

Hunter Lybbert
 Student ID: 2426454
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 AMATH 567

HOMEWORK 4

Collaborators*: Nate Ward

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.4.2 c, e.

Evaluate the integral $\oint_C f(z) dz$, where C is the unit circle enclosing the origin, and $f(z)$ is given as follows:

c)

$$f(z) = \frac{1}{\bar{z}}$$

Solution:

We want to evaluate

$$\oint_C \frac{1}{\bar{z}} dz$$

on the parameterized unit circle $z = e^{i\theta}$ where $\theta \in [0, 2\pi)$, where $\bar{z} = e^{-i\theta}$ on the unit circle. Note, before we do the substitution we need $dz = i e^{i\theta} d\theta$. Now our integral is

$$\begin{aligned} \oint_C \frac{1}{\bar{z}} dz &= \int_0^{2\pi} \frac{1}{e^{-i\theta}} i e^{i\theta} d\theta \\ &= \int_0^{2\pi} i e^{2i\theta} d\theta \\ &= \left(\frac{1}{2} e^{2i\theta} \right) \Big|_0^{2\pi} \\ &= \frac{1}{2} e^{4\pi i} - \frac{1}{2} e^0 \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

□

e)

$$f(z) = e^{\bar{z}}$$

Solution:

We will use the same substitutions from the previous part

$$\begin{aligned}\oint_C e^{\bar{z}} dz &= \oint_0^{2\pi} e^{e^{-i\theta}} i e^{i\theta} d\theta \\ &= \oint_0^{2\pi} \sum_{j=1}^{\infty} \frac{(e^{-i\theta})^j}{j!} i e^{i\theta} d\theta \\ &= \sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta.\end{aligned}$$

We are justified in reordering the integral of the infinite sum to be the infinite sum of the integrals since the original series converges absolutely. Notice,

$$\sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta = \oint_0^{2\pi} i \frac{(e^{-i\theta})^1}{1!} e^{i\theta} d\theta + \sum_{j=2}^{\infty} \oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta$$

We need to look more closely at the first term that we pulled out, which is where $j = 1$. Observe

$$\oint_0^{2\pi} i \frac{(e^{-i\theta})^1}{1!} e^{i\theta} d\theta \text{ **TODO this part**}$$

I will now just look at the integral inside the sum where $j = 2, 3, 4, \dots$

$$\begin{aligned}\oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta &= \oint_0^{2\pi} i \frac{e^{-i\theta j} e^{i\theta}}{j!} d\theta \\ &= \oint_0^{2\pi} i \frac{e^{-i\theta j + i\theta}}{j!} d\theta \\ &= \oint_0^{2\pi} i \frac{e^{i\theta(-j+1)}}{j!} d\theta \\ &= \oint_0^{2\pi} i \frac{e^{i\theta(1-j)}}{j!} d\theta\end{aligned}$$

TODO: Address the $j = 1$.

$$\begin{aligned}\oint_0^{2\pi} i \frac{e^{i\theta(1-j)}}{j!} d\theta &= \frac{1}{1-j} \frac{i e^{i\theta(1-j)}}{j!} \Big|_0^{2\pi} \\ &= \frac{1}{1-j} \frac{i e^{i2\pi(1-j)}}{j!} - \frac{1}{1-j} \frac{i e^0}{j!} \\ &= \frac{i}{(1-j)j!} (e^{i2\pi(1-j)} - 1) \\ &= \frac{i}{(1-j)j!} (1 - 1) \\ &= 0.\end{aligned}$$

I want to clarify why $e^{i2\pi(1-j)} = 1$. Since $j \in \{1, 2, 3, \dots\}$, then $1 - j$ is an integer and we have $e^{i2\pi\ell}$ where $\ell \in \mathbb{Z}$ which is always 1. Now we return to the original problem

$$\sum_{j=1}^{\infty} \oint_0^{2\pi} i \frac{(e^{-i\theta})^j}{j!} e^{i\theta} d\theta = \sum_{j=1}^{\infty} 0 = 0.$$

Now we have completed the requisite task. \square

- 2:** From A&F: 2.4.4 a, b. Use the principal branch where the argument is in $[-\pi, \pi)$. Discuss any ambiguities. Use the principal branch of $\log(z)$ and $z^{\frac{1}{2}}$ where the argument is in $[-\pi, \pi)$ to evaluate the following:

a)

$$\int_{-1}^1 \log z dz$$

Solution:

