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AMATH 567

HOMEWORK 8

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: The Korteweg-de Vries (KdV) equation arises whenever long waves of moderate amplitude in dispersive media are considered. For instance, it describes waves in shallow water, and ion-acoustic waves in plasmas. The equation is given by

$$u_t = 6uu_x + u_{xxx},$$

where indices denote partial differentiation.

(a) By looking for solutions $u(x,t) = U(x)$, derive a first-order ordinary differential equation for $U(x)$. Introduce integration constants as required.

Solution:
Since we want to find time independent ODE such that $\frac{\mathrm{d}}{\mathrm{d}t}U(x) = 0$, then we need

$$0 = 6uu_x + u_{xxx}.$$

Integrating gives us

$$\begin{aligned}\int 0 &= 6\int uu_x \mathrm{d}x + \int u_{xxx}\mathrm{d}x \\ 0 &= 6\int uu_x \mathrm{d}x + u_{xx} + C_2.\end{aligned}$$

Using a substitution $v = u$ and $\mathrm{d}v = u_x\mathrm{d}x$ we have

$$\begin{aligned}0 &= 6\int uu_x \mathrm{d}x + u_{xx} + C_2 \\ 0 &= 6\int v\mathrm{d}v + u_{xx} + C_2 \\ 0 &= 6\left(\frac{1}{2}v^2\right) + C_3 + u_{xx} + C_2 \\ 0 &= 3u^2 + C_3 + u_{xx} + C_2.\end{aligned}$$

Next we multiply through by u_x and use the $v = u$ ($\mathrm{d}v = u_x \mathrm{d}x$) substitution for the first integral and a $\omega = u_x$ ($\mathrm{d}\omega = u_{xx}\mathrm{d}x$) substitution for the third integral

$$\begin{aligned}
0 &= 3u^2 u_x + u_x C_3 + u_x u_{xx} + u_x C_2 \\
0 &= 3\int u^2 u_x \mathrm{d}x + \int u_x C_3 \mathrm{d}x + \int u_x u_{xx} \mathrm{d}x + \int u_x C_2 \mathrm{d}x \\
0 &= 3\int v^2 \mathrm{d}v + \int u_x C_3 \mathrm{d}x + \int \omega \mathrm{d}\omega + \int u_x C_2 \mathrm{d}x \\
0 &= 3\left(\frac{1}{3}v^3\right) + C_4 + uC_3 + C_5 + \frac{1}{2}\omega^2 + C_6 + uC_2 + C_7 \\
0 &= u^3 + \frac{1}{2}(u_x)^2 + u\big(C_2 + C_3\big) + \big(C_4 + C_5 + C_6 + C_7\big) \\
-\frac{1}{2}(u_x)^2 &= u^3 + u\big(C_2 + C_3\big) + \big(C_4 + C_5 + C_6 + C_7\big) \\
(u_x)^2 &= -2u^3 - 2u\big(C_2 + C_3\big) - 2\big(C_4 + C_5 + C_6 + C_7\big).
\end{aligned}$$

Hence

$$u_x^2 = -2u^3 + uC_0 + C_1$$

which is our first-order ordinary differential equation for $U(x)$.

$\square$

(b) Let $U = U_0 \wp(x - x_0)$. Determine U_0 so that $u = U(x)$ solves the KdV equation.

Solution:
Note, we proved last time that $\wp(z + Nw_1 + Mw_2) = \wp(z)$ therefore $\wp(x - x_0) = \wp(x)$. Then plugging $U = U_0\wp(x)$ into our first-order ordinary differential equation for $U(x)$ (while suppressing the argument for $\wp$) we have

$$\begin{aligned}
\big(U_0\wp'\big)^2 &= -2\big(U_0\wp\big)^3 + \big(U_0\wp\big)C_0 + C_1 \\
U_0^2\big(\wp'\big)^2 &= -2U_0^3\wp^3 + U_0 C_0 \wp + C_1 \\
\big(\wp'\big)^2 &= -2U_0\wp^3 + \frac{C_0}{U_0}\wp + \frac{C_1}{U_0^2}.
\end{aligned}$$

Choosing $U_0 = -2$ then we have

$$\big(\wp'\big)^2 = 4\wp^3 - \frac{1}{2}C_0\wp + \frac{1}{4}C_1,$$

which resembles the ode which we proved holds true in the last assignment

$$(\wp')^2 = 4\wp^3 + c\wp + d.$$

The remaining constants can be attained through the initial conditions of the system we are solving and making sure they agree with the values of c, d from the previous assignment.

$\square$

2: From A&F: 3.6.5

Show that if $f(z)$ is meromorphic in the finite z plane, then $f(z)$ must be the ratio of two entire functions.

Solution:

Assume $f(z)$ is a meromorphic function. Then we know all of the singularities of $f(z)$ are poles of some order. If we can multiply $f(z)$ by some entire $g(z)$ function which knocks out all of the poles of $f(z)$ and are left with an entire function $h(z)$, then the original meromorphic function $f(z)$ is a ratio of two entire functions. Now it is left for us to successfully construct such a function $g(z)$. Our construction needs to have zeros at all of the locations where $f(z)$ has poles. Additionally we need to make sure that the multiplicity of the zeros agree with the residue of the poles. We can use the Mittag-Leffler Expansion to assist us here. Suppose $f(z)$ has poles at each $z = z_j$ for $j = 0, 1, 2, ...$ with corresponding residues a_j then let $g(z)$ be

$$g(z) = z^{a_0} \prod_{j=1}^{\infty} \left[(z - z_j) \exp\left(\sum_{k=0}^{m-1} \frac{(z_j)^{k+1}}{k+1} \right) \right]^{a_j}.$$

Then

$$f(z)g(z) = h(z).$$

Since $g(z)$ is an entire function constructed strategically, $h(z)$ has no singularities and is therefore entire in the finite complex plane. Therefore $f(z)$ is the ratio of two entire functions.

$\square$

3: Here's a way to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

due to Euler. We've seen that

$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

(a) Equate the coefficients of z^2 on both sides, to recover the desired sum.

Solution:
Taylor expand on the left to get

$$\begin{aligned}\frac{\sin \pi z}{\pi z} &= \frac{1}{\pi z} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi z)^{2j+1}}{(2j+1)!} \\ &= \frac{1}{\pi z}\left(\pi z - \frac{(z\pi)^3}{6} + \frac{(z\pi)^5}{120} - ...\right) \\ &= 1 - \frac{z^2\pi^2}{6} + \frac{z^4\pi^4}{120} - ...\end{aligned}$$

Now expand out several terms in the product on the right

$$\begin{aligned}\prod_{j=1}^{\infty}\left(1 - \frac{z^2}{j^2}\right) &= (1-z^2)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right)\left(1 - \frac{z^2}{16}\right)\left(1 - \frac{z^2}{25}\right)... \\ &= \left(1 - z^2 - \frac{z^2}{4} + \frac{z^4}{4}\right)\left(1 - \frac{z^2}{9}\right)\left(1 - \frac{z^2}{16}\right)\left(1 - \frac{z^2}{25}\right)... \\ &= \left(1 - z^2 - \frac{z^2}{4} + \frac{z^4}{4} - \frac{z^2}{9} + \frac{z^4}{9} + \frac{z^4}{36} - \frac{z^6}{36}\right)\left(1 - \frac{z^2}{16}\right)\left(1 - \frac{z^2}{25}\right)... \\ &= \left(1 + z^2\Big(-1 - \frac{1}{4} - \frac{1}{9}\Big) + z^4\Big(\frac{1}{4} + \frac{1}{9} + \frac{1}{36}\Big) - \frac{z^6}{36}\right)\left(1 - \frac{z^2}{16}\right)\left(1 - \frac{z^2}{25}\right)...\end{aligned}$$

Then we have the coefficients for z^2 becomes the series $-\sum_{j=0}^{\infty} \frac{1}{j^2}$. Equating the coefficient on the left with the series on the right we have

$$\begin{aligned}-\frac{\pi^2}{6} &= -\sum_{j=0}^{\infty} \frac{1}{j^2} \\ \frac{\pi^2}{6} &= \sum_{j=0}^{\infty} \frac{1}{j^2}\end{aligned}$$

□

(b) Equate the coefficients of z^4 on both sides to recover a different sum.

Solution:
Using the results from the Taylor expansion on the left from part (a) we have the coefficient of the z^4 term is $\frac{\pi^4}{120}$. Additionally, from expanding the first several terms in the product on the right we have that the coefficient of the z^4 term can be written as

$$\sum_{j=0}^{\infty} \sum_{k=1}^{j-1} \frac{1}{j^2} \frac{1}{k^2}.$$

Combining these we have

$$\frac{\pi^4}{120} = \sum_{j=0}^{\infty} \sum_{k=1}^{j-1} \frac{1}{j^2} \frac{1}{k^2}.$$

A little work can be done to relate this to the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^4},$$

I will only come back to this and complete that step if I have time, since Tom expressed that the question was asked in a vague enough way that stopping here is sufficient for grading.

By equating coefficients of higher powers of z, one can recover other identities too.

$\square$

4: For the following, suppose that $f(z)$ is analytic in an open set Ω that contains $[-1, 1]$.

(a) Show that there exists a contour C, encircling $[-1, 1]$, such that

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

Solution:

For convenience, define $h(z)$ to be the integrand $h(z) = \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}$. Siting Homework 5 problem 5, let Σ define the same area as before

$$\Sigma = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \leq r \text{ and } 0 \leq -\operatorname{Im} z \leq R, \ R > 0, \ r > 1\}$$

and let $\partial\Sigma$ be the counterclockwise oriented contour along the boundary of the region Σ. Additionally let $\partial\Sigma \setminus [-1, 1]$ be the contour on the boundary without the section from -1 to 1 on the real line. From that same problem we know

$$\oint_{\partial\Sigma} h(z)\mathrm{d}z = 0.$$

Furthermore, we can say

$$\int_{1}^{-1} h(z)\mathrm{d}z + \oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z = 0$$

$$\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z = \int_{-1}^{1} h(z)\mathrm{d}z. \tag{1}$$

We now define

$$\Sigma' = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \leq r \text{ and } 0 \leq \operatorname{Im} z \leq R, \ R > 0, \ r > 1\}$$

to be the upper half plane analogy of Σ. Therefore, let $\partial\Sigma'$ be the counterclockwise oriented contour along the boundary of the region Σ'.

$$g(z) = \begin{cases} h(z), & \text{if } \Im(z) > 0 \text{ or } |z| > 1 \\ -h(z), & \text{if } \Im(z) = 0 \text{ and } |z| \leq 1 \end{cases}.$$

This helps us preserve the continuity we are concerned with in order to apply the same arguments from Homework 5 problem 5 and making use of lemma 1 from problem 4 of that same assignment. Then we can conclude

$$\oint_{\partial\Sigma'} g(z)\mathrm{d}z = 0.$$

Furthermore, we have

$$\int_{-1}^{1} g(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} g(z)\mathrm{d}z = 0$$

$$\oint_{\partial\Sigma'\setminus[-1,1]} g(z)\mathrm{d}z = -\int_{-1}^{1} g(z)\mathrm{d}z$$

$$\oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = -\int_{-1}^{1} -h(z)\mathrm{d}z$$

$$\oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = \int_{-1}^{1} h(z)\mathrm{d}z. \tag{2}$$

If we add equation (1) and equation (2), we have

$$\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = \int_{-1}^{1} h(z)\mathrm{d}z + \int_{-1}^{1} h(z)\mathrm{d}z$$

$$\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z = 2\int_{-1}^{1} h(z)\mathrm{d}z.$$

Note the contours $\partial\Sigma \setminus [-1,1]$ and $\partial\Sigma' \setminus [-1,1]$ have small overlapping regions on the real axis $z \in (-r-1,-1)$ and $z \in (1, 1+r)$ which cancel out since they are of opposite orientation. We denote the combinations of these contours as $\partial\widehat{\Sigma}$ which is the counterclockwise oriented contour on the boundary of $\widehat{\Sigma}$ with

$$\widehat{\Sigma} = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \le r \text{ and } |\operatorname{Im} z| \le R,\ R > 0,\ r > 1\}.$$

Hence,

$$\begin{aligned}
\oint_{\partial\Sigma\setminus[-1,1]} h(z)\mathrm{d}z + \oint_{\partial\Sigma'\setminus[-1,1]} h(z)\mathrm{d}z &= 2\int_{-1}^{1} h(z)\mathrm{d}z \\
\oint_{\partial\widehat{\Sigma}} h(z)\mathrm{d}z &= 2\int_{-1}^{1} h(z)\mathrm{d}z \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= 2\int_{-1}^{1} \frac{f(x)}{\sqrt{x-1}\sqrt{x+1}}\mathrm{d}x \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= 2\int_{-1}^{1} \frac{f(x)}{(-\mathrm{i})\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= -\frac{2}{\mathrm{i}}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x \\
\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= 2\mathrm{i}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x \\
\frac{1}{2\mathrm{i}}\oint_{\partial\widehat{\Sigma}} \frac{f(z)}{\sqrt{z-1}\sqrt{z+1}}\mathrm{d}z &= \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x}\sqrt{x+1}}\mathrm{d}x.
\end{aligned}$$

Therefore the counterclockwise oriented contour on the boundary of $\widehat{\Sigma}$, denoted as $\partial\widehat{\Sigma}$ is one such contour encircling [-1, 1] such that

$$\int_{-1}^{1} \frac{f(x)\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \frac{1}{2\mathrm{i}}\oint_{C} \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.$$

□

Note we can deform this contour $\partial\widehat{\Sigma}$ into a circle centered at $z = 0$ of radius $\rho > 1$.

(b) Use this to evaluate

$$I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}}, \quad I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x,$$
$$I_3 = \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\mathrm{d}x, \quad I_4 = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\mathrm{d}x,$$

without using any changes of variable (e.g., no trig subs!).

Solution:
Using part (a) and the substitution $z = \rho\,\mathrm{e}^{\mathrm{i}\theta}$ where ρ is very large making z be near ∞. Additionally, our counterclockwise circle contour around $z = 0$ is also a counterclockwise contour around ∞. Therefore

$$\begin{aligned}
I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} &= \frac{1}{2\mathrm{i}} \oint_C \frac{\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= \frac{1}{2\mathrm{i}} \int_0^{2\pi} \frac{\rho\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\sqrt{\rho\,\mathrm{e}^{\mathrm{i}\theta}-1}\sqrt{\rho\,\mathrm{e}^{\mathrm{i}\theta}+1}} \\
&= \frac{1}{2\mathrm{i}} \int_0^{2\pi} \frac{\rho\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\rho\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{1-\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}}}\sqrt{1+\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}}}} \\
&= \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}}}\sqrt{1+\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}}}}.
\end{aligned}$$

Since ρ is large, $\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}} \approx 0$. Thus

$$\begin{aligned}
&= \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}}}\sqrt{1+\frac{1}{\rho\,\mathrm{e}^{\mathrm{i}\theta}}}} \\
&= \frac{1}{2} \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{1}{2}\left(\theta\Big|_0^{2\pi}\right) \\
&= \frac{1}{2}2\pi \\
&= \pi.
\end{aligned}$$

Hence,

$$I_1 = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} = \pi$$

□

Now for I_2 we can begin by converting our integrand into something of the form that will help us use the results from part (a). Notice

$$\sqrt{1-x}\sqrt{1+x} = \frac{(1-x)(1+x)}{\sqrt{1-x}\sqrt{1+x}} = \frac{1-x^2}{\sqrt{1-x}\sqrt{1+x}}.$$

Then we can evaluate the integral as follows

$$
\begin{aligned}
I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x &= \int_{-1}^{1} \frac{1-x^2\mathrm{d}x}{\sqrt{1-x}\sqrt{1+x}} \\
&= \frac{1}{2\mathrm{i}} \oint_C \frac{1-z^2\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}}.
\end{aligned}
$$

Let's take the contour to be a circle centered at $z = 0$ of radius ρ such that ρ is sufficiently large. This integral can also be evaluated at infinity

$$
\begin{aligned}
\frac{1}{2\mathrm{i}} \oint_C \frac{1-z^2\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} &= \frac{1}{2\mathrm{i}} \oint_C \frac{(1-z)(1+z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= -\frac{1}{2\mathrm{i}} \oint_C \frac{(z-1)(1+z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= -\frac{1}{2\mathrm{i}} \oint_C \sqrt{z-1}\sqrt{z+1}\,\mathrm{d}z.
\end{aligned}
$$

From a previous assignment we have that

$$
\sqrt{z-1}\sqrt{z+1} = (z-1)\sqrt{\frac{z+1}{z-1}} = z - \frac{1}{2}z^{-1} + \mathcal{O}(z^{-3}) \tag{3}
$$

and by the Residue theorem we have

$$
-\frac{1}{2\mathrm{i}} \oint_C \sqrt{z-1}\sqrt{z+1}\,\mathrm{d}z = -\pi\big(\mathrm{Res}(f(z);\infty)\big).
$$

Hence,

$$
\begin{aligned}
I_2 &= -\frac{1}{2\mathrm{i}} \oint_C \sqrt{z-1}\sqrt{z+1}\,\mathrm{d}z \\
&= -\pi\left(-\frac{1}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}
$$

Therefore,

$$
I_2 = \int_{-1}^{1} \sqrt{1-x}\sqrt{1+x}\,\mathrm{d}x = \frac{\pi}{2}
$$

$\square$

Notice we can rewrite the integrand from I_3 as follows

$$
\frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{\sqrt{1-x}\sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}} = \frac{1-x}{\sqrt{1-x}\sqrt{1+x}}.
$$

Then we can evaluate I_3 using the tools established in part (a)

$$\begin{aligned}
I_3 &= \int_{-1}^{1} \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1-x}{\sqrt{1-x}\sqrt{1+x}}\,\mathrm{d}x \\
&= \frac{1}{2\mathrm{i}} \oint_C \frac{1-z\,\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= -\frac{1}{2\mathrm{i}} \oint_C \frac{z-1\,\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= -\frac{1}{2\mathrm{i}} \oint_C \frac{\sqrt{z-1}}{\sqrt{z+1}}\,\mathrm{d}z
\end{aligned}$$

The Taylor expansion of our integrand centered at infinity is

$$\frac{1}{H(z)} = \frac{1}{\frac{\sqrt{\frac{1}{z}+1}}{\sqrt{\frac{1}{z}-1}}} = \frac{\sqrt{1-z}}{\sqrt{1+z}} = 1 - \frac{1}{z} + \mathcal{O}(z^{-2}).$$

By the Residue theorem we have

$$\begin{aligned}
I_4 &= -\frac{1}{2\mathrm{i}} \oint_C \frac{\sqrt{z-1}}{\sqrt{z+1}}\,\mathrm{d}z \\
&= -\pi\big(\mathrm{Res}(1/H(z);\infty)\big) \\
&= -\pi\big(-1\big) \\
&= \pi.
\end{aligned}$$

□

Notice we can rewrite the integrand from I_4 as follows

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x}\sqrt{1+x}} = \frac{1+x}{\sqrt{1-x}\sqrt{1+x}}.$$

Then we can evaluate I_4 using the tools established in part (a)

$$\begin{aligned}
I_4 &= \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1+x}{\sqrt{1-x}\sqrt{1+x}}\,\mathrm{d}x \\
&= \frac{1}{2\mathrm{i}} \oint_C \frac{1+z\,\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} \\
&= \frac{1}{2\mathrm{i}} \oint_C \frac{\sqrt{z+1}}{\sqrt{z-1}}\,\mathrm{d}z.
\end{aligned}$$

The Taylor expansion of our integrand centered at infinity is

$$h(z) = \frac{\sqrt{z+1}}{\sqrt{z-1}} = 1 + \frac{1}{z} + \frac{z^{-2}}{2} + \frac{z^{-3}}{2} + \mathcal{O}(z^{-4}).$$

By the Residue theorem we have

$$
\begin{aligned}
I_4 &= \frac{1}{2\mathrm{i}} \oint_C \frac{\sqrt{z+1}}{\sqrt{z-1}}\,\mathrm{d}z \\
&= \pi\big(\mathrm{Res}(h(z); \infty)\big) \\
&= \pi\big(1\big) \\
&= \pi.
\end{aligned}
$$

$\square$

5: Suppose, for $|z| = 1$, that the series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n,$$

converges uniformly.

(a) Compute series representations for

$$F(z) := \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \quad |z| \neq 1, \quad C = \partial B_1(0).$$

Solution:
Jumping right in we can calculate this as follows

$$\begin{aligned}
F(z) &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \\
&= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\sum_{n=-\infty}^{\infty} f_n \xi^n}{\xi - z} \mathrm{d}\xi \\
&= \frac{1}{2\pi \mathrm{i}} \sum_{n=-\infty}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi \\
&= \frac{1}{2\pi \mathrm{i}} \left[\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi + \sum_{n=0}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi \right].
\end{aligned}$$

We can now evaluate this using the two cases when $|z| > 1$ and when $|z| < 1$. Beginning first when $|z| > 1$ we have

$$F(z) = \frac{1}{2\pi \mathrm{i}} \left[\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi + \sum_{n=0}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi \right] \tag{4}$$

Now let's look at solving the term on the left where $n \leq -1$,

$$\begin{aligned}
\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi &= \sum_{n=-\infty}^{-1} f_n \oint_C \frac{\xi^n}{\xi - z} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} f_n \oint_C -\frac{\xi^n}{z} \frac{1}{1 - \frac{\xi}{z}} \mathrm{d}\xi
\end{aligned}$$

On our contour around the unit circle, $|\xi| = 1$. Therefore, $|\xi/z| < 1$, since $|z| > 1$. Thus we can rewrite this using the geometric series

$$\begin{aligned}
&= \sum_{n=-\infty}^{-1} f_n \oint_C -\frac{\xi^n}{z} \sum_{\ell=0}^{\infty} \frac{\xi^\ell}{z^\ell} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} -f_n \oint_C \sum_{\ell=0}^{\infty} \frac{\xi^\ell \xi^n}{z^\ell z} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} -f_n \oint_C \sum_{\ell=0}^{\infty} \frac{\xi^{\ell+n}}{z^{\ell+1}} \mathrm{d}\xi.
\end{aligned}$$

Notice that by the residue theorem, and considering that $n \leq -1$, we only have $\ell + n = -1$ when $\ell = -n-1$. Therefore,

$$\oint_C \sum_{\ell=0}^{\infty} \frac{\xi^{\ell+n}}{z^{\ell+1}} \mathrm{d}\xi = 2\pi\mathrm{i}\ \mathrm{Res}\Big(\sum_{\ell=0}^{\infty} \frac{\xi^{\ell+n}}{z^{\ell+1}}; \infty\Big) = 2\pi\mathrm{i}\, \frac{1}{z^{-n-1+1}} = 2\pi\mathrm{i}\, z^n.$$

Hence,

$$\begin{aligned}
\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi &= \sum_{n=-\infty}^{-1} -f_n \oint_C \sum_{\ell=0}^{\infty} \frac{\xi^{\ell+n}}{z^{\ell+1}} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} -f_n 2\pi\mathrm{i}\, z^n
\end{aligned}$$

Next, let's look at solving the term on the right where $n \geq 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi &= \sum_{n=0}^{\infty} f_n \oint_C \frac{\xi^n}{\xi - z} \mathrm{d}\xi \quad \text{maybe } \to \sum_{n=0}^{\infty} f_n 2\pi\mathrm{i}\, z^n \\
&= \sum_{n=0}^{\infty} f_n \oint_C -\frac{\xi^n}{z} \frac{1}{1 - \frac{\xi}{z}} \mathrm{d}\xi \\
&= \sum_{n=0}^{\infty} f_n \oint_C -\frac{\xi^n}{z} \sum_{\ell=0}^{\infty} \frac{\xi^\ell}{z^\ell} \mathrm{d}\xi \\
&= \sum_{n=0}^{\infty} -f_n \oint_C \sum_{\ell=0}^{\infty} \frac{\xi^{\ell+n}}{z^{\ell+1}} \mathrm{d}\xi \\
&= \sum_{n=0}^{\infty} -f_n \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \oint_C \xi^{\ell+n} \mathrm{d}\xi
\end{aligned}$$

substitute $\xi = \mathrm{e}^{\mathrm{i}\theta}$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} -f_n \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \int_0^{2\pi} \left(\mathrm{e}^{\mathrm{i}\theta}\right)^{\ell+n} \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta \\
&= \sum_{n=0}^{\infty} -f_n \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \left(\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{\ell+n+1} \Big|_0^{2\pi} \right) \\
&= \sum_{n=0}^{\infty} -f_n \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \left(\left(\mathrm{e}^{\mathrm{i}2\pi}\right)^{\ell+n+1} - \left(\mathrm{e}^{\mathrm{i}0}\right)^{\ell+n+1} \right) \\
&= \sum_{n=0}^{\infty} -f_n \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} (1 - 1) \\
&= \sum_{n=0}^{\infty} -f_n \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} 0 \\
&= 0
\end{aligned}$$

Now combining this and our previous result into equation (4) we have

$$
\begin{aligned}
F(z) &= \frac{1}{2\pi \mathrm{i}} \left[\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi + \sum_{n=0}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi \right] \\
&= \frac{1}{2\pi \mathrm{i}} \left[\sum_{n=-\infty}^{-1} -f_n 2\pi \mathrm{i}\, z^n \right] \\
&= \sum_{n=-\infty}^{-1} -f_n\, z^n
\end{aligned}
$$

This is the result for when $|z| > 1$. Continuing on with $|z| < 1$ we have the following Now let's look at solving the term on the left where $n \leq -1$,

$$
\begin{aligned}
\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi &= \sum_{n=-\infty}^{-1} f_n \oint_C \frac{\xi^n}{\xi - z} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} f_n \oint_C \frac{\xi^n}{\xi} \frac{1}{1 - \frac{z}{\xi}} \mathrm{d}\xi
\end{aligned}
$$

On our contour around the unit circle, $|\xi| = 1$. Therefore, $|z/\xi| < 1$, since $|z| < 1$. Thus we can rewrite this using the geometric series

$$
\begin{aligned}
&= \sum_{n=-\infty}^{-1} f_n \oint_C \frac{\xi^n}{\xi} \sum_{\ell=0}^{\infty} \frac{z^\ell}{\xi^\ell} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} f_n \oint_C \sum_{\ell=0}^{\infty} \frac{z^\ell \xi^n}{\xi^{\ell+1}} \mathrm{d}\xi
\end{aligned}
$$

Notice that by the residue theorem, and considering that $n \leq -1$, we only have a ξ^{-1} when $\ell = n$. Therefore,

$$
\oint_C \sum_{\ell=0}^{\infty} \frac{z^\ell \xi^n}{\xi^{\ell+1}} \mathrm{d}\xi = 2\pi \mathrm{i}\ \mathrm{Res}\Big(\sum_{\ell=0}^{\infty} \frac{z^\ell \xi^n}{\xi^{\ell+1}} ; 0 \Big) = 0.
$$

Hence,

$$
\begin{aligned}
\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi &= \sum_{n=-\infty}^{-1} f_n \oint_C \sum_{\ell=0}^{\infty} \frac{z^\ell \xi^n}{\xi^{\ell+1}} \mathrm{d}\xi \\
&= \sum_{n=-\infty}^{-1} f_n 0 \\
&= 0.
\end{aligned}
$$

Now for the case when $n \geq 0$ we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi &= \sum_{n=0}^{\infty} f_n \oint_C \frac{\xi^n}{\xi - z} \mathrm{d}\xi \\
&= \sum_{n=0}^{\infty} f_n 2\pi \mathrm{i}\, z^n
\end{aligned}
$$

by Cauchy's Integral formula. Now combining this and our previous result into equation (4) we have

$$\begin{aligned} F(z) &= \frac{1}{2\pi \mathrm{i}} \left[\sum_{n=-\infty}^{-1} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi + \sum_{n=0}^{\infty} \oint_C \frac{f_n \xi^n}{\xi - z} \mathrm{d}\xi \right] \\ &= \frac{1}{2\pi \mathrm{i}} \left[\sum_{n=0}^{\infty} f_n 2\pi \mathrm{i}\, z^n \right] \\ &= \sum_{n=0}^{\infty} f_n\, z^n \end{aligned}$$

This is the result for when $|z| < 1$.

$\square$

(b) For $|z| = 1$, compute

$$\lim_{\epsilon \to 0^+} F(z(1-\epsilon)) - \lim_{\epsilon \to 0^+} F(z(1+\epsilon)).$$

Solution:

TODO: should expect a "jump" discontinuity at the boundary Use the series representation you arrived at from the previous problem to evaluate this.

$$This$$