AMATH 567 FALL 2024
HOMEWORK 11 — DUE DECEMBER 6 ON GRADESCOPE BY 1:30PM

All solutions must include significant justification to receive full credit. If you handwrite your assignment you must either do so digitally or if it is written on paper you must *scan* your work. A standard photo is not sufficient.

If you work with others on the homework, you must name your collaborators.

1: (a) Let $f : z \to w = f(z)$ be an analytic function on the closed disk $D(z_0, R)$ of radius R centered at z_0. Denote the boundary of $D(z_0, R)$ by $C(z_0, R)$. Assume that $f : D(z_0, R) \to f(D(z_0, R))$ is one-to-one and onto. Show that the inverse function to f is

$$g(w) = \frac{1}{2\pi i} \oint_{C(z_0,R)} \frac{t f'(t)}{f(t) - w} dt,$$

for $w \in f(D(z_0, R))$, not including the boundary.

(b) Use this result to calculate the Taylor series of the inverse function of $w = ze^z$ around $(z_0, w_0) = (0, 0)$. What is the radius of convergence of this series? Does this radius of convergence correspond with what you expect from real analysis?

2: Suppose that $A \in \mathbb{C}^{n\times n}$ and $\mathbf{u}$ is an eigenvector with eigenvalue λ. Show that if no eigenvalue of A is on a simple contour Γ and that λ is in the interior of Γ then

$$\mathbf{u} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} (zI - A)^{-1} \mathbf{u} \mathrm{d}z.$$

Hint: Use $I = (zI - A)^{-1}(wI - A) + (zI - A)^{-1}(z - w)$.

3: Suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A \in \mathbb{C}^{n\times n}$ are real. Show that for any $c > \lambda_n$ that

$$\mathrm{e}^{-ct}\,\mathrm{e}^{At} \to 0, \quad t \to \infty.$$

4: Laplace transform for systems of differential equations. For this problem, recall the Laplace transform

$$\mathcal{L}[f](z) = \int_0^\infty f(t)\,\mathrm{e}^{-tz}\,\mathrm{d}t.$$

For vector-valued functions, we consider it applied component wise.

(a) Show that

$$\mathcal{L}[y'](z) = z\mathcal{L}[y](z) - y(0).$$

You assume as much as you want for y.

(b) If $\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{b}(t)$, show that

$$\mathcal{L}[\mathbf{y}](z) = (zI - A)^{-1}(\mathcal{L}[\mathbf{b}](z) - \mathbf{y}(0)),$$

whenever z is not an eigenvalue of A.

(c) Write an integral formula for $\mathbf{y}(t)$.

5: Define $w(z) = \sqrt{z-1}\sqrt{z+1}$. Suppose $\rho : [-1,1] \to \mathbb{R}$ is smooth and $\rho(x) > 0$. Consider

$$G(z) = \exp\left(-\frac{w(z)}{2\pi}\int_{-1}^{1} \frac{\log \rho(x)}{x-z} \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right), \quad z \notin (-1,1).$$

Define $G_\pm(x) = \lim_{\epsilon \to 0^+} G(x \pm \mathrm{i}\epsilon)$. Show

$$G_+(x)G_-(x) = \rho(x), \quad x \in (-1,1).$$

On a historal note, the function G is an example of a Szegő function, named after Gábor Szegő, and is critical in describing the large-degree behavior of orthogonal polynomials. Specifically, if p_n is the degree n orthonormal polynomial for a weight function $\rho(x)$ supported on $[-1,1]$ then

$$\lim_{n\to\infty} \frac{p_n(z)}{\varphi(z)^n} = \frac{\varphi(z)^{1/2}}{\sqrt{2\pi}(z^2-1)^{1/4}G(z)}, \quad z \in \mathbb{C} \setminus [-1,1].$$