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AMATH 567

## HOMEWORK 2

Collaborators\*: TBD

\*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

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1: From A&F: 1.2.12.

Show that a circle in the  $z$  plane corresponds to a circle on the sphere. (Note the remark following the reference to Figure 1.2.7 in Section 1.2.2)

Hints from Office Hours:

- Use 1.2.25 equations
- perhaps start with a circle centered at the origin for the intuition
- show the intersection of the plane with a sphere is a circle

*Solution:*

Let's first define an arbitrary complex number  $z = x + iy$  and the center of our circle in the complex plane  $c = a + ib$ . The following is the equation for a circle on the complex plane

$$\begin{aligned} |z - c|^2 &= r^2 \\ |x + iy - a - ib|^2 &= r^2 \\ |(x - a) + i(y - b)|^2 &= r^2 \\ \sqrt{(x - a)^2 + (y - b)^2} &= r \\ (x - a)^2 + (y - b)^2 &= r^2 \end{aligned}$$

From the discussion in A&F pg16-18, we know that we have the following formulae for converting between the complex plane and the 3 space  $(X, Y, Z)$

$$X = \frac{4x}{|z|^2 + 4} \quad Y = \frac{4y}{|z|^2 + 4} \quad Z = \frac{2|z|^2}{|z|^2 + 4}.$$

Let's combine these

$$\begin{aligned} (x - a)^2 + (y - b)^2 &= r^2 \\ x^2 - 2ax + a^2 + y^2 - 2by + b^2 &= r^2 \\ x^2 + y^2 - 2ax + a^2 - 2by + b^2 &= r^2 \\ x^2 + y^2 &= r^2 + 2ax - a^2 + 2by - b^2 \\ |z|^2 &= r^2 + 2ax - a^2 + 2by - b^2 \end{aligned}$$

We want to find  $A$ ,  $B$ ,  $C$ , and  $D$  that are real s.t.

$$AX + BY + CZ = D$$

$$\begin{aligned} X &= \frac{4x}{r^2 + 2ax - a^2 + 2by - b^2 + 4} \\ Y &= \frac{4y}{r^2 + 2ax - a^2 + 2by - b^2 + 4} \\ Z &= \frac{2(r^2 + 2ax - a^2 + 2by - b^2)}{r^2 + 2ax - a^2 + 2by - b^2 + 4} \end{aligned}$$

Then our equation is

$$\begin{aligned} & A \frac{4x}{r^2 + 2ax - a^2 + 2by - b^2 + 4} \\ & + B \frac{4y}{r^2 + 2ax - a^2 + 2by - b^2 + 4} \\ & + C \frac{2(r^2 + 2ax - a^2 + 2by - b^2)}{r^2 + 2ax - a^2 + 2by - b^2 + 4} \\ & = D \end{aligned}$$

We have projected the circle from the complex plane onto the sphere. Notice we've constructed this equation showing they all lie in the plane. The intersection of the sphere and this plane is a subset of a circle on the plane. Since we were working with equalities throughout it is actually the whole circle on the sphere. **Almost Complete**

## 2: From A&F: 1.3.5.

Show that the functions  $\Re(z)$  and  $\Im(z)$  are nowhere differentiable.

*Solution:*

**Part 1:**  $f(z) = \Re(z)$

Let's begin with the definition of the derivative

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{f(x_z + x_h + i(y_z + y_h)) - x_z}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_z + x_h - x_z}{x_h + iy_h} \\ &= \lim_{h \rightarrow 0} \frac{x_h}{x_h + iy_h} = \frac{0}{0}. \end{aligned}$$

Now we want to use L'Hôpital's rule so we create two cases fixing  $y_h$  and  $x_h$  each in turn. Starting from where we left off

