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 AMATH 561

PROBLEM SET 6

1. Let $X \sim \text{Binomial}(n, U)$, where $U \sim \text{Uniform}((0, 1))$. What is the probability generating function $G_X(s)$ of X ? What is $P(X = k)$ for $k \in \{0, 1, 2, \dots, n\}$?

Solution: The probability mass function for $X \sim \text{Binomial}(n, U)$, is given by

$$f_X(x) = \binom{n}{x} U^x (1 - U)^{n-x} \quad \text{for } x = 0, 1, 2, \dots, n$$

And the density for the Uniform distribution is

$$g_X(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1 \quad \text{for } x \in (0, 1)$$

Then $G_X(s)$ is

$$\begin{aligned} G_X(s) &= E(s^X) \\ &= E(E(s^X | U)) \\ &= E\left(\sum_{x=0}^n \binom{n}{x} U^x (1 - U)^{n-x} s^x\right) \\ &= E\left(\sum_{x=0}^n \binom{n}{x} (Us)^x (1 - U)^{n-x}\right) \\ &= E((Us + 1 - U)^n) \\ &= \int_0^1 (us + 1 - u)^n du \\ &= \left. \frac{(us + 1 - u)^{n+1}}{(n+1)(s-1)} \right|_0^1 \\ &= \frac{(1s + 1 - 1)^{n+1}}{(n+1)(s-1)} - \frac{(0s + 1 - 0)^{n+1}}{(n+1)(s-1)} \\ &= \frac{s^{n+1}}{(n+1)(s-1)} - \frac{1^{n+1}}{(n+1)(s-1)} \\ &= \frac{s^{n+1} - 1}{(n+1)(s-1)} \end{aligned}$$

Now to calculate $P(X = k)$, notice

$$\begin{aligned}
 G_X(s) &= \frac{s^{n+1} - 1}{(n+1)(s-1)} = \frac{1}{n+1} \frac{s^{n+1} - 1}{s-1} \\
 &= \frac{1}{n+1} \frac{1 - s^{n+1}}{1 - s} \\
 &= \sum_{k=0}^n \frac{1}{n+1} s^k \\
 &= \sum_{k=0}^n P(X = k) s^k \\
 &= G_X(s)
 \end{aligned}$$

Therefore,

$$P(X = k) = \frac{1}{n+1} \quad \text{for all } k \in \{0, 1, 2, \dots, n\}.$$

□

TODO: Prolly delete this section, though it is true

Alternatively, we need to come up with a formula for

$$G_X^{(k)}(0) = \frac{d^k}{ds^k} \left[\frac{s^{n+1} - 1}{(n+1)(s-1)} \right] \Big|_0.$$

I'll begin by calculating this for $k = 0, 1, 2$

$$\begin{aligned}
 G_X^{(0)}(0) &= \frac{d^0}{ds^0} \left[\frac{s^{n+1} - 1}{(n+1)(s-1)} \right] \Big|_0 = \frac{s^{n+1} - 1}{(n+1)(s-1)} \Big|_0 = \frac{0^{n+1} - 1}{(n+1)(0-1)} = \frac{1}{n+1} \\
 G_X^{(1)}(0) &= \frac{d^1}{ds^1} \left[\frac{s^{n+1} - 1}{(n+1)(s-1)} \right] \Big|_0 \\
 &= \frac{s^n(n+1)(s-1) - (s^{n+1} - 1)}{(n+1)(s-1)^2} \Big|_0 = \frac{0^n(n+1)(0-1) - (0^{n+1} - 1)}{(n+1)(0-1)^2} = \frac{1}{n+1} \\
 G_X^{(2)}(0) &= \frac{d^2}{ds^2} \left[\frac{s^{n+1} - 1}{(n+1)(s-1)} \right] \Big|_0 \\
 &= \left[\left((n+1)(ns^{n-1}(s-1) + s^n) - (n+1)s^n \right) (s-1)^2 \right. \\
 &\quad \left. - \left(s^n(n+1)(s-1) - (s^{n+1} - 1) \right) 2(s-1) \right] / ((n+1)(s-1)^4) \Big|_0 \\
 &= \left[\left((n+1)(n0^{n-1}(0-1) + 0^n) - (n+1)0^n \right) (0-1)^2 \right. \\
 &\quad \left. - \left(0^n(n+1)(0-1) - (0^{n+1} - 1) \right) 2(0-1) \right] / ((n+1)(0-1)^4) \\
 &= [0 - 2(0-1)] / ((n+1)(0-1)^4) \\
 &= \frac{2}{n+1}.
 \end{aligned}$$

2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ , the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

Solution:

$$\begin{aligned} G_{Z_{n+1}}(s) &= E(s^{Z_{n+1}}) = E(s^{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}}) \\ &= E(E(s^{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}} | \xi, Y)) \\ &= E(E(s^{\sum_{i=1}^{Z_n} \xi_i^{n+1}} s^{Y_{n+1}} | \xi, Y)) \\ &= E\left(E\left(\left(\prod_{i=1}^{Z_n} s^{\xi_i^{n+1}}\right) s^{Y_{n+1}} \middle| \xi, Y\right)\right) \\ &= E\left(E\left(\prod_{i=1}^{Z_n} s^{\xi_i^{n+1}} \middle| \xi, Y\right) E(s^{Y_{n+1}} | \xi, Y)\right) \\ &= \left(\prod_{i=1}^{Z_n} E(s^{\xi_i^{n+1}} | \xi)\right) E(s^{Y_{n+1}} | Y) \\ &= \left(\prod_{i=1}^{Z_n} G_\xi(s)\right) G_Y(s) \end{aligned}$$

3. (a) Let X be exponentially distributed with parameter λ . Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$. (b) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

Solution:

4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \rightarrow 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} 1_{x \geq 0}.$$

Solution: