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AMATH 567

HOMEWORK 7

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 3.5.1 b, c, d (Only consider singularities in the finite complex plane)

2: From A&F: 3.5.3 a, c, d

3: Introducing the Gamma function: Do A&F: 3.6.6. This is the same Gamma function you may have seen defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

This better known representation is only valid for $\text{Re}(z) > 0$. The representation given here is valid in all of $\mathbb{C}$. It takes a bit of work to show that our representation is an analytic continuation of the integral representation (this requires the Dominated Convergence Theorem), but it is quite doable. Not now though.

4: Consider a sequence of numbers $(a_n)_{n\geq 0}$ such that $|a_n| < 1$ and

$$\sum_{n=0}^\infty (1 - |a_n|) < \infty.$$

Define a Blaschke factor

$$B(a, z) = \begin{cases} \frac{|a|}{a} \frac{a-z}{1-\bar{a}z} & a \neq 0, \\ z & a = 0. \end{cases}$$

- Show that

$$H(z) = \prod_{n=0}^\infty B(a_n, z),$$

  defines an analytic function in the open unit disk $|z| < 1$.
- Show that $H(z)$ has zeros at $z = a_n$ for every n. It might seem that this construction of an analytic function with an infinite number of zeros in a bounded region implies that $H(z) = 0$ for all z. Why is this not the case?

5: We define the Weierstrass $\wp$-function as

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right),$$

where $(j,k) = (0,0)$ is excluded from the double sum. Also, you may assume that ω_1 is a positive real number, and that ω_2 is on the positive imaginary axis. All considerations below are meant for the entire complex plane, except the poles of $\wp(z)$.

(a) Show that $\wp(z + M\omega_1 + N\omega_2) = \wp(z)$, for any two integers M, N. In other words, $\wp(z)$ is a doubly-periodic function: it has two independent periods in the complex plane. Doubly periodic functions are called elliptic functions.

(b) Establish that $\wp(z)$ is an even function: $\wp(-z) = \wp(z)$.

(c) Find Laurent expansions for $\wp(z)$ and $\wp'(z)$ in a neighborhood of the origin in the form

$$\wp(z) = \frac{1}{z^2} + \alpha_0 + \alpha_2 z^2 + \alpha_4 z^4 + \dots$$

and

$$\wp'(z) = -\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \dots$$

Give expressions for the coefficients introduced above.

(d) Show that $\varphi(z)$ satisfies the differential equation

$$(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d,$$

for suitable choices of a, b, c, d. Find these constants. You may need to invoke Liouville's theorem to obtain this final result. It turns out that the function $\varphi(z)$ is determined by the coefficients c and d, implying that it is possible to recover ω_1 and ω_2 from the knowledge of c and d.