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AMATH 567

HOMEWORK 5

Collaborators*:

*Listed in no particular order. And anyone I discussed at least part of one problem with is considered a collaborator.

1: From A&F: 2.6.5
Consider two entire functions with no zeroes and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function.
Solution:
Let's define our two entire functions to be $f(z)$ and $g(z)$. Recall that an entire function is analytic in all of the complex plane. We can focus on the ratio between these two functions $\frac{f(z)}{g(z)}$ since we are also given that $f(z)$ and $g(z)$ have no zeros. Let $h(z)$ be the ratio between f and g

$$h(z) = \frac{f(z)}{g(z)}.$$

If we can use Liouville's theorem to show that $h(z)$ is constant, then $f(z)$ and $g(z)$ are equal everywhere and are thus the same function.

For reference, Liouville's Theorem states that if $f(z)$ is entire and bounded in the z plane (including infinity), then $f(z)$ is a constant. Hence we need to show that $h(z)$ is entire and bounded in the z plane, then $h(z)$ is constant and we will have what we want. We know that the functions $f(z)$ and $g(z)$ are entire. We also know that the function $\frac{1}{z}$ is analytic except when $z = 0$. Since neither f nor g have zeros, then the potential of having 0 in the denominator of $h(z)$ is no longer an issue. Therefore $\frac{1}{z}$, $z \neq 0$ is entire. Therefore $h(z)$ is entire since it is the composition of entire functions.

Now we need to show that $h(z)$ is bounded in the z plane. Since $h(z)$ is entire, then it is analytic interior to and on a simple closed contour C (which we will choose later), then by Theorem 2.6.2, we have

$$h^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} \mathrm{d}\xi.$$

Now we can use the established inequality (2.6.13 in A & F)

$$\left| h^{(n)}(z) \right| \leq \frac{n! M}{R^n}.$$

When $n = 1$ we have

$$|h'(z)| \leq \frac{M}{R}.$$

We can take R to be arbitrarily large to get $|h'(z)| \leq 0$ implying $h'(z) = 0$. Using the fundamental theorem of calculus we can write

$$h(\infty) - h(z) = \int_z^\infty h'(z)\mathrm{d}z = C|_z^\infty = C - C = 0.$$

This gives $h(\infty) = h(z)$, therefore, by Liouville's Theorem $h(z)$ is constant. From the problem's setup we know $h(\infty) = \frac{f(\infty)}{g(\infty)} = 1$. Hence,

$$h(\infty) = h(z) = 1.$$

Therefore, $f(z)$ and $g(z)$ must be the same function, since their ratio is 1 for all z.

$\square$

2: From A&F: 2.6.10

... deduce

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z}\mathrm{d}\theta$$

... explain why we have

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\xi)\xi}{\xi - 1/\bar{z}}\mathrm{d}\theta$$

... use something to show

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\frac{\xi}{\xi - z} \pm \frac{\bar{z}}{\xi - \bar{z}}\right)\mathrm{d}\theta$$

then ...

Solution:

This is a beast of a problem there are **approximately 9** things to show...

3: Suppose Ω is an open simply connected region and $z_0 \in \Omega$. Assume that $f(z)$ is analytic in $\Omega \setminus \{z_0\}$ and satisfies

$$|f(z)| \leq M|z - z_0|^{-\gamma}, \quad \gamma < 1.$$

Show that if the a specific choice for $f(z_0)$ is made then f extends to an analytic function on Ω.

(**1 part**, except maybe if there are multiple things to prove here)

Solution:

4: Establish the following lemma:

> **Lemma 1**
>
> Suppose Ω is an open region and $f(z)$ is continuous on $\overline{\Omega}$. Let Γ be a contour in $\overline{\Omega}$. Suppose a sequence of contours $\Gamma_n \subset \overline{\Omega}$ converge to Γ in the sense that there exists parameterizations $z(t)$ of Γ and $z_n(t)$ of Γ_n defined on $[a,b]$ satisfying
>
> $$z_n(t) \overset{n\to\infty}{\longrightarrow} z(t), \qquad \text{uniformly on } [a,b],$$
> $$z_n'(t) \overset{n\to\infty}{\longrightarrow} z'(t), \qquad \text{uniformly on } [a,b].$$
>
> Then
>
> $$\int_{\Gamma_n} f(z)\mathrm{d}z \overset{n\to\infty}{\longrightarrow} \int_{\Gamma} f(z)\mathrm{d}z.$$

Hint: Use that f is uniformly continuous on $\overline{\Omega}$.
(**1 part**, except maybe if there are multiple things to prove here)
Solution:

5: for any $r, R > 0$, let $C = \partial\Sigma$, $\Sigma = \{z \in \mathbb{C} \ : \ |\operatorname{Re} z| \leq r \text{ and } 0 \leq -\operatorname{Im} z \leq R, \ R > 0\}$. In this problem $\sqrt{z}$ denotes the principal branch with $\arg z \in [-\pi, \pi)$.

- Show that if $f(z)$ is analytic in a region that contains Σ,

$$\oint_C f(z)\sqrt{z-1}\sqrt{z+1}\mathrm{d}z = 0.$$

(**1 part**)
Solution:

- Show that if $f(z)$ is analytic in a region that contains Σ

$$\oint_C \frac{f(z)\mathrm{d}z}{\sqrt{z-1}\sqrt{z+1}} = 0.$$

(**1 part**)
Solution:

6: From A&F: 3.1.1 b,d
In the following we are given sequences. Discuss their limits and whether the convergence is uniform, in the region $\alpha \leq |z| \leq \beta$, for finite $\alpha, \beta > 0$.
b)

$$\left\{\frac{1}{z^n}\right\}_{n=1}^{\infty}$$

(**2 parts**)
Solution:

d)

$$\left\{\frac{1}{1+(nz)^2}\right\}_{n=1}^{\infty}$$

(**2 parts**)
Solution:

7: From A&F: 3.1.2 b,d
For each sequence in problem 1, what can be said if
(a) $\alpha = 0$
(b) $\alpha > 0, \quad \beta = \infty$
(**4 parts** 2x2) *Solution:*

8: From A&F: 3.1.3 Compute the integrals

$$\lim_{n\to\infty} \int_0^1 nz^{n-1} \mathrm{d}z \quad \text{and} \quad \int_0^1 \lim_{n\to\infty} \left(nz^{n-1}\right) \mathrm{d}z$$

and show that they are not equal. Explain why this is not a counter example to Theorem 3.1.1. (A &F pg. 111)
(**3 parts**) *Solution:*

There are approximately 25 things to do