Klausur Mathe II WiSe 2016/17

1) a) i)
$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{2x+3} = \lim_{x\to\infty} \frac{2x\sqrt{1+\frac{1}{4x^2}}}{2x\left(1+\frac{3}{2x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{4x^2}}}{1+\frac{3}{2x}} = \frac{\lim\limits_{x\to\infty} \sqrt{1+\frac{1}{4x^2}}}{\lim\limits_{x\to\infty} 1+\frac{3}{2x}}$$

$$= \frac{\sqrt{\lim\limits_{x\to\infty}\left(1+\frac{4}{4x^4}\right)}}{\lim\limits_{x\to\infty}\left(1+\frac{3}{2x}\right)} = \frac{\sqrt{1}}{1} = \frac{1}{1} = \underline{1}$$

ii)
$$\lim_{x\to 0} \frac{x^2}{\cos(x)^2-1} \underset{x^2\to 0}{\overset{\text{L'Hospital}}{=}} \lim_{x\to 0} \frac{2x}{2\cdot\cos(x)\cdot(-\sin(x))}$$

$$\underset{\substack{2x\to 0\\ -2\cos(x)\sin(x)\to 0}}{\overset{\text{L'Hospital}}{=}} \lim_{x\to 0} \frac{\overset{\cos(x)^2-1\to 0}{2}}{2\sin(x)^2-2\cdot\cos(x)^2} = \frac{2}{-2} = \underline{-1}$$

iii)
$$\lim_{x\to\infty} \cos(\pi x) \quad ?$$

$$\cos(k\pi) = \begin{cases} 1 & k \text{ gerade} \\ -1 & k \text{ ungerade} \end{cases}$$

Definiere $a_k = k$

$$|\cos(a_k) - \cos(a_{k+1})| = |(-1)^k - (-1)^{k+1}|$$
$$= 2$$

$\Rightarrow$ $\cos(a_k)$ keine Cauchyfolge

$\Rightarrow$ $\cos(a_k)$ divergent

b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} x^n$

Konvergenzradius: $a_n = \left(\frac{n+1}{n}\right)^{n^2}$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} = \left(\frac{n+1}{n}\right)^{\frac{n \cdot n}{n}} = \left(\frac{n+1}{n}\right)^n$$

$$\xrightarrow{n \to \infty} e$$

$$\Rightarrow \underline{\underline{\rho = 1/e}}$$

2) $f(x,y) = x^3 + 6xy^2 - 2y^3 - 12x$

a) $\partial_x f(x,y) = 3x^2 + 6y^2 - 12$

$\partial_y f(x,y) = 12xy - 6y^2$

$\nabla f(x,y) = (3x^2 + 6y^2 - 12,\ 12xy - 6y^2)$

$\partial^2_{xx} f(x,y) = 6x$ $\quad\quad$ $\partial^2_{yy} f(x,y) = 12x - 12y$

$\partial_{xy} f(x,y) = 12y$

$$Hf(x,y) = \begin{pmatrix} 6x & 12y \\ 12y & 12x - 12y \end{pmatrix}$$

b) Kritische Punkt $\nabla f(x,y) = 0$

$$\begin{pmatrix} 3x^2 + 6y^2 - 12 = 0 \\ 12xy - 6y^2 = 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 3x^2 + 6y^2 = 12 \\ 6y(2x - y) = 0 \end{pmatrix}$$

Fall 1: $y = 0$: $3x^2 = 12 \Leftrightarrow x^2 = 4 \Leftrightarrow x = \pm 2$

$(x_1, y_1) = (2, 0)$ $\quad\quad$ $(x_2, y_2) = (-2, 0)$

Fall 2: $2x = y$: $3x^2 + 6 \cdot (2x)^2 = 12 \Leftrightarrow (3 + 24)x^2 = 12$

$\Leftrightarrow x^2 = \frac{4}{9} \Leftrightarrow x = \pm 2/3$

$(x_3, y_3) = (2/3, 4/3)$; $(x_4, y_4) = (-2/3, -4/3)$

$$Hf(x_1,y_1) = \begin{pmatrix} 12 & 0 \\ 0 & 24 \end{pmatrix} \det(Hf_{11}) = 12 > 0 \quad \det(Hf) = 12 \cdot 24 > 0$$

$\Rightarrow$ positiv definit $\Rightarrow$ Lokales Minimum

$$Hf(x_2,y_2) = \begin{pmatrix} -12 & 0 \\ 0 & -24 \end{pmatrix} \Rightarrow \det((-12)) = -12 < 0$$

$$\det(Hf) = (-12)\cdot(-24) > 0$$

Hauptminorenkriterium $\Rightarrow$ negativ definit $\Rightarrow$ Lok. Maximum

$$Hf(x_3,y_3) = \begin{pmatrix} 4 & 16 \\ 16 & -8 \end{pmatrix} \Rightarrow \det(Hf) = 4\cdot(-8) - 16^2 < 0$$

$\Rightarrow$ Sattelpunkt

$$Hf(x_4,y_4) = \begin{pmatrix} -4 & -16 \\ -16 & +8 \end{pmatrix} \Rightarrow \det(Hf) = -4\cdot 8 - (-16)^2 < 0$$

$\Rightarrow$ Sattelpunkt

c) $\int_0^1 \partial_y f(e^t,t)\,dt$ $\qquad \partial_y f(x,y) = 12xy - 6y^2$

$$\partial_y f(e^t,t) = 12\,te^t - 6t^2$$

$$\int_0^1 \partial_y f(e^t,t)\,dt = \int_0^1 \underbrace{12t}_{f(t)}\,\underbrace{e^t}_{g'(t)} - 6t^2\,dt \overset{P.I.}{=} [12te^t]_0^1 - [2t^3]_0^1 - \int_0^1 12e^t\,dt$$

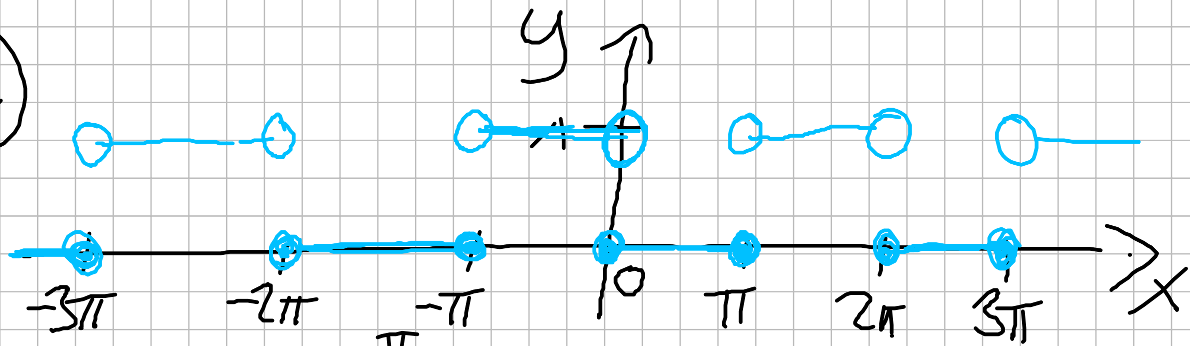
$f'(t) = 12 \quad g(t) = e^t$

$$= [12e - 12\cdot 0\cdot e^0] - [2-0] - [12e^t]_0^1$$

$$= 12e - 2 - [12e - 12] = \underline{\underline{10}}$$

3) $f : [-\pi, \pi] \to \mathbb{R}$, $\quad f(x) = \begin{cases} 1 & x \in [-\pi, 0), \\ 0 & x \in [0, \pi] \end{cases}$

a)



b) $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \qquad k \in \mathbb{N}_0$

$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx \qquad k \in \mathbb{N}$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} 1\,dx + \frac{1}{\pi} \int_{0}^{\pi} 0\,dx = \frac{1}{\pi} \cdot \pi + 0 = \underline{\underline{1}}$$

$k > 0$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} 1 \cdot \cos(kx)\,dx$$

$$= \frac{1}{\pi} \left[\frac{1}{k} \sin(kx) \right]_{-\pi}^{0} = \frac{1}{\pi} [0 - 0] = \underline{\underline{0}}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} 1 \sin(kx)\,dx$$

$$= \frac{1}{\pi} \left[-\frac{1}{k} \cos(kx) \right]_{-\pi}^{0} = \frac{1}{\pi} \left[-\frac{1}{k} \underbrace{\cos(0)}_{1} + \frac{1}{k} \cos(k\pi) \right]$$

N.R.

$$\cos(k\pi) = \begin{cases} 1 & \text{für } k \text{ gerade} \\ -1 & \text{für } k \text{ ungerade} \end{cases}$$

$$= \begin{cases} -\frac{2}{k\pi} & k \text{ ungerade} \\ 0 & k \text{ gerade} \end{cases}$$

$$F_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$= \frac{1}{2} - \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin((2k+1)x)$$

c) Konvergiert für alle $x \in \mathbb{R}$ gegen

$$F_\infty(x) = \begin{cases} 1 & x \in ((2k-1)\pi, (2k\cdot\pi)) \quad k \in \mathbb{Z} \\ \cancel{1/2}\ 0 & x \in ((2k)\pi, (2k+1)\pi) \quad k \in \mathbb{Z} \\ 1/2 & x \in \pi\mathbb{Z}. \end{cases}$$

d) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \cancel{X} = ?$$

$$0 = F_\infty\left(\frac{\pi}{2}\right) = \frac{1}{2} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \cdot \underbrace{\sin\left(\frac{\pi}{2} + k\pi\right)}_{(-1)^k}$$

$$= \frac{1}{2} - \frac{2}{\pi} \cdot X$$

$$\Rightarrow X = \frac{1}{2} / \left(\frac{2}{\pi}\right) = \frac{\pi}{4}$$

4) a) Falsch

Gegenbeispiel: $f(x) = x^2$, $k=1$.

f beliebig oft diffbar in $\mathbb{R}$, $f'(x) = 2x$ erfüllt

$f'(0) = 0$.

$T_{k-1,f}(x) = T_{0,f}(x) = f(0) = 0$

$f(x) = x^2 \neq 0$ $\lightning$.

b) Falsch

Gegenbeispiel: $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

f ist lokal konstant in ± 1

$\Longrightarrow$ in ± 1 diffbar mit $f'(\pm 1) = 0$.

f ist unstetig in $x = 0$ $\Rightarrow$ nicht differenzierbar in 0

c) $p(x) = x^3 + ax^2 + bx$

Dann hat $y'(t) = p(y(t))$ eine konstante Lösung.

Richtig.

$y(t) = 0$ ist konstant und löst

$y'(t) = p(y(t)) = y(t)^3 + a \cdot y(t)^2 + b\,y(t)$

$= 0$ ✓

d) $q \in \mathbb{Q}$, $z \in \mathbb{C}$, $\arg(z) = q \cdot \pi$

Dann existiert $k \in \mathbb{Z}$ mit $z^k \in \mathbb{R}$

Richtig.

- $k = 0$. Dann gilt $z^0 = 1 \in \mathbb{R}$.
- $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

$$z^{2n} = \left(|z| e^{i \cdot \arg(z)}\right)^{2n} = |z|^{2n} \cdot e^{i\pi \frac{m}{n} 2n}$$

$$= \underbrace{|z|^{2n}}_{\in \mathbb{R}} \cdot e^{(2\pi i) \cdot m} = |z|^{2n} \cdot (\underbrace{e^{2\pi i}}_{=1})^m = |z|^{2n} \in \mathbb{R}.$$

5)

a) ✓

b) X $\quad f(x) = x \quad \tan \circ f = \tan$

c) X $\quad f(x) = x^x = \left(e^{\ln(x)}\right)^x = e^{\ln(x) \cdot x}$

$$f'(x) = e^{\ln(x) \cdot x} \cdot \left(\frac{1}{x} \cdot x + \ln(x)\right)$$

$$= x^x \left(1 + \ln(x)\right)$$

d) ✓

e) ✓ Mittelwertsatz!

f) ✓

g) X $\quad$ (z.B. $a_n = 1/n$)

h) X