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AMATH 561

PROBLEM SET 7

Note: Exercises 1-4 are from Matt Lorig's notes (link on course website).

1. Exercise 4.1.
A six-sided die is rolled repeatedly. Which of the following are Markov chains? For those that are, find the one-step transition matrix.

(a) X_n is the largest number rolled up to the nth roll.

Solution:
It helps me to visualize this graphically. Let each state be a node with directed edges going from a given state to all reachable states from the current state. If each directed edge in Figure 1 is equally likely, then we

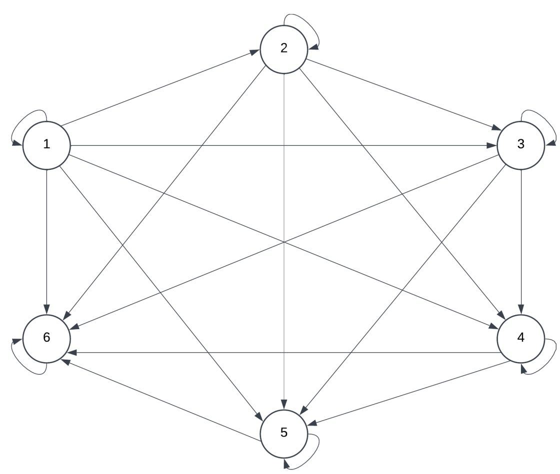

FIGURE 1. A graphical representation of the Markov Chain in problem 1 part (a).

can construct the following transition matrix

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the transition matrix it is easy to see this is a Markov chain. **TODO:**

(b) X_n is the number of sixes rolled in the first n rolls.

Solution:
The transition matrix is given by the following

$$P = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 \dots \\ 0 & 5/6 & 1/6 & 0 \dots \\ 0 & 0 & 5/6 & 1/6 \ddots \\ \vdots & \vdots & \ddots & \ddots \ddots \end{bmatrix}.$$

Since we can write $X_n = X_{n-1} + \xi_n$ where ξ_n is 1 if the nth role is a 6 or 0 otherwise. Therefore ξ_n is Bernoulli distributed with probability of success $p = 1/6$. Let the current state be denoted as ℓ. The state space is $\mathbb{N} \cup \{0\}$. Furthermore we can say if $X_n = \ell$ we denote the following

$$P(X_{n+1} = \ell + 1 | X_n = \ell) = 1/6.$$

Graphically that can be represented as seen in Figure 3 **TODO:**

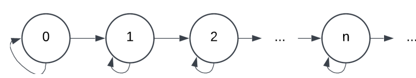

FIGURE 2. A graphical representation of the Markov Chain in problem 1 part (b).

(c) At time n, X_n is the time since the last six was rolled.

Solution:
Each step is 1/6 chance of increasing X_n and a 5/6 chance of returning to the beginning.

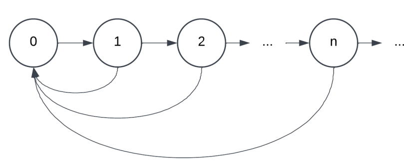

FIGURE 3. A graphical representation of the Markov Chain in problem 1 part (c).

$$P = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 \dots \\ 5/6 & 0 & 1/6 & 0 \dots \\ 5/6 & 0 & 0 & 1/6 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

TODO:

(d) At time n, X_n is the time until the next six is rolled.

Solution:
TODO: ???

2. Exercise 4.2.

Let $Y_n = X_{2n}$. Compute the transition matrix for Y when

(a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q.

Solution:
TODO:

(b) X is a branching process where G is the generating function of the number of offspring from each individual.

Solution:
TODO:

3. Exercise 4.3.
Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k, j) = 0$ for all $j \in S$). Suppose $j \to k$ for all $j \in S$. Show that all states other than k are transient.

Solution:
TODO:

4. Exercise 4.4.
Suppose two distinct states i, j satisfy

$$P(\tau_j < \tau_i | X_0 = i) = P(\tau_i < \tau_j | X_0 = j)$$

where $\tau := \inf\{n \geq 1 : X_n = j\}$. Show that, if $X_0 = i$, the expected number of visits to j prior to re-visiting i is one.

Solution:
TODO:

5. Stationary distribution of Ehrenfest chain. (a) Let X_n be the number of balls in the left urn at time n (total number of balls in both urns is r). At each time step, one of the r balls is picked at random and moved to the other urn.

(a) Let $G_n(s)$ be the generating function of X_n. Derive a formula for G_{n+1} as a function of G_n.

Solution:
TODO:

(b) Let $G(s) = \lim_{n\to\infty} G_n(s)$. Use the relation in part a) to derive an equation for G. Solve it and find G.

Solution:
TODO:

(c) Find the stationary distribution π of Ehrenfest chain. What is the connection between G and π?

Solution:
TODO: