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AMATH 567

HOMEWORK 10

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: I sketched the following in class. Complete the argument. Show that for an integer $j \in (-N, N)$ and $h > 0$,

$$\lim_{h\to\infty} \int_{\mathrm{i}h}^{\mathrm{i}h+\pi} \frac{\mathrm{e}^{2\mathrm{i}jz}}{\tan(Nz)} \mathrm{d}z = \begin{cases} -\mathrm{i}\pi & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution:
TODO:
Following the sketch provided in class let's look at an import representation of $\tan(Nz)$

$$\begin{aligned}
\tan(Nz) &= \frac{\sin(Nz)}{\cos(Nz)} \\
&= \frac{\mathrm{e}^{\mathrm{i}Nz} + \mathrm{e}^{-\mathrm{i}Nz}}{2\mathrm{i}} \left(\frac{\mathrm{e}^{\mathrm{i}Nz} - \mathrm{e}^{-\mathrm{i}Nz}}{2} \right)^{-1} \\
&= \frac{\mathrm{e}^{\mathrm{i}Nz} + \mathrm{e}^{-\mathrm{i}Nz}}{2\mathrm{i}} \left(\frac{2}{\mathrm{e}^{\mathrm{i}Nz} - \mathrm{e}^{-\mathrm{i}Nz}} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{\mathrm{e}^{\mathrm{i}Nz} + \mathrm{e}^{-\mathrm{i}Nz}}{\mathrm{e}^{\mathrm{i}Nz} - \mathrm{e}^{-\mathrm{i}Nz}} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{\mathrm{e}^{\mathrm{i}Nz}}{\mathrm{e}^{\mathrm{i}Nz}} \left(\frac{1 + \mathrm{e}^{-2\mathrm{i}Nz}}{1 - \mathrm{e}^{-2\mathrm{i}Nz}} \right) \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{1 + \mathrm{e}^{-2\mathrm{i}Nz}}{1 - \mathrm{e}^{-2\mathrm{i}Nz}} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{1 + (\cos(Nz) - \mathrm{i}\sin(Nz))^2}{1 - (\cos(Nz) - \mathrm{i}\sin(Nz))^2} \right) \\
&= \frac{1}{\mathrm{i}} \left(\frac{1 + \cos^2(Nz) - 2\mathrm{i}\cos(Nz)\sin(Nz) - \sin^2(Nz)}{1 - \cos^2(Nz) + 2\mathrm{i}\cos(Nz)\sin(Nz) + \sin^2(Nz)} \right) \\
&\dots \\
&= \mathrm{i} + \mathcal{O}(\mathrm{e}^{-2Nh})
\end{aligned}$$

2: From A&F: 4.2.1 (b)

Solution:
See solution to problem 2 from homework set 9.

$\square$

3: From A&F: 4.2.2 (a, h)
Evaluate the following real integrals by residue integration:

(a)

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} \mathrm{d}x, \qquad a^2 > 0$$

Solution:
We can also look at just the imaginary part of another version of this integral

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} \mathrm{d}x &= \operatorname{Im} \int_{-\infty}^{\infty} \frac{x\, \mathrm{e}^{\mathrm{i}x}}{x^2+a^2} \mathrm{d}x \\ &= \operatorname{Im} \int_{-\infty}^{\infty} \frac{x\,(\cos x + \mathrm{i} \sin x)}{x^2+a^2} \mathrm{d}x \\ &= \operatorname{Im} \left[\int_{-\infty}^{\infty} \frac{x \cos x}{x^2+a^2} \mathrm{d}x + \mathrm{i} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} \mathrm{d}x \right] \\ &= \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} \mathrm{d}x. \end{aligned}$$

Therefore, we consider the integral

$$\int_{-\infty}^{\infty} \frac{x\, \mathrm{e}^{\mathrm{i}x}}{x^2+a^2} \mathrm{d}x$$

and then take the imaginary part at the end. In order to evaluate this integral we look at the integral over the contour C which is a counterclockwise semicircle in the upper half plane with radius R. Then applying the Residue Theorem and taking the limit as $R \to \infty$, we have the following

$$\begin{aligned} \oint_C \frac{z\, \mathrm{e}^{\mathrm{i}z}}{z^2+a^2} \mathrm{d}z &= \int_{-R}^{R} \frac{x\, \mathrm{e}^{\mathrm{i}x}}{x^2+a^2} \mathrm{d}x + \oint_{C^R} \frac{z\, \mathrm{e}^{\mathrm{i}z}}{z^2+a^2} \mathrm{d}z \\ 2\pi \mathrm{i} \sum_{w \in S} \left(\frac{z\, \mathrm{e}^{\mathrm{i}z}}{z^2+a^2} \right) &= \int_{-\infty}^{\infty} \frac{x\, \mathrm{e}^{\mathrm{i}x}}{x^2+a^2} \mathrm{d}x + \cancelto{0}{\oint_{C^R} \frac{z\, \mathrm{e}^{\mathrm{i}z}}{z^2+a^2} \mathrm{d}z} \end{aligned}$$

where S is the set of singularities of our function in the upper half plane and I claim the C_R contour integral goes to 0 by Jordan's Lemma. I will verify this second claim next.

To justify using Jordan's Lemma I want to note we are using $k = 1$ in assumptions from the lemma of concern. Moreover, I will be assuming $a > 0$ for now. Additionally, we need to show $f(z) \to 0$ **uniformly** as $R \to \infty$ in C_R, that is if $|f(z)| \leq K_R$, where K_R only depends on R and not arg z and $K_R \to 0$ as $R \to \infty$. Let's first find this bound K_R

$$
\begin{aligned}
|f(z)| &= \left| \frac{z}{z^2 + a^2} \right| \\
&= \left| \frac{R\,\mathrm{e}^{\mathrm{i}\theta}}{R^2\,\mathrm{e}^{2\mathrm{i}\theta} + a^2} \right| \\
&= \frac{R}{|R^2\,\mathrm{e}^{2\mathrm{i}\theta} + a^2|} \\
&\le \frac{R}{\big|\,|R^2\,\mathrm{e}^{2\mathrm{i}\theta}| - |-a^2|\,\big|} \\
&\le \frac{R}{R^2 - a^2}.
\end{aligned}
$$

In the final two steps with inequalities we first apply the inverse triangle inequality, followed by recognizing the following. Since R is becoming arbitrarily large then for a given a eventually R will be larger such that $R^2 > a^2$ and therefore the expression $R^2 - a^2 > 0$ thus we can drop the absolute value in the end. Now, let $K_R = R/(R^2 - a^2)$. Clearly the denominator will win out as we take $R \to \infty$ since it has a squared R in it, while the numerator only has a linear R. Therefore, $K_R \to 0$ as $R \to \infty$ and thus $f(z) \to 0$ **uniformly** as $R \to \infty$ in C_R. Thus we have verified that

$$
\oint_{C^R} \frac{z\,\mathrm{e}^{\mathrm{i}z}}{z^2 + a^2}\,\mathrm{d}z \to 0
$$

by Jordan's Lemma.

TODO: Square away that this is a good argument or not.
We need to evaluate this limit

$$
\lim_{R\to\infty} \frac{z}{z^2 + a^2} = \frac{R}{R^2 + a^2} = \frac{\infty}{\infty}
$$

and then use L'Hôpitals to get

$$
\lim_{R\to\infty} \frac{z}{z^2 + a^2} = \frac{1}{z} = 0
$$

Therefore we no longer need to be concerned with the integral over C_R. Notice the denominator of our function factors to $(z - \mathrm{i}a)(z + \mathrm{i}a)$ therefore the set of singularities in the upper half plane is $S = \{\mathrm{i}a\}$. Hence we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\,\mathrm{e}^{\mathrm{i}x}}{x^2 + a^2}\mathrm{d}x &= 2\pi\mathrm{i} \sum_{w\in S} \operatorname*{Res}_{z=w} \left(\frac{z\,\mathrm{e}^{\mathrm{i}z}}{z^2 + a^2} \right) \\
&= 2\pi\mathrm{i} \operatorname*{Res}_{z=\mathrm{i}a} \left(\frac{z\,\mathrm{e}^{\mathrm{i}z}}{(z - \mathrm{i}a)(z + \mathrm{i}a)} \right) \\
&= 2\pi\mathrm{i} \left(\frac{\mathrm{i}a\,\mathrm{e}^{\mathrm{i}^2 a}}{\mathrm{i}a + \mathrm{i}a} \right) \\
&= 2\pi\mathrm{i} \left(\frac{\mathrm{i}a\,\mathrm{e}^{-a}}{2\mathrm{i}a} \right) \\
&= \pi\mathrm{i}\,\mathrm{e}^{-a}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} \mathrm{d}x &= \operatorname{Im}\left[\int_{-\infty}^{\infty} \frac{x\, \mathrm{e}^{\|x}}{x^2 + a^2} \mathrm{d}x\right] \\
&= \operatorname{Im}\left[\pi \mathrm{i}\, \mathrm{e}^{-a}\right] \\
&= \pi\, \mathrm{e}^{-a}
\end{aligned}$$

TODO: consider doing a case where $a < 0$.

(h)

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(5 - 3\sin\theta)^2}$$

Solution:
Let's begin by reverse parameterizing this into a contour integral around the unit circle. Notice, using the normal parameterization but going the other way we have, $z = \mathrm{e}^{\mathrm{i}\theta}$ and

$$\mathrm{d}z = \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta \implies \frac{1}{\mathrm{i}z} \mathrm{d}z = \mathrm{d}\theta.$$

Additionally, notice,

$$\sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}} = \frac{z - \frac{1}{z}}{2\mathrm{i}}.$$

Hence, our reverse parameterization can get us here (with a little simplification)

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{(5 - 3\sin\theta)^2} &= \oint_{\partial B_1(0)} \left(5 - 3\left(\frac{z - \frac{1}{z}}{2\mathrm{i}}\right)\right)^{-2} \frac{\mathrm{d}z}{\mathrm{i}z} \\
&= \oint_{\partial B_1(0)} \left(5 - \frac{3z}{2\mathrm{i}} + \frac{3}{2\mathrm{i}z}\right)^{-2} \frac{\mathrm{d}z}{\mathrm{i}z} \\
&= \oint_{\partial B_1(0)} \left(\frac{10\mathrm{i}z - 3z^2 + 3}{2\mathrm{i}z}\right)^{-2} \frac{\mathrm{d}z}{\mathrm{i}z} \\
&= \oint_{\partial B_1(0)} \frac{(2\mathrm{i}z)^2}{\left(10\mathrm{i}z - 3z^2 + 3\right)^2 \mathrm{i}z} \mathrm{d}z \\
&= \oint_{\partial B_1(0)} \frac{4\mathrm{i}z}{\left(10\mathrm{i}z - 3z^2 + 3\right)^2} \mathrm{d}z.
\end{aligned}$$

Now we need to factor the quadratic in the denominator to determine the singularities of the integrand

$$\begin{aligned}
10\mathrm{i}z - 3z^2 + 3 &= (\mathrm{i} - 3z)(z - 3\mathrm{i}) \\
&= -3\left(z - \mathrm{i}/3\right)\left(z - 3\mathrm{i}\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
\oint_{\partial B_1(0)} \frac{4\mathrm{i}z}{\left(10\mathrm{i}z - 3z^2 + 3\right)^2} \mathrm{d}z &= \oint_{\partial B_1(0)} \frac{4\mathrm{i}z}{\left(-3\left(z - \mathrm{i}/3\right)\left(z - 3\mathrm{i}\right)\right)^2} \mathrm{d}z \\
&= \oint_{\partial B_1(0)} \frac{4\mathrm{i}z}{9(z - \mathrm{i}/3)^2 (z - 3\mathrm{i})^2} \mathrm{d}z \\
&= 2\pi\mathrm{i} \operatorname*{Res}_{z = \mathrm{i}/3} \left(\frac{4\mathrm{i}z}{9(z - \mathrm{i}/3)^2 (z - 3\mathrm{i})^2}\right).
\end{aligned}$$

Let's compute the residue at the simple pole with this formula

$$
\begin{aligned}
2\pi\mathrm{i}\,\operatorname*{Res}_{z=\mathrm{i}a}\left(\frac{4\mathrm{i}z}{9(z-\mathrm{i}/3)^2(z-3\mathrm{i})^2}\right) &= 2\pi\mathrm{i}\,\frac{1}{(2-1)!}\frac{\mathrm{d}^{2-1}}{\mathrm{d}z^{2-1}}\left(\cancel{(z-\mathrm{i}/3)^2}\frac{4\mathrm{i}z}{9\cancel{(z-\mathrm{i}/3)^2}(z-3\mathrm{i})^2}\right)\Bigg|_{\mathrm{i}/3}\\
&= 2\pi\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{4\mathrm{i}z}{9(z-3\mathrm{i})^2}\right)\Bigg|_{\mathrm{i}/3}\\
&= 2\pi\mathrm{i}\,\left(\frac{4\mathrm{i}9(z-3\mathrm{i})^2-4\mathrm{i}z(9\cdot 2(z-3\mathrm{i}))}{9^2(z-3\mathrm{i})^4}\right)\Bigg|_{\mathrm{i}/3}\\
&= 2\pi\mathrm{i}\,\left(\frac{4\mathrm{i}(z-3\mathrm{i})-8\mathrm{i}z}{9(z-3\mathrm{i})^3}\right)\Bigg|_{\mathrm{i}/3}\\
&= 2\pi\mathrm{i}\,\left(\frac{4\mathrm{i}z+12-8\mathrm{i}z}{9(z-3\mathrm{i})^3}\right)\Bigg|_{\mathrm{i}/3}\\
&= 2\pi\mathrm{i}\,\left(\frac{-4\mathrm{i}z+12}{9(z-3\mathrm{i})^3}\right)\Bigg|_{\mathrm{i}/3}\\
&= 2\pi\mathrm{i}\,\left(\frac{-4\mathrm{i}\frac{\mathrm{i}}{3}+12}{9(\frac{\mathrm{i}}{3}-3\mathrm{i})^3}\right)\\
&= 2\pi\mathrm{i}\,\left(\frac{\frac{4}{3}+12}{9(\frac{-8\mathrm{i}}{3})^3}\right)\\
&= 2\pi\mathrm{i}\,\left(\frac{\frac{40}{3}}{9\frac{(-1)^3 8\cdot 64\mathrm{i}^3}{9\cdot 3}}\right)\\
&= 2\pi\mathrm{i}\,\left(\frac{40}{-8\cdot 64(-1)\mathrm{i}}\right)\\
&= \pi\,\left(\frac{40}{8\cdot 32}\right)\\
&= \frac{5\pi}{32}.
\end{aligned}
$$

□

4: (a) Show that

$$\mathrm{Res}_{z=k}\, f(z)\cot(\pi z) = \frac{1}{\pi} f(k),$$

provided $f(z)$ is analytic at $z = k,\ k \in \mathbb{Z}$.

Solution:

Recall that if $z = z_k$ is a pole of order N of $f(z)\cot \pi z$ then

$$\operatorname*{Res}_{z=z_k} (f(z)\cot \pi z) = \frac{1}{(N-1)!} \lim_{z\to z_k} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \left[(z-z_k)^N f(z)\cot \pi z\right].$$

Notice, that the only places where our function $f(z)\cot \pi z$ only blows up in the locations where $\tan(\pi z) = 0$, therefore, $\sin \pi z = 0$. This holds at all the integers $z = k$. Each $k \in \mathbb{Z}$ will therefore be a simple pole of $f(z)\cot \pi z$. Then let's calculate

$$\begin{aligned}
\operatorname*{Res}_{z=k} (f(z)\cot \pi z) &= \frac{1}{(1-1)!} \lim_{z\to k} \frac{\mathrm{d}^{1-1}}{\mathrm{d}z^{1-1}} \left[(z-k)^1 f(z)\cot \pi z\right] \\
&= \lim_{z\to k} \left[(z-k) f(z)\cot \pi z\right] \\
&= \lim_{z\to k} \left[\frac{(z-k)f(z)}{\tan \pi z}\right] \\
&= \frac{(k-k)f(k)}{\tan \pi k} \\
&= \frac{0}{0}.
\end{aligned}$$

Using L'Hôpital's, we have

$$\begin{aligned}
\lim_{z\to k} \left[\frac{(z-k)f(z)}{\tan \pi z}\right] &= \lim_{z\to k} \left[\frac{\frac{\mathrm{d}}{\mathrm{d}z}(z-k)f(z)}{\frac{\mathrm{d}}{\mathrm{d}z}\tan \pi z}\right] \\
&= \lim_{z\to k} \left[\frac{f(z)+(z-k)f'(z)}{\pi \sec^2(\pi z)}\right] \\
&= \frac{f(k)+(k-k)f'(k)}{\pi \sec^2(\pi k)} \\
&= \frac{1}{\pi}(f(k)+(k-k)f'(k))\cos^2(\pi k) \\
&= \frac{1}{\pi} f(k)\cos^2(\pi k) \\
&= \frac{1}{\pi} f(k).
\end{aligned}$$

$\square$

(b) Let Γ_N be a square contour, with corners at $(N+1/2)(\pm 1 \pm i), N \in \mathbb{Z}^+$. Show that

$$|\cot(\pi z)| \leq 2,$$

for z on Γ_N.

Solution:
Consider the following representation of $\cot \pi z$ with some manipulation

$$\begin{aligned}
|\cot \pi z| = \left|\frac{1}{\tan \pi z}\right| &= \left|\frac{\cos \pi z}{\sin \pi z}\right| \\
&= \left|\cos \pi z (\sin \pi z)^{-1}\right| \\
&= \left|\frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{2}\left(\frac{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}{2\mathrm{i}}\right)^{-1}\right| \\
&= \left|\frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{2}\left(\frac{2\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}\right)\right| \\
&= \left|\frac{\mathrm{i}\left(\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}\right)}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}\right| \\
&\leq |\mathrm{i}| \left|\frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}\right| \\
&= \left|\frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}\right| \\
&= \left|\frac{\mathrm{e}^{\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z}} \frac{1 + \mathrm{e}^{-2\pi \mathrm{i} z}}{1 - \mathrm{e}^{-2\pi \mathrm{i} z}}\right| \\
&= \left|\frac{1 + \mathrm{e}^{-2\pi \mathrm{i} z}}{1 - \mathrm{e}^{-2\pi \mathrm{i} z}}\right|.
\end{aligned}$$

We will analyze this to show $|\cot \pi z| \leq 2$ along the contour Γ_N.
TODO: Parameterize the contour (along the top and bottom pieces of the rectangle where the real part is parameterized but the imaginary part is constant) to get something in terms of $\coth \pi z$ and show that that is bounded by 2 since it is a decreasing function. TODO: Do something similar to this for the sides of the contour where the imaginary part is variable but the real part is constant.

$$f(x) = mx + b$$

(c) Suppose $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, so that the degree of $q(z)$ is at least two more than the degree of $p(z)$. Show that

$$\lim_{N \to \infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| = 0$$

Solution:
TODO:

$$f(x) = mx + b$$

(d) Suppose, in addition, that $q(z)$ has no roots at the integers. Show that

$$\sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = -\pi \sum_j \mathrm{Res}_{z=z_j}\, f(z)\cot(\pi z)$$

where the z_j's are the roots of $q(z)$. Notice that the sum on the right-hand side has a finite number of terms.

Solution:
TODO:

$$f(x) = mx + b$$

(e) Use the result of the previous problem to evaluate the following sums:

(i) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1}$

Solution:
TODO:

$$f(x) = mx + b$$

(ii) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^4+1}$

Solution:
TODO:

$$f(x) = mx + b$$

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2-1/4}$

Solution:
TODO:

$$f(x) = mx + b$$

(iv) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{16k^4-1}$

Solution:
TODO:

$$f(x) = mx + b$$