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11-25-24
AMATH 567

HOMEWORK 9

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

- 1:** From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
Evaluate the integrals $\frac{1}{2i\pi} \oint_C f(z) dz$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C .

(a)

$$\frac{z^2 + 1}{z^2 - a^2}, \quad a^2 < 1$$

Solution:

TODO: gonna do some residues? Maybe principal value integrals...?

$$\frac{1}{2i\pi} \oint_C \frac{z^2 + 1}{z^2 - a^2} dz$$

(b)

$$\frac{z^2 + 1}{z^3}$$

Solution:

TODO:

$$\frac{1}{2i\pi} \oint_C \frac{z^2 + 1}{z^3} dz$$

(c)

$$z^2 e^{-1/z}$$

Solution:

TODO:

$$\frac{1}{2i\pi} \oint_C z^2 e^{-1/z} dz$$

2: From A&F: 4.2.1(b)

Evaluate the following real integral

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2}, \quad a^2 > 0$$

Solution:

TODO: gonna do some residues? Maybe principal value integrals...?

3: Existence and uniqueness of polynomial interpolants.

- (a) Suppose $(z_j)_{j=1}^n$ are distinct points in \mathbb{C} and suppose $f_j \in \mathbb{C}$ for $j = 1, \dots, n$. Show that there is at most one polynomial $p(z)$ of degree $n - 1$ such that $p(z_j) = f_j$ for $j = 1, \dots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant*.

Solution:

Suppose there exists two polynomials $p_1(z)$ and $p_2(z)$ each of degree $n - 1$. Assume both agree with f_j at each z_j such that

$$p_1(z_j) = p_2(z_j) = f_j \quad \text{for each } j = 1, \dots, n.$$

Additionally define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Now let's consider the function

$$g(z) = \frac{p_1(z) - p_2(z)}{\nu(z)}.$$

We want to utilize Liouville's theorem to conclude that $g(z)$ is constant. In order to do this we need to show that $g(z)$ is entire and bounded. Let's begin by demonstrating that it is bounded by taking the limit as $z \rightarrow \infty$

$$\begin{aligned} \lim_{z \rightarrow \infty} g(z) &= \lim_{z \rightarrow \infty} \frac{p_1(z) - p_2(z)}{\nu(z)} \\ &= \lim_{z \rightarrow \infty} \frac{p_1(z) - p_2(z)}{\prod_{j=1}^n (z - z_j)} \\ &= \frac{\infty}{\infty}. \end{aligned}$$

Applying L'Hôpital's rule repeatedly we will end up with $1/z$ which goes to 0 as z goes to infinity since the denominator is an n th degree polynomial while the numerator is a degree $n - 1$ polynomial. Therefore,

$$\lim_{z \rightarrow \infty} g(z) = \lim_{z \rightarrow \infty} \frac{p_1(z) - p_2(z)}{\nu(z)} = 0,$$

which implies that $g(z)$ is bounded. Next, we need to determine if $g(z)$ is entire. Since polynomials are entire in the finite z plane, $p_1(z) - p_2(z)$ is entire. However, $g(z)$ overall requires a little more analysis since it has singularities where $z = z_j$. Notice, since the expression $p_1(z) - p_2(z)$ and $\nu(z)$ are both zero at each z_j , then there exists a factorization of $p_1(z) - p_2(z)$ which would allow us to cancel out each of the factors in the product in the denominator. Therefore, the singularities of $g(z)$ are removable and thus $g(z)$ is entire (or can be made entire, with the right extension at each z_j as we have done in previous assignments). Hence, by Liouville's Theorem, we can conclude that $g(z)$ is constant. Combining with the fact that $p_1(z_j) - p_2(z_j) = 0$ for each $j = 1, \dots, n$, then $g(z)$ must be 0 everywhere, thus implying $p_1(z) = p_2(z)$ everywhere. In conclusion, since these two functions are the same therefore there is at most one polynomial $p(z)$ of degree $n - 1$ such that $p(z_j) = f_j$ for $j = 1, \dots, n$, otherwise known as the interpolant. □

