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 AMATH 561

### PROBLEM SET 3

1. Give an example of a probability space  $(\Omega, \mathcal{F}, P)$ , a random variable  $X$  and a function  $f$  such that  $\sigma(f(X))$  is strictly smaller than  $\sigma(X)$  but  $\sigma(f(X)) \neq \{\emptyset, \Omega\}$ . Give a function  $g$  such that  $\sigma(g(X)) = \{\emptyset, \Omega\}$ . Hint: Look at finite sample spaces with a small number of elements.

*Solution:*

Let our probability space be two independent coin tosses, such that  $\Omega = \{HH, TT, HT, TH\}$ . Define a random variable  $X$  such that  $X(\omega)$  be the number of heads in the outcome  $\omega$  with  $\omega \in \Omega$ . Therefore

$$\begin{aligned} X(HH) &= 2, \\ X(TT) &= 0, \\ X(HT) &= 1, \text{ and} \\ X(TH) &= 1. \end{aligned}$$

Now  $\sigma(X)$  can be written as

$$\sigma(X) = \left\{ \{HH\}, \{TH, HT\}, \{TT\}, \{TT, HH\}, \{HH, HT, TH\}, \{TT, HT, TH\}, \Omega, \emptyset \right\}$$

#### Part one

Random variable  $X$  and  $f$  such that  $\sigma(f(X)) \subsetneq \sigma(X)$  and  $\sigma(f(X))$  is not the trivial  $\sigma$ -algebra.

Define  $f(x)$  as follows

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Let's look at the possible pre-images of  $f(X)$  with respect to a few cases of Borel sets. For convenience, I will define  $\hat{X} = f(X)$ . Now let's look at some cases for the pre-image

Case 1:  $0 \in B$  but  $1 \notin B$

$$\hat{X}^{-1}(B) = \left\{ \omega : \hat{X}(\omega) \in B \right\} = \left\{ \omega : X(\omega) \in (-\infty, 0] \right\} = \{TT\}$$

Case 2:  $0 \notin B$  but  $1 \in B$

$$\hat{X}^{-1}(B) = \left\{ \omega : \hat{X}(\omega) \in B \right\} = \left\{ \omega : X(\omega) \in (0, \infty) \right\} = \{TH, HT, HH\}$$

Case 3:  $0 \in B$  and  $1 \in B$

$$\hat{X}^{-1}(B) = \left\{ \omega : \hat{X}(\omega) \in B \right\} = \left\{ \omega : X(\omega) \in (-\infty, \infty) \right\} = \Omega$$

Case 4:  $0 \notin B$  and  $1 \notin B$

$$\hat{X}^{-1}(B) = \left\{ \omega : \hat{X}(\omega) \in B \right\} = \emptyset.$$

Therefore,

$$\sigma(f(X)) = \left\{ \{TT\}, \{TH, HT, HH\}, \Omega, \emptyset \right\} \neq \{\emptyset, \Omega\}$$

And thus we have  $\sigma(f(X)) \subsetneq \sigma(X)$ .

**Part two** Now also give a function  $g$  such that  $\sigma(g(X))$  is the trivial  $\sigma$ -algebra,  $\{\emptyset, \Omega\}$ .

Define  $g(x)$  to be a constant  $c \in \mathbb{R}$  such that  $g(x) = c$  for all  $x \in \mathbb{R}$ . Once again, for convenience we define  $\tilde{X} = g(X)$ . Let's go through a few cases of what the pre-image may be for any Borel set

*Case 1:*  $c \in B$

$$\tilde{X}^{-1}(B) = \left\{ \omega : \tilde{X}(\omega) \in B \right\} = \left\{ \omega : X(\omega) \in (-\infty, \infty) \right\} = \Omega$$

*Case 2:*  $c \notin B$

$$\tilde{X}^{-1}(B) = \left\{ \omega : \tilde{X}(\omega) \in B \right\} = \emptyset.$$

Therefore,

$$\sigma(g(X)) = \{\Omega, \emptyset\}.$$

□

**2.** Give an example of events  $A$ ,  $B$ , and  $C$ , each of probability strictly between 0 and 1, such that  $P(A \cap B) = P(A)P(B)$ ,  $P(A \cap C) = P(A)P(C)$ , and  $P(A \cap B \cap C) = P(A)P(B)P(C)$  but  $P(B \cap C) \neq P(B)P(C)$ . Are  $A$ ,  $B$  and  $C$  independent? Hint: You can let  $\Omega$  be a set of eight equally likely points.

*Solution:*

Let  $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$ . Define events  $A$ ,  $B$ , and  $C$  as follows

$$A = \{1, 2, 3, 4\}$$

$$B = \{1, 2, 5, 7\}$$

$$C = \{1, 3, 6, 8\}.$$

Then we have

$$P(A \cap B) = P(\{1, 2\}) = \frac{1}{4}$$

and

$$P(A)P(B) = P(\{1, 2, 3, 4\})P(\{1, 2, 5, 7\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Additionally, we have

$$P(A \cap C) = P(\{1, 3\}) = \frac{1}{4}$$

and

$$P(A)P(C) = P(\{1, 2, 3, 4\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Finally, we have

$$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{8}$$

and

$$P(A)P(B)P(C) = P(\{1, 2, 3, 4\})P(\{1, 2, 5, 7\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Notice we also get

$$P(B \cap C) = P(\{1\}) = \frac{1}{8}$$

which is not equal to

$$P(B)P(C) = P(\{1, 2, 5, 7\})P(\{1, 3, 6, 8\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

In class we said two events  $E$  and  $E'$  are independent if  $P(E \cap E') = P(E)P(E')$ . However, since independence of a collection of events  $E_i$  for  $i \in \{1, 2, 3, \dots, n\}$  implies pairwise independence,  $P(E_i \cap E_j) = P(E_i)P(E_j)$  for all  $j \neq i$ , if the collection  $E_i$  fails to be pairwise independent then the collection must not be independent either. We have shown that  $A$  and  $B$  are independent and  $A$  and  $C$  are independent. But  $B$  and  $C$  are not independent, therefore we don't have pairwise independence between each pair of the three events hence  $A$ ,  $B$ , and  $C$  are not independent.  $\square$

**3.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space such that  $\Omega$  is countably infinite, and  $\mathcal{F} = 2^\Omega$ . Show that it is impossible for there to exist a countable collection of events  $A_1, A_2, \dots \in \mathcal{F}$  which are independent, such that  $P(A_i) = 1/2$  for each  $i$ . Hint: First show that for each  $\omega \in \Omega$  and each  $n \in \mathbb{N}$ , we have  $P(\omega) \leq 1/2^n$ . Then derive a contradiction.

*Solution:*

Assume by way of contradiction, there exists a countably infinite collection of independent events  $A_1, A_2, A_3, \dots \in \mathcal{F}$  such that  $P(A_i) = \frac{1}{2}$ . Independence of these events implies that

$$P\left(\bigcap_i^n A_i\right) = \prod_i^n P(A_i) = \prod_i^n \frac{1}{2} = \left(\frac{1}{2}\right)^n.$$

I note that our collection of events is countably infinite so we can take the limit of the previous expression as  $n \rightarrow \infty$ . Their independence also implies the independence of the events  $A_i^c$ , as discussed in class. Next I want to construct a collection of new sets call them  $B_{i,j}$  where  $\omega_j \in B_{i,j}$  (note we can index the  $\omega$ 's since  $\Omega$  is countably infinite). Let  $B_{i,j}$  be

$$B_{i,j} = \begin{cases} A_i, & \omega_j \in A_i \\ A_i^c, & \omega_j \notin A_i. \end{cases}$$

Therefore we can now write each  $\omega_j$  as

$$\bigcap_i^n B_{i,j} = \{\omega_j\}.$$

Then we have

$$P(\{\omega_j\}) = P\left(\bigcap_i^n B_{i,j}\right) = \prod_i^n P(B_{i,j}) = \prod_i^n \frac{1}{2} = \left(\frac{1}{2}\right)^n = 0.$$

Where  $P(\{\omega_j\}) = 0$  since  $n \rightarrow \infty$  because our collection of independent events is countably infinite. Notice, since  $\Omega = \bigcup_{j=1}^\infty \{\omega_j\}$ , then

$$P(\Omega) = P(\bigcup_{j=1}^\infty \{\omega_j\}) = \sum_{j=1}^\infty P(\{\omega_j\}) = \sum_{j=1}^\infty 0 = 0.$$

Which contradicts the fact that if  $(\Omega, \mathcal{F}, P)$  is a probability space then  $P(\Omega) = 1$ . Therefore, it is impossible for there to exist a countable collection of events  $A_1, A_2, \dots \in \mathcal{F}$  which are independent, such that  $P(A_i) = 1/2$  for each  $i$ .  $\square$

4. (a) Let  $X \geq 0$  and  $Y \geq 0$  be independent random variables with distribution functions  $F$  and  $G$ . Find the distribution function of  $XY$ .

*Solution:*

Let  $h(x, y) = \mathbb{1}_{\{xy \leq z\}}$  and  $\mathbb{E}[h(x, y)]$  be

$$\mathbb{E}[h(x, y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x, y) \mu(dx) \nu(dy)$$

where  $\mu$  and  $\nu$  are probability measures with distribution functions  $F$  and  $G$  respectively. We also have

$$\begin{aligned} \mathbb{E}[h(x, y)] &= \mathbb{E}[\mathbb{1}_{\{xy \leq z\}}] \\ &= 1 \cdot P(XY \leq z) + 0 \cdot P(XY > 0) \\ &= P(XY \leq z). \end{aligned}$$

Additionally,

$$\begin{aligned} \int_{\mathbb{R}} h(x, y) \mu(dx) &= \int_{\mathbb{R}} \mathbb{1}_{\{xy \leq z\}} \mu(dx) \\ &= \int_{\mathbb{R}} \mathbb{1}_{\{x \leq \frac{z}{y}\}} \mu(dx) \\ &= P\left(X \leq \frac{z}{y}\right) \\ &= F\left(\frac{z}{y}\right). \end{aligned}$$

Combining these we have

$$\begin{aligned} P(XY \leq z) &= \mathbb{E}[h(x, y)] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x, y) \mu(dx) \nu(dy) \\ &= \int_{\mathbb{R}} F\left(\frac{z}{y}\right) \nu(dy) \\ &= \int_{\mathbb{R}} F\left(\frac{z}{y}\right) dG(y) \end{aligned}$$

**TODO:** Account for when  $y$  is 0 somehow...

$$\int_{\mathbb{R}} F\left(\frac{z}{y}\right) dG(y) = \int_{-\infty}^0 F\left(\frac{z}{y}\right) dG(y) + \int_0^{\infty} F\left(\frac{z}{y}\right) dG(y).$$

**TODO:** Consider doing this again from the beginning but in terms of  $x$  and when  $y$  is 0.

(b) If  $X \geq 0$  and  $Y \geq 0$  are independent continuous random variables with density functions  $f$  and  $g$ , find the density function of  $XY$ .

*Solution:*

**TODO:** Redo this once you know you're starting with the right distribution function. We will follow a similar process as from the class lectures.

$$\int_{\mathbb{R}} F\left(\frac{z}{y}\right) dG(y) = \int_{\mathbb{R}} \int_{-\infty}^{\frac{z}{y}} F(u) du dG(y).$$

Now we need to do something with a change of variables along the lines of  $u = \frac{x}{y}$  then  $du = \frac{dx}{y}$ . We have

$$\begin{aligned} \int_{\mathbb{R}} F\left(\frac{z}{y}\right) dG(y) &= \int_{\mathbb{R}} \int_{-\infty}^{\frac{z}{y}} F(u) du dG(y) \\ &= \int_{\mathbb{R}} \int_{-\infty}^z \frac{1}{y} F\left(\frac{x}{y}\right) dx dG(y) \\ &= \int_{-\infty}^z \int_{\mathbb{R}} \frac{1}{y} F\left(\frac{x}{y}\right) dG(y) dx \\ &= P(XY \leq z). \end{aligned}$$

Therefore the density is

$$f(x) = \int_{\mathbb{R}} \frac{1}{y} F\left(\frac{x}{y}\right) dG(y).$$

(c) If  $X$  and  $Y$  are independent exponentially distributed random variables with parameter  $\lambda$ , find the density function of  $XY$ .

*Solution:*

Recall the density function of an exponentially distributed random variable with parameter  $\lambda$  is the same as a gamma distributed random variable with parameters  $(1, \lambda)$ . Therefore the density function for one of our random variables is

$$f(x) = \frac{\lambda^1}{\Gamma(1)} x^{1-1} e^{-\lambda x} = \lambda e^{-\lambda x}.$$

**TODO:** determine if you are going to use the formula from the previous problem or if there is an easier way to think about this...