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AMATH 567

HOMEWORK 6

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 3.3.2
Given the function

$$f(z) = \frac{z}{a^2 - z^2}, \quad a > 0,$$

expand $f(z)$ in a Laurent series in powers of z in the regions

(a) $|z| < a$
Solution:
We begin with a little algebra

$$f(z) = \frac{z}{a^2 - z^2} = \frac{z}{a^2}\frac{1}{1 - \frac{z^2}{a^2}}.$$

In this case, since $|z| < a$, then $\frac{z^2}{a^2} < 1$. Therefore we can make use of the common geometric series

$$f(z) = \frac{z}{a^2}\frac{1}{1 - \frac{z^2}{a^2}} = \frac{z}{a^2}\sum_{n=0}^{\infty}\left(\frac{z^2}{a^2}\right)^n = \frac{z}{a^2}\sum_{n=0}^{\infty}\frac{z^{2n}}{a^{2n}} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{a^{2n+2}} = \sum_{n=0}^{\infty}\frac{1}{a^{2n+2}}z^{2n+1}.$$

□

(b) $|z| > a$
Solution:
We begin with a little algebra

$$f(z) = \frac{z}{a^2 - z^2} = -\frac{z}{z^2 - a^2} = -\frac{z}{z^2}\frac{1}{1 - \frac{a^2}{z^2}}.$$

In this case, since $|z| > a$, then $\frac{a^2}{z^2} < 1$. Therefore we can make use of the common geometric series

$$\begin{aligned}
f(z) &= -\frac{z}{z^2}\frac{1}{1-\frac{a^2}{z^2}} \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a^2}{z^2}\right)^n \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{a^{2n}}{z^{2n}} \\
&= -\sum_{n=0}^{\infty}\frac{a^{2n}}{z^{2n+1}} \\
&= -\sum_{n=0}^{\infty}a^{2n}\frac{1}{z^{2n+1}} \\
&= -\sum_{n=0}^{\infty}a^{2n}z^{-(2n+1)} \\
&= -\sum_{n=0}^{\infty}a^{2n}z^{-2n-1} \\
&= -\sum_{n=-\infty}^{0}a^{2n}z^{2n-1}.
\end{aligned}$$

$\square$

2: From A&F: 3.3.5
Let

$$\exp\left(\frac{t}{2}\left(z-\frac{1}{z}\right)\right)=\sum_{n=-\infty}^{\infty}J_n(t)z^n.$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned}J_n(t)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}(n\theta-t\sin\theta)}\,\mathrm{d}\theta\\&=\frac{1}{\pi}\int_{0}^{\pi}\cos(n\theta-t\sin\theta)\mathrm{d}\theta.\end{aligned}$$

The functions $J_n(t)$ are called the Bessel function, which are well known special functions in mathematics and physics.

Solution:
Let $f(z)=\exp\left(\frac{t}{2}\left(\frac{z-1}{z}\right)\right)$. We begin by looking at the general Laurent series centered at $z=0$, since our function is undefined at this point it is the only singularity we are concerned with. Therefore we have

$$f(z)=\sum_{n=-\infty}^{\infty}C_n(z-0)^n=\sum_{n=-\infty}^{\infty}C_nz^n.$$

Where the C_n is given by

$$C_n=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{f(\xi)}{\xi^{n+1}}\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{\exp\left(\frac{t}{2}\left(\xi-\frac{1}{\xi}\right)\right)}{\xi^{n+1}}\mathrm{d}\xi.$$

This is really incomplete notationally since our C_n's depend on t so reverting back to the provided notation we have

$$J_n(t)=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{\exp\left(\frac{t}{2}\left(\xi-\frac{1}{\xi}\right)\right)}{\xi^{n+1}}\mathrm{d}\xi.$$

Additionally, I have yet to specify my contour C, but it needs to be within the annulus for which our Laurent series converges. Since, the original function $f(z)$ only has a singularity at $z=0$ the Laurent series really converges uniformly throughout the complex plane except at the origin. Therefore we make the convenient choice for our contour C to be a counterclockwise traversal of the unit circle. Using the parameterization

$\xi = \mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in [-\pi, \pi)$, we have

$$
\begin{aligned}
J_n(t) &= \frac{1}{2\pi\mathrm{i}} \int_{-\pi}^{\pi} \frac{\exp\left(\frac{t}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\right)\right)}{\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{n+1}} \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp\left(\frac{t}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\right)\right)}{\mathrm{e}^{\mathrm{i}n\theta}} \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{t}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right) - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{t}{2}\left(\cancel{\cos\theta} + \mathrm{i}\sin\theta - \cancel{\cos\theta} + \mathrm{i}\sin\theta\right) - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{t}{2}\left(2\mathrm{i}\sin\theta\right) - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(t\mathrm{i}\sin\theta - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta.
\end{aligned}
$$

Therefore

$$
J_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta,
$$

as desired. Furthermore,

$$
\begin{aligned}
J_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{0} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&\quad + \frac{1}{2\pi} \int_{0}^{\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&= -\frac{1}{2\pi} \int_{0}^{-\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&\quad + \frac{1}{2\pi} \int_{0}^{\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta.
\end{aligned}
$$

Now we need to do a substitution for $n\theta - t\sin\theta$ in each of these integrals. For the integral from 0 to $-\pi$ let $\theta = -\theta'$ and for the integral from 0 to π let $\theta = \theta'$. Continuing where we left off we then have

$$
\begin{aligned}
&= -\frac{1}{2\pi}\int_0^{\pi} \cos\big(-n\theta' - t\sin(-\theta')\big) - \mathrm{i}\sin\big(-n\theta' - t\sin(-\theta')\big)(-\mathrm{d}\theta') \\
&\quad + \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin\theta'\big) - \mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\Big(-\big(n\theta' - t\sin(\theta')\big)\Big) - \mathrm{i}\sin\Big(-\big(n\theta' - t\sin(\theta')\big)\Big)\mathrm{d}\theta' \\
&\quad + \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin\theta'\big) - \mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big) + \mathrm{i}\sin\big(n\theta' - t\sin(\theta')\big)\mathrm{d}\theta' + \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin\theta'\big) - \mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big) + \cancel{\mathrm{i}\sin\big(n\theta' - t\sin(\theta')\big)} + \cos\big(n\theta' - t\sin\theta'\big) - \cancel{\mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)}\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big) + \cos\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{2}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big)\mathrm{d}\theta' \\
&= \frac{1}{\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big)\mathrm{d}\theta'.
\end{aligned}
$$

Though we finished in terms of another variable θ' this could easily be changed out with another substitution $\theta' = \theta$. And thus we see

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi} \cos\big(n\theta - t\sin(\theta)\big)\mathrm{d}\theta.$$

$\square$

3: Bernoulli numbers: Consider the function

$$f(z) = \frac{z}{e^z - 1}.$$

(a) Show that $f(z)$ has a removable singularity at $z = 0$. Assume from now on that the definition of $f(z)$ has been extended to remove the singularity.

Solution:
If we can show the limit exists at the potential singularity then we can say it is removable. We can calculate the limit of $f(z)$ as $z \to 0$ explicitly:

$$\begin{aligned}
\lim_{z\to 0} f(z) &= \lim_{z\to 0} \frac{z}{\mathrm{e}^z - 1} = \frac{0}{0} \qquad \text{applying L'Hôpitals rule} \\
&= \lim_{z\to 0} \frac{z}{\mathrm{e}^z - 1} = \lim_{z\to 0} \frac{1}{\mathrm{e}^z} = \frac{1}{1} = 1
\end{aligned}$$

Therefore, we could choose $f(0) = 1$ in order to extend $f(z)$ to be analytic in the region and therefore remove the singularity. Furthermore, we can also show this is a removable singularity by looking at the reciprocal of $f(z)$. If it does not have any zeros, then $f(z)$ will not have any actual singularities or it won't blow up anywhere. We use a Taylor series centered at $z = 0$ for e^z and see the following

$$\begin{aligned}
\frac{1}{f(z)} = \frac{\mathrm{e}^z - 1}{z} = \frac{1}{z}(\mathrm{e}^z - 1) &= \frac{1}{z}\left(\sum_{j=0}^{\infty} \frac{z^j}{j!} - 1\right) \\
&= \frac{1}{z}\left(\sum_{j=1}^{\infty} \frac{z^j}{j!}\right) \\
&= \sum_{j=1}^{\infty} \frac{z^{j-1}}{j!} \\
&= \sum_{j=0}^{\infty} \frac{z^j}{(j+1)!}
\end{aligned}$$

which has no zeros. As an aside I want to highlight that our original function

$$f(z) = \frac{z}{\mathrm{e}^z - 1}$$

blows up at $2\pi\mathrm{i}k$ for $k \in \mathbb{Z}$. Since our Taylor series representation of $1/f(z)$ centered at $z = 0$ has no zeros, then we can say the singularity at $z = 1$ is removable. Finally, we can conclude from these two pieces of evidence that this particular singularity is removable. We will assume from now on that $f(z)$ has been extended to remove the singularity at $z = 0$. $\square$

(b) Suppose you were to find a Taylor series for $f(z)$, centered at $z = 0$. What would be its radius of convergence?

Solution:
We determined there is no singularity for $f(z)$ at $z = 0$, so a Taylor series representation can be constructed centered at $z = 0$. And the radius of convergence will be the distance from $z = 0$ to its nearest singularity. This will be the singularities at $z = -2\pi\mathrm{i}$ or $z = 2\pi\mathrm{i}$, therefore the radius of convergence will be 2π. $\square$

(c) Find the Taylor series in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The numbers B_n are known as the Bernoulli numbers.

Solution:

$$\begin{aligned}
f(z) = \frac{z}{\mathrm{e}^z - 1} &= \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \\
z &= (\mathrm{e}^z - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \\
z &= \left(\sum_{m=0}^{\infty} \frac{z^m}{m!} - 1 \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \\
z &= \left(\sum_{m=1}^{\infty} \frac{z^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \\
z &= \left(\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \right)
\end{aligned}$$

Using the Cauchy Product formula

$$\left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{j=0}^{\infty} b_j \right) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} a_\ell b_{k-\ell}$$

we can continue from where we left off and get

$$\begin{aligned}
z &= \left(\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{z^{k-\ell+1}}{(k-\ell+1)!} \frac{B_\ell z^\ell}{\ell!} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{(k-\ell+1)!} \frac{B_\ell}{\ell!} z^{k+1} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{(k+1)!}{(k-\ell+1)!} \frac{B_\ell}{\ell!} \frac{z^{k+1}}{(k+1)!} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{(k+1)!}{(k+1-\ell)!\ell!} B_\ell \frac{z^{k+1}}{(k+1)!} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell \frac{z^{k+1}}{(k+1)!} \\
&= \sum_{k=1}^{\infty} \sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell \frac{z^k}{k!}.
\end{aligned}$$

Now that we have

$$
\begin{aligned}
z &= \sum_{k=1}^{\infty} \sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell \frac{z^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{z^k}{k!} \sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell
\end{aligned}
$$

we can see

$$
\begin{aligned}
z &= \frac{z}{1!}\binom{1}{0} B_0 + \sum_{k=2}^{\infty} \frac{z^k}{k!} \sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell \\
z &= zB_0 + \sum_{k=2}^{\infty} \frac{z^k}{k!} \sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell .
\end{aligned}
$$

Therefore we need the following to hold

$$
\begin{aligned}
B_0 &= 1 \\
\sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell &= 0, \text{ for } k > 1.
\end{aligned}
$$

Notice,

$$
\begin{aligned}
\sum_{\ell=0}^{k-1} \binom{k}{\ell} B_\ell &= 0 \\
\binom{k}{k-1} B_{k-1} + \sum_{\ell=0}^{k-2} \binom{k}{\ell} B_\ell &= 0 \\
kB_{k-1} &= -\sum_{\ell=0}^{k-2} \binom{k}{\ell} B_\ell \\
B_{k-1} &= -\frac{1}{k} \sum_{\ell=0}^{k-2} \binom{k}{\ell} B_\ell, \text{ for } k > 1.
\end{aligned}
$$

To clean this up let's reindex a little

$$
B_k = -\frac{1}{k+1} \sum_{\ell=0}^{k-1} \binom{k+1}{\ell} B_\ell, \text{ for } k > 0.
$$

Notice this is a recurrence relation for the Bernoulli numbers. We can combine this with the original series representation of $f(z)$ to get

$$\begin{aligned}
f(z) = \frac{z}{\mathrm{e}^z - 1} &= \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} B_n \\
&= -\sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\frac{1}{n+1} \sum_{\ell=0}^{n-1} \binom{n+1}{\ell} B_\ell \right) \\
&= -\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{1}{n+1} \sum_{\ell=0}^{n-1} \binom{n+1}{\ell} B_\ell .
\end{aligned}$$

□

(d) Find a recursion formula for the Bernoulli numbers, and use it to find $B_0, \ldots, B_{12}$.
Solution:
We will use the recurrence relation we found in part (c)

$$B_k = -\frac{1}{k+1} \sum_{\ell=0}^{k-1} \binom{k+1}{\ell} B_\ell, \text{ for } k > 0$$

to calculate the first 12 Bernoulli numbers. We already know that $B_0 = 1$. Then we have

$$\begin{aligned}
B_1 &= -\frac{1}{2} \left(\binom{2}{0} B_0 \right) = -\frac{1}{2} \\
B_2 &= -\frac{1}{3} \left(\binom{3}{0} B_0 + \binom{3}{1} B_1 \right) = -\frac{1}{3} \left(1 - \frac{3}{2} \right) = \frac{1}{6} \\
B_3 &= -\frac{1}{4} \left(\binom{4}{0} B_0 + \binom{4}{1} B_1 + \binom{4}{2} B_2 \right) = -\frac{1}{4} (1 - 2 + 1) = 0 \\
B_4 &= -\frac{1}{5} \left(\binom{5}{0} B_0 + \binom{5}{1} B_1 + \binom{5}{2} B_2 + \binom{5}{3} B_3 \right) \\
&= -\frac{1}{5} \left(1 - \frac{5}{2} + \frac{5}{3} \right) = -\frac{1}{30} \\
B_5 &= -\frac{1}{6} \left(\binom{6}{0} B_0 + \binom{6}{1} B_1 + \binom{6}{2} B_2 + \binom{6}{3} B_3 + \binom{6}{4} B_4 \right) \\
&= -\frac{1}{6} \left(1 - \frac{6}{2} + \frac{15}{6} + 0 - \frac{15}{30} \right) = 0 \\
B_6 &= -\frac{1}{7} \left(\binom{7}{0} B_0 + \binom{7}{1} B_1 + \binom{7}{2} B_2 + \binom{7}{3} B_3 + \binom{7}{4} B_4 + \binom{7}{5} B_5 \right) \\
&= -\frac{1}{7} \left(1 - \frac{7}{2} + \frac{21}{6} + 0 - \frac{35}{30} + 0 \right) = \frac{1}{42}.
\end{aligned}$$

