

Hunter Lybbert  
 Student ID: 2426454  
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 AMATH 567

## HOMEWORK 10

Collaborators\*: Laura Thomas, Cooper Simpson, Nate Ward, Sophia Kamien

\*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

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- 1:** I sketched the following in class. Complete the argument. Show that for an integer  $j \in (-N, N)$  and  $h > 0$ ,

$$\lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} \frac{e^{2ijz}}{\tan(Nz)} dz = \begin{cases} -i\pi & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Solution:*

Consider the following equivalent integral

$$\begin{aligned} \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} \frac{e^{2ijz}}{\tan(Nz)} dz &= \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} e^{2ijz} \left( \left( \frac{1}{\tan(Nz)} - \frac{1}{i} \right) + \frac{1}{i} \right) dz \\ &= \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} \left( e^{2ijz} \left( \frac{1}{\tan(Nz)} - \frac{1}{i} \right) + \frac{e^{2ijz}}{i} \right) dz \\ &= \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} e^{2ijz} \left( \frac{1}{\tan(Nz)} - \frac{1}{i} \right) dz + \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} \frac{e^{2ijz}}{i} dz \\ &= \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} e^{2ijz} \left( \frac{1}{\tan(Nz)} + i \right) dz - \lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} i e^{2ijz} dz \end{aligned}$$

We want to be able to show the first integral on the left goes to 0, then we can show the second integral produces the expected results we are looking for from the problem statement.

$$\lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} e^{2ijz} \left( \frac{1}{\tan(Nz)} + i \right) dz$$

**TODO: Determine how this goes to 0. It might be because  $\tan(Nz)$  behaves approximately like  $1/i + \mathcal{O}(e^{-2Nh})$  because then we would end up with the integral of  $e^{2ijz} / e^{2Nh}$  which would go to 0 as  $h \rightarrow \infty$ . This is kinda tricky.**

Now let's look into the second one more closely. Specifically when  $j \neq 0$  we have

$$\begin{aligned}
-\lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} i e^{2ijz} dz &= -\lim_{h \rightarrow \infty} \left( \frac{1}{2j} e^{2ijz} \Big|_{ih}^{ih+\pi} \right) \\
&= -\lim_{h \rightarrow \infty} \left( \frac{1}{2j} e^{2ij(ih+\pi)} - \frac{1}{2j} e^{2ij(ih)} \right) \\
&= -\lim_{h \rightarrow \infty} \left( \frac{1}{2j} e^{-2jh} e^{2\pi ij} - \frac{1}{2j} e^{-2jh} \right) \\
&= -\lim_{h \rightarrow \infty} \left( \frac{1}{2j e^{2jh}} e^{2\pi ij} - \frac{1}{2j e^{2jh}} \right) \\
&= -\lim_{h \rightarrow \infty} \left( \frac{e^{2\pi ij} - 1}{2j e^{2jh}} \right) \\
&= -\lim_{h \rightarrow \infty} \left( \frac{1 - 1}{2j e^{2jh}} \right) \\
&= -\lim_{h \rightarrow \infty} \left( \frac{0}{2j e^{2jh}} \right) = 0
\end{aligned}$$

Now if  $j = 0$  then we have

$$\begin{aligned}
-\lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} i e^0 dz &= -\lim_{h \rightarrow \infty} \int_{ih}^{ih+\pi} i dz \\
&= -\lim_{h \rightarrow \infty} (iz) \Big|_{ih}^{ih+\pi} \\
&= -\lim_{h \rightarrow \infty} (\cancel{ih} + i\pi \cancel{-h}) \\
&= -\lim_{h \rightarrow \infty} i\pi \\
&= -i\pi
\end{aligned}$$

**TODO: Is this what other people did from office hours?** Following the sketch provided in class let's look at an import representation of  $\tan(Nz)$

$$\begin{aligned}
 \tan(Nz) &= \frac{\sin(Nz)}{\cos(Nz)} \\
 &= \frac{e^{iNz} + e^{-iNz}}{2i} \left( \frac{e^{iNz} - e^{-iNz}}{2} \right)^{-1} \\
 &= \frac{e^{iNz} + e^{-iNz}}{2i} \left( \frac{2}{e^{iNz} - e^{-iNz}} \right) \\
 &= \frac{1}{i} \left( \frac{e^{iNz} + e^{-iNz}}{e^{iNz} - e^{-iNz}} \right) \\
 &= \frac{1}{i} \left( \frac{e^{iNz}}{e^{iNz}} \left( \frac{1 + e^{-2iNz}}{1 - e^{-2iNz}} \right) \right) \\
 &= \frac{1}{i} \left( \frac{1 + e^{-2iNz}}{1 - e^{-2iNz}} \right) \\
 &= \frac{1}{i} \left( \frac{1 + (\cos(Nz) - i \sin(Nz))^2}{1 - (\cos(Nz) - i \sin(Nz))^2} \right) \\
 &= \frac{1}{i} \left( \frac{1 + \cos^2(Nz) - 2i \cos(Nz) \sin(Nz) - \sin^2(Nz)}{1 - \cos^2(Nz) + 2i \cos(Nz) \sin(Nz) + \sin^2(Nz)} \right) \\
 &\dots \\
 &= i + \mathcal{O}(e^{-2Nh})
 \end{aligned}$$

2: From A&F: 4.2.1 (b)

*Solution:*

See solution to problem 2 from homework set 9.

□

3: From A&F: 4.2.2 (a, h)

Evaluate the following real integrals by residue integration:

(a)

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx, \quad a^2 > 0$$

*Solution:*

We can also look at just the imaginary part of another version of this integral

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx &= \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx \\ &= \operatorname{Im} \int_{-\infty}^{\infty} \frac{x (\cos x + i \sin x)}{x^2 + a^2} dx \\ &= \operatorname{Im} \left[ \int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + a^2} dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx \right] \\ &= \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx. \end{aligned}$$

Therefore, we consider the integral

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx$$

and then take the imaginary part at the end. In order to evaluate this integral we look at the integral over the contour  $C$  which is a counterclockwise semicircle in the upper half plane with radius  $R$ . Then applying the Residue Theorem and taking the limit as  $R \rightarrow \infty$ , we have the following

$$\begin{aligned} \oint_C \frac{z e^{iz}}{z^2 + a^2} dz &= \int_{-R}^R \frac{x e^{ix}}{x^2 + a^2} dx + \oint_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz \\ 2\pi i \sum_{w \in S} \left( \frac{z e^{iz}}{z^2 + a^2} \right) &= \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx + \oint_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz \xrightarrow{0} \end{aligned}$$

where  $S$  is the set of singularities of our function in the upper half plane and I claim the  $C_R$  contour integral goes to 0 by Jordan's Lemma. I will verify this second claim next.

To justify using Jordan's Lemma I want to note we are using  $k = 1$  in assumptions from the lemma of concern. Moreover, I will be assuming  $a > 0$  for now. Additionally, we need to show  $f(z) \rightarrow 0$  **uniformly** as  $R \rightarrow \infty$  in  $C_R$ , that is if  $|f(z)| \leq K_R$ , where  $K_R$  only depends on  $R$  and not  $\arg z$  and  $K_R \rightarrow 0$  as  $R \rightarrow \infty$ . Let's first find this

bound  $K_R$

$$\begin{aligned}
 |f(z)| &= \left| \frac{z}{z^2 + a^2} \right| \\
 &= \left| \frac{R e^{i\theta}}{R^2 e^{2i\theta} + a^2} \right| \\
 &= \frac{R}{|R^2 e^{2i\theta} + a^2|} \\
 &\leq \frac{R}{||R^2 e^{2i\theta}| - |-a^2||} \\
 &\leq \frac{R}{R^2 - a^2}.
 \end{aligned}$$

In the final two steps with inequalities we first apply the inverse triangle inequality, followed by recognizing the following. Since  $R$  is becoming arbitrarily large then for a given  $a$  eventually  $R$  will be larger such that  $R^2 > a^2$  and therefore the expression  $R^2 - a^2 > 0$  thus we can drop the absolute value in the end. Now, let  $K_R = R/(R^2 - a^2)$ . Clearly the denominator will win out as we take  $R \rightarrow \infty$  since it has a squared  $R$  in it, while the numerator only has a linear  $R$ . Therefore,  $K_R \rightarrow 0$  as  $R \rightarrow \infty$  and thus  $f(z) \rightarrow 0$  **uniformly** as  $R \rightarrow \infty$  in  $C_R$ . Thus we have verified that

$$\oint_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz \rightarrow 0$$

by Jordan's Lemma.

Therefore we no longer need to be concerned with the integral over  $C_R$ . Notice the denominator of our function factors to  $(z - ia)(z + ia)$  therefore the set of singularities in the upper half plane is  $S = \{ia\}$ . Hence we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx &= 2\pi i \sum_{w \in S} \text{Res} \left( \frac{z e^{iz}}{z^2 + a^2} \right) \\
 &= 2\pi i \text{Res}_{z=ia} \left( \frac{z e^{iz}}{(z - ia)(z + ia)} \right) \\
 &= 2\pi i \left( \frac{ia e^{i^2 a}}{ia + ia} \right) \\
 &= 2\pi i \left( \frac{ia e^{-a}}{2ia} \right) \\
 &= \pi i e^{-a}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx &= \text{Im} \left[ \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx \right] \\
 &= \text{Im} [\pi i e^{-a}] \\
 &= \pi e^{-a}
 \end{aligned}$$

**TODO: consider doing a case where  $a < 0$ .**

(h)

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3 \sin \theta)^2}$$

*Solution:*

Let's begin by reverse parameterizing this into a contour integral around the unit circle. Notice, using the normal parameterization but going the other way we have,  $z = e^{i\theta}$  and

$$dz = i e^{i\theta} d\theta \implies \frac{1}{iz} dz = d\theta.$$

Additionally, notice,

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}.$$

Hence, our reverse parameterization can get us here (with a little simplification)

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(5 - 3 \sin \theta)^2} &= \oint_{\partial B_1(0)} \left( 5 - 3 \left( \frac{z - \frac{1}{z}}{2i} \right) \right)^{-2} \frac{dz}{iz} \\ &= \oint_{\partial B_1(0)} \left( 5 - \frac{3z}{2i} + \frac{3}{2iz} \right)^{-2} \frac{dz}{iz} \\ &= \oint_{\partial B_1(0)} \left( \frac{10iz - 3z^2 + 3}{2iz} \right)^{-2} \frac{dz}{iz} \\ &= \oint_{\partial B_1(0)} \frac{(2iz)^2}{(10iz - 3z^2 + 3)^2} \frac{dz}{iz} \\ &= \oint_{\partial B_1(0)} \frac{4iz}{(10iz - 3z^2 + 3)^2} dz. \end{aligned}$$

Now we need to factor the quadratic in the denominator to determine the singularities of the integrand

$$\begin{aligned} 10iz - 3z^2 + 3 &= (i - 3z)(z - 3i) \\ &= -3(z - i/3)(z - 3i). \end{aligned}$$

Then we have

$$\begin{aligned} \oint_{\partial B_1(0)} \frac{4iz}{(10iz - 3z^2 + 3)^2} dz &= \oint_{\partial B_1(0)} \frac{4iz}{(-3(z - i/3)(z - 3i))^2} dz \\ &= \oint_{\partial B_1(0)} \frac{4iz}{9(z - i/3)^2(z - 3i)^2} dz \\ &= 2\pi i \operatorname{Res}_{z=i/3} \left( \frac{4iz}{9(z - i/3)^2(z - 3i)^2} \right). \end{aligned}$$

Let's compute the residue at the simple pole with this formula

$$\begin{aligned}
2\pi i \operatorname{Res}_{z=ia} \left( \frac{4iz}{9(z-i/3)^2(z-3i)^2} \right) &= 2\pi i \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left( \frac{4iz}{9(z-i/3)^2(z-3i)^2} \right) \Big|_{i/3} \\
&= 2\pi i \frac{d}{dz} \left( \frac{4iz}{9(z-3i)^2} \right) \Big|_{i/3} \\
&= 2\pi i \left( \frac{4i9(z-3i)^2 - 4iz(9 \cdot 2(z-3i))}{9^2(z-3i)^4} \right) \Big|_{i/3} \\
&= 2\pi i \left( \frac{4i(z-3i) - 8iz}{9(z-3i)^3} \right) \Big|_{i/3} \\
&= 2\pi i \left( \frac{4iz + 12 - 8iz}{9(z-3i)^3} \right) \Big|_{i/3} \\
&= 2\pi i \left( \frac{-4iz + 12}{9(z-3i)^3} \right) \Big|_{i/3} \\
&= 2\pi i \left( \frac{-4i\frac{i}{3} + 12}{9(\frac{i}{3} - 3i)^3} \right) \\
&= 2\pi i \left( \frac{\frac{4}{3} + 12}{9(\frac{-8i}{3})^3} \right) \\
&= 2\pi i \left( \frac{\frac{40}{3}}{9\frac{(-1)^3 8 \cdot 64 i^3}{9 \cdot 3}} \right) \\
&= 2\pi i \left( \frac{40}{-8 \cdot 64(-1)i} \right) \\
&= \pi \left( \frac{40}{8 \cdot 32} \right) \\
&= \frac{5\pi}{32}.
\end{aligned}$$

□

4: (a) Show that

$$\operatorname{Res}_{z=k} f(z) \cot(\pi z) = \frac{1}{\pi} f(k),$$

provided  $f(z)$  is analytic at  $z = k$ ,  $k \in \mathbb{Z}$ .

*Solution:*

Recall that if  $z = z_k$  is a pole of order  $N$  of  $f(z) \cot \pi z$  then

$$\operatorname{Res}_{z=z_k} (f(z) \cot \pi z) = \frac{1}{(N-1)!} \lim_{z \rightarrow z_k} \frac{d^{N-1}}{dz^{N-1}} [(z - z_k)^N f(z) \cot \pi z].$$

Notice, that the only places where our function  $f(z) \cot \pi z$  only blows up in the locations where  $\tan(\pi z) = 0$ , therefore,  $\sin \pi z = 0$ . This holds at all the integers  $z = k$ . Each  $k \in \mathbb{Z}$  will therefore be a simple pole of  $f(z) \cot \pi z$ . Then let's calculate

$$\begin{aligned} \operatorname{Res}_{z=k} (f(z) \cot \pi z) &= \frac{1}{(1-1)!} \lim_{z \rightarrow k} \frac{d^{1-1}}{dz^{1-1}} [(z - k)^1 f(z) \cot \pi z] \\ &= \lim_{z \rightarrow k} [(z - k) f(z) \cot \pi z] \\ &= \lim_{z \rightarrow k} \left[ \frac{(z - k) f(z)}{\tan \pi z} \right] \\ &= \frac{(k - k) f(k)}{\tan \pi k} \\ &= \frac{0}{0}. \end{aligned}$$

Using L'Hôpital's, we have

$$\begin{aligned} \lim_{z \rightarrow k} \left[ \frac{(z - k) f(z)}{\tan \pi z} \right] &= \lim_{z \rightarrow k} \left[ \frac{\frac{d}{dz} (z - k) f(z)}{\frac{d}{dz} \tan \pi z} \right] \\ &= \lim_{z \rightarrow k} \left[ \frac{f(z) + (z - k) f'(z)}{\pi \sec^2(\pi z)} \right] \\ &= \frac{f(k) + (k - k) f'(k)}{\pi \sec^2(\pi k)} \\ &= \frac{1}{\pi} (f(k) + (k - k) f'(k)) \cos^2(\pi k) \\ &= \frac{1}{\pi} f(k) \cos^2(\pi k) \\ &= \frac{1}{\pi} f(k). \end{aligned}$$

□



- (b) Let  $\Gamma_N$  be a square contour, with corners at  $(N + 1/2)(\pm 1 \pm i)$ ,  $N \in \mathbb{Z}^+$ . Show that

$$|\cot(\pi z)| \leq 2,$$

for  $z$  on  $\Gamma_N$ .

*Solution:*

Consider the following representation of  $\cot \pi z$  with some manipulation

$$\begin{aligned}
 |\cot \pi z| &= \left| \frac{1}{\tan \pi z} \right| = \left| \frac{\cos \pi z}{\sin \pi z} \right| \\
 &= |\cos \pi z (\sin \pi z)^{-1}| \\
 &= \left| \frac{e^{i\pi z} + e^{-i\pi z}}{2} \left( \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \right)^{-1} \right| \\
 &= \left| \frac{e^{i\pi z} + e^{-i\pi z}}{2} \left( \frac{2i}{e^{i\pi z} - e^{-i\pi z}} \right) \right| \\
 &= \left| \frac{i(e^{i\pi z} + e^{-i\pi z})}{e^{i\pi z} - e^{-i\pi z}} \right| \\
 &\leq |i| \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \\
 &= \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \\
 &= \left| \frac{e^{i\pi z} 1 + e^{-2\pi iz}}{e^{i\pi z} 1 - e^{-2\pi iz}} \right| \\
 (1) \quad &= \left| \frac{1 + e^{-2\pi iz}}{1 - e^{-2\pi iz}} \right|.
 \end{aligned}$$

We will analyze this to show  $|\cot \pi z| \leq 2$  along the contour  $\Gamma_N$ . Let's first do the section of the contour along the top and bottom of the box. The parameterizations for the top and bottom of the box are

$$z(t) = -2t(N + 1/2) + (N + 1/2) + i(N + 1/2)$$

$$z(t) = 2t(N + 1/2) - (N + 1/2) - i(N + 1/2)$$

respectively, where  $t \in [0, 1]$ . Plugging in the parameterization for the top of the box to expression (1) we have

$$\begin{aligned}
 \left| \frac{1 + e^{-2\pi i(-2t(N+1/2)+(N+1/2)+i(N+1/2))}}{1 - e^{-2\pi i(-2t(N+1/2)+(N+1/2)+i(N+1/2))}} \right| &= \left| \frac{1 + e^{4\pi it(N+1/2)-2\pi i(N+1/2)-2\pi i^2(N+1/2)}}{1 - e^{4\pi it(N+1/2)-2\pi i(N+1/2)-2\pi i^2(N+1/2)}} \right| \\
 &= \left| \frac{1 + e^{4\pi it(N+1/2)} e^{-2\pi i(N+1/2)} e^{2\pi(N+1/2)}}{1 - e^{4\pi it(N+1/2)} e^{-2\pi i(N+1/2)} e^{2\pi(N+1/2)}} \right| \\
 &\leq \frac{|1| + |e^{4\pi it(N+1/2)} e^{-2\pi i(N+1/2)} e^{2\pi(N+1/2)}|}{|1 - e^{4\pi it(N+1/2)} e^{-2\pi i(N+1/2)} e^{2\pi(N+1/2)}|} \\
 &\leq \frac{1 + e^{2\pi(N+1/2)}}{|1 - e^{2\pi(N+1/2)}|}.
 \end{aligned}$$

We finished these last few lines using the triangle inequality and noting that the exponential terms on the right two of them have imaginary parts in the exponent therefore the exponential without any imaginary parts in the exponent is the radius or modulus of the overall imaginary number. Furthermore note,

$$e^{2\pi(N+1/2)} > 1$$

implies

$$|1 - e^{2\pi(N+1/2)}| = e^{2\pi(N+1/2)} - 1.$$

Therefore we have,

$$\begin{aligned} \frac{1 + e^{2\pi(N+1/2)}}{|1 - e^{2\pi(N+1/2)}|} &= \frac{1 + e^{2\pi(N+1/2)}}{e^{2\pi(N+1/2)} - 1} \\ &= \frac{1 + e^{2\pi(N+1/2)} e^{-\pi(N+1/2)}}{e^{2\pi(N+1/2)} - 1 e^{-\pi(N+1/2)}} \\ &= \frac{e^{-\pi(N+1/2)} + e^{\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}} \\ &= \coth(\pi(N + 1/2)) \leq 2 \end{aligned}$$

Since  $\coth(\pi/2) \approx 1.09$  and  $\coth$  is a decreasing function for any value of  $N$  we can say

$$|\cot(\pi z)| \leq \coth(\pi(N + 1/2)) \leq 2.$$

Notice, we can plugin the other parameterization for the bottom part of the contour and we would arrive at the same conclusion but when we multiplied by 1 in the form  $e^{-\pi(N+1/2)} / e^{-\pi(N+1/2)}$  we would instead need to multiply by  $e^{\pi(N+1/2)} / e^{\pi(N+1/2)}$  to arrive at  $\coth(\pi(N + 1/2))$  again.

Let's now do the section of the contour along the two sides of the box. The parameterizations for the top and bottom of the box are

$$z(t) = -(N + 1/2) - 2ti(N + 1/2) + i(N + 1/2)$$

$$z(t) = N + 1/2 + 2ti(N + 1/2) - i(N + 1/2)$$

respectively, where  $t \in [0, 1]$ . Plugging in this version of the parameterization for the left side of the box to expression (1) we have

$$\begin{aligned} \left| \frac{1 + e^{-2\pi i(-(N+1/2)-2ti(N+1/2)+i(N+1/2))}}{1 - e^{-2\pi i(-(N+1/2)-2ti(N+1/2)+i(N+1/2))}} \right| &= \left| \frac{1 + e^{2\pi i(N+1/2)+4\pi i^2 t(N+1/2)-2\pi i^2(N+1/2)}}{1 - e^{2\pi i(N+1/2)+4\pi i^2 t(N+1/2)-2\pi i^2(N+1/2)}} \right| \\ &= \left| \frac{1 + e^{2\pi i(N+1/2)} e^{-4\pi t(N+1/2)} e^{2\pi(N+1/2)}}{1 - e^{2\pi i(N+1/2)} e^{-4\pi t(N+1/2)} e^{2\pi(N+1/2)}} \right| \\ &\leq \frac{|1| + |e^{2\pi i(N+1/2)} e^{-4\pi t(N+1/2)} e^{2\pi(N+1/2)}|}{|1| - |e^{2\pi i(N+1/2)} e^{-4\pi t(N+1/2)} e^{2\pi(N+1/2)}|} \\ &\leq \frac{1 + e^{-4\pi t(N+1/2)} e^{2\pi(N+1/2)}}{|1 - e^{-4\pi t(N+1/2)} e^{2\pi(N+1/2)}|} \end{aligned}$$

**TODO: Do something similar to this for the sides of the contour where the imaginary part is variable but the real part is constant.**

$$f(x) = mx + b$$

- (c) Suppose  $f(z) = p(z)/q(z)$ , where  $p(z)$  and  $q(z)$  are polynomials, so that the degree of  $q(z)$  is at least two more than the degree of  $p(z)$ . Show that

$$\lim_{N \rightarrow \infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| = 0$$

*Solution:*

Not sure what the argument is going to be here...but we can use this in part (d).

**TODO:**

$$\begin{aligned} \lim_{N \rightarrow \infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| &\leq \lim_{N \rightarrow \infty} \oint_{\Gamma_N} \left| \frac{p(z)}{q(z)} \right| |\cot(\pi z)| |dz| \\ &\leq \lim_{N \rightarrow \infty} 2 \oint_{\Gamma_N} \left| \frac{p(z)}{q(z)} \right| |dz|. \end{aligned}$$

The largest modulus of any point on the contour is  $|w| \leq (N + 1/2)\sqrt{2}$  for all  $w \in \Gamma_N$  polynomial. Additionally we can determine the smallest modulus is  $|w| \geq N + 1/2$ . Therefore we have the bound

$$N + 1/2 \leq |w| \leq (N + 1/2)\sqrt{2} \quad \text{for all } w \in \Gamma_N.$$

Hence, plugging in the largest term for the numerator and the smallest possible term in the denominator we have

$$\begin{aligned} \lim_{N \rightarrow \infty} 2 \oint_{\Gamma_N} \left| \frac{p(z)}{q(z)} \right| |dz| &\leq \lim_{N \rightarrow \infty} 2 \oint_{\Gamma_N} \left| \frac{p((N + 1/2)\sqrt{2})}{q(N + 1/2)} \right| |dz| \\ &\leq \lim_{N \rightarrow \infty} 2 \oint_{\Gamma_N} \left| \frac{C_0 \left( (N + 1/2)\sqrt{2} \right)^{n-2} + \mathcal{O} \left( \left( (N + 1/2)\sqrt{2} \right)^{n-3} \right)}{C_1 \left( N + 1/2 \right)^n + \mathcal{O} \left( \left( N + 1/2 \right)^{n-1} \right)} \right| |dz|. \end{aligned}$$

Let  $M = N + 1/2$ . We can also write  $|dz|$  as the arc length of the contour which is  $4(2N + 1)$ . Then we rewrite the last inequality as

$$\begin{aligned} &\lim_{N \rightarrow \infty} 2 \oint_{\Gamma_N} \left| \frac{C_0 \left( (N + 1/2)\sqrt{2} \right)^{n-2} + \mathcal{O} \left( \left( (N + 1/2)\sqrt{2} \right)^{n-3} \right)}{C_1 \left( N + 1/2 \right)^n + \mathcal{O} \left( \left( N + 1/2 \right)^{n-1} \right)} \right| |dz| \\ &\leq \lim_{M \rightarrow \infty} 2 \left| \frac{C_0 M^{n-2} \sqrt{2}^{n-2} + \mathcal{O} \left( M^{n-3} \sqrt{2}^{n-3} \right)}{C_1 M^n + \mathcal{O} \left( M^{n-1} \right)} \right| 4(2N + 1) \\ &\leq \lim_{M \rightarrow \infty} 2 \left| \frac{C_0 M^{n-2} \sqrt{2}^{n-2} + \mathcal{O} \left( M^{n-3} \sqrt{2}^{n-3} \right)}{C_1 M^n + \mathcal{O} \left( M^{n-1} \right)} \right| 8M \\ &\leq \lim_{M \rightarrow \infty} 2 \frac{8M \left| C_0 M^{n-2} \sqrt{2}^{n-2} \right| + 8M \left| \mathcal{O} \left( M^{n-3} \sqrt{2}^{n-3} \right) \right|}{|C_1 M^n + \mathcal{O} \left( M^{n-1} \right)|} \end{aligned}$$

**TODO:** multiply by 8M then there is going to just be one final power of M in the denominator which will send it to 0 as M goes to infinity.

(d) Suppose, in addition, that  $q(z)$  has no roots at the integers. Show that

$$\sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = -\pi \sum_j \operatorname{Res}_{z=z_j} f(z) \cot(\pi z)$$

where the  $z_j$ 's are the roots of  $q(z)$ . Notice that the sum on the right-hand side has a finite number of terms.

*Solution:*

**TODO: I think you should argue that this holds for a particular  $N$ , and then take a limit as  $N \rightarrow \infty$**

$$\begin{aligned} \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz & \xrightarrow{0} \sum_{k \in \mathbb{Z}} \operatorname{Res}_{z=k} \left( \frac{p(z)}{q(z)} \cot(\pi z) \right) + \sum_j \operatorname{Res}_{z=z_j} \left( \frac{p(z)}{q(z)} \cot(\pi z) \right) \\ 0 &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} + \sum_j \operatorname{Res}_{z=z_j} f(z) \cot(\pi z) \\ -\frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} &= \sum_j \operatorname{Res}_{z=z_j} f(z) \cot(\pi z) \\ \sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} &= -\pi \sum_j \operatorname{Res}_{z=z_j} f(z) \cot(\pi z) \end{aligned}$$

□

(e) Use the result of the previous problem to evaluate the following sums:

$$(i) \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 1}$$

*Solution:*

Directly applying the conclusion from part (d)

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 1} &= -\pi \sum_j \operatorname{Res}_{z=z_j} \left( \frac{1}{z^2 + 1} \cot(\pi z) \right) \\ &= -\pi \left[ \operatorname{Res}_{z=i} \left( \frac{1}{(z-i)(z+i)} \cot(\pi z) \right) + \operatorname{Res}_{z=-i} \left( \frac{1}{(z-i)(z+i)} \cot(\pi z) \right) \right] \\ &= -\pi \left( \frac{1}{2i} \cot(\pi i) - \frac{1}{2i} \cot(-\pi i) \right) \\ &= -\frac{\pi}{2i} \left( \frac{i(e^{i\pi i} + e^{-i\pi i})}{e^{i\pi i} - e^{-i\pi i}} - \frac{i(e^{-i\pi i} + e^{i\pi i})}{e^{-i\pi i} - e^{i\pi i}} \right) \\ &= -\frac{\pi}{2} \left( \frac{e^{-\pi} + e^{\pi}}{e^{-\pi} - e^{\pi}} - \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} \right) \\ &= -\frac{\pi}{2} \left( -\frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} - \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} \right) \\ &= -\frac{\pi}{2} \left( -2 \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} \right) \\ &= \pi \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} \\ &= \pi \coth \pi. \end{aligned}$$

□

$$(ii) \sum_{k=-\infty}^{\infty} \frac{1}{k^4 + 1}$$

*Solution:*

Before jumping into evaluating this summation using our formula from part (d), let's determine what the singularities of  $1/(k^4 + 1)$  are. In other words we need to find where  $q(z) = 0$  with  $q(z) := z^4 + 1$ . Notice we can factor to arrive at

$$\begin{aligned} z^4 + 1 &= (z^2 + i)(z^2 - i) \\ &= (z - i\sqrt{i})(z + i\sqrt{i})(z - \sqrt{i})(z + \sqrt{i}). \end{aligned}$$

Thus,  $S$ , the set of values  $w \in \mathbb{C}$  such that  $q(z) = 0$ , is  $S = \{i\sqrt{i}, -i\sqrt{i}, \sqrt{i}, -\sqrt{i}\}$ .

Now we can evaluate the sum

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{k^4 + 1} &= -\pi \sum_{w \in S} \operatorname{Res}_{z=w} \left( \frac{1}{z^4 + 1} \cot(\pi z) \right) \\
&= -\pi \left[ \frac{1}{(i\sqrt{i} + i\sqrt{i})(i\sqrt{i} - \sqrt{i})(i\sqrt{i} + \sqrt{i})} \cot(\pi i\sqrt{i}) + \frac{1}{(-i\sqrt{i} - i\sqrt{i})(-i\sqrt{i} - \sqrt{i})(-i\sqrt{i} + \sqrt{i})} \cot(-\pi i\sqrt{i}) \right. \\
&\quad + \frac{1}{(\sqrt{i} - i\sqrt{i})(\sqrt{i} + i\sqrt{i})(\sqrt{i} + \sqrt{i})} \cot(\pi \sqrt{i}) \\
&\quad \left. + \frac{1}{(-\sqrt{i} - i\sqrt{i})(-\sqrt{i} + i\sqrt{i})(-\sqrt{i} - \sqrt{i})} \cot(-\pi \sqrt{i}) \right] \\
&= -\pi \left[ \frac{1}{(2i\sqrt{i})\sqrt{i}(i-1)\sqrt{i}(i+1)} \cot(\pi i\sqrt{i}) + \frac{1}{(-2i\sqrt{i})(-\sqrt{i})(i+1)(-\sqrt{i})(i-1)} \cot(-\pi i\sqrt{i}) \right. \\
&\quad + \frac{1}{\sqrt{i}(1-i)\sqrt{i}(1+i)(2\sqrt{i})} \cot(\pi \sqrt{i}) + \frac{1}{(-\sqrt{i})(1+i)(-\sqrt{i})(1-i)(-2\sqrt{i})} \cot(-\pi \sqrt{i}) \left. \right] \\
&= -\pi \left[ \frac{1}{4\sqrt{i}} \cot(\pi i\sqrt{i}) - \frac{1}{4\sqrt{i}} \cot(-\pi i\sqrt{i}) + \frac{1}{4i\sqrt{i}} \cot(\pi \sqrt{i}) - \frac{1}{4i\sqrt{i}} \cot(-\pi \sqrt{i}) \right]
\end{aligned}$$

**TODO: Check if this is as far as anyone else simplified it.**

$$(iii) \quad \sum_{k=-\infty}^{\infty} \frac{1}{k^2 - 1/4}$$

*Solution:*

The zeros of the denominator (or the singularities of the overall function) are found at  $w \in S = \{\pm 1/2\}$ . Then applying the formula from part (d) we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{k^2 - 1/4} &= -\pi \sum_{w \in S} \operatorname{Res}_{z=w} \left( \frac{1}{z^2 - 1/4} \cot(\pi z) \right) \\
&= -\pi \left( \operatorname{Res}_{z=1/2} \left( \frac{1}{(z-1/2)(z+1/2)} \cot(\pi z) \right) \right. \\
&\quad \left. + \operatorname{Res}_{z=-1/2} \left( \frac{1}{(z-1/2)(z+1/2)} \cot(\pi z) \right) \right) \\
&= -\pi \left( \frac{1}{(1/2+1/2)} \cot(\pi/2) + \frac{1}{(-1/2-1/2)} \cot(-\pi/2) \right) \\
&= -\pi (\cot(\pi/2) - \cot(-\pi/2))
\end{aligned}$$

□

$$(iv) \sum_{k=-\infty}^{\infty} \frac{1}{16k^4 - 1}$$

*Solution:*

Let  $q(z) = 16z^4 - 1$ . We can factor the  $q(z)$  to

$$16z^4 - 1 = (2z - 1)(2z + 1)(2z - i)(2z + i).$$

Therefore, the zeros of this function are found at  $w \in S = \{\pm 1/2, \pm i/2\}$ .

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{1}{16k^4 - 1} &= -\pi \sum_{w \in S} \operatorname{Res}_{z=w} \left( \frac{1}{16z^4 - 1} \cot(\pi z) \right) \\ &= -\pi \sum_{w \in S} \operatorname{Res}_{z=w} \left( \frac{1}{2^4(z - 1/2)(z + 1/2)(z - i/2)(z + i/2)} \cot(\pi z) \right) \\ &= -\pi \left[ \left( \frac{1}{16(1/2 + 1/2)(1/2 - i/2)(1/2 + i/2)} \cot(\pi/2) \right) \right. \\ &\quad + \left( \frac{1}{16(-1/2 - 1/2)(-1/2 - i/2)(-1/2 + i/2)} \cot(-\pi/2) \right) \\ &\quad + \left( \frac{1}{16(i/2 - 1/2)(i/2 + 1/2)(i/2 + i/2)} \cot(\pi i/2) \right) \\ &\quad \left. + \left( \frac{1}{16(-i/2 - 1/2)(-i/2 + 1/2)(-i/2 - i/2)} \cot(-\pi i/2) \right) \right] \\ &= -\pi \left[ \frac{1}{8} \cot(\pi/2) - \frac{1}{8} \cot(-\pi/2) - \frac{1}{8i} \cot(\pi i/2) + \frac{1}{8i} \cot(-\pi i/2) \right] \\ &= -\pi \left[ \frac{1}{8} \cot(\pi/2) - \frac{1}{8} \cot(-\pi/2) + \frac{i}{8} \cot(\pi i/2) - \frac{i}{8} \cot(-\pi i/2) \right] \end{aligned}$$

□