Hunter Lybbert
Student ID: 2426454
10-21-24
AMATH 567

HOMEWORK 4

Collaborators*: TODO

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.4.2 c, e.
Evaluate the integral $\oint_C f(z)\mathrm{d}z$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows:
c)

$$f(z) = \frac{1}{\bar{z}}$$

Solution:
We want to evaluate

$$\oint_C \frac{1}{\bar{z}}\mathrm{d}z$$

on the parameterized unit circle $z = \mathrm{e}^{i\theta}$ where $\theta \in [0, 2\pi)$, where $\bar{z} = \mathrm{e}^{-i\theta}$ on the unit circle. Note, before we do the substitution we need $\mathrm{d}z = i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta$. Now our integral is

$$\begin{aligned}
\oint_C \frac{1}{\bar{z}}\mathrm{d}z &= \oint_0^{2\pi} \frac{1}{\mathrm{e}^{-i\theta}} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \oint_0^{2\pi} i\,\mathrm{e}^{2i\theta}\,\mathrm{d}\theta \\
&= \left(\frac{1}{2}\,\mathrm{e}^{2i\theta} \Big|_0^{2\pi} \right) \\
&= \frac{1}{2}\,\mathrm{e}^{4\pi i} - \frac{1}{2}\,\mathrm{e}^{0} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

□

e)

$$f(z) = \mathrm{e}^{\bar{z}}$$

Solution:
We will use the same substitutions from the previous part

$$\begin{aligned}
\oint_C \mathrm{e}^{\bar{z}}\,\mathrm{d}z &= \oint_0^{2\pi} \mathrm{e}^{\mathrm{e}^{-i\theta}}\, i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \oint_0^{2\pi} \sum_{j=1}^{\infty} \frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\
&= \sum_{j=1}^{\infty} \oint_0^{2\pi} i\frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!}\mathrm{e}^{i\theta}\mathrm{d}\theta.
\end{aligned}$$

We are justified in reordering the integral of the infinite sum to be the infinite sum of the integrals since the original series converges absolutely. I will now just look at the integral inside the sum

$$\begin{aligned}
\oint_0^{2\pi} i\frac{\left(\mathrm{e}^{-i\theta}\right)^j}{j!}\mathrm{e}^{i\theta}\mathrm{d}\theta &= \oint_0^{2\pi} i\frac{\mathrm{e}^{-i\theta j}\,\mathrm{e}^{i\theta}}{j!}\mathrm{d}\theta \\
&= \oint_0^{2\pi} i\frac{\mathrm{e}^{-i\theta j+i\theta}}{j!}\mathrm{d}\theta \\
&= \oint_0^{2\pi} i\frac{\mathrm{e}^{i\theta(-j+1)}}{j!}\mathrm{d}\theta \\
&= \oint_0^{2\pi} \frac{i\,\mathrm{e}^{i\theta(1-j)}}{j!}\mathrm{d}\theta \\
&= \frac{1}{1-j}\frac{i\,\mathrm{e}^{i\theta(1-j)}}{j!}\bigg|_0^{2\pi} \\
&= \frac{1}{1-j}\frac{i\,\mathrm{e}^{i2\pi(1-j)}}{j!} - \frac{1}{1-j}\frac{i\,\mathrm{e}^{0}}{j!} \\
&= \frac{i}{(1-j)\,j!}\left(\mathrm{e}^{i2\pi(1-j)} - 1\right) \\
&= \frac{i}{(1-j)\,j!}(1-1) \\
&= 0.
\end{aligned}$$

I want to clarify why $\mathrm{e}^{i2\pi(1-j)} = 1$. Since $j \in \{1, 2, 3, ...\}$, then $1-j$ is an integer and we have $\mathrm{e}^{i2\pi\ell}$ where $\ell \in \mathbb{Z}$ which is always 1. □

2: From A&F: 2.4.4 a, b. Use the principal branch where the argument is in $[-\pi, \pi)$. Discuss any ambiguities. Use the principal branch of $\log(z)$ and $z^{\frac{1}{2}}$ where the argument is in $[-\pi, \pi)$ to evaluate the following:
a)

$$\int_{-1}^{1} \log z \mathrm{d}z$$

Solution:
We want to parameterize this once again using $z = r\,\mathrm{e}^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our

integral is

$$\begin{aligned}\int_{-1}^{1} \log z \mathrm{d}z &= \int_{-\pi}^{0} \log\left(\mathrm{e}^{i\theta}\right) i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \int_{-\pi}^{0} i\theta i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta.\end{aligned}$$

Let's use integration by parts, woohoo! We will assign the substitutions as follows:

$$\begin{aligned}u &= i\theta \\ \mathrm{d}u &= i\mathrm{d}\theta \\ \\ \mathrm{d}v &= i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ v &= \mathrm{e}^{i\theta}.\end{aligned}$$

Plugging this in we have

$$\begin{aligned}\int_{-\pi}^{0} i\theta i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta &= i\theta\,\mathrm{e}^{i\theta}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} i\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta \\ &= \left(-i\pi\,\mathrm{e}^{-i\pi} - 0\right) - \mathrm{e}^{i\theta}\Big|_{-\pi}^{0} \\ &= -i\pi\,\mathrm{e}^{-i\pi} - \left(\mathrm{e}^{-i\pi} - \mathrm{e}^{0}\right) \\ &= i\pi - \left(\mathrm{e}^{-i\pi} - 1\right) \\ &= i\pi - (-1-1) \\ &= i\pi + 2\end{aligned}$$

This is weird **TODO**.
b)

$$\int_{-1}^{1} z^{\frac{1}{2}}\mathrm{d}z$$

Solution:

3: From A&F: 2.4.7
Let C be an open (upper) semicircle of radius R with its center at the origin, and consider $\int_C f(z)\mathrm{d}z$. Let $f(z) = \frac{1}{z^2+a^2}$ for a real $a > 0$. Show that $|f(z)| \le \frac{1}{R^2-a^2}$, $R > a$, and

$$\left|\int_C f(z)dz\right| \le \frac{\pi R}{R^2-a^2}, \quad R > a.$$

Solution:

4: From A&F: 2.4.8
Let C be an arc of the circle $|z| = R(R > 1)$ of angle $\frac{\pi}{3}$. Show that

$$\left|\int_C \frac{\mathrm{d}z}{z^3+1}\right| \le \frac{\pi}{3}\left(\frac{R}{R^3-1}\right)$$

and deduce

$$\lim_{R\to\infty}\int_C \frac{\mathrm{d}z}{z^3+1} = 0$$

Solution:

5: From A&F: 2.5.1 b, e
Evaluate $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:
b)

$$f(z) = \mathrm{e}^{z^2}$$

Solution:
e)

$$f(z) = \frac{1}{2z^2+1}$$

Solution:

6: Use the ideas from A&F: 2.5.5 to evaluate $\int_0^\infty \mathrm{e}^{\mathrm{i}z^3 t}\,\mathrm{d}z$, $t > 0$. Express the result in terms of $\int_0^\infty \mathrm{e}^{-r^3}\,\mathrm{d}r$.
The ideas we might need to use are ... it's actually really long!
Solution:

7: From A&F: 2.5.6.
Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}.$$

Show how to evaluate this integral by considering

$$\oint_{C_{(\mathbb{R})}} \frac{\mathrm{d}z}{z^2+1},$$

where $C_{(\mathbb{R})}$ is closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x-axis. *Hint:* use

$$\frac{1}{z^2+1} = -\frac{1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables. *Solution:*

Repeat this exercise for

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{\epsilon\mathrm{d}x}{x^2+\epsilon^2}, \quad \epsilon > 0.$$

Seems like I am supposed to do 2.5.6 and then for the given integral as well.
Solution:

8: Use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.
Solution:

9: From A&F: 2.6.1 a, e.

Evaluate the integrals $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin and $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):

a)

$$\frac{\sin z}{z}$$

Solution:

e)

$$\mathrm{e}^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)$$

Solution: