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AMATH 561

PROBLEM SET 3

1. Give an example of a probability space $(\Omega, \mathcal{F}, P)$, a random variable X and a function f such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ but $\sigma(f(X)) \neq \{\emptyset, \Omega\}$. Give a function g such that $\sigma(g(X)) = \{\emptyset, \Omega\}$. Hint: Look at finite sample spaces with a small number of elements.
Solution:
Let our probability space be two independent coin tosses, such that $\Omega = \big\{HH, TT, HT, TH\big\}$. Define a random variable X such that $X(\omega)$ be the number of heads in the outcome ω with $\omega \in \Omega$. Therefore

$$\begin{aligned} X(HH) &= 2, \\ X(TT) &= 0, \\ X(HT) &= 1, \text{ and} \\ X(TH) &= 1. \end{aligned}$$

Now $\sigma(X)$ can be written as

$$\sigma(X) = \Big\{\{HH\}, \{TH, HT\}, \{TT\}, \{TT, HH\}, \{HH, HT, TH\}, \{TT, HT, TH\}, \Omega, \emptyset\Big\}$$

Part one
Random variable X and f such that $\sigma(f(X)) \subsetneq \sigma(X)$ and $\sigma(f(X))$ is not the trivial σ-algebra.

Define $f(x)$ as follows

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Let's look at the possible pre-images of $f(X)$ with respect to a few cases of Borel sets. For convenience, I will define $\hat{X} = f(X)$. Now let's look at some cases for the pre-image
Case 1: $0 \in B$ but $1 \notin B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (-\infty, 0]\ \} = \{TT\}$$

Case 2: $0 \notin B$ but $1 \in B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (0, \infty)\ \} = \{TH, HT, HH\}$$

Case 3: $0 \in B$ and $1 \in B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (-\infty, \infty)\ \} = \Omega$$

Case 4: $0 \notin B$ and $1 \notin B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \emptyset.$$

Therefore,

$$\sigma(f(X)) = \Big\{\{TT\}, \{TH, HT, HH\}, \Omega, \emptyset\Big\} \neq \{\emptyset, \Omega\}$$

And thus we have $\sigma(f(X)) \subsetneq \sigma(X)$.

$\square$

Part two Now also give a function g such that $\sigma(g(X))$ is the trivial σ-algebra, $\{\emptyset, \Omega\}$.
Define $g(x)$ to be a constant $c \in \mathbb{R}$ such that $g(x) = c$ for all $x \in \mathbb{R}$. Once again, for convenience we define $\tilde{X} = g(X)$. Let's go through a few cases of what the pre-image may be for any Borel set
Case 1: $c \in B$

$$\tilde{X}^{-1}(B) = \Big\{\omega : \tilde{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (-\infty, \infty)\ \} = \Omega$$

Case 2: $c \notin B$

$$\tilde{X}^{-1}(B) = \Big\{\omega : \tilde{X}(\omega) \in B\Big\} = \emptyset.$$

Therefore,

$$\sigma(g(X)) = \{\Omega, \emptyset\}.$$

$\square$

2. Give an example of events A, B, and C, each of probability strictly between 0 and 1, such that $P(A\cap B) = P(A)P(B)$, $P(A\cap C) = P(A)P(C)$, and $P(A\cap B\cap C) = P(A)P(B)P(C)$ but $P(B\cap C) \neq P(B)P(C)$. Are A, B and C independent? Hint: You can let Ω be a set of eight equally likely points.
Solution:
Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Define events A, B, and C as follows

$$A = \{1, 2, 3, 4\}$$
$$B = \{1, 2, 5, 7\}$$
$$C = \{1, 3, 6, 8\}.$$

Then we have

$$P(A \cap B) = P(\{1, 2\}) = \frac{1}{4}$$

and

$$P(A)P(B) = P(\{1, 2, 3, 4\})P(\{1, 2, 5, 7\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Additionally, we have

$$P(A \cap C) = P(\{1, 3\}) = \frac{1}{4}$$

and

$$P(A)P(C) = P(\{1, 2, 3, 4\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Finally, we have

$$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{8}$$

and

$$P(A)P(B)P(C) = P(\{1, 2, 3, 4\})P(\{1, 2, 5, 7\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Notice we also get

$$P(B \cap C) = P(\{1\}) = \frac{1}{8}$$

which is not equal to

$$P(B)P(C) = P(\{1,2,5,7\})P(\{1,3,6,8\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

In class we said two events E and E' are independent if $P(E \cap E') = P(E)P(E')$. However, since independence of a collection of events E_i for $i \in \{1,2,3,...,n\}$ implies pairwise independence, $P(E_i \cap E_j) = P(E_i)P(E_j)$ for all $j \neq i$, if the collection E_i fails to be pairwise independent then the collection must not be independent either. We have shown that A and B are independent and A and C are independent. But B and C are not independent, therefore we don't have pairwise independence between each pair of the three events hence A, B, and C are not independent. $\square$

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space such that Ω is countably infinite, and $\mathcal{F} = 2^{\Omega}$. Show that it is impossible for there to exist a countable collection of events $A_1, A_2, ... \in \mathcal{F}$ which are independent, such that $P(A_i) = 1/2$ for each i. Hint: First show that for each $\omega \in \Omega$ and each $n \in \mathbb{N}$, we have $P(\omega) \leq 1/2^n$. Then derive a contradiction.
Solution:
Assume by way of contradiction, there exists a countably infinite collection of independent events $A_1, A_2, A_3, ... \in \mathcal{F}$ such that $P(A_i) = \frac{1}{2}$. Independence of these events implies that

$$P\left(\bigcap_i^n A_i\right) = \prod_i^n P(A_i) = \prod_i^n \frac{1}{2} = \left(\frac{1}{2}\right)^n.$$

I note that our collection of events is countably infinite so we can take the limit of the previous expression as $n \to \infty$. Their independence also implies the independence of the events A_i^c, as discussed in class. Next I want to construct a collection of new sets call them $B_{i,j}$ where $\omega_j \in B_{i,j}$ (note we can index the ω's since Ω is countably infinite). Let $B_{i,j}$ be

$$B_{i,j} = \begin{cases} A_i, & \omega_j \in A_i \\ A_i^c, & \omega_j \notin A_i. \end{cases}$$

Therefore we can now write each ω_j as

$$\bigcap_i^n B_{i,j} = \{\omega_j\}.$$

Then we have

$$P(\{\omega_j\}) = P\left(\bigcap_i^n B_{i,j}\right) = \prod_i^n P(B_{i,j}) = \prod_i^n \frac{1}{2} = \left(\frac{1}{2}\right)^n = 0.$$

Where $P(\{\omega_j\}) = 0$ since $n \to \infty$ because our collection of independent events is countably infinite. Notice, since $\Omega = \cup_{j=1}^{\infty}\{w_j\}$, then

$$P(\Omega) = P(\cup_{j=1}^{\infty}\{w_j\}) = \sum_{j=1}^{\infty} P(\{\omega_j\}) = \sum_{j=1}^{\infty} 0 = 0.$$

Which contradicts the fact that if $(\Omega, \mathcal{F}, P)$ is a probability space then $P(\Omega) = 1$. Therefore, it is impossible for there to exist a countable collection of events $A_1, A_2, ... \in \mathcal{F}$ which are independent, such that $P(A_i) = 1/2$ for each i. $\square$

4. (a) Let $X \geq 0$ and $Y \geq 0$ be independent random variables with distribution functions F and G. Find the distribution function of XY.
Solution:
Let $h(x, y) = \mathbb{1}_{\{xy \leq z\}}$ and $\mathbb{E}[h(x, y)]$ be

$$\mathbb{E}[h(x,y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y)$$

where μ and ν are probability measures with distribution functions F and G respectively. We also have

$$\begin{aligned}\mathbb{E}[h(x,y)] &= \mathbb{E}\left[\mathbb{1}_{\{xy \leq z\}}\right] \\ &= 1 \cdot P(XY \leq z) + 0 \cdot P(XY > 0) \\ &= P(XY \leq z).\end{aligned}$$

Additionally,

$$\begin{aligned}\int_{\mathbb{R}} h(x,y)\mu(\mathrm{d}x) &= \int_{\mathbb{R}} \mathbb{1}_{\{xy \leq z\}}\mu(\mathrm{d}x) \\ &= \int_{\mathbb{R}} \mathbb{1}_{\{x \leq \frac{z}{y}\}}\mu(\mathrm{d}x) \\ &= P\left(X \leq \frac{z}{y}\right) \\ &= F\left(\frac{z}{y}\right).\end{aligned}$$

Combining these we have

$$\begin{aligned}P(XY \leq z) &= \mathbb{E}[h(x,y)] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \\ &= \int_{\mathbb{R}} F\left(\frac{z}{y}\right)\nu(\mathrm{d}y) \\ &= \int_{\mathbb{R}} F\left(\frac{z}{y}\right)\mathrm{d}G(y)\end{aligned}$$

TODO: Account for when y is 0 somehow... End up with $P(Y = 0)$ which is $G(0)$

$$\int_{\mathbb{R}} F\left(\frac{z}{y}\right)\mathrm{d}G(y) = \int_{-\infty}^{0} F\left(\frac{z}{y}\right)\mathrm{d}G(y) + \int_{0}^{\infty} F\left(\frac{z}{y}\right)\mathrm{d}G(y).$$

(b) If $X \geq 0$ and $Y \geq 0$ are independent continuous random variables with density functions f and g, find the density function of XY.
Solution:

$$\int_{\mathbb{R}} F\left(\frac{z}{y}\right) \mathrm{d}G(y) = \int_{\mathbb{R}} \int_{-\infty}^{\frac{z}{y}} f(u)\,\mathrm{d}u\mathrm{d}G(y).$$

Now we need to do something with a change of variables along the lines of $u = \frac{x}{y}$ then $\mathrm{d}u = \frac{\mathrm{d}x}{y}$. We have

$$\begin{aligned}
\int_{\mathbb{R}} F\left(\frac{z}{y}\right) \mathrm{d}G(y) &= \int_{\mathbb{R}} \int_{-\infty}^{\frac{z}{y}} f(u)\,\mathrm{d}u\mathrm{d}G(y) \\
&= \int_{\mathbb{R}} \int_{-\infty}^{z} \frac{1}{y} f\left(\frac{x}{y}\right) \mathrm{d}x\mathrm{d}G(y) \\
&= \int_{-\infty}^{z} \int_{\mathbb{R}} \frac{1}{y} f\left(\frac{x}{y}\right) \mathrm{d}G(y)\mathrm{d}x \\
&= P\left(XY \leq z\right).
\end{aligned}$$

Therefore the density is

$$f(x) = \int_{\mathbb{R}} \frac{1}{y} f\left(\frac{x}{y}\right) \mathrm{d}G(y).$$

Since Y has a density g, we can write the above as

$$f(x) = \int_{\mathbb{R}} \frac{1}{y} f\left(\frac{x}{y}\right) g(y)\mathrm{d}y$$

using Theorem 3 from Lecture 9 slides. □

(c) If X and Y are independent exponentially distributed random variables with parameter λ, find the density function of XY.
Solution:
Recall the density function of an exponentially distributed random variable with parameter λ is the same as a gamma distributed random variable with parameters $(1, \lambda)$. Therefore the density function for one of our random variables is

$$f(x) = \frac{\lambda^1}{\Gamma(1)} x^{1-1}\,\mathrm{e}^{-\lambda x} = \lambda\,\mathrm{e}^{-\lambda x}.$$

Now using the formula we have for the density of the product of two independent random variables we have

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}} \frac{1}{y} f\left(\frac{x}{y}\right) g(y)\mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{1}{y} \lambda\,\mathrm{e}^{-\lambda \frac{x}{y}}\,\lambda\,\mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{1}{y} \lambda^2\,\mathrm{e}^{-\lambda\left(\frac{x}{y}+y\right)}\,\mathrm{d}y.
\end{aligned}$$

TODO: Consider clarifying the bounds of all of your integration (so it matches the support of X, Y)!