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AMATH 567

HOMEWORK 8

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

- 1: The Korteweg-de Vries (KdV) equation arises whenever long waves of moderate amplitude in dispersive media are considered. For instance, it describes waves in shallow water, and ion-acoustic waves in plasmas. The equation is given by

$$u_t = 6uu_x + u_{xxx},$$

where indices denote partial differentiation.

- (a) By looking for solutions $u(x, t) = U(x)$, derive a first-order ordinary differential equation for $U(x)$. Introduce integration constants as required.

Solution:

Since we want to find time independent ODE such that $\frac{d}{dt}U(x) = 0$, then we need

$$0 = 6uu_x + u_{xxx}.$$

Integrating gives us

$$\begin{aligned}\int 0 &= 6 \int uu_x dx + \int u_{xxx} dx \\ 0 &= 6 \int uu_x dx + u_{xx} + C_2.\end{aligned}$$

Using a substitution $v = u$ and $dv = u_x dx$ we have

$$\begin{aligned}0 &= 6 \int uu_x dx + u_{xx} + C_2 \\ 0 &= 6 \int v dv + u_{xx} + C_2 \\ 0 &= 6 \left(\frac{1}{2} v^2 \right) + C_3 + u_{xx} + C_2 \\ 0 &= 3u^2 + C_3 + u_{xx} + C_2.\end{aligned}$$

Next we multiply through by u_x and use the $v = u$ ($dv = u_x dx$) substitution for the first integral and a $\omega = u_x$ ($d\omega = u_{xx} dx$) substitution for the third integral

$$\begin{aligned}
0 &= 3u^2 u_x + u_x C_3 + u_x u_{xx} + u_x C_2 \\
0 &= 3 \int u^2 u_x dx + \int u_x C_3 dx + \int u_x u_{xx} dx + \int u_x C_2 dx \\
0 &= 3 \int v^2 dv + \int u_x C_3 dx + \int \omega d\omega + \int u_x C_2 dx \\
0 &= 3 \left(\frac{1}{3} v^3 \right) + C_4 + u C_3 + C_5 + \frac{1}{2} \omega^2 + C_6 + u C_2 + C_7 \\
0 &= u^3 + \frac{1}{2} (u_x)^2 + u(C_2 + C_3) + (C_4 + C_5 + C_6 + C_7).
\end{aligned}$$

Hence

$$u_x^2 = -2u^3 - 2uC_0 - 2C_1$$

which is our first-order ordinary differential equation for $U(x)$. □

- (b) Let $U = U_0 \wp(x - x_0)$. Determine U_0 so that $u = U(x)$ solves the KdV equation.

Solution:

Note, we proved last time that $\wp(z + Nw_1 + Mw_2) = \wp(z)$ therefore $\wp(x - x_0) = \wp(x)$. Then plugging $U = U_0 \wp(x)$ into our first-order ordinary differential equation for $U(x)$ (while suppressing the argument for \wp) we have

$$\begin{aligned}
(U_0 \wp')^2 &= -2(U_0 \wp)^3 - 2(U_0 \wp)C_0 - 2C_1 \\
U_0^2 (\wp')^2 &= -2U_0^3 \wp^3 - 2U_0 C_0 \wp - 2C_1 \\
(\wp')^2 &= -2U_0 \wp^3 - \frac{2}{U_0} C_0 \wp - \frac{2}{U_0^2} C_1.
\end{aligned}$$

Choosing $U_0 = -2$ then we have

$$(\wp')^2 = 4\wp^3 + C_0 \wp - \frac{1}{2} C_1,$$

which resembles what we showed in the previous assignment holds

$$(\wp')^2 = 4\wp^3 + c\wp + d.$$

The remaining constants can be attained through the initial conditions of the system we are solving and making sure they agree with the values of c, d from the previous assignment.

2: From A&F: 3.6.5

Show that if $f(z)$ is meromorphic in the finite z plane, then $f(z)$ must be the ratio of two entire functions.

Solution:

Assume $f(z)$ is a meromorphic function. Then we know all of the singularities of $f(z)$ are poles of some order. If we can multiply $f(z)$ by some entire $g(z)$ function which knocks out all of the poles of $f(z)$ and are left with an entire function $h(z)$, then the original meromorphic function $f(z)$ is a ratio of two entire functions. Now it is left for us to successfully construct such a function $g(z)$. Our construction needs to have zeros at all of the locations where $f(z)$ has poles. Additionally we need to make sure that the multiplicity of the zeros agree with the residue of the poles. We can use the Mittag-Leffler Expansion to assist us here. Suppose $f(z)$ has poles at each $z = z_j$ for $j = 0, 1, 2, \dots$ with corresponding residues a_j then let $g(z)$ be

$$g(z) = z^{a_0} \prod_{j=1}^{\infty} \left[(z - z_j) \exp \left(\sum_{k=0}^{m-1} \frac{(z_j)^{k+1}}{k+1} \right) \right]^{a_j}.$$

Then

$$f(z)g(z) = h(z).$$

Since $g(z)$ is an entire function constructed strategically, $h(z)$ has no singularities and is therefore entire in the finite complex plane. Therefore $f(z)$ is the ratio of two entire functions.

□

3: Here's a way to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

due to Euler. We've seen that

$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

(a) Equate the coefficients of z^2 on both sides, to recover the desired sum.

Solution:

Taylor expand on the left to get

$$\begin{aligned} \frac{\sin \pi z}{\pi z} &= \frac{1}{\pi z} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi z)^{2j+1}}{(2j+1)!} \\ &= \frac{1}{\pi z} \left(\pi z - \frac{(z\pi)^3}{6} + \frac{(z\pi)^5}{120} - \dots \right) \\ &= 1 - \frac{z^2 \pi^2}{6} + \frac{z^4 \pi^4}{120} - \dots \end{aligned}$$

Now expand out several terms in the product on the right

$$\begin{aligned} \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right) &= (1 - z^2) \left(1 - \frac{z^2}{4}\right) \left(1 - \frac{z^2}{9}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) \dots \\ &= \left(1 - z^2 - \frac{z^2}{4} + \frac{z^4}{4}\right) \left(1 - \frac{z^2}{9}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) \dots \\ &= \left(1 - z^2 - \frac{z^2}{4} + \frac{z^4}{4} - \frac{z^2}{9} + \frac{z^4}{9} + \frac{z^4}{36} - \frac{z^6}{36}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) \dots \\ &= \left(1 + z^2 \left(-1 - \frac{1}{4} - \frac{1}{9}\right) + z^4 \left(\frac{1}{4} + \frac{1}{9} + \frac{1}{36}\right) - \frac{z^6}{36}\right) \left(1 - \frac{z^2}{16}\right) \left(1 - \frac{z^2}{25}\right) \dots \end{aligned}$$

Then we have the coefficients for z^2 becomes the series $-\sum_{j=0}^{\infty} \frac{1}{j^2}$. Equating the coefficient on the left with the series on the right we have

$$\begin{aligned} -\frac{\pi^2}{6} &= -\sum_{j=0}^{\infty} \frac{1}{j^2} \\ \frac{\pi^2}{6} &= \sum_{j=0}^{\infty} \frac{1}{j^2} \end{aligned}$$

□

(b) Equate the coefficients of z^4 on both sides to recover a different sum.

Solution:

Using the results from the Taylor expansion on the left from part (a) we have the coefficient of the z^4 term is $\frac{\pi^4}{120}$. Additionally, from expanding the first several terms in the product on the right we have that the coefficient of the z^4 term can be written as

$$\sum_{j=0}^{\infty} \sum_{k=1}^{j-1} \frac{1}{j^2} \frac{1}{k^2}.$$

Combining these we have

$$\frac{\pi^4}{120} = \sum_{j=0}^{\infty} \sum_{k=1}^{j-1} \frac{1}{j^2} \frac{1}{k^2}.$$

A little work can be done to relate this to the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^4}.$$

By equating coefficients of higher powers of z , one can recover other identities too.

□

- 4: For the following, suppose that $f(z)$ is analytic in an open set Ω that contains $[-1, 1]$.
(a) Show that there exists a contour C , encircling $[-1, 1]$, such that

$$\int_{-1}^1 \frac{f(x)dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i} \oint_C \frac{f(z)dz}{\sqrt{z-1}\sqrt{z+1}}.$$

Solution:

For convenience, define $h(z)$ to be the integrand $h(z) = \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}$. Siting Homework 5 problem 5, let Σ define the same area as before

$$\Sigma = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq r \text{ and } 0 \leq -\operatorname{Im} z \leq R, R > 0, r > 1\}$$

and let $\partial\Sigma$ be the counterclockwise oriented contour along the boundary of the region Σ . Additionally let $\partial\Sigma \setminus [-1, 1]$ be the contour on the boundary without the section from -1 to 1 on the real line. From that same problem we know

$$\oint_{\partial\Sigma} h(z)dz = 0.$$

Furthermore, we can say

$$(1) \quad \begin{aligned} \int_1^{-1} h(z)dz + \oint_{\partial\Sigma \setminus [-1, 1]} h(z)dz &= 0 \\ \oint_{\partial\Sigma \setminus [-1, 1]} h(z)dz &= \int_{-1}^1 h(z)dz. \end{aligned}$$

We now define

$$\Sigma' = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq r \text{ and } 0 \leq \operatorname{Im} z \leq R, R > 0, r > 1\}$$

to be the upper half plane analogy of Σ . Therefore, let $\partial\Sigma'$ be the counterclockwise oriented contour along the boundary of the region Σ' .

$$g(z) = \begin{cases} h(z), & \text{if } \Im(z) > 0 \text{ or } |z| > 1 \\ -h(z), & \text{if } \Im(z) = 0 \text{ and } |z| \leq 1 \end{cases}.$$

This helps us preserve the continuity we are concerned with in order to apply the same arguments from Homework 5 problem 5 and making use of lemma 1 from problem 4 of that same assignment. Then we can conclude

$$\oint_{\partial\Sigma'} g(z)dz = 0.$$

Furthermore, we have

$$(2) \quad \begin{aligned} \int_{-1}^1 g(z)dz + \oint_{\partial\Sigma' \setminus [-1, 1]} g(z)dz &= 0 \\ \oint_{\partial\Sigma' \setminus [-1, 1]} g(z)dz &= - \int_{-1}^1 g(z)dz \\ \oint_{\partial\Sigma' \setminus [-1, 1]} h(z)dz &= - \int_{-1}^1 -h(z)dz \\ \oint_{\partial\Sigma' \setminus [-1, 1]} h(z)dz &= \int_{-1}^1 h(z)dz. \end{aligned}$$

If we add equation (1) and equation (2), we have

$$\begin{aligned} \oint_{\partial\Sigma \setminus [-1,1]} h(z)dz + \oint_{\partial\Sigma' \setminus [-1,1]} h(z)dz &= \int_{-1}^1 h(z)dz + \int_{-1}^1 h(z)dz \\ \oint_{\partial\Sigma \setminus [-1,1]} h(z)dz + \oint_{\partial\Sigma' \setminus [-1,1]} h(z)dz &= 2 \int_{-1}^1 h(z)dz. \end{aligned}$$

Note the contours $\partial\Sigma \setminus [-1, 1]$ and $\partial\Sigma' \setminus [-1, 1]$ have small overlapping regions on the real axis $z \in (-r-1, -1)$ and $z \in (1, 1+r)$ which cancel out since they are of opposite orientation. We denote the combinations of these contours as $\partial\hat{\Sigma}$ which is the counterclockwise oriented contour on the boundary of $\hat{\Sigma}$ with

$$\hat{\Sigma} = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq r \text{ and } |\operatorname{Im} z| \leq R, R > 0, r > 1\}.$$

Hence,

$$\begin{aligned} \oint_{\partial\Sigma \setminus [-1,1]} h(z)dz + \oint_{\partial\Sigma' \setminus [-1,1]} h(z)dz &= 2 \int_{-1}^1 h(z)dz \\ \oint_{\partial\hat{\Sigma}} h(z)dz &= 2 \int_{-1}^1 h(z)dz \\ \oint_{\partial\hat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}dz &= 2 \int_{-1}^1 \frac{f(x)}{\sqrt{x-1}\sqrt{x+1}}dx \\ \oint_{\partial\hat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}dz &= 2 \int_{-1}^1 \frac{f(x)}{(-i)\sqrt{1-x}\sqrt{x+1}}dx \\ \oint_{\partial\hat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}dz &= -\frac{2}{i} \int_{-1}^1 \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}dx \\ \oint_{\partial\hat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}dz &= 2i \int_{-1}^1 \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}dx \\ \frac{1}{2i} \oint_{\partial\hat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}dz &= \int_{-1}^1 \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}dx. \end{aligned}$$

Therefore the counterclockwise oriented contour on the boundary of $\hat{\Sigma}$, denoted as $\partial\hat{\Sigma}$ is one such contour encircling $[-1, 1]$ such that

$$\int_{-1}^1 \frac{f(x)dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i} \oint_C \frac{f(z)dz}{\sqrt{z-1}\sqrt{z+1}}.$$

□

Note we can deform this contour $\partial\hat{\Sigma}$ into a circle centered at $z = 0$ of radius $\rho > 1$.

(b) Use this to evaluate

$$I_1 = \int_{-1}^1 \frac{dx}{\sqrt{1-x}\sqrt{1+x}}, \quad I_2 = \int_{-1}^1 \sqrt{1-x}\sqrt{1+x} dx,$$

$$I_3 = \int_{-1}^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx, \quad I_4 = \int_{-1}^1 \frac{\sqrt{1+x}}{\sqrt{1-x}} dx,$$

without using any changes of variable (e.g., no trig subs!).

Solution:

Using part (a) and the substitution $z = \rho e^{i\theta}$ where ρ is very large making z be near ∞ . Additionally, our counterclockwise circle contour around $z = 0$ is also a counterclockwise contour around ∞ . Therefore

$$\begin{aligned} I_1 &= \int_{-1}^1 \frac{dx}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2i} \oint_C \frac{dz}{\sqrt{z-1}\sqrt{z+1}} \\ &= \frac{1}{2i} \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\sqrt{\rho e^{i\theta}-1}\sqrt{\rho e^{i\theta}+1}} \\ &= \frac{1}{2i} \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta} \sqrt{1-\frac{1}{\rho e^{i\theta}}}\sqrt{1+\frac{1}{\rho e^{i\theta}}}} \\ &= \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{\sqrt{1-\frac{1}{\rho e^{i\theta}}}\sqrt{1+\frac{1}{\rho e^{i\theta}}}}. \end{aligned}$$

Since ρ is large, $\frac{1}{\rho e^{i\theta}} \approx 0$. Thus

$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{\sqrt{1-\frac{1}{\rho e^{i\theta}}}\sqrt{1+\frac{1}{\rho e^{i\theta}}}} \\ &= \frac{1}{2} \int_0^{2\pi} d\theta \\ &= \frac{1}{2} (\theta|_0^{2\pi}) \\ &= \frac{1}{2} 2\pi \\ &= \pi. \end{aligned}$$

Hence,

$$I_1 = \int_{-1}^1 \frac{dx}{\sqrt{1-x}\sqrt{1+x}} = \pi$$

□

Now for I_2 we can begin by converting our integrand into something of the form that will help us use the results from part (a). Notice

$$\sqrt{1-x}\sqrt{1+x} = \frac{(1-x)(1+x)}{\sqrt{1-x}\sqrt{1+x}} = \frac{1-x^2}{\sqrt{1-x}\sqrt{1+x}}.$$

Then we can evaluate the integral as follows

$$\begin{aligned} I_2 &= \int_{-1}^1 \sqrt{1-x}\sqrt{1+x} \, dx = \int_{-1}^1 \frac{1-x^2 \, dx}{\sqrt{1-x}\sqrt{1+x}} \\ &= \frac{1}{2i} \oint_C \frac{1-z^2 \, dz}{\sqrt{z-1}\sqrt{z+1}}. \end{aligned}$$

Let's take the contour to be a circle centered at $z = 0$ of radius ρ such that ρ is sufficiently large. This integral can also be evaluated at infinity

$$\begin{aligned} \frac{1}{2i} \oint_C \frac{1-z^2 \, dz}{\sqrt{z-1}\sqrt{z+1}} &= \frac{1}{2i} \oint_C \frac{(1-z)(1+z) \, dz}{\sqrt{z-1}\sqrt{z+1}} \\ &= -\frac{1}{2i} \oint_C \frac{(z-1)(1+z) \, dz}{\sqrt{z-1}\sqrt{z+1}} \\ &= -\frac{1}{2i} \oint_C \sqrt{z-1}\sqrt{z+1} \, dz. \end{aligned}$$

From a previous assignment we have that

$$(3) \quad \sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}} = z - \frac{1}{2}z^{-1} + \mathcal{O}(z^{-3})$$

and by the Residue theorem we have

$$-\frac{1}{2i} \oint_C \sqrt{z-1}\sqrt{z+1} \, dz = -\pi(\text{Res}(f(z); \infty)).$$

Hence,

$$\begin{aligned} I_2 &= -\frac{1}{2i} \oint_C \sqrt{z-1}\sqrt{z+1} \, dz \\ &= -\pi \left(-\frac{1}{2} \right) \\ &= \frac{\pi}{2} \end{aligned}$$

Therefore

$$I_2 = \int_{-1}^1 \sqrt{1-x}\sqrt{1+x} \, dx = \frac{\pi}{2}$$

□

Notice we can rewrite the integrand from I_3 as follows

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{\sqrt{1-x}\sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}} = \frac{1-x}{\sqrt{1-x}\sqrt{1+x}}.$$

Then we can evaluate I_3 using the tools established in part (a)

$$\begin{aligned}
 I_3 &= \int_{-1}^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \\
 &= \int_{-1}^1 \frac{1-x}{\sqrt{1-x}\sqrt{1+x}} dx \\
 &= \frac{1}{2i} \oint_C \frac{1-z}{\sqrt{z-1}\sqrt{z+1}} dz \\
 &= -\frac{1}{2i} \oint_C \frac{z-1}{\sqrt{z-1}\sqrt{z+1}} dz \\
 &= -\frac{1}{2i} \oint_C \frac{\sqrt{z-1}}{\sqrt{z+1}} dz
 \end{aligned}$$

This should come out to be π .

Notice we can rewrite the integrand from I_4 as follows

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x}\sqrt{1+x}} = \frac{1+x}{\sqrt{1-x}\sqrt{1+x}}.$$

Then we can evaluate I_4 using the tools established in part (a)

$$\begin{aligned}
 I_4 &= \int_{-1}^1 \frac{\sqrt{1+x}}{\sqrt{1-x}} dx \\
 &= \int_{-1}^1 \frac{1+x}{\sqrt{1-x}\sqrt{1+x}} dx \\
 &= \frac{1}{2i} \oint_C \frac{1+z}{\sqrt{z-1}\sqrt{z+1}} dz \\
 &= -\frac{1}{2i} \oint_C \frac{\sqrt{z+1}}{\sqrt{z-1}} dz
 \end{aligned}$$

This should also come out to be π ??

5: Suppose, for $|z| = 1$, that the series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n,$$

converges uniformly.

(a) Compute series representations for

$$F(z) := \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi, \quad |z| \neq 1, \quad C = \partial B_1(0).$$

Solution:

Jumping right in...

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \oint_C \frac{\sum_{n=-\infty}^{\infty} f_n \xi^n}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \oint_C \sum_{n=-\infty}^{\infty} \frac{f_n \xi^n}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \oint_C \sum_{n=-\infty}^{-1} \frac{f_n \xi^n}{\xi - z} + \sum_{n=0}^{\infty} \frac{f_n \xi^n}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \oint_C \sum_{n=1}^{\infty} \frac{f_n}{\xi^n (\xi - z)} + \sum_{n=0}^{\infty} \frac{f_n \xi^n}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \oint_C \sum_{n=1}^{\infty} \frac{f_n}{\xi^n (\xi - z)} d\xi + \oint_C \sum_{n=0}^{\infty} \frac{f_n \xi^n}{\xi - z} d\xi \end{aligned}$$

Do I want to evaluate this sum and then integrate or should I switch the order of integration and summation and integrate the summand to be left with a series? Furthermore, where does the geometric series come into play? What do we know about f_n ? **TODO: two cases where $|z| < 1$ and $|z| > 1$...geometric series???... additional cases where $n < 0$ and $n \geq 0$**

This

(b) For $|z| = 1$, compute

$$\lim_{\epsilon \rightarrow 0^+} F(z(1 - \epsilon)) - \lim_{\epsilon \rightarrow 0^+} F(z(1 + \epsilon)).$$

Solution:

TODO: should expect a “jump” discontinuity at the boundary Use the series representation you arrived at from the previous problem to evaluate this.

This