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AMATH 567

HOMEWORK 3

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.2.4.
Let α be a real number. Show that the set of all values of the multivalued function $\log(z^\alpha)$ is not necessarily the same as that of $\alpha \log z$.
Solution:
Let's begin by looking more closely at the values that the function $\alpha \log z$ can take on. Starting with $w = \alpha \log z$, notice we have

$$\begin{aligned}
\alpha \log z &= \alpha \log\left(r\,\mathrm{e}^{i\theta}\right) \\
&= \alpha\left(\log r + i\theta\right) \text{ where we say } \theta = \theta_p + 2\pi k,\ k \in \mathbb{Z} \\
&= \alpha\left(\log r + i\left(\theta_p + 2\pi k\right)\right),\ k \in \mathbb{Z} \\
&= \alpha\left(\log r + i\theta_p + 2i\pi k\right),\ k \in \mathbb{Z} \\
&= \alpha \log r + \alpha i\theta_p + \alpha 2i\pi k,\ k \in \mathbb{Z}.
\end{aligned}$$

Now considering the other expression

$$\begin{aligned}
\log\left(z^\alpha\right) &= \log\left(\left(r\,\mathrm{e}^{i\theta}\right)^\alpha\right) \\
&= \log\left(r^\alpha\,\mathrm{e}^{i\theta\alpha}\right) \\
&= \log r^\alpha + i\theta\alpha \text{ where we say } \theta = \theta_p + 2\pi k,\ k \in \mathbb{Z} \\
&= \log r^\alpha + i\left(\theta_p + 2\pi k\right)\alpha \text{ where we say } \theta = \theta_p + 2\pi k,\ k \in \mathbb{Z} \\
&= \log r^\alpha + i\left(\theta_p + 2\pi k\right)\alpha,\ k \in \mathbb{Z} \\
&= \log r^\alpha + \alpha i\theta_p + \alpha 2i\pi k,\ k \in \mathbb{Z}.
\end{aligned}$$

Now recall that $w = u + iv$ therefore we also have

$$\begin{aligned}
z &= \mathrm{e}^{\frac{u+iv}{\alpha}} \\
z &= \mathrm{e}^{\frac{u}{\alpha}}\,\mathrm{e}^{\frac{iv}{\alpha}} \\
\log z &= \log\left(\mathrm{e}^{\frac{u}{\alpha}}\,\mathrm{e}^{\frac{iv}{\alpha}}\right) \\
\log z &= \log \mathrm{e}^{\frac{u}{\alpha}} + \log \mathrm{e}^{\frac{iv}{\alpha}} \\
\log z &= \frac{\log r}{\alpha} + \frac{i\theta}{\alpha} + \frac{2i\pi k}{\alpha},\ k \in \mathbb{Z}.
\end{aligned}$$

Where the last line comes from the fact that we have $w = r\,\mathrm{e}^{i\theta}$, $r = \mathrm{e}^{u}$, and $v = \theta + 2\pi k$ again with $k \in \mathbb{Z}$. Now considering the other side

$$\begin{aligned} \log(z^{\alpha}) &= w \\ z^{\alpha} &= \mathrm{e}^{w} \\ z &= (\mathrm{e}^{w})^{\frac{1}{\alpha}}\,. \end{aligned}$$

Therefore this is going to come down to the fact that this is not always going to be equivalent $\mathrm{e}^{\frac{w}{\alpha}} \neq= (\mathrm{e}^{w})^{\frac{1}{\alpha}}$.

2: Describe the Riemann surface on which the multi-valued function $w(z)$, defined by $w^2 = \prod_{j=1}^{n=3}(z - a_j)$ is single-valued. What happens for $n = 4, 5$? For $n > 5$? You may assume that all the a_j are distinct.
Solution:
Let's build up to what the the Riemann surface for $w^2 = \prod_{j=1}^{n=3}(z - a_j)$ will look like. Beginning with $w^2 = z$ the branch point at the origin of the complex plane z $= 0$. is moved to the location a_0 instead of Reimann Surfacebe similar to that of $w^2 = z - a_0$ which is yet again similar to $w^2 = z$ except the branch point is moved to the location a_0 instead of the origin of the complex plane z $= 0$.

3: From A&F: 2.2.5a.
Derive the following formulae:
a)

$$\coth^{-1}(z) = \frac{1}{2}\log\frac{z+1}{z-1}$$

Solution:
We begin with solving for w in $z = \coth w$ with $w, z \in \mathbb{C}$.

$$\begin{aligned} z = \coth w &= \frac{\cosh w}{\sinh w} = \frac{\frac{e^{w}+e^{-w}}{\cancel{2}}}{\frac{e^{w}-e^{-w}}{\cancel{2}}} = \frac{e^{w}+e^{-w}}{e^{w}-e^{-w}} \\ &= \frac{e^{w}}{e^{w}}\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}} = \frac{e^{2w}+1}{e^{2w}-1} \end{aligned}$$

Now let's proceed by multiplying both sides by the denominator

$$\begin{aligned} z\left(e^{2w}-1\right) &= e^{2w}+1 \\ ze^{2w} - z &= e^{2w}+1 \\ ze^{2w} - e^{2w} &= z+1 \\ e^{2w}(z-1) &= z+1 \\ e^{2w} &= \frac{z+1}{z-1} \\ \log\left(e^{2w}\right) &= \log\left(\frac{z+1}{z-1}\right) + 2i\pi k,\ k \in \mathbb{Z} \\ 2w &= \log\left(\frac{z+1}{z-1}\right) + 2i\pi k,\ k \in \mathbb{Z} \\ w &= \frac{1}{2}\log\left(\frac{z+1}{z-1}\right) + i\pi k,\ k \in \mathbb{Z} \end{aligned}$$

This is to show

$$\coth^{-1}(z) = \operatorname{arccot}(\coth w) = w = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right) + i\pi k,\ k \in \mathbb{Z}.$$

More directly we have

$$\coth^{-1}(z) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right) + i\pi k,\ k \in \mathbb{Z}.$$

as required. **TODO:** determine what your choice of k should be $\square$

b)

$$\operatorname{sech}^{-1}(z) = \log\left(\frac{1+(1-z^2)^{\frac{1}{2}}}{z}\right)$$

Solution:

We begin with solving for w in $z = \operatorname{sech} w$ with $w, z \in \mathbb{C}$.

$$\begin{aligned} z = \operatorname{sech} w &= \frac{1}{\cosh w} = \frac{1}{\frac{e^w+e^{-w}}{2}} = \frac{2}{e^w+e^{-w}} \\ &= \frac{e^w}{e^w}\frac{2}{e^w+e^{-w}} = \frac{2e^w}{e^{2w}+1} \end{aligned}$$

Now let's proceed by multiplying both sides by the denominator

$$\begin{aligned} z\left(e^{2w}+1\right) &= 2e^w \\ ze^{2w}+z &= 2e^w \\ ze^{2w}-2e^w+z &= 0 \end{aligned}$$

We now use the quadratic formula to solve for e^w

$$\begin{aligned} e^w &= \frac{2+\left(4-4z^2\right)^{\frac{1}{2}}}{2z} \\ e^w &= \frac{\cancel{2}+\cancel{2}\left(1-z^2\right)^{\frac{1}{2}}}{\cancel{2}z} \\ \log e^w &= \log\left(\frac{1+\left(1-z^2\right)^{\frac{1}{2}}}{z}\right) + 2i\pi k,\ k \in \mathbb{Z} \\ w &= \log\left(\frac{1+\left(1-z^2\right)^{\frac{1}{2}}}{z}\right) + 2i\pi k,\ k \in \mathbb{Z} \end{aligned}$$

This is to show

$$\operatorname{sech}^{-1}(z) = \operatorname{sech}^{-1}(\operatorname{sech} w) = w = \log\left(\frac{1+\left(1-z^2\right)^{\frac{1}{2}}}{z}\right) + 2i\pi k,\ k \in \mathbb{Z}.$$

More directly we have

$$\operatorname{sech}^{-1}(z) = \log\left(\frac{1+\left(1-z^2\right)^{\frac{1}{2}}}{z}\right) + 2i\pi k,\ k \in \mathbb{Z}.$$

as required. **TODO:** determine what your choice of k should be and how to denote the related cuts made for the $\sqrt{\cdot}$ function. $\square$

While you're at it, also derive a formula for $\operatorname{arccot}(z)$ in terms of the logarithm.
Solution:
Let's begin by solving for w in this equation $z = \cot w$ with $w, z \in \mathbb{C}$.

$$\begin{aligned} z = \cot w &= \frac{\cos w}{\sin w} = \frac{\frac{e^{iw}+e^{-iw}}{\cancel{2}}}{\frac{e^{iw}-e^{-iw}}{\cancel{2}i}} = \frac{i\left(e^{iw}+e^{-iw}\right)}{e^{iw}-e^{-iw}} \\ &= \frac{e^{iw}}{e^{iw}}\frac{i\left(e^{iw}+e^{-iw}\right)}{e^{iw}-e^{-iw}} = \frac{i\left(e^{2iw}+1\right)}{e^{2iw}-1} \end{aligned}$$

Now let's proceed by multiplying both sides by the denominator

$$\begin{aligned} z(e^{2iw}-1) &= i\left(e^{2iw}+1\right) \\ ze^{2iw}-z &= ie^{2iw}+i \\ ze^{2iw}-z-ie^{2iw}-i &= 0 \\ e^{2iw}(z-i) &= z+i \\ e^{2iw} &= \frac{z+i}{z-i} \\ \log\left(e^{2iw}\right) &= \log\left(\frac{z+i}{z-i}\right) + 2i\pi k,\ k \in \mathbb{Z} \\ 2iw &= \log\left(\frac{z+i}{z-i}\right) + 2i\pi k,\ k \in \mathbb{Z} \\ w &= \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) + \pi k,\ k \in \mathbb{Z} \end{aligned}$$

This is to show

$$\operatorname{arccot}(z) = \operatorname{arccot}(\cot w) = w = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) + \pi k,\ k \in \mathbb{Z}.$$

More directly we have

$$\operatorname{arccot}(z) = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) + \pi k,\ k \in \mathbb{Z}$$

as required. **TODO:** determine what your choice of k should be $\square$

4: Let

$$s(z) = z^{1/2} = \rho^{1/2}\,\mathrm{e}^{\mathrm{i}\theta/2}, \quad \theta \in [-\pi, \pi),$$

denote the principal branch of the square root. Show that the functions

$$f_1(z) = s(z^2-1), \quad f_2(z) = s(z-1)s(z+1),$$

are not equal as functions on $\mathbb{C}$ — first produce plots and then use a mathematical argument. Determine the branch cut for $f_2(z)$ (Note: My cartoon of what the branch cut for f_1 looks like in lecture was not accurate). Find the relationship between $f_1(z)$ and $f_2(z)$.
Solution:
$z = tz_1 + (1-t)z_2$

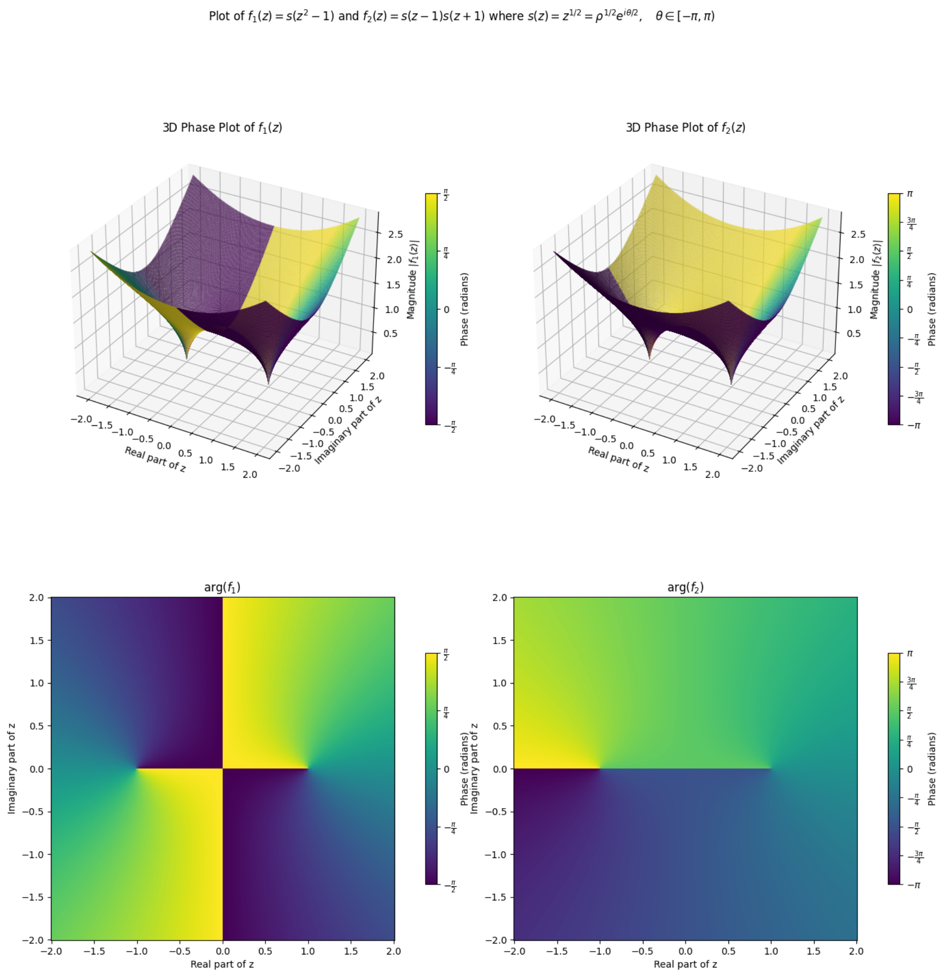

FIGURE 1. Plot $|f_\epsilon(z)| = \left|\frac{\epsilon}{\epsilon^2+z^2}\right|$, for various values of ϵ.

5: Consider the function

$$\psi(z) = \int_1^z \frac{\mathrm{d}w}{(w^2-1)^{1/2}}, \quad z \notin (-\infty, 1),$$

where the path of integration is a straight line from 1 to z.

- Show that

$$\psi(z) = \log \varphi(z), \quad \varphi(z) = z + (z^2-1)^{1/2}, \quad z \notin (-\infty, 1),$$

for an appropriate choice of branch cut for $(z^2-1)^{1/2}$. Here $\log z$ denotes the principal branch.

- Find an expression for

$$\gamma(z) = \int_{-1}^{z} \frac{\mathrm{d}w}{(w^2-1)^{1/2}}, \quad z \notin (-1,\infty),$$

in terms of $\varphi(z)$ and the principal branch of the logarithm. Again, the path of integration is a straight line.

Solution:

6: Show that φ, from the previous problem, maps $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit disk, $\{z \in \mathbb{C} \ : \ |z| > 1\}$. Furthermore

$$\frac{1}{2}\left(\varphi(z) + 1/\varphi(z)\right) = z, \quad \mathbb{C} \setminus [-1,1].$$

Solution:

7: (Sharpness of the Bernstein–Walsh inequality) The Bernstein–Walsh inequality states that if a polynomial p_n of degree n satisfies $\max_{-1\le x\le 1} |p_n(x)| \le 1$ then

$$|p_n(z)| \le |\varphi(z)|^n, \quad z \in \mathbb{C} \setminus [-1,1].$$

Show that

$$T_n(z) = \frac{1}{2}\left(\varphi(z)^n + \varphi(z)^{-n}\right), \quad z \in \mathbb{C} \setminus [-1,1]$$

is a polynomial that satisfies

$$\max_{-1\le x\le 1} |T_n(x)| = 1,$$

$$\lim_{n\to\infty} |T_n(z)|^{1/n} = |\varphi(z)|,$$

for any fixed $z \in \mathbb{C} \setminus [-1,1]$.

Solution: