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AMATH 567

HOMEWORK 4

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*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.4.2 c, e.
Evaluate the integral $\oint_C f(z)\mathrm{d}z$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows:
c)

$$f(z) = \frac{1}{\bar{z}}$$

Solution:
We want to evaluate

$$\oint_C \frac{1}{\bar{z}}\mathrm{d}z$$

on the parameterized unit circle $z = \mathrm{e}^{i\theta}$ where $\theta \in [0, 2\pi)$, where $\bar{z} = \mathrm{e}^{-i\theta}$ on the unit circle. Note, before we do the substitution we need $\mathrm{d}z = i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta$. Now our integral is

$$\begin{aligned}
\oint_C \frac{1}{\bar{z}}\mathrm{d}z &= \oint_0^{2\pi} \frac{1}{\mathrm{e}^{-i\theta}} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \oint_0^{2\pi} i\,\mathrm{e}^{2i\theta}\,\mathrm{d}\theta \\
&= \left(\frac{1}{2}\,\mathrm{e}^{2i\theta}\Big|_0^{2\pi} \right) \\
&= \frac{1}{2}\,\mathrm{e}^{4\pi i} - \frac{1}{2}\,\mathrm{e}^{0} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

□

e)

$$f(z) = \mathrm{e}^{\bar{z}}$$

Solution:
We will use the same substitutions from the previous part

$$\begin{aligned}
\oint_C \mathrm{e}^{\bar{z}}\,\mathrm{d}z &= \oint_0^{2\pi} \mathrm{e}^{\mathrm{e}^{-i\theta}}\, i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \oint_0^{2\pi} \sum_{j=1}^{\infty} \frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!} \mathrm{e}^{i\theta}\mathrm{d}\theta.
\end{aligned}$$

We are justified in reordering the integral of the infinite sum to be the infinite sum of the integrals since the original series converges absolutely. I will now just look at the integral inside the sum

$$\begin{aligned}
\oint_0^{2\pi} i \frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!} \mathrm{e}^{i\theta}\mathrm{d}\theta &= \oint_0^{2\pi} i \frac{\mathrm{e}^{-i\theta j}\,\mathrm{e}^{i\theta}}{j!}\mathrm{d}\theta \\
&= \oint_0^{2\pi} i \frac{\mathrm{e}^{-i\theta j + i\theta}}{j!}\mathrm{d}\theta \\
&= \oint_0^{2\pi} i \frac{\mathrm{e}^{i\theta(-j+1)}}{j!}\mathrm{d}\theta \\
&= \oint_0^{2\pi} \frac{i\,\mathrm{e}^{i\theta(1-j)}}{j!}\mathrm{d}\theta \\
&= \frac{1}{1-j} \frac{i\,\mathrm{e}^{i\theta(1-j)}}{j!}\bigg|_0^{2\pi} \\
&= \frac{1}{1-j} \frac{i\,\mathrm{e}^{i2\pi(1-j)}}{j!} - \frac{1}{1-j} \frac{i\,\mathrm{e}^{0}}{j!} \\
&= \frac{i}{(1-j)\,j!}\left(\mathrm{e}^{i2\pi(1-j)} - 1\right) \\
&= \frac{i}{(1-j)\,j!}(1-1) \\
&= 0.
\end{aligned}$$

I want to clarify why $\mathrm{e}^{i2\pi(1-j)} = 1$. Since $j \in \{1, 2, 3, ...\}$, then $1-j$ is an integer and we have $\mathrm{e}^{i2\pi\ell}$ where $\ell \in \mathbb{Z}$ which is always 1. Now we return to the original problem

$$\sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!} \mathrm{e}^{i\theta}\mathrm{d}\theta = \sum_{j=1}^{\infty} 0 = 0.$$

Now we have completed the requisite task. □

2: From A&F: 2.4.4 a, b. Use the principal branch where the argument is in $[-\pi, \pi)$. Discuss any ambiguities. Use the principal branch of $\log(z)$ and $z^{\frac{1}{2}}$ where the argument is in $[-\pi, \pi)$ to evaluate the following:
a)

$$\int_{-1}^{1} \log z \mathrm{d}z$$

Solution:
We want to parameterize this once again using $z = r\,\mathrm{e}^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our integral is

$$\begin{aligned}\int_{-1}^{1} \log z \mathrm{d}z &= \int_{-\pi}^{0} \log\left(\mathrm{e}^{i\theta}\right) i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \int_{-\pi}^{0} i\theta i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta.\end{aligned}$$

Let's use integration by parts, woohoo! We will assign the substitutions as follows:

$$\begin{aligned}u &= i\theta \\ \mathrm{d}u &= i\mathrm{d}\theta \\ \\ \mathrm{d}v &= i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ v &= \mathrm{e}^{i\theta}.\end{aligned}$$

Plugging this in we have

$$\begin{aligned}\int_{-\pi}^{0} i\theta i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta &= i\theta\,\mathrm{e}^{i\theta}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \left(0 - \left(-i\pi\,\mathrm{e}^{-i\pi}\right)\right) - \mathrm{e}^{i\theta}\Big|_{-\pi}^{0} \\ &= 0 + i\pi\,\mathrm{e}^{-i\pi} - \mathrm{e}^{i\theta}\Big|_{-\pi}^{0} \\ &= i\pi\,\mathrm{e}^{-i\pi} - \left(\mathrm{e}^{0} - \mathrm{e}^{-i\pi}\right) \\ &= -i\pi - (1 - (-1)) \\ &= -i\pi - (2) \\ &= -i\pi - 2.\end{aligned}$$

□

b)

$$\int_{-1}^{1} z^{\frac{1}{2}}\mathrm{d}z$$

Solution:
We want to parameterize this once again using $z = r\,\mathrm{e}^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our

integral is

$$\begin{aligned}
\int_{-1}^{1} z^{\frac{1}{2}} \mathrm{d}z &= \int_{-\pi}^{0} \left(\mathrm{e}^{i\theta}\right)^{\frac{1}{2}} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{0} i\,\mathrm{e}^{\frac{i\theta}{2}}\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{0} i\,\mathrm{e}^{\frac{i3}{2}\theta}\,\mathrm{d}\theta \\
&= \frac{2}{3}\,\mathrm{e}^{\frac{i3}{2}\theta}\Big|_{-\pi}^{0} \\
&= \frac{2}{3}\,\mathrm{e}^{-\frac{i3}{2}\pi} - \frac{2}{3}.
\end{aligned}$$

Now remembering our branch cut limits θ to be within $[-\pi, \pi)$ we change the angle $-\frac{3}{2}\pi$ to be $\frac{1}{2}\pi$. Hence,

$$\begin{aligned}
\frac{2}{3}\,\mathrm{e}^{-\frac{i3}{2}\pi} - \frac{2}{3} &= \frac{2}{3}\,\mathrm{e}^{-\frac{i2}{2}\pi}\,\mathrm{e}^{-\frac{i\pi}{2}} - \frac{2}{3} \\
&= \frac{2}{3}\,\mathrm{e}^{\frac{i\pi}{2}} - \frac{2}{3} \\
&= \frac{2}{3}\left(i - 1\right).
\end{aligned}$$

□

3: From A&F: 2.4.7

Let C be an open (upper) semicircle of radius R with its center at the origin, and consider $\int_C f(z)\mathrm{d}z$. Let $f(z) = \frac{1}{z^2+a^2}$ for a real $a > 0$. Show that $|f(z)| \leq \frac{1}{R^2-a^2}$, $R > a$, and

$$\left|\int_C f(z)dz\right| \leq \frac{\pi R}{R^2 - a^2}, \quad R > a.$$

Solution:

First, we want to show

$$|f(z)| \leq \frac{1}{R^2 - a^2}$$

where $R > a > 0$ and a $\in \mathbb{R}$. Let's consider the function more closely

$$\begin{aligned}
f(z) = \frac{1}{z^2 + a^2} &= \frac{1}{(x + \mathrm{i}y)^2 + a^2} \\
&= \frac{1}{x^2 + 2ixy - y^2 + a^2} \\
&= \frac{1}{x^2 - y^2 + a^2 + i2xy}.
\end{aligned}$$

Notice, we can write the real and imaginary parts of the complex number in the denominator as functions $u(x, y)$ and $v(x, y)$. Where $u(x, y) = x^2 - y^2 + a^2$ and $v(x, y) = 2xy$. Now we get

$$f(z) = \frac{1}{x^2 - y^2 + a^2 + i2xy} = \frac{u - iv}{u - iv}\frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

Then we calculate

$$\begin{aligned}
|f(z)| &= \left|\frac{u-iv}{u^2+v^2}\right| \\
&= \left|\frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}\right| \\
&= \sqrt{\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{v}{u^2+v^2}\right)^2} \\
&= \sqrt{\frac{u^2}{\left(u^2+v^2\right)^2} + \frac{v^2}{\left(u^2+v^2\right)^2}} \\
&= \sqrt{\frac{u^2+v^2}{\left(u^2+v^2\right)^2}} \\
&= \sqrt{\frac{1}{u^2+v^2}} \\
&= \frac{1}{\sqrt{u^2+v^2}}.
\end{aligned}$$

If we plug our substitution back in we see

$$\begin{aligned}
\frac{1}{\sqrt{u^2+v^2}} &= \frac{1}{\sqrt{\left(x^2-y^2+a^2\right)^2+(2xy)^2}} \\
&= \frac{1}{\sqrt{\left(x^2-y^2+a^2\right)\left(x^2-y^2+a^2\right)+4x^2y^2}} \\
&= \frac{1}{\sqrt{x^4-x^2y^2+x^2a^2-x^2y^2+y^4-y^2a^2+x^2a^2-y^2a^2+a^4+4x^2y^2}} \\
&= \frac{1}{\sqrt{x^4+x^2a^2+y^4-y^2a^2+x^2a^2-y^2a^2+a^4+2x^2y^2}}.
\end{aligned}$$

Now we add zero in a particular fashion, namely $-4x^2a^2+4x^2a^2$, so we can regroup the terms and refactor to get closer to what we desire

$$\begin{aligned}
&= \frac{1}{\sqrt{x^4+x^2a^2+y^4-y^2a^2+x^2a^2-y^2a^2+a^4+2x^2y^2+\left(-4x^2a^2+4x^2a^2\right)}} \\
&= \frac{1}{\sqrt{x^4+y^4-y^2a^2-y^2a^2+a^4+2x^2y^2-x^2a^2-x^2a^2+4x^2a^2}} \\
&= \frac{1}{\sqrt{\left(x^2+y^2-a^2\right)^2+(2xa)^2}}.
\end{aligned}$$

Using the fact that $\sqrt{a+b} \geq \sqrt{a}$ for $a, b > 0$, in our next step we get a smaller denominator which makes the overall expression greater or equal to the previous step.

Note, equality only holds when $x = 0$.

$$\frac{1}{\sqrt{(x^2+y^2-a^2)^2+(2xa)^2}} \le \frac{1}{\sqrt{(x^2+y^2-a^2)^2}}$$
$$= \frac{1}{x^2+y^2-a^2}$$
$$= \frac{1}{|z|^2-a^2}$$
$$= \frac{1}{R^2-a^2}$$

Therefore $|f(z)| \le \frac{1}{R^2-a^2}$.

□

Next we wish to show that

$$\left|\int_C f(z)dz\right| \le \frac{\pi R}{R^2-a^2}, \quad R > a.$$

By Theorem 2.4.2 from A&F, if $f(z)$ is continuous on contour C, then

$$\left|\int_C f(z)dz\right| \le ML$$

where L is the length of C and M is an upper bound for $|f(z)|$. We have that C is continuous, since $a > 0$ and $a < R$ there are no singularities or weirdness with $f(z)$ on the specified contour. So we have

$$M = \frac{1}{R^2-a^2}$$

as we calculated in the first part of this problem. Additionally, we know the arc length of C is easy to calculate because it is half the circumference of the circle with radius R. Therefore,

$$L = \int_a^b |z'(t)|\mathrm{d}t = \frac{1}{2}2\pi R = \pi R.$$

And thus

$$\left|\int_C f(z)dz\right| \le ML \le \pi R\frac{1}{R^2-a^2} = \frac{\pi R}{R^2-a^2}.$$

Hence,

$$\left|\int_C f(z)dz\right| \le \frac{\pi R}{R^2-a^2}$$

as desired.

□

4: From A&F: 2.4.8
Let C be an arc of the circle $|z| = R$ with $(R > 1)$ of angle $\frac{\pi}{3}$. Show that

$$\left|\int_C \frac{\mathrm{d}z}{z^3+1}\right| \le \frac{\pi}{3}\left(\frac{R}{R^3-1}\right)$$

and deduce

$$\lim_{R\to\infty}\int_C \frac{\mathrm{d}z}{z^3+1} = 0$$

Solution:
Similar to the previous problem we will utilize Theorem 2.4.2 from A&F. This time, our arc length of the contour C is

$$L = \frac{1}{6} 2\pi R = \frac{\pi}{3} R.$$

Next, we need to calculate M as the upper bound for $\left|\frac{1}{z^3+1}\right|$. Let's follow a similar path as the previous problem

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z^3+1}\right| \\
&= \left|\frac{1}{(x+iy)^3+1}\right| \\
&= \left|\frac{1}{x^3 - 3y^2x + i3x^2y - iy^3 + 1}\right| \\
&= \left|\frac{1}{(x^3 - 3y^2x + 1) + i\left(3x^2y - y^3\right)}\right|.
\end{aligned}$$

Using $u(x,y) = x^3 - 3y^2x + 1$ and $v(x,y) = 3x^2y - y^3$ we have,

$$\begin{aligned}
&= \left|\frac{1}{u+iv}\right| \\
&= \left|\frac{u-iv}{u^2+v^2}\right| \\
&= \frac{1}{\sqrt{u^2+v^2}}.
\end{aligned}$$

Substituting back in we have

$$\begin{aligned}
&\frac{1}{\sqrt{(x^3 - 3y^2x + 1)^2 + (3x^2y - y^3)^2}} \\
&= \frac{1}{\sqrt{x^6 - 3y^2x^4 + x^3 - 3y^2x^4 + 9y^4x^2 - 3y^2x + x^3 - 3y^2x + 1 + (3x^2y - y^3)^2}} \\
&= \frac{1}{\sqrt{x^6 - 3y^2x^4 + x^3 - 3y^2x^4 + 9y^4x^2 - 3y^2x + x^3 - 3y^2x + 1 + 9x^4y^2 - 6x^2y^4 + y^6}} \\
&= \frac{1}{\sqrt{x^6 + x^3 + 9y^4x^2 - 3y^2x + x^3 - 3y^2x + 1 - 3y^2x^4 - 3y^2x^4 + 9x^4y^2 - 6x^2y^4 + y^6}} \\
&= \frac{1}{\sqrt{x^6 + 2x^3 + 9x^2y^4 - 6xy^2 + 1 + 3x^4y^2 - 6x^2y^4 + y^6}} \\
&= \frac{1}{\sqrt{x^6 + 2x^3 + 3x^2y^4 - 6xy^2 + 1 + 3x^4y^2 + y^6}} \\
&= \ldots
\end{aligned}$$

TODO Figure out another way, I don't think this is it. □

5: From A&F: 2.5.1 b, e
Evaluate $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given

by the following:
b)

$$f(z) = e^{z^2}$$

Solution:
e)

$$f(z) = \frac{1}{2z^2+1}$$

Solution:

6: Use the ideas from A&F: 2.5.5 to evaluate $\int_0^\infty e^{iz^3 t}\,dz$, $t > 0$. Express the result in terms of $\int_0^\infty e^{-r^3}\,dr$.
The ideas we might need to use are ... it's actually really long!
Solution:

7: From A&F: 2.5.6.
Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

Show how to evaluate this integral by considering

$$\oint_{C_{(\mathbb{R})}} \frac{dz}{z^2+1},$$

where $C_{(\mathbb{R})}$ is closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x-axis. *Hint:* use

$$\frac{1}{z^2+1} = -\frac{1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables. *Solution:*

Repeat this exercise for

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{\epsilon dx}{x^2+\epsilon^2}, \quad \epsilon > 0.$$

Seems like I am supposed to do 2.5.6 and then for the given integral as well.
Solution:

8: Use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.
Solution:

9: From A&F: 2.6.1 a, e.
Evaluate the integrals $\oint_C f(z)dz$, where C is the unit circle centered at the origin and $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):
a)

$$\frac{\sin z}{z}$$

Solution:

e)

$$\mathrm{e}^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)$$

Solution: