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AMATH 567

HOMEWORK 3

Collaborators*: TBD

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.2.4.

Let α be a real number. Show that the set of all values of the multivalued function $\log(z^\alpha)$ is not necessarily the same as that of $\alpha \log z$.

Solution:

Let's begin by looking more closely at the values that the function $\alpha \log z$ can take on. Starting with $w = \alpha \log z$, notice we have

$$\begin{aligned}\alpha \log z &= w \\ \log z &= \frac{w}{\alpha} \\ z &= e^{\frac{w}{\alpha}}.\end{aligned}$$

Now recall that $w = u + iv$ therefore we also have

$$\begin{aligned}z &= e^{\frac{u+iv}{\alpha}} \\ z &= e^{\frac{u}{\alpha}} e^{\frac{iv}{\alpha}} \\ \log z &= \log \left(e^{\frac{u}{\alpha}} e^{\frac{iv}{\alpha}} \right) \\ \log z &= \log e^{\frac{u}{\alpha}} + \log e^{\frac{iv}{\alpha}} \\ \log z &= \frac{\log r}{\alpha} + \frac{i\theta}{\alpha} + \frac{2i\pi k}{\alpha}, \quad k \in \mathbb{Z}.\end{aligned}$$

Where the last line comes from the fact that we have $w = r e^{i\theta}$, $r = e^u$, and $v = \theta + 2\pi k$ again with $k \in \mathbb{Z}$. Now considering the other side

$$\begin{aligned}\log(z^\alpha) &= w \\ z^\alpha &= e^w \\ z &= (e^w)^{\frac{1}{\alpha}}.\end{aligned}$$

Therefore this is going to come down to the fact that this is not always going to be equivalent $e^{\frac{w}{\alpha}} \neq (e^w)^{\frac{1}{\alpha}}$.

2: Describe the Riemann surface on which the multi-valued function $w(z)$, defined by $w^2 = \prod_{j=1}^{n=3} (z - a_j)$ is single-valued. What happens for $n = 4, 5$? For $n > 5$? You may assume that all the a_j are distinct.

Solution:

Let's build up to what the the Riemann surface for $w^2 = \prod_{j=1}^{n=3} (z - a_j)$ will look like.

Beginning with $w^2 = z$ the branch point at the origin of the complex plane $z = 0$. is moved to the location a_0 instead of Riemann Surface be similar to that of $w^2 = z - a_0$ which is yet again similar to $w^2 = z$ except the branch point is moved to the location a_0 instead of the origin of the complex plane $z = 0$.

3: From A&F: 2.2.5a.

Derive the following formulae:

a)

$$\coth^{-1}(z) = \frac{1}{2} \log \frac{z+1}{z-1}$$

Solution:

We begin with solving for w in $z = \coth w$ with $w, z \in \mathbb{C}$.

$$\begin{aligned} z = \coth w &= \frac{\cosh w}{\sinh w} = \frac{\frac{e^w + e^{-w}}{2}}{\frac{e^w - e^{-w}}{2}} = \frac{e^w + e^{-w}}{e^w - e^{-w}} \\ &= \frac{e^w e^w + e^{-w}}{e^w e^w - e^{-w}} = \frac{e^{2w} + 1}{e^{2w} - 1} \end{aligned}$$

Now let's proceed by multiplying both sides by the denominator

$$\begin{aligned} z(e^{2w} - 1) &= e^{2w} + 1 \\ ze^{2w} - z &= e^{2w} + 1 \\ ze^{2w} - e^{2w} &= z + 1 \\ e^{2w}(z - 1) &= z + 1 \\ e^{2w} &= \frac{z + 1}{z - 1} \\ \log(e^{2w}) &= \log\left(\frac{z + 1}{z - 1}\right) + 2i\pi k, \quad k \in \mathbb{Z} \\ 2w &= \log\left(\frac{z + 1}{z - 1}\right) + 2i\pi k, \quad k \in \mathbb{Z} \\ w &= \frac{1}{2} \log\left(\frac{z + 1}{z - 1}\right) + i\pi k, \quad k \in \mathbb{Z} \end{aligned}$$

This is to show

$$\coth^{-1}(z) = \operatorname{arccot}(\coth w) = w = \frac{1}{2} \log\left(\frac{z + 1}{z - 1}\right) + i\pi k, \quad k \in \mathbb{Z}.$$

More directly we have

$$\coth^{-1}(z) = \frac{1}{2} \log\left(\frac{z + 1}{z - 1}\right) + i\pi k, \quad k \in \mathbb{Z}.$$

as required. **TODO:** determine what your choice of k should be

b)

□

$$\operatorname{sech}^{-1}(z) = \log\left(\frac{1 + (1 - z^2)^{\frac{1}{2}}}{z}\right)$$

Solution:

We begin with solving for w in $z = \operatorname{sech} w$ with $w, z \in \mathbb{C}$.

$$\begin{aligned} z = \operatorname{sech} w &= \frac{1}{\cosh w} = \frac{1}{\frac{e^w + e^{-w}}{2}} = \frac{2}{e^w + e^{-w}} \\ &= \frac{e^w}{e^w} \frac{2}{e^w + e^{-w}} = \frac{2e^w}{e^{2w} + 1} \end{aligned}$$

Now let's proceed by multiplying both sides by the denominator

$$\begin{aligned} z(e^{2w} + 1) &= 2e^w \\ ze^{2w} + z &= 2e^w \\ ze^{2w} - 2e^w + z &= 0 \end{aligned}$$

We now use the quadratic formula to solve for e^w

$$\begin{aligned} e^w &= \frac{2 + (4 - 4z^2)^{\frac{1}{2}}}{2z} \\ e^w &= \frac{2 + 2(1 - z^2)^{\frac{1}{2}}}{2z} \\ \log e^w &= \log \left(\frac{1 + (1 - z^2)^{\frac{1}{2}}}{z} \right) + 2i\pi k, \quad k \in \mathbb{Z} \\ w &= \log \left(\frac{1 + (1 - z^2)^{\frac{1}{2}}}{z} \right) + 2i\pi k, \quad k \in \mathbb{Z} \end{aligned}$$

This is to show

$$\operatorname{sech}^{-1}(z) = \operatorname{sech}^{-1}(\operatorname{sech} w) = w = \log \left(\frac{1 + (1 - z^2)^{\frac{1}{2}}}{z} \right) + 2i\pi k, \quad k \in \mathbb{Z}.$$

More directly we have

$$\operatorname{sech}^{-1}(z) = \log \left(\frac{1 + (1 - z^2)^{\frac{1}{2}}}{z} \right) + 2i\pi k, \quad k \in \mathbb{Z}.$$

as required. **TODO:** determine what your choice of k should be and how to denote the related cuts made for the $\sqrt{\cdot}$ function. \square

While you're at it, also derive a formula for $\operatorname{arccot}(z)$ in terms of the logarithm.

Solution:

Let's begin by solving for w in this equation $z = \cot w$ with $w, z \in \mathbb{C}$.

$$\begin{aligned} z = \cot w &= \frac{\cos w}{\sin w} = \frac{\frac{e^{iw} + e^{-iw}}{2}}{\frac{e^{iw} - e^{-iw}}{2i}} = \frac{i(e^{iw} + e^{-iw})}{e^{iw} - e^{-iw}} \\ &= \frac{e^{iw}}{e^{iw}} \frac{i(e^{iw} + e^{-iw})}{e^{iw} - e^{-iw}} = \frac{i(e^{2iw} + 1)}{e^{2iw} - 1} \end{aligned}$$

Now let's proceed by multiplying both sides by the denominator

$$\begin{aligned}
 z(e^{2iw} - 1) &= i(e^{2iw} + 1) \\
 ze^{2iw} - z &= ie^{2iw} + i \\
 ze^{2iw} - z - ie^{2iw} - i &= 0 \\
 e^{2iw}(z - i) &= z + i \\
 e^{2iw} &= \frac{z + i}{z - i} \\
 \log(e^{2iw}) &= \log\left(\frac{z + i}{z - i}\right) + 2i\pi k, \quad k \in \mathbb{Z} \\
 2iw &= \log\left(\frac{z + i}{z - i}\right) + 2i\pi k, \quad k \in \mathbb{Z} \\
 w &= \frac{1}{2i} \log\left(\frac{z + i}{z - i}\right) + \pi k, \quad k \in \mathbb{Z}
 \end{aligned}$$

This is to show

$$\operatorname{arccot}(z) = \operatorname{arccot}(\cot w) = w = \frac{1}{2i} \log\left(\frac{z + i}{z - i}\right) + \pi k, \quad k \in \mathbb{Z}.$$

More directly we have

$$\operatorname{arccot}(z) = \frac{1}{2i} \log\left(\frac{z + i}{z - i}\right) + \pi k, \quad k \in \mathbb{Z}$$

as required. **TODO:** determine what your choice of k should be □

4: Let

$$s(z) = z^{1/2} = \rho^{1/2} e^{i\theta/2}, \quad \theta \in [-\pi, \pi),$$

denote the principal branch of the square root. Show that the functions

$$f_1(z) = s(z^2 - 1), \quad f_2(z) = s(z - 1)s(z + 1),$$

are not equal as functions on \mathbb{C} — first produce plots and then use a mathematical argument. Determine the branch cut for $f_2(z)$ (Note: My cartoon of what the branch cut for f_1 looks like in lecture was not accurate). Find the relationship between $f_1(z)$ and $f_2(z)$.

Solution:

5: Consider the function

$$\psi(z) = \int_1^z \frac{dw}{(w^2 - 1)^{1/2}}, \quad z \notin (-\infty, 1),$$

where the path of integration is a straight line from 1 to z .

- Show that

$$\psi(z) = \log \varphi(z), \quad \varphi(z) = z + (z^2 - 1)^{1/2}, \quad z \notin (-\infty, 1),$$

for an appropriate choice of branch cut for $(z^2 - 1)^{1/2}$. Here $\log z$ denotes the principal branch.

Plot of $f_1(z) = s(z^2 - 1)$ and $f_2(z) = s(z - 1)s(z + 1)$ where $s(z) = z^{1/2} = \rho^{1/2}e^{i\theta/2}$, $\theta \in [-\pi, \pi)$

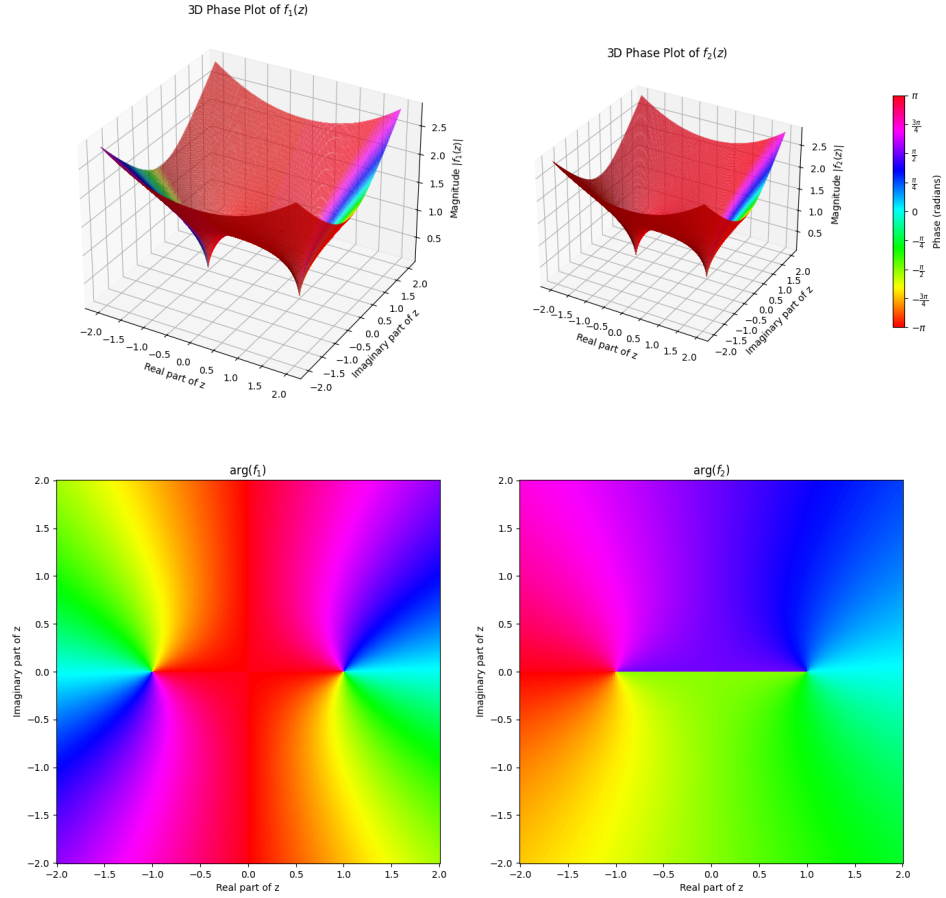


FIGURE 1. Plot $|f_\epsilon(z)| = \left| \frac{\epsilon}{\epsilon^2 + z^2} \right|$, for various values of ϵ .

- Find an expression for

$$\gamma(z) = \int_{-1}^z \frac{dw}{(w^2 - 1)^{1/2}}, \quad z \notin (-1, \infty),$$

in terms of $\varphi(z)$ and the principal branch of the logarithm. Again, the path of integration is a straight line.

Solution:

- 6:** Show that φ , from the previous problem, maps $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit disk, $\{z \in \mathbb{C} : |z| > 1\}$. Furthermore

$$\frac{1}{2}(\varphi(z) + 1/\varphi(z)) = z, \quad \mathbb{C} \setminus [-1, 1].$$

Solution:

- 7:** (Sharpness of the Bernstein–Walsh inequality) The Bernstein–Walsh inequality states that if a polynomial p_n of degree n satisfies $\max_{-1 \leq x \leq 1} |p_n(x)| \leq 1$ then

$$|p_n(z)| \leq |\varphi(z)|^n, \quad z \in \mathbb{C} \setminus [-1, 1].$$

Show that

$$T_n(z) = \frac{1}{2}(\varphi(z)^n + \varphi(z)^{-n}), \quad z \in \mathbb{C} \setminus [-1, 1]$$

is a polynomial that satisfies

$$\begin{aligned} \max_{-1 \leq x \leq 1} |T_n(x)| &= 1, \\ \lim_{n \rightarrow \infty} |T_n(z)|^{1/n} &= |\varphi(z)|, \end{aligned}$$

for any fixed $z \in \mathbb{C} \setminus [-1, 1]$.

Solution: