

Hunter Lybbert
Student ID: 2426454
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AMATH 561

PROBLEM SET 5

1. Let X and Y_0, Y_1, Y_2, \dots be random variables on a probability space (Ω, \mathcal{F}, P) and suppose $E|X| < \infty$. Define $\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n)$ and $X_n = E(X|\mathcal{F}_n)$. Show that the sequence X_0, X_1, X_2, \dots is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$.

Solution:

For my sake, I will review the definition of a martingale then we will show that the sequence X_0, X_1, X_2, \dots satisfies all of the necessary conditions and is thus itself a martingale.

Let \mathcal{F}_n be a filtration, i.e. an increasing sequence of σ -algebras. A sequence of random variables X_n is said to be adapted to \mathcal{F}_n if $X_n \in \mathcal{F}_n$ (that is X_n is \mathcal{F}_n -measurable or for all Borel sets B we have $X_n^{-1}(B) = \{\omega \mid X_n(\omega) \in B\} \in \mathcal{F}_n$) for all n . If X_n is a sequence with:

- (1) $E|X_n| < \infty$
- (2) X_n is adapted to \mathcal{F}_n
- (3) $E(X_{n+1}|\mathcal{F}_n) = X_n$ for all n ,

Then $X = (X_n)_{n \in \mathbb{N}}$ is said to be a martingale with respect to \mathcal{F}_n .

Now I will begin my actual proof. Since X and each Y_n are random variables on the probability space (Ω, \mathcal{F}, P) , then $X \in \mathcal{F}$ and $Y_n \in \mathcal{F}$ for each $n \in \mathbb{N}_0$. By definition of conditional expectation, we have that $X_n \in \mathcal{F}_n$ for all n and thus X_n is adapted to \mathcal{F}_n . Next, let's show $E(X_{n+1}|\mathcal{F}_n) = X_n$ for all n . Recall that if $\mathcal{F}_0 \subset \mathcal{F}_1$, then

$$E(E(X|\mathcal{F}_1)|\mathcal{F}_0) = E(X|\mathcal{F}_0).$$

Therefore, we have,

$$E(X_{n+1}|\mathcal{F}_n) = E(E(X|\mathcal{F}_{n+1})|\mathcal{F}_n) = E(X|\mathcal{F}_n) = X_n$$

since

$$\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n) \subset \sigma(Y_0, Y_1, \dots, Y_n, Y_{n+1}) = \mathcal{F}_{n+1}.$$

Finally, we want to show that $E|X_n| < \infty$ for all n . Additionally, by our definition of conditional expectation in lecture slides 10 we have that for all $A \in \mathcal{F}_n$,

$$\int_A Y dP = \int_A X dP.$$

Since, \mathcal{F}_n is a σ -algebra we can take $A = \Omega$ and then we have

$$\begin{aligned} E|X_n| &= E|E(X|\mathcal{F}_n)| \\ &= \int_{\Omega} |E(X|\mathcal{F}_n)| dP \\ &= \int_{\Omega} |Y| dP \\ &= \int_{\Omega} |X| dP \\ &= E|X| < \infty \end{aligned}$$

Therefore, $E|X_n| < \infty$. Hence, the sequence of random variables $(X_n)_{n \in \mathbb{N}_0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$. □

2. Let X_0, X_1, \dots be i.i.d Bernoulli random variables with parameter p (i.e., $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $S_n = \sum_{i=1}^n X_i$ where $S_0 = 0$. Define

$$Z_n = \left(\frac{1-p}{p} \right)^{2S_n - n}, \quad n = 0, 1, 2, \dots$$

Let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$. Show that Z_n is a martingale with respect to this filtration.

Solution:

We need to verify that the sequence of random variables $(Z_n)_{n \in \mathbb{N}_0}$ satisfies the requisite criteria to be a martingale. Let's begin by showing that $E|Z_n| < \infty$. Let's look more closely at the expected value of the absolute value of Z_n . Since $p \in [0, 1]$, we know that each Z_n is just a nonnegative number raised to some integer power so it is always positive. Therefore, in calculating the expected value we can drop the absolute value

$$\begin{aligned} E|Z_n| &= E\left(\left(\frac{1-p}{p} \right)^{2S_n - n} \right) \\ &= E\left(\left(\frac{1-p}{p} \right)^{2(\sum_{i=1}^n X_i) - n} \right) \\ &= \left(\frac{1-p}{p} \right)^{-n} E\left(\left(\frac{1-p}{p} \right)^{2\sum_{i=1}^n X_i} \right) \\ &= \left(\frac{1-p}{p} \right)^{-n} E\left(\prod_{i=1}^n \left(\frac{1-p}{p} \right)^{2X_i} \right) \\ &= \left(\frac{1-p}{p} \right)^{-n} \prod_{i=1}^n E\left(\left(\frac{1-p}{p} \right)^{2X_i} \right) \\ &= \left(\frac{1-p}{p} \right)^{-n} E\left(\left(\frac{1-p}{p} \right)^{2X} \right)^n. \end{aligned}$$

where the final few lines hold due to the fact that the X_i are i.i.d. It is important to note that we are taking the n th power of the expectation of our expression instead of the expectation of the expression to the n th power. The difference is important. Using the definition of expectation and the fact that the X 's are Bernoulli distributed we have

$$\begin{aligned} E|Z_n| &= \left(\frac{1-p}{p} \right)^{-n} \left(\int_{\Omega} \left(\frac{1-p}{p} \right)^{2X} dP \right)^n \\ &= \left(\frac{1-p}{p} \right)^{-n} \left(\sum_{x \in \{0,1\}} \left(\frac{1-p}{p} \right)^{2x} P(X=x) \right)^n \\ &= \left(\frac{1-p}{p} \right)^{-n} \left(P(X=0) + \left(\frac{1-p}{p} \right)^2 P(X=1) \right)^n \\ &= \left(\frac{1-p}{p} \right)^{-n} \left((1-p) + \left(\frac{1-p}{p} \right)^2 p \right)^n. \end{aligned}$$

I suppose we may be able to say something about the finiteness of the expected value at this point but I will continue with the algebra until it is more obvious to me. Getting common denominators inside the parenthesis on the right, we have

$$\begin{aligned}
E|Z_n| &= \left(\frac{1-p}{p}\right)^{-n} \left(\frac{p(1-p)}{p} + \frac{(1-p)^2}{p}\right)^n \\
&= \left(\frac{1-p}{p}\right)^{-n} \left(\frac{p(1-p) + (1-p)^2}{p}\right)^n \\
&= \frac{(1-p)^{-n}}{p^{-n}} \frac{\left((1-p)(p + (1-p))\right)^n}{p^n} \\
&= (1-p)^{-n} \left((1-p)(p + 1 - p)\right)^n \\
&= (1-p)^{-n} (1-p)^n \\
&= 1
\end{aligned}$$

Thus, $E|Z_n| < \infty$. Now we need to show that Z_n is adapted to \mathcal{F} or that $Z_n \in \mathcal{F}_n$ for all n . The pre-image of Z_n with respect to an arbitrary Borel Set is as follows

$$Z_n^{-1}(B) = \{\omega \mid Z_n(\omega) \in B\}.$$

TODO: Figure out how to show that Z_n is \mathcal{F}_n adapted.

Finally, we need to show, for all n , that

$$E(Z_{n+1}|\mathcal{F}_n) = Z_n.$$

Beginning on the right

$$\begin{aligned}
E(Z_{n+1}|\mathcal{F}_n) &= E\left(\left(\frac{1-p}{p}\right)^{2S_{n+1}-n-1} \middle| \sigma(X_0, X_1, \dots, X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} E\left(\left(\frac{1-p}{p}\right)^{2S_{n+1}} \middle| \sigma(X_0, X_1, \dots, X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} E\left(\left(\frac{1-p}{p}\right)^{2\left(\sum_{i=0}^n X_i\right) + 2X_{n+1}} \middle| \sigma(X_0, X_1, \dots, X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} E\left(\left(\frac{1-p}{p}\right)^{2\sum_{i=0}^n X_i} \left(\frac{1-p}{p}\right)^{2X_{n+1}} \middle| \sigma(X_0, X_1, \dots, X_n)\right).
\end{aligned}$$

Since, we are conditioning on $\sigma(X_0, X_1, \dots, X_n)$ each X_i up to X_n is constant with respect to this given information. Therefore it can be treated like a constant and pulled out of the expected value because of the linearity of expected value. We now proceed with this step and can drop the conditioning since X_{n+1} is independent

from the σ -algebra generated by the collection of X_i 's

$$\begin{aligned}
E(Z_{n+1}|\mathcal{F}_n) &= \left(\frac{1-p}{p}\right)^{-n-1} \left(\frac{1-p}{p}\right)^{2\sum_{i=0}^n X_i} E\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}} \middle| \sigma(X_0, X_1, \dots, X_n)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} \left(\frac{1-p}{p}\right)^{2S_n} E\left(\left(\frac{1-p}{p}\right)^{2X_{n+1}}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} \left(\frac{1-p}{p}\right)^{2S_n} \left(P(X_{n+1}=0) + \left(\frac{1-p}{p}\right)^2 P(X_{n+1}=1)\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} \left(\frac{1-p}{p}\right)^{2S_n} \left((1-p) + \frac{(1-p)^2}{p}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} \left(\frac{1-p}{p}\right)^{2S_n} \left(\frac{(1-p)(p+1-p)}{p}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n-1} \left(\frac{1-p}{p}\right)^{2S_n} \left(\frac{1-p}{p}\right) \\
&= \left(\frac{1-p}{p}\right)^{-n} \left(\frac{1-p}{p}\right)^{2S_n} \\
&= \left(\frac{1-p}{p}\right)^{2S_n-n} \\
&= Z_n.
\end{aligned}$$

Therefore, $E(Z_{n+1}|\mathcal{F}_n) = Z_n$ for all n . And hence, Z_n is a martingale with respect to $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$.

3. Let ξ_i be a sequence of random variables such that the partial sums

$$X_n = \xi_0 + \xi_1 + \dots + \xi_n, \quad n \geq 1,$$

determine a martingale. Show that the summands are mutually uncorrelated, i.e. that $E(\xi_i \xi_j) = E(\xi_i)E(\xi_j)$ for $i \neq j$.

Solution:

This means there exists some filtration \mathcal{F}_n such that X_n is \mathcal{F}_n adapted and both

$$E(X_{n+1}|\mathcal{F}_n) = X_n \text{ and } E|X_n| < \infty$$

hold. Then we also have that

$$\begin{aligned} E(X_{n+1}|\mathcal{F}_n) &= E\left(\sum_{i=0}^{n+1} \xi_i \middle| \mathcal{F}_n\right) \\ &= E\left(\xi_{n+1} + \sum_{i=0}^n \xi_i \middle| \mathcal{F}_n\right) \\ &= E(\xi_{n+1}|\mathcal{F}_n) + E\left(\sum_{i=0}^n \xi_i \middle| \mathcal{F}_n\right) \\ &= E(\xi_{n+1}|\mathcal{F}_n) + \sum_{i=0}^n E(\xi_i|\mathcal{F}_n). \end{aligned}$$

Recall, $E(X_{n+1}|\mathcal{F}_n) = X_n$ thus

$$\begin{aligned} E(X_{n+1}|\mathcal{F}_n) &= X_n \\ E(\xi_{n+1}|\mathcal{F}_n) + \sum_{i=0}^n E(\xi_i|\mathcal{F}_n) &= \sum_{i=0}^n \xi_i \\ E(\xi_{n+1}|\mathcal{F}_n) + \sum_{i=0}^n \xi_i &= \sum_{i=0}^n \xi_i \\ E(\xi_{n+1}|\mathcal{F}_n) &= 0. \end{aligned}$$

Does this get me any closer? **TODO: How does this help me prove that the summands are uncorrelated? From the fact that it's a martingale...**

4. Galton and Watson who invented the process that bears their names were interested in the survival of family names. Suppose each family has exactly 3 children but coin flips determine their sex. In the 1800s, only male children kept the family name so following the male offspring leads to a branching process with $p_0 = 1/8, p_1 = 3/8, p_2 = 3/8, p_3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$. What is ρ if we assume that each family has exactly 2 children?

Solution:

Let $Z_0 = 1$. Furthermore, define $Z_{n+1} = \xi_0^{n+1} + \xi_1^{n+1} + \xi_2^{n+1} + \dots + \xi_{Z_n}^{n+1}$ where this follows the same definition in class. Z_{n+1} represents the number males in the $n+1$ generation which bear the last name. as Z_n total males bearing the last name.

Or, rather, should I be computing the distribution from the probability generating function? Like

$$\begin{aligned}\frac{G_Z(0)}{0!} &= p_0 = \frac{1}{8} \\ \frac{G'_Z(0)}{1!} &= p_1 = \frac{3}{8} \\ \frac{G_Z^{(2)}(0)}{2!} &= p_2 = \frac{3}{8} \\ \frac{G_Z^{(3)}(0)}{3!} &= p_3 = \frac{1}{8}\end{aligned}$$

Possibly...

We really want to compute

$$P(Z_n = 0) = p_0 = G_{Z_n}(0)$$

TODO: Just figure out exactly what the right path forward is here