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 AMATH 567

HOMEWORK 9

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

- 1:** From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
 Evaluate the integrals $\frac{1}{2i\pi} \oint_C f(z) dz$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C .

(a)

$$f(z) = \frac{z^2 + 1}{z^2 - a^2}, \quad a^2 < 1$$

Solution:

Let the set of singularities of $f(z)$ be S

$$\frac{1}{2\pi i} \oint_C \frac{z^2 + 1}{z^2 - a^2} dz = \sum_{w \in S} \operatorname{Res}_{z=w} f(z).$$

The denominator can be factored to $z^2 - a^2 = (z - a)(z + a)$, therefore the singularities of $f(z)$ are $z = \pm a$. Then we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{z^2 + 1}{z^2 - a^2} dz &= \operatorname{Res}_{z=-a} f(z) + \operatorname{Res}_{z=a} f(z) \\ &= \frac{(-a)^2 + 1}{(-a - a)} + \frac{(a)^2 + 1}{(a + a)} \\ &= -\frac{a^2 + 1}{2a} + \frac{a^2 + 1}{2a} = 0 \end{aligned}$$

Fix issue here

□

(b)

$$f(z) = \frac{z^2 + 1}{z^3}$$

Solution:

Looking at this in terms of the residue we have

$$\begin{aligned}\frac{1}{2\pi i} \oint_C \frac{z^2+1}{z^3} dz &= \operatorname{Res}_{z=0} f(z) \\ &= \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left((z-0)^3 \left(\frac{z^2+1}{z^3} \right) \right) \Big|_0 \\ &= \frac{1}{2!} \frac{d^2}{dz^2} (z^2+1) \Big|_0 \\ &= \frac{1}{2} \frac{d}{dz} (2z) \Big|_0 \\ &= \frac{1}{2} 2 \\ &= 1.\end{aligned}$$

TODO: Double check with someone else

□

(c)

$$f(z) = z^2 e^{-1/z}$$

Solution:

Looking at this in terms of the residue we have, what is the order of this pole?? Do I need to do something else?

$$\frac{1}{2\pi i} \oint_C z^2 e^{-1/z} dz = \operatorname{Res}_{z=0} f(z)$$

TODO: Something about the pole order is throwing me off.

2: From A&F: 4.2.1(b)

Evaluate the following real integral

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2}, \quad a^2 > 0$$

Solution:

For convenience throughout this problem let's define

$$f(x) = \frac{1}{(x^2 + a^2)^2}.$$

Since the $f(x)$ is an even function we can say

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{(x^2 + a^2)^2}.$$

Then we can use the principal value integral which is given by

$$\int_{-\infty}^\infty f(x)dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx, \quad \text{if it exists.}$$

Now let's consider the counterclockwise contour C around the closed semicircle centered at the origin in the upper half plane with radius R . We can say

$$(1) \quad \oint_C f(z)dz = \int_{-R}^R f(x)dx + \oint_{C_R} f(z)dz,$$

where $-R$ to R is the section of C along the real axis and C_R is the open semicircle (counterclockwise). We can combine these ideas by taking the limit of both sides as $R \rightarrow \infty$. Let's take the limit of the right and analyze what is going on. This gives us

$$\begin{aligned} \lim_{R \rightarrow \infty} \left(\int_{-R}^R f(x)dx + \oint_{C_R} f(z)dz \right) &= \lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx + \lim_{R \rightarrow \infty} \oint_{C_R} f(z)dz \\ &= \int_{-\infty}^\infty f(x)dx + \lim_{R \rightarrow \infty} \oint_{C_R} \frac{dz}{(z^2 + a^2)^2} \end{aligned}$$

Let's try to bound the integrand to something that depends on R^{-1} . We will use the substitution $z = R e^{i\theta}$ with $\theta \in [0, \pi]$.

$$\begin{aligned} \left| \frac{1}{(z^2 + a^2)^2} \right| &= \left| \frac{1}{(R^2 e^{2i\theta} + a^2)^2} \right| \\ &= \frac{1}{|R^4 e^{4i\theta} + 2R^2 a^2 e^{2i\theta} + a^4|} \\ &\leq \frac{1}{R^2 |R^2 e^{4i\theta} + 2a^2 e^{2i\theta}|} \end{aligned}$$

Since $f(z)$ is continuous on C_R , we can use the ML bound on the integral to say

$$\begin{aligned} \lim_{R \rightarrow \infty} \left| \oint_{C_R} \frac{dz}{(z^2 + a^2)^2} \right| &\leq \lim_{R \rightarrow \infty} \frac{\pi R}{R^2 |R^2 e^{4i\theta} + 2a^2 e^{2i\theta}|} \\ &= \lim_{R \rightarrow \infty} \frac{\pi}{R |R^2 e^{4i\theta} + 2a^2 e^{2i\theta}|} = 0. \end{aligned}$$

Applying a similar squeeze theorem argument from homework 4 problem 4, we can conclude

$$\lim_{R \rightarrow \infty} \oint_{C_R} \frac{dz}{(z^2 + a^2)^2} = 0.$$

Therefore, equation (1) becomes

$$\begin{aligned} \lim_{R \rightarrow \infty} \oint_C f(z) dz &= \int_{-\infty}^{\infty} f(x) dx + \oint_{C_R} f(z) dz \\ \lim_{R \rightarrow \infty} \oint_C f(z) dz &= \int_{-\infty}^{\infty} f(x) dx \end{aligned}$$

Notice, as R goes to infinity the integral of $f(z)$ along the contour C is equal to the sum of all of the residues at all singularities in the upper half plane for $f(z)$. Let S be the collection of singularities in the upper half plane, then we can say

$$\lim_{R \rightarrow \infty} \oint_C f(z) dz = 2\pi i \sum_{w \in S} \text{Res}_{z=w} \left(\frac{1}{(z^2 + a^2)^2} \right).$$

Hence,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} dx = 2\pi i \sum_{w \in S} \text{Res}_{z=w} \left(\frac{1}{(z^2 + a^2)^2} \right).$$

We now need to locate the singularities. The denominator is only 0 when $z^2 + a^2 = 0$ so we know the singularities are at $z = \pm ia$. However, since only $z = ia$ is in the upper half plane we can simplify the previous equation and solve for the one residue we need

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} dx &= 2\pi i \text{Res}_{z=ia} \left(\frac{1}{(z^2 + a^2)^2} \right) \\ &= 2\pi i \frac{1}{(2-1)!} \frac{d}{dz} \left((z - ia)^2 \frac{1}{(z^2 + a^2)^2} \right) \Big|_{ia} \\ &= 2\pi i \frac{d}{dz} \left((z - ia)^2 \frac{1}{(z + ia)^2 (z - ia)^2} \right) \Big|_{ia} \\ &= 2\pi i \frac{d}{dz} \left(\frac{1}{(z + ia)^2} \right) \Big|_{ia} \\ &= 2\pi i \frac{-2}{(ia + ia)^3} \\ &= 2\pi i \frac{-2}{-8ia^3} \\ &= \frac{\pi}{2a^3}. \end{aligned}$$

Recall our original integral was

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2} \frac{\pi}{2a^3} = \frac{\pi}{4a^3}$$

□

3: Existence and uniqueness of polynomial interpolants.

- (a) Suppose $(z_j)_{j=1}^n$ are distinct points in \mathbb{C} and suppose $f_j \in \mathbb{C}$ for $j = 1, \dots, n$. Show that there is at most one polynomial $p(z)$ of degree $n - 1$ such that $p(z_j) = f_j$ for $j = 1, \dots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant*.

Solution:

Suppose there exists two polynomials $p_1(z)$ and $p_2(z)$ each of degree $n - 1$. Assume both agree with f_j at each z_j such that

$$p_1(z_j) = p_2(z_j) = f_j \quad \text{for each } j = 1, \dots, n.$$

Additionally define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Now let's consider the function

$$g(z) = \frac{p_1(z) - p_2(z)}{\nu(z)}.$$

We want to utilize Liouville's theorem to conclude that $g(z)$ is constant. In order to do this we need to show that $g(z)$ is entire and bounded. Let's begin by demonstrating that it is bounded by taking the limit as $z \rightarrow \infty$

$$\begin{aligned} \lim_{z \rightarrow \infty} g(z) &= \lim_{z \rightarrow \infty} \frac{p_1(z) - p_2(z)}{\nu(z)} \\ &= \lim_{z \rightarrow \infty} \frac{p_1(z) - p_2(z)}{\prod_{j=1}^n (z - z_j)} \\ &= \frac{\infty}{\infty}. \end{aligned}$$

Applying L'Hôpital's rule repeatedly we will end up with $1/z$ which goes to 0 as z goes to infinity since the denominator is an n th degree polynomial while the numerator is a degree $n - 1$ polynomial. Therefore,

$$\lim_{z \rightarrow \infty} g(z) = \lim_{z \rightarrow \infty} \frac{p_1(z) - p_2(z)}{\nu(z)} = 0,$$

which implies that $g(z)$ is bounded. Next, we need to determine if $g(z)$ is entire. Since polynomials are entire in the finite z plane, $p_1(z) - p_2(z)$ is entire. However, $g(z)$ overall requires a little more analysis since it has singularities where $z = z_j$. Notice, since the expression $p_1(z) - p_2(z)$ and $\nu(z)$ are both zero at each z_j , then there exists a factorization of $p_1(z) - p_2(z)$ which would allow us to cancel out each of the factors in the product in the denominator. Therefore, the singularities of $g(z)$ are removable and thus $g(z)$ is entire (or can be made entire, with the right extension at each z_j as we have done in previous assignments). Hence, by Liouville's Theorem, we can conclude that $g(z)$ is constant. Combining with the fact that $p_1(z_j) - p_2(z_j) = 0$ for each $j = 1, \dots, n$, then $g(z)$ must be 0 everywhere, thus implying $p_1(z) = p_2(z)$ everywhere. In conclusion, since these two functions are the same therefore there is at most one polynomial $p(z)$ of degree $n - 1$ such that $p(z_j) = f_j$ for $j = 1, \dots, n$, otherwise known as the interpolant.

□

- (b) Define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:

Let's look at $p(z)/\nu(z)$ and consider what happens if we subtract off a specially cooked up collection of terms including the residues r_j for $j = 1, \dots, n$. We can express the residues of $p(z)/\nu(z)$ as

$$\frac{1}{2\pi i} \oint_C \frac{p(z)}{\nu(z)} dz = \sum_{j=0}^n \text{Res} \left(\frac{p(z)}{\nu(z)}; z_j \right) = \sum_{j=0}^n \frac{f_j}{\prod_{k \neq j} (z_k - z_j)}.$$

Recall partial fractions is connected to the residues. We construct the expression to subtract from $p(z)/\nu(z)$ using the partial fraction decomposition relationship to residues

$$\begin{aligned} \frac{p(z)}{\nu(z)} - \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ = \frac{p(z)}{\nu(z)} - \frac{f_1 \left(\prod_{k \neq 1} (z_k - z_1) \right)^{-1}}{z - z_1} - \frac{f_2 \left(\prod_{k \neq 2} (z_k - z_2) \right)^{-1}}{z - z_2} - \dots - \frac{f_n \left(\prod_{k \neq n} (z_k - z_n) \right)^{-1}}{z - z_n} = 0. \end{aligned}$$

TODO: Why is this 0 besides saying it's the partial fraction decomposition? This expression is equal to 0 because the collection of terms we are subtracting is the partial fraction decomposition of $p(z)/\nu(z)$. If we can show that this function is bounded and entire then it is a constant. Therefore, we would be able to state that since it is a constant and 0 then it must be a 0 everywhere. Thus we can say

$$\begin{aligned} \frac{p(z)}{\nu(z)} - \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} &= 0 \\ \frac{p(z)}{\nu(z)} &= \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ p(z) &= \nu(z) \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ p(z) &= \prod_{j=1}^n (z - z_j) \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ p(z) &= \sum_{j=0}^n \frac{f_j \prod_{\ell \neq j} (z - z_\ell)}{\prod_{k \neq j} (z_k - z_j)}. \end{aligned}$$

Therefore we have this expression for $p(z)$.

4: Bernstein interpolation formula. Suppose that $-1 \leq x_1 < x_2 < \dots < x_n \leq 1$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1, 1]$. Show that for any simple contour C inside Ω with $[-1, 1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{dz}{\nu(z)}, \quad x \in [-1, 1],$$

where p is the degree $n-1$ polynomial interpolant satisfying $p(x_j) = f(x_j)$ for $j = 1, 2, \dots, n$. We also have $\nu(x) = \prod_{j=1}^n (x - x_j)$.

Solution:

Starting from the right we have

$$(2) \quad \begin{aligned} \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{dz}{\nu(z)} &= \frac{1}{2\pi i} \int_C \frac{f(z)\nu(x)}{(z-x)\nu(z)} dz \\ &= \text{Res}_{z=x} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right) + \sum_{i=1}^n \text{Res}_{z=x_i} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right). \end{aligned}$$

Calculating the residue at $z = x$ is easy because x is a simple pole

$$\text{Res}_{z=x} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right) = \text{Res}_{z=x} \left(\frac{\frac{f(z)\nu(x)}{\nu(z)}}{(z-x)}; 0 \right) = \frac{f(x)\nu(x)}{\nu(x)} = f(x).$$

Calculating the residue at each $z = x_i$ is similarly quick since they are simple poles

$$(3) \quad \begin{aligned} \sum_{i=1}^n \text{Res}_{z=x_i} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right) &= \sum_{i=1}^n \text{Res}_{z=x_i} \left(\frac{f(z)\nu(x)}{(z-x) \prod_{j=1}^n (z-x_j)}; 0 \right) \\ &= \sum_{i=1}^n \text{Res}_{z=x_i} \left(\left(\frac{f(z)\nu(x)}{(z-x) \prod_{j \neq i}^n (z-x_j)} \right) / (z-x_i); 0 \right) \\ &= \sum_{i=1}^n \frac{f(x_i)\nu(x)}{(x_i-x) \prod_{j \neq i}^n (x_i-x_j)} \\ &= - \sum_{i=1}^n \frac{f(x_i) \prod_{j=1}^n (x-x_j)}{(x-x_i) \prod_{j \neq i}^n (x_i-x_j)} \\ &= - \sum_{i=1}^n \frac{f(x_i) \prod_{j \neq i}^n (x-x_j)}{\prod_{j \neq i}^n (x_i-x_j)} = -p(x) \end{aligned}$$

Where we know this is $p(z)$ from our work in problem 4.

Therefore we have Equation (2) is equal to $f(x) - p(x)$. Furthermore, since $\nu(x_i) = 0$ for each $i = 1, \dots, n$, then

$$\begin{aligned} f(x_i) - p(x_i) &= 0 \\ f(x_i) &= p(x_i) \end{aligned}$$

for all $i = 1, \dots, n$. Finally we can also determine the degree of the polynomial $p(x)$ is $n-1$. This is due the equation (3) being made up of some scalar or weight factor and the product in the numerator which is $\nu(x)$ (a degree n polynomial) but without one of it's factors leaving it as an $n-1$ degree polynomial. \square

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

(a) Show that the polynomial

$$T_n(z) = \frac{1}{2} (\varphi(z)^n + \varphi(z)^{-n}),$$

has all of its roots $x_1 < x_2 < \dots < x_n$ within $[-1, 1]$.

Solution:

Let's begin by looking more closely at $\varphi(z)$ with a specific substitution, namely $z = \cos \theta$ with $\theta \in [0, \pi]$. Then we have

$$\begin{aligned} \varphi(\cos \theta) &= \cos \theta + \sqrt{\cos \theta - 1} \sqrt{\cos \theta + 1} \\ &= \cos \theta - i \sqrt{1 - \cos \theta} \sqrt{1 + \cos \theta} \\ &= \cos \theta - i \sqrt{1 - \cos^2 \theta} \\ &= \cos \theta - i \sqrt{\sin^2 \theta} \\ &= \cos \theta - i \sin \theta \\ &= e^{-i\theta}. \end{aligned}$$

Now putting this thing again we have

$$\begin{aligned} T_n(\cos \theta) &= \frac{1}{2} (\varphi(\cos \theta)^n + \varphi(\cos \theta)^{-n}) \\ &= \frac{1}{2} ((e^{-i\theta})^n + (e^{-i\theta})^{-n}) \\ &= \frac{1}{2} (e^{-ni\theta} + e^{ni\theta}) \\ &= \frac{1}{2} (\cos n\theta - i \sin n\theta + \cos n\theta + i \sin n\theta) \\ &= \frac{1}{2} (2 \cos n\theta) \\ &= \cos n\theta. \end{aligned}$$

Notice, $\theta = \arccos z$, then we have

$$T_n(z) = \cos(n \arccos z).$$

Let's find out what values of z this function $T_n(z) = 0$, let $k \in \mathbb{Z}$, then

$$\begin{aligned} n \arccos z &= \frac{\pi}{2} + \pi k \\ \arccos z &= \frac{1}{n} \left(\frac{\pi}{2} + \pi k \right) \\ z &= \cos \left(\frac{1}{n} \left(\frac{\pi}{2} + \pi k \right) \right). \end{aligned}$$

Therefore, the zeros of $T_n(z)$ are all within the range of \cos which is between $[-1, 1]$. Additionally, because we are dividing by n there will be n zeros between -1 and 1 .

□

- (b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1, 1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:

Let's consider the image of a circle with radius $\rho > 1$, if we parameterize this with $z = \rho e^{i\theta}$ and plug this in to $J(w)$ we have

$$\begin{aligned} J(\rho e^{i\theta}) &= \frac{1}{2} \left(\rho e^{i\theta} + \frac{1}{\rho} e^{-i\theta} \right) \\ &= \frac{1}{2} \left(\rho \cos \theta + \rho \sin \theta + \frac{1}{\rho} \cos \theta - i \frac{1}{\rho} \sin \theta \right) \\ &= \frac{1}{2} \left(\left(\rho + \frac{1}{\rho} \right) \cos \theta + i \left(\rho - \frac{1}{\rho} \right) \sin \theta \right). \end{aligned}$$

Notice this is the equation of an ellipse since it is a slightly stretched and flattened out circle. This is almost a circle of radius ρ but it is slightly stretched in different amounts in the x (real) and y (imaginary) directions. Therefore the image is an ellipse.

We wish to show that $\varphi(J(w)) = w$. Notice,

$$\begin{aligned} \varphi(J(w)) &= J(w) + \sqrt{J(w) - 1} \sqrt{J(w) + 1} \\ &= J(w) - i \sqrt{1 - J(w)} \sqrt{1 + J(w)} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i \sqrt{1 - J(w)^2} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i \sqrt{1 - \frac{1}{4} \left(w + \frac{1}{w} \right)^2} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i \sqrt{1 - \frac{1}{4} \left(w^2 + 2 + \frac{1}{w^2} \right)} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i \sqrt{-\frac{1}{4} w^2 + \frac{1}{2} - \frac{1}{4 w^2}} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i \sqrt{-\frac{1}{4} \left(w^2 - 2 + \frac{1}{w^2} \right)} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i \sqrt{-\frac{1}{4} \left(w - \frac{1}{w} \right)^2} \\ &= \frac{1}{2} \left(w + \frac{1}{w} \right) - i^2 \frac{1}{2} \left(w - \frac{1}{w} \right) \\ &= \frac{1}{2} w + \cancel{\frac{1}{2w}} + \frac{1}{2} w - \cancel{\frac{1}{2w}} \\ &= w. \end{aligned}$$

Hence, we have what we desired. □

- (c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$\begin{aligned} |f(x) - p(x)| &\leq 2 \frac{M|B_\rho|}{\pi} (\rho^n - \rho^{-n})^{-1} (\rho + \rho^{-1} - 2)^{-1} \leq 2 \frac{M|B_\rho|}{\pi} \frac{\rho^{2-n}}{(\rho - 1)^3} \\ &\leq C_\rho \rho^{-n}, \quad \text{for a constant } C_\rho > 0, \end{aligned}$$

where $p(x_j) = f(x_j)$, i.e., p is the degree $n - 1$ interpolant of f at the roots of T_n . Here $|B_\rho|$ denotes the arclength of B_ρ . This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f .

Solution:

Assume f is analytic in a region that contains the ellipse B_ρ and its interior. Additionally, assume $|f(z)| < M$ for z interior to B_ρ . Now consider where $-1 \leq x \leq 1$ and let's look at bounding the following

$$|f(x) - p(x)| = \left| \frac{\nu(x)}{2\pi i} \oint_C \frac{f(z)}{z - x} \frac{dz}{\nu(z)} \right|.$$

First of all, utilizing the conclusions of the Bernstein–Walsh inequality we know that $|\nu(x)| = \frac{1}{2^{n-1}} |T_n(x)|$. Therefore, we change the instances of $\nu(x)$ to $T_n(x)$'s with the appropriate scaling. Hence

$$\begin{aligned} \left| \frac{\nu(x)}{2\pi i} \oint_C \frac{f(z)}{z - x} \frac{dz}{\nu(z)} \right| &= \left| \frac{T_n(x)}{2^n \pi i} \oint_C \frac{f(z)}{z - x} \frac{2^{n-1} dz}{T_n(z)} \right| \\ &= \left| \frac{1}{2\pi i} T_n(x) \oint_C \frac{f(z)}{z - x} \frac{dz}{T_n(z)} \right| \\ &\leq \frac{1}{2\pi i} |T_n(x)| \oint_C \left| \frac{f(z)}{z - x} \right| \frac{1}{|T_n(z)|} |dz|. \end{aligned}$$

We take the contour C to be the ellipse B_ρ and parameterize with $z = J(w)$. Then $dz = J'(w)dw = \frac{1}{2}(1 - 1/w^2)dw$. Now we continue with our from the previous inequality

$$\begin{aligned} &\leq \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \frac{1}{|T_n(J(w))|} \left| \frac{1}{2} \left(1 - \frac{1}{w^2} \right) dw \right| \\ &= \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \frac{1}{\left| \frac{1}{2} (\varphi(J(w))^n + \varphi(J(w))^{-n}) \right|} \left| \frac{1}{2} \left(1 - \frac{1}{w^2} \right) dw \right| \\ &= \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \frac{1}{\left| \frac{1}{2} (w^n + w^{-n}) \right|} \left| \frac{1}{2} \left(1 - \frac{1}{w^2} \right) dw \right| \\ &= \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \left| \frac{1 - \frac{1}{w^2}}{w^n + w^{-n}} \right| |dw| \end{aligned}$$

In the last few steps we used the fact from part (b) which showed $\varphi(J(w)) = w$. Now we will use $w = \rho e^{i\theta}$ and $dw = \rho i e^{i\theta}$ we have

$$\begin{aligned}
&= \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \left| \frac{\left(1 - \frac{1}{\rho^2} e^{2i\theta}\right) \rho i e^{i\theta}}{\rho^n e^{ni\theta} + \rho^{-n} e^{-ni\theta}} \right| |d\theta| \\
&= \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \left| \frac{\left(\frac{\rho^2 e^{2i\theta}}{\rho^2 e^{2i\theta}} - \frac{1}{\rho^2 e^{2i\theta}}\right) \rho i e^{i\theta}}{\rho^n e^{ni\theta} + \rho^{-n} e^{-ni\theta}} \right| |d\theta| \\
&= \frac{1}{2\pi i} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \left| \frac{\left(\frac{\rho^2 e^{2i\theta} - 1}{\rho e^{i\theta}}\right) i}{\rho^n e^{ni\theta} + \rho^{-n} e^{-ni\theta}} \right| |d\theta|
\end{aligned}$$

This is the first fuzzy part for me **TODO: Fix**

$$\begin{aligned}
&\leq \frac{1}{2\pi i} \frac{1}{\rho^n + \rho^{-n}} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \left| \left(\frac{\rho^2 e^{2i\theta} - 1}{\rho e^{i\theta}}\right) i \right| |d\theta| \\
&= \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \left| \frac{\rho^2 e^{2i\theta} - 1}{\rho e^{i\theta}} \right| |d\theta|
\end{aligned}$$

This is another fuzzy part for me **TODO: Fix**

$$\begin{aligned}
&\leq \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \left| \frac{\rho^2 - 1}{\rho} \right| \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| |d\theta| \\
&= \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \left| \frac{\rho^2 - 1}{\rho} \right| \oint_{B_\rho} \left| \frac{f(z)}{\frac{1}{2} \left(\rho e^{i\theta} + \frac{1}{\rho} e^{-i\theta}\right) - x} \right| |d\theta| \\
&\leq \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \left| \frac{\rho^2 - 1}{\rho \left(\frac{1}{2} \left(\rho + \frac{1}{\rho}\right) - 1\right)} \right| \oint_{B_\rho} |f(z)| |d\theta| \\
&= \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \left| \frac{\rho^2 - 1}{\frac{1}{2}\rho^2 + \frac{1}{2} - \rho} \right| \oint_{B_\rho} |f(z)| |d\theta| \\
&= \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \left| \frac{2\rho^2 - 2}{\rho^2 + 1 - 2\rho} \right| \oint_{B_\rho} |f(z)| |d\theta| \\
&= \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \left| \frac{2(\rho - 1)(\rho + 1)}{(\rho - 1)^2} \right| \oint_{B_\rho} |f(z)| |d\theta| \\
&= \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \frac{2\rho + 2}{\rho - 1} \oint_{B_\rho} |f(z)| |d\theta| \\
&\leq \frac{1}{2\pi} \frac{1}{\rho^n + \rho^{-n}} \frac{2\rho + 2}{\rho - 1} |B_\rho| M \\
&= \frac{|B_\rho| M}{\pi} \frac{1}{\rho^n + \rho^{-n}} \frac{\rho + 1}{\rho - 1} \\
&\leq \frac{|B_\rho| M}{\pi} \frac{1}{\rho^n} \frac{\rho + 1}{\rho - 1} \\
&\leq \frac{|B_\rho| M}{\pi} \frac{\rho + 1}{\rho - 1} \rho^{-n}
\end{aligned}$$



6: Compute the following two integrals explicitly for $z \notin [-1, 1]$:

(a)

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-x}\sqrt{1+x}} \frac{dx}{x-z}.$$

Solution:

We first recall that from homework 8 problem 4 part a) we showed

$$(4) \quad \int_{-1}^1 \frac{f(x)dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i} \oint_C \frac{f(z)dz}{\sqrt{z-1}\sqrt{z+1}}.$$

Applying that here we have

$$\frac{1}{\pi} \int_{-1}^1 \frac{\frac{1}{x-z_0}dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\pi i} \oint_C \frac{\frac{1}{z-z_0}dz}{\sqrt{z-1}\sqrt{z+1}}.$$

For notational convenience let

$$g(z) = \frac{\frac{1}{z-z_0}}{\sqrt{z-1}\sqrt{z+1}}.$$

As we expand our contour C outwards we run into the singularity at z_0 , leaving behind a clockwise circular contour around z_0 denoted as $-C_{z_0}$. We also have the normal counterclockwise contour around infinity which we will use in our residue calculation. Hence we have

$$(5) \quad \begin{aligned} \frac{1}{2\pi i} \oint_C g(z)dz &= \frac{1}{2\pi i} \oint_{-C_{z_0}} g(z)dz + \frac{1}{2\pi i} \oint_{C_\infty} g(z)dz \\ &= -\frac{1}{2\pi i} \oint_{C_{z_0}} g(z)dz + \frac{1}{2\pi i} \oint_{C_\infty} g(z)dz \\ &= -\text{Res}_{z=z_0} g(z) + \text{Res}_{z=\infty} g(z) \end{aligned}$$

Now we want to calculate the residues at ∞ and at z_0 . Let

$$h(z) = \frac{1}{\sqrt{z-1}\sqrt{z+1}}$$

and

$$H(z) = h\left(\frac{1}{z}\right) = \frac{1}{\sqrt{1/z-1}\sqrt{1/z+1}} \frac{z}{z} = \frac{z}{\sqrt{1-z}\sqrt{1+z}}.$$

Then we can see $H(0) = 0$. Let's calculate $h'(0)$.

$$H'(z) = \frac{\sqrt{1-z}\sqrt{1+z} - z(-1/2(1-z)^{-1/2}(1+z)^{1/2} + 1/2(1-z)^{1/2}(1+z)^{-1/2})}{(1-z)(1+z)}$$

Hence,

$$H'(0) = \frac{\sqrt{1}\sqrt{1} - 0(-1/2(1)^{-1/2}(1)^{1/2} + 1/2(1)^{1/2}(1)^{-1/2})}{(1)(1)} = 1.$$

Then our Taylor series expansion of $H(z)$ is

$$\begin{aligned} H(z) &= H(0)z^0/0! + H'(0)z^1/1! + \mathcal{O}(z^2) \\ &= 0 + z + \mathcal{O}(z^2) \\ &= z + \mathcal{O}(z^2) \end{aligned}$$

then for $h(z)$ is

$$h(z) = z^{-1} + \mathcal{O}(z^{-2}).$$

We really care about $\frac{1}{z-z_0}h(z)$ so we have

$$\begin{aligned} \frac{1}{z-z_0}h(z) &= \frac{1}{z-z_0} (z^{-1} + \mathcal{O}(z^{-2})) \\ &= \frac{1}{z} \frac{1}{1-z_0/z} (z^{-1} + \mathcal{O}(z^{-2})) \\ &= \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z_0}{z}\right)^k (z^{-1} + \mathcal{O}(z^{-2})) \end{aligned}$$

where $|z_0| < |z|$ since we are on a contour with a large radius R . Then

$$\frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z_0}{z}\right)^k (z^{-1} + \mathcal{O}(z^{-2})) = \frac{1}{z^2} \sum_{k=0}^{\infty} \left(\frac{z_0}{z}\right)^k + \mathcal{O}(z^{-3}) \sum_{k=0}^{\infty} \left(\frac{z_0}{z}\right)^k$$

Therefore the residue of this function at ∞ is trivially

$$\operatorname{Res}_{z=\infty} \left(\frac{1}{(z-z_0)\sqrt{z-1}\sqrt{z+1}} \right) = 0$$

since the coefficient of the $1/z$ is 0. Computing the residue at z_0 is a little easier since it is a simple pole. Therefore

$$\begin{aligned} \operatorname{Res}_{z=z_0} \left(\frac{1}{(z-z_0)\sqrt{z-1}\sqrt{z+1}} \right) &= \operatorname{Res}_{z=z_0} \left(\frac{\frac{1}{\sqrt{z-1}\sqrt{z+1}}}{z-z_0} \right) \\ &= \frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}}. \end{aligned}$$

Plugging these into equation (5) we have

$$\frac{1}{2\pi i} \oint_C g(z) dz = -\operatorname{Res}_{z=z_0} g(z) + \operatorname{Res}_{z=\infty} g(z) = -\frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}} + 0.$$

Hence,

$$\frac{1}{\pi} \int_{-1}^1 \frac{\frac{1}{x-z_0} dx}{\sqrt{1-x}\sqrt{1+x}} = -\frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}}.$$

□

(b)

$$\frac{2}{\pi} \int_{-1}^1 \sqrt{1-x} \sqrt{1+x} \frac{dx}{x-z}.$$

Solution:

Again applying equation (4), we have

TODO: Double check this final manipulation

$$\begin{aligned} \frac{2}{\pi} \int_{-1}^1 \sqrt{1-x} \sqrt{1+x} \frac{dx}{x-z} &= \frac{2}{\pi} \int_{-1}^1 \frac{1-x^2}{\sqrt{1-x} \sqrt{1+x}} \frac{dx}{x-z} \\ &= \frac{1}{\pi i} \oint_C \frac{1-z^2 dz}{\sqrt{z-1} \sqrt{z+1}} \frac{dz}{z-z_0} \\ &= \frac{1}{\pi i} \oint_C \frac{(1-z)(1+z)}{\sqrt{z-1} \sqrt{z+1}} \frac{dz}{z-z_0} \\ &= -\frac{1}{\pi i} \oint_C \sqrt{z-1} \sqrt{z+1} \frac{1}{z-z_0} dz. \end{aligned}$$

Let

$$g(z) = \sqrt{z-1} \sqrt{z+1} \frac{1}{z-z_0}.$$

Then as we expand our contour C outwards we run into the singularity at z_0 , leaving behind a clockwise circular contour around z_0 denoted as $-C_{z_0}$. We also have the normal counterclockwise contour around infinity which we will use in our residue calculation. Hence we have

$$\begin{aligned} -\frac{1}{\pi i} \oint_C g(z) dz &= -\frac{1}{\pi i} \oint_{-C_{z_0}} g(z) dz - \frac{1}{\pi i} \oint_{C_\infty} g(z) dz \\ &= \frac{1}{\pi i} \oint_{C_{z_0}} g(z) dz - \frac{1}{\pi i} \oint_{C_\infty} g(z) dz \\ (6) \qquad &= 2 \operatorname{Res}_{z=z_0} g(z) - 2 \operatorname{Res}_{z=\infty} g(z). \end{aligned}$$

Recall, that we have the Taylor expansion of $\sqrt{z-1} \sqrt{z+1}$ at ∞ is

$$\sqrt{z-1} \sqrt{z+1} = z - \frac{1}{2} z^{-1} + \mathcal{O}(z^{-3}).$$

Then we can multiply through by our extra term in this scenario to get

$$\begin{aligned} \frac{1}{z-z_0} \sqrt{z-1} \sqrt{z+1} &= \frac{1}{z-z_0} \left(z - \frac{1}{2} z^{-1} + \mathcal{O}(z^{-3}) \right) \\ &= \frac{1}{z} \frac{1}{1-z_0/z} \left(z - \frac{1}{2} z^{-1} + \mathcal{O}(z^{-3}) \right) \\ &= \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k \left(z - \frac{1}{2} z^{-1} + \mathcal{O}(z^{-3}) \right) \\ &= z \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k - \frac{1}{2} z^{-1} \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k + \mathcal{O}(z^{-3}) \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k \\ &= \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k - \frac{1}{2} \frac{1}{z^2} \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k + \mathcal{O}(z^{-4}) \sum_{k=0}^{\infty} \left(\frac{z_0}{z} \right)^k. \end{aligned}$$

Therefore,

$$\operatorname{Res}_{z=\infty} \left(\frac{1}{z-z_0} \sqrt{z-1} \sqrt{z+1} \right) = z_0.$$

While the residue at the point z_0 is

$$\operatorname{Res}_{z=z_0} \left(\frac{1}{z-z_0} \sqrt{z-1} \sqrt{z+1} \right) = \sqrt{z_0-1} \sqrt{z_0+1}.$$

Lets plug these in to equation (6) to have

$$-\frac{1}{\pi i} \oint_C g(z) dz = 2 \operatorname{Res}_{z=z_0} g(z) - 2 \operatorname{Res}_{z=\infty} g(z) = 2\sqrt{z_0-1} \sqrt{z_0+1} - 2z_0.$$

Hence,

$$\frac{2}{\pi} \int_{-1}^1 \sqrt{1-x} \sqrt{1+x} \frac{dx}{x-z} = 2(\sqrt{z_0-1} \sqrt{z_0+1} - z_0).$$

□