Homework 3 sample solution

Due 09/02/16

August 30, 2016

Use the *formal definition* of Big-Oh to prove the following.

1. Prove that if $f(n) = n^x + an^y$, where a, x, and y are positive integers such that $x > y$, $f(n) = O(n^x)$.

   Answer:

   Proof. Let $c = 2$ and $n_0 = a$, and let n be an arbitrary number such that $n \geq a$.

   Since $n \geq a$, $an^y \leq n^{y+1}$. Since $x > y$, $x \geq y + 1$, and since $y + 1 \leq x$ and $n \geq 1$, $n^{y+1} \leq n^x$, so $f(n) = n^x + an^y \leq n^x + n^{y+1} \leq n^x + n^x = 2n^x = cn^x$. Hence, $f(n) \leq cn^x$ for any $n \geq n_0$.

   Therefore, $f(n) = O(n^x)$ by the formal definition of Big-Oh. $\square$

2. Use induction to prove that $(n!)^2 = O((2n)!)$ using $c = 1$ and $n_0 = 1$.

   Answer: Let $c = n_0 = 1$. We prove that $(n!)^2 \leq c(2n)!$ for all $n \geq n_0$ by induction.

   (*Base case*) When $n = 1$, $(n!)^2 = 1(1!)^2 = 1$, and $c(2n)! = 2! = 2$, so $(n!)^2 \leq c(2n)!$ when $n = n_0$.

   (*Inductive step*) Suppose that $(k!)^2 \leq c(2k)!$, and consider $n = k + 1$.

$$
\begin{aligned}
c(2(k+1))! &= (2k+2)! \\
&= (2k+2)(2k+1)(2k)! \\
&\geq (2k+2)(2k+1)(k!)^2 \\
&\geq (k+1)(k+1)(k!)^2 \\
&\geq (k+1)^2(k!)^2 \\
&\geq ((k+1)!)^2
\end{aligned}
$$

   Therefore, $(n!)^2 \leq c(2n)!$ for all $n \geq n_0$ by induction. Hence, $(n!)^2 = O((2n)!)$ by the formal definition of Big-Oh.