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AMATH 567

HOMEWORK 9

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 4.1.2 only (i), i.e., only by computing residues inside.
Evaluate the integrals $\frac{1}{2i\pi}\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems (i) enclosing the singular points inside C.

(a)

$$\frac{z^2+1}{z^2-a^2}, \quad a^2<1$$

Solution:
TODO: gonna do some residues? Maybe principal value integrals...?

$$\frac{1}{2i\pi}\oint_C \frac{z^2+1}{z^2-a^2}\mathrm{d}z$$

(b)

$$\frac{z^2+1}{z^3}$$

Solution:
TODO:

$$\frac{1}{2i\pi}\oint_C \frac{z^2+1}{z^3}\mathrm{d}z$$

(c)

$$z^2\,\mathrm{e}^{-1/z}$$

Solution:
TODO:

$$\frac{1}{2i\pi}\oint_C z^2\,\mathrm{e}^{-1/z}\,\mathrm{d}z$$

2: From A&F: 4.2.1(b)
Evaluate the following real integral

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a^2 > 0$$

Solution:
For convenience throughout this problem lets define

$$f(x) = \frac{1}{(x^2+a^2)^2}.$$

Since the $f(x)$ is an even function we can say

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}.$$

Then we can use the principal value integral which is given by

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^R f(x)\mathrm{d}x, \quad \text{if it exists.}$$

Now let's consider the counterclockwise contour C around the closed semicircle centered at the origin in the upper half plane with radius R. We can say

$$\oint_C f(z)\mathrm{d}z = \int_{-R}^R f(x)\mathrm{d}x + \oint_{C_R} f(z)\mathrm{d}z, \tag{1}$$

where $-R$ to R is the section of C along the real axis and C_R is the open semicircle (counterclockwise). We can combine these ideas by taking the limit of both sides as $R\to\infty$. Let's take the limit of the right and analyze what is going on. This gives us

$$\begin{aligned}
\lim_{R\to\infty}\left(\int_{-R}^R f(x)\mathrm{d}x + \oint_{C_R} f(z)\mathrm{d}z\right) &= \lim_{R\to\infty}\int_{-R}^R f(x)\mathrm{d}x + \lim_{R\to\infty}\oint_{C_R} f(z)\mathrm{d}z \\
&= \int_{-\infty}^\infty f(x)\mathrm{d}x + \lim_{R\to\infty}\oint_{C_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}
\end{aligned}$$

Let's try to bound the integrand to something that depends on R^{-1}. We will use the substitution $z = R\,\mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in [0,\pi]$.

$$\begin{aligned}
\left|\frac{1}{(z^2+a^2)^2}\right| &= \left|\frac{1}{(R^2\,\mathrm{e}^{2\mathrm{i}\theta}+a^2)^2}\right| \\
&= \frac{1}{|R^4\,\mathrm{e}^{4\mathrm{i}\theta}+2R^2a^2\,\mathrm{e}^{2\mathrm{i}\theta}+a^4|} \\
&\leq \frac{1}{R^2\,|R^2\,\mathrm{e}^{4\mathrm{i}\theta}+2a^2\,\mathrm{e}^{2\mathrm{i}\theta}|}
\end{aligned}$$

Since $f(z)$ is continuous on C_R, we can use the ML bound on the integral to say

$$\begin{aligned}
\lim_{R\to\infty}\left|\oint_{C_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}\right| &\leq \lim_{R\to\infty}\frac{\pi R}{R^2\,|R^2\,\mathrm{e}^{4\mathrm{i}\theta}+2a^2\,\mathrm{e}^{2\mathrm{i}\theta}|} \\
&= \lim_{R\to\infty}\frac{\pi}{R\,|R^2\,\mathrm{e}^{4\mathrm{i}\theta}+2a^2\,\mathrm{e}^{2\mathrm{i}\theta}|} = 0.
\end{aligned}$$

Applying a similar squeeze theorem argument from homework 4 problem 4, we can conclude

$$\lim_{R\to\infty}\oint_{C_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}=0.$$

Therefore, equation (1) becomes

$$\begin{aligned}
\lim_{R\to\infty}\oint_C f(z)\mathrm{d}z &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x + \cancelto{0}{\oint_{C_R} f(z)\mathrm{d}z}\\
\lim_{R\to\infty}\oint_C f(z)\mathrm{d}z &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x
\end{aligned}$$

