VAR Expectations Basics

March 2014

1. Introduction

In FRB/US simulations, each expectations variable can have either a model-consistent solution or what is known as a VAR solution. Let Z_t be the expectation used in period t of some future economic outcome, such as inflation next quarter or a weighted average of future short-term interest rates. Under VAR expectations, the solution for Z_t comes from a fixed-coefficient formula of the form,

$$Z_t = \sum_{i=n0}^{n1} \Omega_i X_{t-i}, \tag{1}$$

where X is a vector of variables and the $\{\Omega_i\}$, are coefficient matrices. For expectations based on current information, which is the assumption used in asset price equations, $n0 = 0$ and $n1 = 3$. For expectations in other sectors, the information set contains only lagged data, and $n0 = 1$ and $n1 = 4$.

Section 2 discusses in general terms the connection between VAR expectations and the way that FRB/US is estimated. Section 3 describes the VARs that play an important role in the estimation framework and lend their name to this form of expectations. Explicit expectations enter three types of FRB/US equations. Section 4 presents the formula used for the VAR approach to expectations in asset price equations. The VAR formulas associated with PAC (polynomial adjustment cost) equations are shown in Section 5. Section 6 considers the expectations terms in the price and wage equations, which follow the New Keynesian Phillips curve paradigm.

2. FRB/US Estimation and VAR Expectations

FRB/US has too many equations and coefficients for the whole model to be estimated simultaneously. As a result, most of the model is estimated one equation at a time. For each of the equations that contains one or more explicit expectations terms, the approach to estimation involves a small model that contains the structural equation of interest along with a set of non-structural equations whose projections provide proxies for the expectations term(s) in the structural equation.[1] In most instances, the added equations have a VAR-like form. Each small model is itself a linear rational expectation model and thus has a

[1] The only exception to single-equation estimation occurs in the price-wage sector, whose two main structural equations are estimated simultaneously.

backward-looking representation. The specific form of equation (1) for each expectations variable comes from the backward-looking representation of the small model used to estimate the structural equation in which that expectation appears.

3. The VARs

The VARs that appear in the small estimation models all contain a *core* set of equations for three primary macro variables. When an estimation model requires the expectation of a variable not in the core set, equations for *auxiliary* variables are added to its VAR. All of the small estimation models contain the same core equations, which is achieved most easily and parsimoniously by excluding the auxiliary variables from appearing in the core equations. Another aspect of the design of the VARs is that they are constructed around "moving endpoints" for the rate of inflation and the nominal rate of interest.[2]

The core VAR variables are the federal funds rate (r_t), the rate of overall PCE price inflation (π_t), and output gap ($\hat{x}_t$). Let X_t^c denote the vector of these three variables. In addition, let $X_t^{c\infty}$ denote the moving endpoints as measured by long-run expectations of the core variables. For r_t and π_t, long-run expectations are taken from survey data. The endpoint of the output gap is assumed to be zero. Using this notation, the core VAR has the following structure.

$$X_t^c = \begin{bmatrix} r_t \\ \pi_t \\ \hat{x}_t \end{bmatrix} \qquad X_t^{c\infty} = \begin{bmatrix} r_t^\infty \\ \pi_t^\infty \\ 0 \end{bmatrix} \tag{2}$$

$$\Delta X_t^c = \Lambda_0(X_{t-1}^c - X_{t-1}^{c\infty}) + \sum_{i=1}^{3} \Lambda_i \Delta X_{t-i}^c \tag{3}$$

$$X_t^{c\infty} = X_{t-1}^{c\infty} \tag{4}$$

The $\{\Lambda_i\}$ are matrices of estimated coefficients. Given the assumption that the endpoint variables are random walks (equation 4), and as long as the core equations form a stable system, the projection made at the start of period t of X^c into the far future converges to $X_{t-1}^{c\infty}$.

Auxiliary VAR equations may contain lags of both core and auxiliary variables. For auxiliary variables that are rates of interest or inflation, the coefficients in their equations are restricted to ensure that their forecasts converge to the appropriate endpoint. For auxiliary variables that are related to real output growth, coefficient restrictions impose the

[2] The benefits of using moving endpoints in a VAR system are discussed in Kozicki and Tinsley (2001, section 3.3).

condition that their forecasts converge to the rate of potential output growth. For purposes of forming VAR expectations, the latter is modeled as a random walk.

Let X be the combined vector of VAR variables — core, auxiliary (if any), and endoint — in one of the small estimation models. The coefficient matrices $\{\Gamma_i\}$ of its VAR system incorporate the various zero, endpoint-convergence, and random-walk restrictions noted above.

$$X_t = \sum_{i=1}^{4} \Gamma_i X_{t-i} \tag{5}$$

For some uses, it is convenient to express equation 5 in first-order companion form,

$$z_t = H z_{t-1} \tag{6}$$

where $z'_t = [X'_t, X'_{t-1}, X'_{t-2}, X'_{t-3}]$ and H combines the $\{\Gamma_i\}$ coefficient matrices.

4. Asset Valuation

One class of expectations variables in FRB/US is the present value relationships that appear in asset price equations. For example, $ZRFF10$ is the weighted average of expected future values of the funds rate used in equation for the yield on ten-year government bonds ($RG10E$). In the following exposition, let Z^{pv} represent one of the interest-rate present values in FRB/US, such as $ZRFF10$. The formula for Z^{pv} is a infinite-horizon present value that is a convenient approximation to the finite-horizon weighted average of expected future values associated with a bond of given maturity.

$$Z_t^{pv} = (1-w) E_t \left(\sum_{i=0}^{\infty} w^i r_{t+i} \right) \tag{7}$$

Recall that r is the federal funds rate. The value of weight w depends on the average duration of the bond in question which, in turn, depends on the bond's maturity and the average level of nominal interest rates. Equation 7 can also be expressed as,

$$Z_t^{pv} = w E_t Z_{t+1}^{pv} + (1-w) r_t \tag{8}$$

This is the equation that is used for Z^{pv} in simulations of FRB/US under model-consistent expectations.

There are several ways to derive the VAR-expectations equation for Z^{pv}. One is to create a model that consists of equation 8 and the VAR system as given in either equations 5 or 6, use an algorithm (such as AIM) to compute this model's backward-looking representation, and take the equation for Z^{pv} from that representation. Alternatively, because Z^{pv} does not appear in the VAR system, simple matrix algebra yields the following expression,

$$Z_t^{pv} = (1-w)\iota_h'[I - wH]^{-1} z_t \tag{9}$$

in which ι_h is a selector matrix that picks out the element of z_t that corresponds to r_t.

An implication of the endpoint form of the VAR system is that, in the VAR-expectations expression for the interest-rate present value Z^{pv}, the coefficients on r and r^∞ sum to one and the coefficients on π and π^∞ sum to zero. For an inflation present value (such as $ZPI10$), the former sum is zero and the latter sum is one. Both sums are zero for an output gap present value (such as $ZGAP10$).

5. PAC Equations[3]

A second class of expectations variables in FRB/US consists of the expectations terms in PAC equations. Let y be a decision variable whose behavior is modeled using a PAC of order m and y^* its desired level in the absence of adjustment costs. Cost minimization leads to the Euler equation.

$$A(\beta F)A(L)y_t - cy_t^* = 0 \tag{10}$$

In this expressions, A is a polynomial in the lag (L) and lead (F) operators of order m, e.g., $A(L) = 1 + \alpha_1 L + \ldots + \alpha_m L^m$ and $A(\beta F) = 1 + \alpha_1 \beta F + \ldots + \alpha_m \beta^m F^m$, $\beta = .98$ is a discount factor, and $c = A(1)A(\beta)$ is a constant. After decomposing y^* into non-stationary (y^{1*}) and stationary (y^{0*}) components, the Euler equation can be rearranged to form the PAC decision rule.

