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AMATH 567

HOMEWORK 9

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
Evaluate the integrals $\frac{1}{2\mathrm{i}\pi}\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C.

(a)

$$\frac{z^2+1}{z^2-a^2}, \quad a^2 < 1$$

Solution:
TODO:

$$\frac{1}{2\mathrm{i}\pi}\oint_C \frac{z^2+1}{z^2-a^2}\mathrm{d}z$$

(b)

$$\frac{z^2+1}{z^3}$$

Solution:
TODO:

$$\frac{1}{2\mathrm{i}\pi}\oint_C \frac{z^2+1}{z^3}\mathrm{d}z$$

(c)

$$z^2\,\mathrm{e}^{-1/z}$$

Solution:
TODO:

$$\frac{1}{2\mathrm{i}\pi}\oint_C z^2\,\mathrm{e}^{-1/z}\,\mathrm{d}z$$

2: From A&F: 4.2.1(b)
Evaluate the following real integral

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a^2 > 0$$

Solution:
TODO:

3: Existence and uniqueness of polynomial interpolants.

(a) Suppose $(z_i)_{i=1}^n$ are distinct points in $\mathbb{C}$ and suppose $f_i \in \mathbb{C}$ for $i = 1, \ldots, n$. Show that there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_i) = f_i$ for $i = 1, \ldots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant*.

Solution:
Suppose there exists two polynomials $p_1(z)$ and $p_2(z)$ each of degree $n-1$. Assume both agree with f_i at each z_i such that

$$p_1(z_i) = p_2(z_i) = f_i \quad \text{for each } i = 1, ..., n.$$

(b) Define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_i)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:
TODO:

4: Bernstein interpolation formula. Suppose that $x_1 < x_2 < \cdots x_n$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1,1]$. Show that for any simple contour C inside Ω with $[-1,1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{\mathrm{d}z}{\nu(z)}, \quad x \in [-1,1],$$

where $p(x_j) = f(x_j)$ for $j = 1, 2, \ldots, n$, $\nu(x) = \prod_{j=1}^{n}(x - x_j)$.

Solution:
TODO:

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

(a) Show that the polynomial

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right),$$

has all of its roots $x_1 < x_2 < \cdots x_n$ within $[-1, 1]$.

Solution:
TODO:

(b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1, 1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:
TODO:

(c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2\frac{M|B_\rho|}{\pi}(\rho^n - \rho^{-n})^{-1}(\rho + \rho^{-1} - 1)^{-1} \leq 2\frac{M|B_\rho|}{\pi}\frac{\rho^{1-n}}{(\rho-1)^2}.$$

where $p(x_j) = f(x_j)$, i.e., p is the interpolant of f at the roots of T_n. Here $|B_\rho|$ denotes the arclength of B_ρ. This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f.

Solution:
TODO:

6: Compute the following two integrals explicitly for $z \notin [-1, 1]$:

(a)

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x}\sqrt{1+x}} \frac{\mathrm{d}x}{x-z}.$$

Solution:
TODO:

(b)

$$\frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x} \frac{\mathrm{d}x}{x-z}.$$

Solution:
TODO: