Question 1: Prove or disprove that $G = \{a + bi \mid a, b \in \mathbb{Q}\}$ is a field.

Due to the assumption that $\mathbb{C}$ is a field, it is trivial to see we must only verify that G is closed under addition and multiplication so to satisfy the field axioms. This reduces to proving $\forall x, y \in \mathbb{Q} : xy, x + y \in \mathbb{Q}$ due to the definitions of multiplication and addition on the complex field. The previous statement on rationals is well known to be true.

Question 2: Is $M = \{re^{2\pi i\theta} \mid r, \theta \in \mathbb{Q}\}$ a field?

M is intuitively a field, being a subset of complex numbers that would fall on the lines at rational angles.

Question 3: The vector space $V = (\mathbb{F}_2)^2$ has exactly four vectors $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$; so V has exactly $24 = 16$ subsets. How many of these 16 subsets are linearly independent? How many bases does V have?

Question 4: For Each of the following subsets of $\mathbb{F}^3$, determine wether it is a subspace of $\mathbb{F}^3$

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$

- The set gives a plane through the pole and is thus a subspace.

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + 2x_2 + 3x_3 = 4\}$

- The set does not include the zero vector and is thus not a subspace.

(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1x_2x_3 = 0\}$

- The set is not closed under addition and is thus not a subspace.

(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 = 5x_3\}$

- The set gives a line through the pole and is thus a subspace.

Question 5: Prove or give a counter example: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W$$
$$V = U_2 \oplus W$$

then $U_1 = U_2$.

This theorem is false. For one simple counter example let $V = \mathbb{F}^2$

$$\begin{aligned} W &= \{(x, y, 0) \mid x, y \in \mathbb{F}\} \\ U_1 &= \{(x, x, 0) \mid x \in \mathbb{F}\} \\ U_2 &= \{(0, 0, x) \mid x \in \mathbb{F}\} \end{aligned}$$