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AMATH 561

PROBLEM SET 4

1. Let $\Omega = \{a, b, c, d\}$ and let $\mathcal{F} = 2^\Omega$. We define a probability measure P as follows:

$$P(a) = 1/6, \ \ , P(b) = 1/3, \ \ P(c) = 1/4, \ \ P(d) = 1/4.$$

Next, define three random variables:

$$X(a) = 1, \ \ X(b) = 1, \ \ X(c) = -1, \ \ X(d) = -1,$$
$$Y(a) = 1, \ \ Y(b) = -1, \ \ Y(c) = 1, \ \ Y(d) = -1,$$

and $Z = X + Y$.

(a) List the sets in $\sigma(X)$.
Solution:
The pre-image of X is

$$X^{-1}(B) = \begin{cases} \{c, d\}, \text{ if } -1 \in B, 1 \notin B \\ \{a, b\}, \text{ if } 1 \in B, -1 \notin B. \end{cases}$$

Then we have

$$\sigma(X) = \sigma(\{\{a, b\}, \{c, d\}\}) = \Big\{\{a, b\}, \{c, d\}, \Omega, \emptyset\Big\}$$

□

(b) Calculate $E(Y|X)$.
Solution:
We can calculate this as follows

$$\mathbb{E}[Y|X] = \mathbb{E}[Y|\sigma(X)]$$

then we have

$$\mathbb{E}[Y|\sigma(X)] = \frac{\mathbb{E}[Y; \{a, b\}]}{P(\{a, b\})} = \frac{1 \cdot P(a) - 1 \cdot P(b)}{\frac{1}{6} + \frac{1}{3}} = \frac{1 \cdot \frac{1}{6} - 1 \cdot \frac{1}{3}}{\frac{1}{2}} = -\frac{1}{3}$$

and

$$\mathbb{E}[Y|\sigma(X)] = \frac{\mathbb{E}[Y; \{c, d\}]}{P(\{c, d\})} = \frac{1 \cdot P(c) - 1 \cdot P(d)}{\frac{1}{2}} = \frac{1 \cdot \frac{1}{4} - 1 \cdot \frac{1}{4}}{\frac{1}{2}} = 0$$

□

(c) Calculate $E(Z|X)$.
Solution:
Let's first look at the values that $Z(\omega)$ takes on for each $\omega \in \{a, b, c, d\}$.

$$Z(a) = 2, \ Z(b) = 0, \ Z(c) = 0, \ Z(d) = -2$$

Then we have

$$\mathbb{E}[Z|X] = \mathbb{E}[Z|\sigma(X)]$$

giving us

$$\mathbb{E}[Z|\sigma(X)] = \frac{\mathbb{E}[Z;\{a,b\}]}{P(\{a,b\})} = \frac{2 \cdot P(a) + 0 \cdot P(b)}{\frac{1}{6}+\frac{1}{3}} = \frac{2 \cdot \frac{1}{6} + 0 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

and

$$\mathbb{E}[Z|\sigma(X)] = \frac{\mathbb{E}[Z;\{c,d\}]}{P(\{c,d\})} = \frac{2 \cdot P(c) + 0 \cdot P(d)}{\frac{1}{2}} = \frac{2 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$\square$

2. (a) Prove that $E(E(X|\mathcal{F})) = EX$.
Solution:
There is an underlying probability space $(\Omega, \mathcal{F}_\star, P)$. Let $\mathcal{F} = \sigma(\{\Omega_1, \Omega_2, ...\})$, then

$$\mathbb{E}[X|\mathcal{F}] = \frac{\mathbb{E}[X;\Omega_i]}{P(\Omega_i)} = \frac{\int_{\Omega_i} X \mathrm{d}P}{P(\Omega_i)}$$

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\big] = \int_\Omega \mathbb{E}[X|\mathcal{F}] \mathrm{d}P =$$

$$\mathbb{E}[X] = \int_\Omega X \mathrm{d}P = \int_\mathbb{R} x\mu(\mathrm{d}x)$$

(b) Show that if $\mathcal{G} \subset \mathcal{F}$ and $EX^2 < \infty$ then

$$E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) = E(\{X - E(X|\mathcal{G})\}^2)$$

3. An important special case of the previous result (2b) occurs when $\mathcal{G} = \{\emptyset, \Omega\}$. Let $\mathrm{var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

$$\mathrm{var}(X) = E(\mathrm{var}(X|\mathcal{F})) + \mathrm{var}(E(X|\mathcal{F})).$$

4. Let $Y_1, Y_2, ...$ be i.i.d. (independent and identically distributed) random variables with mean μ and variance σ^2, N an independent positive integer valued random variable with $EN^2 < \infty$ and $X = Y_1 + ... + Y_N$. Show that $\mathrm{var}(X) = \sigma^2 EN + \mu^2 \mathrm{var}(N)$. (To understand and help remember the formula, think about the two special cases in which N or Y is constant.)