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AMATH 567

HOMEWORK 6

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 3.3.2
Given the function

$$f(z) = \frac{z}{a^2 - z^2}, \;\; a > 0,$$

expand $f(z)$ in a Laurent series in powers of z in the regions

(a) $|z| < a$
Solution:
We begin with a little algebra

$$f(z) = \frac{z}{a^2 - z^2} = \frac{z}{a^2}\frac{1}{1 - \frac{z^2}{a^2}}.$$

In this case, since $|z| < a$, then $\frac{z^2}{a^2} < 1$. Therefore we can make use of the common geometric series

$$f(z) = \frac{z}{a^2}\frac{1}{1 - \frac{z^2}{a^2}} = \frac{z}{a^2}\sum_{n=0}^{\infty}\left(\frac{z^2}{a^2}\right)^n = \frac{z}{a^2}\sum_{n=0}^{\infty}\frac{z^{2n}}{a^{2n}} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{a^{2n+2}} = \sum_{n=0}^{\infty}\frac{1}{a^{2n+2}}z^{2n+1}.$$

$\square$

(b) $|z| > a$
Solution:
We begin with a little algebra

$$f(z) = \frac{z}{a^2 - z^2} = -\frac{z}{z^2 - a^2} = -\frac{z}{z^2}\frac{1}{1 - \frac{a^2}{z^2}}.$$

In this case, since $|z| > a$, then $\frac{a^2}{z^2} < 1$. Therefore we can make use of the common geometric series

$$
\begin{aligned}
f(z) &= -\frac{z}{z^2}\frac{1}{1-\frac{a^2}{z^2}} \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a^2}{z^2}\right)^n \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{a^{2n}}{z^{2n}} \\
&= -\sum_{n=0}^{\infty}\frac{a^{2n}}{z^{2n+1}} \\
&= -\sum_{n=0}^{\infty}a^{2n}\frac{1}{z^{2n+1}} \\
&= -\sum_{n=0}^{\infty}a^{2n}z^{-(2n+1)} \\
&= -\sum_{n=0}^{\infty}a^{2n}z^{-2n-1} \\
&= -\sum_{n=-\infty}^{0}a^{2n}z^{2n-1}.
\end{aligned}
$$

$\square$

2: From A&F: 3.3.5
Let

$$\exp\left(\frac{t}{2}\left(z-\frac{1}{z}\right)\right)=\sum_{n=-\infty}^{\infty}J_n(t)z^n.$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned}J_n(t)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}(n\theta-t\sin\theta)}\,\mathrm{d}\theta\\&=\frac{1}{\pi}\int_{0}^{\pi}\cos(n\theta-t\sin\theta)\mathrm{d}\theta.\end{aligned}$$

The functions $J_n(t)$ are called the Bessel function, which are well known special functions in mathematics and physics.

Solution:
Let $f(z)=\exp\left(\frac{t}{2}\left(\frac{z-1}{z}\right)\right)$. We begin by looking at the general Laurent series centered at $z=0$, since our function is undefined at this point it is the only singularity we are concerned with. Therefore we have

$$f(z)=\sum_{n=-\infty}^{\infty}C_n(z-0)^n=\sum_{n=-\infty}^{\infty}C_nz^n.$$

Where the C_n is given by

$$C_n=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{f(\xi)}{\xi^{n+1}}\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{\exp\left(\frac{t}{2}\left(\xi-\frac{1}{\xi}\right)\right)}{\xi^{n+1}}\mathrm{d}\xi.$$

This is really incomplete notationally since our C_n's depend on t so reverting back to the provided notation we have

$$J_n(t)=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{\exp\left(\frac{t}{2}\left(\xi-\frac{1}{\xi}\right)\right)}{\xi^{n+1}}\mathrm{d}\xi.$$

Additionally, I have yet to specify my contour C, but it needs to be within the annulus for which our Laurent series converges. Since, the original function $f(z)$ only has a singularity at $z=0$ the Laurent series really converges uniformly throughout the complex plane except at the origin. Therefore we make the convenient choice for our contour C to be a counterclockwise traversal of the unit circle. Using the parameterization

$\xi = \mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in [-\pi, \pi)$, we have

$$\begin{aligned}
J_n(t) &= \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{\exp\left(\frac{t}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\right)\right)}{\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{n+1}} \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp\left(\frac{t}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\right)\right)}{\mathrm{e}^{\mathrm{i}n\theta}} \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{t}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right) - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{t}{2}\left(\cancel{\cos\theta} + \mathrm{i}\sin\theta - \cancel{\cos\theta} + \mathrm{i}\sin\theta\right) - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{t}{2}\left(2\mathrm{i}\sin\theta\right) - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(t\mathrm{i}\sin\theta - \mathrm{i}n\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta.
\end{aligned}$$

Therefore

$$J_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta,$$

as desired. Furthermore,

$$\begin{aligned}
J_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{0} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&\quad + \frac{1}{2\pi} \int_{0}^{\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&= -\frac{1}{2\pi} \int_{0}^{-\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta \\
&\quad + \frac{1}{2\pi} \int_{0}^{\pi} \cos\left(n\theta - t\sin\theta\right) - \mathrm{i}\sin\left(n\theta - t\sin\theta\right)\mathrm{d}\theta.
\end{aligned}$$

Now we need to do a substitution for $n\theta - t\sin\theta$ in each of these integrals. For the integral from 0 to $-\pi$ let $\theta = -\theta'$ and for the integral from 0 to π let $\theta = \theta'$. Continuing where we left off we then have

$$
\begin{aligned}
&= -\frac{1}{2\pi}\int_0^{\pi} \cos\big(-n\theta' - t\sin(-\theta')\big) - \mathrm{i}\sin\big(-n\theta' - t\sin(-\theta')\big)(-\mathrm{d}\theta') \\
&\quad + \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin\theta'\big) - \mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\Bigg(-\big(n\theta' - t\sin(\theta')\big)\Bigg) - \mathrm{i}\sin\Bigg(-\big(n\theta' - t\sin(\theta')\big)\Bigg)\mathrm{d}\theta' \\
&\quad + \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin\theta'\big) - \mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big) + \mathrm{i}\sin\big(n\theta' - t\sin(\theta')\big)\mathrm{d}\theta' + \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin\theta'\big) - \mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big) + \cancel{\mathrm{i}\sin\big(n\theta' - t\sin(\theta')\big)} + \cos\big(n\theta' - t\sin\theta'\big) - \cancel{\mathrm{i}\sin\big(n\theta' - t\sin\theta'\big)}\mathrm{d}\theta' \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big) + \cos\big(n\theta' - t\sin\theta'\big)\mathrm{d}\theta' \\
&= \frac{2}{2\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big)\mathrm{d}\theta' \\
&= \frac{1}{\pi}\int_0^{\pi} \cos\big(n\theta' - t\sin(\theta')\big)\mathrm{d}\theta'.
\end{aligned}
$$

Though we finished in terms of another variable θ' this could easily be changed out with another substitution $\theta' = \theta$. And thus we see

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi} \cos\big(n\theta - t\sin(\theta)\big)\mathrm{d}\theta.$$

□

3: Bernoulli numbers: Consider the function

$$f(z) = \frac{z}{e^z - 1}.$$

(a) Show that $f(z)$ has a removable singularity at $z = 0$. Assume from now on that the definition of $f(z)$ has been extended to remove the singularity.

Solution:
If we can show the limit exists at the potential singularity then we can say it is removable. We can calculate the limit of $f(z)$ as $z \to 0$ explicitly:

$$\begin{aligned} \lim_{z\to 0} f(z) &= \lim_{z\to 0} \frac{z}{\mathrm{e}^z - 1} = \frac{0}{0} \qquad \text{applying L'Hôpitals rule} \\ &= \lim_{z\to 0} \frac{z}{\mathrm{e}^z - 1} = \lim_{z\to 0} \frac{1}{\mathrm{e}^z} = \frac{1}{1} = 1 \end{aligned}$$

Therefore, we could choose $f(0) = 1$ in order to extend $f(z)$ to be analytic in the region and therefore remove the singularity. Furthermore, we can also show this is a removable singularity by looking at the reciprocal of $f(z)$. If it does not have any zeros, then $f(z)$ will not have any actual singularities or it won't blow up anywhere. We use a Taylor series centered at $z = 0$ for e^z and see the following

$$\begin{aligned} \frac{1}{f(z)} = \frac{\mathrm{e}^z - 1}{z} = \frac{1}{z}(\mathrm{e}^z - 1) &= \frac{1}{z}\left(\sum_{j=0}^{\infty} \frac{z^j}{j!} - 1\right) \\ &= \frac{1}{z}\left(\sum_{j=1}^{\infty} \frac{z^j}{j!}\right) \\ &= \sum_{j=1}^{\infty} \frac{z^{j-1}}{j!} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{(j+1)!} \end{aligned}$$

which has no zeros. **TODO: Site the fact that this is a Laurent series with all the negative indices coeffs set to 0, therefore it is removable.** Therefore the original function $f(z)$ has nowhere that the denominator will blow up. Finally, we can conclude from these two pieces of evidence that this singularity is removable. We will assume from now on that $f(z)$ has been extended to remove this singularity. □

(b) Suppose you were to find a Taylor series for $f(z)$, centered at $z = 0$. What would be its radius of convergence?

Solution:
In the part (a) we determined there is no singularity for $f(z)$ therefore, the radius of convergence is infinite. □

(c) Find the Taylor series in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The numbers B_n are known as the Bernoulli numbers.

Solution:

$$\begin{aligned}
f(z) = \frac{z}{\mathrm{e}^z - 1} &= \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \\
z &= (\mathrm{e}^z - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \\
z &= \left(\sum_{m=0}^{\infty} \frac{z^m}{m!} - 1 \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \\
z &= \left(\sum_{m=1}^{\infty} \frac{z^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \\
z &= \left(\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n . \right)
\end{aligned}$$

Using the Cauchy Product formula

$$\left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{j=0}^{\infty} b_j \right) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} a_\ell b_{k-\ell}$$

we can continue from where we left off and get

$$\begin{aligned}
z &= \left(\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{z^{k-\ell+1}}{(k-\ell+1)!} \frac{B_\ell z^\ell}{\ell!} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{(k-\ell+1)!} \frac{B_\ell}{\ell!} z^{k-\ell+1} z^\ell \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{(k-\ell+1)!} \frac{B_\ell}{\ell!} z^{k+1} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{(k+1)!}{(k-\ell+1)!} \frac{B_\ell}{\ell!} \frac{z^{k+1}}{(k+1)!} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{(k+1)!}{(k+1-\ell)!\ell!} B_\ell \frac{z^{k+1}}{(k+1)!} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell \frac{z^{k+1}}{(k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell .
\end{aligned}$$

Now that we have

$$z = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell$$

we can see

$$z = \frac{z}{1!}\binom{1}{0} B_0 + \sum_{k=1}^{\infty} \frac{z^{k+1}}{(k+1)!} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell$$

$$z = zB_0 + \sum_{k=1}^{\infty} \frac{z^{k+1}}{(k+1)!} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell.$$

Therefore we need the following to hold

$$B_0 = 1$$

$$\sum_{k=1}^{\infty} \frac{z^{k+1}}{(k+1)!} \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell = 0.$$

TODO: Now how do I know I have arrived at a formula for the B_n...

(d) Find a recursion formula for the Bernoulli numbers, and use it to find $B_0, \ldots, B_{12}$.
Solution:
put things in terms of taylor series and move them over to the left side of the equation

(e) Show that $B_{2n+1} = 0$ for $n \geq 1$.

(f) Use your result to find a Taylor series for $z \coth z$, in terms of the Bernoulli numbers. Where is this series valid? Using this result, find a Laurent series for $\cot z$. Where is this series valid?

4: Consider $g(z) = 1/f(z)$ where $f(z)$ is as in the previous problem.

(a) Using the formula for $g(z)$, use software that uses double precision floating point arithmetic to compute the errors $e_n := |g(2^{-n}) - g(0)|$ for $n = 1, 2, \ldots, 52$. Produce a plot of these errors.

(b) Derive an approximation $G(z)$ to $g(z)$, near $z = 0$, that does not suffer from the instability you notice. Plot the new errors $E_n := |G(2^{-n}) - g(0)|$ for $n = 1, 2, \ldots, 52$. Ensure that these errors are less than 10^{-10} for all n.

5: Analytic continuation:

(a) Consider

$$F(z) = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n.$$

Where is this function analytic?

(b) Use the above representation to induce a Taylor representation of $F(z)$ centered at $z = -1/2$. Call this representation $G(z)$. Your final result should be of the form

$$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2}\right)^m$$

Where is this series valid?

If you can answer this question without using that both $F(z)$ and $G(z)$ are representations of $1/(1-z)$, you will receive 2 bonus points.

Solution: expansion of the same function allows you justify things and compute the radius of convergence a certain way.

Use the ratio test for a tedious 2 bonus points.

6: This problem is from Whittaker and Watson's "A course of modern analysis": Shew[1] that

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \frac{1}{(1-z)^2}, & |z| < 1 \\ \frac{1}{z(1-z)^2}, & |z| > 1. \end{cases}$$

This might appear to contradict the idea of analytic continuation. Please comment.
Solution:
Do partial fractions on the left. Something telescopes.... something should be the negative of each other in order to telescope they will depend on z likely

[1] Aka "Show".

7: Suppose that f is a function satisfying

$$|f(x)| \leq M, \quad x \in \mathbb{R}.$$

Show that

$$\hat{f}(z) := \int_0^\infty \mathrm{e}^{\mathrm{i}zx} f(x)\mathrm{d}x,$$

is an analytic function of z for $\operatorname{Im} z > 0$. You may assume that f is continuous, but this is not a necessary assumption.

Solution:
Use a theorem, something about this being able to hold if the integral is finite, then take the limit as it becomes infinit

8: Use analytic continuation to show that

$$\sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}},$$

where $\sqrt{\cdot}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$.

Solution:

Consider that they are both analytic everywhere in the same domain (use the form of analytic continuation which depends on the accumulation point)

Choose a contour for which the functions. agree on (positive real axis is a good choice).

Then show that

$$\sqrt{z-1}\sqrt{z+1} = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + O(z^{-3}), \quad z \to \infty,$$

and find b_0, b_1, b_2.