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AMATH 567

HOMEWORK 4

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.4.2 c, e.
Evaluate the integral $\oint_C f(z)\mathrm{d}z$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows:
c)

$$f(z) = \frac{1}{\bar{z}}$$

Solution:
We want to evaluate

$$\oint_C \frac{1}{\bar{z}}\mathrm{d}z$$

on the parameterized unit circle $z = \mathrm{e}^{i\theta}$ where $\theta \in [0, 2\pi)$, where $\bar{z} = \mathrm{e}^{-i\theta}$ on the unit circle. Note, before we do the substitution we need $\mathrm{d}z = i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta$. Now our integral is

$$\begin{aligned}\oint_C \frac{1}{\bar{z}}\mathrm{d}z &= \oint_0^{2\pi} \frac{1}{\mathrm{e}^{-i\theta}} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \oint_0^{2\pi} i\,\mathrm{e}^{2i\theta}\,\mathrm{d}\theta \\ &= \left(\frac{1}{2}\,\mathrm{e}^{2i\theta}\Big|_0^{2\pi} \right) \\ &= \frac{1}{2}\,\mathrm{e}^{4\pi i} - \frac{1}{2}\,\mathrm{e}^{0} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0\end{aligned}$$

$\square$

e)

$$f(z) = \mathrm{e}^{\bar{z}}$$

Solution:
We will use the same substitutions from the previous part **TODO:** maybe change this

back to regular integrals instead of contour integrals after you have parameterized.

$$\begin{aligned}
\oint_C e^{\bar{z}}\, dz &= \oint_0^{2\pi} e^{e^{-i\theta}}\, i\, e^{i\theta}\, d\theta \\
&= \oint_0^{2\pi} \sum_{j=1}^{\infty} \frac{\left(e^{-i\theta}\right)^j}{j!} i\, e^{i\theta}\, d\theta \\
&= \sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^j}{j!} e^{i\theta} d\theta.
\end{aligned}$$

We are justified in reordering the integral of the infinite sum to be the infinite sum of the integrals since the original series converges absolutely. Notice we can pull out the first term where $j = 1$ to get,

$$\sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^j}{j!} e^{i\theta} d\theta = \oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^1}{1!} e^{i\theta} d\theta + \sum_{j=2}^{\infty} \oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^j}{j!} e^{i\theta} d\theta.$$

We need to look more closely at this first term, observe

$$\oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^1}{1!} e^{i\theta}\, d\theta = \oint_0^{2\pi} i \frac{e^{-i\theta}\, e^{i\theta}}{1!} d\theta = \oint_0^{2\pi} i \frac{1}{1} d\theta = i \oint_0^{2\pi} d\theta = 2\pi i.$$

Now, I will focus on the integral inside the sum where $j = 2, 3, 4, ...$

$$\begin{aligned}
\oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^j}{j!} e^{i\theta} d\theta &= \oint_0^{2\pi} i \frac{e^{-i\theta j}\, e^{i\theta}}{j!} d\theta \\
&= \oint_0^{2\pi} i \frac{e^{-i\theta j + i\theta}}{j!} d\theta \\
&= \oint_0^{2\pi} i \frac{e^{i\theta(-j+1)}}{j!} d\theta \\
&= \oint_0^{2\pi} \frac{i\, e^{i\theta(1-j)}}{j!} d\theta \\
&= \frac{1}{1-j} \frac{i\, e^{i\theta(1-j)}}{j!} \Bigg|_0^{2\pi} \\
&= \frac{1}{1-j} \frac{i\, e^{i2\pi(1-j)}}{j!} - \frac{1}{1-j} \frac{i\, e^0}{j!} \\
&= \frac{i}{(1-j)\, j!} \left(e^{i2\pi(1-j)} - 1\right) \\
&= \frac{i}{(1-j)\, j!} (1-1) \\
&= 0.
\end{aligned}$$

I want to clarify why $e^{i2\pi(1-j)} = 1$. Since $j \in \{2, 3, ...\}$, then $1 - j$ is an integer and we have $e^{i2\pi\ell}$ where $\ell \in \mathbb{Z}$, thus $e^{i2\pi\ell} = 1$. Now we return to the original problem

$$\sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{\left(e^{-i\theta}\right)^j}{j!} e^{i\theta} d\theta = 2\pi i + \sum_{j=2}^{\infty} 0 = 2\pi i.$$

Now we have completed the requisite task. $\square$

2: From A&F: 2.4.4 a, b. Use the principal branch where the argument is in $[-\pi, \pi)$. Discuss any ambiguities. Use the principal branch of $\log(z)$ and $z^{\frac{1}{2}}$ where the argument is in $[-\pi, \pi)$ to evaluate the following:

a)

$$\int_{-1}^{1} \log z \mathrm{d}z$$

Solution:

We want to parameterize this once again using $z = r\,\mathrm{e}^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our integral is

$$\begin{aligned}
\int_{-1}^{1} \log z \mathrm{d}z &= \int_{-\pi}^{0} \log\left(\mathrm{e}^{i\theta}\right) i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{0} i\theta i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta.
\end{aligned}$$

Let's use integration by parts, woohoo! We will assign the substitutions as follows:

$$\begin{aligned}
u &= i\theta \\
\mathrm{d}u &= i\mathrm{d}\theta
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}v &= i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
v &= \mathrm{e}^{i\theta}.
\end{aligned}$$

Plugging this in we have

$$\begin{aligned}
\int_{-\pi}^{0} i\theta i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta &= i\theta\,\mathrm{e}^{i\theta}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \left(0 - \left(-i\pi\,\mathrm{e}^{-i\pi}\right)\right) - \mathrm{e}^{i\theta}\Big|_{-\pi}^{0} \\
&= 0 + i\pi\,\mathrm{e}^{-i\pi} - \mathrm{e}^{i\theta}\Big|_{-\pi}^{0} \\
&= i\pi\,\mathrm{e}^{-i\pi} - \left(\mathrm{e}^{0} - \mathrm{e}^{-i\pi}\right) \\
&= -i\pi - (1 - (-1)) \\
&= -i\pi - (2) \\
&= -i\pi - 2.
\end{aligned}$$

$\square$

b)

$$\int_{-1}^{1} z^{\frac{1}{2}} \mathrm{d}z$$

Solution:

We want to parameterize this once again using $z = r\,\mathrm{e}^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our integral is

$$\begin{aligned}
\int_{-1}^{1} z^{\frac{1}{2}} \mathrm{d}z &= \int_{-\pi}^{0} \left(\mathrm{e}^{i\theta}\right)^{\frac{1}{2}} i\, \mathrm{e}^{i\theta}\, \mathrm{d}\theta \\
&= \int_{-\pi}^{0} i\, \mathrm{e}^{\frac{i\theta}{2}}\, \mathrm{e}^{i\theta}\, \mathrm{d}\theta \\
&= \int_{-\pi}^{0} i\, \mathrm{e}^{\frac{i3}{2}\theta}\, \mathrm{d}\theta \\
&= \frac{2}{3}\, \mathrm{e}^{\frac{i3}{2}\theta} \Big|_{-\pi}^{0} \\
&= \frac{2}{3}\, \mathrm{e}^{-\frac{i3}{2}\pi} - \frac{2}{3}.
\end{aligned}$$

Now remembering our branch cut limits θ to be within $[-\pi, \pi)$ we change the angle $-\frac{3}{2}\pi$ to be $\frac{1}{2}\pi$. Hence,

$$\begin{aligned}
\frac{2}{3}\, \mathrm{e}^{-\frac{i3}{2}\pi} - \frac{2}{3} &= \frac{2}{3}\, \mathrm{e}^{-\frac{i2}{2}\pi}\, \mathrm{e}^{-\frac{i\pi}{2}} - \frac{2}{3} \\
&= \frac{2}{3}\, \mathrm{e}^{\frac{i\pi}{2}} - \frac{2}{3} \\
&= \frac{2}{3}\left(i - 1\right).
\end{aligned}$$

$\square$

3: From A&F: 2.4.7
Let C be an open (upper) semicircle of radius R with its center at the origin, and consider $\int_C f(z)\mathrm{d}z$. Let $f(z) = \frac{1}{z^2+a^2}$ for a real $a > 0$. Show that $|f(z)| \le \frac{1}{R^2-a^2}$, $R > a$, and

$$\left|\int_C f(z)dz\right| \le \frac{\pi R}{R^2 - a^2}, \quad R > a.$$

Solution:
First, we want to show

$$|f(z)| \le \frac{1}{R^2 - a^2}$$

where $R > a > 0$ and a $\in \mathbb{R}$. Let's consider the function more closely

$$\begin{aligned}
f(z) = \frac{1}{z^2 + a^2} &= \frac{1}{(x + \mathrm{i}y)^2 + a^2} \\
&= \frac{1}{x^2 + 2ixy - y^2 + a^2} \\
&= \frac{1}{x^2 - y^2 + a^2 + i2xy}.
\end{aligned}$$

Notice, we can write the real and imaginary parts of the complex number in the denominator as functions $u(x, y)$ and $v(x, y)$. Where $u(x, y) = x^2 - y^2 + a^2$ and $v(x, y) = 2xy$. Now we get

$$f(z) = \frac{1}{x^2 - y^2 + a^2 + i2xy} = \frac{u - iv}{u - iv} \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}.$$

Then we calculate

$$
\begin{aligned}
|f(z)| &= \left| \frac{u - iv}{u^2 + v^2} \right| \\
&= \left| \frac{u}{u^2 + v^2} - i \frac{v}{u^2 + v^2} \right| \\
&= \sqrt{\left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{v}{u^2 + v^2}\right)^2} \\
&= \sqrt{\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2}} \\
&= \sqrt{\frac{u^2 + v^2}{(u^2 + v^2)^2}} \\
&= \sqrt{\frac{1}{u^2 + v^2}} \\
&= \frac{1}{\sqrt{u^2 + v^2}}.
\end{aligned}
$$

If we plug our substitution back in we see

$$
\begin{aligned}
\frac{1}{\sqrt{u^2 + v^2}} &= \frac{1}{\sqrt{(x^2 - y^2 + a^2)^2 + (2xy)^2}} \\
&= \frac{1}{\sqrt{(x^2 - y^2 + a^2)(x^2 - y^2 + a^2) + 4x^2y^2}} \\
&= \frac{1}{\sqrt{x^4 - x^2y^2 + x^2a^2 - x^2y^2 + y^4 - y^2a^2 + x^2a^2 - y^2a^2 + a^4 + 4x^2y^2}} \\
&= \frac{1}{\sqrt{x^4 + x^2a^2 + y^4 - y^2a^2 + x^2a^2 - y^2a^2 + a^4 + 2x^2y^2}}.
\end{aligned}
$$

Now we add zero in a particular fashion, namely $-4x^2a^2 + 4x^2a^2$, so we can regroup the terms and refactor to get closer to what we desire

$$
\begin{aligned}
&= \frac{1}{\sqrt{x^4 + y^4 - y^2a^2 + 2x^2a^2 - y^2a^2 + a^4 + 2x^2y^2 + (-4x^2a^2 + 4x^2a^2)}} \\
&= \frac{1}{\sqrt{x^4 + y^4 - 2y^2a^2 + a^4 + 2x^2y^2 - 2x^2a^2 + 4x^2a^2}} \\
&= \frac{1}{\sqrt{(x^2 + y^2 - a^2)^2 + (2xa)^2}}.
\end{aligned}
$$

Using the fact that $\sqrt{a+b} \geq \sqrt{a}$ for $a, b > 0$, in our next step we get a smaller denominator which makes the overall expression greater or equal to the previous step.

Note, equality only holds when $x = 0$.

$$\begin{aligned}
\frac{1}{\sqrt{\left(x^2+y^2-a^2\right)^2+(2xa)^2}} &\leq \frac{1}{\sqrt{\left(x^2+y^2-a^2\right)^2}} \\
&= \frac{1}{x^2+y^2-a^2} \\
&= \frac{1}{|z|^2-a^2} \\
&= \frac{1}{R^2-a^2}
\end{aligned}$$

Therefore $|f(z)| \leq \frac{1}{R^2-a^2}$.

□

Next we wish to show that

$$\left|\int_C f(z)dz\right| \leq \frac{\pi R}{R^2-a^2}, \quad R > a.$$

By Theorem 2.4.2 from A&F, if $f(z)$ is continuous on contour C, then

$$\left|\int_C f(z)dz\right| \leq ML$$

where L is the length of C and M is an upper bound for $|f(z)|$. We have that C is continuous, since $a > 0$ and $a < R$ and there are no singularities or weirdness with $f(z)$ on the specified contour. So we have

$$M = \frac{1}{R^2-a^2}$$

as we calculated in the first part of this problem. Additionally, we know the arc length of C is easy to calculate because it is half the circumference of the circle with radius R. To convince myself of this I will show the general arc length formula also provides this quick calculation. Let our parameterization of this contour be

$$z(\theta) = R\,\mathrm{e}^{i\theta}$$

where $\theta \in [0, \pi)$. Then

$$z'(\theta) = Ri\,\mathrm{e}^{i\theta} = -R\sin\theta + iR\cos\theta.$$

Therefore calculating arc length is as follows,

$$\begin{aligned}
L &= \int_a^b |z'(t)|\mathrm{d}t \\
&= \int_0^\pi \left|-R\sin\theta + iR\cos\theta\right| \mathrm{d}\theta \\
&= \int_0^\pi \sqrt{R^2\sin^2\theta + R^2\cos^2\theta}\mathrm{d}\theta \\
&= \int_0^\pi \sqrt{R^2\left(\sin^2\theta+\cos^2\theta\right)}\mathrm{d}\theta \\
&= \int_0^\pi R\mathrm{d}\theta \\
&= \pi R.
\end{aligned}$$

Which is the same as half the circumference ($\frac{1}{2}2\pi R = \pi R$). And thus

$$\left|\int_C f(z)dz\right| \leq ML \leq \frac{1}{R^2 - a^2}\pi R = \frac{\pi R}{R^2 - a^2}.$$

Hence,

$$\left|\int_C f(z)dz\right| \leq \frac{\pi R}{R^2 - a^2}$$

as desired. □

4: From A&F: 2.4.8

Let C be an arc of the circle $|z| = R$ with $(R > 1)$ of angle $\frac{\pi}{3}$. Show that

$$\left|\int_C \frac{\mathrm{d}z}{z^3 + 1}\right| \leq \frac{\pi}{3}\left(\frac{R}{R^3 - 1}\right)$$

and deduce

$$\lim_{R\to\infty}\int_C \frac{\mathrm{d}z}{z^3 + 1} = 0$$

Solution:

Similar to the previous problem, we will utilize Theorem 2.4.2 from A&F. We are justified in this, since the contour C is continuous on the arc of the circle $|z| = R$. This time, our arc length of the contour C is

$$L = \frac{1}{6}2\pi R = \frac{\pi}{3}R.$$

Next, we need to calculate M as the upper bound for $\left|\frac{1}{z^3+1}\right|$. Let's use a simpler method than in problem 3. Let's get on with it

$$|f(z)| = \left|\frac{1}{z^3 + 1}\right| = \frac{1}{|z^3 + 1|}.$$

Notice, if we can get a lower bound for the denominator then we will have an upper bound for the whole expression. It falls out quickly using the parameterization $z = R\,\mathrm{e}^{i\theta}$ and applying the inverse triangle inequality.

$$\begin{aligned}
\left|z^3 + 1\right| &= \left|R^3\,\mathrm{e}^{i3\theta} + 1\right| \\
&= \left|R^3\,\mathrm{e}^{i3\theta} - (-1)\right| \\
&\geq \left|\left|R^3\,\mathrm{e}^{i3\theta}\right| - |(-1)|\right| \\
&= \left|R^3 - 1\right| \\
&= R^3 - 1
\end{aligned}$$

where the last equality holds because $R > 1$. Therefore, we have our lower bound for the denominator

$$\left|z^3 + 1\right| \geq R^3 - 1,$$

and thus an upper bound for the expression

$$|f(z)| = \frac{1}{|z^3 + 1|} \leq \frac{1}{R^3 - 1} = M.$$

Finally, applying Theorem 2.4.2 from A&F we have

$$\left|\int_C f(z)dz\right| \leq ML \leq \frac{\pi}{3}R\frac{1}{R^3 - 1} = \frac{\pi}{3}\left(\frac{R}{R^3 - 1}\right).$$

Hence,

$$\left| \int_C f(z)dz \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right).$$

□

We will now take the limit of both sides of this inequality as R goes to ∞

$$\lim_{R\to\infty} \left| \int_C f(z)dz \right| \leq \lim_{R\to\infty} \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right) = \frac{\infty}{\infty}.$$

Applying L'Hôpital's rule once, we have

$$\lim_{R\to\infty} \frac{\pi}{3} \left(\frac{1}{3R^2 - 1} \right) = 0.$$

Therefore,

$$\lim_{R\to\infty} \left| \int_C f(z)dz \right| \leq 0.$$

If the limit of the absolute value of something is less than or equal to 0, then the limit of that thing must be zero. This is the case because the absolute value is a non-negative function, so the "≤ 0" must be just an equality. Therefore,

$$\lim_{R\to\infty} \left| \int_C f(z)dz \right| = 0.$$

Note this also implies $\lim_{R\to\infty} - \left| \int_C f(z)dz \right| = 0$. Now we have that

$$- \left| \int_C f(z)dz \right| \leq \int_C f(z)dz \leq \left| \int_C f(z)dz \right|$$

and thus by the squeeze theorem,

$$\lim_{R\to\infty} \int_C f(z)dz = 0.$$

□

5: From A&F: 2.5.1 b, e

Evaluate $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

b)

$$f(z) = \mathrm{e}^{z^2}$$

Solution:

Let's first break $f(z)$ up into real and imaginary parts so we can define $u(x, y)$ and $v(x, y)$ and check if the Cauchy-Riemann (C-R) equations hold.

$$\begin{aligned} f(z) &= \mathrm{e}^{z^2} \\ &= \mathrm{e}^{x^2+2ixy-y^2} \\ &= \mathrm{e}^{x^2}\,\mathrm{e}^{2ixy}\,\mathrm{e}^{-y^2} \\ &= \mathrm{e}^{x^2}\,\mathrm{e}^{-y^2}\,\mathrm{e}^{2ixy} \\ &= \mathrm{e}^{x^2}\,\mathrm{e}^{-y^2}\,(\cos(2xy) + i\sin(2xy)) \\ &= \mathrm{e}^{x^2}\,\mathrm{e}^{-y^2}\cos(2xy) + i\,\mathrm{e}^{x^2}\,\mathrm{e}^{-y^2}\sin(2xy). \end{aligned}$$

Then we can assign $u(x,y) = \mathrm{e}^{x^2}\mathrm{e}^{-y^2}\cos(2xy)$ and $v(x,y) = \mathrm{e}^{x^2}\mathrm{e}^{-y^2}\sin(2xy)$. Now let's calculate the necessary derivatives to verify if $f(z)$ is analytic. We have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \mathrm{e}^{-y^2}\left(2x\,\mathrm{e}^{x^2}\cos(2xy) - \mathrm{e}^{x^2}\sin(2xy)2y\right)\\
&= \mathrm{e}^{x^2}\mathrm{e}^{-y^2}\left(2x\cos(2xy) - \sin(2xy)2y\right)\\
\frac{\partial v}{\partial y} &= \mathrm{e}^{x^2}\left((-2y)\,\mathrm{e}^{-y^2}\sin(2xy) + \mathrm{e}^{-y^2}\cos(2xy)2x\right)\\
&= \mathrm{e}^{x^2}\mathrm{e}^{-y^2}\left(-2y\sin(2xy) + \cos(2xy)2x\right)
\end{aligned}$$

which are equivalent. Additionally, we get

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \mathrm{e}^{-y^2}\left(2x\,\mathrm{e}^{x^2}\sin(2xy) + \mathrm{e}^{x^2}\cos(2xy)2y\right)\\
&= \mathrm{e}^{x^2}\mathrm{e}^{-y^2}\left(2x\sin(2xy) + \cos(2xy)2y\right)\\
-\frac{\partial u}{\partial y} &= -\left(\mathrm{e}^{x^2}\left(-2y\,\mathrm{e}^{-y^2}\cos(2xy) - \mathrm{e}^{-y^2}\sin(2xy)2x\right)\right)\\
&= -\mathrm{e}^{x^2}\mathrm{e}^{-y^2}\left(-2y\cos(2xy) - \sin(2xy)2x\right)\\
&= \mathrm{e}^{x^2}\mathrm{e}^{-y^2}\left(2y\cos(2xy) + \sin(2xy)2x\right)
\end{aligned}$$

which are also equal as desired. Therefore the C-R equations hold and f(z) is analytic everywhere. Now by the Theorem 2.5.2 from A&F or Cauchy's Integral formula, we can conclude

$$\oint_C f(z)\mathrm{d}z = \oint_C \mathrm{e}^{z^2}\,\mathrm{d}z = 0.$$

$\square$

Another method of direct verification.

$$\begin{aligned}
\oint_C \mathrm{e}^{z^2}\,\mathrm{d}z &= \int_0^{2\pi} \mathrm{e}^{\left(\mathrm{e}^{i\theta}\right)^2}\, i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta\\
&= \int_0^{2\pi} i\left(\sum_{j=1}^{\infty}\frac{\left(\mathrm{e}^{2i\theta}\right)^j}{j!}\right)\mathrm{e}^{i\theta}\,\mathrm{d}\theta\\
&= \int_0^{2\pi} i\sum_{j=1}^{\infty}\frac{\mathrm{e}^{i\theta 2j+i\theta}}{j!}\mathrm{d}\theta\\
&= \int_0^{2\pi} i\sum_{j=1}^{\infty}\frac{\mathrm{e}^{i\theta(2j+1)}}{j!}\mathrm{d}\theta\\
&= \sum_{j=1}^{\infty}\int_0^{2\pi}\frac{i\,\mathrm{e}^{i\theta(2j+1)}}{j!}\mathrm{d}\theta
\end{aligned}$$

Again this reordering of the integral and sum is justified by the absolute convergence of the infinite series. We don't need to worry about any terms being undefined by a divide by zero issue. So let's look at the integral inside the sum

$$\int_0^{2\pi} \frac{i\,\mathrm{e}^{i\theta(2j+1)}}{j!}\mathrm{d}\theta = \frac{1}{2j+1}\frac{\mathrm{e}^{i\theta(2j+1)}}{j!}\bigg|_0^{2\pi}$$
$$= \frac{1}{2j+1}\frac{\mathrm{e}^{i2\pi(2j+1)}}{j!} - \frac{1}{2j+1}\frac{\mathrm{e}^{i0(2j+1)}}{j!}$$
$$= \frac{1}{2j+1}\frac{\mathrm{e}^{i2\pi(2j+1)}}{j!} - \frac{1}{(2j+1)\,j!}$$
$$= \frac{1}{(2j+1)\,j!} - \frac{1}{(2j+1)\,j!}$$
$$= 0$$

□

e)

$$f(z) = \frac{1}{2z^2+1}$$

Solution:
TODO: Definitely go the route of checking the analyticity of $f(z)$ with the C-R equations.
Let's determine the real and imaginary parts of this function

$$f(z) = \frac{1}{2z^2+1} = \frac{1}{2\,(x+iy)}$$
$$= \frac{1}{2\,(x^2+2ixy-y^2)+1}$$
$$= \frac{1}{2x^2+4ixy-3y^2+1}$$
$$= \frac{1}{(2x^2-3y^2+1)+i\,(4xy)}$$

Let's assign place holders p and q to be the real and imaginary parts, respectively, of the denominator right now such that $p = 2x^2-3y^2+1$ and $q = 4xy$. These are naturally functions of x and y but I will suppress the arguments for now. Then we multiply the top and bottom by the conjugate

$$= \frac{1}{p+iq}$$
$$= \frac{p-iq}{p^2+q^2}$$
$$= \frac{p}{p^2+q^2} - i\frac{q}{p^2+q^2}.$$

We now assign

$$u = \frac{p}{p^2+q^2} = \frac{2x^2-3y^2+1}{(2x^2-3y^2+1)^2+(4xy)^2}$$

and

$$v = \frac{-q}{p^2+q^2} = \frac{-4xy}{(2x^2-3y^2+1)^2+(4xy)^2}.$$

Similar to p and q, I will suppress the arguments of the functions u and v for simplicity in notation and to ease the book keeping throughout these calculations. Once we calculate the correct partial derivatives, we can see if $f(z)$ is analytic. It may actually be useful to calculate this using the chain rule through the sequence of variables depending on x and y. We need to calculate each of the following partials u_x, v_y, v_x, and u_y.

$$\begin{aligned}
u_x &= u_p p_x + u_q q_x \\
v_y &= v_p p_y + v_q q_y \\
v_x &= v_p p_x + v_q q_x \\
u_y &= u_p p_y + u_q q_y
\end{aligned}$$

Each of these partials is made up of combinations of 8 unique partials. I will calculate those explicitly now, then piece the right ones together to help us verify the C-R equations after.

$$\begin{aligned}
u_p &= \frac{1\left(p^2+q^2\right)-p\,(2p)}{\left(p^2+q^2\right)^2} & u_q &= \frac{0\left(p^2+q^2\right)-p\,(2q)}{\left(p^2+q^2\right)^2} \\
v_p &= \frac{0\left(p^2+q^2\right)-(-q)\,(2p)}{\left(p^2+q^2\right)^2} & v_q &= \frac{(-1)\left(p^2+q^2\right)-(-q)\,(2q)}{\left(p^2+q^2\right)^2} \\
p_x &= 4x & p_y &= -6y \\
q_x &= 4x & q_y &= 4y
\end{aligned}$$

We are now ready to check the C-R equations. Beginning with $u_x = v_y$

$$\begin{aligned}
u_x = u_p p_x + u_q q_x &= \left(\frac{1\left(p^2+q^2\right)-p\,(2p)}{\left(p^2+q^2\right)^2}\right)(4x) + \left(\frac{0\left(p^2+q^2\right)-p\,(2q)}{\left(p^2+q^2\right)^2}\right)(4x) \\
&= \left(\frac{\left(p^2+q^2\right)-p\,(2p)-p\,(2q)}{\left(p^2+q^2\right)^2}\right)(4x) \\
&= \frac{\left(p^2+q^2-2p^2-2pq\right)4x}{\left(p^2+q^2\right)^2} \\
v_y = v_p p_y + v_q q_y &= \left(\frac{0\left(p^2+q^2\right)-(-q)\,(2p)}{\left(p^2+q^2\right)^2}\right)(-6y) + \left(\frac{(-1)\left(p^2+q^2\right)-(-q)\,(2q)}{\left(p^2+q^2\right)^2}\right)(4y) \\
&= \left(\frac{2qp}{\left(p^2+q^2\right)^2}\right)(-6y) + \left(\frac{-p^2-q^2+2q^2}{\left(p^2+q^2\right)^2}\right)(4y) \\
&= \frac{-12qpy+\left(-p^2-q^2+2q^2\right)4y}{\left(p^2+q^2\right)^2}
\end{aligned}$$

TODO: It's not abundantly clear that this is the best route to verify this integrals value...

6: Use the ideas from A&F: 2.5.5 to evaluate $\int_0^\infty \mathrm{e}^{\mathrm{i}z^3 t}\,\mathrm{d}z$, $t > 0$. Express the result in terms of $\int_0^\infty \mathrm{e}^{-r^3}\,\mathrm{d}r$.
The ideas we might need to use are ... it's actually really long!

Solution:

7: From A&F: 2.5.6.
Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}.$$

Show how to evaluate this integral by considering

$$\oint_{C_{(\mathbb{R})}} \frac{\mathrm{d}z}{z^2+1},$$

where $C_{(\mathbb{R})}$ is closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x-axis. *Hint:* use

$$\frac{1}{z^2+1} = -\frac{1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables. *Solution:*

Repeat this exercise for

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{\epsilon \mathrm{d}x}{x^2+\epsilon^2}, \quad \epsilon > 0.$$

Seems like I am supposed to do 2.5.6 and then for the given integral as well.
Solution:

8: Use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.
Solution:

9: From A&F: 2.6.1 a, e.
Evaluate the integrals $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin and $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):
a)

$$\frac{\sin z}{z}$$

Solution:
e)

$$\mathrm{e}^{z^2}\left(\frac{1}{z^2} - \frac{1}{z^3}\right)$$

Solution: