A1)

a) $\frac{\partial f}{\partial x} = \cancel{2}\frac{1}{\cancel{2}}(y-2)(x-1)$ $\qquad \frac{\partial f}{\partial y} = \frac{1}{2}(x-1)^2 + 2y - 4{,}5$

$\frac{\partial^2 f}{\partial x^2} = (y-2)$ $\qquad \frac{\partial^2 f}{\partial y^2} = 2$

$\frac{\partial^2 f}{\partial x \partial y} = (x-1)$.

b) $\nabla f = 0$, $\frac{\partial f}{\partial x} = 0 \Leftrightarrow x = 1$ oder $y = 2$.

$\frac{\partial f}{\partial y}$, $x = 1 \Rightarrow \frac{\partial f}{\partial y} = 2y - 4.5 \Rightarrow y = 9/4$

$y = 2 \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{2}(x-1)^2 + 1/2 \stackrel{!}{=} 0$

$(x-1)^2 = 1 \Rightarrow x-1 = \pm 1 \quad x = 0, 2$

$\vec{x}_1 = (1, 9/4)$ $\quad \vec{x}_2 = (0, 2)$ $\quad \vec{x}_3 = (2, 2)$ kritische Stellen.

$Hf = \begin{pmatrix} (y-2) & (x-1) \\ (x-1) & 2 \end{pmatrix}$, $Hf(\vec{x}_1) = \begin{pmatrix} 1/4 & 0 \\ 0 & 2 \end{pmatrix}$

$Hf(\vec{x}_2) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ $\qquad Hf(\vec{x}_3) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

zu $\vec{x}_1$: $\lambda_1 = 1/4$, $\lambda_2 = 2$, $\geq 0 \longrightarrow \vec{x}_1$ ein Min.

zu $\vec{x}_2$: $-\lambda(2-\lambda)) - 1 = 0 \Rightarrow \lambda^2 - 2\lambda - 1 = 0$, $\lambda_{1,2} = \frac{2 \pm \sqrt{4+4}}{2}$

$= \frac{2 \pm \sqrt{8}}{2}$

$\lambda_1 < 0$

$\lambda_2 > 0$

$= \frac{2 \pm 2\sqrt{2}}{2}$

$\rightarrow$ Indefinit!

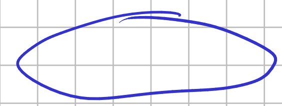
$= 1 \pm \sqrt{2}$

zu $\vec{x}_3$ = s. $\vec{x}_2$ indefinit.

c) $y = 3$ $\quad f(x,3) = \underbrace{\tfrac{1}{2}(x-1)^2}_{>0} + 9 - \tfrac{27}{2} + 1 \xrightarrow{x \to \pm\infty} \infty$

$\rightarrow$ Nein.

d) M - Ellipse ist abgeschlossen!



f stetig diff'bar (da Polynom) $\rightarrow$ nimmt f auf M globales Max an.

A2) a) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

f gerade: $f(-x) = f(x)$

$\sin(nx) = \sin(-nx)$

$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin(nx)dx + \int_{-\pi}^{0} f(x)\sin(nx)dx\right.$

$\overset{x=-x}{=} \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin(nx) - \int_0^{\pi} f(-x)\sin(\cancel{+}-nx)dx\right] = 0$

b) $u^{\perp} = \{v \in C([-\pi,\pi]) : v \perp u\} = \{v \in C([-\pi,\pi]) : (v,u) = 0\}$

ist linearer Unterraum von $C([-\pi,\pi])$.

$f \in U \Rightarrow b_{2021} = \left\langle f, \frac{1}{\pi}\sin(2021x)\right\rangle = 0$

$\Rightarrow U = \left(\frac{1}{\pi}\sin(2021x)\right)^{\perp} \rightarrow$ ein lin. Unterraum.

c) $f(x) = |x| = |-x|$ gerade. $b_n = 0$

$a_n = \frac{2}{\pi}\int_0^{\pi}|x|\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx$

$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{x=0}^{\pi} = \begin{cases} 0 & n \neq 0 \text{ gerade} \\ -\frac{4}{n^2\pi} & n \text{ ungerade} \end{cases}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi .$$

d) Da $f$ stetig ist konvergiert die Fourier Reihe überall gegen $f(x)$. ( 6.9.12 )

A3) a) i) $$\lim_{x\to 0} \frac{\overbrace{x}^{\to 0}\,\overbrace{\cos(x)}^{\to 1}}{\underbrace{x+3}_{\to 3}} = 0$$

ii) $$\lim_{x\to\infty} \frac{\ln(x)+7x}{x^2-2x+5} =$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{\overbrace{1/x}^{\to 0} + 7}{\underbrace{2x-2}_{\to\infty}} = 0$$

b) $$\sum_{n=0}^{\infty} \sqrt{n+1} - \sqrt{n} = \sum_{n=0}^{\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Da $\sqrt{n+1} + \underbrace{\sqrt{n}}_{\leq\sqrt{n+1}} \overset{?}{\leq} 2\sqrt{n+1} \leq n \Rightarrow \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \underline{\frac{1}{n}}$

divergiert da $\sum 1/n$ divergiert.

c) Hadamard: $\sqrt[n]{\frac{n+1}{n^2+n}} = \sqrt[n]{\frac{\cancel{(n+1)}}{n\cancel{(n+1)}}} = \sqrt[n]{1/n} \xrightarrow{n\to\infty} 1$

$x_0 = 0$ $\quad |x| < 1$ Konvergenzradius

d) S WS 18 A5.

A4) a) $y' = \sqrt{y}\, e^{2t} \Rightarrow \frac{y'}{\sqrt{y}} = e^{2t}$

$\int y' y^{-1/2} dy = 2y^{1/2} = \int e^{2t} dt = \frac{1}{2} e^{2t} + C.$

$2y^{1/2} = 1/2\, e^{2t} + C \quad |^2 \quad :2$

$y = \left(\frac{1}{4} e^{2t} + C'\right)^2$

$y(0) = 1 \Rightarrow 1 = (1/4 + C')^2 \Rightarrow \pm 1 = 1/4 + C'$

$\Rightarrow C' = 3/4, -5/4$

$y_1(t) = \left(\frac{1}{4} e^{2t} + 3/4\right)^2, \quad y_2(t) = \frac{1}{16}(e^{2t} - 5)$

$= \frac{1}{16}(e^{2t} + 3)$

b) $y^{(3)} - 3y' - 2y = 0$ $\qquad y = e^{\lambda t}$, $y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$

$e^{\lambda t}(\lambda^3 - 3\lambda - 2) = 0$ $\qquad y^{(3)} = \lambda^3 e^{\lambda t}$

Raten $(0, 1, 2, -1, -2)$ $\qquad \lambda_1 = 2$

$$\begin{array}{l}
\lambda^3 - 3\lambda - 2 : (\lambda - 2) = \lambda^2 + 2\lambda + 1 \\
-(\lambda^3 - 2\lambda^2) \\
\hline
\quad 2\lambda^2 - 3\lambda \\
\quad -(+2\lambda^2 - 4\lambda) \\
\hline
\qquad \lambda - 2
\end{array}$$

$= (\lambda + 1)^2$

$\lambda_2 = -1 \ (2x)$

$F_1 = \{e^{2t}\}$, $F_2 = \{e^{-t}, te^{-t}\}$

$F = F_1 \cup F_2 = \{e^{2t}, e^{-t}, te^{-t}\}$

$y = c_1 e^{2t} + c_2 e^{-t} + c_3 t e^{-t}$.

c) Picard-Lindelöff, $f = -2e^{\sqrt{3}t}$ offensichtlich global Lipschitz-stetig.

$f'(t) = -2\sqrt{3}\, e^{\sqrt{3}t} < L(t)$. Lipschitz Konstant.

$\rightarrow$ global eindeutig Lösung.

A5) a) F gegenbsp $x_n = (-1)^n$ $\quad (|x_n|)_n = 1$

b) R $\quad \frac{d}{dt} g(t) = \frac{d}{dt} f(t,0) = \frac{\partial}{\partial x} f(x,y)\Big|_{y=0} = 0$

c) F $\quad \sum \frac{x^n}{2^n} \quad a_n = 1/2^n$ konvergent, $\rho = 1/2$.

d) F $\quad f = \begin{cases} 1 & x < x_0 \\ -1 & x > x_0 \end{cases}$, $\quad g = \begin{cases} -1 & x < x_0 \\ 1 & x > x_0 \end{cases}$

$f + g = 0$ stetig.

e) R $\quad e^{ia_n} = \underbrace{\cos(a_n)}_{\leq 1} + i\,\underbrace{\sin(a_n)}_{\leq 1} \leq 1 + i$