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AMATH 567

HOMEWORK 5

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1: From A&F: 2.6.5
Consider two entire functions with no zeroes and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.
Solution:
Let's define our two entire functions to be $f(z)$ and $g(z)$. Recall that an entire function is analytic in all of the complex plane. We can focus on the ratio between these two functions $\frac{f(z)}{g(z)}$ since we are also given that $f(z)$ and $g(z)$ have no zeros. Let $h(z)$ be the ratio between f and g

$$h(z) = \frac{f(z)}{g(z)}.$$

If we can use Liouville's theorem to show that $h(z)$ is constant, then $f(z)$ and $g(z)$ are equal everywhere and are thus the same function.

For reference, Liouville's Theorem states that if $f(z)$ is entire and bounded in the z plane (including infinity), then $f(z)$ is a constant. Hence we need to show that $h(z)$ is entire and bounded in the z plane, then $h(z)$ is constant and we will have what we want. We know that the functions $f(z)$ and $g(z)$ are entire. We also know that the function $\frac{1}{z}$ is analytic except when $z = 0$. Since neither f nor g have zeros, then the potential of having 0 in the denominator of $h(z)$ is no longer an issue. Therefore $\frac{1}{z}$, $z \neq 0$ is entire. Therefore $h(z)$ is entire since it is the composition of entire functions.

Now we need to show that $h(z)$ is bounded in the z plane. Since $h(z)$ is entire, then it is analytic interior to and on a simple closed contour C (which we will choose later), then by Theorem 2.6.2, we have

$$h^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} \mathrm{d}\xi.$$

Now we can use the established inequality (2.6.13 in A & F)

$$\left| h^{(n)}(z) \right| \leq \frac{n! M}{R^n}.$$

When $n = 1$ we have

$$|h'(z)| \leq \frac{M}{R}.$$

We can take R to be arbitrarily large to get $|h'(z)| \leq 0$ implying $h'(z) = 0$. Using the fundamental theorem of calculus we can write

$$h(\infty) - h(z) = \int_z^\infty h'(z)\mathrm{d}z = C|_z^\infty = C - C = 0.$$

This gives $h(\infty) = h(z)$, therefore, by Liouville's Theorem $h(z)$ is constant. From the problem's setup we know $h(\infty) = \frac{f(\infty)}{g(\infty)} = 1$. Hence,

$$h(\infty) = h(z) = 1.$$

Therefore, $f(z)$ and $g(z)$ must be the same function, since their ratio is 1 for all z.

$\square$

2: From A&F: 2.6.10 (The *solution* is peppered throught, since there are many things to show and information given between many steps.)
In Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi$$

take the contour to be a circle of unit radius centered at the origin. Let $\xi = \mathrm{e}^{\mathrm{i}\theta}$. We now can plug the substitution in, along with the $\mathrm{d}\xi = \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta$ to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \\ &= \frac{1}{2\pi \cancel{\mathrm{i}}} \int_0^{2\pi} \frac{f(\xi) \cancel{\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\theta}}{\xi - z} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} \mathrm{d}\theta. \end{aligned}$$

□

Since z is inside the unit circle and $z = r\, \mathrm{e}^{\mathrm{i}\theta}$, then $r < 1$. Then we have

$$\frac{1}{\bar{z}} = \frac{1}{r\, \mathrm{e}^{-\mathrm{i}\theta}} = \frac{1}{r} \mathrm{e}^{\mathrm{i}\theta}.$$

Then $\frac{1}{r} > 1$, hence $\frac{1}{\bar{z}}$ is outside the unit circle. Therefore plugging in $\frac{1}{\bar{z}}$ to Cauchy's Formula from the beginning again we have

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - \frac{1}{\bar{z}}} \mathrm{d}\xi &= 0 \\ \frac{1}{2\pi \cancel{\mathrm{i}}} \int_0^{2\pi} \frac{f(\xi) \cancel{\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\theta}}{\xi - \frac{1}{\bar{z}}} \mathrm{d}\theta &= 0 \\ \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - \frac{1}{\bar{z}}} \mathrm{d}\theta &= 0. \end{aligned}$$

□

Notice,

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} \mathrm{d}\theta \mp 0 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} \mathrm{d}\theta \mp \frac{1}{2\pi} \int_0^{2\pi} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \mp \frac{\xi}{\xi - \frac{1}{\bar{z}}} \right) \mathrm{d}\theta. \end{aligned}$$

Now we can use $\xi = 1/\bar{\xi}$ to get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{1/\bar{\xi}}{1/\bar{\xi} - \frac{1}{\bar{z}}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{\bar{\xi}\bar{z}}{\bar{\xi}\bar{z}}\frac{1/\bar{\xi}}{1/\bar{\xi} - \frac{1}{\bar{z}}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \mp \frac{\bar{z}}{\bar{z} - \bar{\xi}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \pm \frac{\bar{z}}{\bar{\xi} - \bar{z}}\right)\mathrm{d}\theta.
\end{aligned}
$$

$\square$

Using the plus sign we see

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} + \frac{\bar{z}}{\bar{\xi} - \bar{z}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)} + \frac{\bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z}) + \bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi\bar{\xi} \cancel{- \xi\bar{z} + \xi\bar{z}} - z\bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - |z|^2}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - |z|^2}{\overline{(\xi - z)}(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\,\mathrm{e}^{\mathrm{i}\theta}\,|^2 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1^2 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1 - |z|^2}{|\xi - z|^2}\right)\mathrm{d}\theta.
\end{aligned}
$$

$\square$

(a) Deduce the "Poisson formula" for the real part of $f(z) : u(r,\phi) = \Re f, z = r\,\mathrm{e}^{i\phi}$.

$$\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta)+\mathrm{i}v(\theta))\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)+\mathrm{i}v(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{i}v(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta + \mathrm{i}\frac{1}{2\pi}\int_0^{2\pi} v(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta
\end{aligned}$$

Thus,

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\frac{1-|z|^2}{|\xi-z|^2}\right)\mathrm{d}\theta.$$

We know $|z| = \left|r\,\mathrm{e}^{i\phi}\right| = r$. Now let's look specifically at the denominator and plugin the substitutions for z to get

$$\begin{aligned}
|\xi-z|^2 &= |\,\mathrm{e}^{\mathrm{i}\theta} - r\,\mathrm{e}^{\mathrm{i}\phi}\,|^2 \\
&= |\cos\theta + \mathrm{i}\sin\theta - r\cos\phi - \mathrm{i}r\sin\phi|^2 \\
&= |\cos\theta - r\cos\phi + \mathrm{i}\,(\sin\theta - r\sin\phi)\,|^2 \\
&= (\cos\theta - r\cos\phi)^2 + (\sin\theta - r\sin\phi)^2 \\
&= \cos^2\theta - 2r\cos\theta\cos\phi + r^2\cos^2\phi + \sin^2\theta - 2\sin\theta r\sin\phi + r^2\sin^2\phi \\
&= \cos^2\theta + \sin^2\theta - 2r\cos\theta\cos\phi - 2\sin\theta r\sin\phi + r^2\cos^2\phi + r^2\sin^2\phi \\
&= 1 - 2r\cos\theta\cos\phi - 2\sin\theta r\sin\phi + r^2\left(\cos^2\phi + \sin^2\phi\right) \\
&= 1 - 2r\cos\theta\cos\phi - 2\sin\theta r\sin\phi + r^2 \\
&= 1 - 2r\,(\cos\theta\cos\phi + \sin\theta\sin\phi) + r^2 \\
&= 1 - 2r\cos(\theta-\phi) + r^2.
\end{aligned}$$

Hence,

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\mathrm{d}\theta$$

□

(b) If we use the minus sign in the formula for $f(z)$ above, we get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} - \frac{\bar{z}}{\bar{\xi} - \bar{z}}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)} - \frac{\bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi(\bar{\xi} - \bar{z}) - \bar{z}(\xi - z)}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi\bar{\xi} - 2\xi\bar{z} + z\bar{z})}{(\bar{\xi} - \bar{z})(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{|\xi|^2 - 2\xi\bar{z} + |z|^2)}{\overline{(\xi - z)}(\xi - z)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{1 - 2\xi\bar{z} + r^2}{|\xi - z|^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2\xi\bar{z} + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2\,\mathrm{e}^{\mathrm{i}\theta}\,r\,\mathrm{e}^{-\mathrm{i}\phi} + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2r\,\mathrm{e}^{\mathrm{i}\theta - \mathrm{i}\phi} + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2r\,\mathrm{e}^{\mathrm{i}(\theta - \phi)} + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\mathrm{d}\theta
\end{aligned}
$$

$\square$

Additionally, if we take the imaginary part this time, we can see

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1 - 2r\,\mathrm{e}^{\mathrm{i}(\theta - \phi)} + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(\frac{1 - 2r\cos(\theta - \phi) - 2r\mathrm{i}\sin(\theta - \phi) + r^2}{1 - 2r\cos(\theta - \phi) + r^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(\frac{1 - 2r\cos(\theta - \phi) + r^2 - 2r\mathrm{i}\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(1 + \frac{-2r\mathrm{i}\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (u(\theta) + \mathrm{i}v(\theta))\left(1 - \frac{\mathrm{i}2r\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2}\right)\mathrm{d}\theta
\end{aligned}
$$

Let's expand out the terms in the integrand

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[u(\theta) - u(\theta)\left(\frac{\mathrm{i}2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right) + \mathrm{i}v(\theta) - \mathrm{i}v(\theta)\left(\frac{\mathrm{i}2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\right]\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[u(\theta) - \mathrm{i}u(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right) + \mathrm{i}v(\theta) - \mathrm{i}^2v(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\right]\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[u(\theta) - \mathrm{i}u(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right) + \mathrm{i}v(\theta) + v(\theta)\left(\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\right]\mathrm{d}\theta.$$

Now the imaginary part of this is

$$\begin{aligned}
\Im(f(z)) &= \frac{1}{2\pi}\int_0^{2\pi}\left(v(\theta) - u(\theta)\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\mathrm{d}\theta - \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\mathrm{d}\theta \\
&= C - \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\mathrm{d}\theta \\
&= C - \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin\big(-(-\theta+\phi)\big)}{1-2r\cos\big(-(-\theta+\phi)\big)+r^2}\mathrm{d}\theta \\
&= C - \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{-r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta \\
&= C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta
\end{aligned}$$

where $C = \frac{1}{2\pi}\int_0^{2\pi} v(1,\theta)\mathrm{d}\theta = v(r=0)$. This last relationship follows from the Cauchy Integral formula at z = 0 – see the first equation in this exercise). Hence,

$$v(r,\phi) = C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta))}{1-2r\cos(\phi-\theta)+r^2}\mathrm{d}\theta$$

□

(c) We wish to show

$$\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2} = \Im\left(\frac{\xi+z}{\xi-z}\right)$$

Let's try a bit

$$\begin{aligned}
\frac{\xi+z}{\xi-z} &= \frac{\overline{(\xi-z)}(\xi+z)}{\overline{(\xi-z)}(\xi-z)} \\
&= \frac{\overline{(\xi-z)}(\xi+z)}{|\xi-z|^2}.
\end{aligned}$$

We have already computed this denominator once. Using our previous result we continue

$$
\begin{aligned}
\frac{\overline{(\xi - z)}(\xi + z)}{|\xi - z|^2} &= \frac{(\bar{\xi} - \bar{z})(\xi + z)}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{\bar{\xi}\xi + \bar{\xi}z - \bar{z}\xi - \bar{z}z}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{|\xi|^2 + \bar{\xi}z - \bar{z}\xi - |z|^2}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 + \bar{\xi}z - \bar{z}\xi - r^2}{1 - 2r\cos(\phi - \theta) + r^2}.
\end{aligned}
$$

Now, lets plugin our parameterizations of ξ and z

$$
\begin{aligned}
&= \frac{1 - r^2 + \mathrm{e}^{-\mathrm{i}\theta}\, r\, \mathrm{e}^{\mathrm{i}\phi} - r\, \mathrm{e}^{-\mathrm{i}\phi}\, \mathrm{e}^{\mathrm{i}\theta}}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + r\, \mathrm{e}^{\mathrm{i}(\phi - \theta)} - r\, \mathrm{e}^{\mathrm{i}(\theta - \phi)}}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + r(\cos(\phi - \theta) + \mathrm{i}\sin(\phi - \theta)) - r(\cos(\theta - \phi) + \mathrm{i}\sin(\theta - \phi))}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + r\cos(\phi - \theta) - r\cos(\theta - \phi) + \mathrm{i}r\sin(\phi - \theta) - \mathrm{i}r\sin(\theta - \phi)}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + \cancel{r\cos(\phi - \theta)} - \cancel{r\cos(\phi - \theta)} + \mathrm{i}r\sin(\phi - \theta) + \mathrm{i}r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2} \\
&= \frac{1 - r^2 + \mathrm{i}2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2}.
\end{aligned}
$$

We have arrived to

$$
\begin{aligned}
\frac{\xi + z}{\xi - z} &= \frac{1 - r^2 + \mathrm{i}2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2} \\
\Im\left[\frac{\xi + z}{\xi - z}\right] &= \Im\left[\frac{1 - r^2 + \mathrm{i}2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2}\right] \\
&= \frac{2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v(r, \phi) &= C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi - \theta))}{1 - 2r\cos(\phi - \theta) + r^2}\mathrm{d}\theta \\
&= C + \frac{\Im}{2\pi}\int_0^{2\pi} u(\theta)\frac{\xi + z}{\xi - z}\mathrm{d}\theta
\end{aligned}
$$

□

This example illustrates that prescribing the real part of $f(z)$ on $|z| = 1$ determines (a) the real part of $f(z)$ everywhere inside the circle and (b) the imaginary part of $f(z)$ inside the circle to within a constant. We *cannot* arbitrarily specify both the real and imaginary parts of an analytic function on $|z| = 1$.

3: Suppose Ω is an open simply connected region and $z_0 \in \Omega$. Assume that $f(z)$ is analytic in $\Omega \setminus \{z_0\}$ and satisfies

$$|f(z)| \leq M|z - z_0|^{-\gamma}, \quad \gamma < 1.$$

Show that if the a specific choice for $f(z_0)$ is made then f extends to an analytic function on Ω.

Solution:

Let's choose the value for $f(z_0)$ to be

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{\xi - z_0} \mathrm{d}\xi.$$

Now we wish to show that $f(z)$ is analytic on Ω after making this choice. The method by which I am going to demonstrate this is by showing that for any contour $C \subset \Omega$ we have

$$\oint_C f(z) \mathrm{d}z = 0.$$

Since $f(z)$ is already analytic away from z_0, this criteria holds for any contour C that z_0 is not contained in the region enclosed by the C. Thus, it suffices to just verify that our criteria still holds in a neighborhood around z_0. Now suppose some contour $\widehat{C}$ encompasses encloses z_0. Using the method of creating channels from our original contour $\widehat{C}$ to cut out z_0 with a new contour C_ϵ, which is a circle of radius $\epsilon = |z - z_0|$ centered at z_0, we can say

$$\oint_{\widehat{C}} f(z) \mathrm{d}z = \oint_{C_\epsilon} f(z) \mathrm{d}z.$$

I will now describe how we know the integral on the right goes to 0 as ϵ goes to 0. Let's begin by looking at the modulus of our integral on the right

$$\begin{aligned} \left| \oint_{C_\epsilon} f(z) \mathrm{d}z \right| &\leq \oint_{C_\epsilon} |f(z)| \, |\mathrm{d}z| \\ &\leq \oint_{C_\epsilon} M|z - z_0|^{-\gamma} \, |\mathrm{d}z| \, . \end{aligned} \tag{1}$$

Now making use of the substitution $z = |z - z_0| \, \mathrm{e}^{\mathrm{i}\theta}$ (and $\mathrm{d}z| = |z - z_0| \mathrm{i} \, \mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta$) and the fact that

$$\begin{aligned} |\mathrm{d}z| &= \left| |z - z_0| \mathrm{i} \, \mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta \right| \\ &= \left| |z - z_0| \, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} \, \mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta \right| \\ &= \left| |z - z_0| \, \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2} + \theta\right)} \, \mathrm{d}\theta \right| \\ &= |z - z_0| \cdot |\mathrm{d}\theta| \end{aligned}$$

Then we continue from inequality (1) we have

$$\begin{aligned}
\oint_{C_\epsilon} M|z-z_0|^{-\gamma}\,|\mathrm{d}z| &= \int_0^{2\pi} M|z-z_0|^{-\gamma}|z-z_0||\mathrm{d}\theta| \\
&= \int_0^{2\pi} M|z-z_0|^{1-\gamma}|\mathrm{d}\theta| \\
&= M|z-z_0|^{1-\gamma}\int_0^{2\pi} |\mathrm{d}\theta| \\
&= M|z-z_0|^{1-\gamma}2\pi.
\end{aligned}$$

Critically, notice since $\gamma < 1$ we have $1-\gamma > 0$. Under this condition as $\epsilon \to 0$, meaning $z \to z_0$ we have that

$$M|z-z_0|^{1-\gamma}2\pi \to 0.$$

Therefore

$$\left|\oint_{C_\epsilon} f(z)\mathrm{d}z\right| \leq 0 \quad \text{as } z \to z_0.$$

Thus the integral over the contour C_ϵ contributes nothing to our original integral over the contour $\widehat{C}$. Hence,

$$\oint_{\widehat{C}} f(z)\mathrm{d}z = 0.$$

Finally, since our original $f(z)$ is analytic everywhere in $\Omega \setminus \{z_0\}$ it satisfies Cauchy's Integral Formula at every $z \neq z_0$ and since our choice of $f(z_0)$ was such that Cauchy's integral formula holds at z_0 we have that $f(z)$ is continuous in all of Ω.

Since, an integral over an arbitrary contour $\widehat{C}$, which encloses z_0 is equal to zero and as established earlier by analyticity of $f(z)$ this is true as well for any contour C which does not enclose z_0, by Morera's Theorem (2.6.5), we have that $f(z)$ is analytic in all of Ω. $\square$

4: Establish the following lemma:

> **Lemma 1**
>
> Suppose Ω is an open region and $f(z)$ is continuous on $\overline{\Omega}$. Let Γ be a contour in $\overline{\Omega}$. Suppose a sequence of contours $\Gamma_n \subset \overline{\Omega}$ converge to Γ in the sense that there exists parameterizations $z(t)$ of Γ and $z_n(t)$ of Γ_n defined on $[a, b]$ satisfying
>
> $$z_n(t) \overset{n\to\infty}{\longrightarrow} z(t), \qquad \text{uniformly on } [a,b],$$
> $$z_n'(t) \overset{n\to\infty}{\longrightarrow} z'(t), \qquad \text{uniformly on } [a,b].$$
>
> Then
>
> $$\int_{\Gamma_n} f(z)\mathrm{d}z \overset{n\to\infty}{\longrightarrow} \int_{\Gamma} f(z)\mathrm{d}z.$$

Hint: Use that f is uniformly continuous on $\overline{\Omega}$.

Solution:

We know the following from the assumptions of this lemma.

- $f(z)$ being continuous on $\overline{\Omega}$ tells us
  - For all $\epsilon > 0$ and for all $z_0 \in \overline{\Omega}$, there exists a $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$.
- $\Gamma_n \subset \overline{\Omega}$ converges to Γ in the sense that there exists parameterizations $z(t)$ of Γ and $z_n(t)$ of Γ_n defined on $[a, b]$ satisfying

  $$z_n(t) \overset{n\to\infty}{\longrightarrow} z(t), \qquad \text{uniformly on } [a,b],$$
  $$z_n'(t) \overset{n\to\infty}{\longrightarrow} z'(t), \qquad \text{uniformly on } [a,b],$$

  tells us
  - For all $\epsilon > 0$ there exists $N(\epsilon)$ such that whenever $n > N$ we have
    $$|z_n(t) - z(t)| < \epsilon.$$
  - For all $\epsilon > 0$ there exists $N(\epsilon)$ such that whenever $n > N$ we have
    $$|z_n'(t) - z'(t)| < \epsilon.$$

    These big N's are not necessarily the same.

Now, in order to establish the lemma, we are going to make use of Theorem 3.1.1 from A & F. In our circumstances, we need to first parameterize the two integrals of interest so we get

$$\int_{\Gamma_n} f(z)\mathrm{d}z = \int_a^b f(z_n(t))z_n'(t)\mathrm{d}t$$
$$\int_{\Gamma} f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

Now if we can show that $f(z_n(t))z_n'(t)$ is a continuous function for all n and $f(z_n(t))z_n'(t)$ converges uniformly to $f(z)$ in $\overline{\Omega}$, then we will be allowed to conclude that

$$\lim_{n\to\infty} \int_a^b f(z_n(t))z_n'(t)\mathrm{d}t = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

Now we need to show that $f(z_n(t))z_n'(t)$ is continuous. Consider the contour's we are interested in, Γ_n and Γ. From page 72 of A & F, we have that the parameterization of each of these contours will be continuous and their respective derivatives will be at least piecewise continuous. Since the composition of continuous functions is continuous, we know $f(z_n(t))$ is continuous. Additionally the product of continuous functions is continuous. Therefore, $f(z_n(t))z_n'(t)$ is at least piecewise continuous as well. For notational clarity, lets define these ℓ piecewise subintervals $I_k = [a_k, b_k]$ of $[a, b]$, where $f(z_n(t))z_n'(t)$ is explicitly continuous, to be

$$I_k = \begin{cases} [a_0, b_0), & \text{if } k = 0 \\ [b_{k-1}, b_k), & \text{if } 0 < k < \ell \\ [b_{\ell-1}, b_\ell], & \text{if } k = \ell \end{cases}$$

Such that $\cup_{k=1}^{\ell} I_k = [a, b]$ and $\cap_{k=1}^{\ell} I_k = \emptyset$.

Recall the image of a continuous function on a compact set is also compact. Since $t \in [a, b]$ is a compact set, the continuity of $z_n(t)$ preserves compactness and thus the image of $z_n(t)$, $t \in [a, b]$ is also compact. Additionally, since the image of $z_n(t)$ is compact and $f(z)$ is continuous, $f(z)$ on our contour (and it's parameterization) is uniformly continuous.

Now, since f is uniformly continuous on our contours, we have that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z_1 - z_2| < \delta \implies |f(z_1) - f(z_2)| < \epsilon$$

for z_1, z_2 on the contour (in the image of $z(t)$), $t \in [a, b]$. Now let's consider $z_n(t)$ and $z(t)$. From the problem statement we know that $|z_n(t) - z(t)| < \epsilon_1$. Let's choose our δ in the uniform continuity of $f(z)$ to be $\delta = \epsilon_1$. Then we have

$$|z_n(t) - z(t)| < \epsilon_1 \implies |f(z_n(t)) - f(z(t))| < \epsilon_2$$

and thus

$$f(z_n(t)) \overset{n\to\infty}{\longrightarrow} f(z(t)) \quad \text{uniformly.}$$

Furthermore, since the image of $z_n(t)$ is compact and $f(z)$ is continuous, the image of $f(z_n(t))$ is also compact. Therefore, there exists some upper bound M such that

$$|f(z_n(t))| \le M$$

when $t \in [a, b]$. Finally, since $z'(t)$ is piecewise continuous on the compact set $t \in [a, b]$ we can say there also exists some upper bound call it L such that

$$|z'(t)| \le L.$$

In summary, we have shown that $f(z_n(t))$ converges uniformly to $f(z(t))$. We have also demonstrated that there exist upper bounds M and L on the functions $|f(z_n(t))|$ and $|z'(t)|$ respectively.

Now we want to show that $f(z_n(t))z_n'(t)$ converges uniformly to $f(z(t))z'(t)$. We begin with what we want on the left and adding zero in a clever way on the right

$$f(z_n(t))z_n'(t) - f(z(t))z'(t) = f(z_n(t))z_n'(t) - f(z(t))z'(t) + f(z_n(t))z'(t) - f(z_n(t))z'(t)$$

$$\begin{aligned}
\Big|f(z_n(t))z_n'(t) - f(z(t))z'(t)\Big| &= \Big|f(z_n(t))z_n'(t) - f(z(t))z'(t) + f(z_n(t))z'(t) - f(z_n(t))z'(t)\Big| \\
&\leq \Big|f(z_n(t))z_n'(t) - f(z_n(t))z'(t)\Big| + \Big|f(z_n(t))z'(t) - f(z(t))z'(t)\Big| \\
&\leq \Big|f(z_n(t))\Big(z_n'(t) - z'(t)\Big)\Big| + \Big|z'(t)\Big(f(z_n(t)) - f(z(t))\Big)\Big| \\
&= \Big|f(z_n(t))\Big|\Big|z_n'(t) - z'(t)\Big| + \Big|z'(t)\Big|\Big|f(z_n(t)) - f(z(t))\Big| \\
&< |f(z_n(t))|\frac{\epsilon}{M2} + |z'(t)|\frac{\epsilon}{L2} \\
&< M\frac{\epsilon}{M2} + L\frac{\epsilon}{L2} \\
&< \epsilon.
\end{aligned}$$

Therefore, $f(z_n(t))z_n'(t)$ converges uniformly to $f(z(t))z'(t)$. Furthermore, since have have this and $f(z_n(t))$ is continuous on a each interval I_k, by Theorem 3.1.1 from A & F we can conclude

$$\lim_{n\to\infty}\int_{a_k}^{b_k} f(z_n(t))z_n'(t)\mathrm{d}t = \int_{a_k}^{b_k} f(z(t))z'(t)\mathrm{d}t$$

for all $k \in \{1, 2, 3, ..., \ell\}$. Just a reminder, ℓ represents the number of finite sub intervals we can break [a,b] up into and where a_k and b_k represent the endpoints of the interval I_k. Hence,

$$\begin{aligned}
\sum_{k=1}^{\ell}\lim_{n\to\infty}\int_{a_k}^{b_k} f(z_n(t))z_n'(t)\mathrm{d}t &= \sum_{k=1}^{\ell}\int_{a_k}^{b_k} f(z(t))z'(t)\mathrm{d}t \\
\sum_{k=1}^{\ell}\lim_{n\to\infty}\int_{a_k}^{b_k} f(z_n(t))z_n'(t)\mathrm{d}t &= \int_{a}^{b} f(z(t))z'(t)\mathrm{d}t \\
\lim_{n\to\infty}\sum_{k=1}^{\ell}\int_{a_k}^{b_k} f(z_n(t))z_n'(t)\mathrm{d}t &= \int_{a}^{b} f(z(t))z'(t)\mathrm{d}t \\
\lim_{n\to\infty}\int_{a}^{b} f(z_n(t))z_n'(t)\mathrm{d}t &= \int_{a}^{b} f(z(t))z'(t)\mathrm{d}t
\end{aligned}$$

where we can move the sum on the left inside the limit on line 3 since the sequence converges uniformly. Additionally, undoing our parameterization we have,

$$\lim_{n\to\infty}\int_{\Gamma_n} f(z)\mathrm{d}z = \int_{\Gamma} f(z)\mathrm{d}z$$

which written another way is

$$\int_{\Gamma_n} f(z)\mathrm{d}z \overset{n\to\infty}{\longrightarrow} \int_{\Gamma} f(z)\mathrm{d}z$$

as desired.

$\square$

5: for any $r, R > 0$, let $C = \partial\Sigma$, $\Sigma = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \leq r \text{ and } 0 \leq -\operatorname{Im} z \leq R, \ R > 0\}$. In this problem $\sqrt{z}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$.

- Show that if $f(z)$ is analytic in a region that contains Σ,

$$\oint_C f(z)\sqrt{z-1}\sqrt{z+1}\mathrm{d}z = 0.$$

(**1 part**)
Solution:
Assume $f(z)$ is an analytic function in a region that contains Σ. Furthermore, let $g(z) = \sqrt{z-1}\sqrt{z+1}$. Using the principal branch for $g(z)$, we also have that $g(z)$ is analytic as well away from it's branch cut. Then we are looking at a contour integral of an analytic function (in our region) and therefore

$$\oint_C f(z)\sqrt{z-1}\sqrt{z+1}\mathrm{d}z = 0$$

by Cauchy's theorem. □

- Show that if $f(z)$ is analytic in a region that contains Σ

$$\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} = 0.$$

(**1 part**)
Solution:
Deal with the singularities on the boundary in some clever way...
perhaps by having a section of the contour be taken to be the limit of a sequence of contours approaching the real axis.

6: From A&F: 3.1.1 b,d

In the following we are given sequences. Discuss their limits and whether the convergence is uniform, in the region $\alpha \leq |z| \leq \beta$, for finite $\alpha, \beta > 0$.

b)

$$\left\{ \frac{1}{z^n} \right\}_{n=1}^{\infty}$$

Solution:

Let $f_n(z) = \frac{1}{z^n}$, for $n \in \mathbb{N}$. Then we notice

$$f_n(z) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0, & \text{if } |z| > 1 \\ 1, & \text{if } |z| = 1 \\ \infty, & \text{if } |z| < 1. \end{cases}$$

Generally, the convergence of this sequence $f_n(z)$ cannot be uniform, since the limit is not continuous. However, this sequence could converge uniformly for specific choices of α and β. For example, if $\alpha > 1$ and $\beta > \alpha$ then for all admissible z we have $f_n(z) \overset{n\to\infty}{\longrightarrow} 0$ and thus the convergence would be uniform. Additionally, if $\beta < 1$ and $\alpha < \beta$ then for all admissible z we have $f_n(z) \overset{n\to\infty}{\longrightarrow} \infty$ and thus the convergence would be uniform. Finally, the only other scenario where $f_n(z)$ converges uniformly is if $\alpha = \beta = 1$. In which case for all admissible z, really just the set $\{1\}$, we have $f_n(z) \overset{n\to\infty}{\longrightarrow} 1$.

TODO:

- Do I need to prove it is uniformly convergent in the cases where that applies?
- Do I need to acknowledge the case where $z = 0$ even though we know the modulus of z is required to be strictly greater than 0 since we have $|z| \geq \alpha > 0$? Do we say that $f_n(z)$ is undefined at $|z| = 0$?

□

d)

$$\left\{ \frac{1}{1 + (nz)^2} \right\}_{n=1}^{\infty}$$

Solution:

Let $f_n(z) = \frac{1}{1+(nz)^2}$, for $n \in \mathbb{N}$. Then we notice

$$f_n(z) \overset{n\to\infty}{\longrightarrow} \begin{cases} 1, & \text{if } |z| = 0 \\ 0, & \text{if } |z| > 0. \end{cases}$$

However, since we have $|z| \geq \alpha > 0$ we know $|z| \neq 0$ and thus

$$f_n(z) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{uniformly.}$$

This result is independent of the choice of upper bound β.

□

7: From A&F: 3.1.2 b,d

For each sequence in problem 6 (book problem 1), what can be said if

Solution:

(b) Now let's begin by reconsidering the sequence from the previous problem, part (b). Let $f_n(z) = \frac{1}{z^n}$, for $n \in \mathbb{N}$. Then we notice

$$f_n(z) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0, & \text{if } |z| > 1 \\ 1, & \text{if } |z| = 1 \\ \infty, & \text{if } |z| < 1. \end{cases}$$

Let's now consider the two cases presented herein.

(a) $\alpha = 0$:

This gives us the following bounds $0 \leq |z| \leq \beta$. First of all, this now includes the point $z = 0$ for which $f_n(z)$ is not defined. Second of all, these bounds include all 3 cases for the limit of $f_n(z)$ and is therefore discontinuous. Hence, our sequence is not uniformly convergent in this case.

(b) $\alpha > 0$ and $\beta = \infty$:

This gives us the following bounds $0 < |z| \leq \infty$. We are still left with a discontinuous limit of $f_n(z)$ meaning the sequence does not converge uniformly.

□

(d) Let $f_n(z) = \frac{1}{1+(nz)^2}$, for $n \in \mathbb{N}$. Then we notice

$$f_n(z) \overset{n\to\infty}{\longrightarrow} \begin{cases} 1, & \text{if } |z| = 0 \\ 0, & \text{if } |z| > 0. \end{cases}$$

However, since we have $|z| \geq \alpha > 0$ we know $|z| \neq 0$ and thus

$$f_n(z) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{uniformly.}$$

This result is independent of the choice of upper bound β.

(a) $\alpha = 0$:

This gives us the following bounds $0 \leq |z| \leq \beta$. Since $|z|$ can be equal to 0, the limit of $f_n(z)$ includes both cases provided before and is therefore discontinuous. Therefore, the sequence $f_n(z)$ cannot be uniformly convergent, given these bounds from α and β.

(b) $\alpha > 0$ and $\beta = \infty$:

This gives us the following bounds $0 < |z| \leq \infty$. Thus

$$f_n(z) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{uniformly.}$$

TODO: Again, do I need to prove uniform convergence with a $N - \epsilon$ proof? It seems sufficient in the negative cases to just demonstrate that the limit is not continuous so it can't be uniformly convergent. The only cases where I claimed (d) was uniformly convergent were when 0 is excluded here in problem 7 and in problem 6. Additionally, I claimed (b) was uniformly convergent in some cases in problem 6.s □

8: From A&F: 3.1.3 Compute the integrals

$$\lim_{n\to\infty}\int_0^1 nz^{n-1}\mathrm{d}z \quad \text{and} \quad \int_0^1 \lim_{n\to\infty}\left(nz^{n-1}\right)\mathrm{d}z$$

and show that they are not equal. Explain why this is not a counter example to Theorem 3.1.1. (A &F pg. 111)

(**3 parts**) *Solution:*

For the integral on the right consider adding a limit outside the integral so the bound doesn't have any issues. We can easily evaluate the left limit to be

$$\lim_{n\to\infty}\int_0^1 nz^{n-1}\mathrm{d}z = \lim_{n\to\infty} z^n\big|_0^1 = \lim_{n\to\infty} 1^n - 0^n = \lim_{n\to\infty} 1 = 1.$$

However, on the right we have (using a limit as we approach the upper bound 1)

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon} \lim_{n\to\infty}\left(nz^{n-1}\right)\mathrm{d}z.$$

There are approximately 34 things to do, 27 down!