P94 4(1)(2)、6、8、12、13、14(1)(3)、17.

4 计算下列函数 $f(x)$ 关于 $C[0,1]$ 的 $\|f\|_\infty$、$\|f\|_1$ 与 $\|f\|_2$.

(1) $f(x)=(x-1)^3$.

解：$\because f'(x)=3(x-1)^2$ 在 $[0,1]$ 上 ≥ 0，$\therefore f(x)$ 在 $[0,1]$ 上是单调增函数，$\therefore \|f\|_\infty = \max\limits_x |f(x)| = \{|f(0)|,|f(1)|\}_{max} = \max\{1,0\}=1$.

$\|f\|_1 = \int_a^b |f(x)|dx = \int_0^1 |(x-1)^3|dx = \left|\frac{1}{4}(x-1)^4\Big|_0^1\right| = \frac{1}{4}$

$\|f\|_2 = \sqrt{\int_a^b \rho(x) f^2(x)dx} = \left(\int_0^1 (x-1)^6 dx\right)^{\frac{1}{2}} = \sqrt{\frac{1}{7}(x-1)^7\Big|_0^1} = \frac{\sqrt{7}}{7}$

$\therefore \|f\|_\infty = 1$，$\|f\|_1 = \frac{1}{4}$，$\|f\|_2 = \frac{\sqrt{7}}{7}$

(2) $f(x)=|x-\frac{1}{2}|$

解：$f(x)$ 在 $[0,1]$ 上的最大值为 $f(0)=f(1)=\frac{1}{2}$ $\therefore \|f\|_\infty = \max\limits_x |f(x)| = \frac{1}{2}$.

$\|f\|_1 = \int_0^1 |x-\frac{1}{2}|dx = \int_{\frac{1}{2}}^1 (x-\frac{1}{2})dx + \int_0^{\frac{1}{2}} (\frac{1}{2}-x)dx$

$= (\frac{1}{2}x^2-\frac{1}{2}x)\Big|_{\frac{1}{2}}^1 + (\frac{1}{2}x-\frac{1}{2}x^2)\Big|_0^{\frac{1}{2}} = \frac{1}{4}$

$\cancel{\|f\|_2 = \left(\int_0^1 |x-1|^2 dx\right)^{\frac{1}{2}} = \left(\int_0^1 (x^2-2x+1)dx\right)^{\frac{1}{2}}}$

$\cancel{= \left(\frac{1}{3}x^3-x^2+x\Big|_0^1\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{3}}$

$\cancel{\therefore \|f\|_\infty = \frac{1}{2},\ \|f\|_1 = \frac{1}{4},\ \|f\|_2 = \frac{\sqrt{3}}{3}}$

$\|f\|_2 = \left(\int_0^1 (x-\frac{1}{2})^2 dx\right)^{\frac{1}{2}} = \left[\int_0^1 (x^2-x+\frac{1}{4})dx\right]^{\frac{1}{2}}$

$= \left(\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{4}x\Big|_0^1\right)^{\frac{1}{2}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

$\therefore \|f\|_\infty = \frac{1}{2}$，$\|f\|_1 = \frac{1}{4}$，$\|f\|_2 = \frac{\sqrt{3}}{6}$.

6. 对 $f(x), g(x) \in C^1[a,b]$ 定义

(1) $(f,g)=\int_a^b f'(x)g'(x)dx$. (2) $(f,g)=\int_a^b f'(x)g'(x)dx+f(a)g(a)$.

问它们是否构成内积

解：(1) ① $\because (f,g)=\int_a^b f'(x)g'(x)dx=\int_a^b g'(x)f'(x)dx=(g,f)$ $\therefore$ 满足对称性.

② $(f_1+f_2,g)=\int_a^b (f_1(x)+f_2(x))'g'(x)\,dx=\int_a^b f_1'(x)g'(x)+f_2'(x)g'(x)dx$

$=\int_a^b f_1'(x)g'(x)dx+\int_a^b f_2'(x)g'(x)\,dx=(f_1,g)+(f_2,g)$.

$(kf,g)=\int_a^b (kf(x))'g'(x)\,dx=\int_a^b kf'(x)g'(x)dx$

$=k\int_a^b f'(x)g'(x)\,dx=k(f,g)$ $\therefore$ 满足线性性.

③ $(f,f)=\int_a^b f'(x)f'(x)dx=\int_a^b f'(x)^2 dx=f'(\xi)^2\int_a^b 1dx$

$=(b-a)f'(\xi)^2\geq 0$ 当且仅当 $f'(\xi)=0$ 时取等号. 其中 $\xi\in[a,b]$.

~~对 $\forall x\in[a,b]$ $f(x)=$~~ 不能推出 $f\equiv 0$ $\therefore$ 不满足 正定性

$\therefore (f,g)$ 不能构成 ~~在~~ $C^1[a,b]$ 上的内积.

(2) ① $\because (f,g)=\int_a^b f'(x)g'(x)dx+f(a)g(a)=\int_a^b g'(x)f'(x)dx+g(a)f(a)=(g,f)$ $\therefore$ 满足对称性

② $(f_1+f_2,g)=\int_a^b (f_1(x)+f_2(x))'g'(x)dx+(f_1(a)+f_2(a))g(a)$

$=\int_a^b f_1'(x)g'(x)+f_2'(x)g'(x)dx+f_1(a)g(a)+f_2(a)g(a)$

$=\int_a^b f_1'(x)g'(x)\,dx+f_1(a)g(a)+\int_a^b f_2'(x)g'(x)dx+f_2(a)g(a)$

$=(f_1,g)+(f_2,g)$

$(kf,g)=\int_a^b (kf(x))'g'(x)dx+kf(a)g(a)=\int_a^b kf'(x)g'(x)dx+kf(a)g(a)$

$=k\left[\int_a^b f'(x)g'(x)dx+f(a)g(a)\right]=k(f,g)$ $\therefore$ 满足线性性.

③ $(f,f)=\int_a^b f(x)f'(x)\,dx+f(a)f(a)=\int_a^b f'(x)^2dx+f(a)^2$

$=(b-a)f'(\xi)^2+f(a)^2\geq 0$ 当且仅当 $f'(\xi)=0$ 且 $f(a)=0$ 时取等。

当 $f(a)=0$，$f'(\xi)=0$，$\xi\in[a,b]$ 时，必有 $f(x)\equiv 0$，$x\in[a,b]$

∴ 满足正定性

∴ (f,g) 可以构成 $C^1[a,b]$ 上的内积。

8. 对权函数 $\rho(x)=1+x^2$，区间 $[-1,1]$，试求首项系数为1的正交多项式 $\varphi_n(x)$，$n=0,1,2,3$.

解：施密特正交化公式：$\varphi_n(x)=x^n-\sum_{j=0}^{n-1}\frac{(\rho(x)x^n,\varphi_j(x))}{(\varphi_j(x),\varphi_j(x))}\varphi_j(x)$

$\varphi_0(x)=1$.

$$\varphi_1(x)=x-\frac{(\rho(x)x,1)}{(1,1)}=x-\frac{\int_{-1}^1 x(1+x^2)dx}{\int_{-1}^1(1+x^2)dx}=x$$

$$\varphi_2(x)=x^2-\frac{(\rho(x)x^2,1)}{(\rho(x),1)}-\frac{(\rho(x)x^2,x)}{(x,x)}x=x^2-\frac{\int_{-1}^1x^2(1+x^2)dx}{\int_{-1}^1(1+x^2)dx}-\frac{\int_{-1}^1x^2(1+x^2)x\,dx}{\int_{-1}^1x^2(1+x^2)dx}x$$

$$=x^2-\frac{2}{5}$$

$$\varphi_3(x)=x^3-\frac{(\rho(x)x^3,1)}{(1,1)}-\frac{(\rho(x)x^3,x)}{(\rho(x)x,x)}x-\frac{(\rho(x)x^3,x^2-\frac{2}{5})}{(\rho(x)(x^2-\frac{2}{5}),x^2-\frac{2}{5})}(x^2-\frac{2}{5})$$

$$=x^3-\frac{\int_{-1}^1x^3(1+x^2)dx}{\int_{-1}^1(1+x^2)dx}-\frac{\int_{-1}^1x^3(1+x^2)x\,dx}{\int_{-1}^1x^2(1+x^2)dx}x-\frac{\int_{-1}^1x^3(1+x^2)(x^2-\frac{1}{3})dx}{\int_{-1}^1(x^2-\frac{1}{3})^2(1+x^2)dx}(x^2-\frac{1}{3})$$

$$=\cancel{x^3-\frac{6}{5}x}\ x^3-\frac{9}{14}x$$

$x^3-\frac{9}{14}x$

∴ 正交多项式可表示为 $\varphi_0(x)=1$，$\varphi_1(x)=x$，$\varphi_2(x)=x^2-\frac{2}{5}$，$\varphi_3(x)=\cancel{\quad}$

12. 设 $f(x)=x^2+3x+2$，$x\in[0,1]$，试求 $f(x)$ 在 $[0,1]$ 上关于 $\rho(x)=1$，

~~~~ $\Phi=\mathrm{span}\{1,x\}$ 的最佳平方逼近多项式，若取 $\Phi=\mathrm{span}\{1,x,x^2\}$，

那么最佳平方逼近多项式是什么？

解：当 $\Phi=\mathrm{span}\{1,x\}$ 时，设 $\varphi(x)=a_0+a_1x$，$\varphi_0(x)=1$，$\varphi_1(x)=x$，

$(\varphi_0,\varphi_0)=\int_0^1 1\,dx=1$，$(\varphi_0,\varphi_1)=\int_0^1 x\,dx=\frac{1}{2}$，$(\varphi_1,\varphi_1)=\int_0^1 x^2\,dx=\frac{1}{3}$，

$(y,\varphi_0)=\int_0^1 x^2+3x+2\,dx=\frac{23}{6}$　$(y,\varphi_1)=\int_0^1(x^2+3x+2)x\,dx=\frac{9}{4}$

$$\therefore \begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}a_0\\ a_1\end{pmatrix}=\begin{pmatrix}\frac{23}{6}\\ \frac{9}{4}\end{pmatrix}\quad\therefore\begin{cases}a_0=\frac{11}{6}\\ a_1=4\end{cases}$$

$\therefore$ 此时的最佳平方逼近多项式为 $\varphi(x)=\frac{11}{6}+4x$

当 $\Phi=\mathrm{span}\{1,x,x^2\}$ 时，设 $\varphi(x)=a_0+a_1x+a_2x^2$，$\varphi_0(x)=1$，$\varphi_1(x)=x$，$\varphi_2(x)=x^2$

$(\varphi_0,\varphi_2)=\int_0^1 x^2\,dx=\frac{1}{3}$，$(\varphi_1,\varphi_2)=\int_0^1 x^3\,dx=\frac{1}{4}$，$(\varphi_2,\varphi_2)=\int_0^1 x^4\,dx=\frac{1}{5}$，

~~~~ $(y,\varphi_2)=\int_0^1(x^2+3x+2)x^2\,dx=\frac{97}{60}$

$$\therefore\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}\frac{23}{6}\\ \frac{9}{4}\\ \frac{97}{60}\end{pmatrix}\quad\therefore\begin{cases}a_0=2\\ a_1=3\\ a_2=1\end{cases}$$

$\therefore$ 此时的最佳平方逼近多项式为 $\varphi(x)=2+3x+x^2$ 即为原方程 $f(x)$

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13. 试求 $f(x)=x^3$ 在 $[-1,1]$ 上关于 $\rho(x)=1$ 的最佳平方逼近二次多项式。

解：$\Phi=\mathrm{span}\{1,x,x^2\}$，设 $\varphi(x)=a_0+a_1x+a_2x^2$．$\varphi_0(x)=1$，$\varphi_1(x)=x$，$\varphi_2(x)=x^2$．

$(\varphi_0,\varphi_0)=\int_{-1}^{1}1\,dx=2$，$(\varphi_0,\varphi_1)=\int_{-1}^{1}x\,dx=0$　$(\varphi_0,\varphi_2)=\int_{-1}^{1}x^2\,dx=\frac{2}{3}$

$(\varphi_1,\varphi_1)=\int_{-1}^{1}x^2\,dx=\frac{2}{3}$，$(\varphi_1,\varphi_2)=\int_{-1}^{1}x^3\,dx=0$　$(\varphi_2,\varphi_2)=\int_{-1}^{1}x^4\,dx=\frac{2}{5}$．

$(y,\varphi_0)=\int_{-1}^{1}x^3\,dx=0$，$(y,\varphi_1)=\int_{-1}^{1}x^4\,dx=\frac{2}{5}$，$(y,\varphi_2)=\int_{-1}^{1}x^5\,dx=0$．

$$\therefore \begin{bmatrix}2 & 0 & \frac{2}{3}\\ 0 & \frac{2}{3} & 0\\ \frac{2}{3} & 0 & \frac{2}{5}\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}=\begin{bmatrix}0\\ \frac{2}{5}\\ 0\end{bmatrix}\qquad \therefore\ \begin{aligned}a_0&=0\\ a_1&=\frac{3}{5}\\ a_2&=0\end{aligned}$$

$\therefore \varphi(x)=\frac{3}{5}x$．

14. 求函数 $f(x)$ 在指定区间上对于 $\Phi=\mathrm{span}\{1,x\}$ 的最佳平方逼近多项式。

(1) $f(x)=\frac{1}{x}$，$x\in[1,3]$

解：$\Phi=\mathrm{span}\{1,x\}$，设 $\varphi(x)=a_0+a_1x$．$\varphi_0(x)=1$，$\varphi_1(x)=x$．

$(\varphi_0,\varphi_0)=\int_{1}^{3}1\,dx=2$　$(\varphi_0,\varphi_1)=\int_{1}^{3}x\,dx=4$．

$(\varphi_1,\varphi_1)=\int_{1}^{3}x^2\,dx=\frac{26}{3}$，$(y,\varphi_0)=\int_{1}^{3}\frac{1}{x}\,dx=\ln 3$，$(y,\varphi_1)=\int_{1}^{3}1\,dx=2$．

$$\therefore \begin{bmatrix}2 & 4\\ 4 & \frac{26}{3}\end{bmatrix}\begin{bmatrix}a_0\\ a_1\end{bmatrix}=\begin{bmatrix}\ln 3\\ 2\end{bmatrix}\qquad \therefore \begin{cases}a_0=\frac{13}{2}\ln 3-6\\ a_1=-3\ln 3+3\end{cases}$$

$\therefore \varphi(x)=(\frac{13}{2}\ln 3-6)+(-3\ln 3+3)x$

(3) $f(x)=\cos\pi x, \quad x\in[0,1]$

解：$\Phi=\text{span}\{1,x\}$，设 $\varphi(x)=a_0+a_1x$，$\varphi_0(x)=1$，$\varphi_1(x)=x$

$(\varphi_0,\varphi_0)=\int_0^1 1\cdot dx=1$，$(\varphi_0,\varphi_1)=\int_0^1 x\,dx=\frac{1}{2}$ $(\varphi_1,\varphi_1)=\int_0^1 x^2dx=\frac{1}{3}$.

$(y,\varphi_0)=\int_0^1\cos\pi x\,dx=\frac{1}{\pi}\sin\pi x\Big|_0^1=0$

$(y,\varphi_1)=\int_0^1 x\cos\pi x\,dx=\frac{1}{\pi}\int_0^1 x\,d\sin\pi x=\frac{1}{\pi}x\sin\pi x\Big|_0^1-\frac{1}{\pi}\int_0^1\sin\pi x\,dx$

$=0+\frac{1}{\pi^2}\cos\pi x\Big|_0^1=-\frac{2}{\pi^2}$

$\therefore\begin{pmatrix}1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{3}\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}=\begin{pmatrix}0\\-\frac{2}{\pi^2}\end{pmatrix}$ $\therefore\begin{cases}a_0=\frac{12}{\pi^2}.\\a_1=-\frac{24}{\pi^2}\end{cases}$

$\therefore\varphi(x)=\cancel{\frac{12}{\pi^2}}\ \frac{12}{\pi^2}-\frac{24}{\pi^2}x$.

17. 已知实验数据如下：

x_i	19	25	31	38	44
y_i	19.0	32.3	49.0	73.3	97.8

用最小二乘法求形如 $y=a+bx^2$ 的经验公式，并计算经验误差。

解：$\Phi=\text{span}\{1,x^2\}$ 设 $\varphi(x)=a+bx^2$，$\varphi_0(x)=1$，$\varphi_1(x)=x^2$.

$(\varphi_0,\varphi_0)=\sum_{i=0}^4 1=5$ $(\varphi_0,\varphi_1)=\sum_{i=0}^4 x_i^2=5327$

$(\varphi_1,\varphi_1)=\sum_{i=0}^4 x^4=7277699$，

$\cancel{(y,\varphi_0)=271}$ $(y,\varphi_0)=\sum_{i=0}^4 y_i=271.4$ $(y,\varphi_1)=\sum_{i=0}^4 y_ix_i^2=369321.5$

$\therefore\begin{pmatrix}5&5327\\5327&7277699\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}271.4\\369321.5\end{pmatrix}$ 解得 $\begin{cases}a=0.9726046\\b=0.0500351\end{cases}$

$\therefore$ 经验公式可以表示为 $y = 0.9726046 + 0.0500351x^2$.

均方误差为 $\delta = \left[\sum_{j=0}^{4} (y(x_j) - y_j)^2\right]^{\frac{1}{2}} = 0.1226$.