

Hunter Lybbert
 Student ID: 2426454
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 AMATH 567

HOMEWORK 10

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

- 1: I sketched the following in class. Complete the argument. Show that for an integer $j \in (-N, N)$ and $h > 0$,

$$\lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} \frac{e^{2ijz}}{\tan(Nz)} dz = \begin{cases} -i\pi & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

TODO:

Following the sketch provided in class let's look at an import representation of $\tan(Nz)$

$$\begin{aligned} \tan(Nz) &= \frac{\sin(Nz)}{\cos(Nz)} \\ &= \frac{e^{iNz} + e^{-iNz}}{2i} \left(\frac{e^{iNz} - e^{-iNz}}{2} \right)^{-1} \\ &= \frac{e^{iNz} + e^{-iNz}}{2i} \left(\frac{2}{e^{iNz} - e^{-iNz}} \right) \\ &= \frac{1}{i} \left(\frac{e^{iNz} + e^{-iNz}}{e^{iNz} - e^{-iNz}} \right) \\ &= \frac{1}{i} \left(\frac{e^{iNz}}{e^{iNz}} \left(\frac{1 + e^{-2iNz}}{1 - e^{-2iNz}} \right) \right) \\ &= \frac{1}{i} \left(\frac{1 + e^{-2iNz}}{1 - e^{-2iNz}} \right) \\ &= \frac{1}{i} \left(\frac{1 + (\cos(Nz) - i \sin(Nz))^2}{1 - (\cos(Nz) - i \sin(Nz))^2} \right) \\ &= \frac{1}{i} \left(\frac{1 + \cos^2(Nz) - 2i \cos(Nz) \sin(Nz) - \sin^2(Nz)}{1 - \cos^2(Nz) + 2i \cos(Nz) \sin(Nz) + \sin^2(Nz)} \right) \\ &\dots \\ &= i + \mathcal{O}(e^{-2Nh}) \end{aligned}$$

2: From A&F: 4.2.1 (b)

Solution:

See solution to problem 2 from homework set 9.

□

3: From A&F: 4.2.2 (a, h)

Solution:

TODO:

$$f(x) = mx + b$$

4: (a) Show that

$$\operatorname{Res}_{z=k} f(z) \cot(\pi z) = \frac{1}{\pi} f(k),$$

provided $f(z)$ is analytic at $z = k$, $k \in \mathbb{Z}$.

Solution:

Recall from homework 6 problem 3 we derived a series representation for $\cot z$

$$\begin{aligned} \cot(\pi z) &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1} \\ &= (-1)^0 \frac{2^0 B_0}{0!} \pi^{0-1} z^{0-1} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1} \\ &= \frac{B_0}{z\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1}. \end{aligned}$$

Therefore, we can compute the residue by multiplying $f(z)$ through this Taylor series and evaluating the expression in the numerator of the simple pole at $z = 0$

$$\begin{aligned} \operatorname{Res}_{z=k} f(z) \cot(\pi z) &= \operatorname{Res}_{z=k} \left[f(z) \left(\frac{B_0}{z\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1} \right) \right] \\ &= \operatorname{Res}_{z=k} \left[\frac{B_0 f(z)}{z\pi} + f(z) \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \pi^{2n-1} z^{2n-1} \right] \\ &= \frac{B_0}{\pi} f(k) \\ &= \frac{1}{\pi} f(k). \end{aligned}$$

We can conclude this because there is no irregularities contributed by the analytic function $f(z)$.

□

(b) Let Γ_N be a square contour, with corners at $(N + 1/2)(\pm 1 \pm i)$, $N \in \mathbb{Z}^+$. Show that

$$|\cot(\pi z)| \leq 2,$$

for z on Γ_N .

Solution:

TODO:

$$f(x) = mx + b$$

(c) Suppose $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, so that the degree of $q(z)$ is at least two more than the degree of $p(z)$. Show that

$$\lim_{N \rightarrow \infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| = 0$$

Solution:

TODO:

$$f(x) = mx + b$$

(d) Suppose, in addition, that $q(z)$ has no roots at the integers. Show that

$$\sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = -\pi \sum_j \operatorname{Res}_{z=z_j} f(z) \cot(\pi z)$$

where the z_j 's are the roots of $q(z)$. Notice that the sum on the right-hand side has a finite number of terms.

Solution:

TODO:

$$f(x) = mx + b$$

(e) Use the result of the previous problem to evaluate the following sums:

(i) $\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 1}$

Solution:

TODO:

$$f(x) = mx + b$$

(ii) $\sum_{k=-\infty}^{\infty} \frac{1}{k^4 + 1}$

Solution:

TODO:

$$f(x) = mx + b$$

(iii) $\sum_{k=-\infty}^{\infty} \frac{1}{k^2 - 1/4}$

Solution:

TODO:

$$f(x) = mx + b$$

(iv) $\sum_{k=-\infty}^{\infty} \frac{1}{16k^4 - 1}$

Solution:

TODO:

$$f(x) = mx + b$$