

# AMATH 561 Autumn 2024

## Problem Set 1

Due: Mon 10/7 at 10am

*Note: Submit electronically to Canvas.*

**1.** Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.

a) *Solution*

Let's assign the probability of getting a heads when tossing this biased coin once as  $p_0$  and the probability of getting a tails as  $1 - p_0$ . Now to describe the probability space we need to define our outcome space  $\Omega$ , the set of relevant events  $\mathcal{F}$  (a  $\sigma$ -algebra) and the probability measure  $P$  then we will have our probability space  $(\Omega, \mathcal{F}, P)$ . In our experiment of 3 tosses of the coin we have

$$\Omega = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$$

And for simplicity let's define  $\mathcal{F} = 2^\Omega$  which means any possible subset of  $\Omega$  is an event. Now let's define  $P : \mathcal{F} \rightarrow \mathbb{R}$ . Let  $A \in \mathcal{F}$  then we say

$$P(A) = \sum_{\omega \in A} p(\omega).$$

Where  $p(\omega) = p_0^{h_\omega} (1 - p_0)^{t_\omega}$  with  $h_\omega$  being the number of heads  $H$  in outcome  $\omega$  and  $t_\omega$  being the number of tails  $T$  in outcome  $\omega$ . Therefore we can rewrite  $P(A)$  as follows:

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} p_0^{h_\omega} (1 - p_0)^{t_\omega}.$$

Let's calculate an example and see if it matches our intuition. Define  $B \in \mathcal{F}$  as the event such that at least one  $H$  occurs in the 3 tosses. Therefore

$B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$ . Now

$$\begin{aligned} P(B) &= \sum_{\omega \in B} p(\omega) = \sum_{\omega \in B} p_0^{h_\omega} (1 - p_0)^{t_\omega} \\ &= p_0^3 (1 - p_0)^0 + p_0^2 (1 - p_0)^1 + p_0^2 (1 - p_0)^1 + p_0^2 (1 - p_0)^1 \\ &\quad + p_0^1 (1 - p_0)^2 + p_0^1 (1 - p_0)^2 + p_0^1 (1 - p_0)^2. \end{aligned}$$

□

b) *Solution*

Now we analyze the experiment where two balls are drawn without replacement from an urn which originally contained two blue and two red balls. It might be helpful to denote ball one and two as  $u_1$  and  $u_2$  respectively. Let's begin by looking at the outcome space  $\Omega$ . In our two step experiment we get the following possible outcomes

$$\Omega = \{bb, br, rb, rr\}.$$

Once again, for simplicity let's define  $\mathcal{F} = 2^\Omega$  which means any possible subset of  $\Omega$  is an event. Now let's define  $P : \mathcal{F} \rightarrow \mathbb{R}$ . Let  $A \in \mathcal{F}$  then we say

$$P(A) = \sum_{\omega \in A} p(\omega)$$

again. Now  $p(\omega)$  is a little trickier to define since the probability for the second draw depends on what the first draw was. We have not described conditional probability in class yet, but I will borrow the notation and language of it here anyway. With this in mind we define

$$p(\omega) = p(\omega_1 \omega_2) = p(u_1 = \omega_1) p(u_2 = \omega_2 | u_1 = \omega_1)$$

with  $\omega$  being broken down in to  $\omega_1 \omega_2$  where  $\omega_1$  and  $\omega_2$  are the particular color ball being drawn first or second respectively in outcome  $\omega$ . We can define  $p(u_1 = \omega_1)$  and  $p(u_2 = \omega_2 | u_1 = \omega_1)$  more explicitly with the following rules.

$$p(u_1 = \omega_1) = \frac{1}{2} \quad \text{for } \omega \in \{b, r\}$$

since there are 2 of each color and 4 balls total. Additionally we determine

$$p(u_2 = \omega_2 | u_1 = \omega_1) = \frac{1}{3} \quad \text{if } \omega_2 = \omega_1$$

and

$$p(u_2 = \omega_2 | u_1 = \omega_1) = \frac{2}{3} \quad \text{if } \omega_2 \neq \omega_1$$

since at the time of the second ball draw there is 1 ball of one color and 2 balls of another color with 3 balls total remaining. Therefore we can rewrite  $P(A)$  as follows:

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} p(\omega_1 \omega_2) = \sum_{\omega \in A} p(u_1 = \omega_1) p(u_2 = \omega_2 | u_1 = \omega_1).$$

Let's calculate an example and see if it matches our intuition. Define  $B \in \mathcal{F}$  as the event where the second ball drawn is red  $u_2 = r$ . Therefore  $B = \{br, rr\}$ . Now

$$\begin{aligned} P(B) &= \sum_{\omega \in B} p(\omega) = \sum_{\omega \in B} p(\omega_1 \omega_2) = \sum_{\omega \in B} p(u_1 = \omega_1) p(u_2 = \omega_2 | u_1 = \omega_1) \\ &= p(u_1 = b) p(u_2 = r | u_1 = b) + p(u_1 = r) p(u_2 = r | u_1 = r) \\ &= \frac{1}{2} \frac{2}{3} + \frac{1}{2} \frac{1}{3} \\ &= \frac{1}{3} + \frac{1}{6} \\ &= \frac{1}{2}. \end{aligned}$$

Thus  $P(B) = P(\{br, rr\}) = \frac{1}{2}$  which is what we'd expect since this particular event  $B$  is comprised of half of the four possible outcomes in  $\Omega$ . □

**2.** (No translation-invariant random integer). Show that there is no probability measure  $P$  on the integers  $\mathbb{Z}$  with the discrete  $\sigma$ -algebra  $2^{\mathbb{Z}}$  with the translation-invariance property  $P(E + n) = P(E)$  for every event  $E \in 2^{\mathbb{Z}}$  and every integer  $n$ .  $E + n$  is obtained by adding  $n$  to every element of  $E$ .

*Solution*

Show by contradiction, eventually  $P(\Omega) = 0$  but it should be 1

**3.** (No translation-invariant random real). Show that there is no probability measure  $P$  on the reals  $\mathbb{R}$  with the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  with the translation-invariance property  $P(E + x) = P(E)$  for every event  $E \in \mathcal{B}(\mathbb{R})$  and every real  $x$ . Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  is the  $\sigma$ -algebra generated by intervals  $(a, b] \subset \mathbb{R}$ .

*Solution*

Also show by contradiction but it is a little trickier. It has to composing the right  $x$  s.t.  $E + x$  is outside the interval  $(a, b]$ .

**4.** Let  $\Omega = \mathbb{R}$ ,  $\mathcal{F} =$  all subsets of  $\mathbb{R}$  so that  $A$  or  $A^c$  is countable. Let  $P(A) = 0$  in the first case and  $P(A) = 1$  in the second. Show that  $(\Omega, \mathcal{F}, P)$

is a probability space. *Solution*

We want to prove something is a probability space. Therefore, we need to prove the following:

(1) prove fancy  $\mathcal{F}$  is a  $\sigma$ -algebra which includes that it contains the complements of each set and that it contains the union of two sets.

(2) prove cap P is a probability measure which includes proving  $P(\Omega) = 1$ ,  $P(A) \geq P(\emptyset) = 0 \quad \forall A \in \mathcal{F}$ , and that  $P(\cup_i A_i) = \sum_i P(A_i)$  for  $A_i \in \mathcal{F}$  that is a countable sequence of disjoint sets.