AMATH 567 FALL 2024
HOMEWORK 1 — DUE SEPT. 30 ON GRADESCOPE BY 1:30PM
THE 48 HOUR LATE PENALTY IS WAIVED FOR THIS ASSIGNMENT

All solutions must include significant justification to receive full credit. If you handwrite your assignment you must either do so digitally or if it is written on paper you must *scan* your work. A standard photo is not sufficient.

If you work with others on the homework, you must name your collaborators.

1: From A&F: 1.1.1: (b, e) Express each of the following complex numbers in exponential form:
b) $-i$
Solution: $-i = 0 - i = 0 + i(-1) = \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = e^{i\frac{3\pi}{2}}$
e) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$
Solution: $\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3}) = e^{i\frac{5\pi}{3}}$

2: From A&F: 1.1.2: b, c, d

3: From A&F: 1.1.3: d

4: From A&F: 1.1.4: d,f

5: For $a, b \in \mathbb{C}$, define

$$a^b = e^{b\log a},$$

where $a = re^{i\theta}$, $-\pi < \theta \leq \pi$ and

$$\log a = \log r + i\theta,$$

is the principal branch of the logarithm. Find the real and imaginary parts of

$$i^i \quad \text{and} \quad (1+i)^i.$$

6: Consider the function $e(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$, which is defined for all $z \in \mathbb{C}$ (you need not show this). Using only the power series, show that $e(z_1 + z_2) = e(z_1)\, e(z_2)$. Can you find other power series with the same property?

7: Consider the complex-valued expression

$$f(z) = z^{1/2}$$

where $z = x + iy$, with $x, y \in \mathbb{R}$. Derive explicit expressions for the real and imaginary part(s) of $f(z)$ in terms of x and y. If you make any choices (e.g. for branch cuts), show how they impact your answer. Your answer should not contain any trig functions.

8: (Solution of the cubic) Consider the cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

where a, b and c are given numbers.

- Use the change of variables $x = y - a/3$ to reduce the equation to the form

  $$y^3 + py + q = 0$$

  Find expressions for p and q.
- Let $y = u+v$. We're replacing one unknown with two, so we get to impose another constraint later. Check that

  $$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$
- Now we impose $3uv + p = 0$, so that

  $$u^3v^3 = -p^3/27$$

  Also, from above, we have

  $$u^3 + v^3 = -q.$$

  Find a quadratic equation satisfied by both u^3 and v^3.
- Solve this quadratic equation, finding expressions for u and v.
- Finally, obtain an expression for x. How many different solutions does your expression give rise to?
- Use your result to solve the cubic $x^3 + 3x^2 + 6x + 8 = 0$.
- (Bombelli's equation) Use your result to solve the cubic $x^3 - 15x - 4 = 0$, writing your result explicitly in terms of real and imaginary parts.