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 AMATH 561

## PROBLEM SET 1

1. Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.

a) *Solution*

Let's assign the probability of getting a heads when tossing this biased coin once as  $p_0$  and the probability of getting a tails as  $1 - p_0$ . Now to describe the probability space we need to define our outcome space  $\Omega$ , the set of relevant events  $\mathcal{F}$  (a  $\sigma$ -algebra) and the probability measure  $P$  then we will have our probability space  $(\Omega, \mathcal{F}, P)$ . In our experiment of 3 tosses of the coin we have

$$\Omega = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$$

And for simplicity let's define  $\mathcal{F} = 2^\Omega$  which means any possible subset of  $\Omega$  is an event. Now let's define  $P : \mathcal{F} \rightarrow \mathbb{R}$ . Let  $A \in \mathcal{F}$  then we say

$$P(A) = \sum_{\omega \in A} p(\omega).$$

Where  $p(\omega) = p_0^{h_\omega} (1 - p_0)^{t_\omega}$  with  $h_\omega$  being the number of heads  $H$  in outcome  $\omega$  and  $t_\omega$  being the number of tails  $T$  in outcome  $\omega$ . Therefore we can rewrite  $P(A)$  as follows:

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} p_0^{h_\omega} (1 - p_0)^{t_\omega}.$$

Let's calculate an example and see if it matches our intuition. Define  $B \in \mathcal{F}$  as the event such that at least one  $H$  occurs in the 3 tosses. Therefore  $B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$ . Now

$$\begin{aligned} P(B) &= \sum_{\omega \in B} p(\omega) = \sum_{\omega \in B} p_0^{h_\omega} (1 - p_0)^{t_\omega} \\ &= p_0^3 (1 - p_0)^0 + p_0^2 (1 - p_0)^1 + p_0^2 (1 - p_0)^1 + p_0^2 (1 - p_0)^1 \\ &\quad + p_0^1 (1 - p_0)^2 + p_0^1 (1 - p_0)^2 + p_0^1 (1 - p_0)^2. \end{aligned}$$

□

b) *Solution*

Now we analyze the experiment where two balls are drawn without replacement from an urn which originally contained two blue and two red balls. It might be helpful to denote ball one and two as  $u_1$  and  $u_2$  respectively. Let's begin by looking at the outcome space  $\Omega$ . In our two step experiment we get the following possible outcomes

$$\Omega = \{bb, br, rb, rr\}.$$

Once again, for simplicity let's define  $\mathcal{F} = 2^\Omega$  which means any possible subset of  $\Omega$  is an event. Now let's define  $P : \mathcal{F} \rightarrow \mathbb{R}$ . Let  $A \in \mathcal{F}$  then we say

$$P(A) = \sum_{\omega \in A} p(\omega)$$

again. Now  $p(\omega)$  is a little trickier to define since the probability for the second draw depends on what the first draw was. We have not described conditional probability in class yet, but I will borrow the notation and language of it here anyway. With this in mind we define

$$p(\omega) = p(\omega_1 \omega_2) = p(u_1 = \omega_1) p(u_2 = \omega_2 | u_1 = \omega_1)$$

with  $\omega$  being broken down in to  $\omega_1 \omega_2$  where  $\omega_1$  and  $\omega_2$  are the particular color ball being drawn first or second respectively in outcome  $\omega$ . We can define  $p(u_1 = \omega_1)$  and  $p(u_2 = \omega_2 | u_1 = \omega_1)$  more explicitly with the following rules.

$$p(u_1 = \omega_1) = \frac{1}{2} \quad \text{for } \omega \in \{b, r\}$$

since there are 2 of each color and 4 balls total. Additionally we determine

$$p(u_2 = \omega_2 | u_1 = \omega_1) = \frac{1}{3} \quad \text{if } \omega_2 = \omega_1$$

and

$$p(u_2 = \omega_2 | u_1 = \omega_1) = \frac{2}{3} \quad \text{if } \omega_2 \neq \omega_1$$

since at the time of the second ball draw there is 1 ball of one color and 2 balls of another color with 3 balls total remaining. Therefore we can rewrite  $P(A)$  as follows:

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} p(\omega_1 \omega_2) = \sum_{\omega \in A} p(u_1 = \omega_1) p(u_2 = \omega_2 | u_1 = \omega_1).$$

Let's calculate an example and see if it matches our intuition. Define  $B \in \mathcal{F}$  as the event where the second ball drawn is red  $u_2 = r$ . Therefore  $B = \{br, rr\}$ . Now

$$\begin{aligned} P(B) &= \sum_{\omega \in B} p(\omega) = \sum_{\omega \in B} p(\omega_1 \omega_2) = \sum_{\omega \in B} p(u_1 = \omega_1) p(u_2 = \omega_2 | u_1 = \omega_1) \\ &= p(u_1 = b) p(u_2 = r | u_1 = b) + p(u_1 = r) p(u_2 = r | u_1 = r) \\ &= \frac{1}{2} \frac{2}{3} + \frac{1}{2} \frac{1}{3} \\ &= \frac{1}{3} + \frac{1}{6} \\ &= \frac{1}{2}. \end{aligned}$$

Thus  $P(B) = P(\{br, rr\}) = \frac{1}{2}$  which is what we'd expect since this particular event  $B$  is comprised of half of the four possible outcomes in  $\Omega$ . □

**2.** (No translation-invariant random integer). Show that there is no probability measure  $P$  on the integers  $\mathbb{Z}$  with the discrete  $\sigma$ -algebra  $2^\mathbb{Z}$  with the translation-invariance property  $P(E + n) = P(E)$  for every event  $E \in 2^\mathbb{Z}$  and every integer  $n$ .  $E + n$  is obtained by adding  $n$  to every element of  $E$ .

*Solution*

Let's assume by way of contradiction, that there is a probability measure  $P$  on the

integers  $\mathbb{Z}$  with a discrete  $\sigma$ -algebra  $2^{\mathbb{Z}}$  and with the translation-invariance property  $P(E + n) = P(E)$  for every event  $E \in 2^{\mathbb{Z}}$  and every integer  $n$ . Now take the event  $E = \{0\}$  which is just a singleton set containing only 0. Using our assumption  $P(E + n) = P(E)$  we get

$$P(\{0\} + n) = P(\{n\}) = P(\{0\}) \forall n \in \mathbb{Z}.$$

Let's assign the value of  $P(\{n\}) = P(\{0\}) = p$  for all  $n \in \mathbb{Z}$ . Now we have the following

$$P(\Omega) = \sum_{\omega \in \Omega} P(\omega) = \sum_{\omega \in \Omega} p.$$

If  $p \neq 0$  this sum will blow up to  $\infty$  therefore  $p = 0$ . However, this means  $P(\Omega) = 0$  but that contradicts the assumption that  $P$  is a probability measure since that would require  $P(\Omega) = 1$ . Therefore there is no probability measure  $P$  on the integers  $\mathbb{Z}$  with the discrete  $\sigma$ -algebra  $2^{\mathbb{Z}}$  with the translation-invariance property.  $\square$

**3.** (No translation-invariant random real). Show that there is no probability measure  $P$  on the reals  $\mathbb{R}$  with the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  with the translation-invariance property  $P(E + x) = P(E)$  for every event  $E \in \mathcal{B}(\mathbb{R})$  and every real  $x$ . Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  is the  $\sigma$ -algebra generated by intervals  $(a, b] \subset \mathbb{R}$ .

*Solution*

Let's assume by way of contradiction, that there is a probability measure  $P$  on the reals  $\mathbb{R}$  with the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  with the translation-invariance property  $P(E + x) = P(E)$  for every event  $E \in \mathcal{B}(\mathbb{R})$  and every real  $x$ . Let  $I = (0, 1]$  be the interval of real numbers between 0 and 1. Consider the sets  $I_n = I + n$  for  $n \in \mathbb{Z}$ , where each  $I_n$  is a translation of the interval  $I$ :  $I_n = (n, n + 1]$ . The sets  $I_n$  are disjoint for different values of  $n$  that is

$$I_n \cap I_m = \emptyset \quad \text{for } n \neq m.$$

The way these sets are constructed we also get that, the union of these disjoint sets covers the entire real line:

$$\mathbb{R} = \cup_{n \in \mathbb{Z}} I_n.$$

By translation invariance of  $P$ , we have  $P(I_n) = P(I)$  for all  $n \in \mathbb{Z}$ . Let  $p = P(I)$ . Then, since  $P$  is a probability measure we have

$$P(\mathbb{R}) = P(\cup_{n \in \mathbb{Z}} I_n) = \sum_{n \in \mathbb{Z}} P(I_n) = \sum_{n \in \mathbb{Z}} p.$$

Since the sum  $\sum_{n \in \mathbb{Z}} p$  includes infinitely many terms, it will go to  $\infty$  unless  $p = 0$ . Thus,  $p = 0$ , meaning  $P(I) = 0$ . Since  $P(I) = 0$ , by translation invariance,  $P(I_n) = 0$  for all  $n \in \mathbb{Z}$ . Therefore:

$$P(\cup_{n \in \mathbb{Z}} I_n) = P(\mathbb{R}) = 0.$$

However, this contradicts our assumptions which required the fact that  $P(\mathbb{R}) = 1$ . Therefore, no probability measure  $P$  on the reals  $\mathbb{R}$  with the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  with the translation-invariance property.  $\square$

**4.** Let  $\Omega = \mathbb{R}$ ,  $\mathcal{F} =$  all subsets of  $\mathbb{R}$  so that  $A$  or  $A^c$  is countable. Let  $P(A) = 0$  in the first case and  $P(A) = 1$  in the second. Show that  $(\Omega, \mathcal{F}, P)$  is a probability space.

*Solution:*

We want to show  $(\Omega, \mathcal{F}, P)$  is a probability space. Therefore, we need to prove the following:

- $\mathcal{F}$  is a  $\sigma$ -algebra
  - Is non-empty  
Showing  $\mathcal{F}$  is non empty is trivial since we know there exists at least one set  $A$  s.t.  $A$  or  $A^c$  is countable. For example, take the singleton set  $B = \{0\}$  since  $B \in \mathbb{R}$  and is countable then  $B \in \mathcal{F}$ .
  - If  $A \in \mathcal{F}$  then  $A^c \in \mathcal{F}$   
Let  $A \in \mathcal{F}$  then either  $A$  is countable or  $A^c$  is countable.  
**Case 1:** ( $A$  is countable) then  $A^c$  is uncountable. However,  $(A^c)^c$  is countable therefore by construction of  $\mathcal{F}$   $A^c$  we can conclude  $A^c \in \mathcal{F}$  as well.  
**Case 2:** ( $A$  is uncountable) then  $A^c$  is countable, therefore by construction of  $\mathcal{F}$ , it immediately follows that  $A^c \in \mathcal{F}$ .
  - Let  $A_i, i \in \mathbb{N}$  be a countable collection of sets in  $\mathcal{F}$ , then their union is also in  $\mathcal{F}$ .

We will argue this by going through cases where every  $A_i$  is countable, at least one  $A_i$  has a countable compliment, and every  $A_i$  has a countable compliment.

**Case 1:** (Every  $A_i$  is countable) which implies  $P(A_i) = 0$  for all  $i$ . The union  $\cup_{i=1}^{\infty} A_i$  is also countable (the countable union of countable sets is countable), then we have

$$P(\cup_{i=1}^{\infty} A_i) = 0 \quad \text{and} \quad \cup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

**Case 2:** (At least one  $A_i$  has a countable compliment) Without loss of generality, let  $A_1$  be such that  $A_1^c$  is countable. Then  $P(A_1) = 1$ . Therefore,  $\cup_{i=1}^{\infty} A_i$  must also be uncountable (since it contains  $A_1$ ) and then we get

$$P(\cup_{i=1}^{\infty} A_i) = 1 \quad \text{and} \quad \cup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

**Case 3:** (Every  $A_i$  has a countable compliment) then  $P(A_i) = 1$  for all  $i$ . Then  $P(A_1) = 1$ . Therefore,  $\cup_{i=1}^{\infty} A_i$  must also be uncountable (since it is the union of uncountable sets) and then we get

$$P(\cup_{i=1}^{\infty} A_i) = 1 \quad \text{and} \quad \cup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Therefore  $\mathcal{F}$  is a  $\sigma$ -algebra. □

- $P$  is a probability measure
  - $P(A) \geq P(\emptyset) = 0 \quad \forall A \in \mathcal{F}$   
Take  $A \in \mathcal{F}$  to be countable, then  $P(A) = 0$ . If  $A$  is instead uncountable, then  $P(A) = 1$ . In any case  $P(A) \geq 0$ . Therefore by definition of  $P$  we have this first property satisfied.
  - $P(\Omega) = 1$   
The set  $\Omega = \mathbb{R}$  is uncountable, therefore  $P(\Omega) = 1$ , again by the definition of  $P$  as given in the problem statement.
  - $P(\cup_i A_i) = \sum_i P(A_i)$  for a countable sequence of disjoint sets  $A_i \in \mathcal{F}$   
We will need to look at this in terms of similar cases as we did in the third part of the  $\sigma$ -algebra proof. Recall those cases are where every  $A_i$  is countable, at least one  $A_i$  has a countable compliment, and every  $A_i$  has a countable compliment.

**Case 1:** (Every  $A_i$  is countable) which implies  $P(A_i) = 0$  for all  $i$ . And the countable union of countable sets is countable so we have

$$P(\cup_{i=1}^{\infty} A_i) = 0$$

which is the left side of what we want to show. We also have the right side is equal to 0 as well since

$$\sum_i P(A_i) = \sum_i 0 = 0$$

. Therefore in this case we have

$$P(\cup_i A_i) = \sum_i P(A_i)$$

**Case 2:** (At least one  $A_i$  has a countable compliment) Without loss of generality, let  $A_1$  be such that  $A_1^c$  is countable. We then have  $P(A_1) = 1$ . Now since each of the  $A_i$  s.t.  $i \neq 1$  are disjoint from  $A_1$  then they can't have a countable compliment. This is due to the fact that if an  $A_k$  had a countable compliment then  $A_k$  would be uncountable and therefore intersect with  $A_1$  at some point. Now, since each  $A_i$  s.t.  $i \neq 1$  does not have a countable compliment then they are countable and thus  $P(A_i) = 0 \forall i \neq 1$ . Therefore we have

$$P(\cup_i A_i) = P(A_1) = 1$$

Since the union  $\cup_i A_i$  is uncountable. Now

$$\sum_i P(A_i) = 1 + \sum_{i \neq 1} P(A_i) = 1 + \sum_{i \neq 1} 0 = 1$$

Therefore in this case we also have

$$P(\cup_i A_i) = \sum_i P(A_i)$$

**Case 3:** (Every  $A_i$  has a countable compliment) then  $P(A_i) = 1$  for all  $i$ . We also have that the countable union of uncountable sets is uncountable  $\cup_i A_i$ . And thus  $P(\cup_i A_i) = 1$ . Now looking at the sum

$$\sum_i P(A_i) = \sum_i 1 = \infty.$$

However, since all  $A_i$  sets are disjoint this case would never actually occur. When all sets have a countable compliment, because then the intersection of these uncountable sets would be none empty and they would thus not be disjoint. This case leads to a contradiction. Therefore have thus shown all valid cases lead to

$$P(\cup_i A_i) = \sum_i P(A_i)$$

And thus  $P$  is a probability measure.

We have now shown  $(\Omega, \mathcal{F}, P)$  is a probability space.