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AMATH 561

**PROBLEM SET 3**

**1.** Give an example of a probability space $(\Omega, \mathcal{F}, P)$, a random variable $X$ and a function $f$ such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ but $\sigma(f(X)) \neq \{\emptyset, \Omega\}$. Give a function $g$ such that $\sigma(g(X)) = \{\emptyset, \Omega\}$. Hint: Look at finite sample spaces with a small number of elements.

*Solution:*

Let our probability space be two independent coin tosses, such that $\Omega = \big\{HH, TT, HT, TH\big\}$. Define a random variable $X$ such that $X(\omega)$ be the number of heads in the outcome $\omega$ with $\omega \in \Omega$. Therefore

$$\begin{aligned} X(HH) &= 2, \\ X(TT) &= 0, \\ X(HT) &= 1, \text{ and} \\ X(TH) &= 1. \end{aligned}$$

Now $\sigma(X)$ can be written as

$$\sigma(X) = \Big\{\{HH\}, \{TH, HT\}, \{TT\}, \{TT, HH\}, \{HH, HT, TH\}, \{TT, HT, TH\}, \Omega, \emptyset\Big\}$$

**Part one**

Random variable $X$ and $f$ such that $\sigma(f(X)) \subsetneq \sigma(X)$ and $\sigma(f(X))$ is not the trivial $\sigma$-algebra.

Define $f(x)$ as follows

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Let's look at the possible pre-images of $f(X)$ with respect to a few cases of Borel sets. For convenience, I will define $\hat{X} = f(X)$. Now let's look at some cases for the pre-image

*Case 1:* $0 \in B$ but $1 \notin B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (-\infty, 0]\ \} = \{TT\}$$

*Case 2:* $0 \notin B$ but $1 \in B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (0, \infty)\ \} = \{TH, HT, HH\}$$

*Case 3:* $0 \in B$ and $1 \in B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (-\infty, \infty)\ \} = \Omega$$

*Case 4:* $0 \notin B$ and $1 \notin B$

$$\hat{X}^{-1}(B) = \Big\{\omega : \hat{X}(\omega) \in B\Big\} = \emptyset.$$

Therefore,

$$\sigma(f(X)) = \Big\{\{TT\}, \{TH, HT, HH\}, \Omega, \emptyset\Big\} \neq \{\emptyset, \Omega\}$$

And thus we have $\sigma(f(X)) \subsetneq \sigma(X)$.
**Part two** Now also give a function $g$ such that $\sigma(g(X))$ is the trivial $\sigma$-algebra, $\{\emptyset, \Omega\}$.
Define $g(x)$ to be a constant $c \in \mathbb{R}$ such that $g(x) = c$ for all $x \in \mathbb{R}$. Once again, for convenience we define $\tilde{X} = g(X)$. Let's go through a few cases of what the pre-image may be for any Borel set
*Case 1:* $c \in B$

$$\tilde{X}^{-1}(B) = \Big\{\omega : \tilde{X}(\omega) \in B\Big\} = \{\omega : X(\omega) \in (-\infty, \infty)\ \} = \Omega$$

*Case 2:* $c \notin B$

$$\tilde{X}^{-1}(B) = \Big\{\omega : \tilde{X}(\omega) \in B\Big\} = \emptyset.$$

Therefore,

$$\sigma(g(X)) = \{\Omega, \emptyset\}.$$

$\square$

**2.** Give an example of events $A$, $B$, and $C$, each of probability strictly between 0 and 1, such that $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, and $P(A \cap B \cap C) = P(A)P(B)P(C)$ but $P(B \cap C) \neq P(B)P(C)$. Are $A$, $B$ and $C$ independent? Hint: You can let $\Omega$ be a set of eight equally likely points.
*Solution:*
Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Define events $A$, $B$, and $C$ as follows

$$A = \{1, 2, 3, 4\}$$
$$B = \{1, 2, 5, 7\}$$
$$C = \{1, 3, 6, 8\}.$$

Then we have

$$P(A \cap B) = P(\{1, 2\}) = \frac{1}{4}$$

and

$$P(A)P(B) = P(\{1, 2, 3, 4\})P(\{1, 2, 5, 7\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Additionally, we have

$$P(A \cap C) = P(\{1, 3\}) = \frac{1}{4}$$

and

$$P(A)P(C) = P(\{1, 2, 3, 4\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Finally, we have

$$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{8}$$

and

$$P(A)P(B)P(C) = P(\{1, 2, 3, 4\})P(\{1, 2, 5, 7\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Notice we also get

$$P(B \cap C) = P(\{1\}) = \frac{1}{8}$$

which is not equal to

$$P(B)P(C) = P(\{1,2,5,7\})P(\{1,3,6,8\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

In class we said two events $E$ and $E'$ are independent if $P(E \cap E') = P(E)P(E')$. Therefore we have shown that $A$ and $B$ are independent, $A$ and $C$ are independent but $B$ and $C$ are not independent. □

**3.** Let $(\Omega, \mathcal{F}, P)$ be a probability space such that $\Omega$ is countably infinite, and $\mathcal{F} = 2^{\Omega}$. Show that it is impossible for there to exist a countable collection of events $A_1, A_2, ... \in \mathcal{F}$ which are independent, such that $P(A_i) = 1/2$ for each $i$. Hint: First show that for each $\omega \in \Omega$ and each $n \in \mathbb{N}$, we have $P(\omega) \leq 1/2^n$. Then derive a contradiction.
*Solution:*
Literally just use the hint...

**4.** (a) Let $X \geq 0$ and $Y \geq 0$ be independent random variables with distribution functions $F$ and $G$. Find the distribution function of $XY$.
*Solution:*
These are not explicitly dealing with discrete or continuous. Definitely review lecture notes. Since these are independent try using the formulae from the lecture on 10-16-24.

(b) If $X \geq 0$ and $Y \geq 0$ are independent continuous random variables with density functions $f$ and $g$, find the density function of $XY$.
*Solution:*
Notice these are continuous and you're dealing with densities.

(c) If $X$ and $Y$ are independent exponentially distributed random variables with parameter $\lambda$, find the density function of $XY$.
*Solution:*
TBD