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AMATH 567

# HOMEWORK 8

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

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**1:** The Korteweg-de Vries (KdV) equation arises whenever long waves of moderate amplitude in dispersive media are considered. For instance, it describes waves in shallow water, and ion-acoustic waves in plasmas. The equation is given by

$$u_t = 6uu_x + u_{xxx},$$

where indices denote partial differentiation.

(a) By looking for solutions $u(x,t) = U(x)$, derive a first-order ordinary differential equation for $U(x)$. Introduce integration constants as required.

*Solution:*
**TODO:**

$$This$$

(b) Let $U = U_0\wp(x - x_0)$. Determine $U_0$ so that $u = U(x)$ solves the KdV equation.

*Solution:*
**TODO:**

$$This$$

**2:** From A&F: 3.6.5

*Solution:*
**TODO:**

$$This$$

**3:** Here's a way to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

due to Euler. We've seen that

$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

(a) Equate the coefficients of $z^2$ on both sides, to recover the desired sum.

*Solution:*
**TODO:**

$$This$$

(b) Equate the coefficients of $z^4$ on both sides to recover a different sum.

*Solution:*
**TODO:**

$$This$$

By equating coefficients of higher powers of $z$, one can recover other identities too.

**4:** For the following, suppose that $f(z)$ is analytic in an open set $\Omega$ that contains $[-1,1]$.

(a) Show that there exists a contour $C$, encircling $[-1,1]$, such that

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i}\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

*Solution:*
**TODO:**

$$This$$

(b) Use this to evaluate

$$I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}}, \quad I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x,$$
$$I_3 = \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\mathrm{d}x, \quad I_4 = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\mathrm{d}x,$$

without using any changes of variable (e.g., no trig subs!).

*Solution:*
**TODO:**

$$This$$

**5:** Suppose, for $|z| = 1$, that the series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n,$$

converges uniformly.

(a) Compute series representations for

$$F(z) := \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \quad |z| \neq 1, \quad C = \partial B_1(0).$$

*Solution:*
**TODO:**

$$This$$

(b) For $|z| = 1$, compute

$$\lim_{\epsilon \to 0^+} F(z(1-\epsilon)) - \lim_{\epsilon \to 0^+} F(z(1+\epsilon)).$$

*Solution:*
**TODO:**

$$This$$