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 AMATH 561

PROBLEM SET 8

Note: Exercises are from Matt Lorig's notes (link on course website).

1. Exercise 5.1. Patients arrive at an emergency room as a Poisson process with intensity λ . The time to treat each patient is an independent exponential random variable with parameter μ . Let $X = (X_t)_{t \geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X . Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π . What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

Hint: You can use Little's law, which states that the expected number of people in the hospital at steady-state is equal to the average arrival rate multiplied by the average processing time.

Solution:

The generator for X is

$$\mathbf{G} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -(\mu + \lambda) & \lambda & 0 & 0 & \dots \\ 0 & \mu & -(\mu + \lambda) & \lambda & 0 & \dots \\ 0 & 0 & \mu & -(\mu + \lambda) & \lambda & \ddots \\ 0 & 0 & 0 & \mu & -(\mu + \lambda) & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Now if the invariant distribution π exists then $\pi \mathbf{G} = \mathbf{0}$. Let's look at what conditions would need to hold for this π to exist. First, looking at $\pi \mathbf{G}$ we have

$$\begin{aligned} 0 &= -\lambda\pi(0) + \mu\pi(1) \\ 0 &= \lambda\pi(0) - (\mu + \lambda)\pi(1) + \mu\pi(2) \\ 0 &= \lambda\pi(1) - (\mu + \lambda)\pi(2) + \mu\pi(3) \\ &\vdots \\ 0 &= \lambda\pi(n-1) - (\mu + \lambda)\pi(n) + \mu\pi(n+1). \end{aligned}$$

Then we can say

$$\pi(1) = \frac{\lambda}{\mu}\pi(0), \quad \pi(2) = \frac{\lambda^2}{\mu^2}\pi(0), \quad \dots, \quad \pi(n) = \frac{\lambda^n}{\mu^n}\pi(0).$$

If π is a stationary distribution then the row vector needs to sum to one so we have the condition

$$\sum_{n=0}^{\infty} \pi(n) \left(\frac{\lambda}{\mu} \right)^n = 1.$$

This sum is finite if and only if

$$\left| \frac{\lambda}{\mu} \right| < 1 \implies \lambda < \mu.$$

Furthermore, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \pi(n) \left(\frac{\lambda}{\mu} \right)^n &= 1 \\ \pi(0) \frac{1}{1 - \frac{\lambda}{\mu}} &= 1 \\ \pi(0) &= 1 - \frac{\lambda}{\mu}. \end{aligned}$$

Hence,

$$\pi(n) = \left(1 - \frac{\lambda}{\mu} \right) \left(\frac{\lambda}{\mu} \right)^n.$$

Therefore, we have found the stationary distribution π which only exists if and only if the condition that $\lambda < \mu$ since the sum of the entries of the vector π is only finite in this scenario.

Now I need to find the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution using Little's law which gives

$$E(X_t) = \lambda E(\tau).$$

Then we have

$$\begin{aligned}
 E(\tau) &= \frac{1}{\lambda} E(X_t) \\
 &= \frac{1}{\lambda} \sum_{n=0}^{\infty} n \pi(n) \\
 &= \frac{1}{\lambda} \sum_{n=0}^{\infty} n \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n \\
 &= \frac{1}{\lambda} \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \\
 &= \frac{1}{\lambda} \left(1 - \frac{\lambda}{\mu}\right) \frac{\frac{\lambda}{\mu}}{\left(1 - \frac{\lambda}{\mu}\right)^2} \\
 &= \frac{1}{\lambda} \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} \\
 &= \frac{\frac{1}{\mu}}{1 - \frac{\lambda}{\mu}} \\
 &= \frac{1}{\mu - \lambda}.
 \end{aligned}$$

Therefore, the expected time (waiting + treatment) a patient waits when the system is in its invariant distribution is $1/(\mu - \lambda)$.

□

2. Exercise 5.3. Let $X = (X_t)_{t \geq 0}$ be a Markov chain with state space $S = \{0, 1, 2, \dots\}$ and with a generator \mathbf{G} whose i th row has entries

$$g_{i,i-1} = i\mu, \quad g_{i,i} = -i\mu - \lambda, \quad g_{i,i+1} = \lambda,$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $X_0 = j$. Find $G_{X_T}(s) := E(s^{X_t})$. What is the distribution of X_t as $t \rightarrow \infty$?

Solution:

For my sake I am going to write down the generator \mathbf{G}

$$\mathbf{G} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\mu + \lambda) & \lambda & 0 & \dots \\ 0 & 2\mu & -(2\mu + \lambda) & \lambda & \ddots \\ 0 & 0 & 3\mu & -(3\mu + \lambda) & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then we want to calculate $G_{X_T}(s) := E(s^{X_t})$. This takes the form

$$E(s^{X_t}) = \sum_{n=0}^{\infty} s^n P(X_t = n),$$

but we need to use the Kolmogorov forward equation to find these $P(X_t = n)$ terms. The forward equation is

$$\frac{d}{dt} \mathbf{P}_t = \mathbf{P}_t \mathbf{G}$$

which more explicitly is

$$\begin{bmatrix} p'_t(0,0) & p'_t(0,1) & p'_t(0,2) & \dots \\ p'_t(1,0) & p'_t(1,1) & p'_t(1,2) & \dots \\ p'_t(2,0) & p'_t(2,1) & p'_t(2,2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} p_t(0,0) & p_t(0,1) & p_t(0,2) & \dots \\ p_t(1,0) & p_t(1,1) & p_t(1,2) & \dots \\ p_t(2,0) & p_t(2,1) & p_t(2,2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -\lambda & \lambda & 0 & \dots \\ \mu & -(\mu + \lambda) & \lambda & \ddots \\ 0 & 2\mu & -(2\mu + \lambda) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

This results in the following system of differential equations

$$\begin{aligned} p'_t(0,0) &= -\lambda p_t(0,0) + \mu p_t(0,1) \\ p'_t(0,1) &= \lambda p_t(0,0) - (\mu + \lambda) p_t(0,1) + 2\mu p_t(0,2) \\ p'_t(0,2) &= \lambda p_t(0,1) - (2\mu + \lambda) p_t(0,2) + 3\mu p_t(0,3) \\ &\vdots \\ p'_t(0,n) &= \lambda p_t(0,n-1) - (n\mu + \lambda) p_t(0,n) + (n+1)\mu p_t(0,n+1). \end{aligned}$$

As it turns out this is true for any starting point instead of just 0, therefore, we have

$$\begin{aligned}
 p'_t(j, 0) &= -\lambda p_t(j, 0) + \mu p_t(j, 1) \\
 p'_t(j, 1) &= \lambda p_t(j, 0) - (\mu + \lambda) p_t(j, 1) + 2\mu p_t(j, 2) \\
 p'_t(j, 2) &= \lambda p_t(j, 1) - (2\mu + \lambda) p_t(j, 2) + 3\mu p_t(j, 3) \\
 &\vdots \\
 p'_t(j, n) &= \lambda p_t(j, n-1) - (n\mu + \lambda) p_t(j, n) + (n+1)\mu p_t(j, n+1).
 \end{aligned}$$

Multiplying through by s^n and summing over all n we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} s^n p'_t(j, n) &= \lambda \sum_{n=0}^{\infty} s^n p_t(j, n-1) - \sum_{n=0}^{\infty} s^n (n\mu + \lambda) p_t(j, n) + \sum_{n=0}^{\infty} s^n (n+1)\mu p_t(j, n+1) \\
 \frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s \sum_{n=0}^{\infty} s^{n-1} p_t(j, n-1) - \sum_{n=0}^{\infty} s^n (n\mu + \lambda) p_t(j, n) + \mu \frac{\partial G_{X_T}(s)}{\partial s} \\
 \frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \sum_{n=0}^{\infty} s^n (n\mu + \lambda) p_t(j, n) + \mu \frac{\partial G_{X_T}(s)}{\partial s} \\
 \frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \sum_{n=0}^{\infty} s^n n\mu p_t(j, n) - \sum_{n=0}^{\infty} s^n \lambda p_t(j, n) + \mu \frac{\partial G_{X_T}(s)}{\partial s} \\
 \frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \mu s \sum_{n=0}^{\infty} n s^{n-1} p_t(j, n) - \lambda G_{X_T}(s) + \mu \frac{\partial G_{X_T}(s)}{\partial s} \\
 \frac{\partial G_{X_T}(s)}{\partial t} &= \lambda s G_{X_T}(s) - \mu s \frac{\partial G_{X_T}(s)}{\partial s} - \lambda G_{X_T}(s) + \mu \frac{\partial G_{X_T}(s)}{\partial s} \\
 \frac{\partial G_{X_T}(s)}{\partial t} &= (s-1)\lambda G_{X_T}(s) + (1-s)\mu \frac{\partial G_{X_T}(s)}{\partial s}.
 \end{aligned}$$

Suppressing some notation for convenience, we now need to solve the following differential equation with initial condition of $G_{X_0}(s) = s^j$

$$\frac{\partial}{\partial t} G = (s-1)\lambda G + (1-s)\mu \frac{\partial}{\partial s} G.$$

Using mathematica to solve this pde I ended up with

$$G_{X_t}(s) = \exp\left(\frac{e^{-t\mu}(-1 + e^{t\mu})(-1 + s)\lambda}{\mu}\right) (1 + e^{-t\mu}(-1 + s))^j.$$

Simplifying a little we have

$$G_{X_t}(s) = \exp\left(\frac{(1 - e^{-t\mu})(s-1)\lambda}{\mu}\right) (1 + e^{-t\mu}(s-1))^j.$$

The generating function as

$$\begin{aligned}\lim_{t \rightarrow \infty} G_{X_t}(s) &= \lim_{t \rightarrow \infty} \exp\left(\frac{(1 - e^{-t\mu})(s-1)\lambda}{\mu}\right) (1 + e^{-t\mu}(s-1))^j \\ &= \exp\left(\frac{(1-0)(s-1)\lambda}{\mu}\right) (1+0(s-1))^j \\ &= \exp\left(\frac{(s-1)\lambda}{\mu}\right).\end{aligned}$$

Let $G_X(s) = \exp\left(\frac{(s-1)\lambda}{\mu}\right)$ denote the generating function after taking the limit as t goes to infinity. We can compute the distribution from the generating function as

$$\begin{aligned}p_n(t) &= \frac{1}{n!} \frac{\partial^n}{\partial s^n} G_X(s) \Big|_{s=0} \\ &= \frac{1}{n!} \frac{\partial^n}{\partial s^n} \exp\left(\frac{(s-1)\lambda}{\mu}\right) \Big|_{s=0} \\ &= \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n \exp\left(\frac{(s-1)\lambda}{\mu}\right) \Big|_{s=0} \\ &= \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n \exp\left(-\frac{\lambda}{\mu}\right) \\ &= \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}.\end{aligned}$$

Which is a poisson distribution with rate λ/μ .

□

3. Exercise 5.4. Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jump of size one in the time interval $(t, t+dt)$ is $\lambda(t)dt$ and the probability of two jumps in that interval of time is $\mathcal{O}(dt^2)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $N_0 = 0$ and let τ_1 be the time of the first jump of N . If $\lambda(t) = c/(1+t)$ show that $E(\tau_1) < \infty$ if and only if $c > 1$.

Solution:

Once again I think it is very helpful to write down the generator \mathbf{G} for this scenario

$$\mathbf{G} = \begin{bmatrix} -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

For the forward Kolmogorov equation we have

$$\frac{d}{dt} \mathbf{P}_t = \mathbf{P}_t \mathbf{G}$$

which more explicitly is

$$\begin{bmatrix} p'_t(0,0) & p'_t(0,1) & \dots \\ p'_t(1,0) & p'_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} p_t(0,0) & p_t(0,1) & \dots \\ p_t(1,0) & p_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

This results in the following system of differential equations

$$\begin{aligned} p'_t(0,0) &= -\lambda(t)p_t(0,0) \\ p'_t(0,1) &= \lambda(t)p_t(0,0) - \lambda(t)p_t(0,1) \\ &\vdots \\ p'_t(0,n) &= \lambda(t)p_t(0,n-1) - \lambda(t)p_t(0,n) \end{aligned}$$

Multiplying through by s^n and summing over all n we have

$$\begin{aligned} \sum_{n=0}^{\infty} s^n p'_t(0,n) &= \lambda(t) \sum_{n=0}^{\infty} s^n p_t(0,n-1) - \lambda(t) \sum_{n=0}^{\infty} s^n p_t(0,n) \\ \frac{d}{dt} G_{N_t}(s) &= \lambda(t)s \sum_{n=0}^{\infty} s^{n-1} p_t(0,n-1) - \lambda(t) G_{N_t}(s) \\ \frac{d}{dt} G_{N_t}(s) &= \lambda(t)s G_{N_t}(s) - \lambda(t) G_{N_t}(s) \\ \frac{d}{dt} G_{N_t}(s) &= (s-1)\lambda(t) G_{N_t}(s). \end{aligned}$$

Now we have the pde (again suppressing extra notation) $G_{N_0}(s) = s^0 = 1$

$$\frac{d}{dt} G = (s-1)\lambda(t)G.$$

Let's solve this directly

$$\begin{aligned}\frac{d}{dt}G &= (s-1)\lambda(t)G \\ \frac{\frac{d}{dt}G}{G} &= (s-1)\lambda(t) \\ \int \frac{\frac{d}{dt}G}{G} dt &= \int (s-1)\lambda(t) dt \\ \log(G) &= \int (s-1)\lambda(t) dt \\ G &= e^{\int (s-1)\lambda(t) dt}\end{aligned}$$

For the backward Kolmogorov equation we have

$$\frac{d}{dt}\mathbf{P}_t = \mathbf{G}\mathbf{P}_t$$

which more explicitly is

$$\begin{bmatrix} p'_t(0,0) & p'_t(0,1) & \dots \\ p'_t(1,0) & p'_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} -\lambda(t) & \lambda(t) & 0 & \dots \\ 0 & -\lambda(t) & \lambda(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} p_t(0,0) & p_t(0,1) & \dots \\ p_t(1,0) & p_t(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

This results in the following system of differential equations

$$\begin{aligned}p'_t(0,0) &= -\lambda(t)p_t(0,0) + \lambda(t)p_t(1,0) \\ p'_t(0,1) &= -\lambda(t)p_t(0,1) + \lambda(t)p_t(1,1) \\ p'_t(0,2) &= -\lambda(t)p_t(0,2) + \lambda(t)p_t(1,2) \\ &\vdots \\ p'_t(0,n) &= -\lambda(t)p_t(0,n) + \lambda(t)p_t(1,n).\end{aligned}$$

We also have

$$\begin{aligned}p'_t(1,0) &= -\lambda(t)p_t(1,0) + \lambda(t)p_t(2,0) \\ p'_t(1,1) &= -\lambda(t)p_t(1,1) + \lambda(t)p_t(2,1) \\ p'_t(1,2) &= -\lambda(t)p_t(1,2) + \lambda(t)p_t(2,2) \\ &\vdots \\ p'_t(1,n) &= -\lambda(t)p_t(1,n) + \lambda(t)p_t(2,n).\end{aligned}$$

I'm not totally sure how to combine these, however... Multiplying through by s^n and summing over all n for the set of pde's where $N_0 = 0$

$$\begin{aligned}\sum_{n=0}^{\infty} s^n p'_t(0,n) &= -\lambda(t) \sum_{n=0}^{\infty} s^n p_t(0,n) + \lambda(t) \sum_{n=0}^{\infty} s^n p_t(1,n) \\ \frac{\partial}{\partial t} G_{N_t}(s) &= -\lambda(t) G_{N_t}(s) + \lambda(t) \sum_{n=0}^{\infty} s^n p_t(1,n)\end{aligned}$$

How do I handle this thing with the incrementing in the first index not the second?

4. Exercise 5.5. Let N be a poisson process with a random intensity Λ witch is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $G_{N_t}(s) = E(s^{N_t})$. What is the mean and variance of N_t ?

Solution:

TODO:

Here you