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AMATH 561

PROBLEM SET 2

1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.
Solution:
We need to show that Z is a random variable as it is defined. That is we need to show it is a function that maps from a sample space Ω to the real numbers and that for every Borel set $B \subset \mathbb{R}$ we have

$$Z^{-1}(B) = \{\, \omega \,|Z(\omega) \in B\} \in \mathcal{F}.$$

Starting from knowing X and Y are random variables that means we have:

$$X : \Omega \to \mathbb{R}, \quad Y : \Omega \to \mathbb{R}.$$

Now rewriting Z a little more mathematically we have

$$Z(\omega) = \begin{cases} X(\omega), & \omega \in A, \\ Y(\omega), & \omega \in A^c. \end{cases}$$

Since $A \in \mathcal{F}$, every $\omega \in A$ must also be in Ω since $\mathcal{F}$ is made up of subsets of Ω which means $A \subseteq \Omega$ and thus $A^c \subseteq \Omega$ as well. By definition of the compliment $A \cap A^c = \emptyset$. Therefore A and A^c are a partition on Ω. Since Z is defined on $\omega \in A$ or $\omega \in A^C$ then Z is defined on all of Ω. Now we have shown that the domain of Z is Ω. Additionally, since X and Y each map from Ω to $\mathbb{R}$, Z must also map to $\mathbb{R}$ since it's output is determined by the output of X and Y. Therefore Z is function such that $Z : \Omega \to \mathbb{R}$.

Now we begin the argument that $Z^{-1}(B) = \{\omega | X(\omega) \in B\} \in \mathcal{F}$. First, since X and Y are random variables on our probability space we have that for every Borel set B

$$X^{-1}(B) = \{\, \omega \,|X(\omega) \in B\} \in \mathcal{F}$$

and

$$Y^{-1}(B) = \{\, \omega \,|Y(\omega) \in B\} \in \mathcal{F}.$$

Now it is important to observe that the $Z{-}1(B)$ is going to be some combination of the $X^{-1}(B)$ and $Y^{-1}(B)$. Let's take for example some $\omega^* \in A \subset \Omega$, then $Z(\omega^*) = X(\omega^*) = c$ for some constant $c \in \mathbb{R}$. Then if $c \in B$ then $\omega^* \in X^{-1}(B)$

and thus $\omega^* \in Z^{-1}(B)$. Therefore part of $Z^{-1}(B)$ can be written as

$$A \cap X^{-1}(B).$$

Additionally, we can also write part of $Z^{-1}(B)$ as

$$A^c \cap Y^{-1}(B).$$

Since A and A^c are a partition on Ω we know $A^c \cap Y^{-1}(B)$ and $A \cap X^{-1}(B)$ are disjoint. And they actually contain all of $Z^{-1}(B)$ since Z is only defined by X and Y in each of those scenarios respecting $\omega \in A$ or $\omega \in A^c$. Therefore

$$Z^{-1}(B) = \big(A \cap X^{-1}(B)\big) \cup \big(A^c \cap Y^{-1}(B)\big)$$

Now we need to finally demonstrate that $Z^{-1}(B) \in \mathcal{F}$. Recall we are given that $A \in \mathcal{F}$, and since X is a R.V. then $X^{-1}(B) \in \mathcal{F}$ therefore

$$A \cap X^{-1}(B) \in \mathcal{F}.$$

By a σ-algebra being closed under compliments we know $A^c \in \mathcal{F}$ and similar to X since Y is a R.V. then $Y^{-1}(B) \in \mathcal{F}$, therefore

$$A^c \cap Y^{-1}(B) \in \mathcal{F}.$$

And lastly the countable union of elements of $\mathcal{F}$ is therefore also in $\mathcal{F}$ hence

$$Z^{-1}(B) = \big(A \cap X^{-1}(B)\big) \cup \big(A^c \cap Y^{-1}(B)\big) \in \mathcal{F}.$$

And thus Z is a random variable on the probability space $(\Omega, \mathcal{F}, P)$. $\square$

2. Suppose X is a continuous random variable with distribution function F_X. Let g be a strictly increasing continuous function. Define $Y = g(X)$.

a) What is F_Y, the distribution function of Y?
Solution:
We know that there is some probability space that the random variable X is defined on, let that be $(\Omega, \mathcal{F}, P)$. Therefore $X : \Omega \to \mathbb{R}$ and since g is a strictly increasing continuous function $g : \mathbb{R} \to L$ where L is the output space of g, L could be $\mathbb{R}$ for example, then $g(X) : \Omega \to \mathbb{R}$ (we take $L = \mathbb{R}$ for now as the most likely assumption). Note that since $Y = g(X)$ then $Y : \Omega \to \mathbb{R}$ is also true. In order to construct F_Y we need to determine the relationship they have.

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Now we need to argue that g is invertible as we claim above.**TODO** $\square$

b) What is f_Y, the density function of Y?
Solution:
Since

$$F_Y(y) = \int_{-\infty}^{y} f_Y(x)\mathrm{d}x$$

we just need to differentiate F_Y as follows

$$\frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_X(g^{-1}(y)) = \frac{f_X(g^{-1}(y))}{g\prime(g^{-1}(y))}.$$

□

3. Suppose X is a continuous random variable with distribution function F_X. Find F_Y where Y is given by

a) X^2
Solution:
That is to say $Y = X^2$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

□

b) $\sqrt{|X|}$
Solution:
That is to say $Y = \sqrt{|X|}$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\sqrt{|X|} \leq y) \\ &= P(|X| \leq y^2) \\ &= P(-y^2 \leq X \leq y^2) \\ &= P(X \leq y^2) - P(X \leq -y^2) \\ &= F_X(y^2) - F_X(-y^2) \end{aligned}$$

□

c) $\sin X$ *Solution:*

That is to say $Y = \sin X$

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(\sin X \le y) \\
&= P(X \le \arcsin y) \\
&= \sum_{k \in \mathbb{Z}} P(\arcsin y + 2\pi k \le X \le \arcsin y + 2\pi(k+1)) \\
&= \sum_{k \in \mathbb{Z}} \left[P(X \le \arcsin y + 2\pi(k+1)) - P(X \le \arcsin y + 2\pi k) \right] \\
&= \sum_{k \in \mathbb{Z}} \left[F_X(\arcsin y + 2\pi(k+1)) - F_X(\arcsin y + 2\pi k) \right].
\end{aligned}$$

d) $F_X(X)$ *Solution:*
That is to say $Y = F_X(X)$

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(F_X(X) \le y) \\
&= P(X \le F_X^{-1}(y)) \\
&= F_X(F_X^{-1}(y)) \\
&= y
\end{aligned}$$

Now there is a bit more to be said to ensure we are covering all of our bases here as we try to invert the nondecreasing but not necessarily always increasing function $F_X(x)$. We need to discuss how we will take the

$$\sup_y \{\text{over the values in interval where } F_X(x) \text{ is constant}\}$$

Something like this...**TODO** □

4. Let $X : [0,1] \to \mathbf{R}$ be a function that maps every rational number in the interval $[0,1]$ to 0, and every irrational number to 1. We assume that the probability space where X is defined is $([0,1], \mathcal{B}[0,1], P)$, where $\mathcal{B}[0,1]$ is the Borel σ-algebra on [0,1], and P is the Lebesgue measure.

(a) Is the set of rational numbers in [0,1] a Borel set? Show using definition of the Borel σ-algebra on $[0,1]$.
Solution:
I will argue that yes the set of rational numbers in $[0,1]$ is a Borel set. We will construct the set of rational numbers in a way such a that it is a countable union of sets, which are themselves the countable intersection of open sets and thus we will have a Borel set. First note we can write any number $x \in [0,1]$ as

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap [0,1]$$

That is to say this countably infinity intersection of open sets is the singleton set $\{x\}$. Therefore we can also represent each of the rational numbers in $[0,1]$ in this way as well. We do have to be careful that when near the boundary of [0, 1] n has to be sufficiently large. Now we construct the set of all rationals in $[0, 1]$ as follows:

$$\mathbb{Q} \cap [0,1] = \bigcup_{q \in \mathbb{Q} \cap [0,1]}^{\infty} \{q\}.$$

Now we have that the rationals between 0 and 1, $\mathbb{Q} \cap [0,1]$, can be written in the form of a countably infinite union of sets which themselves are countably infinite intersections of open sets, which is a Borel set. Hence $\mathbb{Q} \cap [0,1]$ is a Borel set.

$\square$

(b) Is X a random variable (and why)? If it is, what are its distribution function and expectation? Does X have a density function? Is X discrete? **TODO**