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AMATH 561

PROBLEM SET 6

1. Let $X \sim Binomial(n, U)$, where $U \sim Uniform((0, 1))$. What is the probability generating function $G_X(s)$ of X? What is $P(X = k)$ for $k \in \{0, 1, 2, ..., n\}$?

Solution: The probability mass function for $X \sim Binomial(n, U)$, is given by

$$f_X(x) = \binom{n}{x} U^x (1-U)^{n-x} \quad \text{for } x = 0, 1, 2, ..., n$$

And the density for the Uniform distribution is

$$g_X(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1 \quad \text{for } x \in (0, 1)$$

Then $G_X(s)$ is

$$\begin{aligned}
G_X(s) &= E(s^X) \\
&= E(E(s^X|U)) \\
&= E\left(\sum_{x=0}^{n} \binom{n}{x} U^x (1-U)^{n-x} s^x\right) \\
&= E\left(\sum_{x=0}^{n} \binom{n}{x} (Us)^x (1-U)^{n-x}\right) \\
&= E\big((Us + 1 - U)^n\big) \\
&= \int_0^1 (us + 1 - u)^n du \\
&= \frac{(us + 1 - u)^{n+1}}{(n+1)(s-1)}\bigg|_0^1 \\
&= \frac{(1s + 1 - 1)^{n+1}}{(n+1)(s-1)} - \frac{(0s + 1 - 0)^{n+1}}{(n+1)(s-1)} \\
&= \frac{s^{n+1}}{(n+1)(s-1)} - \frac{1^{n+1}}{(n+1)(s-1)} \\
&= \frac{s^{n+1} - 1}{(n+1)(s-1)}
\end{aligned}$$

Now to calculate $P(X=k)$, notice

$$\begin{aligned} G_X(s) = \frac{s^{n+1}-1}{(n+1)(s-1)} &= \frac{1}{n+1}\frac{s^{n+1}-1}{s-1} \\ &= \frac{1}{n+1}\frac{1-s^{n+1}}{1-s} \\ &= \sum_{k=0}^{n} \frac{1}{n+1}s^k \\ &= \sum_{k=0}^{n} P(X=k)s^k \\ &= G_X(s) \end{aligned}$$

Therefore,

$$P(X=k) = \frac{1}{n+1} \quad \text{for all } k \in \{0,1,2,..,n\}.$$

$\square$

2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ, the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

Solution:
We can write the generating function $G_{Z_{n+1}}(s)$ as follows

$$\begin{aligned}
G_{Z_{n+1}}(s) &= G_{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}}(s) \\
&= G_{\xi_1^{n+1}+\xi_2^{n+1}+\xi_3^{n+1}+\ldots+\xi_{Z_n}^{n+1}+Y_{n+1}}(s) \\
&= G_{\xi_1^{n+1}+\xi_2^{n+1}+\xi_3^{n+1}+\ldots+\xi_{Z_n}^{n+1}} G_{Y_{n+1}}(s) \\
&= G_{Z_n}(G_\xi(s)) G_{Y_{n+1}}(s).
\end{aligned}$$

The second to third equality comes from the fact that the Y_{n+1} and ξ_i^{n+1} are independent. Finally, the last equality comes from an application of the Theorem 3 from Lecture 15.

Next to calculate $G_{Z_2}(s)$ explicitly we get

$$\begin{aligned}
G_{Z_2}(s) &= G_{Z_1}(G_\xi(s)) G_Y(s) \\
&= \Big(G_\xi\Big(G_\xi(s)\Big) G_Y\Big(G_\xi(s)\Big)\Big) G_Y(s).
\end{aligned}$$

$\square$

3. (a) Let X be exponentially distributed with parameter λ. Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$.

Solution:
We can begin by looking directly at the expectation we want to calculate

$$\begin{aligned} E(\mathrm{e}^{\mathrm{i}\,tX}) = \int_\Omega \mathrm{e}^{\mathrm{i}\,tX}\,dP &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\,tx}\,\lambda\,\mathrm{e}^{-\lambda x}\,dx \\ &= \int_0^\infty \mathrm{e}^{\mathrm{i}\,tx}\,\lambda\,\mathrm{e}^{-\lambda x}\,dx \\ &= \int_0^\infty \lambda\,\mathrm{e}^{(\mathrm{i}\,t-\lambda)x}\,dx \\ &= \int_0^\infty \lambda\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx \end{aligned}$$

Notice this integral is off by a scale factor to the density of an exponentially distributed random variable with parameter $\lambda - \mathrm{i}\,t$. Additionally, we know the integral of a probability density function is equal to 1, therefore,

$$\begin{aligned} \int_0^\infty (\lambda - \mathrm{i}\,t)\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= 1 \\ \frac{\lambda}{\lambda-\mathrm{i}\,t}\int_0^\infty (\lambda - \mathrm{i}\,t)\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= \frac{\lambda}{\lambda-\mathrm{i}\,t} \\ \int_0^\infty \frac{\lambda}{\lambda-\mathrm{i}\,t}(\lambda - \mathrm{i}\,t)\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= \frac{\lambda}{\lambda-\mathrm{i}\,t} \\ \int_0^\infty \lambda\,\mathrm{e}^{-(\lambda-\mathrm{i}\,t)x}\,dx &= \frac{\lambda}{\lambda-\mathrm{i}\,t}. \end{aligned}$$

Which is indeed the integral we wanted to compute.

□

(b) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$.

Solution: The characteristic function is (skipping directly to the change of variable form of the expectation)

$$\begin{aligned} \phi_X(t) = E(\mathrm{e}^{\mathrm{i}\,tX}) &= \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,tx}\,\frac{1}{2}\,\mathrm{e}^{-|x|}\,dx \\ &= \frac{1}{2}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx. \end{aligned}$$

Let's split up the integral into cases in order to handle the absolute value. Then we have

$$
\begin{aligned}
\phi_X(t) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx &= \frac{1}{2}\left[\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx + \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\,tx-|x|}\,dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\,tx+x}\,dx + \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\,tx-x}\,dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} \mathrm{e}^{(\mathrm{i}\,t+1)x}\,dx + \int_{0}^{\infty} \mathrm{e}^{-(1-\mathrm{i}\,t)x}\,dx\right] \\
&= \frac{1}{2}\left[\left(\frac{1}{\mathrm{i}\,t+1}\,\mathrm{e}^{(\mathrm{i}\,t+1)x}\Big|_{-\infty}^{0}\right) + \left(-\frac{1}{1-\mathrm{i}\,t}\,\mathrm{e}^{-(1-\mathrm{i}\,t)x}\Big|_{0}^{\infty}\right)\right].
\end{aligned}
$$

Now, as we evaluate these expressions at the their respective bounds of integration, notice the terms evaluated at $-\infty$ and at ∞ in the left and right integrals both go to 0. Then we have

$$
\begin{aligned}
\phi_X(t) &= \frac{1}{2}\left[\left(\frac{1}{\mathrm{i}\,t+1}\,\mathrm{e}^{(\mathrm{i}\,t+1)0} - \cancelto{0}{\frac{1}{\mathrm{i}\,t+1}\,\mathrm{e}^{(\mathrm{i}\,t+1)(-\infty)}}\right) + \left(\cancelto{0}{-\frac{1}{1-\mathrm{i}\,t}\,\mathrm{e}^{-(1-\mathrm{i}\,t)\infty}} + \frac{1}{1-\mathrm{i}\,t}\,\mathrm{e}^{-(1-\mathrm{i}\,t)0}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{\mathrm{i}\,t+1} + \frac{1}{1-\mathrm{i}\,t}\right] \\
&= \frac{1}{2}\left(\frac{1-\mathrm{i}\,t+\mathrm{i}\,t+1}{(\mathrm{i}\,t+1)(1-\mathrm{i}\,t)}\right) \\
&= \frac{1}{2}\left(\frac{2}{1-\mathrm{i}^2\,t^2}\right) \\
&= \frac{1}{1+t^2}.
\end{aligned}
$$

Hence, the characteristic function for a random variable with it's density given by $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$ is

$$\phi_X(t) = \frac{1}{1+t^2}.$$

4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} 1_{x \geq 0}.$$

Solution:
Recall a geometric distributed variable represents the number of failures before one success. Therefore we can represent N as the sum of k i.i.d geometrically distributed random variables $N = \sum_{i=0}^{k} X_i$ where each $X_i \sim Geo(p)$. Note, from the lectures we have

$$\phi_X(t) = \frac{p\,\mathrm{e}^{\mathrm{i}\,t}}{1-(1-p)\,\mathrm{e}^{\mathrm{i}\,t}}.$$

Therefore, the characteristic function for $2Np$ is given by

$$\begin{aligned}
\phi_{2Np}(t) &= \phi_N(2pt) \\
&= \phi_{\sum_{i=0}^{k} X_i}(2pt) \\
&= \prod_{i=0}^{k} \phi_{X_i}(2pt) \\
&= (\phi_X(2pt))^k \\
&= \left(\frac{p\,\mathrm{e}^{\mathrm{i}\,2pt}}{1-(1-p)\,\mathrm{e}^{\mathrm{i}\,2pt}} \right)^k.
\end{aligned}$$

Now let $p = \frac{1}{n}$. Then we can conclude following the example 6 from lecture 16 that this converges to

$$\left(\frac{\frac{1}{2}}{\frac{1}{2} - \mathrm{i}\,t} \right)^k$$

Which is equal to a product of exponential distributed characteristic functions. Thus defining the characteristic function of random variable which is the sum of exponential random variables. From lecture 9 the sum of exponential random variables is gamma distributed.

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