Hunter Lybbert
Student ID: 2426454
11-22-24
AMATH 561

PROBLEM SET 7

Note: Exercises 1-4 are from Matt Lorig's notes (link on course website).

1. Exercise 4.1.
A six-sided die is rolled repeatedly. Which of the following are Markov chains? For those that are, find the one-step transition matrix.

(a) X_n is the largest number rolled up to the nth roll.

Solution:
It helps me to visualize this graphically. Let each state be a node with directed edges going from a given state to all reachable states from the current state. If each directed edge in Figure 1 is equally likely, then we

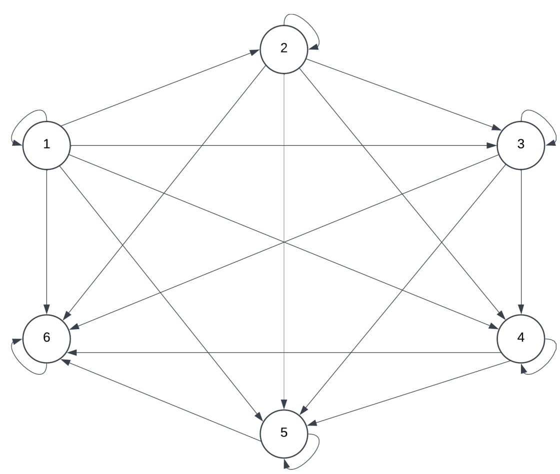

FIGURE 1. A graphical representation of the Markov Chain in problem 1 part (a).

can construct the following transition matrix

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the transition matrix it is easy to see this is a Markov chain. This is due to the fact that the rows sum to 1 and the probability of moving to a new state only depends on our current state.

□

(b) X_n is the number of sixes rolled in the first n rolls.

Solution:
The transition matrix for this scenario can be written as

$$P = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & \dots \\ 0 & 5/6 & 1/6 & 0 & \dots \\ 0 & 0 & 5/6 & 1/6 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Since we can write $X_n = X_{n-1} + \xi_n$ where ξ_n is 1 if the nth role is a 6 or 0 otherwise. Therefore ξ_n is Bernoulli distributed with probability of success $p = 1/6$. Let the current state be denoted as ℓ. The state space is $\mathbb{N} \cup \{0\}$. Furthermore we can say if $X_n = \ell$ we denote the following

$$P(X_{n+1} = \ell + 1 | X_n = \ell) = 1/6$$

if the next role is a 6 and

$$P(X_{n+1} = \ell | X_n = \ell) = 5/6$$

if the next role is not a 6. Graphically that can be represented as seen in Figure 2 Where the self returning edge has probability 5/6 and the edge

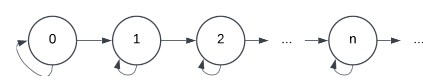

FIGURE 2. A graphical representation of the Markov Chain in problem 1 part (b).

advancing from one state to the next higher state has probability 1/6. Now we can conclude this is a Markov chain because the probability of moving to a particular state is only determined by the current state we are in. Additionally the rows of our matrix sum to 1 which satisfies our other criteria.

□

(c) At time n, X_n is the time since the last six was rolled.

Solution:
Each step is 5/6 chance of not rolling a 6 therefore increasing X_n and a 1/6 chance of rolling a 6 therefore returning to the beginning state. If I may say so myself, this is depicted clearly in the directed graph in Figure 3.

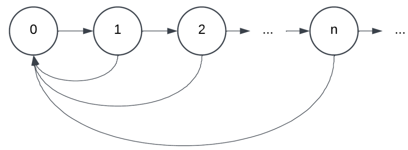

FIGURE 3. A graphical representation of the Markov Chain in problem 1 part (c).

The one step transition matrix can be written as

$$P = \begin{bmatrix} 1/6 & 5/6 & 0 & 0 & \dots \\ 1/6 & 0 & 5/6 & 0 & \dots \\ 1/6 & 0 & 0 & 5/6 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Once again, observe we have rows that sum to 1 and the probability of moving from one state to another only depends on which state we are currently in.

□

(d) At time n, X_n is the time until the next six is rolled.

Solution:
Suppose $X_1 = k$ meaning we are going to role a 6 in k roles. Then we know that $X_2 = k-1, ..., X_k = 1$. Thus the $k+1$ role is a 6 and we no longer have definitive information about time until the next 6 will be rolled. Let's consider $X_{k+1} = \ell$, $\ell \in \mathbb{N}$. Then there are going to be $\ell - 1$ non 6 roles followed by a single role of 6. The probability of $\ell - 1$ non 6 roles is $\left(\frac{5}{6}\right)^{\ell-1}$ then we would multiply by an additional 1/6. We can represent this uncertainty for the state of X_{k+1} as

$$P(X_{k+1} = \ell) = \left(\frac{5}{6}\right)^{\ell-1} \frac{1}{6}.$$

Notice, this is just a geometric distribution. Therefore, the one step transition matrix can be written as

$$P = \begin{bmatrix} 1/6 & (5/6)1/6 & (5/6)^2 1/6 & (5/6)^3 1/6 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Since the first row of the matrix is just a geometric distribution it sums to 1 and the others sum to 1 trivially. Additionally, the probability of moving from one state to another is fully determined by the current state, this is indeed a Markov Chain. □

2. Exercise 4.2.
Let $Y_n = X_{2n}$. Compute the transition matrix for Y when

(a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q.

Solution:
Let's define the one step transition matrix for the random variable X which is a random walk increasing by 1 or decreasing by 1 with probability p and q respectively. The matrix is

$$\begin{bmatrix} 0 & p & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & \dots \\ 0 & q & 0 & p & 0 & \dots \\ 0 & 0 & q & 0 & p & \dots \\ 0 & 0 & 0 & q & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

Looking at the graph in Figure 4 Let P be transition matrix for X_n, then

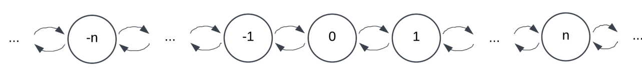

Figure 4. A graphical representation of the Markov Chain in problem 2 part (a).

transition matrix for Y_n should be P^2, also could use kolmogorov equation or something... **TODO:**

(b) X is a branching process where G is the generating function of the number of offspring from each individual.

Solution:
TODO: Office Hour notes

- G_{i+1} conditioning?
- ξ_i
- $\sum_i \xi_i \implies E(s^{\sum_i \xi_i})$
- ends up being G^i maybe
- (random) $G^i(s) = E(s^{X_{2n+1}} | X_{2n} = i)$

3. Exercise 4.3.
Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k, j) = 0$ for all $j \in S$). Suppose $j \to k$ for all $j \in S$. Show that all states other than k are transient.

Solution:
TODO:
This is supposedly a clean quick proof...
argue with words...? No needs to be written in mathematical probability terms ...
chapman? no
Consider a proof by contradiction... actually maybe not though

4. Exercise 4.4.
Suppose two distinct states i, j satisfy

$$P(\tau_j < \tau_i | X_0 = i) = P(\tau_i < \tau_j | X_0 = j)$$

where $\tau := \inf\{n \geq 1 : X_n = j\}$. Show that, if $X_0 = i$, the expected number of visits to j prior to re-visiting i is one.

Solution:
TODO:
Layer cake representation

$$E(X) = \int_0^\infty P(X > t)\mathrm{d}t$$

but this is a summation in our scenario

5. Stationary distribution of Ehrenfest chain. (a) Let X_n be the number of balls in the left urn at time n (total number of balls in both urns is r). At each time step, one of the r balls is picked at random and moved to the other urn.

(a) Let $G_n(s)$ be the generating function of X_n. Derive a formula for G_{n+1} as a function of G_n.

Solution:
TODO:

(b) Let $G(s) = \lim_{n\to\infty} G_n(s)$. Use the relation in part a) to derive an equation for G. Solve it and find G.

Solution:
TODO:

(c) Find the stationary distribution π of Ehrenfest chain. What is the connection between G and π?

Solution:
Stationary distr?
TODO: