Quantum Computing Assignment 9 - Group 18

Rallabhandi, Anand Krishna Mustafa, Syed Husain
, Mohammed Kamran

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Exercise 9.1

(Phase Estimation)

(a) The Quantum Fourier Transform acts on a quantum state $\sum_{k=0}^{N-1} x_i |i\rangle$ and maps it to the quantum state $\sum_{k=0}^{N-1} y_i |i\rangle$ according to the formula:

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N} \qquad \textbf{(A)}$$

Similarly the formula for Inverse Quantum Fourier Transform is given by:

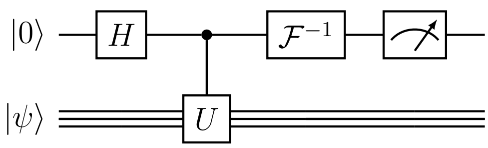
$$x_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k e^{-2\pi ijk/N} \qquad \textbf{(B)}$$

Applying **(A)** to a single qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:
$y_0 = \frac{1}{\sqrt{2}}\left(\alpha e^{2\pi i \frac{0\times 0}{2}} + \beta e^{2\pi i \frac{1\times 0}{2}}\right) = \frac{1}{\sqrt{2}}(\alpha+\beta)$ & $y_1 = \frac{1}{\sqrt{2}}\left(\alpha e^{2\pi i \frac{0\times 1}{2}} + \beta e^{2\pi i \frac{1\times 1}{2}}\right) = \frac{1}{\sqrt{2}}(\alpha-\beta)$
$U_{QFT}|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha+\beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha-\beta)|1\rangle = |\tilde{\psi}\rangle$ **(C)**

The above result is the same as applying the Hadamard Operator **(H)** on the qubit.
$\therefore$ Circuit for Forward Fourier Transform: $|\psi\rangle \; -\boxed{H}- \; |\tilde{\psi}\rangle$

Applying **(B)** on **(C)** :
$x_0 = \frac{1}{\sqrt{2}}\left(\frac{\alpha+\beta}{\sqrt{2}} e^{-2\pi i \frac{0\times 0}{2}} + \frac{\alpha-\beta}{\sqrt{2}} e^{-2\pi i \frac{0\times 1}{2}}\right) = \alpha$ & $x_1 = \frac{1}{\sqrt{2}}\left(\frac{\alpha+\beta}{\sqrt{2}} e^{-2\pi i \frac{1\times 0}{2}} + \frac{\alpha-\beta}{\sqrt{2}} e^{-2\pi i \frac{1\times 1}{2}}\right) = \beta$
$U_{IQFT}|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ **(D)**
The above result is the same as applying Hadamard Gate to the **(C)**.
$\therefore$ Circuit for Inverse Fourier Transform: $|\tilde{\psi}\rangle \; -\boxed{H}- \; |\psi\rangle$

(b)



Let $|-1\rangle$ & $|1\rangle$ be the eigen states of U with eigen values ± 1. We apply the above circuit operations and obtain the following states:
$|\psi_0\rangle = \sum\limits_{j=-1,1} c_i |0\rangle |j\rangle$: Before Applying Any Operation.
$|\psi_1\rangle = \sum\limits_{j=-1,1} c_i \frac{1}{\sqrt{2}} \sum\limits_{k=0,1} |k\rangle |j\rangle$: After Applying the Hadamard Gate.
$|\psi_2\rangle = \sum\limits_{j=-1,1} c_i \frac{1}{\sqrt{2}} \Big(|0\rangle + e^{2\pi i \varphi_i} |1\rangle \Big) |j\rangle$: After Applying Controlled-U Operator.
$|\psi_3\rangle = \sum\limits_{j=-1,1} c_i |\varphi_i\rangle |j\rangle$: After Applying Inverse Fourier Transform to obtain Phase φ

When performing the measuring operation on $|\psi_3\rangle$ the second qubit collapses into one of the eigenstates of U. When the first qubit is measured to be $|0\rangle$ the final state of second qubit is $|+1\rangle$, and when the first qubit is $|1\rangle$ it is $|-1\rangle$. This implies that Phase Estimation applied to a one bit register is equivalent to "Measuring an Operator" as shown in Tutorial 3.

Exercise 9.2

(**Order Finding**)

(a)

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{2\pi i s k/r} |u_s\rangle =$$

$$\frac{1}{r} \sum_{s=0}^{r-1} \sum_{k_2=0}^{r-1} e^{2\pi i s(k-k_2)/r} |x^{k_2}(modN)\rangle$$

$$= \sum_{k_2=0}^{r-1} \delta |x^{k_2}(modN)\rangle$$

$$= |x^k(modN)\rangle$$

(b)

$$7^1 \equiv 7 \, mod \, 15$$

$$7^2 \equiv 4 \, mod \, 15$$

$$7^3 \equiv 13 \, mod \, 15$$

$$7^4 \equiv 1 \, mod \, 15$$

Therefore $r = 4$

$$|\chi\rangle = \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle |x^j modN\rangle$$

For $t = 5$, we get

$$|\chi\rangle = \frac{1}{\sqrt{2^5}}[|0\rangle|7^0 mod15\rangle + |1\rangle|7^1 mod15\rangle + |2\rangle|7^2 mod15\rangle + |3\rangle|7^3 mod15\rangle + ...$$
$$.... + |31\rangle|7^{31} mod15\rangle]$$

$$|\chi\rangle = \frac{1}{\sqrt{2^5}}[|0\rangle|7^0 mod15\rangle + |1\rangle|7^1 mod15\rangle + |10\rangle|7^2 mod15\rangle + |11\rangle|7^2 mod15\rangle + ...$$
$$.... + |11111\rangle|7^{31} mod15\rangle]$$

(c) When we select $|4\rangle$ as the basis state for the second qubit, then we can only select values that produce 4 mod 15.
We know $7^2 = 4$ mod 15, as the period of function $f : k \to 7^k$ mod 31 is 4, then $\forall k \geq 1$ we select 7^{2+4k} mod 15.
From **(b)** we have 8 such powers of 7, ie; $7^2, 7^6, 7^{10}, 7^{14}, 7^{18}, 7^{22}, 7^{26}, \& \ 7^{30}$.
$|\chi'\rangle = \frac{1}{\sqrt{2^5}} \sum_{j=0}^{7} |2 + 4j\rangle |4\rangle$
$\|\chi'\| = \sqrt{\frac{1}{32} + \frac{1}{32} + \frac{1}{32} + 8 \text{ times } ...} = \sqrt{\frac{8}{32}} = \frac{1}{2}$
$\underline{\text{Normalized } |\chi'\rangle}$: $\frac{|\chi'\rangle}{\|\chi'\|} = \frac{2}{\sqrt{2^5}} \sum_{j=0}^{7} |2 + 4j\rangle |4\rangle = \frac{1}{\sqrt{2^3}} \sum_{j=0}^{7} |2 + 4j\rangle |4\rangle$

(d)

In [1]:
```python
import numpy as np
import math
import matplotlib.pyplot as plt
```

In [2]:
```python
%matplotlib inline
```

$chip = |\chi'\rangle = \frac{1}{\sqrt{8}} \sum_{j=0}^{7} |2+4j\rangle \otimes |4\rangle$

In [3]:
```python
chip = np.zeros(32)
for i in range(2, 31, 4):
    chip[i] = 1
#Normalizing the vector
chip = chip/math.sqrt(8)
print(chip)
```

```
[0.         0.         0.35355339 0.         0.         0.
 0.35355339 0.         0.         0.         0.35355339 0.
 0.         0.         0.35355339 0.         0.         0.
 0.35355339 0.         0.         0.         0.35355339 0.
 0.         0.         0.35355339 0.         0.         0.
 0.35355339 0.        ]
```

In [4]:
```python
Fchip = np.fft.ifft(chip, norm='ortho')
plt.plot(np.abs(Fchip)**2, '.')
```

Out[4]: [<matplotlib.lines.Line2D at 0x7fe51250ca60>]