Notice, as R goes to infinity the integral of $f(z)$ along the contour C is equal to the sum of all of the residues at all singularities in the upper half plane for $f(z)$. Let S be the collection of singularities in the upper half plane, then we can say

$$\lim_{R\to\infty}\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{w\in S}\operatorname*{Res}_{z=w}\left(\frac{1}{(z^2+a^2)^2}\right).$$

Hence,

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}\mathrm{d}x = 2\pi\mathrm{i}\sum_{w\in S}\operatorname*{Res}_{z=w}\left(\frac{1}{(z^2+a^2)^2}\right).$$

We now need to locate the singularities. The denominator is only 0 when $z^2+a^2=0$ so we know the singularities are at $z=\pm\mathrm{i}a$. However, since only $z=\mathrm{i}a$ is in the upper half plane we can simplify the previous equation and solve for the one residue we need

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}\mathrm{d}x &= 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}a}\left(\frac{1}{(z^2+a^2)^2}\right)\\
&= 2\pi\mathrm{i}\frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-\mathrm{i}a)^2\frac{1}{(z^2+a^2)^2}\right)\Bigg|_{\mathrm{i}a}\\
&= 2\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-\mathrm{i}a)^2\frac{1}{(z+\mathrm{i}a)^2(z-\mathrm{i}a)^2}\right)\Bigg|_{\mathrm{i}a}\\
&= 2\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{(z+\mathrm{i}a)^2}\right)\Bigg|_{\mathrm{i}a}\\
&= 2\pi\mathrm{i}\frac{-2}{(\mathrm{i}a+\mathrm{i}a)^3}\\
&= 2\pi\mathrm{i}\frac{-2}{-8\mathrm{i}a^3}\\
&= \frac{\pi}{2a^3}.
\end{aligned}$$

Recall our original integral was

$$\int_0^{\infty}\frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{1}{2}\frac{\pi}{2a^3}=\frac{\pi}{4a^3}$$

$\square$

3: Existence and uniqueness of polynomial interpolants.

(a) Suppose $(z_j)_{j=1}^n$ are distinct points in $\mathbb{C}$ and suppose $f_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Show that there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_j) = f_j$ for $j = 1, \ldots, n$ using Liouville's theorem. Such a polynomial p is called an *interpolant.*

Solution:
Suppose there exists two polynomials $p_1(z)$ and $p_2(z)$ each of degree $n-1$. Assume both agree with f_j at each z_j such that

$$p_1(z_j) = p_2(z_j) = f_j \quad \text{for each } j = 1, ..., n.$$

Additionally define the node polynomial $\nu(z) = \prod_{j=1}^n (z - z_j)$. Now let's consider the function

$$g(z) = \frac{p_1(z) - p_2(z)}{\nu(z)}.$$

We want to utilize Liouville's theorem to conclude that $g(z)$ is constant. In order to do this we need to show that $g(z)$ is entire and bounded. Let's begin by demonstrating that it is bounded by taking the limit as $z \to \infty$

$$\begin{aligned}
\lim_{z\to\infty} g(z) &= \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\nu(z)} \\
&= \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\prod_{j=1}^n (z - z_j)} \\
&= \frac{\infty}{\infty}.
\end{aligned}$$

Applying L'Hôpitals rule repeatedly we will end up with $1/z$ which goes to 0 as z goes to infinity since the denominator is an nth degree polynomial while the numerator is a degree $n-1$ polynomial. Therefore,

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{p_1(z) - p_2(z)}{\nu(z)} = 0,$$

which implies that $g(z)$ is bounded. Next, we need to determine if $g(z)$ is entire. Since polynomials are entire in the finite z plane, $p_1(z) - p_2(z)$ is entire. However, $g(z)$ overall requires a little more analysis since it has singularities where $z = z_j$. Notice, since the expression $p_1(z) - p_2(z)$ and $\nu(z)$ are both zero at each z_j, then there exists a factorization of $p_1(z) - p_2(z)$ which would allows us to cancel out each of the factors in the product in the denominator. Therefore, the singularities of $g(z)$ are removable and thus $g(z)$ is entire (or can be made entire, with the right extension at each z_j as we have done in previous assignments). Hence, by Liouville's Theorem, we can conclude that $g(z)$ is constant. Combining with the fact that $p_1(z_j) - p_2(z_j) = 0$ for each $j = 1, ..., n$, then $g(z)$ must be 0 everywhere, thus implying $p_1(z) = p_2(z)$ everywhere. In conclusion, since these two functions are the same therefore there is at most one polynomial $p(z)$ of degree $n-1$ such that $p(z_j) = f_j$ for $j = 1, \ldots, n$, otherwise known at the interpolant.

