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AMATH 561

PROBLEM SET 8

Note: Exercises are from Matt Lorig's notes (link on course website).

1. Exercise 5.1. Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $X = (X_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution $\boldsymbol{\pi}$ if and only if $\lambda < \mu$. Find $\boldsymbol{\pi}$. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Hint: You can use Little's law, which states that the expected number of people in the hospital at steady-state is equal to the average arrival rate multiplied by the average processing time.

Solution:
The generator for X is

$$\boldsymbol{G} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \cdots \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \ddots \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Now if the invariant distribution $\boldsymbol{\pi}$ exists then $\boldsymbol{\pi G} = \boldsymbol{0}$. Let's look at what conditions would need to hold for this $\boldsymbol{\pi}$ to exist. First, looking at $\boldsymbol{\pi G}$ we have

$$\begin{aligned} 0 &= -\lambda\pi(0) + \mu\pi(1) \\ 0 &= \lambda\pi(0) - (\mu+\lambda)\pi(1) + \mu\pi(2) \\ 0 &= \lambda\pi(1) - (\mu+\lambda)\pi(2) + \mu\pi(3) \\ &\vdots \\ 0 &= \lambda\pi(n-1) - (\mu+\lambda)\pi(n) + \mu\pi(n+1). \end{aligned}$$

Then we can say

$$\pi(1) = \frac{\lambda}{\mu}\pi(0), \quad \pi(2) = \frac{\lambda^2}{\mu^2}\pi(0), \quad \ldots, \quad \pi(n) = \frac{\lambda^n}{\mu^n}\pi(0).$$

If $\boldsymbol{\pi}$ is a stationary distribution then the row vector needs to sum to one so we have the condition

$$\sum_{n=0}^{\infty} \pi(0)\left(\frac{\lambda}{\mu}\right)^{n} = 1.$$

This sum is finite if and only if

$$\left|\frac{\lambda}{\mu}\right| < 1 \implies \lambda < \mu.$$

Furthermore, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \pi(0)\left(\frac{\lambda}{\mu}\right)^{n} &= 1 \\ \pi(0)\frac{1}{1-\frac{\lambda}{\mu}} &= 1 \\ \pi(0) &= 1-\frac{\lambda}{\mu}. \end{aligned}$$

Hence,

$$\pi(n) = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n}.$$

Therefore, we have found the stationary distribution $\boldsymbol{\pi}$ which only exists if and only if the condition that $\lambda < \mu$ since the sum of the entries of the vector $\boldsymbol{\pi}$ is only finite in this scenario.

Now I need to find the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution using Little's law which gives

$$E\left(X_t\right) = \lambda E\left(\tau\right).$$

Then we have

$$\begin{aligned}
E(\tau) &= \frac{1}{\lambda} E(X_t) \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} n\pi(n) \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} n \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n \\
&= \frac{1}{\lambda} \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \\
&= \frac{1}{\lambda} \left(1 - \frac{\lambda}{\mu}\right) \frac{\frac{\lambda}{\mu}}{\left(1 - \frac{\lambda}{\mu}\right)^2} \\
&= \frac{1}{\lambda} \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} \\
&= \frac{\frac{1}{\mu}}{1 - \frac{\lambda}{\mu}} \\
&= \frac{1}{\mu - \lambda}.
\end{aligned}$$

Therefore, the expected time (waiting + treatment) a patient waits when the system is in its invariant distribution is $1/(\mu - \lambda)$.

$\square$

2. Exercise 5.3. Let $X = (X_t)_{t\geq 0}$ be a Markov chain with state space $S = \{0,1,2,\dots\}$ and with a generator $\mathbf{G}$ whose ith row has entries

$$g_{i,i-1} = i\mu, \quad g_{i,i} = -i\mu - \lambda, \quad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_T}(s) := E(s^{X_t})$. What is the distribution of X_t as $t \to \infty$?

Solution:
For my sake I am going to write down the generator $\boldsymbol{G}$

$$\boldsymbol{G} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \ddots \\ 0 & 0 & 3\mu & -(3\mu+\lambda) & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then we want to calculate $G_{X_T}(s) := E(s^{X_t})$. This takes the form

$$E(s^{X_t}) = \sum_{n=0}^{\infty} s^n P(X_t = n),$$

but we need to use the Kolmogorov forward equation to find these $P(X_t = n)$ terms. The forward equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_t = \boldsymbol{P}_t\boldsymbol{G}$$

which more explicitly is

$$\begin{bmatrix} p_t'(0,0) & p_t'(0,1) & p_t'(0,2) & \dots \\ p_t'(1,0) & p_t'(1,1) & p_t'(1,2) & \dots \\ p_t'(2,0) & p_t'(2,1) & p_t'(2,2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} p_t(0,0) & p_t(0,1) & p_t(0,2) & \dots \\ p_t(1,0) & p_t(1,1) & p_t(1,2) & \dots \\ p_t(2,0) & p_t(2,1) & p_t(2,2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -\lambda & \lambda & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & \ddots \\ 0 & 2\mu & -(2\mu+\lambda) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

This results in the following system of differential equations

$$\begin{aligned} p_t'(0,0) &= -\lambda p_t(0,0) + \mu p_t(0,1) \\ p_t'(0,1) &= \lambda p_t(0,0) - (\mu+\lambda)p_t(0,1) + 2\mu p_t(0,2) \\ p_t'(0,2) &= \lambda p_t(0,1) - (2\mu+\lambda)p_t(0,2) + 3\mu p_t(0,3) \\ &\vdots \\ p_t'(0,n) &= \lambda p_t(0,n-1) - (n\mu+\lambda)p_t(0,n) + (n+1)\mu p_t(0,n+1). \end{aligned}$$

As it turns out this is true for any starting point instead of just 0, therefore, we have

$$\begin{aligned}
p_t'(j,0) &= -\lambda p_t(j,0) + \mu p_t(j,1)\\
p_t'(j,1) &= \lambda p_t(j,0) - (\mu+\lambda)p_t(j,1) + 2\mu p_t(j,2)\\
p_t'(j,2) &= \lambda p_t(j,1) - (2\mu+\lambda)p_t(j,2) + 3\mu p_t(j,3)\\
&\ \vdots\\
p_t'(j,n) &= \lambda p_t(j,n-1) - (n\mu+\lambda)p_t(j,n) + (n+1)\mu p_t(j,n+1).
\end{aligned}$$

Multiplying through by s^n and summing over all n we have

$$\begin{aligned}
\sum_{n=0}^{\infty} s^n p_t'(j,n) &= \lambda \sum_{n=0}^{\infty} s^n p_t(j,n-1) - \sum_{n=0}^{\infty} s^n (n\mu+\lambda)p_t(j,n) + \sum_{n=0}^{\infty} s^n (n+1)\mu p_t(j,n+1)\\
\frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s \sum_{n=0}^{\infty} s^{n-1} p_t(j,n-1) - \sum_{n=0}^{\infty} s^n (n\mu+\lambda)p_t(j,n) + \mu \frac{\partial G_{X_T}(s)}{\partial s}\\
\frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \sum_{n=0}^{\infty} s^n (n\mu+\lambda)p_t(j,n) + \mu \frac{\partial G_{X_T}(s)}{\partial s}\\
\frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \sum_{n=0}^{\infty} s^n n\mu p_t(j,n) - \sum_{n=0}^{\infty} s^n \lambda p_t(j,n) + \mu \frac{\partial G_{X_T}(s)}{\partial s}\\
\frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \mu s \sum_{n=0}^{\infty} n s^{n-1} p_t(j,n) - \lambda G_{X_T}(s) + \mu \frac{\partial G_{X_T}(s)}{\partial s}\\
\frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \mu s \frac{\partial G_{X_T}(s)}{\partial s} - \lambda G_{X_T}(s) + \mu \frac{\partial G_{X_T}(s)}{\partial s}\\
\frac{\partial G_{X_T}(s)}{\partial t} &= (s-1)\lambda G_{X_T}(s) + (1-s)\mu \frac{\partial}{\partial s} G_{X_T}(s).
\end{aligned}$$

Suppressing some notation for convenience, we now need to solve the following differential equation with initial condition of $G_{X_0}(s) = s^j$

$$\frac{\partial}{\partial t} G = (s-1)\lambda G + (1-s)\mu \frac{\partial}{\partial s} G.$$

Using mathematica to solve this pde I ended up with

$$G_{X_t}(s) = \exp\left(\frac{\mathrm{e}^{-t\mu}(-1+\mathrm{e}^{t\mu})(-1+s)\lambda}{\mu}\right)\left(1+\mathrm{e}^{-t\mu}(-1+s)\right)^j.$$

Simplifying a little we have

$$G_{X_t}(s) = \exp\left(\frac{(1-\mathrm{e}^{-t\mu})(s-1)\lambda}{\mu}\right)\left(1+\mathrm{e}^{-t\mu}(s-1)\right)^j.$$

The generating function as

$$\begin{aligned}
\lim_{t\to\infty} G_{X_t}(s) &= \lim_{t\to\infty} \exp\left(\frac{(1-\mathrm{e}^{-t\mu})(s-1)\lambda}{\mu}\right)\left(1+\mathrm{e}^{-t\mu}(s-1)\right)^j \\
&= \exp\left(\frac{(1-0)(s-1)\lambda}{\mu}\right)(1+0(s-1))^j \\
&= \exp\left(\frac{(s-1)\lambda}{\mu}\right).
\end{aligned}$$

Let $G_X(s) = \exp\left(\frac{(s-1)\lambda}{\mu}\right)$ denote the generating function after taking the limit as t goes to infinity. We can compute the distribution from the generating function as

$$\begin{aligned}
p_n(t) &= \frac{1}{n!}\frac{\partial^n}{\partial s^n} G_X(s)\bigg|_{s=0} \\
&= \frac{1}{n!}\frac{\partial^n}{\partial s^n} \exp\left(\frac{(s-1)\lambda}{\mu}\right)\bigg|_{s=0} \\
&= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \exp\left(\frac{(s-1)\lambda}{\mu}\right)\bigg|_{s=0} \\
&= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \exp\left(-\frac{\lambda}{\mu}\right) \\
&= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \mathrm{e}^{-\frac{\lambda}{\mu}}.
\end{aligned}$$

Which is a poisson distribution with rate λ/μ.

$\square$

3. Exercise 5.4. Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t+\mathrm{d}t)$ is $\lambda(t)\mathrm{d}t$ and the probability of two jumps in that interval of time is $\mathcal{O}(\mathrm{d}t^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1+t)$ show that $E(\tau_1) < \infty$ if and only if $c > 1$.

Solution:
Once again I think it is very helpful to write down the generator $\boldsymbol{G}$ for this scenario

$$\boldsymbol{G} = \begin{bmatrix} -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

For the forward Kolmogorov equation we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_t = \boldsymbol{P}_t\boldsymbol{G}$$

which more explicitly is

$$\begin{bmatrix} p'_t(0,0) & p'_t(0,1) & \dots \\ p'_t(1,0) & p'_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} p_t(0,0) & p_t(0,1) & \dots \\ p_t(1,0) & p_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

This results in the following system of differential equations

$$\begin{aligned} p'_t(0,0) &= -\lambda(t)p_t(0,0) \\ p'_t(0,1) &= \lambda(t)p_t(0,0) - \lambda(t)p_t(0,1) \\ &\vdots \\ p'_t(0,n) &= \lambda(t)p_t(0,n-1) - \lambda(t)p_t(0,n) \end{aligned}$$

Multiplying through by s^n and summing over all n we have

$$\begin{aligned} \sum_{n=0}^{\infty} s^n p'_t(0,n) &= \lambda(t)\sum_{n=0}^{\infty} s^n p_t(0,n-1) - \lambda(t)\sum_{n=0}^{\infty} s^n p_t(0,n) \\ \frac{\mathrm{d}}{\mathrm{d}t}G_{N_t}(s) &= \lambda(t)s\sum_{n=0}^{\infty} s^{n-1}p_t(0,n-1) - \lambda(t)G_{N_t}(s) \\ \frac{\mathrm{d}}{\mathrm{d}t}G_{N_t}(s) &= \lambda(t)sG_{N_t}(s) - \lambda(t)G_{N_t}(s) \\ \frac{\mathrm{d}}{\mathrm{d}t}G_{N_t}(s) &= (s-1)\lambda(t)G_{N_t}(s). \end{aligned}$$

Now we have the pde (again suppressing extra notation) $G_{N_0}(s) = s^0 = 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}G = (s-1)\lambda(t)G.$$

Let's solve this directly

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}G &= (s-1)\lambda(t)G \\
\frac{\frac{\mathrm{d}}{\mathrm{d}t}G}{G} &= (s-1)\lambda(t) \\
\int \frac{\frac{\mathrm{d}}{\mathrm{d}t}G}{G}\mathrm{d}t &= \int (s-1)\lambda(t)\mathrm{d}t \\
\log(G) &= \int (s-1)\lambda(t)\mathrm{d}t \\
G &= \mathrm{e}^{\int (s-1)\lambda(t)\mathrm{d}t}.
\end{aligned}$$

TODO: You need to solve this!! Use the initial condition $N_0 = 0$ and the function given for $\lambda(t)$. For the backward Kolmogorov equation we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_t = \boldsymbol{G}\boldsymbol{P}_t$$

which more explicitly is

$$\begin{bmatrix} p_t'(0,0) & p_t'(0,1) & \dots \\ p_t'(1,0) & p_t'(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} p_t(0,0) & p_t(0,1) & \dots \\ p_t(1,0) & p_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

This results in the following system of differential equations

$$\begin{aligned}
p_t'(0,0) &= -\lambda(t)p_t(0,0) + \lambda(t)p_t(1,0) \\
p_t'(0,1) &= -\lambda(t)p_t(0,1) + \lambda(t)p_t(1,1) \\
p_t'(0,2) &= -\lambda(t)p_t(0,2) + \lambda(t)p_t(1,2) \\
&\vdots \\
p_t'(0,n) &= -\lambda(t)p_t(0,n) + \lambda(t)p_t(1,n).
\end{aligned}$$

We also have

$$\begin{aligned}
p_t'(1,0) &= -\lambda(t)p_t(1,0) + \lambda(t)p_t(2,0) \\
p_t'(1,1) &= -\lambda(t)p_t(1,1) + \lambda(t)p_t(2,1) \\
p_t'(1,2) &= -\lambda(t)p_t(1,2) + \lambda(t)p_t(2,2) \\
&\vdots \\
p_t'(1,n) &= -\lambda(t)p_t(1,n) + \lambda(t)p_t(1,n).
\end{aligned}$$

I'm not totally sure how to combine these, however... Multiplying through by s^n and summing over all n for the set of pde's where $N_0 = 0$

$$\begin{aligned}
\sum_{n=0}^{\infty} s^n p_t'(0,n) &= -\lambda(t)\sum_{n=0}^{\infty} s^n p_t(0,n) + \lambda(t)\sum_{n=0}^{\infty} s^n p_t(1,n) \\
\frac{\partial}{\partial t}G_{N_t}(s) &= -\lambda(t)G_{N_t}(s) + \lambda(t)\sum_{n=0}^{\infty} s^n p_t(1,n)
\end{aligned}$$

How do I handle this thing with the incrementing in the first index not the second?

4. Exercise 5.5. Let N be a poisson process with a random intensity Λ witch is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = E(s^{N_t})$. What is the mean and variance of N_t?

Solution:
TODO:

$$Here you$$