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ON THE CLASSIFICATION OF QUANDLES OF LOW ORDER

L. VENDRAMIN

Departamento de Matemática, FCEN,
Universidad de Buenos Aires,
Pab. I – Ciudad Universitaria,
(1428), Buenos Aires, Argentina
lvendramin@dm.uba.ar

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ABSTRACT

Using the classification of transitive groups we classify indecomposable quandles of size < 36. This classification is available in Rig, a GAP package for computations related to racks and quandles. As an application, the list of all indecomposable quandles of size < 36 not of type D is computed.

Keywords: Quandles; transitive groups; computer software.

Mathematics Subject Classification 2010: 57M27

1. Introduction

Racks appeared for the first time in [11] and quandles appeared in [18, 21]. Racks and quandles are used in modern knot theory because they provide good knot invariants [18]. They are also useful for the classification problem of pointed Hopf algebras because they provide a powerful tool to understand Yetter–Drinfeld modules over groups, see [5]. Of course, the classification of finite racks (or quandles) is a very difficult problem. Several papers about classifications of different subcategories of racks have appeared, see for example [5, 8, 10, 15, 16, 18, 19].

In this paper, we use the classification of transitive groups and the program described in [9] to classify indecomposable quandles. With this method, we complete the classification of all non-isomorphic indecomposable quandles of size < 36. This classification is available in Rig, a GAP [1] package designed for computations related to racks and quandles. Rig is a free software and it is available at `http://code.google.com/p/rig/`.

2. Definitions and Examples

We recall basic notions and facts about racks. For additional information we refer for example to [5]. A *rack* is a pair $(X, \triangleright)$, where X is a nonempty set and $\triangleright : X \times X \to X$ is a map (considered as a binary operation on X) such that

(1) the map $\varphi_i : X \to X$, where $x \mapsto i \triangleright x$, is bijective for all $i \in X$, and
(2) $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$.

A rack $(X, \triangleright)$, or shortly X, is a *quandle* if $i \triangleright i = i$ for all $i \in X$. A *subrack* of a rack X is a nonempty subset $Y \subseteq X$ such that $(Y, \triangleright)$ is also a rack.

Example 2.1. A group G is a quandle with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If a subset $X \subseteq G$ is stable under conjugation by G, then it is a subquandle of G.

To construct racks associated to (union of) conjugacy classes of groups use the Rig function `Rack`. For example, to construct the quandle of three elements associated to the conjugacy class of transpositions in $\mathbb{S}_3$:

```
gap> r := Rack(SymmetricGroup(3), (1,2));;
gap> Size(r);
3
```

Example 2.2. Let G be a group and $s \in \mathrm{Aut}(G)$. Define $x \triangleright y = xs(x^{-1}y)$ for $x, y \in G$. Then $(G, \triangleright)$ is a quandle. Further, let $H \subseteq G$ be a subgroup such that $s(h) = h$ for all $h \in H$. Then G/H is a quandle with $xH \triangleright yH = xs(x^{-1}y)H$. It is called the *homogeneous quandle* (G, H, s).

Example 2.3. Let $n \geq 2$. The *dihedral quandle* of order n is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with $i \triangleright j = 2i - j \pmod{n}$.

The package provides several functions to construct racks and quandles. See the documentation for more information.

Let X be a finite rack. Assume that $X = \{x_1, x_2, \ldots, x_n\}$. With the identification $x_i \equiv i$ the rack X can be presented as a square matrix $M \in \mathbb{N}^{n \times n}$ such that $M_{ij} = (i \triangleright j)$. This matrix is called *the table* of the rack. See [16].

Example 2.4. The matrix (or table) of the rack $\mathbb{D}_4$ is

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

The files of the matrix are the permutations of the quandle: $\varphi_1 = \varphi_3 = (2\,4)$ and $\varphi_2 = \varphi_4 = (1\,3)$.

```
gap> D4 := DihedralQuandle(4);;
gap> Permutations(D4);
[ (2,4), (1,3), (2,4), (1,3) ]
gap> Table(D4);
[ [  1,  4,  3,  2 ],
  [  3,  2,  1,  4 ],
  [  1,  4,  3,  2 ],
  [  3,  2,  1,  4 ] ]
```

Let $(X, \triangleright)$ and $(Y, \triangleright)$ be racks. A map $f : X \to Y$ is a *morphism* of racks if $f(i \triangleright j) = f(i) \triangleright f(j)$ for all $i, j \in X$.

Notation 2.5. We write g^G for the conjugacy class of g in G.

Example 2.6. Let $\mathcal{T}_1 = (1\,2\,3)^{\mathbb{A}_4}$ and $\mathcal{T}_2 = (1\,3\,2)^{\mathbb{A}_4}$. Then the quandles $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic.

```
gap> T1 := Rack(AlternatingGroup(4), (1,2,3));;
gap> T2 := Rack(AlternatingGroup(4), (1,3,2));;
gap> IsomorphismRacks(T1, T2);
(3,4)
```

Hence $\mathcal{T}_1 \simeq \mathcal{T}_2$ and the isomorphism is given by the permutation $\sigma = (3\,4)$. More precisely, assume that $\mathcal{T}_1 = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{T}_2 = \{y_1, y_2, y_3, y_4\}$. Then the map $f : \mathcal{T}_1 \to \mathcal{T}_2$, $f(x_i) = y_{\sigma(i)}$, is an isomorphism of racks.

Example 2.7. Let A be an abelian group, and let $T \in \mathrm{Aut}(A)$. We have a quandle structure on A given by

$$a \triangleright b = (1 - T)a + Tb$$

for $a, b \in A$. The quandle $(A, \triangleright)$ is called *affine* (*or Alexander*) *quandle* and it will be denoted by $\mathrm{Aff}(A, T)$. In particular, let p be a prime number, q be a power of p and $\alpha \in \mathbb{F}_q^\times = \mathbb{F}_q \setminus \{0\}$. We write $\mathrm{Aff}(\mathbb{F}_q, \alpha)$, or simply $\mathrm{Aff}(q, \alpha)$, for the affine quandle $\mathrm{Aff}(A, g)$, where $A = \mathbb{F}_q$ and g is the automorphism given by $x \mapsto \alpha x$ for all $x \in \mathbb{F}_q$.

Example 2.8. The *tetrahedron quandle* is the quandle $\mathcal{T} = (1\,2\,3)^{\mathbb{A}_4}$. It is easy to see that this quandle is isomorphic to an affine quandle over $\mathbb{F}_4$.

The *inner group* of a rack X is the group generated by the permutations φ_i of X, where $i \in X$. We write $\mathrm{Inn}(X)$ for the inner group of X. A rack is said to be *faithful* if the map

$$\varphi : X \to \mathrm{Inn}(X), \quad i \mapsto \varphi_i,$$

is injective. We say that a rack X is *indecomposable* (or connected) if the inner group $\mathrm{Inn}(X)$ acts transitively on X. Also, X is *decomposable* if it is not indecomposable. Any finite rack X is the disjoint union of indecomposable subracks [5, Proposition 1.17] called the *components of* X.

Example 2.9. The dihedral quandle $\mathbb{D}_4$ is decomposable: $\mathbb{D}_4 = \{1,3\} \sqcup \{2,4\}$.

```
gap> D4 := DihedralQuandle(4);;
gap> IsIndecomposable(D4);
false
gap> Components(D4);
[ [ 1, 3 ], [ 2, 4 ] ]
```

For any rack X, the *enveloping group* of X is

$$G_X = F(X)/\langle iji^{-1} = i \triangleright j,\ i,j \in X\rangle,$$

where $F(X)$ denotes the free group generated by X. This group is also called the *associated group* of X, see [11]. Let

$$\overline{G_X} = G_X/\langle x^{\mathrm{ord}(\varphi_x)} \mid x \in X\rangle.$$

If X is finite then the group $\overline{G_X}$ is finite and it is called the *finite enveloping group of* X, see [14].

Example 2.10. Let $X = \mathcal{T}$ be the tetrahedron rack. Then $\mathrm{Inn}(X) \simeq \mathbb{A}_4$ and $\overline{G_X} \simeq \mathbf{SL}(2,3)$.

```
gap> T := Rack(AlternatingGroup(4), (1,2,3));;
gap> inn := InnerGroup(T);;
gap> StructureDescription(inn);
A4
gap> env := FiniteEnvelopingGroup(T);;
gap> StructureDescription(env);
SL(2,3)
```

Table 1 contains the inner group and the finite enveloping groups associated to some particular racks. These racks appear in the classification of finite-dimensional Nichols algebras, see for example [2, Table 6].

Table 1. Some finite enveloping groups.

