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AMATH 567

HOMEWORK 2

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a colaborator.

1: From A&F: 1.2.12.
Show that a circle in the z plane corresponds to a circle on the sphere. (Note the remark following the reference to Figure 1.2.7 in Section 1.2.2)
Hints from Office Hours:

- Use 1.2.25 equations
- perhaps start with a circle centered at the origin for the intuition
- show the intersection of the plane with a sphere is a circle

Solution:
Incomplete

2: From A&F: 1.3.5.
Show that the functions $\Re(z)$ and $\Im(z)$ are nowhere differentiable.
Solution:
Part 1: $f(z) = \Re(z)$
Let's begin with the definition of the derivative

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{f(x_z+x_h+i(y_z+y_h))-x_z}{h}\\ &= \lim_{h\to 0}\frac{x_z+x_h-x_z}{x_h+iy_h}\\ &= \lim_{h\to 0}\frac{x_h}{x_h+iy_h}=\frac{0}{0}.\end{aligned}$$

Now we want to use L'Hôpital's rule so we create two cases fixing y_h and x_h each in turn. Starting from where we left off

$$\lim_{h\to 0}\frac{x_h}{x_h+iy_h}=\lim_{h\to 0}\frac{\frac{d}{dx_h}x_h}{\frac{d}{dx_h}(x_h+iy_h)}=\lim_{h\to 0}\frac{1}{1}=1 \text{ when } y_h \text{ is fixed}$$

$$\lim_{h\to 0}\frac{x_h}{x_h+iy_h}=\lim_{h\to 0}\frac{\frac{d}{dy_h}x_h}{\frac{d}{dy_h}(x_h+iy_h)}=\lim_{h\to 0}\frac{0}{i}=0 \text{ when } x_h \text{ is fixed.}$$

Since we get a different result at an arbitrary z depending on which x_h and y_h is currently fixed the limit does not exist and therefore $f(z) = \Re(z)$ is nowhere differentiable.
□

Part 2: $f(z) = \Im(z)$
For this one we will use a different method. Recall that the Cauchy-Riemann equations are a necessary condition that must hold if $f(z)$ is differentiable (A&F pg. 33). Therefore we can show that the Cauchy-Riemann equations do not hold and therefore the function is non differentiable. We use the fact that $f(z) = u(x, y) + iv(x, y)$ and

$$f(z) = \Im(z) = y \quad \text{since } z = x + iy$$

to get

$$\begin{aligned} u(x, y) &= y \\ v(x, y) &= 0. \end{aligned}$$

Now we want to check if both of the following hold

$$u_x = v_y$$

$$v_x = -u_y.$$

Let's calculate them

$$u_x = 0, v_y = 0, v_x = 0, u_y = 1$$

Therefore the first condition holds

$$\begin{aligned} u_x &= v_y \\ 0 &= 0 \end{aligned}$$

however, the second does not

$$\begin{gathered} v_x = -u_y \\ 0 \neq -1. \end{gathered}$$

Therefore $f(z) = \Im(z)$ is nowhere differentiable.

□

3: Consider the function

$$\varphi(z) = z + \sqrt{z^2 - 1}, \quad z > 1.$$

Show that

$$\log \varphi(z) = \int_1^z \frac{\mathrm{d}x}{\sqrt{x^2 - 1}}.$$

Office Hour Hints:
- z is real!

Solution:
Incomplete

4: Find all zeroes of $\tan(z), z \in \mathbb{C}$. What can you conclude about the zeroes of $\tanh(z) = \sinh(z)/\cosh(z), z \in \mathbb{C}$?
Solution:
Incomplete

5: Consider $f_\epsilon(z) = \epsilon/\left(\epsilon^2 + z^2\right)$, where ϵ is a small positive number, and $z \in \mathbb{C}/\{i\epsilon, -i\epsilon\}$. Plot $|f_\epsilon(z)|$, for various values of ϵ. Discuss the influence the singularities of a function in the complex plane have on its behavior on the real line. Compute

$$\int_{-\infty}^{\infty} f_\epsilon(x)\mathrm{d}x.$$

Solution:
Incomplete

6: Visualizing complex functions is not as easy as visualizing real-valued functions, since we need 4 dimensions: two for the input, two for the output. Different visualizations are commonly used, such as showing 3-dimensional plots of the real and imaginary parts. Plotting the modulus is informational, but it eliminates a lot of information.

- To see this, plot the real and imaginary part of the exponential function $\exp(z) = \exp(x + \mathrm{i}y)$, for $x \in [-1, 1], y \in [-2\pi, 2\pi]$. Now plot the modulus over the same region, and compare.
- A "new" popular way to do this is to plot the modulus of the function with the color defined by the phase. The Digital Library of Mathematical Functions has lots of examples. Create a plot of the $|\exp(z)| = |\exp(x + \mathrm{i}y)|$, for $x \in [-1, 1]$, $y \in [-2\pi, 2\pi]$. colored by the argument. Experimenting with other functions is highly encouraged! The book visual Complex Functions: An Introduction with Phase Portraits by Elias Wegert (Birkhäuser, 2012) is a good companion to our textbook, if you think geometrically.

Solution:
Incomplete

7: From A&F: 2.1.1
Which of the following satisfy the Cauchy-Riemann (C-R) equations? If they satisfy the C-R equations, give the analytic function of z.
a) $f(x, y) = x - iy + 1$
Solution:
Identify u and v and their partial derivatives

$$u(x, y) = x + 1 \implies u_x = 1,\ u_y = 0$$
$$v(x, y) = -y \implies v_x = 0,\ v_y = -1$$

Therefore, $v_x = 0 = -0 = -u_y$ holds, however $u_x = 1 \neq -1 = v_y$ does not hold. In conclusion $f(x, y) = x - iy + 1$ does not satisfy the Cauchy-Riemann (C-R) equations.
□

b) $f(x, y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$
Solution:
Identify u and v and their partial derivatives

$$u(x, y) = y^3 - 3x^2y \implies u_x = -6xy,\ u_y = 3y^2 - 3x^2$$
$$v(x, y) = x^3 - 3xy^2 + 2 \implies v_x = 3x^2 - 3y^2,\ v_y = -6xy$$

Therefore, $v_x = 3x^2 - 3y^2 = -(3y^2 - 3x^2) = -u_y$ and $u_x = -6xy = -6xy = v_y$ both hold. In conclusion $f(x, y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$ satisfies the Cauchy-Riemann (C-R) equations. And the analytic function of z is **To Be Continued after you know what the analytic form of this function is**.
□

c) $f(x, y) = e^y(\cos\theta + i\sin\theta)$
Solution:
Identify u and v and their partial derivatives

$$u(x, y) = ... \implies u_x = ..., \, u_y = ...$$
$$v(x, y) = ... \implies v_x = ..., \, v_y = ...$$

Therefore, $v_x = ... = ... = -u_y$ and $u_x = ... = ... = v_y$ both hold. In conclusion $f(x, y) = e^y(\cos\theta + i\sin\theta)$?satisfies? the Cauchy-Riemann (C-R) equations. And the analytic function of z is **To Be Continued after you know what the analytic form of this function is**.

□

8: From A&F: 2.1.7
Hint: No Hints yet...
Solution:
Incomplete

9: Show that the derivative of $f(z) = |z|^2$ is defined at $z = 0$, but nowhere else.
Solution:
Once again we are going to use the C-R equations and the fact that satisfying them is a necessary condition for differentiability.

$$\begin{aligned} f(z) &= |z|^2 \\ &= \sqrt{x^2 + y^2}^2 \\ &= x^2 + y^2 \\ &= x^2 + y^2 + i \cdot 0 \end{aligned}$$

Therefore $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Now let's calculate the necessary partials

$$u_x = 2x$$
$$v_y = 0$$
$$v_x = 0$$
$$u_y = 2y.$$

Now we need the following to hold for $f(z)$ to be differentiable

$$\begin{aligned} u_x &= v_y \\ 2x &= 0 \end{aligned}$$

and

$$\begin{aligned} v_x &= -u_y \\ 0 &= -2y. \end{aligned}$$

Both of these only hold if $x = 0$ and $y = 0$ or in other words if $z = 0$. Which means the C-R equations only hold at $z = 0$ and the derivative of $f(z) = |z|^2$ is defined at $z = 0$ but nowhere else.

□

10: Derive the polar-coordinates form of the Cauchy-Riemann equations

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta.$$

where $x = r\cos\theta$ and $y = r\sin\theta$
Solution:

We know we can represent $f(z) = u(x, y) + iv(x, y)$. Let's make a substitution to polar coordinates

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ &= u(r\cos\theta, r\sin\theta) + iv(r\cos\theta, r\sin\theta) \\ &= U(r, \theta) + iV(r, \theta) \end{aligned}$$

We need to calculate the derivatives using the chain rule

$$U_r = u_x x_r + u_y y_r$$

$$U_\theta = u_x x_\theta + u_y y_\theta$$

$$V_r = v_x x_r + v_y y_r$$

$$V_\theta = v_x x_\theta + v_y y_\theta.$$

Now we compute the necessary partials and plug them back in

$$\begin{aligned} x_r &= \cos\theta \\ x_\theta &= -r\sin\theta \\ y_r &= \sin\theta \\ y_\theta &= r\cos\theta \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} U_r &= u_x\cos\theta + u_y\sin\theta \\ U_\theta &= -u_x r\sin\theta + u_y r\cos\theta \\ V_r &= v_x\cos\theta + v_y\sin\theta \\ V_\theta &= -v_x r\sin\theta + v_y r\cos\theta. \end{aligned}$$

Now try to make them look like each other with the C-R equations

$$\begin{aligned} U_r &= u_x\cos\theta + u_y\sin\theta \\ &= v_y\cos\theta + u_y\sin\theta \end{aligned}$$

then

$$\begin{aligned} V_\theta &= -v_x r\sin\theta + v_y r\cos\theta \\ &= u_y r\sin\theta + v_y r\cos\theta \\ &= r(u_y\sin\theta + v_y\cos\theta) \\ &= r(v_y\cos\theta + u_y\sin\theta) \\ &= rU_r \end{aligned}$$

Therefore $U_r = \frac{1}{r}V_\theta$.

$$\begin{aligned} V_r &= v_x\cos\theta + v_y\sin\theta \\ &= -u_y\cos\theta + v_y\sin\theta \end{aligned}$$

then

$$\begin{aligned} U_\theta &= -u_x r\sin\theta + u_y r\cos\theta \\ &= -v_y r\sin\theta + u_y r\cos\theta \\ &= -r(v_y\sin\theta - u_y\cos\theta) \\ &= -r(-u_y\cos\theta + v_y\sin\theta) \\ &= -rV_r \end{aligned}$$

Therefore $V_r = -\frac{1}{r}U_\theta$. And with this we have derived the polar-coordinates form of the Cauchy-Riemann equations.

$\square$