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AMATH 567

HOMEWORK 5

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.6.5
Consider two entire functions with no zeroes and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.
Solution:
Let's define our two entire functions to be $f(z)$ and $g(z)$. Recall that an entire function is analytic in all of the complex plane. We can focus on the ratio between these two functions $\frac{f(z)}{g(z)}$ since we are also given that $f(z)$ and $g(z)$ have no zeros. Let $h(z)$ be the ratio between f and g

$$h(z) = \frac{f(z)}{g(z)}.$$

If we can use Liouville's theorem to show that $h(z)$ is constant, then $f(z)$ and $g(z)$ are equal everywhere and are thus the same function.

For reference, Liouville's Theorem states that if $f(z)$ is entire and bounded in the z plane (including infinity), then $f(z)$ is a constant. Hence we need to show that $h(z)$ is entire and bounded in the z plane, then $h(z)$ is constant and we will have what we want. We know that the functions $f(z)$ and $g(z)$ are entire. We also know that the function $\frac{1}{z}$ is analytic except when $z = 0$. Since neither f nor g have zeros, then the potential of having 0 in the denominator of $h(z)$ is no longer an issue. Therefore $\frac{1}{z}$, $z \neq 0$ is entire. Therefore $h(z)$ is entire since it is the composition of entire functions.

Now we need to show that $h(z)$ is bounded in the z plane. Since $h(z)$ is entire, then it is analytic interior to and on a simple closed contour C (which we will choose later), then by Theorem 2.6.2, we have

$$h^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} \mathrm{d}\xi.$$

Now we can use the established inequality (2.6.13 in A & F)

$$\left| h^{(n)}(z) \right| \leq \frac{n! M}{R^n}.$$

When $n = 1$ we have

$$|h'(z)| \leq \frac{M}{R}.$$

We can take R to be arbitrarily large to get $|h'(z)| \leq 0$ implying $h'(z) = 0$. Using the fundamental theorem of calculus we can write

$$h(\infty) - h(z) = \int_z^\infty h'(z)\mathrm{d}z = C|_z^\infty = C - C = 0.$$

This gives $h(\infty) = h(z)$, therefore, by Liouville's Theorem $h(z)$ is constant. From the problem's setup we know $h(\infty) = \frac{f(\infty)}{g(\infty)} = 1$. Hence,

$$h(\infty) = h(z) = 1.$$

Therefore, $f(z)$ and $g(z)$ must be the same function, since their ratio is 1 for all z.

$\square$

2: From A&F: 2.6.10 (The *solution* is peppered throught, since there are many things to show and information given between many steps.)
In Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z}\mathrm{d}\xi$$

take the contour to be a circle of unit radius centered at the origin. Let $\xi = \mathrm{e}^{\mathrm{i}\theta}$. We now can plug the substitution in, along with the $\mathrm{d}\xi = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$ to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z}\mathrm{d}\xi \\ &= \frac{1}{2\pi\cancel{\mathrm{i}}} \int_0^{2\pi} \frac{f(\xi)\cancel{\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\theta}}{\xi - z}\mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z}\mathrm{d}\theta. \end{aligned}$$

$\square$

Since z is inside the unit circle and $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, then $r < 1$. Then we have

$$\frac{1}{\bar{z}} = \frac{1}{r\,\mathrm{e}^{-\mathrm{i}\theta}} = \frac{1}{r}\mathrm{e}^{\mathrm{i}\theta}.$$

Then $\frac{1}{r} > 1$, hence $\frac{1}{\bar{z}}$ is outside the unit circle. Therefore plugging in $\frac{1}{\bar{z}}$ to Cauchy's Formula from the beginning again we have

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(\xi)}{\xi - \frac{1}{\bar{z}}}\mathrm{d}\xi &= 0 \\ \frac{1}{2\pi\cancel{\mathrm{i}}} \int_0^{2\pi} \frac{f(\xi)\cancel{\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\theta}}{\xi - \frac{1}{\bar{z}}}\mathrm{d}\theta &= 0 \\ \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - \frac{1}{\bar{z}}}\mathrm{d}\theta &= 0. \end{aligned}$$

$\square$

Notice,

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z}\mathrm{d}\theta \mp 0 \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z}\mathrm{d}\theta \mp \frac{1}{2\pi}\int_0^{2\pi} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{\xi}{\xi - \frac{1}{\bar{z}}}\right)\mathrm{d}\theta.
\end{aligned}
$$

Now we can use $\xi = 1/\bar{\xi}$ to get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{1/\bar{\xi}}{1/\bar{\xi} - \frac{1}{\bar{z}}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{\bar{\xi}\bar{z}}{\bar{\xi}\bar{z}}\frac{1/\bar{\xi}}{1/\bar{\xi} - \frac{1}{\bar{z}}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{\bar{z}}{\bar{z} - \bar{\xi}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \pm \frac{\bar{z}}{\bar{\xi} - \bar{z}}\right)\mathrm{d}\theta.
\end{aligned}
$$

$\square$

Using the plus sign we see

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} + \frac{\bar{z}}{\bar{\xi} - \bar{z}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)} + \frac{\bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z}) + \bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi\bar{\xi}\cancel{-\xi\bar{z} + \xi\bar{z}} - z\bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - |z|^2}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - |z|^2}{\overline{(\xi - z)}(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\,\mathrm{e}^{\mathrm{i}\theta}\,|^2 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1^2 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta.
\end{aligned}
$$

$\square$

(a) Deduce the "Poisson formula" for the real part of $f(z) : u(r,\phi) = \Re f, z = r\,\mathrm{e}^{i\phi}$.

$$\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta)+\mathrm{i}v(\theta))\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)+\mathrm{i}v(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{i}v(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta + \mathrm{i}\frac{1}{2\pi}\int_0^{2\pi} v(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta
\end{aligned}$$

Thus,

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta.$$

We know $|z| = \left|r\,\mathrm{e}^{i\phi}\right| = r$. Now let's look specifically at the denominator and plugin the substitutions for z to get

$$\begin{aligned}
|\xi-z|^2 &= |\,\mathrm{e}^{\mathrm{i}\theta} - r\,\mathrm{e}^{\mathrm{i}\phi}\,|^2 \\
&= |\cos\theta + \mathrm{i}\sin\theta - r\cos\phi - \mathrm{i}r\sin\phi|^2 \\
&= |\cos\theta - r\cos\phi + \mathrm{i}\,(\sin\theta - r\sin\phi)\,|^2 \\
&= (\cos\theta - r\cos\phi)^2 + (\sin\theta - r\sin\phi)^2 \\
&= \cos^2\theta - 2r\cos\theta\cos\phi + r^2\cos^2\phi + \sin^2\theta - 2\sin\theta r\sin\phi + r^2\sin^2\phi \\
&= \cos^2\theta + \sin^2\theta - 2r\cos\theta\cos\phi - 2\sin\theta r\sin\phi + r^2\cos^2\phi + r^2\sin^2\phi \\
&= 1 - 2r\cos\theta\cos\phi - 2\sin\theta r\sin\phi + r^2\left(\cos^2\phi + \sin^2\phi\right) \\
&= 1 - 2r\cos\theta\cos\phi - 2\sin\theta r\sin\phi + r^2 \\
&= 1 - 2r\,(\cos\theta\cos\phi + \sin\theta\sin\phi) + r^2 \\
&= 1 - 2r\cos(\theta-\phi) + r^2.
\end{aligned}$$

Hence,

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\mathrm{d}\theta$$

□

(b) If we use the minus sign in the formula for $f(z)$ above, we get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} - \frac{\bar{z}}{\bar{\xi} - \bar{z}}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)} - \frac{\bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z}) - \bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi\bar{\xi} - 2\xi\bar{z} + z\bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - 2\xi\bar{z} + |z|^2)}{\overline{(\xi - z)}(\xi - z)}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1 - 2\xi\bar{z} + r^2}{|\xi - z|^2}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2\xi\bar{z} + r^2}{1 - 2r\cos(\theta - \phi) + r^2} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2\,\mathrm{e}^{\mathrm{i}\theta}\, r\,\mathrm{e}^{-\mathrm{i}\phi} + r^2}{1 - 2r\cos(\theta - \phi) + r^2} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2r\,\mathrm{e}^{\mathrm{i}\theta - \mathrm{i}\phi} + r^2}{1 - 2r\cos(\theta - \phi) + r^2} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2r\,\mathrm{e}^{\mathrm{i}(\theta - \phi)} + r^2}{1 - 2r\cos(\theta - \phi) + r^2} \mathrm{d}\theta
\end{aligned}
$$

□

Additionally, if we take the imaginary part this time, we can see

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2r\,\mathrm{e}^{\mathrm{i}(\theta - \phi)} + r^2}{1 - 2r\cos(\theta - \phi) + r^2} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(\frac{1 - 2r\cos(\theta - \phi) - 2r\mathrm{i}\sin(\theta - \phi) + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(\frac{1 - 2r\cos(\theta - \phi) + r^2 - 2r\mathrm{i}\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(1 + \frac{-2r\mathrm{i}\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(1 - \frac{\mathrm{i}2r\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2}\right) \mathrm{d}\theta
\end{aligned}
$$

Let's expand out the terms in the integrand

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[u(\theta) - u(\theta)\left(\frac{\mathrm{i}2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right) + \mathrm{i}v(\theta) - \mathrm{i}v(\theta)\left(\frac{\mathrm{i}2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\right]\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[u(\theta) - \mathrm{i}u(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right) + \mathrm{i}v(\theta) - \mathrm{i}^2v(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\right]\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[u(\theta) - \mathrm{i}u(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right) + \mathrm{i}v(\theta) + v(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\right]\mathrm{d}\theta.$$

Now the imaginary part of this is

$$\begin{aligned}
\Im(f(z)) &= \frac{1}{2\pi}\int_0^{2\pi}\left(v(\theta) - u(\theta)\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\mathrm{d}\theta - \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\mathrm{d}\theta \\
&= C - \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\mathrm{d}\theta \\
&= C - \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin\big(-(-\theta+\phi)\big)}{1-2r\cos\big(-(-\theta+\phi)\big)+r^2}\mathrm{d}\theta \\
&= C - \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{-r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta \\
&= C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta
\end{aligned}$$

where $C = \frac{1}{2\pi}\int_0^{2\pi} v(1,\theta)\mathrm{d}\theta = v(r=0)$. This last relationship follows from the Cauchy Integral formula at z = 0 – see the first equation in this exercise). Hence,

$$v(r,\phi) = C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta))}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta$$

□

(c) We wish to show

$$\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2} = \Im\left(\frac{\xi+z}{\xi-z}\right)$$

Let's try a bit

$$\begin{aligned}
\frac{\xi+z}{\xi-z} &= \frac{\overline{(\xi-z)}(\xi+z)}{\overline{(\xi-z)}(\xi-z)} \\
&= \frac{\overline{(\xi-z)}(\xi+z)}{|\xi-z|^2}.
\end{aligned}$$

We have already computed this denominator once. Using our previous result we continue

$$\begin{aligned}
\frac{\overline{(\xi - z)}(\xi + z)}{|\xi - z|^2} &= \frac{(\bar{\xi} - \bar{z})(\xi + z)}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{\bar{\xi}\xi + \bar{\xi}z - \bar{z}\xi - \bar{z}z}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{|\xi|^2 + \bar{\xi}z - \bar{z}\xi - |z|^2}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 + \bar{\xi}z - \bar{z}\xi - r^2}{1 - 2r\cos(\phi - \theta) + r^2}.
\end{aligned}$$

Now, lets plugin our parameterizations of ξ and z

$$\begin{aligned}
&= \frac{1 - r^2 + \mathrm{e}^{-\mathrm{i}\theta}\, r\, \mathrm{e}^{\mathrm{i}\phi} - r\, \mathrm{e}^{-\mathrm{i}\phi}\, \mathrm{e}^{\mathrm{i}\theta}}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + r\, \mathrm{e}^{\mathrm{i}(\phi - \theta)} - r\, \mathrm{e}^{\mathrm{i}(\theta - \phi)}}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + r(\cos(\phi - \theta) + \mathrm{i}\sin(\phi - \theta)) - r(\cos(\theta - \phi) + \mathrm{i}\sin(\theta - \phi))}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + r\cos(\phi - \theta) - r\cos(\theta - \phi) + \mathrm{i}r\sin(\phi - \theta) - \mathrm{i}r\sin(\theta - \phi)}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + \cancel{r\cos(\phi - \theta)} - \cancel{r\cos(\phi - \theta)} + \mathrm{i}r\sin(\phi - \theta) + \mathrm{i}r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + \mathrm{i}2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2}.
\end{aligned}$$

We have arrived to

$$\begin{aligned}
\frac{\xi + z}{\xi - z} &= \frac{1 - r^2 + \mathrm{i}2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2} \\
\Im\left[\frac{\xi + z}{\xi - z}\right] &= \Im\left[\frac{1 - r^2 + \mathrm{i}2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2}\right] \\
&= \frac{2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
v(r, \phi) &= C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi - \theta))}{1 - 2r\cos(\phi - \theta) + r^2}\mathrm{d}\theta \\
&= C + \frac{\Im}{2\pi}\int_0^{2\pi} u(\theta)\frac{\xi + z}{\xi - z}\mathrm{d}\theta
\end{aligned}$$

□

This example illustrates that prescribing the real part of $f(z)$ on $|z| = 1$ determines (a) the real part of $f(z)$ everywhere inside the circle and (b) the imaginary part of $f(z)$ inside the circle to within a constant. We *cannot* arbitrarily specify both the real and imaginary parts of an analytic function on $|z| = 1$.

3: Suppose Ω is an open simply connected region and $z_0 \in \Omega$. Assume that $f(z)$ is analytic in $\Omega \setminus \{z_0\}$ and satisfies

$$|f(z)| \leq M|z - z_0|^{-\gamma}, \quad \gamma < 1.$$

Show that if the a specific choice for $f(z_0)$ is made then f extends to an analytic function on Ω.
(**1 part**, except maybe if there are multiple things to prove here)
Solution:
Let's choose the value for $f(z_0)$ to be

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z_0} \mathrm{d}\xi.$$

Now we wish to show that $f(z)$ is analytic on Ω after making this choice. Since $f(z)$ is already analytic away from z_0, it suffices to just verify that it is now analytic in a neighborhood around z_0. I will demonstrate this by showing $f(z)$ satisfies the Cauchy Integral Formula.

$$|f(z)| \leq M|z - z_0|^{-\gamma}, \quad \gamma < 1$$

...
Since $f(z)$ is analytic we know

(a) $f(z)$ satisfies the Cauchy Riemann (C-R) equations.
(b) then on a contour $C \subset \Omega$ f(z) satisfies

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^{k+1}} \mathrm{d}\xi$$

Do I need to use the Maximum Principles??
Should I make a choice of $f(z_0)$ and prove that f with that condition is analytic or should I just prove that one exists?

4: Establish the following lemma:

> **Lemma 1**
>
> Suppose Ω is an open region and $f(z)$ is continuous on $\overline{\Omega}$. Let Γ be a contour in $\overline{\Omega}$. Suppose a sequence of contours $\Gamma_n \subset \overline{\Omega}$ converge to Γ in the sense that there exists parameterizations $z(t)$ of Γ and $z_n(t)$ of Γ_n defined on $[a, b]$ satisfying
>
> $$z_n(t) \overset{n\to\infty}{\longrightarrow} z(t), \quad \text{uniformly on } [a, b],$$
> $$z_n'(t) \overset{n\to\infty}{\longrightarrow} z'(t), \quad \text{uniformly on } [a, b].$$
>
> Then
>
> $$\int_{\Gamma_n} f(z)\mathrm{d}z \overset{n\to\infty}{\longrightarrow} \int_\Gamma f(z)\mathrm{d}z.$$

Hint: Use that f is uniformly continuous on $\overline{\Omega}$.
(**1 part**, except maybe if there are multiple things to prove here)
Solution:
We want to show that

$$\lim_{n\to\infty} \int_{\Gamma^n} f(z)\mathrm{d}z = \int_\Gamma f(z)\mathrm{d}z.$$

In order to demonstrate this, we need to find that there exists an N for every $\epsilon > 0$ such that when $n > N$ we have

$$\left| \int_{\Gamma^n} f(z)\mathrm{d}z - \int_\Gamma f(z)\mathrm{d}z \right| < \epsilon.$$

Notice we can write each of these integrals using the appropriate parameterization

$$\int_{\Gamma^n} f(z)\mathrm{d}z = \int_a^b f(z_n(t))z_n'(t)\mathrm{d}t$$
$$\int_\Gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

Since f is uniformly continuous on $\overline{\Omega}$, then $f(z_n(t)) \overset{n\to\infty}{\longrightarrow} f(z(t))$. Since, we have the parameterizations $z(t)$ of Γ and $z_n(t)$ of Γ_n such that

$$\lim_{n\to\infty} z_n(t) = z(t)$$
$$\lim_{n\to\infty} z_n'(t) = z'(t)$$

Does the convergence of these sequences of functions imply the convergence of their integrals of [a, b]? Is there an established real analysis theorem I am missing?

5: for any $r, R > 0$, let $C = \partial\Sigma$, $\Sigma = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \le r \text{ and } 0 \le -\operatorname{Im} z \le R,\ R > 0\}$. In this problem $\sqrt{z}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$.

- Show that if $f(z)$ is analytic in a region that contains Σ,

$$\oint_C f(z)\sqrt{z-1}\sqrt{z+1}\mathrm{d}z = 0.$$

(**1 part**)
Solution:
Assume $f(z)$ is an analytic function in a region that contains Σ. Furthermore, let $g(z) = \sqrt{z-1}\sqrt{z+1}$. Using the principal branch for $g(z)$, we also have that $g(z)$

is analytic as well away from it's branch cut. Then we are looking at a contour integral of an analytic function (in our region) and therefore

$$\oint_C f(z)\sqrt{z-1}\sqrt{z+1}\mathrm{d}z = 0$$

by Cauchy's theorem. $\square$

- Show that if $f(z)$ is analytic in a region that contains Σ

$$\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} = 0.$$

(**1 part**)
Solution:
Deal with the singularities on the boundary in some clever way...
perhaps by having a section of the contour be taken to be the limit of a sequence of contours approaching the real axis.

6: From A&F: 3.1.1 b,d
In the following we are given sequences. Discuss their limits and whether the convergence is uniform, in the region $\alpha \leq |z| \leq \beta$, for finite $\alpha, \beta > 0$.
b)

$$\left\{\frac{1}{z^n}\right\}_{n=1}^{\infty}$$

(**2 parts**)
Solution:
This one is weird, seems like it's divergent since the limit is ∞ when $|z| < 1$ but if $|z| > 1$ the limit is 0, it's also 1 if $|z| = 1$. Consider following the outline for example 3.1.6 on page 110 in A & F

d)

$$\left\{\frac{1}{1+(nz)^2}\right\}_{n=1}^{\infty}$$

(**2 parts**)
Solution:
The limit of this function is 0. The sequence converges uniformly (maybe need to show this).

$$\lim_{n\to\infty} \frac{1}{1+(nz)^2} = \lim_{n\to\infty} \frac{1}{1+n^2z^2} = 0$$

I'm not sure that there is anything to think about with the limits α and β.

7: From A&F: 3.1.2 b,d
For each sequence in problem 1, what can be said if
(a) $\alpha = 0$
(b) $\alpha > 0, \quad \beta = \infty$
(**4 parts** 2x2) *Solution:*

8: From A&F: 3.1.3 Compute the integrals

$$\lim_{n\to\infty} \int_0^1 nz^{n-1}\mathrm{d}z \quad \text{and} \quad \int_0^1 \lim_{n\to\infty} \left(nz^{n-1}\right)\mathrm{d}z$$

and show that they are not equal. Explain why this is not a counter example to Theorem 3.1.1. (A &F pg. 111)
(**3 parts**) *Solution:*
For the integral on the right consider adding a limit outside the integral so the bound doesn't have any issues. We can easily evaluate the left limit to be

$$\lim_{n\to\infty}\int_0^1 nz^{n-1}\mathrm{d}z = \lim_{n\to\infty} z^n\big|_0^1 = \lim_{n\to\infty} 1^n - 0^n = \lim_{n\to\infty} 1 = 1.$$

However, on the right we have (using a limit as we approach the upper bound 1)

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon} \lim_{n\to\infty}\left(nz^{n-1}\right)\mathrm{d}z.$$

There are approximately 25 things to do, 10 down!