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AMATH 567

HOMEWORK 4

Collaborators*: TODO

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.4.2 c, e.
Evaluate the integral $\oint_C f(z)\mathrm{d}z$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows:
c)

$$f(z) = \frac{1}{\bar{z}}$$

Solution:
e)

$$f(z) = e^{\bar{z}}$$

Solution:

2: From A&F: 2.4.4 a, b. Use the principal branch where the argument is in $[-\pi, \pi)$. Discuss any ambiguities. Use the principal branch of $\log(z)$ and $z^{\frac{1}{2}}$ where the argument is in $[-\pi, \pi)$ to evaluate the following:
a)

$$\int_{-1}^{1} \log z \mathrm{d}z$$

Solution:
b)

$$\int_{-1}^{1} z^{\frac{1}{2}} \mathrm{d}z$$

Solution:

3: From A&F: 2.4.7
Let C be an open (upper) semicircle of radius R with its center at the origin, and consider $\int_C f(z)\mathrm{d}z$. Let $f(z) = \frac{1}{z^2+a^2}$ for a real $a > 0$. Show that $|f(z)| \leq \frac{1}{R^2-a^2}$, $R > a$, and

$$\left| \int_C f(z)dz \right| \leq \frac{\pi R}{R^2 - a^2}, \quad R > a.$$

Solution:

4: From A&F: 2.4.8
Let C be an arc of the circle $|z| = R(R > 1)$ of angle $\frac{\pi}{3}$. Show that

$$\left| \int_C \frac{\mathrm{d}z}{z^3+1} \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3-1} \right)$$

and deduce

$$\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{z^3+1} = 0$$

Solution:

5: From A&F: 2.5.1 b, e
Evaluate $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:
b)

$$f(z) = \mathrm{e}^{z^2}$$

Solution:
e)

$$f(z) = \frac{1}{2z^2+1}$$

Solution:

6: Use the ideas from A&F: 2.5.5 to evaluate $\int_0^\infty \mathrm{e}^{\mathrm{i}z^3 t}\,\mathrm{d}z$, $t > 0$. Express the result in terms of $\int_0^\infty \mathrm{e}^{-r^3}\,\mathrm{d}r$.
The ideas we might need to use are ... it's actually really long!
Solution:

7: From A&F: 2.5.6.
Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}.$$

Show how to evaluate this integral by considering

$$\oint_{C_{(\mathbb{R})}} \frac{\mathrm{d}z}{z^2+1},$$

where $C_{(\mathbb{R})}$ is closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x-axis. *Hint:* use

$$\frac{1}{z^2+1} = -\frac{1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables. *Solution:*

Repeat this exercise for

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{\epsilon \mathrm{d}x}{x^2+\epsilon^2}, \quad \epsilon > 0.$$

Seems like I am supposed to do 2.5.6 and then for the given integral as well.

Solution:

8: Use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.
Solution:

9: From A&F: 2.6.1 a, e.
Evaluate the integrals $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin and $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):
a)

$$\frac{\sin z}{z}$$

Solution:
e)

$$\mathrm{e}^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)$$

Solution: