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AMATH 561

PROBLEM SET 1

1. Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.
a) *Solution*
Let's assign the probability of getting a heads when tossing this biased coin once as p_0 and the probability of getting a tails as $1 - p_0$. Now to describe the probability space we need to define our outcome space Ω, the set of relevant events $\mathcal{F}$ (a σ-algebra) and the probability measure P then we will have our probability space $(\Omega, \mathcal{F}, P)$. In our experiment of 3 tosses of the coin we have

$$\Omega = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$$

And for simplicity let's define $\mathcal{F} = 2^\Omega$ which means any possible subset of Ω is an event. Now let's define $P : \mathcal{F} \to \mathbb{R}$. Let $A \in \mathcal{F}$ then we say

$$P(A) = \sum_{\omega \in A} p(\omega).$$

Where $p(\omega) = p_0^{h_\omega}(1 - p_0)^{t_\omega}$ with h_ω being the number of heads H in outcome ω and t_ω being the number of tails T in outcome ω. Therefore we can rewrite $P(A)$ as follows:

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} p_0^{h_\omega}(1 - p_0)^{t_\omega}.$$

Let's calculate an example and see if it matches our intuition. Define $B \in \mathcal{F}$ as the event such that at least one H occurs in the 3 tosses. Therefore $B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$. Now

$$\begin{aligned} P(B) &= \sum_{\omega \in B} p(\omega) = \sum_{\omega \in B} p_0^{h_\omega}(1 - p_0)^{t_\omega} \\ &= p_0^3(1 - p_0)^0 + p_0^2(1 - p_0)^1 + p_0^2(1 - p_0)^1 + p_0^2(1 - p_0)^1 \\ &\quad + p_0^1(1 - p_0)^2 + p_0^1(1 - p_0)^2 + p_0^1(1 - p_0)^2. \end{aligned}$$

□

b) *Solution*
Now we analyze the experiment where two balls are drawn without replacement from an urn which originally contained two blue and two red balls. It might be helpful to denote ball one and two as u_1 and u_2 respectively. Let's begin by looking at the outcome space Ω. In our two step experiment we get the following possible outcomes

$$\Omega = \{bb, br, rb, rr\}.$$

Once again, for simplicity let's define $\mathcal{F} = 2^\Omega$ which means any possible subset of Ω is an event. Now let's define $P : \mathcal{F} \to \mathbb{R}$. Let $A \in \mathcal{F}$ then we say

$$P(A) = \sum_{\omega \in A} p(\omega)$$

again. Now $p(\omega)$ is a little tricker to define since the probability for the second draw depends on what the first draw was. We have not described conditional probability in class yet, but I will borrow the notation and language of it here anyway. With this in mind we define

$$p(\omega) = p(\omega_1\omega_2) = p(u_1 = \omega_1)p(u_2 = \omega_2|u_1 = \omega_1)$$

with ω being broken down in to $\omega_1\omega_2$ where ω_1 and ω_2 are the particular color ball being drawn first or second respectively in outcome ω. We can define $p(u_1 = \omega_1)$ and $p(u_2 = \omega_2|u_1 = \omega_1)$ more explicitly with the following rules.

$$p(u_1 = \omega_1) = \frac{1}{2} \quad \text{for } \omega \in \{b, r\}$$

since there are 2 of each color and 4 balls total. Additionally we determine

$$p(u_2 = \omega_2|u_1 = \omega_1) = \frac{1}{3} \quad \text{if } \omega_2 = \omega_1$$

and

$$p(u_2 = \omega_2|u_1 = \omega_1) = \frac{2}{3} \quad \text{if } \omega_2 \neq \omega_1$$

since at the time of the second ball draw there is 1 ball of one color and 2 balls of another color with 3 balls total remaining. Therefore we can rewrite $P(A)$ as follows:

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} p(\omega_1\omega_2) = \sum_{\omega \in A} p(u_1 = \omega_1)p(u_2 = \omega_2|u_1 = \omega_1).$$

Let's calculate an example and see if it matches our intuition. Define $B \in \mathcal{F}$ as the event where the second ball drawn is red $u_2 = r$. Therefore $B = \{br, rr\}$. Now

$$\begin{aligned}
P(B) &= \sum_{\omega \in B} p(\omega) = \sum_{\omega \in B} p(\omega_1\omega_2) = \sum_{\omega \in B} p(u_1 = \omega_1)p(u_2 = \omega_2|u_1 = \omega_1) \\
&= p(u_1 = b)p(u_2 = r|u_1 = b) + p(u_1 = r)p(u_2 = r|u_1 = r) \\
&= \frac{1}{2}\frac{2}{3} + \frac{1}{2}\frac{1}{3} \\
&= \frac{1}{3} + \frac{1}{6} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus $P(B) = P(\{br, rr\}) = \frac{1}{2}$ which is what we'd expect since this particular event B is comprised of half of the four possible outcomes in Ω.

$\square$

2. (No translation-invariant random integer). Show that there is no probability measure P on the integers $\mathbb{Z}$ with the discrete σ-algebra $2^{\mathbb{Z}}$ with the translation-invariance property $P(E + n) = P(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer n. $E + n$ is obtained by adding n to every element of E.
Solution
Let's assume by way of contradiction, that there is a probability measure P on the

integers $\mathbb{Z}$ with a discrete σ-algebra $2^{\mathbb{Z}}$ and with the translation-invariance property $P(E+n) = P(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer n. Now take the event $E = \{0\}$ which is just a singleton set containing only 0. Using our assumption $P(E+n) = P(E)$ we get

$$P(\{0\}+n) = P(\{n\}) = P(\{0\})\ \forall n \in \mathbb{Z}.$$

Let's assign the value of $P(\{n\}) = P(\{0\}) = p$ for all $n \in \mathbb{Z}$. Now we have the following

$$P(\Omega) = \sum_{\omega\in\Omega} P(\omega) = \sum_{\omega\in\Omega} p.$$

If $p \neq 0$ this sum will blow up to ∞ therefore $p = 0$. However, this means $P(\Omega) = 0$ but that contradicts the assumption that P is a probability measure since that would require $P(\Omega) = 1$. Therefore there is no probability measure P on the integers $\mathbb{Z}$ with the discrete σ-algebra $2^{\mathbb{Z}}$ with the translation-invariance property.

□

3. (No translation-invariant random real). Show that there is no probability measure P on the reals $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property $P(E+x) = P(E)$ for every event $E \in \mathcal{B}(\mathbb{R})$ and every real x. Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by intervals $(a,b] \subset \mathbb{R}$.

Solution

Let's assume by way of contradiction, that there is a probability measure P on the reals $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property $P(E+x) = P(E)$ for every event $E \in \mathcal{B}(\mathbb{R})$ and every real x. Let $I = (0,1]$ be the interval of real numbers between 0 and 1. Consider the sets $I_n = I + n$ for $n \in \mathbb{Z}$, where each I_n is a translation of the interval I: $I_n = (n, n+1]$. The sets I_n are disjoint for different values of n that is

$$I_n \cap I_m = \emptyset \quad \text{for } n \neq m.$$

The way these sets are constructed we also get that, the union of these disjoint sets covers the entire real line:

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n.$$

By translation invariance of P, we have $P(I_n) = P(I)$ for all $n \in \mathbb{Z}$. Let $p = P(I)$. Then, since P is a probability measure we have

$$P(\mathbb{R}) = P\left(\bigcup_{n\in\mathbb{Z}} I_n\right) = \sum_{n\in\mathbb{Z}} P(I_n) = \sum_{n\in\mathbb{Z}} p.$$

Since the sum $\sum_{n\in\mathbb{Z}} p$ includes infinitely many terms, it will go to ∞ unless $p = 0$. Thus, $p = 0$, meaning $P(I) = 0$. Since $P(I) = 0$, by translation invariance, $P(I_n) = 0$ for all $n \in \mathbb{Z}$. Therefore:

$$P\left(\bigcup_{n\in\mathbb{Z}} I_n\right) = P(\mathbb{R}) = 0.$$

However, this contradicts our assumptions which required the fact that $P(\mathbb{R}) = 1$. Therefore, no probability measure P on the reals $\mathbb{R}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property.

□

4. Let $\Omega = \mathbb{R}$, $\mathcal{F} =$ all subsets of $\mathbb{R}$ so that A or A^c is countable. Let $P(A) = 0$ in the first case and $P(A) = 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Solution:

We want to show $(\Omega, \mathcal{F}, P)$ is a probability space. Therefore, we need to prove the following:

- $\mathcal{F}$ is a σ-algebra
  - Is non-empty
    Showing $\mathcal{F}$ is non empty is trivial since we know there exists at least one set A s.t. A or A^c is countable. For example, take the singleton set $B = \{0\}$ since $B \in \mathbb{R}$ and is countable then $B \in \mathcal{F}$. $\square$
  - If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
    Let $A \in \mathcal{F}$ then either A is countable or A^c is countable.
    Case 1: (A is countable) then A^c is uncountable. However, $(A^c)^c$ is countable therefore by construction of $\mathcal{F}$ A^c we can conclude $A^c \in \mathcal{F}$ as well.
    Case 2: (A is uncountable) then A^c is countable, therefore by construction of $\mathcal{F}$, it immediately follows that $A^c \in \mathcal{F}$. $\square$
  - Let A_i, $i \in \mathbb{N}$ be a countable collection of sets in $\mathcal{F}$, then their union is also in $\mathcal{F}$.
    We will argue this by going through cases of having two sets A and B each be countable, uncountable, and one of each. Suppose
    Incomplete
- P is a probability measure
  - $P(A) \geq P(\emptyset) = 0 \, \forall A \in \mathcal{F}$
    Incomplete
  - $P(\Omega) = 1$
    Incomplete
  - $P(\cup_i A_i) = \sum_i P(A_i)$ for a countable sequence of disjoint sets $A_i \in \mathcal{F}$
    Incomplete