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AMATH 561

PROBLEM SET 6

1. Let $X \sim Binomial(n, U)$, where $U \sim Uniform((0, 1))$. What is the probability generating function $G_X(s)$ of X? What is $P(X = k)$ for $k \in \{0, 1, 2, ..., n\}$?

Solution: The probability mass function for $X \sim Binomial(n, U)$, is given by

$$f_X(x) = \binom{n}{x} U^x (1-U)^{n-x} \quad \text{for } x = 0, 1, 2, ..., n$$

And the density for the Uniform distribution is

$$g_X(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1 \quad \text{for } x \in (0, 1)$$

Then $G_X(s)$ is

$$\begin{aligned}
G_X(s) &= E(s^X) \\
&= E(E(s^X|U)) \\
&= E\left(\sum_{x=0}^{n} \binom{n}{x} U^x (1-U)^{n-x} s^x\right) \\
&= E\left(\sum_{x=0}^{n} \binom{n}{x} (Us)^x (1-U)^{n-x}\right) \\
&= E\big((Us + 1 - U)^n\big) \\
&= \int_0^1 (us + 1 - u)^n du \\
&= \left.\frac{(us+1-u)^{n+1}}{(n+1)(s-1)}\right|_0^1 \\
&= \frac{(1s+1-1)^{n+1}}{(n+1)(s-1)} - \frac{(0s+1-0)^{n+1}}{(n+1)(s-1)} \\
&= \frac{s^{n+1}}{(n+1)(s-1)} - \frac{1^{n+1}}{(n+1)(s-1)} \\
&= \frac{s^{n+1}-1}{(n+1)(s-1)}
\end{aligned}$$

Now to calculate $P(X = k)$, notice

$$\begin{aligned}
G_X(s) = \frac{s^{n+1} - 1}{(n+1)(s-1)} &= \frac{1}{n+1}\frac{s^{n+1}-1}{s-1} \\
&= \frac{1}{n+1}\frac{1-s^{n+1}}{1-s} \\
&= \sum_{k=0}^{n} \frac{1}{n+1} s^k \\
&= \sum_{k=0}^{n} P(X = k) s^k \\
&= G_X(s)
\end{aligned}$$

Therefore,

$$P(X = k) = \frac{1}{n+1} \quad \text{for all } k \in \{0, 1, 2, .., n\}.$$

$\square$

2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ, the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

Solution:
We can write the generating function $G_{Z_{n+1}}(s)$ as follows

$$\begin{aligned} G_{Z_{n+1}}(s) &= G_{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}}(s) \\ &= G_{\xi_1^{n+1} + \xi_2^{n+1} + \xi_3^{n+1} + \ldots + \xi_{Z_n}^{n+1} + Y_{n+1}}(s) \\ &= G_{\xi_1^{n+1} + \xi_2^{n+1} + \xi_3^{n+1} + \ldots + \xi_{Z_n}^{n+1}} G_{Y_{n+1}}(s) \\ &= G_{Z_n}(G_\xi(s)) G_{Y_{n+1}}(s). \end{aligned}$$

The second to third equality comes from the fact that the Y_{n+1} and ξ_i^{n+1} are independent. Finally, the last equality comes from an application of the Theorem 3 from Lecture 15.

Next to calculate $G_{Z_2}(s)$ explicitly we get

$$\begin{aligned} G_{Z_2}(s) &= G_{Z_1}(G_\xi(s)) G_Y(s) \\ &= \Big(G_\xi \Big(G_\xi(s) \Big) G_Y \Big(G_\xi(s) \Big) \Big) G_Y(s). \end{aligned}$$

$\square$

3. (a) Let X be exponentially distributed with parameter λ. Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$.

Solution:
We can begin by looking directly at the expectation we want to calculate

$$\begin{aligned}
E(\mathrm{e}^{\mathrm{i}\,tX}) = \int_\Omega \mathrm{e}^{\mathrm{i}\,tX}\,dP &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\,tx}\,\lambda\,\mathrm{e}^{-\lambda x}\,dx \\
&= \int_0^\infty \mathrm{e}^{\mathrm{i}\,tx}\,\lambda\,\mathrm{e}^{-\lambda x}\,dx \\
&= \int_0^\infty \lambda\,\mathrm{e}^{(\mathrm{i}\,t-\lambda)x}\,dx \\
&= \int_0^\infty \lambda\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx
\end{aligned}$$

Notice this integral is off by a scale factor to the density of an exponentially distributed random variable with parameter $\lambda - \mathrm{i}\,t$. Additionally, we know the integral of a probability density function is equal to 1, therefore,

$$\begin{aligned}
\int_0^\infty (\lambda - \mathrm{i}\,t)\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= 1 \\
\frac{\lambda}{\lambda - \mathrm{i}\,t}\int_0^\infty (\lambda - \mathrm{i}\,t)\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= \frac{\lambda}{\lambda - \mathrm{i}\,t} \\
\int_0^\infty \frac{\lambda}{\lambda - \mathrm{i}\,t}(\lambda - \mathrm{i}\,t)\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= \frac{\lambda}{\lambda - \mathrm{i}\,t} \\
\int_0^\infty \lambda\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= \frac{\lambda}{\lambda - \mathrm{i}\,t}.
\end{aligned}$$

Which is indeed the integral we wanted to compute.

□

(b) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

Solution: The characteristic function is (skipping directly to the change of variable form of the expectation)

$$\begin{aligned}
\phi_X(t) = E(\mathrm{e}^{\mathrm{i}\,tX}) &= \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,tx}\,\frac{1}{2}\,\mathrm{e}^{-|x|}\,dx \\
&= \frac{1}{2}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx.
\end{aligned}$$

Let's split up the integral into cases in order to handle the absolute value. Then we have

$$\begin{aligned}
\phi_X(t) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx &= \frac{1}{2}\left[\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx + \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\,tx+x}\,dx + \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\,tx-x}\,dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} \mathrm{e}^{(\mathrm{i}\,t+1)x}\,dx + \int_{0}^{\infty} \mathrm{e}^{-(1-\mathrm{i}\,t)x}\,dx\right] \\
&= \frac{1}{2}\left[\left(\frac{1}{\mathrm{i}\,t+1}\,\mathrm{e}^{(\mathrm{i}\,t+1)x}\Big|_{-\infty}^{0}\right) + \left(-\frac{1}{1-\mathrm{i}\,t}\,\mathrm{e}^{-(1-\mathrm{i}\,t)x}\Big|_{0}^{\infty}\right)\right].
\end{aligned}$$

Now, as we evaluate these expressions at the their respective bounds of integration, notice the terms evaluated at $-\infty$ and at ∞ in the left and right integrals both go to 0. Then we have

$$\begin{aligned}
\phi_X(t) &= \frac{1}{2}\left[\left(\frac{1}{\mathrm{i}\,t+1}\,\mathrm{e}^{(\mathrm{i}\,t+1)0} \cancelto{0}{-\frac{1}{\mathrm{i}\,t+1}\,\mathrm{e}^{(\mathrm{i}\,t+1)(-\infty)}}\right) + \left(\cancelto{0}{-\frac{1}{1-\mathrm{i}\,t}\,\mathrm{e}^{-(1-\mathrm{i}\,t)\infty}} + \frac{1}{1-\mathrm{i}\,t}\,\mathrm{e}^{-(1-\mathrm{i}\,t)0}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{\mathrm{i}\,t+1} + \frac{1}{1-\mathrm{i}\,t}\right] \\
&= \frac{1}{2}\left(\frac{1-\mathrm{i}\,t+\mathrm{i}\,t+1}{(\mathrm{i}\,t+1)(1-\mathrm{i}\,t)}\right) \\
&= \frac{1}{2}\left(\frac{2}{1-\mathrm{i}^2\,t^2}\right) \\
&= \frac{1}{1+t^2}.
\end{aligned}$$

Hence, the characteristic function for a random variable with it's density given by $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$ is

$$\phi_X(t) = \frac{1}{1+t^2}.$$

4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}\, 1_{x \geq 0}.$$

Solution:
Notice our scenario of N being the minimum number of tosses required to obtain k heads (call a heads a "succcess") is related to the negative binomial distribution. This distribution describes the probability of ℓ failures occurring before k successes, the p.m.f. is given by

$$P(N = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}.$$

To further connect this distribution to our scenario $N = \ell + k$. I think...

$$\begin{aligned}
\lim_{p \to 0} \binom{n-1}{k-1} p^k (1-p)^{n-k} &= \lim_{p \to 0} \frac{(n-1)!}{(n-k)!(k-1)!} p^k (1-p)^{n-k} \\
&= \lim_{p \to 0} \frac{(n-1)!}{(n-k)!(k-1)!} p^k \left[(1-p)^{1/p} \right]^{(n-k)p} \\
&= \frac{(n-1)!}{(n-k)!(k-1)!} p^k \, \mathrm{e}^{(n-k)p} \\
&= \frac{\frac{(n-1)!}{(k-1)!}}{\Gamma(n-k+1)} p^k \, \mathrm{e}^{(n-k)p} \ldots
\end{aligned}$$

TODO: Come back to this, you're so close We wish to show that as $p \to 0$ the distribution of $2Np$ converges to that of a gamma distribution. Recall that the density of the gamma distribution is