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 AMATH 567

HOMEWORK 4

Collaborators*: Nate Ward

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.4.2 c, e.

Evaluate the integral $\oint_C f(z)dz$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows:

c)

$$f(z) = \frac{1}{\bar{z}}$$

Solution:

We want to evaluate

$$\oint_C \frac{1}{\bar{z}} dz$$

on the parameterized unit circle $z = e^{i\theta}$ where $\theta \in [0, 2\pi)$, where $\bar{z} = e^{-i\theta}$ on the unit circle. Note, before we do the substitution we need $dz = i e^{i\theta} d\theta$. Now our integral is

$$\begin{aligned} \oint_C \frac{1}{\bar{z}} dz &= \int_0^{2\pi} \frac{1}{e^{-i\theta}} i e^{i\theta} d\theta \\ &= \int_0^{2\pi} i e^{2i\theta} d\theta \\ &= \left(\frac{1}{2} e^{2i\theta} \right) \Big|_0^{2\pi} \\ &= \frac{1}{2} e^{4\pi i} - \frac{1}{2} e^0 \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

□

e)

$$f(z) = e^{\bar{z}}$$

Solution:

We will use the same substitutions from the previous part

$$\begin{aligned}\oint_C e^{\bar{z}} dz &= \oint_0^{2\pi} e^{e^{-i\theta}} i e^{i\theta} d\theta \\ &= \oint_0^{2\pi} \sum_{j=1}^{\infty} \frac{(e^{-i\theta})^j}{j!} i e^{i\theta} d\theta \\ &= \sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta.\end{aligned}$$

We are justified in reordering the integral of the infinite sum to be the infinite sum of the integrals since the original series converges absolutely. I will now just look at the integral inside the sum

$$\begin{aligned}\oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta &= \oint_0^{2\pi} i \frac{e^{-i\theta j} e^{i\theta}}{j!} d\theta \\ &= \oint_0^{2\pi} i \frac{e^{-i\theta j + i\theta}}{j!} d\theta \\ &= \oint_0^{2\pi} i \frac{e^{i\theta(-j+1)}}{j!} d\theta \\ &= \oint_0^{2\pi} i \frac{e^{i\theta(1-j)}}{j!} d\theta \\ &= \frac{1}{1-j} \frac{i e^{i\theta(1-j)}}{j!} \Big|_0^{2\pi} \\ &= \frac{1}{1-j} \frac{i e^{i2\pi(1-j)}}{j!} - \frac{1}{1-j} \frac{i e^0}{j!} \\ &= \frac{i}{(1-j)j!} (e^{i2\pi(1-j)} - 1) \\ &= \frac{i}{(1-j)j!} (1 - 1) \\ &= 0.\end{aligned}$$

I want to clarify why $e^{i2\pi(1-j)} = 1$. Since $j \in \{1, 2, 3, \dots\}$, then $1-j$ is an integer and we have $e^{i2\pi\ell}$ where $\ell \in \mathbb{Z}$ which is always 1. Now we return to the original problem

$$\sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta = \sum_{j=1}^{\infty} 0 = 0.$$

Now we have completed the requisite task. □

- 2:** From A&F: 2.4.4 a, b. Use the principal branch where the argument is in $[-\pi, \pi)$. Discuss any ambiguities. Use the principal branch of $\log(z)$ and $z^{\frac{1}{2}}$ where the argument is in $[-\pi, \pi)$ to evaluate the following:
- a)

$$\int_{-1}^1 \log z dz$$

Solution:

We want to parameterize this once again using $z = r e^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our integral is

$$\begin{aligned}\int_{-1}^1 \log z dz &= \int_{-\pi}^0 \log(e^{i\theta}) i e^{i\theta} d\theta \\ &= \int_{-\pi}^0 i\theta i e^{i\theta} d\theta.\end{aligned}$$

Let's use integration by parts, woohoo! We will assign the substitutions as follows:

$$\begin{aligned}u &= i\theta \\ du &= i d\theta\end{aligned}$$

$$\begin{aligned}dv &= i e^{i\theta} d\theta \\ v &= e^{i\theta}.\end{aligned}$$

Plugging this in we have

$$\begin{aligned}\int_{-\pi}^0 i\theta i e^{i\theta} d\theta &= i\theta e^{i\theta} \Big|_{-\pi}^0 - \int_{-\pi}^0 i e^{i\theta} d\theta \\ &= (0 - (-i\pi e^{-i\pi})) - e^{i\theta} \Big|_{-\pi}^0 \\ &= 0 + i\pi e^{-i\pi} - e^{i\theta} \Big|_{-\pi}^0 \\ &= i\pi e^{-i\pi} - (e^0 - e^{-i\pi}) \\ &= -i\pi - (1 - (-1)) \\ &= -i\pi - (2) \\ &= -i\pi - 2.\end{aligned}$$

□

b)

$$\int_{-1}^1 z^{\frac{1}{2}} dz$$

Solution:

We want to parameterize this once again using $z = r e^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our

integral is

$$\begin{aligned}
 \int_{-1}^1 z^{\frac{1}{2}} dz &= \int_{-\pi}^0 (e^{i\theta})^{\frac{1}{2}} i e^{i\theta} d\theta \\
 &= \int_{-\pi}^0 i e^{\frac{i\theta}{2}} e^{i\theta} d\theta \\
 &= \int_{-\pi}^0 i e^{\frac{i3}{2}\theta} d\theta \\
 &= \frac{2}{3} e^{\frac{i3}{2}\theta} \Big|_{-\pi}^0 \\
 &= \frac{2}{3} e^{-\frac{i3}{2}\pi} - \frac{2}{3}.
 \end{aligned}$$

Now remembering our branch cut limits θ to be within $[-\pi, \pi)$ we change the angle $-\frac{3}{2}\pi$ to be $\frac{1}{2}\pi$. Hence,

$$\begin{aligned}
 \frac{2}{3} e^{-\frac{i3}{2}\pi} - \frac{2}{3} &= \frac{2}{3} e^{\frac{i\pi}{2}} - \frac{2}{3} \\
 &= \frac{2}{3} (i - 1).
 \end{aligned}$$

□

3: From A&F: 2.4.7

Let C be an open (upper) semicircle of radius R with its center at the origin, and consider $\int_C f(z) dz$. Let $f(z) = \frac{1}{z^2 + a^2}$ for a real $a > 0$. Show that $|f(z)| \leq \frac{1}{R^2 - a^2}$, $R > a$, and

$$\left| \int_C f(z) dz \right| \leq \frac{\pi R}{R^2 - a^2}, \quad R > a.$$

Solution:

4: From A&F: 2.4.8

Let C be an arc of the circle $|z| = R$ ($R > 1$) of angle $\frac{\pi}{3}$. Show that

$$\left| \int_C \frac{dz}{z^3 + 1} \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right)$$

and deduce

$$\lim_{R \rightarrow \infty} \int_C \frac{dz}{z^3 + 1} = 0$$

Solution:

5: From A&F: 2.5.1 b, e

Evaluate $\oint_C f(z) dz$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

b)

$$f(z) = e^{z^2}$$

Solution:

e)

$$f(z) = \frac{1}{2z^2 + 1}$$

Solution:

- 6:** Use the ideas from A&F: 2.5.5 to evaluate $\int_0^\infty e^{iz^3 t} dz$, $t > 0$. Express the result in terms of $\int_0^\infty e^{-r^3} dr$.

The ideas we might need to use are ... it's actually really long!

Solution:

- 7:** From A&F: 2.5.6.

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

Show how to evaluate this integral by considering

$$\oint_{C(\mathbb{R})} \frac{dz}{z^2 + 1},$$

where $C(\mathbb{R})$ is closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x -axis. *Hint:* use

$$\frac{1}{z^2 + 1} = -\frac{1}{2i} \left(\frac{1}{z + i} - \frac{1}{z - i} \right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \rightarrow \infty$. Verify your answer by usual integration in real variables. *Solution:*

Repeat this exercise for

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{\epsilon dx}{x^2 + \epsilon^2}, \quad \epsilon > 0.$$

Seems like I am supposed to do 2.5.6 and then for the given integral as well.

Solution:

- 8:** Use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

Solution:

- 9:** From A&F: 2.6.1 a, e.

Evaluate the integrals $\oint_C f(z) dz$, where C is the unit circle centered at the origin and $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):

a)

$$\frac{\sin z}{z}$$

Solution:

e)

$$e^{z^2} \left(\frac{1}{z^2} - \frac{1}{z^3} \right)$$

Solution: