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AMATH 567

HOMEWORK 5

Collaborators*:

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.6.5

Consider two entire functions with no zeroes and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.

Solution:

Let's define our two entire functions to be $f(z)$ and $g(z)$. Recall that an entire function is analytic in all of the complex plane. We can focus on the ratio between these two functions $\frac{f(z)}{g(z)}$ since we are also given that $f(z)$ and $g(z)$ have no zeros. Let $h(z)$ be the ratio between f and g

$$h(z) = \frac{f(z)}{g(z)}.$$

If we can use Liouville's theorem to show that $h(z)$ is constant, then $f(z)$ and $g(z)$ are equal everywhere and are thus the same function.

For reference, Liouville's Theorem states that if $f(z)$ is entire and bounded in the z plane (including infinity), then $f(z)$ is a constant. Hence we need to show that $h(z)$ is entire and bounded in the z plane, then $h(z)$ is constant and we will have what we want. We know that the functions $f(z)$ and $g(z)$ are entire. We also know that the function $\frac{1}{z}$ is analytic except when $z = 0$. Since neither f nor g have zeros, then the potential of having 0 in the denominator of $h(z)$ is no longer an issue. Therefore $\frac{1}{z}$, $z \neq 0$ is entire. Therefore $h(z)$ is entire since it is the composition of entire functions.

Now we need to show that $h(z)$ is bounded in the z plane. Since $h(z)$ is entire, then it is analytic interior to and on a simple closed contour C (which we will choose later), then by Theorem 2.6.2, we have

$$h^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

Now we can use the established inequality (2.6.13 in A & F)

$$|h^{(n)}(z)| \leq \frac{n!M}{R^n}.$$

When $n = 1$ we have

$$|h'(z)| \leq \frac{M}{R}.$$

We can take R to be arbitrarily large to get $|h'(z)| \leq 0$ implying $h'(z) = 0$. Using the fundamental theorem of calculus we can write

$$h(\infty) - h(z) = \int_z^\infty h'(z) dz = C|_z^\infty = C - C = 0.$$

This gives $h(\infty) = h(z)$, therefore, by Liouville's Theorem $h(z)$ is constant. From the problem's setup we know $h(\infty) = \frac{f(\infty)}{g(\infty)} = 1$. Hence,

$$h(\infty) = h(z) = 1.$$

Therefore, $f(z)$ and $g(z)$ must be the same function, since their ratio is 1 for all z . □

- 2:** From A&F: 2.6.10 (The *solution* is peppered through, since there are many things to show and information given between many steps.)

In Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi$$

take the contour to be a circle of unit radius centered at the origin. Let $\xi = e^{i\theta}$. We now can plug the substitution in, along with the $d\xi = ie^{i\theta} d\theta$ to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\xi) i e^{i\theta}}{\xi - z} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi) \xi}{\xi - z} d\theta. \end{aligned}$$

□

Since z is inside the unit circle and $z = r e^{i\theta}$, then $r < 1$. Then we have

$$\frac{1}{\bar{z}} = \frac{1}{r e^{-i\theta}} = \frac{1}{r} e^{i\theta}.$$

Then $\frac{1}{r} > 1$, hence $\frac{1}{\bar{z}}$ is outside the unit circle. Therefore plugging in $\frac{1}{\bar{z}}$ to Cauchy's Formula from the beginning again we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - \frac{1}{\bar{z}}} d\xi &= 0 \\ \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\xi) i e^{i\theta}}{\xi - \frac{1}{\bar{z}}} d\theta &= 0 \\ \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi) \xi}{\xi - \frac{1}{\bar{z}}} d\theta &= 0. \end{aligned}$$

□

Notice,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} d\theta \mp 0 \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} d\theta \mp \frac{1}{2\pi} \int_0^{2\pi} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \mp \frac{\xi}{\xi - \frac{1}{\bar{z}}} \right) d\theta.
\end{aligned}$$

Now we can use $\xi = 1/\bar{\xi}$ to get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \mp \frac{1/\bar{\xi}}{1/\bar{\xi} - \frac{1}{\bar{z}}} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \mp \frac{\bar{\xi}\bar{z}}{\bar{\xi}\bar{z}} \frac{1/\bar{\xi}}{1/\bar{\xi} - \frac{1}{\bar{z}}} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \mp \frac{\bar{z}}{\bar{z} - \xi} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \pm \frac{\bar{z}}{\xi - \bar{z}} \right) d\theta.
\end{aligned}$$

□

Using the plus sign we see

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} + \frac{\bar{z}}{\xi - \bar{z}} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi(\bar{\xi} - \bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)} + \frac{\bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi(\bar{\xi} - \bar{z}) + \bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi\bar{\xi} - \xi\bar{z} + \bar{z}\xi - z\bar{z}}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{|\xi|^2 - |z|^2}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{|\xi|^2 - |z|^2}{(\xi - z)(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{|\xi|^2 - |z|^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{|e^{i\theta}|^2 - |z|^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{1^2 - |z|^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta.
\end{aligned}$$

□

(a) Deduce the “Poisson formula” for the real part of $f(z)$: $u(r, \phi) = \Re f, z = r e^{i\phi}$.

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (u(\theta) + iv(\theta)) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(u(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) + iv(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta + \frac{1}{2\pi} \int_0^{2\pi} iv(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta + i \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta
\end{aligned}$$

Thus,

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left(\frac{1 - |z|^2}{|\xi - z|^2} \right) d\theta.$$

We know $|z| = |r e^{i\phi}| = r$. Now let's look specifically at the denominator and plugin the substitutions for z to get

$$\begin{aligned}
|\xi - z|^2 &= |e^{i\theta} - r e^{i\phi}|^2 \\
&= |\cos \theta + i \sin \theta - r \cos \phi - ir \sin \phi|^2 \\
&= |\cos \theta - r \cos \phi + i(\sin \theta - r \sin \phi)|^2 \\
&= (\cos \theta - r \cos \phi)^2 + (\sin \theta - r \sin \phi)^2 \\
&= \cos^2 \theta - 2r \cos \theta \cos \phi + r^2 \cos^2 \phi + \sin^2 \theta - 2 \sin \theta r \sin \phi + r^2 \sin^2 \phi \\
&= \cos^2 \theta + \sin^2 \theta - 2r \cos \theta \cos \phi - 2 \sin \theta r \sin \phi + r^2 \cos^2 \phi + r^2 \sin^2 \phi \\
&= 1 - 2r \cos \theta \cos \phi - 2 \sin \theta r \sin \phi + r^2 (\cos^2 \phi + \sin^2 \phi) \\
&= 1 - 2r \cos \theta \cos \phi - 2 \sin \theta r \sin \phi + r^2 \\
&= 1 - 2r (\cos \theta \cos \phi + \sin \theta \sin \phi) + r^2 \\
&= 1 - 2r \cos(\theta - \phi) + r^2.
\end{aligned}$$

Hence,

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta$$

□

(b) If we use the minus sign in the formula for $f(z)$ above, we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} - \frac{\bar{z}}{\bar{\xi} - \bar{z}} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi(\bar{\xi} - \bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)} - \frac{\bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi(\bar{\xi} - \bar{z}) - \bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi\bar{\xi} - 2\xi\bar{z} + z\bar{z}}{(\bar{\xi} - \bar{z})(\xi - z)} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{|\xi|^2 - 2\xi\bar{z} + |z|^2}{(\xi - z)(\bar{\xi} - \bar{z})} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{1 - 2\xi\bar{z} + r^2}{|\xi - z|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \frac{1 - 2\xi\bar{z} + r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \frac{1 - 2e^{i\theta} r e^{-i\phi} + r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \frac{1 - 2r e^{i\theta - i\phi} + r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \frac{1 - 2r e^{i(\theta - \phi)} + r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta
\end{aligned}$$

□

Additionally, if we take the imaginary part of

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \frac{1 - 2r e^{i(\theta-\phi)} + r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (u(\theta) + iv(\theta)) \left(\frac{1 - 2r \cos(\theta - \phi) - 2ri \sin(\theta - \phi) + r^2}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (u(\theta) + iv(\theta)) \left(\frac{1 - 2r \cos(\theta - \phi) + r^2 - 2ri \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (u(\theta) + iv(\theta)) \left(1 + \frac{-2ri \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (u(\theta) + iv(\theta)) \left(1 - \frac{i2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\theta) d\theta \\
&\quad - \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left(\frac{i2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} iv(\theta) d\theta \\
&\quad - \frac{1}{2\pi} \int_0^{2\pi} iv(\theta) \left(\frac{i2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\theta) d\theta \\
&\quad - i \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left(\frac{2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&\quad + i \frac{1}{2\pi} \int_0^{2\pi} v(\theta) d\theta \\
&\quad - i^2 \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \left(\frac{2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\theta) d\theta \\
&\quad - i \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left(\frac{2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta \\
&\quad + i \frac{1}{2\pi} \int_0^{2\pi} v(\theta) d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \left(\frac{2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \right) d\theta
\end{aligned}$$

Now the imaginary part of this is

$$\begin{aligned}\Im(f(z)) &= \frac{1}{2\pi} \int_0^{2\pi} v(\theta) d\theta - \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{2r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} d\theta \\ &= C - \frac{1}{\pi} \int_0^{2\pi} u(\theta) \frac{r \sin(\theta - \phi)}{1 - 2r \cos(\theta - \phi) + r^2} d\theta\end{aligned}$$

This is a beast of a problem there are **approximately 9** things to show...

$$\mathbb{R} \setminus \{0\}$$

- 3:** Suppose Ω is an open simply connected region and $z_0 \in \Omega$. Assume that $f(z)$ is analytic in $\Omega \setminus \{z_0\}$ and satisfies

$$|f(z)| \leq M|z - z_0|^{-\gamma}, \quad \gamma < 1.$$

Show that if the a specific choice for $f(z_0)$ is made then f extends to an analytic function on Ω .

(**1 part**, except maybe if there are multiple things to prove here)

Solution:

4: Establish the following lemma:

Lemma 1

Suppose Ω is an open region and $f(z)$ is continuous on $\overline{\Omega}$. Let Γ be a contour in $\overline{\Omega}$. Suppose a sequence of contours $\Gamma_n \subset \overline{\Omega}$ converge to Γ in the sense that there exists parameterizations $z(t)$ of Γ and $z_n(t)$ of Γ_n defined on $[a, b]$ satisfying

$$\begin{aligned} z_n(t) &\xrightarrow{n \rightarrow \infty} z(t), & \text{uniformly on } [a, b], \\ z'_n(t) &\xrightarrow{n \rightarrow \infty} z'(t), & \text{uniformly on } [a, b]. \end{aligned}$$

Then

$$\int_{\Gamma_n} f(z) dz \xrightarrow{n \rightarrow \infty} \int_{\Gamma} f(z) dz.$$

Hint: Use that f is uniformly continuous on $\overline{\Omega}$.

(1 part, except maybe if there are multiple things to prove here)

Solution:

5: for any $r, R > 0$, let $C = \partial\Sigma$, $\Sigma = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq r \text{ and } 0 \leq -\operatorname{Im} z \leq R, R > 0\}$.

In this problem \sqrt{z} denotes the principal branch with $\arg z \in [-\pi, \pi)$.

- Show that if $f(z)$ is analytic in a region that contains Σ ,

$$\oint_C f(z) \sqrt{z-1} \sqrt{z+1} dz = 0.$$

(1 part)

Solution:

Consider just citing Cauchy's Theorem...

- Show that if $f(z)$ is analytic in a region that contains Σ

$$\oint_C \frac{f(z) dz}{\sqrt{z-1} \sqrt{z+1}} = 0.$$

(1 part)

Solution:

Deal with the singularities on the boundary.

6: From A&F: 3.1.1 b,d

In the following we are given sequences. Discuss their limits and whether the convergence is uniform, in the region $\alpha \leq |z| \leq \beta$, for finite $\alpha, \beta > 0$.

b)

$$\left\{ \frac{1}{z^n} \right\}_{n=1}^{\infty}$$

(2 parts)

Solution:

d)

$$\left\{ \frac{1}{1 + (nz)^2} \right\}_{n=1}^{\infty}$$

(2 parts)

Solution:

7: From A&F: 3.1.2 b,d

For each sequence in problem 1, what can be said if

(a) $\alpha = 0$

(b) $\alpha > 0, \quad \beta = \infty$

(4 parts 2x2) *Solution:*

8: From A&F: 3.1.3 Compute the integrals

$$\lim_{n \rightarrow \infty} \int_0^1 n z^{n-1} dz \quad \text{and} \quad \int_0^1 \lim_{n \rightarrow \infty} (n z^{n-1}) dz$$

and show that they are not equal. Explain why this is not a counter example to Theorem 3.1.1. (A &F pg. 111)

(3 parts) *Solution:*

For the integral on the right consider adding a limit outside the integral so the bound doesn't have any issues.

There are approximately 25 things to do