$\square$

(b) Define the node polynomial $\nu(z) = \prod_{j=1}^{n}(z - z_j)$. Supposing that p is an interpolant, as above, express $p(z)/\nu(z)$ as a rational function. Find an expression for $p(z)$. This shows existence.

Solution:
Let's look at $p(z)/\nu(z)$ and consider what happens if we subtract off a specially cooked up collection of terms including the residues r_j for $j = 1, ..., n$. We can express the residues of $p(z)/\nu(z)$ as

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{p(z)}{\nu(z)} \mathrm{d}z = \sum_{j=0}^{n} \mathrm{Res}\left(\frac{p(z)}{\nu(z)}; z_j\right) = \sum_{j=0}^{n} \frac{f_j}{\prod_{k \neq j}(z_k - z_j)}.$$

Recall partial fractions is connected to the residues. We construct the expression to subtract from $p(z)/\nu(z)$ using the partial fraction decomposition relationship to residues

$$\frac{p(z)}{\nu(z)} - \sum_{j=0}^{n} \frac{f_j\Big(\prod_{k\neq j}(z_k - z_j)\Big)^{-1}}{z - z_j}$$

$$= \frac{p(z)}{\nu(z)} - \frac{f_1\Big(\prod_{k\neq 1}(z_k - z_1)\Big)^{-1}}{z - z_1} - \frac{f_2\Big(\prod_{k\neq 2}(z_k - z_2)\Big)^{-1}}{z - z_2} - \dots - \frac{f_n\Big(\prod_{k\neq n}(z_k - z_n)\Big)^{-1}}{z - z_n} = 0.$$

TODO: Why is this 0 besides saying it's the partial fraction decomposition? This expression is equal to 0 because the collection of terms we are subtracting is the partial fraction decomposition of $p(z)/\nu(z)$. If we can show that this function is bounded and entire then it is a constant. Therefore, we would be able to state that since it is a constant and 0 then it must be a 0 everywhere. Thus we can say

$$\frac{p(z)}{\nu(z)} - \sum_{j=0}^{n} \frac{f_j\Big(\prod_{k\neq j}(z_k - z_j)\Big)^{-1}}{z - z_j} = 0$$

$$\frac{p(z)}{\nu(z)} = \sum_{j=0}^{n} \frac{f_j\Big(\prod_{k\neq j}(z_k - z_j)\Big)^{-1}}{z - z_j}$$

$$p(z) = \nu(z) \sum_{j=0}^{n} \frac{f_j\Big(\prod_{k\neq j}(z_k - z_j)\Big)^{-1}}{z - z_j}$$

$$p(z) = \prod_{j=1}^{n}(z - z_j) \sum_{j=0}^{n} \frac{f_j\Big(\prod_{k\neq j}(z_k - z_j)\Big)^{-1}}{z - z_j}$$

$$p(z) = \sum_{j=0}^{n} \frac{f_j \prod_{\ell\neq j}(z - z_\ell)}{\prod_{k\neq j}(z_k - z_j)}.$$

Therefore we have this expression for $p(z)$.

4: Bernstein interpolation formula. Suppose that $-1 \le x_1 < x_2 < \cdots x_n \le 1$. And suppose that $f(z)$ is analytic in a region Ω that contains $[-1,1]$. Show that for any simple contour C inside Ω with $[-1,1]$ in its interior that

$$f(x) - p(x) = \frac{\nu(x)}{2\pi i}\int_C \frac{f(z)}{z-x}\frac{\mathrm{d}z}{\nu(z)}, \quad x \in [-1,1],$$

where p is the degree $n-1$ polynomial interpolant satisfying $p(x_j) = f(x_j)$ for $j = 1, 2, \ldots, n$. We also have $\nu(x) = \prod_{j=1}^{n}(x - x_j)$.

Solution:
Starting from the right we have

$$\begin{aligned}\frac{\nu(x)}{2\pi i}\int_C \frac{f(z)}{z-x}\frac{\mathrm{d}z}{\nu(z)} &= \frac{1}{2\pi i}\int_C \frac{f(z)\nu(x)}{(z-x)\nu(z)}\mathrm{d}z \\ &= \operatorname*{Res}_{z=x}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)};0\right) + \sum_{i=1}^{n}\operatorname*{Res}_{z=x_i}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)};0\right). \qquad (2)\end{aligned}$$

Calculating the residue at $z = x$ is easy because x is a simple pole

$$\operatorname*{Res}_{z=x}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)};0\right) = \operatorname*{Res}_{z=x}\left(\frac{\frac{f(z)\nu(x)}{\nu(z)}}{(z-x)};0\right) = \frac{f(x)\nu(x)}{\nu(x)} = f(x).$$

Calculating the residue at each $z = x_i$ is similarly quick since they are simple poles

$$\begin{aligned}\sum_{i=1}^{n}\operatorname*{Res}_{z=x_i}\left(\frac{f(z)\nu(x)}{(z-x)\nu(z)};0\right) &= \sum_{i=1}^{n}\operatorname*{Res}_{z=x_i}\left(\frac{f(z)\nu(x)}{(z-x)\prod_{j=1}^{n}(z-x_j)};0\right) \\ &= \sum_{i=1}^{n}\operatorname*{Res}_{z=x_i}\left(\left(\frac{f(z)\nu(x)}{(z-x)\prod_{\substack{j=1\\ j\neq i}}^{n}(z-x_j)}\right)\Big/(z-x_i)\ ;\ 0\right) \\ &= \sum_{i=1}^{n}\frac{f(x_i)\nu(x)}{(x_i-x)\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)} \\ &= -\sum_{i=1}^{n}\frac{f(x_i)\prod_{j=1}^{n}(x-x_j)}{(x-x_i)\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)} \\ &= -\sum_{i=1}^{n}\frac{f(x_i)\prod_{\substack{j=1\\ j\neq i}}^{n}(x-x_j)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)} = -p(x) \qquad (3)\end{aligned}$$

Where we know this is $p(z)$ from our work in problem 4.
Therefore we have Equation (2) is equal to $f(x) - p(x)$. Furthermore, since $\nu(x_i) = 0$ for each $i = 1, ..., n$, then

$$\begin{aligned}f(x_i) - p(x_i) &= 0 \\ f(x_i) &= p(x_i)\end{aligned}$$

for all $i = 1, ..., n$. Finally we can also determine the degree of the polynomial $p(x)$ is $n-1$. This is due the equation (3) being made up of some scalar or weight factor and the product in the numerator which is $\nu(x)$ (a degree n polynomial) but without one of it's factors leaving it as an $n-1$ degree polynomial. $\square$

5: Chebyshev polynomial interpolants. Recall

$$\varphi(z) = z + \sqrt{z-1}\sqrt{z+1}, \quad z \in \mathbb{C} \setminus [-1,1].$$

(a) Show that the polynomial

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right),$$

has all of its roots $x_1 < x_2 < \cdots x_n$ within $[-1,1]$.

Solution:
Let's begin by looking more closely at $\varphi(z)$ with a specific substitution, namely $z = \cos\theta$ with $\theta \in [0,\pi]$. **TODO:**

(b) Consider $J(w) = 1/2(w + 1/w)$. Show that the image of the circle of radius $\rho > 1$ under J is an ellipse B_ρ that contains $[-1,1]$ in its interior. Then show $\varphi(J(w)) = w$.

Solution:
TODO: apply things from hw3 problem 7 or 8?

(c) Show that if f is analytic in a region that contains B_ρ and its interior, and $|f(z)| \leq M$ for z interior to B_ρ then for $-1 \leq x \leq 1$,

$$|f(x) - p(x)| \leq 2\frac{M|B_\rho|}{\pi}(\rho^n - \rho^{-n})^{-1}(\rho + \rho^{-1} - 1)^{-1} \leq 2\frac{M|B_\rho|}{\pi}\frac{\rho^{1-n}}{(\rho-1)^2}.$$

where $p(x_j) = f(x_j)$, i.e., p is the interpolant of f at the roots of T_n. Here $|B_\rho|$ denotes the arclength of B_ρ. This shows that the exponential rate of convergence of Chebyshev interpolants is governed by the proximity of the nearest singularity of f.

