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AMATH 561

PROBLEM SET 8

Note: Exercises are from Matt Lorig's notes (link on course website).

1. Exercise 5.1. Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $X = (X_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find μ. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Hint: You can use Little's law, which states that the expected number of people in the hospital at steady-state is equal to the average arrival rate multiplied by the average processing time.

Solution:
TODO:

$$Hereyou$$

2. Exercise 5.3. Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = 0, 1, 2, \ldots$ and with a generator $\mathbf{G}$ whose ith row has entries

$$g_{i,i-1} = i\mu, \quad g_{i,i} = -i\mu - \lambda, \quad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_T}(s) := E(s^{X_t})$. What is the distribution of X_t as $t \to \infty$?

Solution:
TODO:

$$Hereyou$$

3. Exercise 5.4. Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t+\mathrm{d}t)$ is $\lambda(t)\mathrm{d}t$ and the probability of two jumps in that interval of time is $\mathcal{O}(\mathrm{d}t^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1+t)$ show that $E(\tau) < \infty$ if and only if $c > 1$.

Solution:
TODO:

$$Hereyou$$

4. Exercise 5.5. Let N be a poisson process with a random intensity Λ witch is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = E(s^{N_t})$. What is the mean and variance of N_t?

Solution:
TODO:

$$Hereyou$$