We're half way there! Continuing on, we have

$$\begin{aligned}
B_7 &= -\frac{1}{8}\left(\binom{8}{0}B_0 + \binom{8}{1}B_1 + \binom{8}{2}B_2 + \binom{8}{3}B_3 + \binom{8}{4}B_4 + \binom{8}{5}B_5 + \binom{8}{6}B_6\right) \\
&= -\frac{1}{8}\left(1 - \frac{8}{2} + \frac{28}{6} + 0 - \frac{70}{30} + 0 + \frac{28}{42}\right) = 0 \\
B_8 &= -\frac{1}{9}\left(\binom{9}{0}B_0 + \binom{9}{1}B_1 + \binom{9}{2}B_2 + \binom{9}{3}B_3 + \binom{9}{4}B_4 + \binom{9}{5}B_5 + \binom{9}{6}B_6 + \binom{9}{7}B_7\right) \\
&= -\frac{1}{9}\left(1 - \frac{9}{2} + \frac{36}{6} + 0 - \frac{126}{30} + 0 + \frac{84}{42} + 0\right) = -\frac{1}{30} \\
B_9 &= -\frac{1}{10}\left(\binom{10}{0}B_0 + \binom{10}{1}B_1 + \binom{10}{2}B_2 + \binom{10}{3}B_3 + \binom{10}{4}B_4 + \binom{10}{5}B_5 + \binom{10}{6}B_6 + \binom{10}{7}B_7 + \binom{10}{8}B_8\right) \\
&= -\frac{1}{10}\left(1 - \frac{10}{2} + \frac{45}{6} + 0 - \frac{210}{30} + 0 + \frac{210}{42} + 0 - \frac{45}{30}\right) = 0 \\
B_{10} &= -\frac{1}{11}\left(\binom{11}{0}B_0 + \binom{11}{1}B_1 + \binom{11}{2}B_2 + \binom{11}{3}B_3 + \binom{11}{4}B_4\right. \\
&\qquad \left. + \binom{11}{5}B_5 + \binom{11}{6}B_6 + \binom{11}{7}B_7 + \binom{11}{8}B_8 + \binom{11}{9}B_9\right) \\
&= -\frac{1}{11}\left(1 - \frac{11}{2} + \frac{55}{6} + 0 - \frac{330}{30} + 0 + \frac{462}{42} + 0 - \frac{165}{30} + 0\right) = \frac{5}{66} \\
B_{11} &= -\frac{1}{12}\left(\binom{12}{0}B_0 + \binom{12}{1}B_1 + \binom{12}{2}B_2 + \binom{12}{3}B_3 + \binom{12}{4}B_4\right. \\
&\qquad \left. + \binom{12}{5}B_5 + \binom{12}{6}B_6 + \binom{12}{7}B_7 + \binom{12}{8}B_8 + \binom{12}{9}B_9 + \binom{12}{10}B_{10}\right) \\
&= -\frac{1}{12}\left(1 - \frac{12}{2} + \frac{66}{6} + 0 - \frac{495}{30} + 0 + \frac{924}{42} + 0 - \frac{495}{30} + 0 + \frac{66\cdot 5}{66}\right) = 0 \\
B_{12} &= -\frac{1}{13}\left(\binom{13}{0}B_0 + \binom{13}{1}B_1 + \binom{13}{2}B_2 + \binom{13}{3}B_3 + \binom{13}{4}B_4 + \binom{13}{5}B_5\right. \\
&\qquad \left. + \binom{13}{6}B_6 + \binom{13}{7}B_7 + \binom{13}{8}B_8 + \binom{13}{9}B_9 + \binom{13}{10}B_{10} + \binom{13}{11}B_{11}\right) \\
&= -\frac{1}{13}\left(1 - \frac{13}{2} + \frac{78}{6} + 0 - \frac{715}{30} + 0 + \frac{1716}{42} + 0 - \frac{1287}{30} + 0 + \frac{186\cdot 5}{66} + 0\right) = -\frac{691}{2730}
\end{aligned}$$

□

(e) Show that $B_{2n+1} = 0$ for $n \geq 1$.
Solution:
TODO: Here is the idea...just do the thingy from the larson pdf. □

(f) Use your result to find a Taylor series for $z \coth z$, in terms of the Bernoulli numbers. Where is this series valid? Using this result, find a Laurent series for $\cot z$. Where is this series valid?
Solution:
TODO:

4: Consider $g(z) = 1/f(z)$ where $f(z)$ is as in the previous problem.

(a) Using the formula for $g(z)$, use software that uses double precision floating point arithmetic to compute the errors $e_n := |g(2^{-n}) - g(0)|$ for $n = 1, 2, \ldots, 52$. Produce a plot of these errors. *Solution:*
TODO:

(b) Derive an approximation $G(z)$ to $g(z)$, near $z = 0$, that does not suffer from the instability you notice. Plot the new errors $E_n := |G(2^{-n}) - g(0)|$ for $n = 1, 2, \ldots, 52$. Ensure that these errors are less than 10^{-10} for all n.
Solution:
TODO:

5: Analytic continuation:

(a) Consider

$$F(z) = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n.$$

Where is this function analytic?

Solution:

Looking at this the ratio test we have

$$\left| \frac{z^{n+1}}{z^n} \right| = |z|.$$

Which implies the Taylor series only converges inside the unit disc or when $|z| < 1$. Therefore, our series is analytic inside the unit disc. $\square$

(b) Use the above representation to induce a Taylor representation of $F(z)$ centered at $z = -1/2$. Call this representation $G(z)$. Your final result should be of the form

$$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2} \right)^m$$

Where is this series valid?

If you can answer this question without using that both $F(z)$ and $G(z)$ are representations of $1/(1-z)$, you will receive 2 bonus points.

Solution:

TODO:

Follow the path forged in class by doing the following

$$\begin{aligned}
\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (z-0)^n &= \sum_{n=0}^{\infty} \left(z + \frac{1}{2} - \frac{1}{2} - 0 \right)^n \\
&= \sum_{n=0}^{\infty} \left((z + \frac{1}{2}) + (-\frac{1}{2}) \right)^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \left(z + \frac{1}{2} \right)^m \left(-\frac{1}{2} \right)^{n-m} \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \binom{n}{m} \left(z + \frac{1}{2} \right)^m \left(-\frac{1}{2} \right)^{n-m} \\
&= \sum_{m=0}^{\infty} \left(z + \frac{1}{2} \right)^m \sum_{n=m}^{\infty} \binom{n}{m} \left(-\frac{1}{2} \right)^{n-m} \\
&= \sum_{m=0}^{\infty} \left(z + \frac{1}{2} \right)^m c_m.
\end{aligned}$$

Where our coeffecients c_m are

$$c_m = \sum_{n=m}^{\infty} \binom{n}{m} \left(-\frac{1}{2} \right)^{n-m}.$$

Now do the ratio test to determine the radius of convergence

$$\lim_{m\to\infty}\left|\frac{\sum_{n=m+1}^{\infty}\binom{n}{m+1}\left(-\frac{1}{2}\right)^{n-(m+1)}}{\sum_{n=m}^{\infty}\binom{n}{m}\left(-\frac{1}{2}\right)^{n-m}}\right|$$

$$\lim_{m\to\infty}\lim_{M\to\infty}\left|\frac{\sum_{n=m+1}^{M+1}\binom{n}{m+1}\left(-\frac{1}{2}\right)^{n-(m+1)}}{\sum_{n=m}^{M+1}\binom{n}{m}\left(-\frac{1}{2}\right)^{n-m}}\right|$$

we want the ratio test limit to come out to $\frac{2}{3}$ otherwise known as $1/r$ where $r=\frac{3}{2}$ the expected ratio of convergence

6: This problem is from Whittaker and Watson's "A course of modern analysis": Shew[1] that

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \frac{1}{(1-z)^2}, & |z| < 1 \\ \frac{1}{z(1-z)^2}, & |z| > 1. \end{cases}$$

This might appear to contradict the idea of analytic continuation. Please comment.
Solution:
We will begin by using partial fractions on the summand term

$$\begin{aligned} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} &= \frac{A}{1-z^n} + \frac{B}{1-z^{n+1}} \\ z^{n-1} &= A(1-z^{n+1}) + B(1-z^n) \\ z^{n-1} &= A - Az^{n+1} + B - Bz^n \end{aligned}$$

Notice if we plug in $z = 0$ we get $0 = A + B$ implying $A = -B$. Using these facts we get

$$\begin{aligned} z^{n-1} &= A - Az^{n+1} + B - Bz^n \\ z^{n-1} &= -Az^{n+1} - Bz^n \\ z^{n-1} &= z^n(-Az - B) \\ \frac{1}{z} &= -Az - B \\ \frac{1}{z} &= -Az + A \\ \frac{1}{z} &= A(1-z) \\ \frac{1}{z(1-z)} &= A. \end{aligned}$$

Therefore,

$$A = \frac{1}{z(1-z)} \quad \text{and} \quad B = -\frac{1}{z(1-z)}.$$

Thus we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} &= \sum_{n=1}^{\infty} \frac{\frac{1}{z(1-z)}}{1-z^n} + \frac{-\frac{1}{z(1-z)}}{1-z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{\frac{1}{z(1-z)}}{1-z^n} - \frac{\frac{1}{z(1-z)}}{1-z^{n+1}}. \end{aligned}$$

[1] Aka "Show".

Let's look at a finite sum of this form

$$\sum_{n=1}^{N} \frac{\frac{1}{z(1-z)}}{1-z^n} - \frac{\frac{1}{z(1-z)}}{1-z^{n+1}}$$

$$= \frac{\frac{1}{z(1-z)}}{1-z} - \cancel{\frac{\frac{1}{z(1-z)}}{1-z^2}} + \cancel{\frac{\frac{1}{z(1-z)}}{1-z^2}} - \cancel{\frac{\frac{1}{z(1-z)}}{1-z^3}} + \cancel{\frac{\frac{1}{z(1-z)}}{1-z^3}} - \cancel{\frac{\frac{1}{z(1-z)}}{1-z^4}} + \ldots + \cancel{\frac{\frac{1}{z(1-z)}}{1-z^N}} - \frac{\frac{1}{z(1-z)}}{1-z^{N+1}}$$

$$= \frac{\frac{1}{z(1-z)}}{1-z} - \frac{\frac{1}{z(1-z)}}{1-z^{N+1}}$$

$$= \frac{1}{z(1-z)^2} - \frac{1}{z(1-z)(1-z^{N+1})}$$

Now we want to take the limit of this and see the results

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} &= \sum_{n=1}^{\infty} \frac{\frac{1}{z(1-z)}}{1-z^n} - \frac{\frac{1}{z(1-z)}}{1-z^{n+1}} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\frac{1}{z(1-z)}}{1-z^n} - \frac{\frac{1}{z(1-z)}}{1-z^{n+1}} \\
&= \lim_{N\to\infty} \frac{1}{z(1-z)^2} - \frac{1}{z(1-z)(1-z^{N+1})} = \begin{cases} \frac{1}{z(1-z)^2} - \frac{1}{z(1-z)}, & |z| < 1 \\ \frac{1}{z(1-z)^2}, & |z| > 1. \end{cases}
\end{aligned}$$

Looking more closely at the resulting expression in the case $|z| < 1$, we have

$$\begin{aligned}
\frac{1}{z(1-z)^2} - \frac{1}{z(1-z)} &= \frac{1}{z(1-z)^2} - \frac{1-z}{z(1-z)^2} \\
&= \frac{1-1+z}{z(1-z)^2} \\
&= \frac{z}{z(1-z)^2} \\
&= \frac{1}{(1-z)^2}.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \frac{1}{(1-z)^2}, & |z| < 1 \\ \frac{1}{z(1-z)^2}, & |z| > 1. \end{cases}$$

Now this might appear to contradict the idea of analytic continuation, however, the dense collection of singularities on the unit circle does not allow for us to apply analytic continuation here. **TODO: How should I talk through this** It looks like it should have some section of the unit circle where they can use a common contour to agree, however, it cannot since there is no continuous contour not broken up by some of the singularities due to the roots of unity

7: Suppose that f is a function satisfying

$$|f(x)| \leq M, \quad x \in \mathbb{R}.$$

Show that

$$\hat{f}(z) := \int_0^\infty \mathrm{e}^{\mathrm{i}zx} f(x)\mathrm{d}x,$$

is an analytic function of z for $\operatorname{Im} z > 0$. You may assume that f is continuous, but this is not a necessary assumption.

Solution:
Use a theorem, something about this being able to hold if the integral is finite, then take the limit as it becomes infinit

8: Use analytic continuation to show that

$$\sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}},$$

where $\sqrt{\cdot}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$.

Solution:

Consider that they are both analytic everywhere in the same domain (use the form of analytic continuation which depends on the accumulation point)

Choose a contour for which the functions. agree on (positive real axis is a good choice).

Then show that

$$\sqrt{z-1}\sqrt{z+1} = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + O(z^{-3}), \quad z \to \infty,$$

and find b_0, b_1, b_2.