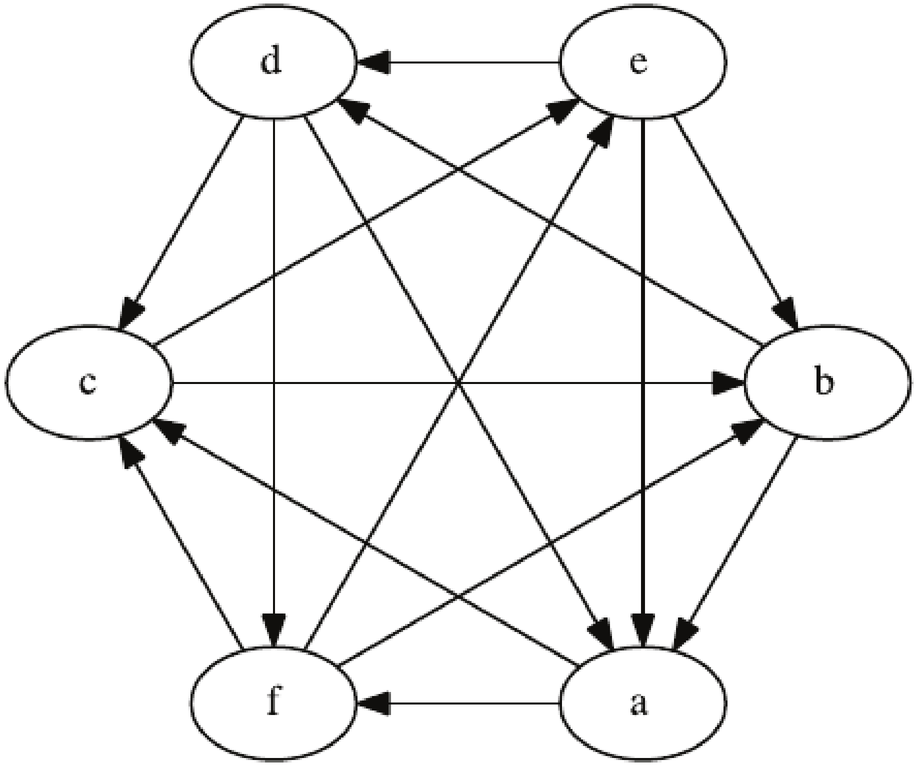
Fair and Strategy-proof Tournament Rules Design

Tournament

A *tournament* is $T = (A, \succ_T)$

- A : (finite) set of *agents/ alternatives/participants/players*
- $\succ_T$: complete asymmetric binary relation over A, describes match outcomes

Can be represented as a complete oriented graph.



Tournament

Notation

$\mathcal{T}(A)$: set of tournaments on A

$\mathcal{T}_n$: set of tournaments on $[n]$

$$\#A = n \quad \Rightarrow \quad A \cong [n]$$

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Goal

how to choose a winner in a way that is both fair and non-manipulable?

fair : selects a meritable alternative
non-manipulable : alternatives do not benefit from match-fixing

Applications

- sports ranking/seeding
- voting (social choice theory)
- decision making
- argumentation theory
- multi-agent systems
  - mixture of experts
  - compare models

Deterministic Rules

Tournament rule

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Fairness

Idea: win every match → win the tournament

$x \in A$ is *Condorcet winner* in T if $x \succ_T y$ for all $y \in A \setminus \{x\}$.

A rule is *Condorcet consistent* if it selects the Condorcet winner (if it exists).

Non-manipulability

Manipulations:

- one player loses on purpose / reverse outgoing edges
- two or more players reverse the outcomes of matches in the coalition

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Two tournaments are S-*adjacent* if they differ only on edges in S.

Non-manipulability

Monotonicity

Manipulation 1 is successful if a losing player becomes the winner after reversing an outgoing edge.

Non-manipulability

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A rule is *monotone* if a player cannot do this:
he must win/lose in all $\{x, y\}$-adjacent tournaments to T where $x \succ_T y$.

Non-manipulability

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Example. *Copeland rule* : wins who has maximum outdegree

Non-manipulability

Pairwise non-manipulability

Manipulation 2 is successful if one of the player becomes the winner after reversing the outcome of their match.

Non-manipulability

Pairwise non-manipulability

Manipulation 2 is successful if one of the player becomes the winner after reversing the outcome of their match.

A rule is *pairwise non-manipulable* if $r(T) \in \{x, y\} \Leftrightarrow r(T') \in \{x, y\}$ for every T, T' which are $\{x, y\}$-adjacent.

Impossibility result

Theorem.

Any pairwise non-manipulable rule is not Condorcet consistent.

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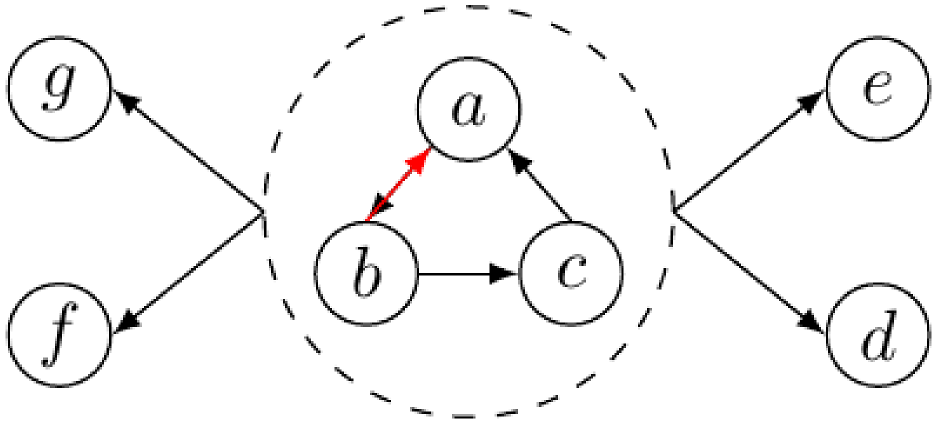
Proof.



WLOG $r(T) \notin \{a, b\}$

$b \succ_{T'} a \quad \Rightarrow$
b Condorcet winner

$r(T') = b \quad \lightning$

$\square$

Non-imposition

A rule is *non-imposing* if every player has a chance to win:
for every $x \in A$ there exists a tournament T such that $r(T) = x$.

Non-imposition

Theorem.

If $\#A = 3$, then every PNM rule is not non-imposing.

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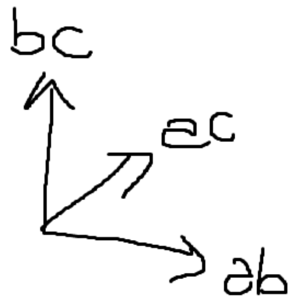
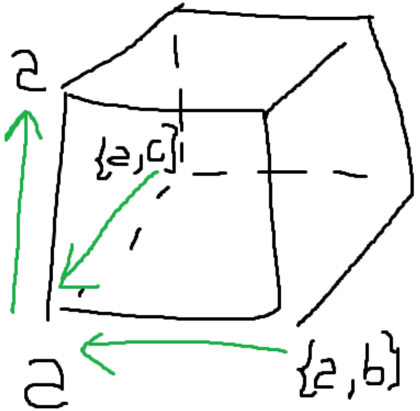
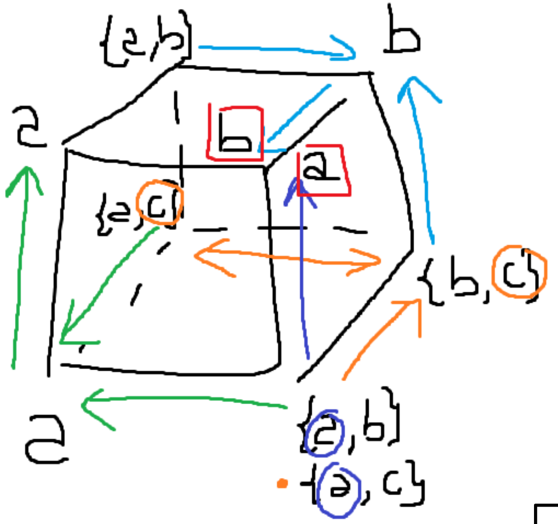




caso 1



$\square$

Non-imposition

Theorem.

If $\#A \neq 3$, then there exists a tournament rule that is monotone, PNM and non-imposing.

Probabilistic Rules

A *(probabilistic/randomized) tournament rule over* A is a map $r : \mathcal{T}(A) \to \Delta(A)$, that maps each tournament to a probability distribution over the players.

