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AMATH 561

PROBLEM SET 2

1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.
Solution:
We need to show that Z is a random variable as it is defined. That is we need to show it is a function that maps from a sample space Ω to the real numbers and that for every Borel set $B \subset \mathbb{R}$ we have

$$Z^{-1}(B) = \{\, \omega \,|Z(\omega) \in B\} \in \mathcal{F}.$$

Starting from knowing X and Y are random variables that means we have:

$$X : \Omega \to \mathbb{R}, \quad Y : \Omega \to \mathbb{R}.$$

Now rewriting Z a little more mathematically we have

$$Z(\omega) = \begin{cases} X(\omega), & \omega \in A, \\ Y(\omega), & \omega \in A^c. \end{cases}$$

Since $A \in \mathcal{F}$, every $\omega \in A$ must also be in Ω since $\mathcal{F}$ is made up of subsets of Ω which means $A \subseteq \Omega$ and thus $A^c \subseteq \Omega$ as well. By definition of the compliment $A \cap A^c = \emptyset$. Therefore A and A^c are a partition on Ω. Since Z is defined on $\omega \in A$ or $\omega \in A^C$ then Z is defined on all of Ω. Now we have shown that the domain of Z is Ω. Additionally, since X and Y each map from Ω to $\mathbb{R}$, Z must also map to $\mathbb{R}$ since it's output is determined by the output of X and Y. Therefore Z is function such that $Z : \Omega \to \mathbb{R}$.

Now we begin the argument that $Z^{-1}(B) = \{\omega | X(\omega) \in B\} \in \mathcal{F}$. First, since X and Y are random variables on our probability space we have that for every Borel set B

$$X^{-1}(B) = \{\, \omega \,|X(\omega) \in B\} \in \mathcal{F}$$

and

$$Y^{-1}(B) = \{\, \omega \,|Y(\omega) \in B\} \in \mathcal{F}.$$

Now it is important to observe that the $Z{-}1(B)$ is going to be some combination of the $X^{-1}(B)$ and $Y^{-1}(B)$. Let's take for example some $\omega^* \in A \subset \Omega$, then $Z(\omega^*) = X(\omega^*) = c$ for some constant $c \in \mathbb{R}$. Then if $c \in B$ then $\omega^* \in X^{-1}(B)$

and thus $\omega^* \in Z^{-1}(B)$. Therefore part of $Z^{-1}(B)$ can be written as

$$A \cap X^{-1}(B).$$

Additionally, we can also write part of $Z^{-1}(B)$ as

$$A^c \cap Y^{-1}(B).$$

Since A and A^c are a partition on Ω we know $A^c \cap Y^{-1}(B)$ and $A \cap X^{-1}(B)$ are disjoint. And they actually contain all of $Z^{-1}(B)$ since Z is only defined by X and Y in each of those scenarios respecting $\omega \in A$ or $\omega \in A^c$. Therefore

$$Z^{-1}(B) = \big(A \cap X^{-1}(B)\big) \cup \big(A^c \cap Y^{-1}(B)\big)$$

Now we need to finally demonstrate that $Z^{-1}(B) \in \mathcal{F}$. Recall we are given that $A \in \mathcal{F}$, and since X is a R.V. then $X^{-1}(B) \in \mathcal{F}$ therefore

$$A \cap X^{-1}(B) \in \mathcal{F}.$$

By a σ-algebra being closed under compliments we know $A^c \in \mathcal{F}$ and similar to X since Y is a R.V. then $Y^{-1}(B) \in \mathcal{F}$, therefore

$$A^c \cap Y^{-1}(B) \in \mathcal{F}.$$

And lastly the countable union of elements of $\mathcal{F}$ is therefore also in $\mathcal{F}$ hence

$$Z^{-1}(B) = \big(A \cap X^{-1}(B)\big) \cup \big(A^c \cap Y^{-1}(B)\big) \in \mathcal{F}.$$

And thus Z is a random variable on the probability space $(\Omega, \mathcal{F}, P)$. $\square$

2. Suppose X is a continuous random variable with distribution function F_X. Let g be a strictly increasing continuous function. Define $Y = g(X)$.

a) What is F_Y, the distribution function of Y?
Solution:
We know that there is some probability space that the random variable X is defined on, let that be $(\Omega, \mathcal{F}, P)$. Therefore $X : \Omega \to \mathbb{R}$ and since g is a strictly increasing continuous function $g : \mathbb{R} \to L$ where L is the output space of g, L could be $\mathbb{R}$ for example, then $g(X) : \Omega \to \mathbb{R}$ (we take $L = \mathbb{R}$ for now as the most likely assumption). Note that since $Y = g(X)$ then $Y : \Omega \to \mathbb{R}$ is also true. In order to construct F_Y we need to determine the relationship they have.

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Now since g is only given to be a strictly increasing continuous function there is some technicalities to address with respect to inverting it. We instead define the following:

$$h(y) = \begin{cases} g^{-1}(y) & \text{if } y \in (a,b) \\ -\infty & \text{if } y \leq a \\ \infty & \text{if } y \geq b \end{cases}.$$

Where (a,b) is an arbitrary open interval. Now our expression for $F_Y(y)$ holds on these arbitrary intervals. $\square$

b) What is f_Y, the density function of Y?
Solution:
Since

$$F_Y(y) = \int_{-\infty}^{y} f_Y(x)\mathrm{d}x$$

we just need to differentiate F_Y as follows

$$\frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_X(g^{-1}(y)) = \frac{f_X(g^{-1}(y))}{g\prime(g^{-1}(y))}.$$

$\square$

3. Suppose X is a continuous random variable with distribution function F_X. Find F_Y where Y is given by

a) X^2
Solution:
That is to say $Y = X^2$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

$\square$

b) $\sqrt{|X|}$
Solution:
That is to say $Y = \sqrt{|X|}$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\sqrt{|X|} \leq y) \\ &= P(|X| \leq y^2) \\ &= P(-y^2 \leq X \leq y^2) \\ &= P(X \leq y^2) - P(X \leq -y^2) \\ &= F_X(y^2) - F_X(-y^2) \end{aligned}$$

$\square$

c) $\sin X$ *Solution:*
That is to say $Y = \sin X$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\sin X \leq y) \\ &= P(X \leq \arcsin y) \\ &= \sum_{k \in \mathbb{Z}} P(\arcsin y + 2\pi k \leq X \leq \arcsin y + 2\pi(k+1)) \\ &= \sum_{k \in \mathbb{Z}} \left[P(X \leq \arcsin y + 2\pi(k+1)) - P(X \leq \arcsin y + 2\pi k) \right] \\ &= \sum_{k \in \mathbb{Z}} \left[F_X(\arcsin y + 2\pi(k+1)) - F_X(\arcsin y + 2\pi k) \right]. \end{aligned}$$

d) $F_X(X)$ *Solution:*
That is to say $Y = F_X(X)$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X(X) \leq y) \\ &= P(X \leq F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \\ &= y \end{aligned}$$

Now there is a bit more to be said to ensure we are covering all of our bases here as we try to invert the nondecreasing but not necessarily always increasing function $F_X(x)$. We define the inverse $F_X^{-1}(y)$ as follows

$$F_X^{-1}(y) = \sup \{x \in D : F_X(x) \leq y\}$$

Once we have defined the inverse above, then we are done justifying the expression for the distribution function $F_Y(y)$. $\square$

4. Let $X : [0,1] \to \mathbf{R}$ be a function that maps every rational number in the interval $[0,1]$ to 0, and every irrational number to 1. We assume that the probability space where X is defined is $([0,1], \mathcal{B}[0,1], P)$, where $\mathcal{B}[0,1]$ is the Borel σ-algebra on [0,1], and P is the Lebesgue measure.

(a) Is the set of rational numbers in [0,1] a Borel set? Show using definition of the Borel σ-algebra on $[0,1]$.
Solution:
I will argue that yes the set of rational numbers in $[0,1]$ is a Borel set. We will construct the set of rational numbers in a way such a that it is a countable union of sets, which are themselves the countable intersection of open sets and thus we will have a Borel set. First note we can write any number $x \in [0,1]$ as

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap [0,1]$$

That is to say this countably infinity intersection of open sets is the singleton set $\{x\}$. Therefore we can also represent each of the rational numbers in $[0,1]$ in this way as well. We do have to be careful that when near the boundary of [0, 1] n has to be sufficiently large. Now we construct the set of all rationals in $[0,1]$ as follows:

$$\mathbb{Q} \cap [0,1] = \bigcup_{q \in \mathbb{Q}\cap[0,1]}^{\infty} \{q\}.$$

Now we have that the rationals between 0 and 1, $\mathbb{Q} \cap [0,1]$, can be written in the form of a countably infinite union of sets which themselves are countably infinite intersections of open sets, which is a Borel set. Hence $\mathbb{Q} \cap [0,1]$ is a Borel set.

$\square$

(b) Is X a random variable (and why)? If it is, what are its distribution function and expectation? Does X have a density function? Is X discrete?
Solution:
Yes X is a random variable on the probability space $([0,1], \mathcal{B}[0,1], P)$ because the $X^{-1}(B) \in \mathcal{F} = \mathcal{B}[0,1]$, for every Borel set B. Notice we can equivalently think about X as follows

$$X(\omega) = \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega)$$

Now let's define what exactly the pre-image would look like

$$\begin{aligned} X^{-1}(B) &= \{\omega \,|X(\omega) \in B\,\} \\ &= \left\{\omega \,\middle|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) \in B\,\right\} \end{aligned}$$

Let B_0 denote any Borel set s.t. it contains 0 but not 1, B_1 denote any Borel set s.t. it contains 1 but not 0, $B_{\{01\}}$ denote any Borel set s.t. it contains both 0 and 1, and lastly B_* denote any Borel set s.t. it does not contain either 0 or 1. Then we have

$$\begin{aligned} X^{-1}(B_1) &= \left\{\omega \,\middle|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) \in B_1\,\right\} \\ &= \left\{\omega \,\middle|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) = 1\,\right\} \\ &= \{\omega \,|\omega \in \mathbb{Q} \cap [0,1]\,\} \\ &= \mathbb{Q} \cap [0,1] \in \mathcal{F}. \end{aligned}$$

Which we already proved the rationals contained in $[0,1]$ is a Borel set in the interval $[0,1]$, therefore it is contained in $\mathcal{F}$. Next, we have

$$\begin{aligned} X^{-1}(B_0) &= \left\{\omega \,\middle|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) \in B_0\,\right\} \\ &= \left\{\omega \,\middle|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) = 0\,\right\} \\ &= \{\omega \,|\omega \notin \mathbb{Q} \cap [0,1]\,\} \\ &= \{\omega \,|\omega \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]\,\} \\ &= (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \in \mathcal{F}. \end{aligned}$$

The set of irrational numbers in $[0,1]$ is contained in $\mathcal{F}$, since $\mathcal{F}$ is closed under compliments. The irrational numbers are a complement to the rationals with respect to the reals, since any real number is either rational or irrational. Continuing on, we have

$$\begin{aligned} X^{-1}(B_{\{01\}}) &= \big\{\omega \,\big|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) \in B_{\{01\}} \big\} \\ &= \big\{\omega \,\big|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) = 1 \text{ or } 0 \big\} \\ &= \{\omega \,| \omega \in (\mathbb{Q}\cap[0,1]) \cup ((\mathbb{R}\setminus\mathbb{Q})\cap[0,1]) \} \\ &= \{\omega \,| \omega \in [0,1] \} \\ &= [0,1] \in \mathcal{F}, \end{aligned}$$

since $\mathcal{F}$ always contains Ω which is the interval $[0,1]$ in our case. Lastly,

$$X^{-1}(B_*) = \big\{\omega \,\big|\, \mathbb{1}_{\mathbb{Q}\cap[0,1]}(\omega) \in B_* \big\} = \emptyset \in \mathcal{F},$$

since $\emptyset$ is also always contained in $\mathcal{F}$ and it is the compliment of Ω with respect to Ω. Therefore X is a random variable on the given probability space.

The distribution function of X is like a heavy side function where the jump is at x = 1. This would look like

$$F_X(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 1) \\ 1 & \text{if } x \in [1, \infty) \end{cases} .$$

Since we can represent the random variable X as the indicator function $\mathbb{1}_{\mathbb{Q}\cap[0,1]}$, we can calculate the expectation of X as follows:

$$\mathbb{E}[\mathbb{1}_{\mathbb{Q}\cap[0,1]}] = \int_0^1 \mathbb{1}_{\mathbb{Q}\cap[0,1]}(x)\,dx = 0.$$

This makes sense because once again th rationals between 0 and 1 form a set of measure zero. Since $\mathbb{Q}\cap[0,1]$ is built out of the union of singleton sets which each have measure zero, then they each individually have a probability of 0. Therefore it does not make sense to have a a density function for X. Now, X can be considered a a discrete random variable, since there exists a set $S \subset \mathbb{R}$ with $\mu(S^c) = 0$. As the example given before.