We want to parameterize this once again using $z = r e^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our integral is

$$\begin{aligned} \int_{-1}^1 \log z dz &= \int_{-\pi}^0 \log(e^{i\theta}) i e^{i\theta} d\theta \\ &= \int_{-\pi}^0 i\theta i e^{i\theta} d\theta. \end{aligned}$$

Let's use integration by parts, woohoo! We will assign the substitutions as follows:

$$u = i\theta$$

$$du = i d\theta$$

$$dv = i e^{i\theta} d\theta$$

$$v = e^{i\theta}.$$

Plugging this in we have

$$\begin{aligned} \int_{-\pi}^0 i\theta i e^{i\theta} d\theta &= i\theta e^{i\theta} \Big|_{-\pi}^0 - \int_{-\pi}^0 i e^{i\theta} d\theta \\ &= (0 - (-i\pi e^{-i\pi})) - e^{i\theta} \Big|_{-\pi}^0 \\ &= 0 + i\pi e^{-i\pi} - e^{i\theta} \Big|_{-\pi}^0 \\ &= i\pi e^{-i\pi} - (e^0 - e^{-i\pi}) \\ &= -i\pi - (1 - (-1)) \\ &= -i\pi - (2) \\ &= -i\pi - 2. \end{aligned}$$

\square

b)

$$\int_{-1}^1 z^{\frac{1}{2}} dz$$

Solution:

We want to parameterize this once again using $z = r e^{i\theta}$ where $\theta \in [-\pi, \pi)$. Now our integral is

$$\begin{aligned} \int_{-1}^1 z^{\frac{1}{2}} dz &= \int_{-\pi}^0 (e^{i\theta})^{\frac{1}{2}} i e^{i\theta} d\theta \\ &= \int_{-\pi}^0 i e^{\frac{i\theta}{2}} e^{i\theta} d\theta \\ &= \int_{-\pi}^0 i e^{\frac{i3}{2}\theta} d\theta \\ &= \frac{2}{3} e^{\frac{i3}{2}\theta} \Big|_{-\pi}^0 \\ &= \frac{2}{3} e^{-\frac{i3}{2}\pi} - \frac{2}{3}. \end{aligned}$$

Now remembering our branch cut limits θ to be within $[-\pi, \pi)$ we change the angle $-\frac{3}{2}\pi$ to be $\frac{1}{2}\pi$. Hence,

$$\begin{aligned} \frac{2}{3} e^{-\frac{i3}{2}\pi} - \frac{2}{3} &= \frac{2}{3} e^{-\frac{i2}{2}\pi} e^{-\frac{i\pi}{2}} - \frac{2}{3} \\ &= \frac{2}{3} e^{\frac{i\pi}{2}} - \frac{2}{3} \\ &= \frac{2}{3} (i - 1). \end{aligned}$$

□

3: From A&F: 2.4.7

Let C be an open (upper) semicircle of radius R with its center at the origin, and consider $\int_C f(z) dz$. Let $f(z) = \frac{1}{z^2 + a^2}$ for a real $a > 0$. Show that $|f(z)| \leq \frac{1}{R^2 - a^2}$, $R > a$, and

$$\left| \int_C f(z) dz \right| \leq \frac{\pi R}{R^2 - a^2}, \quad R > a.$$

Solution:

First, we want to show

$$|f(z)| \leq \frac{1}{R^2 - a^2}$$

where $R > a > 0$ and $a \in \mathbb{R}$. Let's consider the function more closely

$$\begin{aligned} f(z) &= \frac{1}{z^2 + a^2} = \frac{1}{(x + iy)^2 + a^2} \\ &= \frac{1}{x^2 + 2ixy - y^2 + a^2} \\ &= \frac{1}{x^2 - y^2 + a^2 + i2xy}. \end{aligned}$$

Notice, we can write the real and imaginary parts of the complex number in the denominator as functions $u(x, y)$ and $v(x, y)$. Where $u(x, y) = x^2 - y^2 + a^2$ and $v(x, y) = 2xy$.

Now we get

$$f(z) = \frac{1}{x^2 - y^2 + a^2 + i2xy} = \frac{u - iv}{u - iv} \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

Then we calculate

$$\begin{aligned} |f(z)| &= \left| \frac{u - iv}{u^2 + v^2} \right| \\ &= \left| \frac{u}{u^2 + v^2} - i \frac{v}{u^2 + v^2} \right| \\ &= \sqrt{\left(\frac{u}{u^2 + v^2} \right)^2 + \left(\frac{v}{u^2 + v^2} \right)^2} \\ &= \sqrt{\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2}} \\ &= \sqrt{\frac{u^2 + v^2}{(u^2 + v^2)^2}} \\ &= \sqrt{\frac{1}{u^2 + v^2}} \\ &= \frac{1}{\sqrt{u^2 + v^2}}. \end{aligned}$$

If we plug our substitution back in we see

$$\begin{aligned} \frac{1}{\sqrt{u^2 + v^2}} &= \frac{1}{\sqrt{(x^2 - y^2 + a^2)^2 + (2xy)^2}} \\ &= \frac{1}{\sqrt{(x^2 - y^2 + a^2)(x^2 - y^2 + a^2) + 4x^2y^2}} \\ &= \frac{1}{\sqrt{x^4 - x^2y^2 + x^2a^2 - x^2y^2 + y^4 - y^2a^2 + x^2a^2 - y^2a^2 + a^4 + 4x^2y^2}} \\ &= \frac{1}{\sqrt{x^4 + x^2a^2 + y^4 - y^2a^2 + x^2a^2 - y^2a^2 + a^4 + 2x^2y^2}}. \end{aligned}$$

Now we add zero in a particular fashion, namely $-4x^2a^2 + 4x^2a^2$, so we can regroup the terms and refactor to get closer to what we desire

$$\begin{aligned} &= \frac{1}{\sqrt{x^4 + x^2a^2 + y^4 - y^2a^2 + x^2a^2 - y^2a^2 + a^4 + 2x^2y^2 + (-4x^2a^2 + 4x^2a^2)}} \\ &= \frac{1}{\sqrt{x^4 + y^4 - y^2a^2 - y^2a^2 + a^4 + 2x^2y^2 - x^2a^2 - x^2a^2 + 4x^2a^2}} \\ &= \frac{1}{\sqrt{(x^2 + y^2 - a^2)^2 + (2xa)^2}}. \end{aligned}$$

Using the fact that $\sqrt{a+b} \geq \sqrt{a}$ for $a, b > 0$, in our next step we get a smaller denominator which makes the overall expression greater or equal to the previous step.

Note, equality only holds when $x = 0$.

$$\begin{aligned}\frac{1}{\sqrt{(x^2 + y^2 - a^2)^2 + (2xa)^2}} &\leq \frac{1}{\sqrt{(x^2 + y^2 - a^2)^2}} \\ &= \frac{1}{x^2 + y^2 - a^2} \\ &= \frac{1}{|z|^2 - a^2} \\ &= \frac{1}{R^2 - a^2}\end{aligned}$$

Therefore $|f(z)| \leq \frac{1}{R^2 - a^2}$.

□

Next we wish to show that

$$\left| \int_C f(z) dz \right| \leq \frac{\pi R}{R^2 - a^2}, \quad R > a.$$

By Theorem 2.4.2 from A&F, if $f(z)$ is continuous on contour C , then

$$\left| \int_C f(z) dz \right| \leq ML$$

where L is the length of C and M is an upper bound for $|f(z)|$. We have that C is continuous, since $a > 0$ and $a < R$ there are no singularities or weirdness with $f(z)$ on the specified contour. So we have

$$M = \frac{1}{R^2 - a^2}$$

as we calculated in the first part of this problem. Additionally, we know the arc length of C is easy to calculate because it is half the circumference of the circle with radius R . To convince myself of this I will show the general arc length formula also provides this quick calculation. Let our parameterization of this contour be

$$z(\theta) = R e^{i\theta}$$

where $\theta \in [0, \pi)$. Then

$$z'(\theta) = R i e^{i\theta} = -R \sin \theta + i R \cos \theta.$$

Therefore calculating arc length is as follows,

$$\begin{aligned}L &= \int_a^b |z'(t)| dt \\ &= \int_0^\pi |-R \sin \theta + i R \cos \theta| d\theta \\ &= \int_0^\pi \sqrt{R^2 \sin^2 \theta + R^2 \cos^2 \theta} d\theta \\ &= \int_0^\pi \sqrt{R^2 (\sin^2 \theta + \cos^2 \theta)} d\theta \\ &= \int_0^\pi R d\theta \\ &= \pi R.\end{aligned}$$

Which is the same as half the circumference ($\frac{1}{2}2\pi R = \pi R$). And thus

$$\left| \int_C f(z) dz \right| \leq ML \leq \pi R \frac{1}{R^2 - a^2} = \frac{\pi R}{R^2 - a^2}.$$

Hence,

$$\left| \int_C f(z) dz \right| \leq \frac{\pi R}{R^2 - a^2}$$

as desired. □

4: From A&F: 2.4.8

Let C be an arc of the circle $|z| = R$ with ($R > 1$) of angle $\frac{\pi}{3}$. Show that

$$\left| \int_C \frac{dz}{z^3 + 1} \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right)$$

and deduce

$$\lim_{R \rightarrow \infty} \int_C \frac{dz}{z^3 + 1} = 0$$

Solution:

Similar to the previous problem, we will utilize Theorem 2.4.2 from A&F. We are justified in this, since the contour C is continuous on the arc of the circle $|z| = R$. This time, our arc length of the contour C is

$$L = \frac{1}{6}2\pi R = \frac{\pi}{3}R.$$

Next, we need to calculate M as the upper bound for $\left| \frac{1}{z^3 + 1} \right|$. Let's use a simpler method than in problem 3.

$$|f(z)| = \left| \frac{1}{z^3 + 1} \right| = \frac{1}{|z^3 + 1|}.$$

If we can get a lower bound for the denominator then we will have an upper bound for the whole expression. It falls out quickly from the inverse triangle inequality and the parameterization $z = R e^{i\theta}$.

$$\begin{aligned} |z^3 + 1| &= |R^3 e^{i3\theta} + 1| \\ &= |R^3 e^{i3\theta} - (-1)| \\ &\geq ||R^3 e^{i3\theta}| - |(-1)|| \\ &= |R^3 - 1| \\ &= R^3 - 1 \end{aligned}$$

where the last equality holds because $R > 1$. Therefore, we have our lower bound for the denominator

$$|z^3 + 1| \geq R^3 - 1.$$

And thus we have an upper bound for the expression as follows:

$$|f(z)| = \frac{1}{|z^3 + 1|} \leq \frac{1}{R^3 - 1} = M.$$

Once again leaning on Theorem 2.4.2 from A&F we have

$$\left| \int_C f(z) dz \right| \leq ML \leq \frac{\pi}{3} R \frac{1}{R^3 - 1} = \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right).$$

Hence,

$$\left| \int_C f(z) dz \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right).$$

□

5: From A&F: 2.5.1 b, e

Evaluate $\oint_C f(z) dz$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

b)

$$f(z) = e^{z^2}$$

Solution:

e)

$$f(z) = \frac{1}{2z^2 + 1}$$

Solution:

6: Use the ideas from A&F: 2.5.5 to evaluate $\int_0^\infty e^{iz^3 t} dz$, $t > 0$. Express the result in terms of $\int_0^\infty e^{-r^3} dr$.

The ideas we might need to use are ... it's actually really long!

Solution:

7: From A&F: 2.5.6.

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

Show how to evaluate this integral by considering

$$\oint_{C(\mathbb{R})} \frac{dz}{z^2 + 1},$$

where $C(\mathbb{R})$ is closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x -axis. *Hint:* use

$$\frac{1}{z^2 + 1} = -\frac{1}{2i} \left(\frac{1}{z + i} - \frac{1}{z - i} \right),$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \rightarrow \infty$. Verify your answer by usual integration in real variables. *Solution:*

Repeat this exercise for

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{\epsilon dx}{x^2 + \epsilon^2}, \quad \epsilon > 0.$$

Seems like I am supposed to do 2.5.6 and then for the given integral as well.

Solution:

8: Use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

Solution:

9: From A&F: 2.6.1 a, e.

Evaluate the integrals $\oint_C f(z)dz$, where C is the unit circle centered at the origin and $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):

a)

$$\frac{\sin z}{z}$$

Solution:

e)

$$e^{z^2} \left(\frac{1}{z^2} - \frac{1}{z^3} \right)$$

Solution: