Hunter Lybbert
Student ID: 2426454
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AMATH 561

PROBLEM SET 5

1. Let X and $Y_0, Y_1, Y_2, ...$ be random variables on a probability space $(\Omega, \mathcal{F}, P)$ and suppose $E|X| < \infty$. Define $\mathcal{F}_n = \sigma(Y_0, Y_1, ..., Y_n)$ and $X_n = E(X|\mathcal{F}_n)$. Show that the sequence $X_0, X_1, X_2, ...$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.
Solution:
Learn about the filtration... and how to show something satisfies the definition of a martingale.

2. Let $X_0, X_1, ...$ be i.i.d Bernoulli random variables with parameter p (i.e., $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $S_n = \sum_{i=1}^{n} X_i$ where $S_0 = 0$. Define

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}, \quad n = 0, 1, 2, ...$$

Let $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$. Show that Z_n is a martingale with respect to this filtration.

Solution:

Learn about the filtration... and how to show something satisfies the definition of a martingale.

3. Let ξ_i be a sequence of random variables such that the partial sums

$$X_n = \xi_0 + \xi_1 + ... + \xi_n, \quad n \geq 1,$$

determine a martingale. Show that the summands are mutually uncorrelated, i.e. that $E(\xi_i \xi_j) = E(\xi_i)E(\xi_j)$ for $i \neq j$.

Solution:

So we are given something that "determines" a martingale and we need to show that something is uncorrelated does this mean they are independent?

4. Galton and Watson who invented the process that bears their names were interested in the survival of family names. Suppose each family has exactly 3 children but coin flips determine their sex. In the 1800s, only male children kept the family name so following the male offspring leads to a branching process with $p_0 = 1/8, p_1 = 3/8, p_2 = 3/8, p_3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$. What is ρ if we assume that each family has exactly 2 children?

Solution:

Sompute the probability that they will die out... this seems related to using the generating function when $s = 0$ from the lecture on friday November 1^{st}.