- (b) Define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:

Let's look at $p(z)/\nu(z)$ and consider what happens if we subtract off a specially cooked up collection of terms including the residues r_j for $j = 1, \dots, n$. We can express the residues of $p(z)/\nu(z)$ as

$$\frac{1}{2\pi i} \oint_C \frac{p(z)}{\nu(z)} dz = \sum_{j=0}^n \text{Res} \left(\frac{p(z)}{\nu(z)}; z_j \right) = \sum_{j=0}^n \frac{f_j}{\prod_{k \neq j} (z_k - z_j)}.$$

Recall partial fractions is connected to the residues. We construct the expression to subtract from $p(z)/\nu(z)$ using the partial fraction decomposition relationship to residues

$$\begin{aligned} \frac{p(z)}{\nu(z)} - \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ = \frac{p(z)}{\nu(z)} - \frac{f_1 \left(\prod_{k \neq 1} (z_k - z_1) \right)^{-1}}{z - z_1} - \frac{f_2 \left(\prod_{k \neq 2} (z_k - z_2) \right)^{-1}}{z - z_2} - \dots - \frac{f_n \left(\prod_{k \neq n} (z_k - z_n) \right)^{-1}}{z - z_n} = 0. \end{aligned}$$

TODO: Why is this 0 besides saying it's the partial fraction decomposition? This expression is equal to 0 because the collection of terms we are subtracting is the partial fraction decomposition of $p(z)/\nu(z)$. If we can show that this function is bounded and entire then it is a constant. Therefore, we would be able to state that since it is a constant and 0 then it must be a 0 everywhere. Thus we can say

$$\begin{aligned} \frac{p(z)}{\nu(z)} - \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} &= 0 \\ \frac{p(z)}{\nu(z)} &= \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ p(z) &= \nu(z) \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ p(z) &= \prod_{j=1}^n (z - z_j) \sum_{j=0}^n \frac{f_j \left(\prod_{k \neq j} (z_k - z_j) \right)^{-1}}{z - z_j} \\ p(z) &= \sum_{j=0}^n \frac{f_j \prod_{\ell \neq j} (z - z_\ell)}{\prod_{k \neq j} (z_k - z_j)}. \end{aligned}$$

Therefore we have this expression for $p(z)$.

4: Bernstein interpolation formula. Suppose that $-1 \leq x_1 < x_2 < \dots < x_n \leq 1$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1, 1]$. Show that for any simple contour C inside Ω with $[-1, 1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{dz}{\nu(z)}, \quad x \in [-1, 1],$$

where p is the degree $n-1$ polynomial interpolant satisfying $p(x_j) = f(x_j)$ for $j = 1, 2, \dots, n$. We also have $\nu(x) = \prod_{j=1}^n (x - x_j)$.

Solution:

Starting from the right we have

$$(1) \quad \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z-x} \frac{dz}{\nu(z)} = \frac{1}{2\pi i} \int_C \frac{f(z)\nu(x)}{(z-x)\nu(z)} dz = \text{Res}_{z=x} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right) + \sum_{i=1}^n \text{Res}_{z=x_i} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right).$$

Calculating the residue at $z = x$ is easy because x is a simple pole

$$\text{Res}_{z=x} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right) = \text{Res}_{z=x} \left(\frac{\frac{f(z)\nu(x)}{\nu(z)}}{(z-x)}; 0 \right) = \frac{f(x)\nu(x)}{\nu(x)} = f(x).$$

Calculating the residue at each $z = x_i$ is similarly quick since they are simple poles

$$(2) \quad \begin{aligned} \sum_{i=1}^n \text{Res}_{z=x_i} \left(\frac{f(z)\nu(x)}{(z-x)\nu(z)}; 0 \right) &= \sum_{i=1}^n \text{Res}_{z=x_i} \left(\frac{f(z)\nu(x)}{(z-x) \prod_{j=1}^n (z-x_j)}; 0 \right) \\ &= \sum_{i=1}^n \text{Res}_{z=x_i} \left(\left(\frac{f(z)\nu(x)}{(z-x) \prod_{j \neq i}^n (z-x_j)} \right) / (z-x_i); 0 \right) \\ &= \sum_{i=1}^n \frac{f(x_i)\nu(x)}{(x_i-x) \prod_{j \neq i}^n (x_i-x_j)} \\ &= - \sum_{i=1}^n \frac{f(x_i) \prod_{j=1}^n (x-x_j)}{(x-x_i) \prod_{j \neq i}^n (x_i-x_j)} \\ &= - \sum_{i=1}^n \frac{f(x_i) \prod_{j \neq i}^n (x-x_j)}{\prod_{j \neq i}^n (x_i-x_j)} = -p(x) \end{aligned}$$

Where we know this is $p(z)$ from our work in problem 4.

Therefore we have Equation (1) is equal to $f(x) - p(x)$. Furthermore, since $\nu(x_i) = 0$ for each $i = 1, \dots, n$, then

$$\begin{aligned} f(x_i) - p(x_i) &= 0 \\ f(x_i) &= p(x_i) \end{aligned}$$

for all $i = 1, \dots, n$. Finally we can also determine the degree of the polynomial $p(x)$ is $n-1$. This is due the equation (2) being made up of some scalar or weight factor and the product in the numerator which is $\nu(x)$ (a degree n polynomial) but without one of it's factors leaving it as an $n-1$ degree polynomial. \square

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

- (a) Show that the polynomial

$$T_n(z) = \frac{1}{2} (\varphi(z)^n + \varphi(z)^{-n}),$$

has all of its roots $x_1 < x_2 < \dots < x_n$ within $[-1, 1]$.

Solution:

TODO:

- (b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1, 1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:

TODO: apply things from hw3 problem 7 or 8?

- (c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2 \frac{M|B_\rho|}{\pi} (\rho^n - \rho^{-n})^{-1} (\rho + \rho^{-1} - 1)^{-1} \leq 2 \frac{M|B_\rho|}{\pi} \frac{\rho^{1-n}}{(\rho - 1)^2}.$$

where $p(x_j) = f(x_j)$, i.e., p is the interpolant of f at the roots of T_n . Here $|B_\rho|$ denotes the arclength of B_ρ . This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f .

Solution:

TODO: p is the polynomial interpolant of f of degree n - 1, lots of varphi stuff hw 3 prob 6/7/8

6: Compute the following two integrals explicitly for $z \notin [-1, 1]$:

(a)

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-x}\sqrt{1+x}} \frac{dx}{x-z}.$$

Solution:

We first recall that from homework 8 problem 4 part a) we showed

$$(3) \quad \int_{-1}^1 \frac{f(x)dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i} \oint_C \frac{f(z)dz}{\sqrt{z-1}\sqrt{z+1}}.$$

Applying that here we have

$$\frac{1}{\pi} \int_{-1}^1 \frac{\frac{1}{x-z_0} dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\pi i} \oint_C \frac{\frac{1}{z-z_0} dz}{\sqrt{z-1}\sqrt{z+1}}.$$

Now we want to calculate the integral on the right as the sum of the residues at ∞ and at z_0 .

TODO: Finish calculating the residues

□

(b)

$$\frac{2}{\pi} \int_{-1}^1 \sqrt{1-x}\sqrt{1+x} \frac{dx}{x-z}.$$

Solution:

Again applying equation (3), we have

$$\frac{2}{\pi} \int_{-1}^1 \sqrt{1-x}\sqrt{1+x} \frac{dx}{x-z} = \frac{1}{\pi i} \oint_C \sqrt{z-1}\sqrt{z+1} \frac{1}{z-z_0} dz$$

TODO: Finish calculating the residues

□