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AMATH 561

PROBLEM SET 6

1. Let $X \sim Binomial(n, U)$, where $U \sim Uniform((0,1))$. What is the probability generating function $G_X(s)$ of X? What is $P(X = k)$ for $k \in \{0, 1, 2, ..., n\}$?

Solution: The probability mass function for $X \sim Binomial(n, U)$, is given by

$$f_X(x) = \binom{n}{x} U^x (1-U)^{n-x} \qquad \text{for } x = 0, 1, 2, ..., n$$

And the density for the Uniform distribution is

$$g_X(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1 \qquad \text{for } x \in (0,1)$$

Then $G_X(s)$ is

$$\begin{aligned}
G_X(s) &= E(s^X) \\
&= E(E(s^X|U)) \\
&= E\left(\sum_{x=0}^{n} \binom{n}{x} U^x (1-U)^{n-x} s^x\right) \\
&= E\left(\sum_{x=0}^{n} \binom{n}{x} (Us)^x (1-U)^{n-x}\right) \\
&= E\big((Us + 1 - U)^n\big) \\
&= \int_0^1 (us + 1 - u)^n du \\
&= \left.\frac{(us+1-u)^{n+1}}{(n+1)(s-1)}\right|_0^1 \\
&= \frac{(1s+1-1)^{n+1}}{(n+1)(s-1)} - \frac{(0s+1-0)^{n+1}}{(n+1)(s-1)} \\
&= \frac{s^{n+1}}{(n+1)(s-1)} - \frac{1^{n+1}}{(n+1)(s-1)} \\
&= \frac{s^{n+1} - 1}{(n+1)(s-1)}
\end{aligned}$$

Now to calculate $P(X=k)$, notice

$$\begin{aligned}
G_X(s) = \frac{s^{n+1}-1}{(n+1)(s-1)} &= \frac{1}{n+1}\frac{s^{n+1}-1}{s-1} \\
&= \frac{1}{n+1}\frac{1-s^{n+1}}{1-s} \\
&= \sum_{k=0}^{n} \frac{1}{n+1} s^k \\
&= \sum_{k=0}^{n} P(X=k) s^k \\
&= G_X(s)
\end{aligned}$$

Therefore,

$$P(X=k) = \frac{1}{n+1} \qquad \text{for all } k \in \{0,1,2,..,n\}.$$

$\square$

TODO: Prolly delete this section, though it is true

Alternatively, we need to come up with a formula for

$$G_X^{(k)}(0) = \frac{d^k}{ds^k}\left[\frac{s^{n+1}-1}{(n+1)(s-1)}\right]\Bigg|_0 .$$

I'll begin by calculating this for $k = 0, 1, 2$

$$G_X^{(0)}(0) = \frac{d^0}{ds^0}\left[\frac{s^{n+1}-1}{(n+1)(s-1)}\right]\Bigg|_0 = \frac{s^{n+1}-1}{(n+1)(s-1)}\Bigg|_0 = \frac{0^{n+1}-1}{(n+1)(0-1)} = \frac{1}{n+1}$$

$$\begin{aligned}
G_X^{(1)}(0) &= \frac{d^1}{ds^1}\left[\frac{s^{n+1}-1}{(n+1)(s-1)}\right]\Bigg|_0 \\
&= \frac{s^n(n+1)(s-1)-(s^{n+1}-1)}{(n+1)(s-1)^2}\Bigg|_0 = \frac{0^n(n+1)(0-1)-(0^{n+1}-1)}{(n+1)(0-1)^2} = \frac{1}{n+1}
\end{aligned}$$

$$\begin{aligned}
G_X^{(2)}(0) &= \frac{d^2}{ds^2}\left[\frac{s^{n+1}-1}{(n+1)(s-1)}\right]\Bigg|_0 \\
&= \Bigg[\Big((n+1)\big(ns^{n-1}(s-1)+s^n\big)-(n+1)s^n\Big)(s-1)^2 \\
&\qquad -\Big(s^n(n+1)(s-1)-(s^{n+1}-1)\Big)2(s-1)\Bigg] \Big/ \big((n+1)(s-1)^4\big)\Bigg|_0 \\
&= \Bigg[\Big((n+1)\big(n0^{n-1}(0-1)+0^n\big)-(n+1)0^n\Big)(0-1)^2 \\
&\qquad -\Big(0^n(n+1)(0-1)-(0^{n+1}-1)\Big)2(0-1)\Bigg] \Big/ \big((n+1)(0-1)^4\big) \\
&= [0-2(0-1)] \,/\, \big((n+1)(0-1)^4\big) \\
&= \frac{2}{n+1}.
\end{aligned}$$

2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ, the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

Solution: **TODO: incorporate something about the Z_n generating function being around**

$$\begin{aligned}
G_{Z_{n+1}}(s) = E(s^{Z_{n+1}}) &= E(s^{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}}) \\
&= E(E(s^{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}} | \xi, Y)) \\
&= E(E(s^{\sum_{i=1}^{Z_n} \xi_i^{n+1}} s^{Y_{n+1}} | \xi, Y)) \\
&= E\left(E\left(\left(\prod_{i=1}^{Z_n} s^{\xi_i^{n+1}} \right) s^{Y_{n+1}} \middle| \xi, Y \right) \right) \\
&= E\left(\left(\prod_{i=1}^{Z_n} E\left(s^{\xi_i^{n+1}} \middle| \xi, Y \right) \right) E\left(s^{Y_{n+1}} \middle| \xi, Y \right) \right) \\
&= \left(\prod_{i=1}^{Z_n} E\left(s^{\xi_i^{n+1}} \middle| \xi \right) \right) E\left(s^{Y_{n+1}} \middle| Y \right) \\
&= \left(\prod_{i=1}^{Z_n} G_\xi(s) \right) G_Y(s)
\end{aligned}$$

3. (a) Let X be exponentially distributed with parameter λ. Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$. (b) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

Solution:

4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} \, 1_{x \geq 0}.$$

Solution: