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AMATH 561

PROBLEM SET 6

1. Let $X \sim Binomial(n, U)$, where $U \sim Uniform((0,1))$. What is the probability generating function $G_X(s)$ of X? What is $P(X = k)$ for $k \in \{0, 1, 2, ..., n\}$?

Solution: This is it

2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ, the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

Solution:

3. (a) Let X be exponentially distributed with parameter λ. Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$. (b) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

Solution:

4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} \, 1_{x \geq 0}.$$

Solution: