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AMATH 567

HOMEWORK 9

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
Evaluate the integrals $\frac{1}{2\mathrm{i}\pi}\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C.

(a)

$$f(z) = \frac{z^2+1}{z^2-a^2}, \quad a^2 < 1$$

Solution:
Let the set of singularities of $f(z)$ be S

$$\frac{1}{2\pi\mathrm{i}}\oint_C \frac{z^2+1}{z^2-a^2}\mathrm{d}z = \sum_{w\in S}\operatorname*{Res}_{z=w} f(z).$$

The denominator can be factored to $z^2-a^2 = (z-a)(z+a)$, therefore the singularities of $f(z)$ are $z = \pm a$. Then we have

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\oint_C \frac{z^2+1}{z^2-a^2}\mathrm{d}z &= \operatorname*{Res}_{z=-a} f(z) + \operatorname*{Res}_{z=a} f(z)\\ &= \frac{(-a)^2+1}{(-a-a)} + \frac{(a)^2+1}{(a+a)}\\ &= -\frac{a^2+1}{2a} + \frac{a^2+1}{2a} = 0.\end{aligned}$$

□

(b)

$$f(z) = \frac{z^2+1}{z^3}$$

Solution:
Looking at this in terms of the residue we have

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\oint_C \frac{z^2+1}{z^3}\mathrm{d}z &= \operatorname*{Res}_{z=0} f(z)\\ &= \frac{1}{(3-1)!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left((z-0)^3\left(\frac{z^2+1}{z^3}\right)\right)\Big|_0\\ &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(z^2+1\right)\Big|_0\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left(2z\right)\Big|_0\\ &= \frac{1}{2}2\\ &= 1.\end{aligned}$$

□

(c)

$$f(z) = z^2\,\mathrm{e}^{-1/z}$$

Solution:
Looking at this in terms of the residue we have, what is the order of this pole?? Do I need to do something else?

$$\frac{1}{2\pi\mathrm{i}}\oint_C z^2\,\mathrm{e}^{-1/z}\,\mathrm{d}z = \operatorname*{Res}_{z=0} f(z)$$

Let's look at things, in terms of the Taylor series expansion

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\oint_C z^2\,\mathrm{e}^{-1/z}\,\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}}\oint_C z^2\sum_{j=0}^{\infty}\left(-\frac{1}{z}\right)^j\frac{1}{j!}\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\oint_C \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{z^2}{z^j}\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\oint_C \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{1}{z^{j-2}}\mathrm{d}z\\ &= \operatorname*{Res}_{z=0}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{1}{z^{j-2}}\right)\\ &= \operatorname*{Res}_{z=0}\left(\frac{(-1)^0}{0!}\frac{1}{z^{0-2}}+\frac{(-1)^1}{1!}\frac{1}{z^{1-2}}+\frac{(-1)^2}{2!}\frac{1}{z^{2-2}}+\frac{(-1)^3}{3!}\frac{1}{z^{3-2}}+\sum_{j=4}^{\infty}\frac{(-1)^j}{j!}\frac{1}{z^{j-2}}\right)\\ &= \operatorname*{Res}_{z=0}\left(z^2-z+\frac{1}{2}-\frac{1}{6z}+\sum_{j=4}^{\infty}\frac{(-1)^j}{j!}\frac{1}{z^{j-2}}\right)\\ &= -\frac{1}{6}.\end{aligned}$$

□

2: From A&F: 4.2.1(b)
Evaluate the following real integral

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a^2 > 0$$

Solution:
For convenience throughout this problem lets define

$$f(x) = \frac{1}{(x^2+a^2)^2}.$$

Since the $f(x)$ is an even function we can say

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}.$$

Then we can use the principal value integral which is given by

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^R f(x)\mathrm{d}x, \quad \text{if it exists.}$$

Now let's consider the counterclockwise contour C around the closed semicircle centered at the origin in the upper half plane with radius R. We can say

(1) $$\oint_C f(z)\mathrm{d}z = \int_{-R}^R f(x)\mathrm{d}x + \oint_{C_R} f(z)\mathrm{d}z,$$

where $-R$ to R is the section of C along the real axis and C_R is the open semicircle (counterclockwise). We can combine these ideas by taking the limit of both sides as $R\to\infty$. Let's take the limit of the right and analyze what is going on. This gives us

$$\begin{aligned}\lim_{R\to\infty}\left(\int_{-R}^R f(x)\mathrm{d}x + \oint_{C_R} f(z)\mathrm{d}z\right) &= \lim_{R\to\infty}\int_{-R}^R f(x)\mathrm{d}x + \lim_{R\to\infty}\oint_{C_R} f(z)\mathrm{d}z \\ &= \int_{-\infty}^\infty f(x)\mathrm{d}x + \lim_{R\to\infty}\oint_{C_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}\end{aligned}$$

Let's try to bound the integrand to something that depends on R^{-1}. We will use the substitution $z = R\,\mathrm{e}^{\mathrm{i}\theta}$ with $\theta\in[0,\pi]$.

$$\begin{aligned}\left|\frac{1}{(z^2+a^2)^2}\right| &= \left|\frac{1}{(R^2\,\mathrm{e}^{2\mathrm{i}\theta}+a^2)^2}\right| \\ &= \frac{1}{|R^4\,\mathrm{e}^{4\mathrm{i}\theta}+2R^2a^2\,\mathrm{e}^{2\mathrm{i}\theta}+a^4|} \\ &\le \frac{1}{R^2\,|R^2\,\mathrm{e}^{4\mathrm{i}\theta}+2a^2\,\mathrm{e}^{2\mathrm{i}\theta}|}\end{aligned}$$

Since $f(z)$ is continuous on C_R, we can use the ML bound on the integral to say

$$\begin{aligned}\lim_{R\to\infty}\left|\oint_{C_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}\right| &\le \lim_{R\to\infty}\frac{\pi R}{R^2\,|R^2\,\mathrm{e}^{4\mathrm{i}\theta}+2a^2\,\mathrm{e}^{2\mathrm{i}\theta}|} \\ &= \lim_{R\to\infty}\frac{\pi}{R\,|R^2\,\mathrm{e}^{4\mathrm{i}\theta}+2a^2\,\mathrm{e}^{2\mathrm{i}\theta}|} = 0.\end{aligned}$$

Applying a similar squeeze theorem argument from homework 4 problem 4, we can conclude

$$\lim_{R\to\infty}\oint_{C_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}=0.$$

Therefore, equation (1) becomes

$$\begin{aligned}\lim_{R\to\infty}\oint_C f(z)\mathrm{d}z &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x+\cancelto{0}{\oint_{C_R} f(z)\mathrm{d}z}\\ \lim_{R\to\infty}\oint_C f(z)\mathrm{d}z &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x\end{aligned}$$

Notice, as R goes to infinity the integral of $f(z)$ along the contour C is equal to the sum of all of the residues at all singularities in the upper half plane for $f(z)$. Let S be the collection of singularities in the upper half plane, then we can say

$$\lim_{R\to\infty}\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{w\in S}\operatorname*{Res}_{z=w}\left(\frac{1}{(z^2+a^2)^2}\right).$$

Hence,

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}\mathrm{d}x = 2\pi\mathrm{i}\sum_{w\in S}\operatorname*{Res}_{z=w}\left(\frac{1}{(z^2+a^2)^2}\right).$$

We now need to locate the singularities. The denominator is only 0 when $z^2+a^2=0$ so we know the singularities are at $z=\pm\mathrm{i}a$. However, since only $z=\mathrm{i}a$ is in the upper half plane we can simplify the previous equation and solve for the one residue we need

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}\mathrm{d}x &= 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}a}\left(\frac{1}{(z^2+a^2)^2}\right)\\ &= 2\pi\mathrm{i}\frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-\mathrm{i}a)^2\frac{1}{(z^2+a^2)^2}\right)\Bigg|_{\mathrm{i}a}\\ &= 2\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-\mathrm{i}a)^2\frac{1}{(z+\mathrm{i}a)^2(z-\mathrm{i}a)^2}\right)\Bigg|_{\mathrm{i}a}\\ &= 2\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{(z+\mathrm{i}a)^2}\right)\Bigg|_{\mathrm{i}a}\\ &= 2\pi\mathrm{i}\frac{-2}{(\mathrm{i}a+\mathrm{i}a)^3}\\ &= 2\pi\mathrm{i}\frac{-2}{-8\mathrm{i}a^3}\\ &= \frac{\pi}{2a^3}.\end{aligned}$$

Recall our original integral was

$$\int_0^{\infty}\frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{1}{2}\frac{\pi}{2a^3}=\frac{\pi}{4a^3}$$

$\square$

3: Existence and uniqueness of polynomial interpolants.

(a) Suppose $(z_j)_{j=1}^n$ are distinct points in $\mathbb{C}$ and suppose $f_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Show that there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_j) = f_j$ for $j = 1, \ldots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant*.

Solution:
Suppose there exists two polynomials $p_1(z)$ and $p_2(z)$ each of degree $n-1$. Assume both agree with f_j at each z_j such that

$$p_1(z_j) = p_2(z_j) = f_j \quad \text{for each } j = 1, ..., n.$$

Additionally define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Now let's consider the function

$$g(z) = \frac{p_1(z) - p_2(z)}{\nu(z)}.$$

We want to utilize Liouville's theorem to conclude that $g(z)$ is constant. In order to do this we need to show that $g(z)$ is entire and bounded. Let's begin by demonstrating that it is bounded by taking the limit as $z \to \infty$

$$\begin{aligned} \lim_{z\to\infty} g(z) &= \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\nu(z)} \\ &= \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\prod_{j=1}^n (z - z_j)} \\ &= \frac{\infty}{\infty}. \end{aligned}$$

Applying L'Hôpitals rule repeatedly we will end up with $1/z$ which goes to 0 as z goes to infinity since the denominator is an nth degree polynomial while the numerator is a degree $n-1$ polynomial. Therefore,

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\nu(z)} = 0,$$

which implies that $g(z)$ is bounded. Next, we need to determine if $g(z)$ is entire. Since polynomials are entire in the finite z plane, $p_1(z) - p_2(z)$ is entire. However, $g(z)$ overall requires a little more analysis since it has singularities where $z = z_j$. Notice, since the expression $p_1(z) - p_2(z)$ and $\nu(z)$ are both zero at each z_j, then there exists a factorization of $p_1(z) - p_2(z)$ which would allows us to cancel out each of the factors in the product in the denominator. Therefore, the singularities of $g(z)$ are removable and thus $g(z)$ is entire (or can be made entire, with the right extension at each z_j as we have done in previous assignments). Hence, by Liouville's Theorem, we can conclude that $g(z)$ is constant. Combining with the fact that $p_1(z_j) - p_2(z_j) = 0$ for each $j = 1, ..., n$, then $g(z)$ must be 0 everywhere, thus implying $p_1(z) = p_2(z)$ everywhere. In conclusion, since these two functions are the same therefore there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_j) = f_j$ for $j = 1, \ldots, n$, otherwise known at the interpolant.

$\square$

(b) Define the node polynomial $\nu(z) = \prod_{j=1}^{n}(z - z_j)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:
Let's look at $p(z)/\nu(z)$ and consider what happens if we subtract off a specially cooked up collection of terms including the residues r_j for $j = 1, ..., n$. We can express the residues of $p(z)/\nu(z)$ as

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{p(z)}{\nu(z)} \mathrm{d}z = \sum_{j=0}^{n} \mathrm{Res}\left(\frac{p(z)}{\nu(z)}; z_j\right) = \sum_{j=0}^{n} \frac{f_j}{\prod_{k \neq j}(z_k - z_j)}.$$

Recall partial fractions is connected to the residues. We construct the expression to subtract from $p(z)/\nu(z)$ using the partial fraction decomposition relationship to residues

$$\frac{p(z)}{\nu(z)} - \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$= \frac{p(z)}{\nu(z)} - \frac{f_1\left(\prod_{k \neq 1}(z_k - z_1)\right)^{-1}}{z - z_1} - \frac{f_2\left(\prod_{k \neq 2}(z_k - z_2)\right)^{-1}}{z - z_2} - \dots - \frac{f_n\left(\prod_{k \neq n}(z_k - z_n)\right)^{-1}}{z - z_n} = 0.$$

TODO: Why is this 0 besides saying it's the partial fraction decomposition? This expression is equal to 0 because the collection of terms we are subtracting is the partial fraction decomposition of $p(z)/\nu(z)$. If we can show that this function is bounded and entire then it is a constant. Therefore, we would be able to state that since it is a constant and 0 then it must be a 0 everywhere. Thus we can say

$$\frac{p(z)}{\nu(z)} - \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j} = 0$$

$$\frac{p(z)}{\nu(z)} = \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$p(z) = \nu(z) \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$p(z) = \prod_{j=1}^{n}(z - z_j) \sum_{j=0}^{n} \frac{f_j\left(\prod_{k \neq j}(z_k - z_j)\right)^{-1}}{z - z_j}$$

$$p(z) = \sum_{j=0}^{n} \frac{f_j \prod_{\ell \neq j}(z - z_\ell)}{\prod_{k \neq j}(z_k - z_j)}.$$

Therefore we have this expression for $p(z)$.

4: Bernstein interpolation formula. Suppose that $-1 \leq x_1 < x_2 < \cdots x_n \leq 1$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1,1]$. Show that for any simple contour C inside Ω with $[-1,1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{\mathrm{d}z}{\nu(z)}, \quad x \in [-1,1],$$

where p is the degree $n-1$ polynomial interpolant satisfying $p(x_j) = f(x_j)$ for $j = 1, 2, \ldots, n$. We also have $\nu(x) = \prod_{j=1}^{n}(x - x_j)$.

Solution:
Starting from the right we have

$$\begin{aligned}\frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{\mathrm{d}z}{\nu(z)} &= \frac{1}{2\pi i} \int_C \frac{f(z)\nu(x)}{(z-x)\nu(z)} \mathrm{d}z \\ &= \operatorname*{Res}_{z=x}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0\right) + \sum_{i=1}^{n} \operatorname*{Res}_{z=x_i}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0\right). \qquad (2)\end{aligned}$$

Calculating the residue at $z = x$ is easy because x is a simple pole

$$\operatorname*{Res}_{z=x}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0\right) = \operatorname*{Res}_{z=x}\left(\frac{\frac{f(z)\nu(x)}{\nu(z)}}{(z-x)}; 0\right) = \frac{f(x)\nu(x)}{\nu(x)} = f(x).$$

Calculating the residue at each $z = x_i$ is similarly quick since they are simple poles

$$\begin{aligned}\sum_{i=1}^{n} \operatorname*{Res}_{z=x_i}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0\right) &= \sum_{i=1}^{n} \operatorname*{Res}_{z=x_i}\left(\frac{f(z)\nu(x)}{(z-x)\prod_{j=1}^{n}(z-x_j)}; 0\right) \\ &= \sum_{i=1}^{n} \operatorname*{Res}_{z=x_i}\left(\left(\frac{f(z)\nu(x)}{(z-x)\prod_{\substack{j=1 \\ j\neq i}}^{n}(z-x_j)}\right) \Big/ (z-x_i) \;;\; 0\right) \\ &= \sum_{i=1}^{n} \frac{f(x_i)\nu(x)}{(x_i - x)\prod_{\substack{j=1 \\ j\neq i}}^{n}(x_i - x_j)} \\ &= -\sum_{i=1}^{n} \frac{f(x_i)\prod_{j=1}^{n}(x - x_j)}{(x - x_i)\prod_{\substack{j=1 \\ j\neq i}}^{n}(x_i - x_j)} \\ &= -\sum_{i=1}^{n} \frac{f(x_i)\prod_{\substack{j=1 \\ j\neq i}}^{n}(x - x_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(x_i - x_j)} = -p(x) \qquad (3)\end{aligned}$$

Where we know this is $p(z)$ from our work in problem 4.
Therefore we have Equation (2) is equal to $f(x) - p(x)$. Furthermore, since $\nu(x_i) = 0$ for each $i = 1, ..., n$, then

$$\begin{aligned}f(x_i) - p(x_i) &= 0 \\ f(x_i) &= p(x_i)\end{aligned}$$

for all $i = 1, ..., n$. Finally we can also determine the degree of the polynomial $p(x)$ is $n-1$. This is due the equation (3) being made up of some scalar or weight factor and the product in the numerator which is $\nu(x)$ (a degree n polynomial) but without one of it's factors leaving it as an $n-1$ degree polynomial. $\square$

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1,1].$$

(a) Show that the polynomial

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right),$$

has all of its roots $x_1 < x_2 < \cdots x_n$ within $[-1,1]$.

Solution:
Let's begin by looking more closely at $\varphi(z)$ with a specific substitution, namely $z = \cos\theta$ with $\theta \in [0,\pi]$. Then we have

$$\begin{aligned}
\varphi(\cos\theta) &= \cos\theta + \sqrt{\cos\theta - 1}\sqrt{\cos\theta + 1} \\
&= \cos\theta - \mathrm{i}\sqrt{1-\cos\theta}\sqrt{1+\cos\theta} \\
&= \cos\theta - \mathrm{i}\sqrt{1-\cos^2\theta} \\
&= \cos\theta - \mathrm{i}\sqrt{\sin^2\theta} \\
&= \cos\theta - \mathrm{i}\sin\theta \\
&= \mathrm{e}^{-\mathrm{i}\theta}.
\end{aligned}$$

Now putting this thing again we have

$$\begin{aligned}
T_n(\cos\theta) &= \frac{1}{2}\left(\varphi(\cos\theta)^n + \varphi(\cos\theta)^{-n}\right) \\
&= \frac{1}{2}\left(\left(\mathrm{e}^{-\mathrm{i}\theta}\right)^n + \left(\mathrm{e}^{-\mathrm{i}\theta}\right)^{-n}\right) \\
&= \frac{1}{2}\left(\mathrm{e}^{-n\mathrm{i}\theta} + \mathrm{e}^{n\mathrm{i}\theta}\right) \\
&= \frac{1}{2}\left(\cos n\theta - \cancel{\mathrm{i}\sin n\theta} + \cos n\theta + \cancel{\mathrm{i}\sin n\theta}\right) \\
&= \frac{1}{2}(2\cos n\theta) \\
&= \cos n\theta.
\end{aligned}$$

Notice, $\theta = \arccos z$, then we have

$$T_n(z) = \cos\left(n \arccos z\right).$$

Let's find out what values of z this function $T_n(z) = 0$, let $k \in \mathbb{Z}$, then

$$\begin{aligned}
n \arccos z &= \frac{\pi}{2} + \pi k \\
\arccos z &= \frac{1}{n}\left(\frac{\pi}{2} + \pi k\right) \\
z &= \cos\left(\frac{1}{n}\left(\frac{\pi}{2} + \pi k\right)\right).
\end{aligned}$$

Therefore, the zeros of $T_n(z)$ are all within the range of cos which is between $[-1,1]$. Additionally, because we are dividing by n there will be n zeros between -1 and 1.

$\square$

(b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1, 1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:
Let's consider the image of a circle with radius $\rho > 1$, if we parameterize this with $z = \rho\,\mathrm{e}^{\mathrm{i}\theta}$ and plug this in to $J(w)$ we have

$$\begin{aligned}
J(\rho\,\mathrm{e}^{\mathrm{i}\theta}) &= \frac{1}{2}\left(\rho\,\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{\rho}\,\mathrm{e}^{-\mathrm{i}\theta}\right) \\
&= \frac{1}{2}\left(\rho\cos\theta + \rho\sin\theta + \frac{1}{\rho}\cos\theta - \mathrm{i}\frac{1}{\rho}\sin\theta\right) \\
&= \frac{1}{2}\left(\left(\rho + \frac{1}{\rho}\right)\cos\theta + \mathrm{i}\left(\rho - \frac{1}{\rho}\right)\sin\theta\right).
\end{aligned}$$

Notice this is the equation of an ellipse since it is a slightly stretched and flattened out circle. This is almost a circle of radius ρ but it is slightly stretched in different amounts in the x (real) and y (imaginary) directions. Therefore the image is an ellipse.

We wish to show that $\varphi(J(w)) = w$. Notice,

$$\begin{aligned}
\varphi(J(w)) &= J(w) + \sqrt{J(w) - 1}\sqrt{J(w) + 1} \\
&= J(w) + \sqrt{1/2(w + 1/w) - 1}\sqrt{1/2(w + 1/w) + 1} \\
&= J(w) + \sqrt{\frac{w}{2} + \frac{1}{2w} - 1}\sqrt{\frac{w}{2} + \frac{1}{2w} + 1} \\
&= \frac{1}{2}\left(w + \frac{1}{w}\right) + \sqrt{\frac{w^2}{2w} + \frac{1}{2w} - \frac{2w}{2w}}\sqrt{\frac{w^2}{2w} + \frac{1}{2w} + \frac{2w}{2w}} \\
&= \frac{1}{2}\left(w + \frac{1}{w}\right) + \sqrt{\frac{(w-1)^2}{2w}}\sqrt{\frac{(w+1)^2}{2w}} \\
&= \frac{1}{2}\left(w + \frac{1}{w}\right) + \frac{(w-1)(w+1)}{2w} \\
&= \frac{1}{2}\left(w + \frac{1}{w}\right) + \frac{w^2 - 1}{2w} \\
&= \frac{1}{2}\left(w + \frac{1}{w}\right) + \frac{1}{2}\left(w - \frac{1}{w}\right) \\
&= \frac{1}{2}w + \cancel{\frac{1}{2w}} + \frac{1}{2}w - \cancel{\frac{1}{2w}} \\
&= w.
\end{aligned}$$

Hence, we have what we desired.

$\square$

(c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2\frac{M|B_\rho|}{\pi}(\rho^n - \rho^{-n})^{-1}(\rho + \rho^{-1} - 2)^{-1} \leq 2\frac{M|B_\rho|}{\pi}\frac{\rho^{2-n}}{(\rho-1)^3}$$
$$\leq C_\rho \rho^{-n}, \quad \text{for a constant} \quad C_\rho > 0,$$

where $p(x_j) = f(x_j)$, i.e., p is the degree $n-1$ interpolant of f at the roots of T_n. Here $|B_\rho|$ denotes the arclength of B_ρ. This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f.

Solution:
Assume f is analytic in a region that contains the ellipse B_ρ and it's interior. Additionally, assume $|f(z)| < M$ for z interior to B_ρ. Now consider where $-1 \leq x \leq 1$ and let's look at bounding the following

$$|f(x) - p(x)| = \left| \frac{\nu(x)}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z-x}\frac{\mathrm{d}z}{\nu(z)} \right|.$$

First of all, utilizing the conclusions of the Bernstein–Walsh inequality we know that $|\nu(x)| = \frac{1}{2^{n-1}}|T_n(x)|$. Therefore, we change the instances of $\nu(x)$ to $T_n(x)$'s with the appropriate scaling. Hence

$$\begin{aligned}
\left| \frac{\nu(x)}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z-x}\frac{\mathrm{d}z}{\nu(z)} \right| &= \left| \frac{T_n(x)}{2^n \pi \mathrm{i}} \oint_C \frac{f(z)}{z-x}\frac{2^{n-1}\mathrm{d}z}{T_n(z)} \right| \\
&= \left| \frac{1}{2\pi \mathrm{i}} T_n(x) \oint_C \frac{f(z)}{z-x}\frac{\mathrm{d}z}{T_n(z)} \right| \\
&\leq \frac{1}{2\pi}|T_n(x)| \oint_C \left| \frac{f(z)}{z-x} \right| \frac{1}{|T_n(z)|}|\mathrm{d}z|.
\end{aligned}$$

Recall in homework 3 we showed $|T_n(z)| \leq 1$ on the interval $[-1, 1]$. We take the contour C to be the ellipse B_ρ and parameterize with $z = J(w) = \frac{1}{2}(w + \frac{1}{w})$. Then $\mathrm{d}z = J'(w)\mathrm{d}w = \frac{1}{2}(1 - 1/w^2)\mathrm{d}w$. Now we continue with our from the previous inequality

$$\begin{aligned}
&= \frac{1}{2\pi} \oint_{B_\rho} \left| \frac{f(z)}{J(w) - x} \right| \frac{1}{|T_n(J(w))|} \left| \frac{1}{2}\left(1 - \frac{1}{w^2}\right) \mathrm{d}w \right| \\
&= \frac{1}{2\pi} \oint_{B_\rho} \left| \frac{f(z)}{\frac{1}{2}(w + \frac{1}{w}) - x} \right| \frac{1}{\left|\frac{1}{2}\left(\varphi(J(w))^n + \varphi(J(w))^{-n}\right)\right|} \left| \frac{1}{2}\left(1 - \frac{1}{w^2}\right) \mathrm{d}w \right| \\
&= \frac{1}{2\pi} \oint_{B_\rho} \left| \frac{f(z)}{\frac{1}{2}(w + \frac{1}{w}) - x} \right| \frac{1}{\left|\frac{1}{2}\left(w^n + w^{-n}\right)\right|} \left| \frac{1}{2}\left(1 - \frac{1}{w^2}\right) \mathrm{d}w \right|
\end{aligned}$$

In the last few steps we used the fact from part (b) which showed $\varphi(J(w)) = w$. Now we will use $w = \rho\, \mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{d}w = \rho \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}$ we have

$$= \frac{1}{2\pi} \oint_{B_\rho} \frac{|f(z)|}{|\frac{1}{2}(\rho\, \mathrm{e}^{\mathrm{i}\theta} + \frac{1}{\rho}\, \mathrm{e}^{-\mathrm{i}\theta}) - x|} \frac{1}{\left|\frac{1}{2}\left(\rho^n\, \mathrm{e}^{n\mathrm{i}\theta} + \rho^{-n}\, \mathrm{e}^{-n\mathrm{i}\theta}\right)\right|} \left| \frac{1}{2}\left(1 - \frac{1}{\rho^2\, \mathrm{e}^{2\mathrm{i}\theta}}\right) \rho \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta \right|.$$

Notice, applying the reverse triangle inequality to this term we have

$$\frac{1}{\frac{1}{2}\left|\rho^n\, \mathrm{e}^{n\mathrm{i}\theta} + \rho^{-n}\, \mathrm{e}^{-n\mathrm{i}\theta}\right|} \leq \frac{1}{\frac{1}{2}\Big| |\rho^n\, \mathrm{e}^{n\mathrm{i}\theta}| - |-\rho^{-n}\, \mathrm{e}^{-n\mathrm{i}\theta}| \Big|} \leq \frac{1}{\frac{1}{2}\left|\rho^n - \rho^{-n}\right|}.$$

Additionally, using a well known ellipse fact, we have

$$\begin{aligned}
\frac{1}{|\frac{1}{2}(\rho\,e^{i\theta}+\frac{1}{\rho}\,e^{-i\theta})-x|} &\le \frac{1}{\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-1} \\
&\le \frac{1}{\frac{\rho+\frac{1}{\rho}-2}{2}} \\
&= \frac{2}{\rho+\frac{1}{\rho}-2}
\end{aligned}$$

Now we can move forward with our original statement, applying these two inequalities at once,

$$\begin{aligned}
&= \frac{1}{2\pi}\oint_{B_\rho}\frac{|f(z)|}{|\frac{1}{2}(\rho\,e^{i\theta}+\frac{1}{\rho}\,e^{-i\theta})-x|}\frac{1}{\left|\frac{1}{2}\left(\rho^n\,e^{ni\theta}+\rho^{-n}\,e^{-ni\theta}\right)\right|}\left|\frac{1}{2}\left(1-\frac{1}{\rho^2\,e^{2i\theta}}\right)\rho i\,e^{i\theta}\,d\theta\right| \\
&\le \frac{1}{2\pi}\frac{1}{\frac{1}{2}\left|\rho^n-\rho^{-n}\right|}\frac{2}{\left(\rho+\frac{1}{\rho}-2\right)}\oint_{B_\rho}|f(z)|\left|\frac{1}{2}\left(1-\frac{1}{\rho^2\,e^{2i\theta}}\right)\rho i\,e^{i\theta}\,d\theta\right| \\
&\le \frac{1}{\pi}\frac{1}{|\rho^n-\rho^{-n}|}\frac{2}{\left(\rho+\frac{1}{\rho}-2\right)}\oint_{B_\rho}|f(z)|\left|\frac{1}{2}\left(1-\frac{1}{\rho^2\,e^{2i\theta}}\right)\rho i\,e^{i\theta}\,d\theta\right| \\
&\le \frac{2M\left|B_\rho\right|}{\pi}\frac{1}{|\rho^n-\rho^{-n}|}\frac{1}{\left(\rho+\frac{1}{\rho}-2\right)} \\
&\le \frac{2M\left|B_\rho\right|}{\pi}(\rho^n-\rho^{-n})^{-1}\left(\rho+\rho^{-1}-2\right)^{-1}.
\end{aligned}$$