Quandle	$\mathrm{Inn}(Q)$	$\overline{G_X}$
$\mathbb{D}_3$	$\mathbb{S}_3$	$\mathbb{S}_3$
$\mathcal{T}$	$\mathbb{A}_4$	$\mathbf{SL}(2,3)$
$\mathrm{Aff}(5,2)$, $\mathrm{Aff}(5,3)$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
$(1\,2)^{\mathbb{S}_4}$	$\mathbb{S}_4$	$\mathbb{S}_4$
$\mathrm{Aff}(7,3)$, $\mathrm{Aff}(7,5)$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
$(1\,2\,3\,4)^{\mathbb{S}_4}$	$\mathbb{S}_4$	$\mathbf{SL}(2,3) \rtimes \mathbb{Z}_4$
$(1\,2)^{\mathbb{S}_5}$	$\mathbb{S}_5$	$\mathbb{S}_5$

3. The Classification of Indecomposable Quandles of Low Order

The main tool for the classification of indecomposable quandles is the following theorem of [9]. Our proof is heavily based on [18, Theorem 7.1]. For completeness we give a proof in the context of this paper.

Theorem 3.1. *Let X be an indecomposable quandle of n elements. Let $x_0 \in X$, $z = \varphi_{x_0}$, $G = \operatorname{Inn}(X)$ and $H = \operatorname{Stab}_G(x_0) = \{g \in G \mid g \cdot x_0 = x_0\}$. Then*

(1) *G is a transitive group of degree n,*
(2) *z is a central element of H,*
(3) *X is isomorphic to the homogeneous quandle (G, H, I_z), where $I_z : G \to G$ is the conjugation $x \mapsto zxz^{-1}$.*

Proof. The claim (1) follows by definition. The claim (2) follows from [9, Theorem 4.3]. We now prove (3). We consider the quandle structure over G given by $x \triangleright y = xI_z(x^{-1}y)$ for all $x, y \in G$, and let $e : G \to X$, $x \mapsto x \cdot x_0$, be the evaluation map. Since G acts transitively on X, the map e is surjective. We claim that e is a rack morphism. Indeed,

$$\begin{aligned} e(x \triangleright y) = e(xs(x^{-1}y)) = e(xzx^{-1}yz^{-1}) &= zxx^{-1}yz^{-1} \cdot x_0 \\ &= xzx^{-1}y \cdot x_0 = x \cdot (x_0 \triangleright (x^{-1}y \cdot x_0)) = e(x) \triangleright e(y) \end{aligned}$$

for all $x, y \in G$. Further, $e(x) = e(y)$ if and only if $xH = yH$. Then e induces the isomorphism $G/H \to X$, $xH \mapsto e(x)$. Hence the claim follows. □

Algorithm 1: Indecomposable quandles of size n

Result: The list L of all non-isomorphic indecomposable quandles $L \longleftarrow \emptyset$;
for *all transitive groups G of degree n* **do**
  Compute $H = \operatorname{Stab}_G(x_0)$;
  Compute $Z(H)$, the center of H;
  for $z \in Z(H) \setminus \{1\}$ **do**
    Compute the homogeneous quandle $Q = (G, H, I_z)$;
    if *Q is indecomposable and $Q \not\simeq X$ for all $X \in L$* **then**
      Add the quandle Q to L;
    end
  end
end

Recall that all indecomposable quandles of prime order p are affine, see [10]. Let $n \in \mathbb{N}, n < 36$, and n not being a prime number. Using Theorem 3.1 and Algorithm 1, the list of all non-isomorphic indecomposable quandles can be

Table 2. The number of non-isomorphic indecomposable quandles.

n	1	2	3	4	5	6	7	8	9	10	11	12
$q(n)$	1	0	1	1	3	2	5	3	8	1	9	10
n	13	14	15	16	17	18	19	20	21	22	23	24
$q(n)$	11	0	7	9	15	12	17	10	9	0	21	42
n	25	26	27	28	29	30	31	32	33	34	35	
$q(n)$	34	0	65	13	27	24	29	17	11	0	15	

constructed. The only requirement is the classification of transitive groups. The complete list of transitive groups up to degree < 32 is included in GAP. Hulpke classified several of these transitive groups, see [17]. Further, Hulpke classified transitive groups of degree 33, 34 and 35. Transitive groups of degree 32 were classified in [6].

For $n \in \mathbb{N}$ let $q(n)$ be the number of non-isomorphic indecomposable quandles or size n. In Example 3.2, $q(20)$ is computed. Further, Table 2 shows the value of $q(n)$ for $n \in \{1, 2, \ldots, 35\}$.

Example 3.2. There are 10 isomorphism classes of indecomposable quandles of order 20.

```
gap> NrSmallQuandles(20);
10
```

Rig contains a huge database with the set of representatives of isomorphism classes of indecomposable quandles of size < 36. Let $n \in \{1, 2, \ldots, 35\}$ such that $q(n) \neq 0$, and let

$$Q_{n,1}, Q_{n,2}, \ldots, Q_{n,q(n)}$$

be the set of representatives of isomorphism classes of indecomposable quandles of size n. In the package, a representative $Q_{n,i}, 1 \leq i \leq q(n)$, can be obtained with the function `SmallQuandle`.

Example 3.3. There exists only one (up to isomorphism) indecomposable quandle of order 10. Further, this quandle is isomorphic to the conjugacy class of transpositions in $\mathbb{S}_5$.

```
gap> NrSmallQuandles(10);
1
gap> Q := SmallQuandle(10, 1);;
gap> R := Rack(SymmetricGroup(5), (1,2));;
gap> IsomorphismRacks(Q, R);
(3,5,6,10,8,4,9,7)
```

Recall that a *crossed set* is a quandle $(X, \triangleright)$ which further satisfies $j \triangleright i = i$ whenever $i \triangleright j = j$ for all $i, j \in X$.

Example 3.4. It is easy to see that the only indecomposable quandles of size < 36 which are not crossed sets are $Q_{30,4}$ and $Q_{30,5}$.

Conjeture 3.5. *Let p be an odd prime number and let Q be an indecomposable quandle of $2p$ elements. Then $p \in \{3, 5\}$.*

4. Rack Homology

Let X be a rack. For $n \geq 0$ let $C_n(X, \mathbb{Z}) = \mathbb{Z}X^n$. Consider $C_*(X, \mathbb{Z})$ as a complex with boundary $\partial_0 = \partial_1 = 0$ and $\partial_{n+1} : C_{n+1}(X, \mathbb{Z}) \to C_n(X, \mathbb{Z})$ defined by

$$\begin{aligned}\partial_{n+1}(x_1, x_2, \dots, x_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} [&(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \\ &- (x_1, \dots, x_{i-1}, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_{n+1})]\end{aligned}$$

for $n \geq 1$. It is straightforward to prove that $\partial^2 = 0$. The *homology* $H_*(X, \mathbb{Z})$ of X is the homology of the complex $C_*(X, \mathbb{Z})$. See for example [7, 12, 13] for applications to the theory of knots and [5] for applications to the theory of Hopf algebras.

Example 4.1. Let $X = \mathbb{D}_5$. Then $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$.

```
gap> RackHomology(DihedralQuandle(5), 2);
[ 1, [  ] ]
```

Example 4.2. Let $X = (1\,2)(3\,4\,5)^{\mathbb{S}_5}$. Then $H_2(X, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_6$.

```
gap> r := Rack(SymmetricGroup(5), (1,2)(3,4,5));;
gap> RackHomology(r, 2);
[ 1, [ 6 ] ]
```

Example 4.3. Recall that $\mathcal{T}$ is the tetrahedron quandle defined in Example 2.8. Then $H_2(\mathcal{T}, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2$ and $H_3(\mathcal{T}, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Further, the torsion subgroup of $H_2(\mathcal{T}, \mathbb{Z})$ is generated by

$$\chi = \chi_{(1,2)} + \chi_{(1,3)} + \chi_{(2,1)} + \chi_{(2,3)} + \chi_{(3,1)} + \chi_{(3,2)},$$

where

$$\chi_{(i,j)}(a, b) = \begin{cases} 1 & \text{if } (i, j) = (a, b), \\ 0 & \text{otherwise.} \end{cases}$$

Indeed,

```
gap> T := Rack(AlternatingGroup(3), (1,2,3));;
gap> RackHomology(T, 2);
[ 1, [ 2 ] ]
```

```
gap> RackHomology(T, 3);
[ 1, [ 2, 2, 4 ] ]
gap> TorsionGenerators(T, 2);
[ [ 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0 ] ]
```

Table 3 contains the second rack homology group of all the indecomposable quandles of size ≤ 21. Quandles with a prime number of elements were not included in Table 3 because of the following lemma of [15].

Lemma 4.4. *Let p be a prime number. Let X be an indecomposable quandle of p elements. Then $H_2(X,\mathbb{Z}) \simeq \mathbb{Z}$.*

Proof. It follows from [15, Lemma 5.1; 20, Theorem 2.2]. □

Table 3. Some homology groups.

Indecomposable quandle Q	$H_2(Q,\mathbb{Z})$
$Q_{4,1}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{6,1}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{6,2}$	$\mathbb{Z}\times\mathbb{Z}_4$
$Q_{8,1}$, $Q_{8,2}$, $Q_{8,3}$	$\mathbb{Z}$
$Q_{9,1}$, $Q_{9,4}$, $Q_{9,5}$, $Q_{9,7}$, $Q_{9,8}$	$\mathbb{Z}$
$Q_{9,2}$, $Q_{9,3}$, $Q_{9,6}$	$\mathbb{Z}\times\mathbb{Z}_3$
$Q_{10,1}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{12,1}$, $Q_{12,2}$, $Q_{12,4}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{12,3}$	$\mathbb{Z}\times\mathbb{Z}_{10}$
$Q_{12,5}$, $Q_{12,6}$	$\mathbb{Z}\times\mathbb{Z}_4$
$Q_{12,7}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$
$Q_{12,8}$	$\mathbb{Z}\times\mathbb{Z}_2^3$
$Q_{12,9}$	$\mathbb{Z}\times\mathbb{Z}_4^2$
$Q_{12,10}$	$\mathbb{Z}\times\mathbb{Z}_6$
$Q_{15,1}$, $Q_{15,3}$, $Q_{15,4}$	$\mathbb{Z}$
$Q_{15,2}$	$\mathbb{Z}\times\mathbb{Z}_2^2$
$Q_{15,5}$, $Q_{15,6}$	$\mathbb{Z}\times\mathbb{Z}_5$
$Q_{15,7}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{16,1}$,$Q_{16,7}$	$\mathbb{Z}\times\mathbb{Z}_4$
$Q_{16,2}$	$\mathbb{Z}\times\mathbb{Z}_2^4$
$Q_{16,3}$, $Q_{16,4}$	$\mathbb{Z}\times\mathbb{Z}_2^2$
$Q_{16,5}$, $Q_{16,6}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{16,8}$, $Q_{16,9}$	$\mathbb{Z}$
$Q_{18,1}$, $Q_{18,8}$, $Q_{18,11}$, $Q_{18,12}$	$\mathbb{Z}\times\mathbb{Z}_6$
$Q_{18,2}$, $Q_{18,9}$, $Q_{18,10}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{18,3}$, $Q_{18,6}$, $Q_{18,7}$	$\mathbb{Z}\times\mathbb{Z}_4$
$Q_{18,4}$, $Q_{18,5}$	$\mathbb{Z}\times\mathbb{Z}_{12}$
$Q_{20,1}$, $Q_{20,2}$, $Q_{20,3}$	$\mathbb{Z}\times\mathbb{Z}_6$
$Q_{20,4}$, $Q_{20,7}$, $Q_{20,8}$	$\mathbb{Z}\times\mathbb{Z}_2$
$Q_{20,5}$, $Q_{20,9}$	$\mathbb{Z}\times\mathbb{Z}_2^2$
$Q_{20,6}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$
$Q_{20,10}$	$\mathbb{Z}\times\mathbb{Z}_4$
$Q_{21,1}$, $Q_{21,2}$, $Q_{21,3}$, $Q_{21,4}$, $Q_{21,5}$	$\mathbb{Z}$
$Q_{21,6}$	$\mathbb{Z}\times\mathbb{Z}_2^2$
$Q_{21,7}$, $Q_{21,8}$	$\mathbb{Z}\times\mathbb{Z}_7$
$Q_{21,9}$	$\mathbb{Z}\times\mathbb{Z}_2$

5. Racks of Type D

Recall from [3] that a finite rack X is of type D if there exists an indecomposable subrack $Y = R \sqcup S$ (here R and S are the components of Y) such that

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s$$

for some $r \in R$ and $s \in S$.

Quandles of type D are very important for the classification of finite-dimensional pointed Hopf algebras, see for example the program described in [2, Sec. 2.6]. For some interesting applications we refer to [3, 4].

Proposition 5.1. *Let Q be an indecomposable quandle of size < 36. Then Q is of type D if and only if Q is isomorphic to one of the following quandles*:

(1) $Q_{12,1}$,
(2) $Q_{18,i}$ *for* $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
(3) $Q_{20,3}$,
(4) $Q_{24,i}$ *for* $i \in \{1, 2, 3, 4, 5, 6, 8, 10, 11, 16, 17, 21, 22, 23, 26, 27, 28, 32\}$,
(5) $Q_{27,i}$ *for* $i \in \{1, 14\}$,
(6) $Q_{30,i}$ *for* $i \in \{1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15, 16\}$,
(7) $Q_{32,i}$ *for* $i \in \{1, 2, 3, 5, 6, 7, 8, 9\}$.

Proof. By [10], indecomposable quandles of size p are affine. Further, [2, Proposition 4.2] implies that affine quandles with p elements are not of type D. Therefore, we may assume that the size of Q is not a prime number. Now the claim follows from a straightforward computer calculation. □

Corollary 5.2. *Let Q be an indecomposable simple quandle of size < 36. Assume that Q is of type D. Then $Q \simeq Q_{30,3}$.*

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