Introduction to Quantum Computing
Assignment 3 - Group 18

Rallabhandi, Anand Krishna Mustafa, Syed Husain
, Mohammed Kamran

November 2020

Ex 3.1 Basis Transformation and Measurement

(a)

We need to show that,

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \tag{1}$$

is orthogonal, this implies we need to prove that

$$\langle+|+\rangle = 1, \langle-|-\rangle = 1, \langle+|-\rangle = 0$$

$$\langle+|+\rangle = \langle+\rangle^{\intercal}\langle+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)^{\intercal} * \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1$$

$$\langle-|-\rangle = \langle-\rangle^{\intercal}\langle-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)^{\intercal} * \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1$$

$$\langle+|-\rangle = \langle+\rangle^{\intercal}\langle-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)^{\intercal} * \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$$

(b)

$$|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|U_1\rangle = \frac{3}{5}|0\rangle + \frac{4i}{5}|1\rangle$$

$$|U_2\rangle = \frac{4}{5}|0\rangle - \frac{3i}{5}|1\rangle$$

$$\frac{5}{3} * |U_1\rangle + \frac{4}{5} * |U_2\rangle = \frac{25}{12}i|1\rangle, \quad i|1\rangle = \frac{4}{5}|U_1\rangle - \frac{3}{5}|U_2\rangle$$

$$\frac{5}{4} * |U_1\rangle + \frac{5}{3} * |U_2\rangle = \frac{25}{12}|0\rangle, \quad |0\rangle = \frac{3}{5}|U_1\rangle + \frac{4}{5}|U_2\rangle$$

$$|\psi\rangle = -(\frac{i}{5\sqrt{2}})|U_1\rangle + (\frac{7i}{5\sqrt{2}})|U_2\rangle$$

$$P_1 = |\frac{i}{5\sqrt{2}}^2| = \frac{1}{50}, \quad P_2 = |\frac{7i}{5\sqrt{2}}|^2 = \frac{49}{50}$$

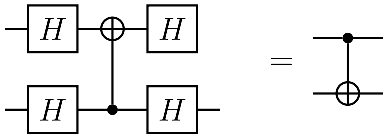
(c)

$$CNOT - INV = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

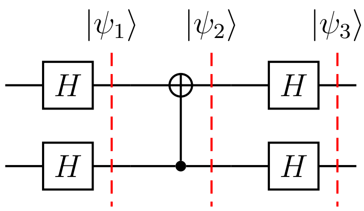
The left hand side of the diagram equation is $(H \otimes H)(CNOT - INV)(H \otimes H)$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$
$$= \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{bmatrix} = CNOT$$

Hence the following circuits are equivalent



Using the equivalent CNOT circuit to derive the following relations



For $|+\rangle |+\rangle \xrightarrow{\text{CNOT}} |+\rangle |+\rangle$

$$|\psi_1\rangle = (H|+\rangle) \otimes (H|+\rangle) \to |0\rangle |0\rangle \qquad (\ From\ \ Hint\ )$$

$$|\psi_2\rangle = (CNOT - INV)(|\psi_1\rangle) \to\ |0\rangle |0\rangle \quad (\ Target\ \ Qubit\ \ is\ \ 0\ )$$

$$|\psi_3\rangle = (H|0\rangle) \otimes (H|0\rangle) \to |+\rangle |+\rangle \qquad (\ From\ \ Hint\ )$$

For $|-\rangle |+\rangle \xrightarrow{\text{CNOT}} |-\rangle |+\rangle$

$$|\psi_1\rangle = (H|-\rangle) \otimes (H|+\rangle) \to |1\rangle |0\rangle \qquad (\ From\ \ Hint\ )$$

$$|\psi_2\rangle = (CNOT - INV)(|\psi_1\rangle) \to\ |1\rangle |0\rangle \quad (\ Target\ \ Qubit\ \ is\ \ 0\ )$$

$$|\psi_3\rangle = (H|1\rangle) \otimes (H|0\rangle) \to |-\rangle |+\rangle \qquad (\ From\ \ Hint\ )$$

For $|+\rangle |-\rangle \xrightarrow{\text{CNOT}} |-\rangle |-\rangle$

$$|\psi_1\rangle = (H|+\rangle) \otimes (H|-\rangle) \to |0\rangle |1\rangle \qquad (\ From\ \ Hint\ )$$

$$|\psi_2\rangle = (CNOT{-}INV)(|\psi_1\rangle) = (CNOT{-}INV\,|0\rangle) \otimes |1\rangle \to\ |1\rangle |1\rangle \quad (\ Target\ \ Qubit\ \ is\ \ 1\ )$$

$$|\psi_3\rangle = (H|1\rangle) \otimes (H|1\rangle) \to |-\rangle |-\rangle \qquad (\ From\ \ Hint\ )$$