Solution:
TODO: TODO: p is the polynomial interpolant of f of degree n - 1, lots of varphi stuff hw 3 prob 6/7/8

6: Compute the following two integrals explicitly for $z \notin [-1,1]$:

(a)

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x}\sqrt{1+x}}\frac{\mathrm{d}x}{x-z}.$$

Solution:

We first recall that from homework 8 problem 4 part a) we showed

$$\int_{-1}^{1}\frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\mathrm{i}}\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}. \tag{4}$$

Applying that here we have

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\frac{1}{x-z_0}\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\frac{1}{z-z_0}\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

For notational convenience let

$$g(z) = \frac{\frac{1}{z-z_0}}{\sqrt{z-1}\sqrt{z+1}}.$$

As we expand our contour C outwards we run into the singularity at z_0, leaving behind a clockwise circular contour around z_0 denoted as $-C_{z_0}$. We also have the normal counterclockwise contour around infinity which we will use in our residue calculation. Hence we have

$$\begin{aligned}
\frac{1}{2\pi\mathrm{i}}\oint_C g(z)\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}}\oint_{-C_{z_0}} g(z)\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\oint_{C_\infty} g(z)\mathrm{d}z \\
&= -\frac{1}{2\pi\mathrm{i}}\oint_{C_{z_0}} g(z)\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\oint_{C_\infty} g(z)\mathrm{d}z \\
&= -\operatorname*{Res}_{z=z_0} g(z) + \operatorname*{Res}_{z=\infty} g(z) \qquad (5)
\end{aligned}$$

Now we want to calculate the residues at ∞ and at z_0. Let

$$h(z) = \frac{1}{\sqrt{z-1}\sqrt{z+1}}$$

and

$$H(z) = h\left(\frac{1}{z}\right) = \frac{1}{\sqrt{1/z-1}\sqrt{1/z+1}}\frac{z}{z} = \frac{z}{\sqrt{1-z}\sqrt{1+z}}.$$

Then we can see $H(0) = 0$. Let's calculate $h'(0)$.

$$H'(z) = \frac{\sqrt{1-z}\sqrt{1+z} - z\big(-1/2(1-z)^{-1/2}(1+z)^{1/2} + 1/2(1-z)^{1/2}(1+z)^{-1/2}\big)}{(1-z)(1+z)}$$

Hence,

$$H'(0) = \frac{\sqrt{1}\sqrt{1} - 0\big(-1/2(1)^{-1/2}(1)^{1/2} + 1/2(1)^{1/2}(1)^{-1/2}\big)}{(1)(1)} = 1.$$

Then our Taylor series expansion of $H(z)$ is

$$\begin{aligned}
H(z) &= H(0)z^0/0! + H'(0)z^1/1! + \mathcal{O}(z^2) \\
&= 0 + z + \mathcal{O}(z^2) \\
&= z + \mathcal{O}(z^2)
\end{aligned}$$

then for $h(z)$ is

$$h(z) = z^{-1} + \mathcal{O}(z^{-2}).$$

We really care about $\frac{1}{z-z_0}h(z)$ so we have

$$\begin{aligned}
\frac{1}{z-z_0}h(z) &= \frac{1}{z-z_0}\left(z^{-1} + \mathcal{O}(z^{-2})\right) \\
&= \frac{1}{z}\frac{1}{1-z_0/z}\left(z^{-1} + \mathcal{O}(z^{-2})\right) \\
&= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\left(z^{-1} + \mathcal{O}(z^{-2})\right)
\end{aligned}$$

where $|z_0| < |z|$ since we are on a contour with a large radius R. Then

$$\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\left(z^{-1} + \mathcal{O}(z^{-2})\right) = \frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k + \mathcal{O}(z^{-3})\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k$$

