Hunter Lybbert
Student ID: 2426454
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AMATH 567

HOMEWORK 3

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a colaborator.

1: From A&F: 2.2.4.
Let α be a real number. Show that the set of all values of the multivalued function $\log(z^a)$ is not necessarily the same as that of $\alpha \log z$.
Solution:

2: Describe the Riemann surface on which the multi-valued function $w(z)$, defined by $w^2 = \prod_{j=1}^{n=3} (z - a_j)$ is single-valued. What happens for $n = 4, 5$? For $n > 5$? You may assume that all the a_j are distinct.
Solution:

3: From A&F: 2.2.5a. While you're at it, also derive a formula for $\text{arccot}(z)$ in terms of the logarithm.
Derive the following formulae:
a)

$$\coth^{-1}(z) = \frac{1}{2} \log \frac{z+1}{z-1}$$

Solution:
b)

$$\sec h^{-1}(z) = \log \left(\frac{1 + (1 - z^2)^{\frac{1}{2}}}{z} \right)$$

Solution:

4: Let

$$s(z) = z^{1/2} = \rho^{1/2} \, e^{i\theta/2}, \quad \theta \in [-\pi, \pi),$$

denote the principal branch of the square root. Show that the functions

$$f_1(z) = s(z^2 - 1), \quad f_2(z) = s(z-1)s(z+1),$$

are not equal as functions on $\mathbb{C}$ — first produce plots and then use a mathematical argument. Determine the branch cut for $f_2(z)$ (Note: My cartoon of what the branch cut for f_1 looks like in lecture was not accurate). Find the relationship between $f_1(z)$ and $f_2(z)$.
Solution:

5: Consider the function

$$\psi(z) = \int_1^z \frac{\mathrm{d}w}{(w^2-1)^{1/2}}, \quad z \notin (-\infty, 1),$$

where the path of integration is a straight line from 1 to z.

- Show that

  $$\psi(z) = \log \varphi(z), \quad \varphi(z) = z + (z^2-1)^{1/2}, \quad z \notin (-\infty, 1),$$

  for an appropriate choice of branch cut for $(z^2-1)^{1/2}$. Here $\log z$ denotes the principal branch.
- Find an expression for

  $$\gamma(z) = \int_{-1}^z \frac{\mathrm{d}w}{(w^2-1)^{1/2}}, \quad z \notin (-1, \infty),$$

  in terms of $\varphi(z)$ and the principal branch of the logarithm. Again, the path of integration is a straight line.

Solution:

6: Show that φ, from the previous problem, maps $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit disk, $\{z \in \mathbb{C} \ : \ |z| > 1\}$. Furthermore

$$\frac{1}{2}\left(\varphi(z) + 1/\varphi(z)\right) = z, \quad \mathbb{C} \setminus [-1,1].$$

Solution:

7: (Sharpness of the Bernstein–Walsh inequality) The Bernstein–Walsh inequality states that if a polynomial p_n of degree n satisfies $\max_{-1 \leq x \leq 1} |p_n(x)| \leq 1$ then

$$|p_n(z)| \leq |\varphi(z)|^n, \quad z \in \mathbb{C} \setminus [-1,1].$$

Show that

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right), \quad z \in \mathbb{C} \setminus [-1,1]$$

is a polynomial that satisfies

$$\max_{-1 \leq x \leq 1} |T_n(x)| = 1,$$

$$\lim_{n \to \infty} |T_n(z)|^{1/n} = |\varphi(z)|,$$

for any fixed $z \in \mathbb{C} \setminus [-1,1]$.

Solution: