Hunter Lybbert
Student ID: 2426454
10-07-24
AMATH 561

PROBLEM SET 2

1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.
Solution:
We need to show that Z is a random variable as it is defined. That is we need to show it is a function that maps from a sample space Ω to the real numbers. Starting from knowing X and Y are random variables that means we have:

$$X : \Omega \to \mathbb{R}$$

and

$$Y : \Omega \to \mathbb{R}.$$

Now rewriting Z a little more mathematically we have

$$Z(\omega) = \begin{cases} X(\omega), & \omega \in A, \\ Y(\omega), & \omega \in A^c \end{cases}$$

where $A \in \mathcal{F}$. Note, since A is an event in $\mathcal{F}$, every $\omega \in A$ must also be in Ω since $\mathcal{F}$ is a collection of events composed of outcomes from Ω. In other words, $\mathcal{F}$ is made up of subsets of Ω which means $A \subseteq \Omega$ and thus $A^c \subseteq \Omega$ as well. By definition of the compliment $A \cap A^c = \emptyset$. Therefore A and A^c are a partition on Ω. Since Z is defined on $\omega \in A$ or $\omega \in A^C$ then Z is defined on all of Ω. Now we have shown that the domain of Z is Ω. Additionally, since X and Y each map from Ω to $\mathbb{R}$, Z must also map to $\mathbb{R}$ since it's output is determined by the output of X and Y. Therefore Z is function such that

$$Z : \Omega \to \mathbb{R}$$

and thus Z is a random variable. $\square$

2. Suppose X is a continuous random variable with distribution function F_X. Let g be a strictly increasing continuous function. Define $Y = g(X)$.

a) What is F_Y, the distribution function of Y?
Solution:
We know that there is some probability space that the random variable X is defined on, let that be $(\Omega, \mathcal{F}, P)$. Therefore $X : \Omega \to \mathbb{R}$ and since g is a strictly increasing continuous function $g : \mathbb{R} \to L$ where L is the output space of g, L could be $\mathbb{R}$ for example, then $g(X) : \Omega \to \mathbb{R}$ (we take $L = \mathbb{R}$ for now as the most likely assumption). Note that since $Y = g(X)$ then $Y : \Omega \to \mathbb{R}$ is also true. In order to construct F_Y we need to determine the relationship they have.

$$F_X = \int_{-\infty}^{x} f_X(x) \mathrm{d}x$$

Therefore we get

$$\begin{aligned} g(F_X) &= \int_{-\infty}^{x} g(f_X(x))\mathrm{d}x \\ F_Y &= \int_{-\infty}^{x} F_y(x)\mathrm{d}x \end{aligned}$$

b) What is f_Y, the density function of Y?
Solution:
And thus $f_Y = g(f_X(x))$.

3. Suppose X is a continuous random variable with distribution function F_X. Find F_Y where Y is given by

a) X^2 *Solution:*
That is to say $Y = X^2$ This is going to be easy once I know how to change from X to Y.
b) $\sqrt{|X|}$ *Solution:*
That is to say $Y = \sqrt{|X|}$
c) $\sin X$ *Solution:*
That is to say $Y = \sin X$
d) $F_X(X)$ *Solution:*
That is to say $Y = F_X(X)$

4. Let $X : [0,1] \to \mathbf{R}$ be a function that maps every rational number in the interval $[0,1]$ to 0, and every irrational number to 1. We assume that the probability space where X is defined is $([0,1], \mathcal{B}[0,1], P)$, where $\mathcal{B}[0,1]$ is the Borel σ-algebra on [0,1], and P is the Lebesgue measure.

(a) Is the set of rational numbers in [0,1] a Borel set? Show using definition of the Borel σ-algebra on $[0,1]$.

(b) Is X a random variable (and why)? If it is, what are its distribution function and expectation? Does X have a density function? Is X discrete?