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{x_h}{x_h + iy_h} &= \lim_{h \rightarrow 0} \frac{\frac{d}{dx_h} x_h}{\frac{d}{dx_h} (x_h + iy_h)} = \lim_{h \rightarrow 0} \frac{1}{1} = 1 \text{ when } y_h \text{ is fixed} \\ \lim_{h \rightarrow 0} \frac{x_h}{x_h + iy_h} &= \lim_{h \rightarrow 0} \frac{\frac{d}{dy_h} x_h}{\frac{d}{dy_h} (x_h + iy_h)} = \lim_{h \rightarrow 0} \frac{0}{i} = 0 \text{ when } x_h \text{ is fixed.} \end{aligned}$$

Since we get a different result at an arbitrary  $z$  depending on which  $x_h$  and  $y_h$  is currently fixed the limit does not exist and therefore  $f(z) = \Re(z)$  is nowhere differentiable.  $\square$

**Part 2:**  $f(z) = \Im(z)$

For this one we will use a different method. Recall that the Cauchy-Riemann equations are a necessary condition that must hold if  $f(z)$  is differentiable (A&F pg. 33).

Therefore we can show that the Cauchy-Riemann equations do not hold and therefore the function is non differentiable. We use the fact that  $f(z) = u(x, y) + iv(x, y)$  and

$$f(z) = \Im(z) = y \quad \text{since } z = x + iy$$

to get

$$\begin{aligned} u(x, y) &= y \\ v(x, y) &= 0. \end{aligned}$$

Now we want to check if both of the following hold

$$u_x = v_y$$

$$v_x = -u_y.$$

Let's calculate them

$$u_x = 0, v_y = 0, v_x = 0, u_y = 1$$

Therefore the first condition holds

$$\begin{aligned} u_x &= v_y \\ 0 &= 0 \end{aligned}$$

however, the second does not

$$\begin{aligned} v_x &= -u_y \\ 0 &\neq -1. \end{aligned}$$

Therefore  $f(z) = \Im(z)$  is nowhere differentiable.

□

**3:** Consider the function

$$\varphi(z) = z + \sqrt{z^2 - 1}, \quad z > 1.$$

Show that

$$\log \varphi(z) = \int_1^z \frac{dx}{\sqrt{x^2 - 1}}.$$

*Solution:*

Let's begin from evaluating the integral on the right and showing that the result is

what we have on the left. First we substitute  $x = \sec \theta$ ,  $dx = \sec \theta \tan \theta d\theta$

$$\begin{aligned}
 \int_1^z \frac{dx}{\sqrt{x^2 - 1}} &= \int_0^{\operatorname{arcsec} z} \frac{\sec \theta \tan \theta d\theta}{\sqrt{\sec^2 \theta - 1}} \\
 &= \int_0^{\operatorname{arcsec} z} \frac{\sec \theta \tan \theta d\theta}{\sqrt{\tan^2 \theta}} \\
 &= \int_0^{\operatorname{arcsec} z} \frac{\sec \theta \tan \theta d\theta}{\tan \theta} \\
 &= \int_0^{\operatorname{arcsec} z} \sec \theta d\theta \\
 &= \log |\sec \theta + \tan \theta| \Big|_0^{\operatorname{arcsec} z} \\
 &= \log |\sec(\operatorname{arcsec} z) + \tan(\operatorname{arcsec} z)| - \log |\sec 0 + \tan 0| \\
 &= \log \left| z + \sqrt{\tan^2(\operatorname{arcsec} z)} \right| - \log |1 + 0| \\
 &= \log \left| z + \sqrt{\sec^2(\operatorname{arcsec} z) - 1} \right| \\
 &= \log \left| z + \sqrt{z^2 - 1} \right| \\
 &= \log \left| z + \sqrt{z^2 - 1} \right| \\
 &= \log |\varphi(z)| \\
 &= \log(\varphi(z)) \quad \text{since } z > 1 \text{ then } \varphi(z) > 0
 \end{aligned}$$

□

- 4:** Find all zeroes of  $\tan(z)$ ,  $z \in \mathbb{C}$ . What can you conclude about the zeroes of  $\tanh(z) = \sinh(z)/\cosh(z)$ ,  $z \in \mathbb{C}$ ?

*Solution:*

Let's have some fun with trig identities

$$\begin{aligned}
 \tan(z) &= \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin x \cos iy + \cos x \sin iy}{\cos x \cos iy - \sin x \sin iy} \\
 &= \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y} \\
 &= \frac{\sin x \cosh y + i \cos x \sinh y (\cos x \cosh y + i \sin x \sinh y)}{\cos x \cosh y - i \sin x \sinh y (\cos x \cosh y + i \sin x \sinh y)} \\
 &= \frac{\sin x \cosh y + i \cos x \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} (\cos x \cosh y + i \sin x \sinh y) \\
 &= \frac{\sin x \cos x \cosh^2 y + i \cos^2 x \sinh y \cosh y + i \sin^2 x \cosh y \sinh y - \cos x \sin x \sinh^2 y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\
 &= \frac{(\sin x \cos x \cosh^2 y - \cos x \sin x \sinh^2 y) + i(\cos^2 x \sinh y \cosh y + \sin^2 x \cosh y \sinh y)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\
 &= \frac{\sin x \cos x (\cosh^2 y - \sinh^2 y) + i \sinh y \cosh y (\cos^2 x + \sin^2 x)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\
 &= \frac{\sin x \cos x (\cosh^2 y - \sinh^2 y) + i \sinh y \cosh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}
 \end{aligned}$$

is this true  $\cosh^2 y - \sinh^2 y = 1$  ?

□

- 5:** Consider  $f_\epsilon(z) = \epsilon / (\epsilon^2 + z^2)$ , where  $\epsilon$  is a small positive number, and  $z \in \mathbb{C} \setminus \{i\epsilon, -i\epsilon\}$ . Plot  $|f_\epsilon(z)|$ , for various values of  $\epsilon$ . Discuss the influence the singularities of a function in the complex plane have on its behavior on the real line. Compute

$$\int_{-\infty}^{\infty} f_\epsilon(x) dx.$$

*Solution:*

**Incomplete**

- 6:** Visualizing complex functions is not as easy as visualizing real-valued functions, since we need 4 dimensions: two for the input, two for the output. Different visualizations are commonly used, such as showing 3-dimensional plots of the real and imaginary parts. Plotting the modulus is informational, but it eliminates a lot of information.
- To see this, plot the real and imaginary part of the exponential function  $\exp(z) = \exp(x+iy)$ , for  $x \in [-1, 1], y \in [-2\pi, 2\pi]$ . Now plot the modulus over the same region, and compare.
  - A "new" popular way to do this is to plot the modulus of the function with the color defined by the phase. The Digital Library of Mathematical Functions has lots of examples. Create a plot of the  $|\exp(z)| = |\exp(x+iy)|$ , for  $x \in [-1, 1], y \in [-2\pi, 2\pi]$ . colored by the argument. Experimenting with other functions is highly encouraged! The book *visual Complex Functions: An Introduction with*

Phase Portraits by Elias Wegert (Birkhäuser, 2012) is a good companion to our textbook, if you think geometrically.

*Solution:*

**Incomplete**

**7:** From A&F: 2.1.1

Which of the following satisfy the Cauchy-Riemann (C-R) equations? If they satisfy the C-R equations, give the analytic function of  $z$ .

a)  $f(x, y) = x - iy + 1$

*Solution:*

Identify  $u$  and  $v$  and their partial derivatives

$$u(x, y) = x + 1 \implies u_x = 1, u_y = 0$$

$$v(x, y) = -y \implies v_x = 0, v_y = -1$$

Therefore,  $v_x = 0 = -0 = -u_y$  holds, however  $u_x = 1 \neq -1 = v_y$  does not hold. In conclusion  $f(x, y) = x - iy + 1$  does not satisfy the Cauchy-Riemann (C-R) equations.  $\square$

b)  $f(x, y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$

*Solution:*

Identify  $u$  and  $v$  and their partial derivatives

$$u(x, y) = y^3 - 3x^2y \implies u_x = -6xy, u_y = 3y^2 - 3x^2$$

$$v(x, y) = x^3 - 3xy^2 + 2 \implies v_x = 3x^2 - 3y^2, v_y = -6xy$$

Therefore,  $v_x = 3x^2 - 3y^2 = -(3y^2 - 3x^2) = -u_y$  and  $u_x = -6xy = -6xy = v_y$  both hold. In conclusion  $f(x, y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$  satisfies the Cauchy-Riemann (C-R) equations. To determine the analytic function notice

$$\begin{aligned} f(x, y) &= y^3 - 3x^2y + i(x^3 - 3xy^2 + 2) \\ &= y^3 - 3x^2y + ix^3 - i3xy^2 + i2 \end{aligned}$$

$$\begin{aligned} z^3 &= (x + iy)^3 \\ &= (x + iy)(x + iy)(x + iy) \\ &= (x^2 + i2xy - y^2)(x + iy) \\ &= x^3 + i3x^2y - 3xy^2 - iy^3 \\ &= -i(ix^3 - 3x^2y - i3xy^2 + y^3) \\ &= -i(y^3 - 3x^2y + ix^3 - i3xy^2) \\ &= -i(f(x, y) - 2i). \end{aligned}$$

Continuing

$$\begin{aligned} z^3 &= -i(f(x, y) - 2i) \\ \frac{z^3}{-i} &= f(x, y) - 2i \\ \frac{z^3}{-i} \cdot \frac{i}{i} &= f(x, y) - 2i \\ iz^3 &= f(x, y) - 2i \\ iz^3 + 2i &= f(x, y). \end{aligned}$$

Therefore, the analytic function of  $z$  is  $f(z) = iz^3 + 2i$ .

□

c)  $f(x, y) = e^y(\cos x + i \sin y)$

*Solution:*

Identify  $u$  and  $v$  and their partial derivatives

$$f(x, y) = e^y(\cos x + i \sin y) = e^y \cos x + ie^y \sin y$$

$$u(x, y) = e^y \cos x \implies u_x = -e^y \sin x, u_y = e^y \cos x$$

$$v(x, y) = e^y \sin y \implies v_x = 0, v_y = e^y \sin y + e^y \cos y$$

Therefore neither

$$v_x = 0 \neq -e^y \cos x = -u_y$$

nor

$$u_x = -e^y \sin x \neq e^y \sin y + e^y \cos y = v_y$$

hold. In conclusion  $f(x, y) = e^y(\cos x + i \sin y)$  does not satisfy the Cauchy-Riemann (C-R) equations.

□

### 8: From A&F: 2.1.7

Consider the complex analytic function,  $\Omega(z) = \phi(x, y) + i\psi(x, y)$ , in a domain  $D$ . Let us transform from  $z$  to  $w$  using  $w = f(z)$ ,  $w = u + iv$  where  $f(z)$  is analytic in  $D$ , with the corresponding domain in the  $w$  plane,  $D'$ . Establish the following:

$$\frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial \phi}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial \phi}{\partial v}$$

*Solution:*

Does this mean

$$\Omega(f(z)) = \phi(u(x, y), v(x, y)) + i\psi(u(x, y), v(x, y))$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \frac{\partial \phi}{\partial u} - \frac{\partial^2 u}{\partial x \partial y} \frac{\partial \phi}{\partial v} + \left( \frac{\partial v}{\partial x} \right)^2 \frac{\partial^2 \phi}{\partial u^2} - 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial^2 \phi}{\partial u \partial v} + \left( \frac{\partial v}{\partial y} \right)^2 \frac{\partial^2 \phi}{\partial v^2}$$

Also find the corresponding formulae for  $\frac{\partial \phi}{\partial y}$  and  $\frac{\partial^2 \phi}{\partial y^2}$ . Recall that  $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y}$ , and  $u(x, y)$  satisfies Laplace's equation the domain  $D$ . Show that

$$\nabla_{x,y}^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u_x^2 + y_y^2) \left( \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right) = |f'(z)|^2 \nabla_{u,v}^2 \phi$$

Consequently, we find that if  $\phi$  satisfies Laplace's equation  $\nabla_{x,y}^2 \phi = 0$  in the domain  $D$ , then so long as  $f'(z) \neq 0$  in  $D$  it also satisfies Laplace's equation  $\nabla_{u,v}^2 \phi = 0$  in domain  $D'$ .

**Hint:** No Hints yet...

*Solution:*

**Incomplete**

**9:** Show that the derivative of  $f(z) = |z|^2$  is defined at  $z = 0$ , but nowhere else.

*Solution:*

Once again we are going to use the C-R equations and the fact that satisfying them is a necessary condition for differentiability.

$$\begin{aligned} f(z) &= |z|^2 \\ &= \sqrt{x^2 + y^2}^2 \\ &= x^2 + y^2 \\ &= x^2 + y^2 + i \cdot 0 \end{aligned}$$

Therefore  $u(x, y) = x^2 + y^2$  and  $v(x, y) = 0$ . Now let's calculate the necessary partials

$$u_x = 2x$$

$$v_y = 0$$

$$v_x = 0$$

$$u_y = 2y.$$

Now we need the following to hold for  $f(z)$  to be differentiable

$$u_x = v_y$$

$$2x = 0$$

and

$$v_x = -u_y$$

$$0 = -2y.$$

Both of these only hold if  $x = 0$  and  $y = 0$  or in other words if  $z = 0$ . Which means the C-R equations only hold at  $z = 0$  and the derivative of  $f(z) = |z|^2$  is defined at  $z = 0$  but nowhere else.

□

**10:** Derive the polar-coordinates form of the Cauchy-Riemann equations

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta.$$

where  $x = r \cos \theta$  and  $y = r \sin \theta$

*Solution:*

We know we can represent  $f(z) = u(x, y) + iv(x, y)$ . Let's make a substitution to polar coordinates

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ &= u(r \cos \theta, r \sin \theta) + iv(r \cos \theta, r \sin \theta) \\ &= U(r, \theta) + iV(r, \theta) \end{aligned}$$

We need to calculate the derivatives using the chain rule

$$U_r = u_x x_r + u_y y_r$$

$$U_\theta = u_x x_\theta + u_y y_\theta$$

$$V_r = v_x x_r + v_y y_r$$

$$V_\theta = v_x x_\theta + v_y y_\theta.$$



Now we compute the necessary partials and plug them back in

$$\begin{aligned}x_r &= \cos \theta \\x_\theta &= -r \sin \theta \\y_r &= \sin \theta \\y_\theta &= r \cos \theta\end{aligned}$$

$\downarrow$

$$\begin{aligned}U_r &= u_x \cos \theta + u_y \sin \theta \\U_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\V_r &= v_x \cos \theta + v_y \sin \theta \\V_\theta &= -v_x r \sin \theta + v_y r \cos \theta.\end{aligned}$$

Now try to make them look like each other with the C-R equations

$$\begin{aligned}U_r &= u_x \cos \theta + u_y \sin \theta \\&= v_y \cos \theta + u_y \sin \theta\end{aligned}$$

then

$$\begin{aligned}V_\theta &= -v_x r \sin \theta + v_y r \cos \theta \\&= u_y r \sin \theta + v_y r \cos \theta \\&= r(u_y \sin \theta + v_y \cos \theta) \\&= r(v_y \cos \theta + u_y \sin \theta) \\&= rU_r\end{aligned}$$

Therefore  $U_r = \frac{1}{r}V_\theta$ .

$$\begin{aligned}V_r &= v_x \cos \theta + v_y \sin \theta \\&= -u_y \cos \theta + v_y \sin \theta\end{aligned}$$

then

$$\begin{aligned}U_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\&= -v_y r \sin \theta + u_y r \cos \theta \\&= -r(v_y \sin \theta - u_y \cos \theta) \\&= -r(-u_y \cos \theta + v_y \sin \theta) \\&= -rV_r\end{aligned}$$

Therefore  $V_r = -\frac{1}{r}U_\theta$ . And with this we have derived the polar-coordinates form of the Cauchy-Riemann equations.

□