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A *tournament rule on* n *agents* is a map $r^{(n)} : \mathcal{T}_n \to \Delta^n$.
A *tournament rule* r is a family of tournament rules on n players for every $n \in \mathbb{N}$: $\left\{r^{(n)}\right\}_{n=1}^{\infty}$.

Notation. $r(T) := r^{(n)}(T) \qquad r_i(T) := (r(T))_i$

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A rule is *deterministic* if has values in $\{0, 1\}$.

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A rule is *anonymous* if $\pi(r(T)) = r(\pi(T))$.

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The *top cycle* $TC(T)$ is the minimal subset of players that beats all the others outside it.
A rule is *top cycle consistent* if $r_x(T) > 0$ implies $x \in TC(T)$.
Equivalently, if $z \notin TC(T)$ then $r_z(T) = 0$.

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Example. *Top cycle rule*: $r_i(T) = \begin{cases} \frac{1}{\#TC(T)}, & i \in TC(T) \\ 0, & i \notin TC(T) \end{cases}$

Non-manipulability

Monotonicity

A rule is *monotone* if $x \succ_T y$ implies $r_x(T) \geq r_x(T')$ for every T, T' which are $\{x, y\}$-adjacent.

Non-manipulability

Pareto non-manipulability

non-transferable utility $\rightarrow$ neither are worse off and one is better off

A rule is *Pareto non-manipulable for coalitions of size up to k* (k-PNM) if for every $S \subseteq A,\ \#S \leq k$ and for every T, T' which are S-adjacent:

- $\exists x \in S : r_x(T') < r_x(T)$ or
- $\forall x \in S : r_x(T') \leq r_x(T)$.

Non-manipulability

Strong non-manipulability

transferable utility $\rightarrow$ coalition is better off

A rule is *strongly non-manipulable for coalitions of size up to k* (k-SNM) if for every $S \subseteq A,\ \#S \leq k$ and for every T, T' which are S-adjacent:

$$\sum_{x \in S} r_x(T) = \sum_{x \in S} r_x(T')$$

Non-manipulability

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Remark. For deterministic rules they are equivalent.

Results

Proposition.

The top cycle rule is monotone and PNM.

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Proof. **Monotone.** Losing a match

- inside the top cycle → may be kicked out
- outside the top cycle → does not make inside

PNM. Successful manipulation

- reduce top cycle size while keeping all manipulators
- add new manipulator while keeping same size

$\Rightarrow$ kick a non-manipulator from top cycle
needs to remove also all dominated player
among them there must be a manipulator ↯ $\square$

Proposition.

There exist no 2-SNM Condorcet consistent tournament rules.

Proof. Same as before. $\square$

Need to weaken either non-manipulability or Condorcet consistency.

Approximate Condorcet Consistency

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A rule has *Condorcet consistency value* $\gamma \in [0, 1]$ if for a Condorcet winner c we have $r_c(T) \geq \gamma$.

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Proposition.

There exists a monotone SNM tournament rule which has Condorcet consistency value of $\frac{2}{n}$.

Proof. Choose two at random, who wins the match is the winner.

$$r_x(T) = \frac{2}{n(n-1)} \cdot \#\{y : x \succ_T y\}$$

□

Approximate Condorcet consistency

Theorem.

There exists no 3-SNM rule with CC value $\gamma > \frac{2}{n}$.

Approximate Condorcet consistency

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There exists no 3-SNM rule with CC value $\gamma > \frac{2}{n}$.

Theorem.

There exists no 2-SNM rule with CC value $\gamma > \frac{2}{3}$.

Proof. top cycle = $\{a, b, c\}$

$$r_a(T) + r_b(T) > \tfrac{2}{3}, \quad r_a(T) + r_c(T) > \tfrac{2}{3}, \quad r_b(T) + r_c(T) > \tfrac{2}{3}$$

$$2[r_a(T) + r_b(T) + r_c(T)] > 3 \cdot \tfrac{2}{3} = 2 \quad ⚡$$

□

Approximate Non-Manipulability

Approximate non-manipulability

A rule is 2-*strongly non-manipulable at probability* α (2-SNM-α) if, for all i and j and pairs of $\{i, j\}$-adjacent tournaments T and T',

$$r_i(T') + r_j(T') - r_i(T) - r_j(T) \leq \alpha$$

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A rule is k*-strongly non-manipulable at probability* α (k-SNM-α) if, for all subsets S of players of size at most k, for all pairs of S-adjacent tournaments T and T',

$$\sum_{i \in S} r_i(T') - \sum_{i \in S} r_i(T) \leq \alpha$$

Approximate non-manipulability

Theorem.

There is no Condorcet-consistent rule on n players (for $n \geq 3$) that is 2-SNM-α for $\alpha < \frac{1}{3}$.

Approximate non-manipulability

Theorem.

There is no Condorcet-consistent rule on n players (for $n \geq 3$) that is 2-SNM-α for $\alpha < \frac{1}{3}$.

Proof. top cycle = $\{a, b, c\}$

$$1 - \sum r_i \leq \alpha$$

$$r_a(T) + r_b(T) \geq 1 - \alpha$$

$$r_a(T) + r_c(T) \geq 1 - \alpha$$

$$r_b(T) + r_c(T) \geq 1 - \alpha$$

$$r_a(T) + r_b(T) + r_c(T) \geq \tfrac{3}{2}(1 - \alpha)$$

$$r_a(T) + r_b(T) + r_c(T) = 1$$

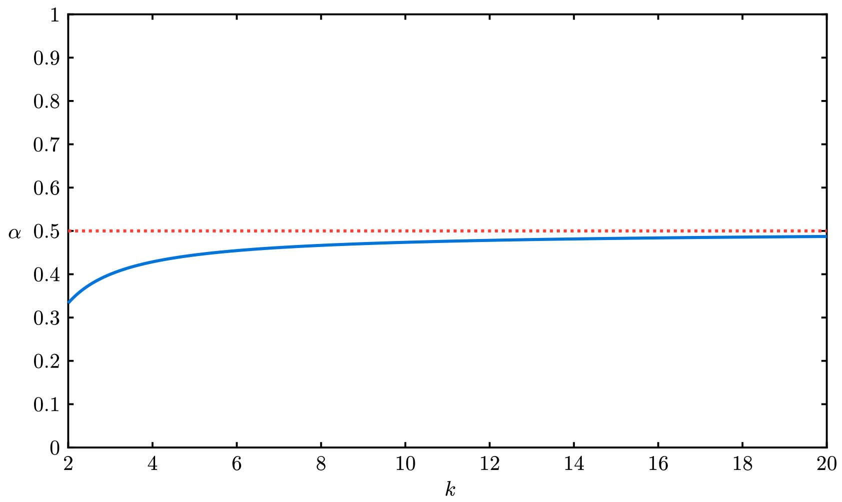
$$\Rightarrow \ \alpha \geq \tfrac{1}{3}$$

$\square$

Approximate non-manipulability

Theorem.

There is no Condorcet-consistent rule on n players (for $n \geq 2k-1$) that is k-SNM-α for $\alpha < \frac{k-1}{2k-1}$.



Random Single-Elimination Bracket Rule

A *single-elimination bracket* is binary tree labelled with the players: permutation of all on the leaves, winner of the match on the parent, label of the root $\rightarrow$ winner of the bracket

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r_i = fraction of winning brackets

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Remark. RSEB is CC and monotone.

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Remark. RSEB is CC and monotone.

Theorem.

RSEB is 2-SNM-$1/3$.

Partially Transferable Utility

Non-manipulability

non-transferable utility

$$\max\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} > 0 \text{ and}$$

$$\min\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} \geq 0.$$

2-Pareto non-manipulable

- $\min\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} < 0$ or
- $\max\{r_i(T') - r_i(T), r_j(T') - r_j(T)\} \leq 0$

Non-manipulability

transferable utility

$$r_i(T') + r_j(T') > r_i(T) + r_j(T)$$

2-strongly non-manipulable

$$\sum_{x \in S} r_x(T) = \sum_{x \in S} r_x(T')$$

Non-manipulability

partially transferable utility

$$r_i(T') + r_j(T') > r_i(T) + r_j(T) + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

Non-manipulability

partially transferable utility

$$r_i(T') + r_j(T') > r_i(T) + r_j(T) + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

A tournament rule r is 2-*non-manipulable for* $\lambda \geq 0$ (2-NM$_\lambda$) if

$$r_i(T') + r_j(T') \leq r_i(T) + r_j(T) + \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

We say r is 2-NM$_\infty$ if

$$r_i(T') + r_j(T') \leq r_i(T) + r_j(T) + \lim_{\lambda \to \infty} \lambda \max\{r_i(T) - r_i(T'), r_j(T) - r_j(T')\}$$

Non-manipulability

$$\sum_{i \in S} r_i(T') - \sum_{i \in S} r_i(T) \leq \lambda \max_{i \in S}\{r_i(T) - r_i(T')\}$$

Remark.
- For $\lambda = 0$, we get back SNM.
- For $\lambda = +\infty$, we get back PNM.

Non-manipulability

$$\sum_{i \in S} r_i(T') - \sum_{i \in S} r_i(T) \leq \lambda \max_{i \in S}\{r_i(T) - r_i(T')\}$$

Remark.
- For $\lambda = 0$, we get back SNM.
- For $\lambda = +\infty$, we get back PNM.

2-NM$_\lambda$-α

$$\sum_{i \in S} r_i(T') - \sum_{i \in S} r_i(T) \leq \lambda \max_{i \in S}\{r_i(T) - r_i(T')\} + \alpha$$

no Condorcet + 2-NM_0-α for $\alpha < 1/3$

Theorem.

No Condorcet consistent tournament rule is 2-NM_λ-α for $\lambda < 1 - 3\alpha$.

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Theorem.

No Condorcet consistent tournament rule is 2-NM_λ-α for $\lambda < 1 - 3\alpha$.

Proof. top cycle: $\{1,2,3\}$, coalition: $\{1,2\}$, 2 is Condorcet in T'

CC $\Rightarrow$ $r_2(T') = 1,\ r_1(T') = 0$; 2-NM_λ-α $\Rightarrow$

$$\underbrace{r_1(T') + r_2(T')}_{1} - r_1(T) - r_2(T) \leq \lambda \max\{\underbrace{r_1(T) - r_1(T')}_{r_1(T)\geq 0}, \underbrace{r_2(T) - r_2(T')}_{r_2(T)-1\leq 0}\} + \alpha$$

$$1 - r_1(T) - r_2(T) \leq \lambda\, r_1(T) + \alpha$$

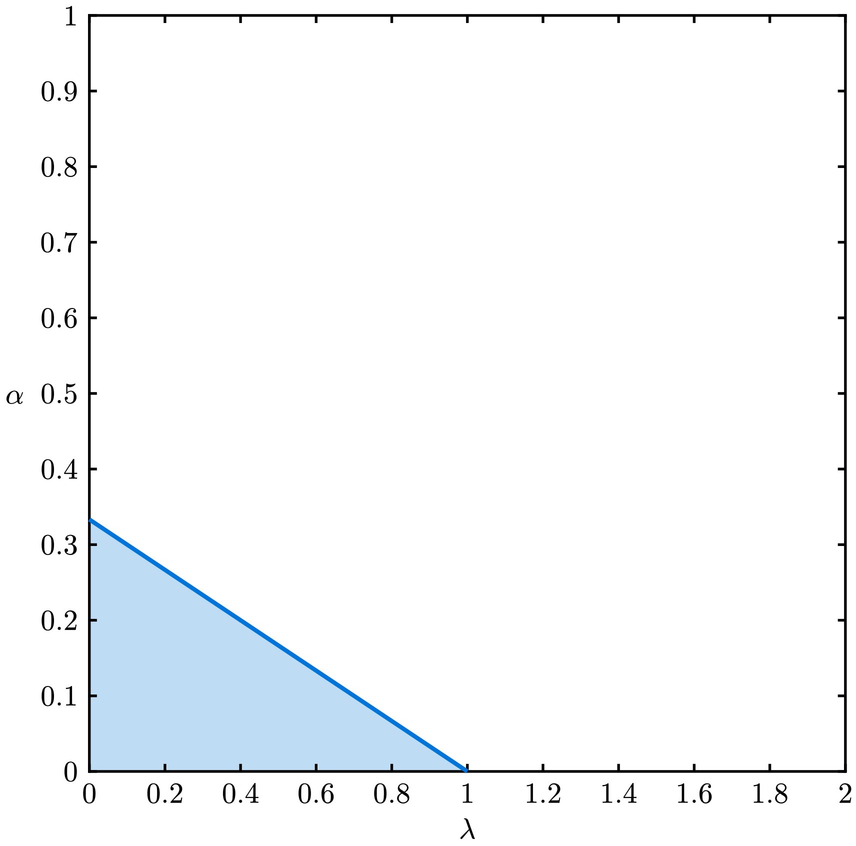
$$1 - r_2(T) \leq (\lambda + 1) r_1(T) + \alpha$$
$$1 - r_3(T) \leq (\lambda + 1) r_2(T) + \alpha$$
$$1 - r_1(T) \leq (\lambda + 1) r_3(T) + \alpha$$

$$3 - [r_1(T) + r_2(T) + r_3(T)] \leq (\lambda + 1)[r_1(T) + r_2(T) + r_3(T)] + 3\alpha$$
$$3 - 3\alpha \leq (\lambda + 2)[r_1(T) + r_2(T) + r_3(T)]$$
$$\underbrace{\frac{3 - 3\alpha}{r_1(T) + r_2(T) + r_3(T)}}_{\leq 1} - 2 \leq \lambda$$
$$3 - 3\alpha - 2 \leq \lambda \quad \Rightarrow \quad \lambda \geq 1 - 3\alpha \qquad \square$$

Corollary.

No Condorcet consistent tournament rule is 2-NM_λ for $\lambda < 1$.

Conjecture. There exists a tournament rule that is monotone, Condorcet consistent, and 2-NM_1.



Beyond Condorcet

Condorcet = bad

- Condorcet winner may not be so dominant
- superman-kryptonite

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`where are the graphs???`

Condorcet = bad

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- superman-kryptonite

Solution: consider *weigthed* tournaments !

weight = # points / goals / proportion of wins

A *weighted tournament* is a tournament with pairs of weights for each match $w_{ij}, w_{ji} \in [0, 1]$ such that $w_{ij} + w_{ji} = 1$.
It is represented as a complete directed weighted graph.

Proportional Score Rule

Idea: probability should be proportional to the number of matches won

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aggregate the weights → *score* : outdegree of the node

$$s_i = \sum_{j=1}^{n} w_{ij}$$

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aggregate the weights $\quad \rightarrow \quad$ *score* : outdegree of the node

$$s_i = \sum_{j=1}^{n} w_{ij}$$

how to turn into probabilities? softmax! $\quad r_i = \dfrac{e^{s_i}}{\sum_{j=1}^{n} e^{s_j}}$

$$(s_1, \ldots, s_n) \rightsquigarrow (r_1, \ldots, r_n), \qquad r_i \in [0, 1], \quad \sum_{i=1}^{n} r_i = 1$$

Proportional Score Rule

Fairness

Intuitively fair (see geogebra).

NOT Condorcet

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Non-manipulability

Theorem.

PSR is 2-SNM-$\overbrace{\tanh\left(\frac{1}{4}\right)}^{\approx 0.245}$.

Proportional Score Rule

Fairness

Intuitively fair (see geogebra).

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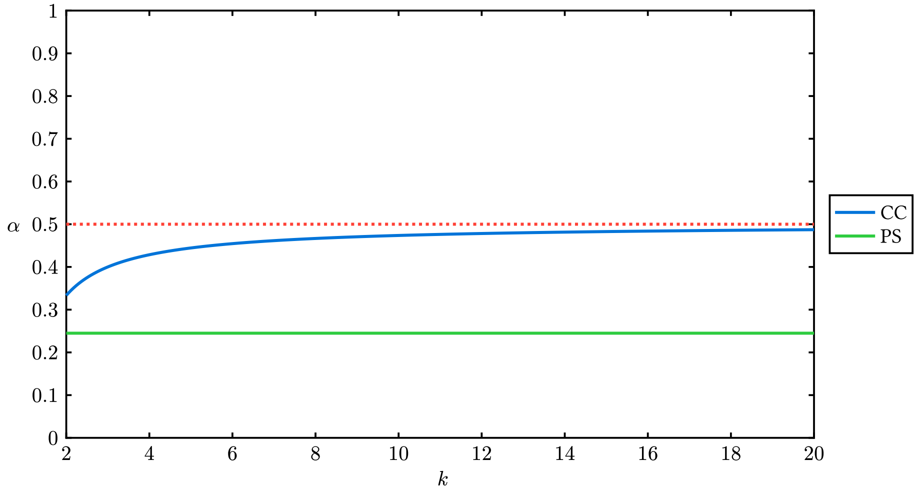
Theorem.

$$\text{PSR is 2-SNM-}\overbrace{\tanh\left(\tfrac{1}{4}\right)}^{\approx 0.245}.$$

Theorem.

$$\text{PSR is } k\text{-SNM-}\tanh\left(\tfrac{1}{4}\right)\ \forall k\ !$$



Even further beyond

- margin of win
- Bradley-Terry model (ELO)
- network flow
- hypergraphs