/2}} \approx \hat{\mathbf{F}}_{i+1/2} - \hat{\mathbf{F}}_{i-1/2} \tag{31}$$

3.3 Time integrator

Now we have a way to evaluate the flux difference, $\mathbf{F}(\mathbf{U})\big|_{x_{i-1/2}}^{x_{i+1/2}}$, the R.H.S. of Eq.(27) is known. We can apply the usual time integration technique to update the state variable $\mathbf{U}_i$. Since the spatial accuracy is second-order, so second-order accurate time integration method, RK2, is chosen.

Let $\mathbf{U}_i^n$ denotes the mean value of $\mathbf{U}$ in control volume i at time t_n. Then the R.H.S. of Eq.(27) can be denoted $\mathbf{f}(\mathbf{U}_i^n)$. The update rule of RK2 can be described as

$$\begin{aligned} \mathbf{U}_i^* &= \Delta t \mathbf{f}(\mathbf{U}_i^n) \\ \mathbf{U}_i^{**} &= \Delta t \mathbf{f}\left(\mathbf{U}_i^n + \frac{1}{2}\mathbf{U}_i^*\right) \\ \mathbf{U}_i^{n+1} &= \mathbf{U}_i^n + \mathbf{U}_i^{**} \end{aligned} \tag{32}$$

Table 1: The first three entries are the frequencies of pump wave, ω_0, seed wave, ω_1, and electron plasma frequency, ω_0. The last three entries are the initial amplitudes of pump wave, ϕ_0, seed wave ϕ_1 and plasma, ϕ_2.

ω_0	ω_1	ω_p	ϕ_p	ϕ_s	ϕ_{pl}
1.885PHz	1.6965PHz	188.5THz	0.05	0.1	0.01

Table 2: The grid resolution is set to be $\Delta x = 0.2\mu$m and the time step is $\Delta t = 0.9\Delta x/v_{g,0}$ and the two boundaries are fixed. Meaning the values of the boundaries are unchanged. The system length is set to be L =2mm, so that we can minimize the boundary effect on the result.

L	Δx	$\Delta t v_{g,0}/\Delta x$	B.C.
2mm	0.2μm	0.9	Fixed

4 Numerical Experiment

The plasma frequencies and initial wave amplitudes are taken from [2] and summarized in Table.1.

The coupling coefficients $\beta_j (j = 0, 1, 2)$ can be obtained using Eq.(21). Moreover, plasma frequency, ω_2, and the wavenumber of electron plasma wave, k_p, can be determined from the resonance conditions Eq.(14) for the three-wave interaction.

Knowing wavenumbers for pump and seed waves, $k_j (j = 0, 1)$, we can determine their group velocities $v_{g,j} = k_j c^2/\omega_j (j = 0, 1)$. For simplicity, a cold plasma is assumed, $v_{g,2} = 0$.

Now, the initial conditions are known given the above parameters.

$$\begin{aligned} \phi_0(x,0) &= \phi_p H((x-x_0) - (v_0+v_1)t_r) \\ \phi_1(x,0) &= \phi_s \exp\left(-\frac{(x-x_0)^2}{(ct_s)^2}\right) \\ \phi_2(x,0) &= \phi_{pl} \end{aligned} \tag{33}$$

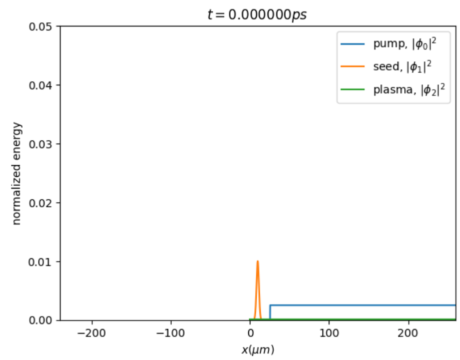
where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

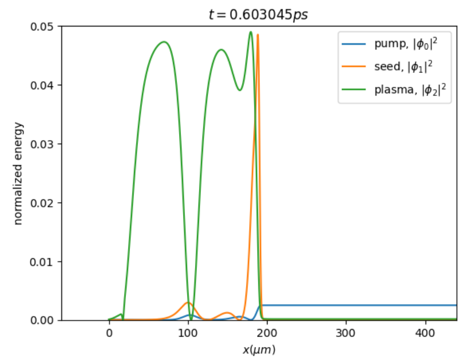
and $t_s = 0.1$ps is the duration of the initial seed pulse, and $t_r = 2.627 t_s$ is the rising time of the pulse. Finally, the parameter $x_0 = 10\mu$m controls the initial position of the waves.

In this simulation, the grid resolution is $\Delta x = 0.2\mu$m, the time step is set to be $\Delta t = 0.9\Delta x/v_{g,0}$. During the simulation, the boundary values are fixed to its initial value to maintain the left-going pump wave. To minimize the effect of the right boundary on the result, we set the length of the region to be 2mm.

The parameter $x_0 = 10\mu\text{m}$ is chosen in a way such that the waves interact at a position far away from the right end of the region.



(a) Before the waves interact, the seed pulse moving to the right has amplitude 0.1. And the pump wave moving to the left has amplitude 0.05.



(b) After the seed pulse encounter the oppositely propagating pump wave in the plasma region, energy of pump wave is transferred to the seed wave.

5 Conclusion

From the numerical experiment, we conclude that, in a cold plasma, by sending a short laser pulse (seed) to a counter-propagating pump laser, the energy of the pump wave is transferred to the seed wave under the resonance conditions Eq.(14). Hence, we see the increase in the energy of the seed wave and depletion of the energy in the pump wave near the position interaction.

References

[1] Stephen Jardin. *Computational methods in plasma physics.* Chapman & Hall/CRC computational science series. CRC Press, Boca Raton, 2010. OCLC: ocn426810763.

[2] Yeun Jung Kim, Minsoo Lee, and Hae June Lee. Machine Learning Analysis for the Soliton Formation in Resonant Nonlinear Three-Wave Interactions. *J. Korean Phys. Soc.*, 75(11):909–916, December 2019.

[3] H.J. Lee, J. Kim, C. Kim, G.-H. Kim, J.-U. Kim, and H. Suk. Solitary waves in a plasma interacting with two counter-propagating laser pulses. In *The 30th International Conference on Plasma Science, 2003. ICOPS 2003. IEEE Conference Record - Abstracts.*, page 366, Jeju, South Korea, 2003. IEEE.