# AMATH 561 Autumn 2024
# Problem Set 1

Due: Mon 10/7 at 10am

*Note: Submit electronically to Canvas.*

**1.** Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.
a) *Solution*
Actual math here
b) *Solution*
Actual math here

**2.** (No translation-invariant random integer). Show that there is no probability measure $P$ on the integers $\mathbb{Z}$ with the discrete $\sigma$-algebra $2^{\mathbb{Z}}$ with the translation-invariance property $P(E + n) = P(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer $n$. $E + n$ is obtained by adding $n$ to every element of $E$.
*Solution*
Show by contradiction, eventually $P(\Omega) = 0$ but it should be 1

**3.** (No translation-invariant random real). Show that there is no probability measure $P$ on the reals $\mathbb{R}$ with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property $P(E + x) = P(E)$ for every event $E \in \mathcal{B}(\mathbb{R})$ and every real $x$. Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ is the $\sigma$-algebra generated by intervals $(a, b] \subset \mathbb{R}$.
*Solution*
Also show by contradiction but it is a little trickier. It has to composing the right $x$ s.t. $E + x$ is outside the interval $(a, b]$.

**4.** Let $\Omega = \mathbb{R}$, $\mathcal{F}$ = all subsets of $\mathbb{R}$ so that $A$ or $A^c$ is countable. Let $P(A) = 0$ in the first case and $P(A) = 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$

is a probability space. *Solution*
We want to prove something is a probability space. Therefore, we need to prove the following:
(1) prove fancy $\mathcal{F}$ is a $\sigma$-algebra which includes that it contains the compliments of each set and that it contains the union of two sets.
(2) prove cap P is a probability measure which includes proving $P(\Omega) = 1$, $P(A) \geq P(\emptyset) = 0 \quad \forall A \in \mathcal{F}$, and that $P(\cup_i A_i) = \sum_i P(A_i)$ for $A_i \in \mathcal{F}$ that is a countable sequence of disjoint sets.