Therefore the residue of this function at ∞ is trivially

$$\operatorname*{Res}_{z=\infty}\left(\frac{1}{(z-z_0)\sqrt{z-1}\sqrt{z+1}}\right) = 0$$

since the coefficient of the $1/z$ is 0. Computing the residue at z_0 is a little easier since it is a simple pole. Therefore

$$\begin{aligned}
\operatorname*{Res}_{z=z_0}\left(\frac{1}{(z-z_0)\sqrt{z-1}\sqrt{z+1}}\right) &= \operatorname*{Res}_{z=z_0}\left(\frac{\frac{1}{\sqrt{z-1}\sqrt{z+1}}}{z-z_0}\right) \\
&= \frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}}.
\end{aligned}$$

Plugging these into equation (5) we have

$$\frac{1}{2\pi \mathrm{i}}\oint_C g(z)\mathrm{d}z = -\operatorname*{Res}_{z=z_0} g(z) + \operatorname*{Res}_{z=\infty} g(z) = -\frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}} + 0.$$

Hence,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\frac{1}{x-z_0}\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = -\frac{1}{\sqrt{z_0-1}\sqrt{z_0+1}}.$$

$\square$

(b)

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x}\sqrt{1+x}\frac{\mathrm{d}x}{x-z}.$$

Solution:
Again applying equation (4), we have

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x}\sqrt{1+x}\frac{\mathrm{d}x}{x-z}=\frac{1}{\pi \mathrm{i}}\oint_{C}\sqrt{z-1}\sqrt{z+1}\frac{1}{z-z_0}\mathrm{d}z.$$

Let

$$g(z)=\sqrt{z-1}\sqrt{z+1}\frac{1}{z-z_0}.$$

Then as we expand our contour C outwards we run into the singularity at z_0, leaving behind a clockwise circular contour around z_0 denoted as $-C_{z_0}$. We also have the normal counterclockwise contour around infinity which we will use in our residue calculation. Hence we have

$$\begin{aligned}\frac{1}{\pi \mathrm{i}}\oint_{C}g(z)\mathrm{d}z&=\frac{1}{\pi \mathrm{i}}\oint_{-C_{z_0}}g(z)\mathrm{d}z+\frac{1}{\pi \mathrm{i}}\oint_{C_\infty}g(z)\mathrm{d}z\\&=-\frac{1}{\pi \mathrm{i}}\oint_{C_{z_0}}g(z)\mathrm{d}z+\frac{1}{\pi \mathrm{i}}\oint_{C_\infty}g(z)\mathrm{d}z\\&=-2\operatorname*{Res}_{z=z_0}g(z)+2\operatorname*{Res}_{z=\infty}g(z).\end{aligned}\tag{6}$$

Recall, that we have the Taylor expansion of $\sqrt{z-1}\sqrt{z+1}$ at ∞ is

$$\sqrt{z-1}\sqrt{z+1}=z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3}).$$

Then we can multiply through by our extra term in this scenario to get

$$\begin{aligned}\frac{1}{z-z_0}\sqrt{z-1}\sqrt{z+1}&=\frac{1}{z-z_0}\left(z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3})\right)\\&=\frac{1}{z}\frac{1}{1-z_0/z}\left(z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3})\right)\\&=\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\left(z-\frac{1}{2}z^{-1}+\mathcal{O}(z^{-3})\right)\\&=z\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k-\frac{1}{2}z^{-1}\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k+\mathcal{O}(z^{-3})\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k\\&=\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k-\frac{1}{2}\frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k+\mathcal{O}(z^{-4})\sum_{k=0}^{\infty}\left(\frac{z_0}{z}\right)^k.\end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z=\infty}\left(\frac{1}{z-z_0}\sqrt{z-1}\sqrt{z+1}\right)=z_0.$$

While the residue at the point z_0 is

$$\operatorname*{Res}_{z=z_0}\left(\frac{1}{z-z_0}\sqrt{z-1}\sqrt{z+1}\right)=\sqrt{z_0-1}\sqrt{z_0+1}.$$

Lets plug these in to equation (6) to have

$$\frac{1}{\pi \mathrm{i}} \oint_C g(z) \mathrm{d}z = -2 \operatorname*{Res}_{z=z_0} g(z) + 2 \operatorname*{Res}_{z=\infty} g(z) = -2\sqrt{z_0 - 1}\sqrt{z_0 + 1} + 2z_0.$$

Hence,

$$\frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x} \frac{\mathrm{d}x}{x - z} = 2(z_0 - \sqrt{z_0 - 1}\sqrt{z_0 + 1}).$$

$\square$