$$\Delta y_t = a_0(y_{t-1}^{1*} - y_{t-1}) + \sum_{i=1}^{m-1} a_i \Delta y_{t-i} + \sum_{i=0}^{\infty} d_i E_{t-1} \Delta y_{t+i}^{1*} + \sum_{i=0}^{\infty} h_i E_{t-1} y_{t+i}^{0*} \tag{11}$$

The $\{a_i\}$ coefficients in the decision rule are linear functions of the $\{\alpha_i\}$ coefficients in the Euler equation. The coefficients that appear in the two expecations summations are also functions of the Euler coefficients, but the relationships are nonlinear, as well as the discount factor, β.

$$h_i = A(1)A(\beta)\iota_m' G^i \iota_m \tag{12}$$

$$d_i = A(1)A(\beta)\iota_m'[I - G]^{-1} G^i \iota_m \tag{13}$$

[3] PAC equations are discussed in more detail in the *PAC Basics* document and in Brayton, Davis, and Tulip (2000).

Here, ι_m is a $m \times 1$ vector with a one in the mth element and zeroes elsewhere, and G is the $m \times m$ matrix

$$G = \left[\begin{array}{cccc} 0 & & I_{m-1} & \\ -\alpha_m \beta^m & -\alpha_{m-1}\beta^{m-1} & \ldots & -a_1\beta \end{array} \right].$$

The sums of the expectations coefficients are not one: $\sum h_i = A(1)A(\beta)\iota'_m[I_m - G]^{-1}\iota_m$ and $\sum d_i = A(1)A(\beta)\iota'_m[I_m - G]^{-2}\iota_m$.

To be able to form projections of the two components of y^*, the VAR system must include y_t^{0*} and Δy^{1*}.[4] Let y_t^{0*} be element $h0$ in vector of VAR variables z_t and Δy^{1*} be element $h1$, and define ι_{h0} and ι_{h1} as the corresponding selector vectors. With this notation, each future expectation can be expressed as as linear function of the VAR coefficients, H, and the VAR variables, z.

$$E_{t-1}y_{t+i}^{0*} = \iota'_{h0}H^i z_{t-1} \quad (14)$$
$$E_{t-1}\Delta y_{t+i}^{1*} = \iota'_{h1}H^i z_{t-1}. \quad (15)$$

Finally, let the variables Z^1 and Z^0 to represent the pair of expectations sums. Additional matrix algebra yields the VAR-expectations equations,

$$Z_t^0 = h'_0 z_{t-1} \quad (16)$$
$$Z_t^1 = h'_1 z_{t-1} \quad (17)$$

whose coefficient vectors h_0 and h_1 are defined as follows.

$$h_0 = A(1)A(\beta)[(\iota'_m I_m) \otimes H'][I_{nm} - (G \otimes H')]^{-1}[\iota_m \otimes \iota_{h0}] \quad (18)$$
$$h_1 = A(1)A(\beta)[(\iota'_m (I_m - G)^{-1}) \otimes H'][I_{nm} - (G \otimes H')]^{-1}[\iota_m \otimes \iota_{h1}] \quad (19)$$

6. New Keynesian Phillips curve

The FRB/US NKPC equations for core consumer prices and the ECI measure of hourly compensation are estimated simultaneously in a small model that also contains a number of non-structural relationships.[5] Because the small model contains more than one structural

[4] Alternatively, the VAR could contain the level of y^{1*}, but the standard practice in FRB/US is to use its first difference.

[5] See Brayton (2013).

equation with explicit expectations terms, analytic represenations of their VAR-expectations coefficients are not available. Numerical values of the coefficients can be obtained directly using AIM or from a regression of the time series of the expectations variables computed in Dynare on the information set as determined by the set of variables and lags in the estimation model.

References

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