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AMATH 567

HOMEWORK 8

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: The Korteweg-de Vries (KdV) equation arises whenever long waves of moderate amplitude in dispersive media are considered. For instance, it describes waves in shallow water, and ion-acoustic waves in plasmas. The equation is given by

$$u_t = 6uu_x + u_{xxx},$$

where indices denote partial differentiation.

(a) By looking for solutions $u(x,t) = U(x)$, derive a first-order ordinary differential equation for $U(x)$. Introduce integration constants as required.

Solution:
Since we want to find time independent ODE such that $\frac{\mathrm{d}}{\mathrm{d}t}U(x) = 0$, then we need

$$0 = 6uu_x + u_{xxx}.$$

Integrating gives us

$$\begin{aligned} \int 0 &= 6\int uu_x \mathrm{d}x + \int u_{xxx}\mathrm{d}x \\ 0 &= 6\int uu_x \mathrm{d}x + u_{xx} + C_1. \end{aligned}$$

Using a substitution $v = u$ and $\mathrm{d}v = u_x \mathrm{d}x$ we have

$$\begin{aligned} 0 &= 6\int uu_x \mathrm{d}x + u_{xx} + C_1 \\ 0 &= 6\int v\mathrm{d}v + u_{xx} + C_1 \\ 0 &= 6(3v^2) + C_2 + u_{xx} + C_1 \\ 0 &= 18u^2 + C_2 + u_{xx} + C_1 \\ 0 &= 18u^2 + u_{xx} + C. \end{aligned}$$

Hence

$$U(x) = 18u^2 + u_{xx} + C.$$

□

(b) Let $U = U_0\wp(x - x_0)$. Determine U_0 so that $u = U(x)$ solves the KdV equation.

Solution:

$$18u^2 + u_{xx} + C = U_0 \wp(x - x_0)$$

$$This$$

2: From A&F: 3.6.5
Show that if $f(z)$ is meromorphic in the finite z plane, then $f(z)$ must be the ratio of two entire functions.

Solution:
TODO: multiply by something that is going to knock out all of the poles and leave you something that is entire. You need to construct something with zeros of the appropriate order to give you what you want. Definitely use the mittag lefler expansion to have zeros where you want and the right residual that you want.

this

3: Here's a way to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

due to Euler. We've seen that

$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

(a) Equate the coefficients of z^2 on both sides, to recover the desired sum.

Solution:
Taylor expand on the left to get

$$\begin{aligned}
\frac{\sin \pi z}{\pi z} &= \frac{1}{\pi z} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi z)^{2j+1}}{(2j+1)!} \\
&= \frac{1}{\pi z} \left(\pi z - \frac{(z\pi)^3}{6} + \frac{(z\pi)^5}{120} - ...\right) \\
&= 1 - \frac{z^2 \pi^2}{6} + \frac{z^4 \pi^4}{120} - ...
\end{aligned}$$

Now expand out several terms in the product on the right

$$\begin{aligned}
\prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right) &= (1 - z^2) \left(1 - \frac{z^2}{4}\right) \left(1 - \frac{z^2}{9}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) ... \\
&= \left(1 - z^2 - \frac{z^2}{4} + \frac{z^4}{4}\right) \left(1 - \frac{z^2}{9}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) ... \\
&= \left(1 - z^2 - \frac{z^2}{4} + \frac{z^4}{4} - \frac{z^2}{9} + \frac{z^4}{9} + \frac{z^4}{36} - \frac{z^6}{36}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) ... \\
&= \left(1 + z^2\Big(-1 - \frac{1}{4} - \frac{1}{9}\Big) + z^4\Big(\frac{1}{4} + \frac{1}{9} + \frac{1}{36}\Big) - \frac{z^6}{36}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) ...
\end{aligned}$$

Then we have the coefficients for z^2 becomes the series $-\sum_{j=0}^{\infty} \frac{1}{j^2}$. Equating the coefficient on the left with the series on the right we have

$$\begin{aligned}
-\frac{\pi^2}{6} &= -\sum_{j=0}^{\infty} \frac{1}{j^2} \\
\frac{\pi^2}{6} &= \sum_{j=0}^{\infty} \frac{1}{j^2}
\end{aligned}$$

$\square$

(b) Equate the coefficients of z^4 on both sides to recover a different sum.

Solution:
Using the results from the Taylor expansion on the left from part (a) we have the coefficient of the z^4 term is $\frac{\pi^4}{120}$. Additionally, from expanding the first several terms in the product on the right we have that the coefficient of the z^4 term can be written as

$$\sum_{j=0}^{\infty}\sum_{k=1}^{j-1}\frac{1}{j^2}\frac{1}{k^2}.$$

Combining these we have

$$\frac{\pi^4}{120}=\sum_{j=0}^{\infty}\sum_{k=1}^{j-1}\frac{1}{j^2}\frac{1}{k^2}.$$

A little work can be done to relate this to the sum

$$\sum_{k=1}^{\infty}\frac{1}{k^4}.$$

By equating coefficients of higher powers of z, one can recover other identities too.

$\square$

4: For the following, suppose that $f(z)$ is analytic in an open set Ω that contains $[-1,1]$.

(a) Show that there exists a contour C, encircling $[-1,1]$, such that

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

Solution:
For convenience, define $h(z)$ to be the integrand $h(z) = \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}$. Siting Homework 5 problem 5, let Σ define the same area as before

$$\Sigma = \{z \in \mathbb{C} \;:\; |\operatorname{Re} z| \le r \text{ and } 0 \le -\operatorname{Im} z \le R,\ R > 0,\ r > 1\}$$

and let $\partial\Sigma$ be the clockwise oriented contour along the boundary of the region Σ. Additionally let $\partial\Sigma \setminus [-1,1]$ bet he contour on the boundary without the section from -1 to 1 on the real line. From that same problem we know

$$\oint_{\partial\Sigma} h(z)\mathrm{d}z = 0.$$

Furthermore, we can say

$$\int_{-1}^{1} h(z)\mathrm{d}z + \oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z = 0$$

$$\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z = -\int_{-1}^{1} h(z)\mathrm{d}z. \tag{1}$$

We now define

$$\Sigma' = \{z \in \mathbb{C} \;:\; |\operatorname{Re} z| \le r \text{ and } 0 \le \operatorname{Im} z \le R,\ R > 0,\ r > 1\}$$

to be the upper half plane analogy of Σ. Therefore, let $\partial\Sigma'$ be the clockwise oriented contour along the boundary of the region Σ'.

$$g(z) = \begin{cases} h(z), & \text{if } \Im(z) > 0 \text{ or } |z| > 1 \\ -h(z), & \text{if } \Im(z) = 0 \text{ and } |z| \le 1 \end{cases}.$$

This helps us preserve the continuity we are concerned with in order to apply the same arguments from Homework 5 problem 5 and making use of lemma 1 from problem 4 of that same assignment. Then we can conclude

$$\oint_{\partial\Sigma'} g(z)\mathrm{d}z = 0.$$

Furthermore, we have

$$\int_{1}^{-1} g(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} g(z)\mathrm{d}z = 0$$

$$\oint_{\partial\Sigma'\setminus[-1,1]} g(z)\mathrm{d}z = -\int_{1}^{-1} g(z)\mathrm{d}z$$

$$\oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = -\int_{1}^{-1} -h(z)\mathrm{d}z$$

$$\oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = -\int_{-1}^{1} h(z)\mathrm{d}z. \tag{2}$$

If we add equation (1) and equation (2), we have

$$\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = -\int_{-1}^{1} h(z)\mathrm{d}z - \int_{-1}^{1} h(z)\mathrm{d}z$$

$$\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = -2\int_{-1}^{1} h(z)\mathrm{d}z.$$

Note the contours $\partial\Sigma \setminus [-1,1]$ and $\partial\Sigma' \setminus [-1,1]$ have small overlapping regions on the real axis which cancel out since they are of opposite orientation. We denote the combinations of these contours as $\partial\widehat{\Sigma}$ which is the clockwise oriented contour on the boundary of $\widehat{\Sigma}$ with

$$\widehat{\Sigma} = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \le r \text{ and } |\operatorname{Im} z| \le R,\ R > 0,\ r > 1\}.$$

Hence,

$$\begin{aligned}
\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z &= -2\int_{-1}^{1} h(z)\mathrm{d}z \\
\oint_{\partial\widehat{\Sigma}} h(z)\mathrm{d}z &= -2\int_{-1}^{1} h(z)\mathrm{d}z \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= -2\int_{-1}^{1} \frac{f(x)}{\sqrt{x-1}\sqrt{x+1}}\mathrm{d}x \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= -\frac{2}{\mathrm{i}}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= 2\mathrm{i}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x \\
\frac{1}{2\mathrm{i}}\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x.
\end{aligned}$$

Therefore the clockwise oriented contour on the boundary of $\widehat{\Sigma}$, denoted as $\partial\widehat{\Sigma}$ is one such contour encircling [-1, 1] such that

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\mathrm{i}}\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

$\square$

Note we can deform this contour $\partial\widehat{\Sigma}$ into a circle centered at $z = 0$ of radius $\rho > 1$.

(b) Use this to evaluate

$$I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}}, \quad I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x,$$
$$I_3 = \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\mathrm{d}x, \quad I_4 = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\mathrm{d}x,$$

without using any changes of variable (e.g., no trig subs!).

Solution:

TODO: Kind of use problem 8 from hw 6. Now deform the contour of a rectangle into a circle... maybe at infinity

TODO: Something weird is happening with orientation

Using part (a) and the substitution $z = \rho\,\mathrm{e}^{\mathrm{i}\theta}$

$$\begin{aligned}
I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} &= \frac{1}{2\mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= \frac{1}{2\mathrm{i}} \oint_0^{2\pi} \frac{\rho \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\sqrt{\rho\,\mathrm{e}^{\mathrm{i}\theta} - 1}\sqrt{\rho\,\mathrm{e}^{\mathrm{i}\theta} + 1}} \\
&= \oint_0^{2\pi} \frac{1}{2} \frac{\rho\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\sqrt{\rho^2\,\mathrm{e}^{2\mathrm{i}\theta} - 1}} \\
&= \left(\frac{1}{2\mathrm{i}\rho} \sqrt{\rho^2\,\mathrm{e}^{2\mathrm{i}\theta} - 1} \right) \Bigg|_0^{2\pi} \\
&= \left(\frac{1}{2\mathrm{i}\rho} \sqrt{\rho^2\,\mathrm{e}^{4\pi\mathrm{i}} - 1} - \frac{1}{2\mathrm{i}\rho} \sqrt{\rho^2\,\mathrm{e}^{0} - 1} \right) \\
&= \left(\frac{1}{2\mathrm{i}\rho} \sqrt{\rho^2 - 1} - \frac{1}{2\mathrm{i}\rho} \sqrt{\rho^2 - 1} \right) \\
&= \frac{1}{2\mathrm{i}\rho} \left(\sqrt{\rho^2 - 1} - \sqrt{\rho^2 - 1} \right) \\
&= 0...
\end{aligned}$$

This is wrong...

$$I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x...$$

Another one

$$I_3 = \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\mathrm{d}x...$$

one more

$$I_4 = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\,\mathrm{d}x...$$

5: Suppose, for $|z| = 1$, that the series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n,$$

converges uniformly.

(a) Compute series representations for

$$F(z) := \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \quad |z| \neq 1, \quad C = \partial B_1(0).$$

Solution:

TODO: two cases where $|z| < 1$ and $|z| > 1$

$$This$$

(b) For $|z| = 1$, compute

$$\lim_{\epsilon \to 0^+} F(z(1-\epsilon)) - \lim_{\epsilon \to 0^+} F(z(1+\epsilon)).$$

Solution:

TODO: should expect a "jump" discontinuity at the boundary

$$This$$