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AMATH 567

HOMEWORK 10

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: I sketched the following in class. Complete the argument. Show that for an integer $j \in (-N, N)$ and $h > 0$,

$$\lim_{h\to\infty} \int_{\mathrm{i}h}^{\mathrm{i}h+\pi} \frac{\mathrm{e}^{2\mathrm{i}jz}}{\tan(Nz)} \mathrm{d}z = \begin{cases} -\mathrm{i}\pi & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution:
TODO:
Following the sketch provided in class let's look at an import representation of $\tan(Nz)$

$$\begin{aligned}
\tan(Nz) &= \frac{\sin(Nz)}{\cos(Nz)} \\
&= \frac{\mathrm{e}^{\mathrm{i}Nz} + \mathrm{e}^{-\mathrm{i}Nz}}{2\mathrm{i}} \left(\frac{\mathrm{e}^{\mathrm{i}Nz} - \mathrm{e}^{-\mathrm{i}Nz}}{2} \right)^{-1} \\
&= \frac{\mathrm{e}^{\mathrm{i}Nz} + \mathrm{e}^{-\mathrm{i}Nz}}{2\mathrm{i}} \left(\frac{2}{\mathrm{e}^{\mathrm{i}Nz} - \mathrm{e}^{-\mathrm{i}Nz}} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{\mathrm{e}^{\mathrm{i}Nz} + \mathrm{e}^{-\mathrm{i}Nz}}{\mathrm{e}^{\mathrm{i}Nz} - \mathrm{e}^{-\mathrm{i}Nz}} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{\mathrm{e}^{\mathrm{i}Nz}}{\mathrm{e}^{\mathrm{i}Nz}} \left(\frac{1 + \mathrm{e}^{-2\mathrm{i}Nz}}{1 - \mathrm{e}^{-2\mathrm{i}Nz}} \right) \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{1 + \mathrm{e}^{-2\mathrm{i}Nz}}{1 - \mathrm{e}^{-2\mathrm{i}Nz}} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{1 + (\cos(Nz) - \mathrm{i}\sin(Nz))^2}{1 - (\cos(Nz) - \mathrm{i}\sin(Nz))^2} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{1 + \cos^2(Nz) - 2\mathrm{i}\cos(Nz)\sin(Nz) - \sin^2(Nz)}{1 - \cos^2(Nz) + 2\mathrm{i}\cos(Nz)\sin(Nz) + \sin^2(Nz)} \right) \\
&\dots \\
&= \mathrm{i} + \mathcal{O}(\mathrm{e}^{-2Nh})
\end{aligned}$$

2: From A&F: 4.2.1 (b)

Solution:
TODO:

$$f(x) = mx + b$$

3: From A&F: 4.2.2 (a, h)

Solution:
TODO:

$$f(x) = mx + b$$

4: (a) Show that

$$\operatorname{Res}_{z=k} f(z)\cot(\pi z) = \frac{1}{\pi} f(k),$$

provided $f(z)$ is analytic at $z = k,\ k \in \mathbb{Z}$.

Solution:
Recall from homework 6 problem 3 we derived a series representation for $\cot z$

$$\begin{aligned}\cot(\pi z) &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1}\\ &= (-1)^0 \frac{2^0 B_0}{0!} \pi^{0-1} z^{0-1} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1}\\ &= \frac{B_0}{z\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1}.\end{aligned}$$

Therefore, we can compute the residue by multiplying $f(z)$ through this Taylor series and evaluating the expression in the numerator of the simple pole at $z = 0$

$$\begin{aligned}\operatorname{Res}_{z=k} f(z)\cot(\pi z) &= \operatorname{Res}_{z=k} \left[f(z) \left(\frac{B_0}{z\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1} \right) \right]\\ &= \operatorname{Res}_{z=k} \left[\frac{B_0 f(z)}{z\pi} + f(z) \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1} \right]\\ &= \frac{B_0}{\pi} f(k)\\ &= \frac{1}{\pi} f(k).\end{aligned}$$

We can conclude this because there is no irregularities contributed by the analytic function $f(z)$.

$\square$

(b) Let Γ_N be a square contour, with corners at $(N + 1/2)(\pm 1 \pm i), N \in \mathbb{Z}^+$. Show that

$$|\cot(\pi z)| \leq 2,$$

for z on Γ_N.

Solution:
TODO:

$$f(x) = mx + b$$

(c) Suppose $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, so that the degree of $q(z)$ is at least two more than the degree of $p(z)$. Show that

$$\lim_{N\to\infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| = 0$$

Solution:
TODO:

$$f(x) = mx + b$$

(d) Suppose, in addition, that $q(z)$ has no roots at the integers. Show that

$$\sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = -\pi \sum_{j} \operatorname{Res}_{z=z_j} f(z) \cot(\pi z)$$

where the z_j's are the roots of $q(z)$. Notice that the sum on the right-hand side has a finite number of terms.

Solution:
TODO:

$$f(x) = mx + b$$

(e) Use the result of the previous problem to evaluate the following sums:

(i) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1}$

Solution:
TODO:

$$f(x) = mx + b$$

(ii) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^4+1}$

Solution:
TODO:

$$f(x) = mx + b$$

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2-1/4}$

Solution:
TODO:

$$f(x) = mx + b$$

(iv) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{16k^4-1}$

Solution:
TODO:

$$f(x) = mx + b$$