数学推导

火兴辉

2025 年 3 月 9 日

1 无量纲量

1.1 密度比

$$s = \frac{\rho_p}{\rho_f} \tag{1}$$

where ρ_p is the particle density and ρ_f is the fluid density.

1.2 考虑浮力的有效重力加速度

$$\hat{g} = g(1 - \frac{1}{s}) \tag{2}$$

where g is the gravity acceleration.

$\sqrt{s}$ separates the settling velocity scale $\sqrt{s\hat{g}d}$ from the escape velocity scale $\sqrt{\hat{g}d}$ that grains need to exceed in order to escape the potential traps of the bed surface.

1.3 Galileo Number

$$Ga = \frac{d\sqrt{s\hat{g}d}}{\nu_f} \tag{3}$$

where d is the particle diameter and ν_f is the kinematic viscosity.

Ga controls the scaling of the dimensionless terminal grain settling velocity in quiescent flow $v_s^-/\sqrt{s\hat{g}d}$ (Camenen, 2007).

1.4 Shields Number

$$\Theta = \frac{\tau}{\rho_p\hat{g}d} \tag{4}$$

where τ is the fluid shear stress on the bed surface.

The Shields Number Θ is a measure for the ratio of the shear stress on the bed surface to the effective weight of the sediment particle (Pähtz et al., 2020).

1.5 空中颗粒承载量

The transport load M is the total mass of transported particles in the air per unit bed area. Non-dimensionalized by the particle density and diameter, it is defined as

$$M_* = \frac{M}{\rho_p d}. \tag{5}$$

1.6 流向颗粒通量

$$Q = M\overline{v_x} \tag{6}$$

where $\overline{v_x}$ is the streamwise average particle velocity. Non-dimensionalized as

$$\overline{v_{x*}} = \frac{\overline{v_x}}{\sqrt{s\hat{g}d}}. \tag{7}$$

while the non-dimensional streamwise particle transport rate is defined as

$$Q_* = \frac{Q}{\rho_p d\sqrt{s\hat{g}d}}. \tag{8}$$

2 输沙率表达式

Pähtz et al. (2020) proposed new expressions for M_* and Q_*.

$$M_* = (\Theta - \Theta_t)/\mu_b, \tag{9}$$

$$Q_* = M_*\overline{v_{x*t}}(1 + c_M M_*), \tag{10}$$

where μ_b is the bed friction coefficient, which approximates the ratio between the average streamwise momentum loss and vertical momentum of transported particles during their contacts with the bed surface (Pähtz et al., 2018). The subscripts t refers to the threshold value and Θ_t is the partical transport threshold (i.e., $M_* \to 0$ when $\Theta \to \Theta_t$).

Equation 10 consists of two additive contributions to Q_*: the term $M_*\overline{v_{x*t}}$ corresponds to the scaling of Q_* in the hypothetical case that collisions between transported particles are ommitted, and the term $c_M M_*\overline{v_{x*t}}$ accounts for such collisions.

Pähtz et al. (2020) found that Eq. 10 with $c_M = 1.7$ is universally valid across a wide range of equilibrium non-suspended transport simulations ($s \in [2.65, 2000] and Ga \in [5, 100]$) that satisfy $s^{1/2}Ga \geq 80$ for $s \leq 10$ (typical for fluvial environments) or $s^{1/2}Ga \geq$

200 for $s \leq 10$ (typical for aeolian environments). The value of Θ_t is fitted to experimental and numerical data, while μ_b and $\overline{v_{x*t}}$ are derived from semi-empirical relations: $\mu_b = 0.63$ (Pähtz and Durán, 2018) and $\overline{v_{x*t}} \approx 2\kappa^{-1}\sqrt{\Theta_t}$ (limited to $s^{1/4}Ga \geq 40$, Pähtz and Durán, 2018).

3 颗粒跃移模型

3.1 流场廓线

$$u_x = \frac{u^*}{\kappa} ln \frac{z}{z_0}, \tag{11}$$

where u_x is the wind speed, u^* is the friction velocity, $\kappa = 0.41$ is the von Karman constant, z is the height, and $z_0 = d_{90}/30$ is the roughness length.

We use non-dimensional arguments to rewrite Eq. 11 as

$$u_x = u^* f_u(z), \tag{12}$$

with

$$f_u(z) = \frac{1}{\kappa} ln \frac{z}{z_0}. \tag{13}$$

3.2 流场对颗粒的拖曳力

Define the fluid velocity as $\vec{u}$ and the particle velocity as $\vec{v}$. Considering the particle as a rigid sphere, the drag force exerted by the fluid on the particle can be expressed as

$$F_d = \frac{\pi}{8} \rho_f C_d d^2 |\vec{u} - \vec{v}| (\vec{u} - \vec{v}), \tag{14}$$

where C_d is the drag coefficient.

$$C_d = \left(\sqrt{C_d^\infty} + \sqrt{\frac{R_u^c}{R_u}} \right)^2, \tag{15}$$

where $R_u = |\vec{u} - \vec{v}| d/\nu_f$ is the particle Reynolds number based on the fluid-particle relative velocity, $R_u^c \approx 24$ is the critical Reynolds number above which the drag coefficient becomes almost constant, and $C_d^\infty \approx 0.5$ is the drag coefficient in the high Reynolds number limit.

In the fluid field, the grain acceleration $\vec{a}$ consists of a drag (superscript d), a gravity (superscript g), and a buoyancy (superscript b) component:

$$\vec{a} = \vec{a^d} + \vec{a^g} + \vec{a^b}. \tag{16}$$

In particular, if we only consider a 2D problem, acceleration in each direction writes

$$\begin{cases} a_x = a_x^d, \\ a_z = a_z^d + a_z^g + a_z^b = a_z^d - \hat{g}. \end{cases} \tag{17}$$

3.3 床面摩擦系数

The bed friction coefficient μ_b is linked to the streamwise and vertical components of the particle acceleration due to the non-contact forces via (Pähtz and Durán, 2018; Pähtz et al., 2020)

$$\mu_b = -\frac{\overline{a_x}}{\overline{a_z}} = \frac{\overline{a_x^d}}{\overline{a_z^d} - \hat{g}}. \tag{18}$$

The overbar denotes the particle concentration-weighted average, which is defined as

$$\overline{A} = \frac{\int_0^{z_{max}} \phi A \mathrm{d}z}{\int_0^{z_{max}} \phi \mathrm{d}z}. \tag{19}$$

where ϕ is the particle volume fraction, and z_{max} is the top of the transport layer. In fact, one can derive that $M = \int_0^{z_{max}} \phi \mathrm{d}z$.

Now we check the expressions for $\overline{a_x^d}$ and $\overline{a_z^d}$. To simplify them, a linearization is applied by approximating the difference $|\vec{u} - \vec{v}|$ by $\overline{u_x} - \overline{v_x}$. Then the drag acceleration is given by

$$\vec{a^d} = \frac{F_{dx}}{(1/6)\pi d^3 \rho_p} = \frac{3\Delta V}{4sd}\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{24\sqrt{s\hat{g}d}}{\Delta V Ga}}\right)^2 (\vec{u} - \vec{v}), \tag{20}$$

with

$$\Delta V = |\overline{u_x} - \overline{v_x}|. \tag{21}$$

So we have the weighted average of the x-component of the drag acceleration as

$$\overline{a_x^d} = \frac{3(\Delta V)^2}{4sd}\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{24\sqrt{s\hat{g}d}}{\Delta V Ga}}\right)^2. \tag{22}$$

Since the vertical component of the fluid velocity $u_z = 0$, and $\overline{v_z} = 0$ due to the mass conservation, we have

$$\overline{a_z^d} = \frac{3\Delta V}{4sd}\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{24\sqrt{s\hat{g}d}}{\Delta V Ga}}\right)^2 (\overline{u_z} - \overline{v_z}) = 0. \tag{23}$$

Finally, μ_b is given by

$$\mu_b = \frac{\overline{a_x^d}}{\hat{g}} = \frac{\Delta V}{v_s}, \tag{24}$$

where v_s is the particle velocity when $a_z^d = \hat{g}$. In other words, it is the terminal settling velocity.

V_s is the dimensionless form of v_s, which is useful in subsequent derivations. As $v_s = \Delta V/\mu_b$, V_s can be written as

$$V_s = \frac{v_s}{\sqrt{s\hat{g}d}} = \frac{\Delta V}{\mu_b\sqrt{s\hat{g}d}}. \tag{25}$$

Substituting Eq. 24 into Eq. 22, we have a quadratic equation for $\sqrt{\Delta V}$:

$$\Delta V\sqrt{\frac{1}{2}}+\sqrt{\frac{24\Delta V\sqrt{s\hat{g}d}}{Ga}}=\sqrt{\frac{4}{3}\mu_b s\hat{g}d}. \tag{26}$$

With the solution of ΔV, V_s can be derived as

$$V_s=\frac{1}{2\mu_b}\left(-\sqrt{\frac{24}{Ga}}+\sqrt{\frac{24}{Ga}+4\sqrt{\frac{2\mu_b}{3}}}\right)^2. \tag{27}$$

Note that v_s^- is distinct from v_s, which obeys a modified version of Eq. 27 with $\mu_b=1$ (Camenen, 2007).

$$\begin{aligned}\frac{\partial f}{\partial t}&=\frac{\partial}{\partial t}\int_a^b g(x,t)\mathrm{d}x && (28)\\ &=\int_a^b\frac{\partial g}{\partial t}\mathrm{d}x && (29)\end{aligned}$$

$$f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$$

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

4 推导过程