AMATH 567 FALL 2024
HOMEWORK 2 — DUE OCTOBER 7 ON GRADESCOPE BY 1:30PM

All solutions must include significant justification to receive full credit. If you handwrite your assignment you must either do so digitally or if it is written on paper you must *scan* your work. A standard photo is not sufficient.

If you work with others on the homework, you must name your collaborators.

1: From A&F: 1.2.12.

2: From A&F: 1.3.5.

3: Consider the function

$$\varphi(z) = z + \sqrt{z^2 - 1}, \quad z > 1.$$

Show that

$$\log \varphi(z) = \int_1^z \sqrt{x^2 - 1}\, \mathrm{d}x.$$

4: Find all zeroes of $\tan(z), z \in \mathbb{C}$. What can you conclude about the zeroes of $\tanh(z) = \sinh(z)/\cosh(z), z \in \mathbb{C}$?

5: Consider $f_\epsilon(z) = \epsilon/\left(\epsilon^2 + z^2\right)$, where ϵ is a small positive number, and $z \in \mathbb{C}/\{i\epsilon, -i\epsilon\}$. Plot $|f_\epsilon(z)|$, for various values of ϵ. Discuss the influence the singularities of a function in the complex plane have on its behavior on the real line. Compute

$$\int_{-\infty}^{\infty} f_\epsilon(x)\mathrm{d}x.$$

6: Visualizing complex functions is not as easy as visualizing real-valued functions, since we need 4 dimensions: two for the input, two for the output. Different visualizations are commonly used, such as showing 3-dimensional plots of the real and imaginary parts. Plotting the modulus is informational, but it eliminates a lot of information.

- To see this, plot the real and imaginary part of the exponential function $\exp(z) = \exp(x + \mathrm{i}y)$, for $x \in [-1, 1], y \in [-2\pi, 2\pi]$. Now plot the modulus over the same region, and compare.
- A "new" popular way to do this is to plot the modulus of the function with the color defined by the phase. The Digital Library of Mathematical Functions has lots of examples. Create a plot of the $|\exp(z)| = |\exp(x + \mathrm{i}y)|$, for $x \in [-1, 1]$, $y \in [-2\pi, 2\pi]$. colored by the argument. Experimenting with other functions is highly encouraged! The book visual Complex Functions: An Introduction with Phase Portraits by Elias Wegert (Birkhäuser, 2012) is a good companion to our textbook, if you think geometrically.

7: From A&F: 2.1.1

8: From A&F: 2.1.7

9: Show that the derivative of $f(z) = |z|^2$ is defined at $z = 0$, but nowhere else.

10: Derive the polar-coordinates form of the Cauchy-Riemann equations

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta.$$

where $x = r \cos\theta$ and $y = r \sin\theta$