A 1)

a) $\lim_{x\to 0} \frac{\ln(1+5x)}{x} = \lim_{x\to 0} \frac{1}{1+5x} \cdot 5 \cdot \frac{1}{1} = \lim_{x\to 0} \frac{5}{\underbrace{1+5x}_{\to 0}} = 5$

b) s.v. Hadamard.

$\frac{(2x-1)^n}{n} \overset{(*)}{=} \frac{2^n (x-1/2)^n}{n} \qquad a_n = \frac{2^n}{n}, \; x_0 = 1/2$

$\lim_{n\to\infty} \left(\frac{2^n}{n}\right)^{1/n} = 2 \cdot \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 2 \lim \frac{1}{\sqrt[n]{n}} = \underline{2}$

$\rho = 1/2 \qquad |x - 1/2| < 1/2$ d.h. $x \, (0,1)$

(*) $(2x-1)^n = (2(x-1/2))^n = 2^n (x-1/2)^n$

$\sum a_n (x-x_0)^n$ Wenn $\rho = \lim \sqrt[n]{|a_n|}$ ex oder

a) $\sqrt[n]{|a_n|}$ unbeschränkt. konvergiert $\sum a_n (x-x_0)$ nur wenn $x = x_0$

b) $\rho = 0 \Rightarrow \sum a_n (x-x_0)^n$ absolut für jede x

c) $\rho \in (0,\infty)$ so konvergent $|x-x_0| < 1/\rho$, divergent sonst.

A 2)

$$\int_0^{2\pi} \underset{u}{x}\ \underset{v'}{\cos(x/2)}\,dx = \left[\overset{u\,v}{x(-2\sin(x/2))}\right]_0^{2\pi} - \int_0^{2\pi} \overset{-u'v}{-2\sin(x/2)}\,dx$$

$$= (0-0) + 2\int_0^{2\pi} \sin(x/2)\,dx = 2\left[2\cos(x/2)\right]_0^{2\pi}$$

$$= 4\,(\underbrace{\cos(\pi)}_{=-1} - \underbrace{\cos(0)}_{=1}) = -8$$

$$\int_0^4 \frac{2x}{\underline{\sqrt{x^2+9}}}\,dx = \int_9^{25} \frac{1}{\sqrt{z}}\,dz = \int_9^{25} z^{-1/2}\,dz = \left[2z^{1/2}\right]_9^{25}$$

$z = x^2+9$

$\boxed{dz = 2x\,dx}$

$z_1 = 9$

$z_2 = 25$

$$= 2[5-3] = 4.$$

A3) $f: \mathbb{R}^2 \to \mathbb{R}$:

$$Df = \begin{pmatrix} 3x^2 + y \\ -\frac{1}{3}y + x \end{pmatrix} \overset{!}{=} 0 \Rightarrow$$

$\mathrm{II} \Rightarrow x = \frac{1}{3}y, \quad y = 3x$

$\mathrm{I} \Rightarrow -3x^2 + 3x = 0$

$x(x-1) = 0$

$x = 0, 1$

$\vec{x}_1 = (0,0), \quad \vec{x}_2 = (1,3)$

$$Hf = \begin{pmatrix} 6x & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} \quad Hf(\vec{x}_1) = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix}$$

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\frac{1}{3}-\lambda \end{pmatrix} = \lambda(\lambda + \frac{1}{3}) - 1 \overset{!}{=} 0$$

$ax^2 + bx + c = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\underline{\lambda^2} + \frac{1}{3}\lambda - 1 = 0 \Rightarrow \underline{3x}^2 + \lambda - 3 = 0$

$\lambda_{1,2} = \frac{-1 \pm \sqrt{1 + 36}}{6} = \frac{-1 \pm \sqrt{37}}{6}, \quad \lambda_1 < 0, \; \lambda_2 > 0$

da $\sqrt{37} > 1$

$\to$ indefinit.

$\det(A) = \prod \lambda_i < 0 \Rightarrow$

$\det(Hf(\vec{x}_1)) = -1 \Rightarrow$ indefinit

$$Hf\,\vec{x}_2 = \begin{pmatrix} -6 & 1 \\ 1 & -1/3 \end{pmatrix} \Rightarrow (-6-\lambda)(-1/3-\lambda) - 1 = 0$$

$$(\lambda+6)(\lambda+1/3) - 1 = 0$$

$$\det(Hf(\vec{x}_2)) = 1 \quad ?$$

$$\lambda^2 + 13/3\,\lambda + 1 = 0$$

$$\longrightarrow 3\lambda^2 + 13\lambda + 3 = 0$$

$$\lambda_{1,2} = \frac{-13 \pm \sqrt{13^2 - 36}}{6}$$

$$\Rightarrow \sqrt{13^2-36} < |-13| = 13$$

$$\Rightarrow -13 \pm \sqrt{13^2-36} < 0$$

$$\lambda_{1,2} < 0 \rightarrow \text{Max.}$$

A4) a) $z'(t) = \frac{2t}{e^{z(t)}}$, $z(0) = 0$

$z'e^z = 2t$ $\quad : \quad \int_0^t z'e^z \, dt = t^2 + C$

$$e^z = t^2 + C$$

$$z(t) = \ln(t^2 + \underline{\underline{C}})$$

$z(0) = 0 \qquad 0 = \ln(C) \Rightarrow C = 1$

$$\Rightarrow z(t) = \ln(t^2+1)$$

b) $y = e^{\lambda t}$, $y^{(4)} = \lambda^4 e^{\lambda t}$

$e^{\lambda t}(\underbrace{\lambda^4 + \lambda^3 - 2\lambda^2}_{=0}) = 0$ $\qquad \lambda^2(\lambda^2 + \lambda - 2) = 0$

$\lambda_1 = 0 \; (2x)$

$\lambda_2 = 1$, $\lambda_3 = -2$

$F_1 = \{ e^{0t}, te^{0t} \} = \{1, t\}$

$F_2 = \{ e^t \}$, $F_3 = \{ e^{-2t} \}$ $\qquad y = c_1 + c_2 t + c_3 e^t + c_4 e^{-2t}$

$F = \{ 1, t, e^t, e^{-2t} \}$

A5) a) $f(x) = \sqrt{x}$

Hinweis

$a = \sqrt{x}, \quad b = \sqrt{y} \Rightarrow \quad (\sqrt{x} - \sqrt{y})^2 \leq |x - y|$

$$|\sqrt{x} - \sqrt{y}| \leq |x - y|^{1/2}$$

$$\Rightarrow |f(x) - f(y)| \leq |x - y|^{1/2}$$

b)

$$|f(x) - f(y)| \leq C\,|x - y|^2$$

$$\frac{|f(x) - f(x)|}{|x - y|} \leq C\,|x - y|$$

$$f'(x) = \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{y \to x} C\,|x - y| = 0$$

$$= 0$$

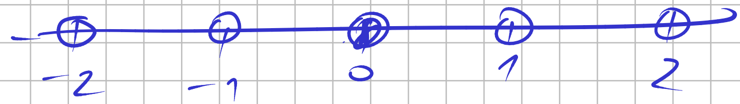
$\Rightarrow \quad f \equiv \text{const}.$ $\qquad f'(x) = 0 \quad \forall x$

$$\Rightarrow f(x) = \int f'(x) = \int 0 = C$$

6) a) F Leibniz, brauchen Monotonie von a_n

b) W $\mathbb{R} \setminus \mathbb{Z}$ ist offen $\rightarrow$ Menge der offenen Intervalle

$(\mathbb{R} \setminus \mathbb{Z})^c = \mathbb{Z}$ abgeschl.

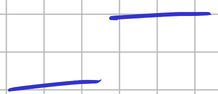


c) F $f(x) = 1/x$ $\lim\limits_{x \to 0} f = \infty$

d) W $\int_0^{\pi} \underbrace{\sin^2 + \cos^2}_{=1} dx = \pi - 0$

e) W $f'(x) > 0 \quad \forall x \in [0,1] \Rightarrow (f^{-1})' = \dfrac{1}{\underbrace{f'(f^{-1}(x))}_{>0}}$

f) F

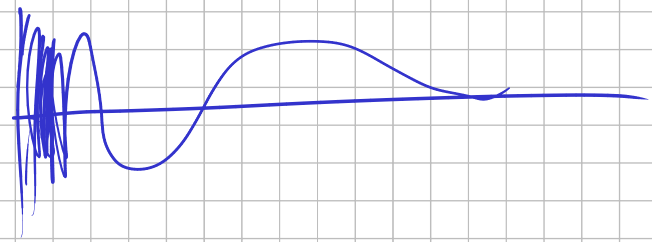


$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

auf $[-1, 1]$

$\int f = 1$

g) F $y \equiv 0$ h) W notw. Bed.

$\frac{\sin(x)}{x} \longrightarrow$  nicht Riemann integrierbar