2- and 3-cocycles of RIG Quandles

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Abstract

We compute 2- and 3-cocycles of RIG Quandles......

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1 Introduction

Quandles and racks are algebraic structures whose axiomatization comes from Reidemeister moves in knot theory. They appeared in the literature under many different names (automorphic sets, crystals, kei, rack, etc). Around 1982, Joyce [5] (used the term quandle) and Matveev [6] (who call them distributive groupoids) introduced independently the notion of a quandle. They used it to construct representations of the braid groups. Joyce and Matveev associated to each knot a quandle that determines the knot up to isotopy and mirror image. Since then quandles and racks have been investigated by topologists in order to construct knot and link invariants and their higher analogues (see for example [2] and references therein).

In this paper, we 2- and 3-cocycles of RIG Quandles......
In Section 2, we review the basics of quandles, give examples and describes
in section 3......

Notations Through the paper, ...

2 Basics of quandles

We start by reviewing the basics of quandles.

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(1) For any $a \in X$, $a * a = a$.

(2) For any $a, b \in X$, there is a unique $x \in X$ such that $a = x * b$.

(3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (2) and (3).

Racks and quandles have been studied in, for example, [3, 5, 6].

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The axioms for a quandle correspond respectively to the Reidemeister moves of type I, II, and III (see [3], for example).

Here are some typical examples of quandles.

- A group $X = G$ with n-fold conjugation as the quandle operation: $a * b = b^{-n}ab^n$.
- Let n be a positive integer. For elements $i, j \in \mathbb{Z}_n$ (integers modulo n), define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. This set can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.
- Any $\Lambda(= \mathbb{Z}[T, T^{-1}])$-module M is a quandle with $a * b = Ta + (1 - T)b$, $a, b \in M$, called an *Alexander quandle*. Furthermore for a positive integer n, a *mod-n Alexander quandle* $\mathbb{Z}_n[T, T^{-1}]/(h(T))$ is a quandle for a Laurent polynomial $h(T)$. The mod-n Alexander quandle is finite if the coefficients of the highest and lowest degree terms of h are units in $\mathbb{Z}_n$.

3. Cohomology of Quandles

For X quandle and A an abelian group.

2-cocycle: A function $\Phi : X \times X \to A$ such that $\forall x \in X, \Phi(x, x) = 0$ and

$$\Phi(x, y) + \Phi(x * y, z) = \Phi(x, z) + \Phi(x * z, y * z).$$

Coboundary: A 2-cocycle $\Phi : X \times X \to A$ is a coboundary if there exists a function $g : X \to A$, such that $\forall x, y \in X$ we have $\Phi(x, y) = g(x) - g(x * y)$.

3 The Problem

Fix a RIG quandle X of cardinality n from the list of 431 connected quandles. Given an abelian group A, say $\mathbb{Z}_3$ for example, we want to compute explicit 2-cocycles $f : X \times X \to A$. Let $f(x, y) = \lambda_{(x,y)} \in A$. Since the quandle is finite we can write

$f = \sum_{x,y \in X} \lambda_{(x,y)} \chi_{(x,y)}$.

The functions $\chi_{(x,y)} : X \times X \to A$ are the characteristic functions taking value 1 at (x, y) and zero otherwise.

Now by pluging this formula of the function f into the 2-cocycle condition

We obtain the equation:

$$\lambda_{(x,y)} + \lambda_{(x*y,z)} = \lambda_{(x,z)} + \lambda_{(x*z,y*z)}.$$

Now finding f is exactly the same as solving this system of equations in which we plug ALL possible n^3 values of (x, y, z).

We know that $\lambda_{(x,x)} = 0$ since $f(x, x) = 0$.

BRIAN: Here is an example so it will help you to start.

Example 3.1 Consider the dihedral quandle R_3. We will show in this example that every 2-cocycle $\Phi : X \times X \to \mathbb{Z}$ with coefficients in $\mathbb{Z}$ is a coboundary. First we write $f = \sum_{x,y \in R_3} \lambda_{(x,y)}\ \chi_{(x,y)}$. By substituting this expression of f in the 2-cocycle equation, we obtain $\lambda_{(x,x)} = 0$ for all $x \in R_3$, and

$$\lambda_{(x,y)} + \lambda_{(x*y,z)} = \lambda_{(x,z)} + \lambda_{(x*z,y*z)}.$$

Now we write $R_3 = \{0, 1, 2\}$ with $x * y = 2y - x \pmod 3$, and substitute the values $0, 1, 2$ for all possibilities of the variables x, y, z. We obtain after simplification the following equations

$$\begin{aligned}
\lambda_{(0,0)} = \lambda_{(1,1)} = \lambda_{(2,2)} &= 0 \\
\lambda_{(0,1)} + \lambda_{(2,1)} &= 0 \\
\lambda_{(1,0)} + \lambda_{(2,0)} &= 0 \\
\lambda_{(0,2)} + \lambda_{(1,2)} &= 0 \\
\lambda_{(0,2)} + \lambda_{(2,1)} - \lambda_{(2,0)} &= 0.
\end{aligned}$$

Again by substitution we can write the function f in the following form

$$\begin{aligned}
f &= \lambda_{(0,1)}[\chi_{(0,1)} - \chi_{(2,1)} + \chi_{(0,2)} - \chi_{(1,2)}] + \lambda_{(1,0)}[\chi_{(1,0)} - \chi_{(2,0)} + \chi_{(0,2)} - \chi_{(1,2)}], \\
f &= \lambda_{(0,1)}\delta(\chi_0) + \lambda_{(1,0)}\delta(\chi_1)
\end{aligned}$$

making it a coboundary. This proves that every 2-cocycle is a coboundary. Here $\chi_0 : R_3 \to \mathbb{Z}$ such that $\chi_0(0) = 1$ and zero otherwise, and $\chi_1 : R_3 \to \mathbb{Z}$ such that $\chi_1(1) = 1$ and zero otherwise. Recall that $\delta(\chi_0)(x, y) = \chi_0(x) - \chi_0(x * y)$ and the same for χ_1.

References

[1] Carter S., Elhamdadi M., Nikifourou M., and Saito M., 2003. *Extensions of quandles and cocycle knot invariants,* J. Knot Theory Ramifications 12, no. 6, 725-738.

[2] Carter S., Kamada S., and Saito M., 2004 *Surfaces in 4-space,* Encyclopaedia of Mathematical Sciences, 142. Low-Dimensional Topology, III. Springer-Verlag, Berlin.

[3] Fenn R., and Rourke C., 1992. *Racks and links in codimension two,* J. Knot Theory Ramifications, 1, 343-406.

[4] Ho, B.; Nelson S. *Matrices and finite quandles,* Homology Homotopy Appl. 7 (2005), no. 1, 197–208.

[5] Joyce, D., *A classifying invariant of knots, the knot quandle,* J. Pure Appl. Alg., 23, 37–65.

[6] Matveev, S., *Distributive groupoids in knot theory,* (Russian) Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78–88, 160.