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AMATH 567

HOMEWORK 9

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
Evaluate the integrals $\frac{1}{2\mathrm{i}\pi}\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C.

(a)

$$\frac{z^2+1}{z^2-a^2}, \quad a^2<1$$

Solution:
TODO:

$$\frac{1}{2\mathrm{i}\pi}\oint_C \frac{z^2+1}{z^2-a^2}\mathrm{d}z$$

(b)

$$\frac{z^2+1}{z^3}$$

Solution:
TODO:

$$\frac{1}{2\mathrm{i}\pi}\oint_C \frac{z^2+1}{z^3}\mathrm{d}z$$

(c)

$$z^2\,\mathrm{e}^{-1/z}$$

Solution:
TODO:

$$\frac{1}{2\mathrm{i}\pi}\oint_C z^2\,\mathrm{e}^{-1/z}\,\mathrm{d}z$$

2: From A&F: 4.2.1(b)

Evaluate the following real integral

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a^2 > 0$$

Solution:

TODO:

3: Existence and uniqueness of polynomial interpolants.

(a) Suppose $(z_j)_{j=1}^n$ are distinct points in $\mathbb{C}$ and suppose $f_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Show that there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_j) = f_j$ for $j = 1, \ldots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant.*

Solution:
Suppose there exists two polynomials $p_1(z)$ and $p_2(z)$ each of degree $n-1$. Assume both agree with f_j at each z_j such that

$$p_1(z_j) = p_2(z_j) = f_j \quad \text{for each } j = 1, ..., n.$$

Additionally define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Now let's consider the function

$$g(z) = \frac{p_1(z) - p_2(z)}{\nu(z)}.$$

We want to utilize Liouville's theorem to conclude that $g(z)$ is constant. In order to do this we need to show that $g(z)$ is entire and bounded. Let's begin by demonstrating that it is bounded by taking the limit as $z \to \infty$

$$\begin{aligned}
\lim_{z\to\infty} g(z) &= \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\nu(z)} \\
&= \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\prod_{j=1}^n (z - z_j)} \\
&= \frac{\infty}{\infty}.
\end{aligned}$$

Applying L'Hôpitals rule repeatedly we will end up with $1/z$ which goes to 0 as z goes to infinity since the denominator is an nth degree polynomial while the numerator is a degree $n-1$ polynomial. Therefore,

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\nu(z)} = 0,$$

which implies that $g(z)$ is bounded. Next, we need to determine if $g(z)$ is entire. Since polynomials are entire in the finite z plane, $p_1(z) - p_2(z)$ is entire. However, $g(z)$ overall requires a little more analysis since it has singularities where $z = z_j$. Notice, since the expression $p_1(z) - p_2(z)$ and $\nu(z)$ are both zero at each z_j, then there exists a factorization of $p_1(z) - p_2(z)$ which would allows us to cancel out each of the factors in the product in the denominator. Therefore, the singularities of $g(z)$ are removable and thus $g(z)$ is entire (or can be made entire, with the right extension at each z_j as we have done in previous assignments). Hence, by Liouville's Theorem, we can conclude that $g(z)$ is constant. Combining with the fact that $p_1(z_j) - p_2(z_j) = 0$ for each $j = 1, ..., n$, then $g(z)$ must be 0 everywhere, thus implying $p_1(z) = p_2(z)$ everywhere. In conclusion, since these two functions are the same therefore there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_j) = f_j$ for $j = 1, \ldots, n$, otherwise known at the interpolant.

$\square$

(b) Define the node polynomial $\nu(z) = \prod_{j=1}^{n}(z - z_j)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:
Let's look at $p(z)/\nu(z)$ and consider what happens if we subtract off a specially cooked up collection of terms including the residues r_j for $j = 1, ..., n$. We can express the residues of $p(z)/\nu(z)$ as

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{p(z)}{\nu(z)} \mathrm{d}z = \sum_{j=0}^{n} \mathrm{Res}\left(\frac{p(z)}{\nu(z)}; z_j\right) = \sum_{j=0}^{n} \frac{f_j}{\prod_{k \neq j}(z_k - z_j)}.$$

Recall partial fractions is connected to the residues. We construct the expression to subtract from $p(z)/\nu(z)$ using the partial fraction decomposition relationship to residues

$$\frac{p(z)}{\nu(z)} - \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$= \frac{p(z)}{\nu(z)} - \frac{f_1\left(\prod_{k \neq 1}(z_k - z_1)\right)^{-1}}{z - z_1} - \frac{f_2\left(\prod_{k \neq 2}(z_k - z_2)\right)^{-1}}{z - z_2} - \dots - \frac{f_n\left(\prod_{k \neq n}(z_k - z_n)\right)^{-1}}{z - z_n} = 0.$$

TODO: Why is this 0 besides saying it's the partial fraction decomposition? This expression is equal to 0 because the collection of terms we are subtracting is the partial fraction decomposition of $p(z)/\nu(z)$. If we can show that this function is bounded and entire then it is a constant. Therefore, we would be able to state that since it is a constant and 0 then it must be a 0 everywhere. Thus we can say

$$\frac{p(z)}{\nu(z)} - \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j} = 0$$

$$\frac{p(z)}{\nu(z)} = \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$p(z) = \nu(z) \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$p(z) = \prod_{j=1}^{n}(z - z_j) \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$p(z) = \sum_{j=0}^{n} \frac{f_j \prod_{\ell \neq j}(z - z_\ell)}{\prod_{k \neq j}(z_k - z_j)}.$$

Therefore we have this expression for $p(z)$.

4: Bernstein interpolation formula. Suppose that $-1 \le x_1 < x_2 < \cdots x_n \le 1$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1,1]$. Show that for any simple contour C inside Ω with $[-1,1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{\mathrm{d}z}{\nu(z)}, \quad x \in [-1,1],$$

where p is the degree $n-1$ polynomial interpolant satisfying $p(x_j) = f(x_j)$ for $j = 1, 2, \ldots, n$. We also have $\nu(x) = \prod_{j=1}^{n}(x - x_j)$.

Solution:
TODO: the x_j are in -1,1...residue on the right, and p is degree n-1

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1,1].$$

(a) Show that the polynomial

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right),$$

has all of its roots $x_1 < x_2 < \cdots x_n$ within $[-1,1]$.

Solution:
TODO:

(b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1,1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:
TODO: apply things from hw3 problem 7 or 8?

(c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2\frac{M|B_\rho|}{\pi}(\rho^n - \rho^{-n})^{-1}(\rho + \rho^{-1} - 1)^{-1} \leq 2\frac{M|B_\rho|}{\pi}\frac{\rho^{1-n}}{(\rho-1)^2}.$$

where $p(x_j) = f(x_j)$, i.e., p is the interpolant of f at the roots of T_n. Here $|B_\rho|$ denotes the arclength of B_ρ. This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f.

Solution:
TODO: TODO: p is the polynomial interpolant of f of degree n - 1, lots of varphi stuff hw 3 prob 6/7/8

6: Compute the following two integrals explicitly for $z \notin [-1, 1]$:

(a)

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x}\sqrt{1+x}} \frac{\mathrm{d}x}{x-z}.$$

Solution:

TODO:

(b)

$$\frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x} \frac{\mathrm{d}x}{x-z}.$$

Solution:

TODO: