Homework 1: Induction practice

Due 08/26/16

August 23, 2016

1. Use induction to prove that if a_n is a sequence such that $a_0 = 0$ and $a_n = 2a_{n-1} + 2^n$ for $n > 0$, then $a_n = n2^n$ for all $n \geq 0$.

   Answer:

   Proof. (*Base case*) When $n = 0$, $a_n = 0$ and $n2^n = 0(2^0) = 0$, so $a_n = n2^n$ when $n = 0$.

   (*Induction step*) Suppose that $a_k = k2^k$ for some $k \geq 0$, and consider a_{k+1}. By the definition of a_n, $a_{k+1} = 2a_k + 2^{k+1}$. By the Inductive Hypothesis, $a_k = k2^k$, so $a_{k+1} = 2k2^k + 2^{k+1} = k2^{k+1} + 2^{k+1} = (k+1)2^{k+1}$.

   Hence, $a_n = n2^n$ for all $n \geq 0$ by induction. □

2. Use strong induction to prove that if b_n is a sequence such that $b_0 = 1$, $b_1 = 6$, and $b_n = 4b_{n-2}$, then $b_n = 2^{n+1} - (-2)^n$ for all $n \geq 0$.

   Answer:

   Proof. (*Base case*) When $n = 0$, $a_n = 1$ and $2^{n+1} - (-2)^n = 2^1 - (-2)^0 = 2 - 1 = 1$, so $a_n = 2^{n+1} - (-2)^n$ when $n = 0$.

   When $n = 1$, $a_n = 6$ and $2^{n+1} - (-2)^n = 2^2 - (-2)^1 = 4 - (-2) = 6$, so $a_n = 2^{n+1} - (-2)^n$ when $n = 1$.

   (*Inductive step*) Suppose that $a_n = 2^{n+1} - (-2)^n$ for all n from 0 up to k, for some $k \geq 1$, and consider a_{k+1}. Since $k \geq 1$, $a_{k+1} = 4a_{k-1}$ by the definition of a_n. Since $k - 1 < k$ and $k - 1 \geq 0$, $a_{k-1} = 2^k - (-2)^{k-1}$ by the induction hypothesis. So:

$$\begin{aligned} a_{k+1} &= 4(2^k - (-2)^{k-1}) \\ &= 4(2^k) - 4(-2)^{k-1} \\ &= 2^2(2^k) - (-2)^2(-2)^{k-1} \\ &= 2^{k+2} - (-2)^{k+1} \end{aligned}$$

   Hence, $a_n = 2^{n+1} - (-2)^n$ for all $n \geq 0$ by strong induction. □