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AMATH 561

PROBLEM SET 5

1. Let X and $Y_0, Y_1, Y_2, ...$ be random variables on a probability space $(\Omega, \mathcal{F}, P)$ and suppose $E|X| < \infty$. Define $\mathcal{F}_n = \sigma(Y_0, Y_1, ..., Y_n)$ and $X_n = E(X|\mathcal{F}_n)$. Show that the sequence $X_0, X_1, X_2, ...$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$.

Solution:
For my sake, I will review the definition of a martingale then we will show that the sequence $X_0, X_1, X_2, ...$ satisfies all of the necessary conditions and is thus itself a martingale.

Let $\mathcal{F}_n$ be a filtration, i.e. an increasing sequence of σ-algebras. A sequence of random variables X_n is said to be adapted to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ (that is X_n is $\mathcal{F}_n$-measurable or for all Borel sets B we have $X_n^{-1}(B) = \{\omega \mid X(\omega) \in B\} \in \mathcal{F}_n$) for all n. If X_n is a sequence with:

(1) $E|X_n| < \infty$
(2) X_n is adapted to $\mathcal{F}_n$
(3) $E(X_{n+1}|\mathcal{F}_n) = X_n$ for all n,

Then $X = (X_n)_{n \in \mathbb{N}}$ is said to be a martingale with respect to $\mathcal{F}_n$.

Now I will begin my actual proof. Since X and each Y_n are random variables on the probability space $(\Omega, \mathcal{F}, P)$, then $X \in \mathcal{F}$ and $Y_n \in \mathcal{F}$ for each $n \in \mathbb{N}_0$. By definition of conditional expectation, we have that $X_n \in \mathcal{F}_n$ for all n and thus X_n is adapted to $\mathcal{F}_n$. Next, let's show $E(X_{n+1}|\mathcal{F}_n) = X_n$ for all n. Recall that if $\mathcal{F}_0 \subset \mathcal{F}_1$, then

$$E\big(E(X|\mathcal{F}_1)\big|\mathcal{F}_0\big) = E(X|\mathcal{F}_0).$$

Therefore, we have,

$$E(X_{n+1}|\mathcal{F}_n) = E\big(E(X|\mathcal{F}_{n+1})\big|\mathcal{F}_n\big) = E(X|\mathcal{F}_n) = X_n$$

since

$$\mathcal{F}_n = \sigma(Y_0, Y_1, ..., Y_n) \subset \sigma(Y_0, Y_1, ..., Y_n, Y_{n+1}) = \mathcal{F}_{n+1}.$$

Finally, we want to show that $E|X_n| < \infty$ for all n. Additionally, by our definition of conditional expectation in lecture slides 10 we have that for all $A \in \mathcal{F}_n$,

$$\int_A Y \mathrm{d}P = \int_A X \mathrm{d}P.$$

Since, $\mathcal{F}_n$ is a σ-algebra we can take $A = \Omega$ and then we have

$$\begin{aligned} E|X_n| &= E\big|E(X|\mathcal{F}_n)\big| \\ &= \int_\Omega \big|E(X|\mathcal{F}_n)\big| \mathrm{d}P \\ &= \int_\Omega |Y| \mathrm{d}P \\ &= \int_\Omega |X| \mathrm{d}P \\ &= E|X| < \infty \end{aligned}$$

Therefore, $E|X_n| < \infty$. Hence, the sequence of random variables $(X_n)_{n\in\mathbb{N}_0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.

$\square$

2. Let $X_0, X_1, ...$ be i.i.d Bernoulli random variables with parameter p (i.e., $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $S_n = \sum_{i=1}^n X_i$ where $S_0 = 0$. Define

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}, \quad n = 0, 1, 2, ...$$

Let $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$. Show that Z_n is a martingale with respect to this filtration.

Solution:
We need to verify that the sequence of random variables $(Z_n)_{n\in\mathbb{N}_0}$ satisfies the requisite criteria to be a martingale. Let's begin by showing that $E|Z_n| < \infty$. Let's look more closely at the expected value of the absolute value of Z_n. Since $p \in [0, 1]$, we know that each Z_n is just a nonnegative number raised to some integer power so it is always positive. Therefore, in calculating the expected value we can drop the absolute value

$$\begin{aligned}
E|Z_n| &= E\left(\left(\frac{1-p}{p}\right)^{2S_n - n}\right) \\
&= E\left(\left(\frac{1-p}{p}\right)^{2(\sum_{i=1}^n X_i) - n}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n} E\left(\left(\frac{1-p}{p}\right)^{2\sum_{i=1}^n X_i}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n} E\left(\prod_{i=1}^n \left(\frac{1-p}{p}\right)^{2X_i}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n} \prod_{i=1}^n E\left(\left(\frac{1-p}{p}\right)^{2X_i}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n} E\left(\left(\frac{1-p}{p}\right)^{2X}\right)^n.
\end{aligned}$$

where the final few lines hold due to the fact that the X_i are i.i.d. It is important to note that we are taking the nth power of the expectation of our expression instead of the expectation of the expression to the nth power. The difference is important. Using the definition of expectation and the fact that the X's are Bernoulli distributed we have

$$\begin{aligned}
E|Z_n| &= \left(\frac{1-p}{p}\right)^{-n} \left(\int_\Omega \left(\frac{1-p}{p}\right)^{2X} \mathrm{d}P\right)^n \\
&= \left(\frac{1-p}{p}\right)^{-n} \left(\sum_{x\in\{0,1\}} \left(\frac{1-p}{p}\right)^{2x} P(X = x)\right)^n \\
&= \left(\frac{1-p}{p}\right)^{-n} \left(P(X = 0) + \left(\frac{1-p}{p}\right)^2 P(X = 1)\right)^n \\
&= \left(\frac{1-p}{p}\right)^{-n} \left((1-p) + \left(\frac{1-p}{p}\right)^2 p\right)^n.
\end{aligned}$$

I suppose we may be able to say something about the finiteness of the expected value at this point but I will continue with the algebra until it is more obvious to me. Getting common denominators inside the parenthesis on the right, we have

$$\begin{aligned}
E|Z_n| &= \left(\frac{1-p}{p}\right)^{-n}\left(\frac{p(1-p)}{p}+\frac{(1-p)^2}{p}\right)^n \\
&= \left(\frac{1-p}{p}\right)^{-n}\left(\frac{p(1-p)+(1-p)^2}{p}\right)^n \\
&= \frac{(1-p)^{-n}}{p^{-n}}\frac{\Big((1-p)\big(p+(1-p)\big)\Big)^n}{p^n} \\
&= (1-p)^{-n}\Big((1-p)\big(p+1-p\big)\Big)^n \\
&= (1-p)^{-n}(1-p)^n \\
&= 1
\end{aligned}$$

Thus, $E|Z_n| < \infty$. Now we need to show that Z_n is adapted to $\mathcal{F}$ or that $Z_n \in \mathcal{F}_n$ for all n. Each $X_n \in \mathcal{F}_n$ for all n and thus, $S_n \in \mathcal{F}_n$. Observe that Z_n is a nonnegative real number raised to the power of S_n (an $\mathcal{F}_n$-measurable random variable). Since Z_n is of the form $Z_n = g(S_n)$ with the g afore described, then $Z_n \in \mathcal{F}_n$. Finally, we need to show, for all n, that

$$E(Z_{n+1}|\mathcal{F}_n) = Z_n.$$

Beginning on the right

$$\begin{aligned}
E(Z_{n+1}|\mathcal{F}_n) &= E\left(\left(\frac{1-p}{p}\right)^{2S_{n+1}-n-1}\middle|\sigma(X_0,X_1,...,X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}E\left(\left(\frac{1-p}{p}\right)^{2S_{n+1}}\middle|\sigma(X_0,X_1,...,X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}E\left(\left(\frac{1-p}{p}\right)^{2\left(\sum_{i=0}^n X_i\right)+2X_{n+1}}\middle|\sigma(X_0,X_1,...,X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}E\left(\left(\frac{1-p}{p}\right)^{2\sum_{i=0}^n X_i}\left(\frac{1-p}{p}\right)^{2X_{n+1}}\middle|\sigma(X_0,X_1,...,X_n)\right).
\end{aligned}$$

Since, we are conditioning on $\sigma(X_0, X_1, ..., X_n)$ each X_i up to X_n is constant with respect to this given information. Therefore it can be treated like a constant and pulled out of the expected value because of the linearity of expected value. We now proceed with this step and can drop the conditioning since X_{n+1} is independent from the σ-algebra generated by the collection of X_i's

$$
\begin{aligned}
E(Z_{n+1}|\mathcal{F}_n) &= \left(\frac{1-p}{p}\right)^{-n-1}\left(\frac{1-p}{p}\right)^{2\sum_{i=0}^{n} X_i} E\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}}\middle|\sigma(X_0, X_1, ..., X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}\left(\frac{1-p}{p}\right)^{2S_n} E\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}\left(\frac{1-p}{p}\right)^{2S_n}\left(P(X_{n+1}=0) + \left(\frac{1-p}{p}\right)^2 P(X_{n+1}=1)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}\left(\frac{1-p}{p}\right)^{2S_n}\left((1-p) + \frac{(1-p)^2}{p}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}\left(\frac{1-p}{p}\right)^{2S_n}\left(\frac{(1-p)\big(p+1-p\big)}{p}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1}\left(\frac{1-p}{p}\right)^{2S_n}\left(\frac{1-p}{p}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n}\left(\frac{1-p}{p}\right)^{2S_n} \\
&= \left(\frac{1-p}{p}\right)^{2S_n-n} \\
&= Z_n.
\end{aligned}
$$

Therefore, $E(Z_{n+1}|\mathcal{F}_n) = Z_n$ for all n. And hence, Z_n is a martingale with respect to $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$.

$\square$

3. Let ξ_i be a sequence of random variables such that the partial sums

$$X_n = \xi_0 + \xi_1 + ... + \xi_n, \quad n \geq 1,$$

determine a martingale. Show that the summands are mutually uncorrelated, i.e. that $E(\xi_i \xi_j) = E(\xi_i)E(\xi_j)$ for $i \neq j$.

Solution:
This means there exists some filtration $\mathcal{F}_n = \sigma(X_0, X_1, X_2, ..., X_n)$ built out of all the information from previous steps in the martingale such that X_n is $\mathcal{F}_n$ adapted and both

$$E(X_{n+1}|\mathcal{F}_n) = X_n \text{ and } E|X_n| < \infty$$

hold. Then we also have that

$$\begin{aligned}
E(X_{n+1}|\mathcal{F}_n) &= E\left(\sum_{i=0}^{n+1} \xi_i \middle| \mathcal{F}_n\right) \\
&= E\left(\xi_{n+1} + \sum_{i=0}^{n} \xi_i \middle| \mathcal{F}_n\right) \\
&= E(\xi_{n+1}|\mathcal{F}_n) + E\left(\sum_{i=0}^{n} \xi_i \middle| \mathcal{F}_n\right) \\
&= E(\xi_{n+1}|\mathcal{F}_n) + \sum_{i=0}^{n} E(\xi_i|\mathcal{F}_n).
\end{aligned}$$

Recall, $E(X_{n+1}|\mathcal{F}_n) = X_n$ thus

$$\begin{aligned}
E(X_{n+1}|\mathcal{F}_n) &= X_n \\
E(\xi_{n+1}|\mathcal{F}_n) + \sum_{i=0}^{n} E(\xi_i|\mathcal{F}_n) &= \sum_{i=0}^{n} \xi_i \\
E(\xi_{n+1}|\mathcal{F}_n) + \sum_{i=0}^{n} \xi_i &= \sum_{i=0}^{n} \xi_i \\
E(\xi_{n+1}|\mathcal{F}_n) &= 0 \\
E(\xi_{n+1}) &= 0.
\end{aligned}$$

We arrive at the final equality, since $\mathcal{F}_n$ has no information about X_{n+1} let alone ξ_{n+1}. Therefore, without loss of generality let $i < n+1$, then

$$\begin{aligned}
E(\xi_i \xi_{n+1}) &= E(\xi_i|\xi_{n+1})E(\xi_{n+1}) \\
&= E(\xi_i|\xi_{n+1}) \cdot 0 \\
&= 0 \\
&= E(\xi_i) \cdot 0 \\
&= E(\xi_i)E(\xi_{n+1}).
\end{aligned}$$

Hence, ξ_i and ξ_j $(i \neq j)$ are uncorrelated, since

$$E(\xi_i \xi_j) = E(\xi_i)E(\xi_j) = 0.$$

□

4. Galton and Watson who invented the process that bears their names were interested in the survival of family names. Suppose each family has exactly 3 children but coin flips determine their sex. In the 1800s, only male children kept the family name so following the male offspring leads to a branching process with $p_0 = 1/8, p_1 = 3/8, p_2 = 3/8, p_3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$. What is ρ if we assume that each family has exactly 2 children?

Solution:
Let $Z_0 = 1$. Furthermore, define $Z_{n+1} = \xi_0^{n+1} + \xi_1^{n+1} + \xi_2^{n+1} + ... + \xi_{Z_n}^{n+1}$ where this follows the same definition in class. Z_{n+1} represents the number males in the $n+1$ generation which bear the last name.

We are going to go about this thinking about the branching process with respect to the birth death process. We are given that $Z_0 = 1$. At the beginning there is a $\frac{1}{8}$ probability the name dies out right away, a $\frac{3}{8}$ probability 1 male child is born and can bear the name, a $\frac{3}{8}$ probability 2 male children are born and can bear the name and finally, a $\frac{1}{8}$ probability that 3 male children are born and can bear the name. This gives us

$$\rho = \frac{1}{8} + \frac{3}{8}\rho + \frac{3}{8}\rho^2 + \frac{1}{8}\rho^3.$$

Solving this for ρ gives us

$$\rho = 1, \quad \rho = -2 + \sqrt{5}, \quad \rho = -2 - \sqrt{5}.$$

Since,

$$E(\xi_i) = 0\frac{1}{8} + 1\frac{3}{8} + 2\frac{3}{8} + 3\frac{1}{8} = \frac{12}{8} > 1$$

we can say the process is superciritical. Therefore can determine that we need $0 < \rho < 1$ and thus we have

$$\rho = -2 + \sqrt{5}.$$

TODO: If I have time I will address the case if the family has exactly 2 children.