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AMATH 567

HOMework 9

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

- 1: From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
Evaluate the integrals $\frac{1}{2i\pi} \oint_C f(z) dz$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C .

(a)

$$\frac{z^2 + 1}{z^2 - a^2}, \quad a^2 < 1$$

Solution:

TODO:

$$\frac{1}{2i\pi} \oint_C \frac{z^2 + 1}{z^2 - a^2} dz$$

(b)

$$\frac{z^2 + 1}{z^3}$$

Solution:

TODO:

$$\frac{1}{2i\pi} \oint_C \frac{z^2 + 1}{z^3} dz$$

(c)

$$z^2 e^{-1/z}$$

Solution:

TODO:

$$\frac{1}{2i\pi} \oint_C z^2 e^{-1/z} dz$$

2: From A&F: 4.2.1(b)

Evaluate the following real integral

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2}, \quad a^2 > 0$$

Solution:

TODO:

3: Existence and uniqueness of polynomial interpolants.

- (a) Suppose $(z_i)_{i=1}^n$ are distinct points in \mathbb{C} and suppose $f_i \in \mathbb{C}$ for $i = 1, \dots, n$. Show that there is at most one polynomial $p(z)$ of degree $n - 1$ such that $p(z_i) = f_i$ for $i = 1, \dots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant*.

Solution:

TODO:

- (b) Define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:

TODO:

4: Bernstein interpolation formula. Suppose that $x_1 < x_2 < \cdots x_n$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1, 1]$. Show that for any simple contour C inside Ω with $[-1, 1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i} \int_C \frac{f(z)}{z - x} \frac{dz}{\nu(z)}, \quad x \in [-1, 1],$$

where $p(x_j) = f(x_j)$ for $j = 1, 2, \dots, n$, $\nu(x) = \prod_{j=1}^n (x - x_j)$.

Solution:

TODO:

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

- (a) Show that the polynomial

$$T_n(z) = \frac{1}{2} (\varphi(z)^n + \varphi(z)^{-n}),$$

has all of its roots $x_1 < x_2 < \cdots < x_n$ within $[-1, 1]$.

Solution:

TODO:

- (b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1, 1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:

TODO:

- (c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2 \frac{M|B_\rho|}{\pi} (\rho^n - \rho^{-n})^{-1} (\rho + \rho^{-1} - 1)^{-1} \leq 2 \frac{M|B_\rho|}{\pi} \frac{\rho^{1-n}}{(\rho - 1)^2}.$$

where $p(x_j) = f(x_j)$, i.e., p is the interpolant of f at the roots of T_n . Here $|B_\rho|$ denotes the arclength of B_ρ . This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f .

Solution:

TODO:

6: Compute the following two integrals explicitly for $z \notin [-1, 1]$:

(a)

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-x}\sqrt{1+x}} \frac{dx}{x-z}.$$

Solution:

TODO:

(b)

$$\frac{2}{\pi} \int_{-1}^1 \sqrt{1-x}\sqrt{1+x} \frac{dx}{x-z}.$$

Solution:

TODO: