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AMATH 567

HOMEWORK 6

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 3.3.2
Given the function

$$f(z) = \frac{z}{a^2 - z^2}, \;\; a > 0,$$

expand $f(z)$ in a Laurent series in powers of z in the regions

(a) $|z| < a$
Solution:

(b) $|z| > a$
Solution:

2: From A&F: 3.3.5

Let

$$\exp\left(\frac{t}{2}\left(\frac{z-1}{z}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(n\theta - t\sin\theta)}\,\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(n\theta - t\sin\theta)\mathrm{d}\theta. \end{aligned}$$

The functions $J_n(t)$ are called the Bessel function, which are well known special functions in mathematics and physics.

Solution:

3: Bernoulli numbers: Consider the function

$$f(z) = \frac{z}{e^z - 1}.$$

(a) Show that $f(z)$ has a removable singularity at $z = 0$. Assume from now on that the definition of $f(z)$ has been extended to remove the singularity.

(b) Suppose you were to find a Taylor series for $f(z)$, centered at $z = 0$. What would be its radius of convergence?

(c) Find the Taylor series in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The numbers B_n are known as the Bernoulli numbers.

(d) Find a recursion formula for the Bernoulli numbers, and use it to find $B_0, \ldots, B_{12}$.

(e) Show that $B_{2n+1} = 0$ for $n \geq 1$.

(f) Use your result to find a Taylor series for $z \coth z$, in terms of the Bernoulli numbers. Where is this series valid? Using this result, find a Laurent series for $\cot z$. Where is this series valid?

4: Consider $g(z) = 1/f(z)$ where $f(z)$ is as in the previous problem.

(a) Using the formula for $g(z)$, use software that uses double precision floating point arithmetic to compute the errors $e_n := |g(2^{-n}) - g(0)|$ for $n = 1, 2, \ldots, 52$. Produce a plot of these errors.

(b) Derive an approximation $G(z)$ to $g(z)$, near $z = 0$, that does not suffer from the instability you notice. Plot the new errors $E_n := |G(2^{-n}) - g(0)|$ for $n = 1, 2, \ldots, 52$. Ensure that these errors are less than 10^{-10} for all n.

5: Analytic continuation: (a) Consider

$$F(z) = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n.$$

Where is this function analytic? (b) Use the above representation to induce a Taylor representation of $F(z)$ centered at $z = -1/2$. Call this representation $G(z)$. Your final result should be of the form

$$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2}\right)^m$$

Where is this series valid? If you can answer this question without using that both $F(z)$ and $G(z)$ are representations of $1/(1-z)$, you will receive 2 bonus points.

6: This problem is from Whittaker and Watson's "A course of modern analysis": Shew[1] that

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{\left(1-z^{n}\right)\left(1-z^{n+1}\right)} = \begin{cases} \frac{1}{(1-z)^2}, & |z|<1 \\ \frac{1}{z(1-z)^2}, & |z|>1. \end{cases}$$

This might appear to contradict the idea of analytic continuation. Please comment.

[1] Aka "Show".

7: Suppose that f is a function satisfying

$$|f(x)| \leq M, \quad x \in \mathbb{R}.$$

Show that

$$\hat{f}(z) := \int_0^\infty \mathrm{e}^{\mathrm{i}zx} f(x) \mathrm{d}x,$$

is an analytic function of z for $\operatorname{Im} z > 0$. You may assume that f is continuous, but this is not a necessary assumption.

8: Use analytic continuation to show that

$$\sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}},$$

where $\sqrt{\cdot}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$. Then show that

$$\sqrt{z-1}\sqrt{z+1} = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + O(z^{-3}), \quad z \to \infty,$$

and find b_0, b_1, b_2.