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AMATH 567

HOMEWORK 7

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 3.5.1 b, c, d (Only consider singularities in the finite complex plane) Discuss the type of singularity (removable, pole and order, essential, branch cluster, natural barrier, etc.); if the type is a pole give the strength of the pole, and give the nature (isolated or not) of all singular points associated with the following functions.

For my reference, here are the definitions of types of singularities. A point $z = z_0$ is a **singularity** of $f(z)$ if $f'(z_0)$ does not exist. Suppose $f(z)$ is analytic in a region $0 < |z - z_0| < R$ (a neighborhood of $z = z_0$) but not at the point z_0, then z_0 is called an **isolated singular point** of $f(z)$. An isolated singularity $z = z_0$ is **removable**, if $f(z)$ is bounded in some neighborhood of $z = z_0$ s.t. $|f(z)| < M$. Additionally, an isolated singularity at $z = z_0$ of $f(z)$ is called a **pole** if $f(z)$ has the following representation

$$f(z) = \frac{\phi(z)}{(z - z_0)^N}$$

where N is a positive integer, $N \geq 1$, $\phi(z)$ is analytic in a neighborhood of z_0, and $\phi(z_0) \neq 0$. If $N \geq 2$ we say the pole is an $\mathbf{N^{th}}$ **order pole** and if $N = 1$ it is a **simple pole**. Therefore, the Laurent expansion of $f(z)$ takes the form

$$f(z) = \sum_{n=-N}^{\infty} C_n (z - z_0)^n.$$

The first coefficient C_{-N} is the **strength of the pole**.
(b)

$$f(z) = \frac{\mathrm{e}^{2z} - 1}{z^2}$$

Solution:

(c)

$$f(z) = \mathrm{e}^{\tan z}$$

Solution:

(d)

$$f(z) = \frac{z^3}{z^2 + z + 1}$$

Solution:

2: From A&F: 3.5.3 a, c, d

Show that the functions below are meromorphic; that is, the only singularities in the finite z plane are poles. Determine the location, order and strength of the poles.

(a)

$$f(z) = \frac{z}{z^4 + 2}$$

Solution:

(c)

$$f(z) = \frac{z}{\sin^2 z}$$

Solution:

(d)

$$f(z) = \frac{\mathrm{e}^z - 1 - z}{z^4}$$

Solution:

3: Introducing the Gamma function: Do A&F: 3.6.6.

Solution:
This is tough...

This is the same Gamma function you may have seen defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

This better known representation is only valid for $\mathrm{Re}(z) > 0$. The representation given here is valid in all of $\mathbb{C}$. It takes a bit of work to show that our representation is an analytic continuation of the integral representation (this requires the Dominated Convergence Theorem), but it is quite doable. Not now though.

4: Consider a sequence of numbers $(a_n)_{n\geq 0}$ such that $|a_n| < 1$ and

$$\sum_{n=0}^{\infty}(1 - |a_n|) < \infty.$$

Define a Blaschke factor

$$B(a, z) = \begin{cases} \frac{|a|}{a}\frac{a-z}{1-\bar{a}z} & a \neq 0, \\ z & a = 0. \end{cases}$$

- Show that

  $$H(z) = \prod_{n=0}^{\infty} B(a_n, z),$$

  defines an analytic function in the open unit disk $|z| < 1$.
- Show that $H(z)$ has zeros at $z = a_n$ for every n. It might seem that this construction of an analytic function with an infinite number of zeros in a bounded region implies that $H(z) = 0$ for all z. Why is this not the case?

Solution:

5: We define the Weierstrass $\wp$-function as

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right),$$

where $(j,k) = (0,0)$ is excluded from the double sum. Also, you may assume that ω_1 is a positive real number, and that ω_2 is on the positive imaginary axis. All considerations below are meant for the entire complex plane, except the poles of $\wp(z)$.

(a) Show that $\wp(z + M\omega_1 + N\omega_2) = \wp(z)$, for any two integers M, N. In other words, $\wp(z)$ is a doubly-periodic function: it has two independent periods in the complex plane. Doubly periodic functions are called elliptic functions.

(b) Establish that $\wp(z)$ is an even function: $\wp(-z) = \wp(z)$.

(c) Find Laurent expansions for $\wp(z)$ and $\wp'(z)$ in a neighborhood of the origin in the form

$$\wp(z) = \frac{1}{z^2} + \alpha_0 + \alpha_2 z^2 + \alpha_4 z^4 + \dots$$

and

$$\wp'(z) = -\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \dots$$

Give expressions for the coefficients introduced above.

(d) Show that $\wp(z)$ satisfies the differential equation

$$(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d,$$

for suitable choices of a, b, c, d. Find these constants. You may need to invoke Liouville's theorem to obtain this final result. It turns out that the function $\varphi(z)$ is determined by the coefficients c and d, implying that it is possible to recover ω_1 and ω_2 from the knowledge of c and d.

Solution:

OH Notes: For c take the derivative of the terms in the original sum of the $\wp(z)$ function, apply Taylor's theorem to get something with a $\mathcal{O}(z^3)$. Do we need to show any uniform convergence in (c) to go from the representation of $\wp(z)$ to the representation of $\wp'(z)$. First do everything assuming no issues with convergence or uniformity. Get what you are looking for, then go back and justify things after. Consider using Liouville theorem for part (d).