Now to get to the final inequality requested we have

$$\begin{aligned}
\frac{1}{\rho^n-\rho^{-n}}\frac{1}{\rho+\rho^{-1}-2} &= \frac{1}{\rho^n-\rho^{-n}}\frac{\rho}{\rho^2-2\rho+1} \\
&= \frac{1}{\rho^n-\rho^{-n}}\frac{\rho}{(\rho-1)^2}\frac{\rho^{-n}}{\rho^{-n}} \\
&= \frac{\rho^{-n}}{1-\rho^{-2n}}\frac{\rho}{(\rho-1)^2} \\
&= \frac{1}{1-\rho^{-2n}}\frac{\rho^{1-n}}{(\rho-1)^2}
\end{aligned}$$

Now consider

$$\begin{gathered}
\rho^{2n}\ge\rho \implies \\
\rho^{-2n}\le\rho^{-1} \implies \\
-\rho^{-2n}\ge-\rho^{-1} \implies \\
1-\rho^{-2n}\ge 1-\rho^{-1}
\end{gathered}$$

thus

$$\frac{1}{1-\rho^{-2n}}\le\frac{1}{1-\rho^{-1}}$$

Apply this to the term on the left then you get

$$\frac{1}{1-\rho^{-2n}}\frac{\rho^{1-n}}{(\rho-1)^2}\le\frac{1}{1-\rho^{-1}}\frac{\rho^{1-n}}{(\rho-1)^2}=\frac{\rho}{\rho-1}\frac{\rho^{1-n}}{(\rho-1)^2}=\frac{\rho^{2-n}}{(\rho-1)^3}$$

$\square$

6: Compute the following two integrals explicitly for $z \notin [-1,1]$:

(a)

$$\frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x}\sqrt{1+x}}\frac{\mathrm{d}x}{x-z}.$$

Solution:

We first recall that from homework 8 problem 4 part a) we showed

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\mathrm{i}}\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}. \tag{4}$$

Applying that here we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\frac{1}{x-z_0}\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\frac{1}{z-z_0}\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

For notational convenience let

$$g(z) = \frac{\frac{1}{z-z_0}}{\sqrt{z-1}\sqrt{z+1}}.$$

As we expand our contour C outwards we run into the singularity at z_0, leaving behind a clockwise circular contour around z_0 denoted as $-C_{z_0}$. We also have the normal counterclockwise contour around infinity which we will use in our residue calculation. Hence we have

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\oint_C g(z)\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}}\oint_{-C_{z_0}} g(z)\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\oint_{C_\infty} g(z)\mathrm{d}z \\ &= -\frac{1}{2\pi\mathrm{i}}\oint_{C_{z_0}} g(z)\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\oint_{C_\infty} g(z)\mathrm{d}z \\ &= -\operatorname*{Res}_{z=z_0} g(z) + \operatorname*{Res}_{z=\infty} g(z) \qquad (5)\end{aligned}$$

Now we want to calculate the residues at ∞ and at z_0. Let

$$h(z) = \frac{1}{\sqrt{z-1}\sqrt{z+1}}$$

and

$$H(z) = h\left(\frac{1}{z}\right) = \frac{1}{\sqrt{1/z-1}\sqrt{1/z+1}}\frac{z}{z} = \frac{z}{\sqrt{1-z}\sqrt{1+z}}.$$

Then we can see $H(0)=0$. Let's calculate $h'(0)$.

$$H'(z) = \frac{\sqrt{1-z}\sqrt{1+z} - z\big(-1/2(1-z)^{-1/2}(1+z)^{1/2} + 1/2(1-z)^{1/2}(1+z)^{-1/2}\big)}{(1-z)(1+z)}$$

Hence,

$$H'(0) = \frac{\sqrt{1}\sqrt{1} - 0\big(-1/2(1)^{-1/2}(1)^{1/2} + 1/2(1)^{1/2}(1)^{-1/2}\big)}{(1)(1)} = 1.$$

Then our Taylor series expansion of $H(z)$ is

$$\begin{aligned}H(z) &= H(0)z^0/0! + H'(0)z^1/1! + \mathcal{O}(z^2) \\ &= 0 + z + \mathcal{O}(z^2) \\ &= z + \mathcal{O}(z^2)\end{aligned}$$

then for $h(z)$ is

$$h(z) = z^{-1} + \mathcal{O}(z^{-2}).$$

We really care about $\frac{1}{z-z_0}h(z)$ so we have

$$\begin{aligned}\frac{1}{z-z_0}h(z) &= \frac{1}{z-z_0}\left(z^{-1} + \mathcal{O}(z^{-2})\right)\\ &= \frac{1}{z}\frac{1}{1-z_0/z}\left(z^{-1} + \mathcal{O}(z^{-2})\right)\\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\left(z^{-1} + \mathcal{O}(z^{-2})\right)\end{aligned}$$

where $|z_0| < |z|$ since we are on a contour with a large radius R. Then

$$\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\left(z^{-1} + \mathcal{O}(z^{-2})\right) = \frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k + \mathcal{O}(z^{-3})\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k$$

Therefore the residue of this function at ∞ is trivially

$$\operatorname*{Res}_{z=\infty}\left(\frac{1}{(z-z_0)\sqrt{z-1}\sqrt{z+1}}\right) = 0$$

since the coefficient of the $1/z$ is 0. Computing the residue at z_0 is a little easier since it is a simple pole. Therefore

$$\begin{aligned}\operatorname*{Res}_{z=z_0}\left(\frac{1}{(z-z_0)\sqrt{z-1}\sqrt{z+1}}\right) &= \operatorname*{Res}_{z=z_0}\left(\frac{\frac{1}{\sqrt{z-1}\sqrt{z+1}}}{z-z_0}\right)\\ &= \frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}}.\end{aligned}$$

Plugging these into equation (5) we have

$$\frac{1}{2\pi\mathrm{i}}\oint_C g(z)\mathrm{d}z = -\operatorname*{Res}_{z=z_0}g(z) + \operatorname*{Res}_{z=\infty}g(z) = -\frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}} + 0.$$

Hence,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\frac{1}{x-z_0}\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = -\frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}}.$$

$\square$

(b)

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x}\sqrt{1+x}\frac{\mathrm{d}x}{x-z}.$$

Solution:
Again applying equation (4), we have

$$\begin{aligned}\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x}\sqrt{1+x}\,\frac{\mathrm{d}x}{x-z} &= \frac{2}{\pi}\int_{-1}^{1}\frac{1-x^2}{\sqrt{1-x}\sqrt{1+x}}\frac{\mathrm{d}x}{x-z} \\ &= \frac{1}{\pi \mathrm{i}}\oint_C \frac{1-z^2\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}\frac{\mathrm{d}z}{z-z_0} \\ &= \frac{1}{\pi \mathrm{i}}\oint_C \frac{(1-z)(1+z)}{\sqrt{z-1}\sqrt{z+1}}\frac{\mathrm{d}z}{z-z_0} \\ &= -\frac{1}{\pi \mathrm{i}}\oint_C \sqrt{z-1}\sqrt{z+1}\frac{1}{z-z_0}\mathrm{d}z.\end{aligned}$$

Let

$$g(z)=\sqrt{z-1}\sqrt{z+1}\frac{1}{z-z_0}.$$

Then as we expand our contour C outwards we run into the singularity at z_0, leaving behind a clockwise circular contour around z_0 denoted as $-C_{z_0}$. We also have the normal counterclockwise contour around infinity which we will use in our residue calculation. Hence we have

$$\begin{aligned}-\frac{1}{\pi \mathrm{i}}\oint_C g(z)\mathrm{d}z &= -\frac{1}{\pi \mathrm{i}}\oint_{-C_{z_0}} g(z)\mathrm{d}z - \frac{1}{\pi \mathrm{i}}\oint_{C_\infty} g(z)\mathrm{d}z \\ &= \frac{1}{\pi \mathrm{i}}\oint_{C_{z_0}} g(z)\mathrm{d}z - \frac{1}{\pi \mathrm{i}}\oint_{C_\infty} g(z)\mathrm{d}z \\ &= 2\operatorname*{Res}_{z=z_0} g(z) - 2\operatorname*{Res}_{z=\infty} g(z). \qquad (6)\end{aligned}$$

Recall, that we have the Taylor expansion of $\sqrt{z-1}\sqrt{z+1}$ at ∞ is

$$\sqrt{z-1}\sqrt{z+1}=z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3}).$$

Then we can multiply through by our extra term in this scenario to get

$$\begin{aligned}\frac{1}{z-z_0}\sqrt{z-1}\sqrt{z+1} &= \frac{1}{z-z_0}\left(z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3})\right) \\ &= \frac{1}{z}\frac{1}{1-z_0/z}\left(z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3})\right) \\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\left(z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3})\right) \\ &= z\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k-\frac{1}{2}z^{-1}\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k+\mathcal{O}(z^{-3})\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k \\ &= \sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k-\frac{1}{2}\frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k+\mathcal{O}(z^{-4})\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k.\end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z=\infty}\left(\frac{1}{z-z_0}\sqrt{z-1}\sqrt{z+1}\right) = z_0.$$

While the residue at the point z_0 is

$$\operatorname*{Res}_{z=z_0}\left(\frac{1}{z-z_0}\sqrt{z-1}\sqrt{z+1}\right) = \sqrt{z_0-1}\sqrt{z_0+1}.$$

Lets plug these in to equation (6) to have

$$-\frac{1}{\pi \mathrm{i}}\oint_C g(z)\mathrm{d}z = 2\operatorname*{Res}_{z=z_0} g(z) - 2\operatorname*{Res}_{z=\infty} g(z) = 2\sqrt{z_0-1}\sqrt{z_0+1} - 2z_0.$$

Hence,

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x}\sqrt{1+x}\frac{\mathrm{d}x}{x-z} = 2(\sqrt{z_0-1}\sqrt{z_0+1} - z_0).$$

$\square$