Journal of Algebra and Its Applications
Vol. 11, No. 1 (2012) 1250008 (9 pages)

DOI: 10.1142/S0219498812500089

AUTOMORPHISM GROUPS OF QUANDLES

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Received 24 December 2010
Accepted 6 March 2011

Communicated by X.-D. Hou

We prove that the automorphism group of the dihedral quandle with n elements is isomorphic to the affine group of the integers mod n, and also obtain the inner automorphism group of this quandle. In [B. Ho and S. Nelson, Matrices and finite quandles, *Homology Homotopy Appl.* **7**(1) (2005) 197–208.], automorphism groups of quandles (up to isomorphisms) of order less than or equal to 5 were given. With the help of the software Maple, we compute the inner and automorphism groups of all seventy three quandles of order six listed in the appendix of [S. Carter, S. Kamada and M. Saito, *Surfaces in 4-Space*, Encyclopaedia of Mathematical Sciences, Vol. 142, Low-Dimensional Topology, III (Springer-Verlag, Berlin, 2004)]. Since computations of automorphisms of quandles relate to the problem of classification of quandles, we also describe an algorithm implemented in C for computing all quandles (up to isomorphism) of order less than or equal to nine.

Keywords: Quandles; isomorphisms; automorphism groups; inner automorphism groups.

Mathematics Subject Classification: 20B25, 20B20

1. Introduction

Quandles and racks are algebraic structures whose axiomatization comes from Reidemeister moves in knot theory. The earliest known work on racks is contained in the 1959 correspondence between John Conway and Gavin Wraith who studied racks in the context of the conjugation operation in a group. Around 1982, the notion of a quandle was introduced independently by Joyce [13] and Matveev [14]. They used it to construct representations of the braid groups. Joyce and Matveev associated to each knot a quandle that determines the knot up to isotopy and mirror image. Since then quandles and racks have been investigated by topologists in order to construct knot and link invariants and their higher analogues (see, for example, [5] and references therein).

In this paper, we prove that the automorphism group of the dihedral quandle with n elements is isomorphic to the affine group of the integers mod n. In [11], Ho and Nelson gave the list of quandles (up to isomorphism) of orders $n = 3, n = 4$ and $n = 5$ and determined their automorphism groups. In this paper, with the help of the software Maple, we extend their results by computing the inner and automorphism groups of all quandles of order six listed in the appendix of [5]. Since computations of automorphisms of quandles relates to the problem of classification of quandles, we also describe an algorithm implemented in C for computing all quandles (up to isomorphism) of order up to nine.

In Sec. 2, we review the basics of quandles, give examples and describe the automorphisms and inner automorphisms of dihedral quandles. The inner and automorphism groups of all quandles of order 6 are computed in Sec. 3. A description of an algorithm which generates all quandles of order up to 9 (up to isomorphisms) is contained in Sec. 4.

Notations. Throughout the paper, the symbol $\mathbb{Z}_n$ will denote the set of integers modulo n and $\mathbb{Z}_n^{\times}$ will stand for the group of its units. The dihedral group of order $2m$ will be denoted by D_m. The symbol Σ_n will stand for the symmetric group on the set $\{1, 2, \ldots, n\}$ and A_n will be its alternating subgroup (even permutations).

2. Automorphism Groups of Quandles

We start this section by reviewing the basics of quandles and give examples.

A *quandle* X is a set with a binary operation $(a, b) \mapsto a * b$ such that

(1) For any $a \in X$, $a * a = a$.
(2) For any $a, b \in X$, there is a unique $x \in X$ such that $a = x * b$.
(3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

Axiom (2) states that for each $u \in X$, the map $S_u : X \to X$ with $S_u(x) := x * u$ is a bijection. Its inverse will be denoted by the mapping $\overline{S}_u : X \to X$ with $\overline{S}_u(x) = x \overline{*} u$, so that $(x * u) \overline{*} u = x = (x \overline{*} u) * u$.

A *rack* is a set with a binary operation that satisfies (2) and (3). Racks and quandles have been studied in, for example, [9, 13, 14]. The axioms for a quandle correspond respectively to the Reidemeister moves of types I, II and III (see [9], for example).

Here are some typical examples of quandles.

- Any set X with the operation $x * y = x$ for any $x, y \in X$ is a quandle called the *trivial* quandle. The trivial quandle of n elements is denoted by T_n.
- A group $X = G$ with n-fold conjugation as the quandle operation: $a * b = b^n a b^{-n}$.

- Let n be a positive integer. For elements $i, j \in \mathbb{Z}_n$ (integers modulo n), define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. This set can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.
- Any $\Lambda(= \mathbb{Z}[T, T^{-1}])$-module M is a quandle with $a * b = Ta + (1 - T)b$, $a, b \in M$, called an *Alexander quandle*. Furthermore for a positive integer n, a *mod-n Alexander quandle* $\mathbb{Z}_n[T, T^{-1}]/(h(T))$ is a quandle for a Laurent polynomial $h(T)$. The mod-n Alexander quandle is finite if the coefficients of the highest and lowest degree terms of h are units in $\mathbb{Z}_n$.

A function $f : (X, *) \to (Y, \triangleright)$ between quandles X and Y is a *homomorphism* if $f(a * b) = f(a) \triangleright f(b)$ for any $a, b \in X$. We will denote the group of automorphisms of the quandle X by $\mathrm{Aut}(X)$. Axioms (2) and (3) respectively state that for each $u \in X$, the map $S_u : X \to X$ is respectively a bijection and a quandle homomorphism. Let us call the subgroup of $\mathrm{Aut}(X)$, generated by the *symmetries* S_x, the *inner* automorphism group of X denoted by $\mathrm{Inn}(X)$. By axiom (3), the map $S : X \to \mathrm{Inn}(X)$ sending u to S_u satisfies the equation $S_z\ S_y = S_{y*z}\ S_z$, $\forall\, y, z \in X$, which can be written as $S_z\ S_y\ {S_z}^{-1} = S_{y*z}$. Thus, if the group $\mathrm{Inn}(X)$ is considered as a quandle with conjugation then the map S becomes a quandle homomorphism. As noted in [1, p. 184], the map S is not injective in general. The quandle $(X, *)$ is called *faithful* when the map S is injective. If $(X, *)$ is *faithful* then the center of $\mathrm{Inn}(X)$ is trivial.

2.1. *Automorphism groups and inner automorphism groups of dihedral quandles*

Now we characterize the automorphisms of the dihedral quandles. Recall that the affine group of $\mathbb{Z}_n$ is the group of all invertible affine transformations of $\mathbb{Z}_n$,

$$\mathrm{Aff}(\mathbb{Z}_n) := \{f_{a,b} : \mathbb{Z}_n \to \mathbb{Z}_n, f_{a,b}(x) = ax + b, a \in \mathbb{Z}_n^\times, b \in \mathbb{Z}\},$$

The element $f_{a,b}$ is identified with the pair (a, b) and the group multiplication is given by $(a, b)(c, d) = (ac, ad + b)$. The identity is $(1, 0)$ and the inverse is given by $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$. Usually the element (a, b) is represented in a matrix notation as $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ so group multiplication corresponds to multiplication of matrices.

Theorem 2.1. *Let $R_n = \mathbb{Z}_n$ be the dihedral quandle with the operation $i * j = 2j - i \pmod{n}$. Then the automorphism group $\mathrm{Aut}(R_n)$ is isomorphic to the affine group $\mathrm{Aff}(\mathbb{Z}_n)$.*

Proof. It is clear that the mapping $f_{a,b}$ (with $f_{a,b}(x) = ax + b$) is a quandle homomorphism. It is a bijective mapping if and only if $a \in \mathbb{Z}_n^\times$. Now we show that any quandle automorphism of $\mathbb{Z}_n$ (with the operation $x * y = 2y - x$) is an affine transformation $f_{a,b}$ for some $a \in \mathbb{Z}_n^\times$ and $b \in \mathbb{Z}_n$. Let $f \in \mathrm{Aut}(\mathbb{Z}_n)$, then

$\forall\, x, y \in \mathbb{Z}_n, f(2y - x) = 2f(y) - f(x)$. Now consider the mapping $g : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $g(x) = f(x) - f(0)$. By subtracting $f(0)$ from both sides of the equation $f(2y - x) = 2f(y) - f(x)$ we obtain $g(2y - x) = 2g(y) - g(x)$. We have $g(0) = 0$ and thus $g(-a) = -g(a)$. We now prove linearity of g, that is $g(\lambda x) = \lambda g(x)$ for any $\lambda \in \mathbb{Z}_n$. We have $g(2b - a) = 2g(b) - g(a)$, thus $g(2b) = 2g(b)$ and by induction on even integers $g(2ka) = 2kg(a)$, for all k. Now we do induction on odd integers: $g[(2k + 1)a] = g[2ka - (-a)] = 2kg(a) - g(-a) = 2kg(a) + g(a) = (2k + 1)g(a)$. Now g is a bijection if and only if $g(1) \in \mathbb{Z}_n^{\times}$ which ends the proof. □

Since the affine group $\mathrm{Aff}(\mathbb{Z}_n)$ is semi-direct product group $\mathbb{Z}_n \rtimes \mathbb{Z}_n^{\times}$, we have

Corollary 2.1. *The cardinal of* $\mathrm{Aut}(R_n)$ *is* $n\ \phi(n)$, *where* ϕ *denotes the Euler function.*

For the dihedral quandle $R_n = \mathbb{Z}_n$ and for each $i \in \mathbb{Z}_n$ the symmetry S_i given by $S_i(j) = 2i - j \pmod n$, can be thought of as a reflection of a regular n-gon. If n is odd, the axis of symmetry of S_i connects the vertex i to the mid-point of the side opposite to i. If $n = 2m$ is even, the axis of symmetry of S_i passes through the opposite vertices i and $i + m \pmod{2m}$. From these observations, we have the easy characterization of the inner automorphism group of dihedral quandles given by the following.

Theorem 2.2. *The inner automorphism group* $\mathrm{Inn}(R_n)$ *of the dihedral quandle* R_n *is isomorphic to the dihedral group* $D_{\frac{m}{2}}$ *of order* m *where* m *is the least common multiple of* n *and* 2.

Theorem 2.3. *Let* G *be a group and let the quandle* X *be the group* G *as a set with the conjugation* $x * y = yxy^{-1}$ *as operation. This quandle is usually denoted by* $\mathrm{Conj}(G)$. *Then* $\mathrm{Inn}(X) \cong G/Z(G) \cong \mathrm{Inn}(G)$.

Proof. The proof is straightforward from the fact that in this case the surjective map $S : G \to \mathrm{Inn}(G)$ sending $a \in G$ to S_a is a group homomorphism with kernel the center $Z(G)$ of G. □

If $(X, *)$ is a quandle for which the map $S : X \to \mathrm{Inn}(X)$ is one-to-one and onto then $(X, *) \cong \mathrm{Conj}(\mathrm{Inn}(X))$ with $Z(\mathrm{Inn}(X))$ being the trivial group. An interesting question would be to calculate the automorphism groups $\mathrm{Aut}(\mathrm{Conj}(G))$. Obviously for the symmetric group Σ_3, we have $\mathrm{Aut}(\mathrm{Conj}(\Sigma_3)) \cong \mathrm{Inn}(\mathrm{Conj}(\Sigma_3)) \cong \Sigma_3$.

3. Automorphism and Inner Automorphism Groups of Quandles of Order 6

In this section, we compute the automorphism groups and the inner automorphism groups of all quandles of order six. The computation is accomplished with the help of the software Maple which also allows the computation of the inner and automorphism groups for quandles of order 7 and 8. Since the numbers of isomorphism classes of quandles of order 7 and 8 are respectively 298 and 1581, we decided not to include these two cases in this paper.

We describe each quandle Q_j of order 6 for $1 \leq j \leq 73$ by explicitly giving each symmetry S_k for $1 \leq k \leq 6$, in terms of products of disjoint cycles. The symmetries are the columns in the Cayley table of the quandle. For example the quandle, denoted Q_{46} in Table 1, with the Cayley table

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 5 & 2 & 5 \\ 3 & 4 & 3 & 3 & 4 & 4 \\ 4 & 3 & 4 & 4 & 3 & 3 \\ 5 & 5 & 2 & 2 & 5 & 2 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

is described by the permutations of the six elements set $\{1, 2, 3, 4, 5, 6\}$, $S_1 = (1)$, $S_2 = (34)$, $S_3 = (25)$, $S_4 = (25)$, $S_5 = (34)$, $S_6 = (25)(34)$. In this example $\mathrm{Aut}(Q_{46}) = D_4$, the dihedral group of eight elements and $\mathrm{Inn}(Q_{46}) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Another example given in Table 2 is $\mathrm{Inn}(Q_{49}) = D_5$ the dihedral group of order 10 and $\mathrm{Aut}(Q_{49}) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, the semidirect product of the cyclic group $\mathbb{Z}_5$ by $\mathbb{Z}_4$.

4. Algorithm Description

In the quest of finding computationally the quandles of certain order up to isomorphism, we are cursed by the fact that any sort of naive algorithm will take an exponential time (in the order of the quandle) to do such task. Therefore, we are required to exploit structural or logical aspects of the quandle theory to reduce the running time at least by a proportional factor. In particular, there are two different phases to focus in: (i) the generation of quandles and (ii) the isomorphic distinction of them.

Phase 1: *List generation.* In initial versions of the *quandles algorithm* [10], the set of all matrices such that every row is a permutation of $[n] = \{1, \ldots, n\}$, is generated. After this, the matrices that do not correspond to the operation table of a quandle (i.e. such that do not satisfy the quandle axioms) are ruled out. We call this initial process the *list generation*, and its purpose is to list a set of quandles such that among then, we are guaranteed to find representatives for all isomorphic classes of quandles of order n. A further improvement in this process consists in verifying the quandles axioms *online*, this means that, during the generation of the matrices, the axioms are immediately verified, a process that was also carried out in [10]. We elaborate this improvement to a higer level: Besides verifying the quandle axioms online, we also fill in online, entries that are implied by the quandle axioms. Another easy improvement, which certainly reduces considerably the size of the list of quandles to output in this first step of the quandles algorithm, comes from elementary logic: When you are trying to generate all the models of cardinality n of a theory (in our case the theory of quandles), we can start introducing constants

Table 1. Quandles of order 6 in term of disjoint cycles of columns.

Quandle	Disjoint cycle notation for the columns of the quandle
Q_1	(1), (1), (1), (1), (1), (1)
Q_2	(1), (1), (1), (1), (1), (12)
Q_3	(1), (1), (1), (1), (1), (132)
Q_4	(1), (1), (1), (1), (1), (1243)
Q_5	(1), (1), (1), (1), (1), (12)(34)
Q_6	(1), (1), (1), (1), (1), (15234)
Q_7	(1), (1), (1), (1), (1), (134)(25)
Q_8	(1), (1), (1), (1), (12), (12)
Q_9	(1), (1), (1), (1), (12), (12)(34)
Q_{10}	(1), (1), (1), (1), (12), (34)
Q_{11}	(1), (1), (1), (1), (132), (132)
Q_{12}	(1), (1), (1), (1), (132), (123)
Q_{13}	(1), (1), (1), (1), (1243), (1243)
Q_{14}	(1), (1), (1), (1), (1243), (1342)
Q_{15}	(1), (1), (1), (1), (1243), (14)(23)
Q_{16}	(1), (1), (1), (1), (12)(34), (12)(34)
Q_{17}	(1), (1), (1), (1), (12)(34), (13)(24)
Q_{18}	(1), (1), (1), (12), (12), (12)
Q_{19}	(1), (1), (1), (12), (12), (12)(45)
Q_{20}	(1), (1), (1), (12), (12), (45)
Q_{21}	(1), (1), (1), (132), (132), (132)
Q_{22}	(1), (1), (1), (132), (132), (123)
Q_{23}	(1), (1), (1), (132), (132), (45)
Q_{24}	(1), (1), (1), (132), (132), (123)(45)
Q_{25}	(1), (1), (1), (132), (132), (132)(45)
Q_{26}	(1), (1), (1), (12)(56), (12)(46), (12)(45)
Q_{27}	(1), (1), (1), (12)(56), (13)(46), (23)(45)
Q_{28}	(1), (1), (1), (56), (46), (45)
Q_{29}	(1), (1), (1), (123)(56), (123)(46), (123)(45)
Q_{30}	(1), (1), (12), (12), (12), (12)
Q_{31}	(1), (1), (12), (12), (12), (12)(34)
Q_{32}	(1), (1), (12), (12), (12), (34)
Q_{33}	(1), (1), (12), (12), (12), (345)
Q_{34}	(1), (1), (12), (12), (12), (12)(345)
Q_{35}	(1), (1), (12), (12), (12)(34), (12)(34)
Q_{36}	(1), (1), (12), (12), (12)(34), (34)
Q_{37}	(1), (1), (12), (12), (34), (34)
Q_{38}	(1), (1), (12), (12)(56), (12)(46), (12)(45)
Q_{39}	(1), (1), (12), (56), (46), (45)
Q_{40}	(1), (1), (12)(45), (12)(36), (12)(36), (12)(45)
Q_{41}	(1), (1), (12)(45), (36), (36), (12)(45)
Q_{42}	(1), (1), (45), (36), (36), (45)
Q_{43}	(1), (1), (456), (365), (346), (354)
Q_{44}	(1), (1), (12)(456), (12)(365), (12)(346), (12)(354)
Q_{45}	(1), (34), (25), (25), (34), (34)
Q_{46}	(1), (34), (25), (25), (34), (25)(34)
Q_{47}	(1), (34), (256), (256), (34), (34)
Q_{48}	(1), (354), (26)(45), (26)(35), (26)(34), (345)
Q_{49}	(1), (36)(45), (25)(46), (23)(56), (26)(34), (24)(35)
Q_{50}	(1), (3546), (2456), (2365), (2643), (2534)

Table 1. (*Continued*)

Quandle	Disjoint cycle notation for the columns of the quandle
Q_{51}	(1), (3546), (2564), (2653), (2436), (2345)
Q_{52}	(23), (13), (12), (56), (46), (45)
Q_{53}	(23), (14), (14), (23), (23), (23)
Q_{54}	(23), (14), (14), (23), (23), (14)(23)
Q_{55}	(23), (14), (14), (23), (23), (14)
Q_{56}	(23), (14), (14), (23), (14)(23), (14)(23)
Q_{57}	(23), (154), (154), (23), (23), (23)
Q_{58}	(23), (154), (154), (23), (23), (154)(23)
Q_{59}	(23), (154), (154), (23), (23), (154)
Q_{60}	(23), (154), (154), (23), (23), (145)
Q_{61}	(23), (154), (154), (23), (23), (145)(23)
Q_{62}	(23), (45), (45), (16)(23), (16)(23), (23)
Q_{63}	(23), (45), (45), (16), (16), (23)
Q_{64}	(23), (1564), (1564), (23), (23), (23)
Q_{65}	(23), (15)(46), (15)(46), (23), (23), (23)
Q_{66}	(23), (15)(46), (15)(46), (15)(23), (23), (15)(23)
Q_{67}	(243), (165), (165), (165), (243), (243)
Q_{68}	(2354), (1463), (1265), (1562), (1364), (2453)
Q_{69}	(2354), (16)(34), (16)(25), (16)(25), (16)(34), (2453)
Q_{70}	(23)(45), (15)(36), (14)(26), (15)(36), (14)(26), (23)(45)
Q_{71}	(23)(45), (15)(46), (14)(56), (16)(23), (16)(23), (23)(45)
Q_{72}	(23)(45), (13)(46), (12)(56), (15)(26), (14)(36), (24)(35)
Q_{73}	(23)(45), (16)(45), (16)(45), (16)(23), (16)(23), (23)(45)

and the corresponding relations between these constants one by one (in a valid way), until we get n constants (so, the possible ways to generate the relations between constants will correspond to the models of the theory). This is exactly what any algorithm will do, just in the language of logic, but the point to emphasize is that, when a new constant is introduced, the name of such constant is irrelevant. This is a trivial logic fact, but one that was not used in previous versions of this listing procedure.

Phase 2: *Isomorphic comparison.* After the previous listing procedure has been elaborated (or more precisely, *while* the listing procedure is elaborated), we want to eliminate irrelevant quandles, that is, we want to leave only one representative per isomorphism class. For such comparison process, instead of doing a brute force algorithm that takes all possible bijections and checks for isomorphic equivalence, we can do two things:

(1) Use simple invariant checks, like number of cycles in every row action, to discard rapidly some nonisomorphic pairs of quandles.
(2) Use the quandle axioms to reduce the complexity of the isomorphic comparison process.

Regarding (2), we employ the quandle axioms to extend appropriately a partial isomorphism among valid possibilities, using a prescribed set of rules that iterates the quandle axioms to full account. While for (1), the idea is to avoid unnecessary

Table 2. Inner and automorphism groups of quandles of order 6.

Quandle X	$\mathrm{Inn}(X)$	$\mathrm{Aut}(X)$	Quandle X	$\mathrm{Inn}(X)$	$\mathrm{Aut}(X)$
Q_1	$\{1\}$	Σ_6	Q_{38}	$D_3 \times \mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$
Q_2	$\mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$	Q_{39}	$D_3 \times \mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$
Q_3	$\mathbb{Z}_3$	$\mathbb{Z}_6$	Q_{40}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$
Q_4	$\mathbb{Z}_4$	$\mathbb{Z}_4$	Q_{41}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_5	$\mathbb{Z}_2$	D_4	Q_{42}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$
Q_6	$\mathbb{Z}_5$	$\mathbb{Z}_5$	Q_{43}	A_4	$A_4 \times \mathbb{Z}_2$
Q_7	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{44}	$A_4 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$
Q_8	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{45}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_9	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{46}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4
Q_{10}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{47}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{11}	$\mathbb{Z}_3$	$\mathbb{Z}_6$	Q_{48}	D_3	D_3
Q_{12}	$\mathbb{Z}_3$	D_3	Q_{49}	D_5	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
Q_{13}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{50}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
Q_{14}	$\mathbb{Z}_4$	D_4	Q_{51}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
Q_{15}	$\mathbb{Z}_4$	$\mathbb{Z}_4$	Q_{52}	$D_3 \times D_3$	$(D_3 \times D_3) \rtimes \mathbb{Z}_2$
Q_{16}	$\mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$	Q_{53}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{17}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{54}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{18}	$\mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$	Q_{55}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4
Q_{19}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{56}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$
Q_{20}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{57}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{21}	$\mathbb{Z}_3$	$D_3 \times \mathbb{Z}_3$	Q_{58}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{22}	$\mathbb{Z}_3$	$\mathbb{Z}_6$	Q_{59}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{23}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{60}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{24}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{61}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{25}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{62}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{26}	D_3	$D_3 \times \mathbb{Z}_2$	Q_{63}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$
Q_{27}	D_3	D_3	Q_{64}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{28}	D_3	$D_3 \times D_3$	Q_{65}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$
Q_{29}	$D_3 \times \mathbb{Z}_3$	$D_3 \times \mathbb{Z}_3$	Q_{66}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{30}	$\mathbb{Z}_2$	$\Sigma_4 \times \mathbb{Z}_2$	Q_{67}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$D_3 \times \mathbb{Z}_3$
Q_{31}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{68}	Σ_4	Σ_4
Q_{32}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{69}	D_4	D_4
Q_{33}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{70}	D_3	$D_3 \times \mathbb{Z}_2$
Q_{34}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{71}	D_4	D_4
Q_{35}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{72}	Σ_4	Σ_4
Q_{36}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{73}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\Sigma_4 \times \mathbb{Z}_2$
Q_{37}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$			

isomorphism comparisons for quandles that we know before hand that are 'too different'. For early versions of the algorithm, we used invariants based on the permutation structure of the columns of the quandle table, like total number of cycles among all columns of the quandle table, number of cycles of every column or the more elaborate superset of supersets of cycle lengths of every column.

Using the previous ideas we could list isomorphic classes of quandles up to order $n = 9$ (order $n = 8$ took roughly half hour while order $n = 9$ took 16 h). The number of isomorphism of quandles of order 3, 4, 5, 6, 7, 8 and 9 we obtain are respectively 3, 7, 22, 73, 298, 1581, 11079. These same numbers are obtained by McCarron in [15].

Further information regarding the algorithm including its code, output and benchmarks is available at the web address http://people.math.gatech.edu/˜restrepo/quandles.html. Another set of invariants suggested by professor Edwin Clark, which according to his experiments seem to distinguish isomorphic classes effectively, take in account the structure of the rows of the quandle. We hope to add such functionality to our algorithm and also modify it to find particular kinds of quandles with more restrictive properties.

Acknowledgments

The authors would like to thank Professor Edwin Clark for his help and fruitful suggestions.

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