Hunter Lybbert
Student ID: 2426454
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AMATH 561

PROBLEM SET 4

1. Let $\Omega = \{a, b, c, d\}$ and let $\mathcal{F} = 2^\Omega$. We define a probability measure P as follows:

$$P(a) = 1/6, \ \ , P(b) = 1/3, \ \ P(c) = 1/4, \ \ P(d) = 1/4.$$

Next, define three random variables:

$$X(a) = 1, \ \ X(b) = 1, \ \ X(c) = -1, \ \ X(d) = -1,$$
$$Y(a) = 1, \ \ Y(b) = -1, \ \ Y(c) = 1, \ \ Y(d) = -1,$$

and $Z = X + Y$.

(a) List the sets in $\sigma(X)$.
Solution:
The pre-image of X is

$$X^{-1}(B) = \begin{cases} \{c, d\}, & \text{if } -1 \in B, 1 \notin B \\ \{a, b\}, & \text{if } 1 \in B, -1 \notin B. \end{cases}$$

Then we have

$$\sigma(X) = \sigma(\{\{a, b\}, \{c, d\}\}) = \Big\{ \{a, b\}, \{c, d\}, \Omega, \emptyset \Big\}$$

□

(b) Calculate $E(Y|X)$.
Solution:
We can calculate this as follows

$$\mathbb{E}[Y|X] = \mathbb{E}[Y|\sigma(X)]$$

then we have

$$\mathbb{E}[Y|\sigma(X)] = \frac{\mathbb{E}[Y; \{a, b\}]}{P(\{a, b\})} = \frac{1 \cdot P(a) - 1 \cdot P(b)}{\frac{1}{6} + \frac{1}{3}} = \frac{1 \cdot \frac{1}{6} - 1 \cdot \frac{1}{3}}{\frac{1}{2}} = -\frac{1}{3}$$

and

$$\mathbb{E}[Y|\sigma(X)] = \frac{\mathbb{E}[Y; \{c, d\}]}{P(\{c, d\})} = \frac{1 \cdot P(c) - 1 \cdot P(d)}{\frac{1}{2}} = \frac{1 \cdot \frac{1}{4} - 1 \cdot \frac{1}{4}}{\frac{1}{2}} = 0$$

□

(c) Calculate $E(Z|X)$.
Solution:
Let's first look at the values that $Z(\omega)$ takes on for each $\omega \in \{a, b, c, d\}$.

$$Z(a) = 2, \ Z(b) = 0, \ Z(c) = 0, \ Z(d) = -2$$

Then we have

$$\mathbb{E}[Z|X] = \mathbb{E}[Z|\sigma(X)]$$

giving us

$$\mathbb{E}[Z|\sigma(X)] = \frac{\mathbb{E}[Z;\{a,b\}]}{P(\{a,b\})} = \frac{2\cdot P(a) + 0\cdot P(b)}{\frac{1}{6}+\frac{1}{3}} = \frac{2\cdot\frac{1}{6}+0\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

and

$$\mathbb{E}[Z|\sigma(X)] = \frac{\mathbb{E}[Z;\{c,d\}]}{P(\{c,d\})} = \frac{0\cdot P(c) + -2\cdot P(d)}{\frac{1}{2}} = \frac{0\cdot\frac{1}{4} - 2\cdot\frac{1}{4}}{\frac{1}{2}} = -1$$

$\square$

2. (a) Prove that $E(E(X|\mathcal{F})) = EX$.
Solution:
There is an underlying probability space $(\Omega, \mathcal{F}_0, P)$. And all we know about $\mathcal{F}$ is that it is a subset of the σ-algebra which X is defined on, $\mathcal{F} \subset \mathcal{F}_0$. Let $Y = \mathbb{E}[X|\mathcal{F}]$ be a random variable, then by our definition in lecture slides 10 we have

(1) $Y \in \mathcal{F}$ that is Y is $\mathcal{F}$ measurable and
(2) For all $A \in \mathcal{F}$, we have

$$\int_A Y \mathrm{d}P = \int_A X \mathrm{d}P.$$

Since, $\mathcal{F}$ is a σ-algebra we can take $A = \Omega$ and then we have

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\big] = \int_\Omega \mathbb{E}[X|\mathcal{F}]\mathrm{d}P = \int_\Omega Y \mathrm{d}P = \int_\Omega X \mathrm{d}P = \mathbb{E}[X]$$

□

(b) Show that if $\mathcal{G} \subset \mathcal{F}$ and $EX^2 < \infty$ then

$$E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) = E(\{X - E(X|\mathcal{G})\}^2)$$

Solution:
Assume $\mathcal{G} \subset \mathcal{F}$ and $\mathbb{E}[X^2] < \infty$. Let's begin by expanding the terms on the left

$$\begin{aligned}
&\mathbb{E}\left[\big(X - \mathbb{E}[X|\mathcal{F}]\big)^2\right] + \mathbb{E}\left[\big(\mathbb{E}[X|\mathcal{F}] - \mathbb{E}[X|\mathcal{G}]\big)^2\right] \\
&= \mathbb{E}\left[X^2 - 2X\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[X|\mathcal{F}]^2\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]^2 - 2\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}] + \mathbb{E}[X|\mathcal{G}]^2\right] \\
&= \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]^2\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]^2\right] - 2\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right].
\end{aligned}$$

Let's look specifically at this term $\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]^2\big]$ which shows up twice. We make use of the theorem from class which states if $X \in \mathcal{F}$ then

$$\mathbb{E}[XY|\mathcal{F}] = X\mathbb{E}[Y|\mathcal{F}].$$

And finish by applying part (a) as well. Thus

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]^2\right] &= \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{F}]\right] \\
&= \mathbb{E}\left[\mathbb{E}\big[X\mathbb{E}[X|\mathcal{F}]\big|\mathcal{F}\big]\right] \\
&= \mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right].
\end{aligned}$$

Picking up where we left off we have

$$= \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]^2\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]^2\right] - 2\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right]$$

$$= \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right] + \mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right] + \mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right] - 2\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right]$$

$$= \mathbb{E}\left[X^2\right] \cancel{- 2\mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right]} \cancel{+ 2\mathbb{E}\left[X\mathbb{E}[X|\mathcal{F}]\right]} - 2\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right]$$

$$= \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right].$$

Pausing again, to look more closely at $\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\big]$, we can apply the same theorem and part (a) to get

$$\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] = \mathbb{E}\left[\mathbb{E}\big[X\mathbb{E}[X|\mathcal{G}]\big|\mathcal{F}\big]\right] = \mathbb{E}\left[X\mathbb{E}[X|\mathcal{G}]\right]$$

Therefore we have

$$\begin{aligned}
&= \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right] \\
&= \mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[X\mathbb{E}[X|\mathcal{G}]\right] + \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]^2\right] \\
&= \mathbb{E}\left[X^2 - 2X\mathbb{E}[X|\mathcal{G}] + \mathbb{E}[X|\mathcal{G}]^2\right] \\
&= \mathbb{E}\left[\big(X - \mathbb{E}[X|\mathcal{G}]\big)^2\right].
\end{aligned}$$

Therefore,

$$\mathbb{E}\big[\big(X - \mathbb{E}[X|\mathcal{F}]\big)^2\big] + \mathbb{E}\big[\big(\mathbb{E}[X|\mathcal{F}] - \mathbb{E}[X|\mathcal{G}]\big)^2\big] = \mathbb{E}\big[\big(X - \mathbb{E}[X|\mathcal{G}]\big)^2\big]$$

as desired. $\square$

3. An important special case of the previous result (2b) occurs when $\mathcal{G} = \{\emptyset, \Omega\}$. Let $\mathrm{var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

$$\mathrm{var}(X) = E(\mathrm{var}(X|\mathcal{F})) + \mathrm{var}(E(X|\mathcal{F})).$$

Solution

This can be shown directly by starting from the right and applying 2(a)

$$\begin{aligned}
&\mathbb{E}\big[\mathrm{Var}(X|\mathcal{F})\big] + \mathrm{Var}\big(\mathbb{E}[X|\mathcal{F}]\big) \\
&= \mathbb{E}\Bigg[\mathbb{E}[X^2|\mathcal{F}] - \mathbb{E}[X|\mathcal{F}]^2\Bigg] + \mathbb{E}\Bigg[\mathbb{E}[X|\mathcal{F}]^2\Bigg] - \mathbb{E}\Bigg[\mathbb{E}[X|\mathcal{F}]\Bigg]^2 \\
&= \mathbb{E}\Bigg[\mathbb{E}[X^2|\mathcal{F}]\Bigg] - \cancel{\mathbb{E}\Bigg[\mathbb{E}[X|\mathcal{F}]^2\Bigg]} + \cancel{\mathbb{E}\Bigg[\mathbb{E}[X|\mathcal{F}]^2\Bigg]} - \mathbb{E}\Bigg[\mathbb{E}[X|\mathcal{F}]\Bigg]^2 \\
&= \mathbb{E}\Bigg[\mathbb{E}[X^2|\mathcal{F}]\Bigg] - \mathbb{E}\Bigg[\mathbb{E}[X|\mathcal{F}]\Bigg]^2 \\
&= \mathbb{E}\big[X^2\big] - \mathbb{E}\big[X\big]^2 \\
&= \mathrm{Var}(X).
\end{aligned}$$

Therefore,

$$\mathrm{Var}(X) = \mathbb{E}\big[\mathrm{Var}(X|\mathcal{F})\big] + \mathrm{Var}\big(\mathbb{E}[X|\mathcal{F}]\big)$$

as desired. $\square$

4. Let $Y_1, Y_2, ...$ be i.i.d. (independent and identically distributed) random variables with mean μ and variance σ^2, N an independent positive integer valued random variable with $EN^2 < \infty$ and $X = Y_1 + ... + Y_N$. Show that

$$\mathrm{var}(X) = \sigma^2 EN + \mu^2 \mathrm{var}(N).$$

(To understand and help remember the formula, think about the two special cases in which N or Y is constant.)

Solution:

Let's begin by using the formula we proved in problem 3 $\mathrm{Var}(X) = \mathbb{E}\big[\mathrm{Var}(X|\mathcal{F})\big] + \mathrm{Var}\big(\mathbb{E}[X|\mathcal{F}]\big)$. Conditioning on the random variable N we have

$$\mathrm{Var}(X) = \mathbb{E}\big[\mathrm{Var}(X|N)\big] + \mathrm{Var}\big(\mathbb{E}[X|N]\big).$$

Now we can look at this piece by piece beginning with $\mathrm{Var}(X|N)$

$$\mathrm{Var}(X|N) = \mathrm{Var}\left(\sum_{i}^{N} Y_i\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{Cov}(Y_i, Y_j) = \sum_{i=1}^{N} \mathrm{Var}(Y_i) + \cancelto{0}{\sum_{j\neq i}^{N} \mathrm{Cov}(Y_i, Y_j)} = \sigma^2 N.$$

Where the sum of the covariances is 0, since the Y_i's are independent. Now looking at $\mathbb{E}[X|N]$

$$\mathbb{E}[X|N] = \mathbb{E}\left[\sum_{i}^{N} Y_i\right] = \sum_{i}^{N} \mathbb{E}\big[Y_i\big] = \mu N.$$

Furthermore, we have

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\big[\mathrm{Var}(X|N)\big] + \mathrm{Var}\big(\mathbb{E}[X|N]\big) \\
&= \mathbb{E}\big[\sigma^2 N\big] + \mathrm{Var}\big(\mu N\big) \\
&= \sigma^2 \mathbb{E}\big[N\big] + \mathrm{Var}\big(\mu N\big) \\
&= \sigma^2 \mathbb{E}\big[N\big] + \mathbb{E}\big[\mu N - \mathbb{E}[\mu N]\big]^2 \\
&= \sigma^2 \mathbb{E}\big[N\big] + \mathbb{E}\big[\mu(N - \mathbb{E}[N])\big]^2 \\
&= \sigma^2 \mathbb{E}\big[N\big] + \mu^2 \mathbb{E}\big[N - \mathbb{E}[N]\big]^2 \\
&= \sigma^2 \mathbb{E}[N] + \mu^2 \mathrm{Var}(N)
\end{aligned}$$

Hence,

$$\mathrm{Var}(X) = \sigma^2 \mathbb{E}[N] + \mu^2 \mathrm{Var}(N)$$

as desired. $\square$