Three Wave Interaction

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Abstract

Resonant three-wave interaction is investigated by doing finite-volume simulation. The energy transfer between waves is observed.

1 Introduction

Resonant three-wave interaction is a well known non-dispersive system where nonlinear interactions play an important role in energy transfer. One of interesting applications is the amplification of a short laser pulse in a plasma with a counter-propagating pump laser using the transient Raman backscattering (RBS). With this method, it is possible to achieve not only a highpower laser but also very short laser pulse, which is applicable to the ignition of inertial confinement fusion. [2]

2 Three Wave Interaction

The electromagnetic wave in a plasma is modeled as

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \quad \mathbf{E} = E_{\parallel}\mathbf{e}_x - \frac{\partial \mathbf{A}}{\partial t} \tag{1}$$

where the Coulomb gauge is used ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), and the longitudinal electric field $E_{\parallel}$ is derived from Gauss's law,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\partial E_{\parallel}}{\partial x} = \frac{e}{\epsilon}(n_i - n_e) \tag{2}$$

Here we assume ions are immobile and the ion density $n_i = n$ is constant and is the same as the equilibrium electron density during the considered time scale. Ampere's law gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial E_{\parallel}}{\partial t}\mathbf{e}_x = \mu_0 n_e e \mathbf{v}, \tag{3}$$

and the equation of motion is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4}$$

Here, we consider nonrelativistic motion because the laser intensity is low enough to be in the nonrelativistic region. In the one-dimensional model, the perpendicular and the longitudinal velocities are derived from Eq.(4) as

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right)\left(\mathbf{v}_\perp - \frac{e}{m}\mathbf{A}\right) = 0 \tag{5}$$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = -\frac{e}{m}\left(E_\parallel + \mathbf{v}_\perp \cdot \frac{\partial \mathbf{A}}{\partial x}\right) \tag{6}$$

Combine Eq.(2) and Eq.(3), we get

$$\mathbf{v}_\perp \left(1 - \frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial x}\right) = \frac{ec^c}{m\omega_p^2}\left(\frac{\partial^2 \mathbf{A}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \tag{7}$$

$$v_x \left(1 - \frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial x}\right) = \frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial t} \tag{8}$$

From Eq.(5) we get $\mathbf{v}_\perp = e\mathbf{A}/m$, therefore we have

$$E_\parallel + \frac{1}{\omega_p^2}\frac{\partial^2 E_\parallel}{\partial t^2} \simeq -\frac{e}{m}\mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial x} \tag{9}$$

$$\mathbf{A}\left(1 - \frac{e}{m\omega_p^2}\frac{\partial E_\parallel}{\partial x}\right) = \frac{c^2}{\omega_p^2}\left(\frac{\partial^2 \mathbf{A}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \tag{10}$$

Here the second-order nonlinear terms are small enough to be neglected for electrostatic quantities; thus, $v_x\ \partial v_x/\partial x$ and $v_x\ \partial E_\parallel \big/ \partial x$ are approximated by zero. With lineraly polarized waves, the seed and the pump awaves are expressed, respectively, as

$$\mathbf{A}_s(x,t) = \mathbf{e}_y \frac{mc}{e}\,\mathrm{Re}\{\phi_1(x,t)\exp(i\Phi_1)\} \tag{11}$$

$$\mathbf{A}_p(x,t) = \mathbf{e}_y \frac{mc}{e}\,\mathrm{Re}\{\phi_0(x,t)\exp(i\Phi_0)\} \tag{12}$$

where $\Phi_0 = -k_0 x - \omega_0 t, \Phi_1 = k_1 x - \omega_1 t$, and $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_p$. The longitudinal electric field is expressed as

$$E_\parallel = \frac{mc\omega_p}{e}\,\mathrm{Re}\{\phi_2 \exp(i\Phi_{PM})\} \tag{13}$$

where the ponderomotive phas, $\Phi_{PM} = \Phi_0 - \Phi_1$, has a shorter wavelength and lower frequency than the electromagnetic waves. The exact matching conditions for RBS are $k_p = k_0 + k_1$ and $\omega_0 - \omega_1 = \omega_p$ where k_p is the wave number of the longitudinal wave. The equation for the longitudinal field is obtained from Eq.(9). When we consider only the terms having the ponderomotive phase $\exp(i\Phi_{PM})$ in order to treat the coupling with the longitudinal field,

$$\mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial x} = -\,\mathrm{Re}\left\{\frac{i}{2}\left(\frac{mc}{2}\right)^2 k_p \phi_1^* \phi_0 \exp(i\Phi_{PM})\right\} \tag{14}$$

Also, the left-hand side of Eq.(9) can be converted to

$$\left(1+\frac{1}{\omega_p^2}\frac{\partial^2}{\partial t^2}\right)E_{\parallel}=\mathrm{Re}\left\{\frac{mc}{e}\left(\frac{1}{\omega_p}\frac{\partial^2\phi_2}{\partial t^2}-2i\frac{\partial\phi_2}{\partial t}\right)\exp(i\Phi_{PM})\right\} \tag{15}$$

Therefore, Eq.(9) becomes

$$\frac{\partial\phi_2}{\partial t}+\frac{i}{2\omega_p}\frac{\partial^2\phi_2}{\partial t^2}=-\frac{c}{4}k_p\phi_1^*\phi_0 \tag{16}$$

The group velocity of the plasma wave is zero in a cold plasma, and the second term on the left-hand side can be neglected because ϕ_2 varies very slowly in time. In the same way, Eq.(10) gives the equations for the pump and the seed pulses, respectively, with phase terms of $\exp(i\Phi_0)$ and $\exp(i\Phi_1)$. For the pump wave,

$$-\omega_p\frac{c}{4}k_p\phi_1\phi_2\simeq k_0c^2\frac{\partial\phi_0}{\partial x}-\omega_0\frac{\partial\phi_0}{\partial t} \tag{17}$$

and for the seed wave,

$$\omega_p\frac{c}{4}k_p\phi_0\phi_2\simeq -k_1c^2\frac{\partial\phi_1}{\partial x}-\omega_0\frac{\partial\phi_1}{\partial t} \tag{18}$$

With the definitions of the group velocities, $v_{g,0}=k_0c^2/\omega_0$ and $v_{g,1}=k_1c^2/\omega_1$, we finally get the governing equations of the nonlinear three-wave interaction as [3]

$$\begin{aligned}\frac{\partial\phi_0}{\partial t}-v_{g,0}\frac{\partial\phi_0}{\partial x}&=\beta_0\phi_1\phi_2\\ \frac{\partial\phi_1}{\partial t}+v_{g,1}\frac{\partial\phi_1}{\partial x}&=-\beta_1\phi_0\phi_2^*\\ \frac{\partial\phi_2}{\partial t}&=-\beta_2\phi_0\phi_1*\end{aligned} \tag{19}$$

where

$$\beta_0=\beta_2\frac{\omega_p}{\omega_0},\quad \beta_1=\beta_2\frac{\omega_p}{\omega_1},\quad \beta_2=\frac{c}{4}k_p \tag{20}$$

Here, $\phi_0=eE_0/mc\omega_0$ and $\phi_1=eE_1/mc\omega-1$ are the normalized electric fields for the pump and the seed pulses, respectively, and $\phi_2=eE_{\parallel}/mc\omega_p$ is the normalized longitudinal electric field. The seed ($j=1$) and the pump ($j=0$) laser satisfy the dispersio relation of an electromagnetic wave in a plasma:

$$\omega_j^2=\omega_p^2+c^2k_j^2 \tag{21}$$

3 Methodology

We will use finite-volume simulation to verify the amplification of seed laser.

3.1 Finite Volume

The three-wave interaction is governed by the PDE system Eq.(19). The PDE system can be written as the non-conservative form [1]

$$\frac{\partial \mathbf{U}}{\partial t} + A\frac{\partial \mathbf{U}}{\partial x} = \mathbf{S}(\mathbf{U}), \tag{22}$$

or in the conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}(\mathbf{U}) \tag{23}$$

where $\mathbf{U} = [\phi_0, \phi_1, \phi_2]^T$ is the state variable, $A = \mathrm{diag}\{-v_{g,0}, v_{g,1}, 0\}$ is the Jacobian matrix $\partial\mathbf{F}/\partial\mathbf{U}$, and $\mathbf{S}(\mathbf{U}) = [\beta_0\phi_0\phi_2, -\beta_1\phi_0\phi_2^*, -\beta_2\phi_0\phi_1^*]^T$ is the source term.

Then we discretize the conservative form Eq.(23) by applying integrals on a control volume (a cell) $[x_{i-1/2}, x_{i+1/2}]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}dx + \mathbf{F}(\mathbf{U})\Big|_{x_{i-1/2}}^{x_{i+1/2}} = \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{S}(\mathbf{U})dx \tag{24}$$

Define the mean value of $\mathbf{U}$ in control volume i as

$$\mathbf{U}_i = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}dx, \quad \Delta x_i = x_{i+1/2} - x_{i-1/2} \tag{25}$$

Then Eq.(24) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_i \approx -\frac{1}{\Delta x_i}\,\mathbf{F}(\mathbf{U})\Big|_{x_{i-1/2}}^{x_{i+1/2}} + \mathbf{S}(\mathbf{U}_i) \tag{26}$$

The mean value of source term $\mathbf{S}(\mathbf{U})_i$ is approximated by $\mathbf{S}(\mathbf{U}_i)$. This is the semidiscrete form of Eq.(23).

3.2 Evaluate flux difference

3.2.1 Spatial reconstruction and flux limiter

The first task we need to do is to evaluate $\mathbf{F}(\mathbf{U})\big|_{x_{i-1/2}}^{x_{i+1/2}}$. We see that this requires the value at the interface of the cell i. We need to reconstruct the value of $\mathbf{U}(x,t)$ within the control volume so that we can extrapolate the value of $\mathbf{U}(x_{i\pm1/2}, t)$ at the interface.

In this experiment, we do linear reconstruction of $\mathbf{U}$. Together with the use of flux limiter, we can achieve second-order spatial accuracy. In this experiment, minmod limiter is chosen. It is defined as

$$\phi^{(m)}(r_i) = \max[0, \min(1, r)], \quad r_i = \frac{u_i^{(m)} - u_{i-1}^{(m)}}{u_{i+1}^{(m)} - u_i^{(m)}} \tag{27}$$

Here $u_i^{(m)}$ means the m-th component of $\mathbf{U}_i$. The linear extrapolation of $\mathbf{U}$ at interface $x_{i+1/2}$ is therefore,

$$\begin{aligned} u_{i+1/2}^{(m)L} &= u_i^{(m)} + \phi^{(m)}(r_i)\frac{u_{i+1}^{(m)} - u_i^{(m)}}{2} \\ u_{i-1/2}^{(m)R} &= u_i^{(m)} + \phi^{(m)}(r_i)\frac{u_i^{(m)} - u_{i-1}^{(m)}}{2} \end{aligned} \tag{28}$$

where L and R stands for the left limit and right limit of the linear reconstruction.

3.2.2 Lax-Friedrichs flux

The Lax-Friedrichs method is an alternative to Godunov's scheme, where one avoids solving a Riemann problem at each cell interface, at the expense of adding artificial viscosity.

$$\hat{\mathbf{F}}_{i+1/2} = \frac{1}{2}(\mathbf{F}_{i+1/2}^L + \mathbf{F}_{i+1/2}^R) - \frac{\Delta x}{2\Delta t}(\mathbf{U}_{i+1/2}^R - \mathbf{U}_{i+1/2}^L) \tag{29}$$

Once we compute the Lax-Friedrichs flux, the flux difference can be estimated,

$$\mathbf{F}(\mathbf{U})\Big|_{x_{i-1/2}}^{x_{i+1/2}} \approx \hat{\mathbf{F}}_{i+1/2} - \hat{\mathbf{F}}_{i-1/2} \tag{30}$$

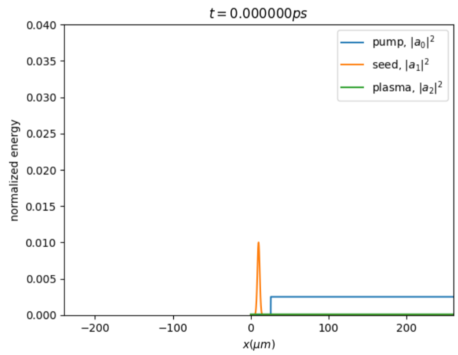
3.3 Time integrator

Now we have a way to evaluate the flux difference, $\mathbf{F}(\mathbf{U})\big|_{x_{i-1/2}}^{x_{i+1/2}}$, the R.H.S. of Eq.(26) is known. We can apply the usual time integration technique to update the state variable $\mathbf{U}_i$. Since the spatial accuracy is second-order, so second-order accurate time integration method, RK2, is chosen.

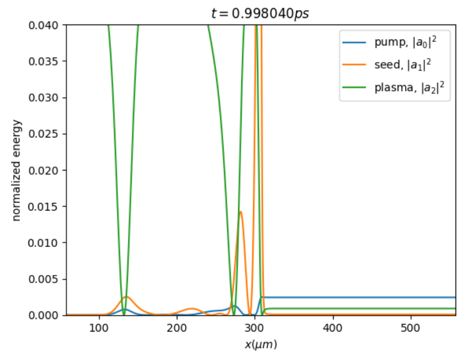
Let $\mathbf{U}_i^n$ denotes the mean value of $\mathbf{U}$ in control volume i at time t_n. Then the R.H.S. of Eq.(26) can be denoted $\mathbf{f}(\mathbf{U}_i^n)$. The update rule of RK2 can be described as

$$\begin{aligned} \mathbf{U}_i^* &= \Delta t \mathbf{f}(\mathbf{U}_i^n) \\ \mathbf{U}_i^{**} &= \Delta t \mathbf{f}\left(\mathbf{U}_i^n + \frac{1}{2}\mathbf{U}_i^*\right) \\ \mathbf{U}_i^{n+1} &= \mathbf{U}_i^n + \mathbf{U}_i^{**} \end{aligned} \tag{31}$$

4 Result



(a) Before waves interact.



(b) Energy of pump is transfer to seed.

5 Conclusion

By sending a short laser pulse (seed) to a plasma with a counter-propagating pump laser, we observed the amplification of the short laser from numerical experiment. The energy is transferred from the pump wave to the seed wave.

References

[1] Stephen Jardin. *Computational methods in plasma physics*. Chapman & Hall/CRC computational science series. CRC Press, Boca Raton, 2010. OCLC: ocn426810763.

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[3] H.J. Lee, J. Kim, C. Kim, G.-H. Kim, J.-U. Kim, and H. Suk. Solitary waves in a plasma interacting with two counter-propagating laser pulses. In *The 30th International Conference on Plasma Science, 2003. ICOPS 2003. IEEE Conference Record - Abstracts.*, page 366, Jeju, South Korea, 2003. IEEE.