

Hunter Lybbert
Student ID: 2426454
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AMATH 567

HOMework 6

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 3.3.2

2: From A&F: 3.3.5

3: Bernoulli numbers: Consider the function

$$f(z) = \frac{z}{e^z - 1}.$$

- (a) Show that $f(z)$ has a removable singularity at $z = 0$. Assume from now on that the definition of $f(z)$ has been extended to remove the singularity.
- (b) Suppose you were to find a Taylor series for $f(z)$, centered at $z = 0$. What would be its radius of convergence?
- (c) Find the Taylor series in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The numbers B_n are known as the Bernoulli numbers.

- (d) Find a recursion formula for the Bernoulli numbers, and use it to find B_0, \dots, B_{12} .
 - (e) Show that $B_{2n+1} = 0$ for $n \geq 1$.
 - (f) Use your result to find a Taylor series for $z \coth z$, in terms of the Bernoulli numbers. Where is this series valid? Using this result, find a Laurent series for $\cot z$. Where is this series valid?
- 4:** Consider $g(z) = 1/f(z)$ where $f(z)$ is as in the previous problem.
- (a) Using the formula for $g(z)$, use software that uses double precision floating point arithmetic to compute the errors $e_n := |g(2^{-n}) - g(0)|$ for $n = 1, 2, \dots, 52$. Produce a plot of these errors.
 - (b) Derive an approximation $G(z)$ to $g(z)$, near $z = 0$, that does not suffer from the instability you notice. Plot the new errors $E_n := |G(2^{-n}) - g(0)|$ for $n = 1, 2, \dots, 52$. Ensure that these errors are less than 10^{-10} for all n .

5: Analytic continuation: (a) Consider

$$F(z) = 1 + z + z^2 + z^3 + \dots = \sum_{n=0}^{\infty} z^n.$$

Where is this function analytic? (b) Use the above representation to induce a Taylor representation of $F(z)$ centered at $z = -1/2$. Call this representation $G(z)$. Your final result should be of the form

$$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2}\right)^m$$

Where is this series valid? If you can answer this question without using that both $F(z)$ and $G(z)$ are representations of $1/(1-z)$, you will receive 2 bonus points.

- 6:** This problem is from Whittaker and Watson's "A course of modern analysis": Shew¹ that

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \frac{1}{(1-z)^2}, & |z| < 1 \\ \frac{1}{z(1-z)^2}, & |z| > 1. \end{cases}$$

This might appear to contradict the idea of analytic continuation. Please comment.

- 7:** Suppose that f is a function satisfying

$$|f(x)| \leq M, \quad x \in \mathbb{R}.$$

Show that

$$\hat{f}(z) := \int_0^{\infty} e^{izx} f(x) dx,$$

is an analytic function of z for $\text{Im } z > 0$. You may assume that f is continuous, but this is not a necessary assumption.

- 8:** Use analytic continuation to show that

$$\sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}},$$

where $\sqrt{\cdot}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$. Then show that

$$\sqrt{z-1}\sqrt{z+1} = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + O(z^{-3}), \quad z \rightarrow \infty,$$

and find b_0, b_1, b_2 .

¹Aka "Show".