For $|-\rangle |-\rangle \xrightarrow{\text{CNOT}} |+\rangle |-\rangle$

$$|\psi_1\rangle = (H|-\rangle) \otimes (H|-\rangle) \to |1\rangle |1\rangle \qquad (\ From\ \ Hint\ )$$

$$|\psi_2\rangle = (CNOT{-}INV)(|\psi_1\rangle) = (CNOT{-}INV\,|1\rangle) \otimes |1\rangle \to\ |0\rangle |1\rangle \quad (\ Target\ \ Qubit\ \ is\ \ 1\ )$$

$$|\psi_3\rangle = (H|0\rangle) \otimes (H|1\rangle) \to |+\rangle |-\rangle \qquad (\ From\ \ Hint\ )$$

Ex 3.2 The Stern-Gerlach Experiment

(a)

For a square matrix **A** the Eigen Values and Eigen Vector satisfy:

$$Av = \lambda v \qquad \textbf{(1)}$$

Where **v** represents the Eigen Vectors and $\boldsymbol{\lambda}$ represents the Eigen Values.

Assuming there exists a non zero Eigen Vector we first solve for $\boldsymbol{\lambda}$ with the determinant

$$|A - \lambda I| = 0 \qquad \textbf{(2)}$$

For Pauli-X Matrix :

$$\left| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \right| = 0$$

$$\lambda^2 - 1 = 0 \implies \lambda = \pm 1$$

Substitute $\lambda = 1$ in **(1)**

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\implies$ The Eigen Vector is a non-zero multiple of y=x

$\implies \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a possible eigen vector for $\lambda = +1$

Similarly, for $\lambda = -1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix}$$

$\implies$ The Eigen Vector is a non-zero multiple of y= - x

$\implies \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a possible eigen vector for $\lambda = -1$

$\therefore$ Resultant normalized Eigen Vectors for Pauli-X Matrix are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \& \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Similarly, for Pauli-Y Matrix:

$$\left| \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\implies \lambda = \pm 1$$

Substituting $\lambda = \pm 1$ in **(1)** $\implies$ normalized Eigen Vectors for Pauli-Y Matrix are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \quad \& \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

Similarly, for Pauli-Z Matrix:

$$\left|\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$$

$$\implies \lambda = \pm 1$$

Substituting $\lambda = \pm 1$ in **(1)** $\implies$ normalized Eigen Vectors for Pauli-Z Matrix are

$$\begin{pmatrix}1\\ 0\end{pmatrix} \quad \& \quad \begin{pmatrix}0\\ 1\end{pmatrix}$$

(b)

$|\psi\rangle = |0\rangle$ (Given)

Using Pauli-X Eigen Vectors as orthonormal basis

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = |+\rangle \quad \& \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = |-\rangle$$

Calculating Measurement Operator w.r.t computational basis $\{+,-\}$

$$M_+ := |+\rangle\langle+| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$M_- := |-\rangle\langle-| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

$$p(+) = \langle 0| M_+^\dagger M_+ |0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2} = 50\%$$

$$p(-) = \langle 0| M_-^\dagger M_- |0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2} = 50\%$$

State after measurement corresponds to

$$\frac{M_m|\psi\rangle}{\|M_m|\psi\rangle\|}$$

$$\implies \frac{M_+|0\rangle}{\|M_+|0\rangle\|} = \frac{\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}}{\left\|\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\right\|} = \frac{\frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) = |+\rangle$$

$$\implies \frac{M_-|0\rangle}{\|M_-|0\rangle\|} = \frac{\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}}{\left\|\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\right\|} = \frac{\frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big) = |-\rangle$$

$\therefore$ The computed probabilities and output states are consistent w.r.t. the outputs of the non-uniform magnet applied in the X-direction

(c)

In Schematic Diagram 3: When electrons in $|+\rangle$ superposition-state are passed through a non-uniform magnetic field along the Z-direction it is observed that 50% of the electrons deflect up onto the metal-screen, i.e.; demonstrating a positive-spin ($|0\rangle$), while 50% deflect downward, demonstrating a negative spin ($|1\rangle$).

These results can be verified numerically as follows:

$|\psi\rangle = |+\rangle$ (Given)

Using Pauli-Z Eigen Vectors as orthonormal basis

$$u_1 = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle \quad \& \quad u_2 = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$$

Calculating Measurement Operator w.r.t computational basis $\{u_1, u_2\}$

$$M_0 := |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$M_1 := |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$p(0) = \langle +| M_0^\dagger M_0 |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2} = 50\%$$

$$p(1) = \langle +| M_1^\dagger M_1 |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2} = 50\%$$

State after measurement corresponds to

$$\frac{M_m |\psi\rangle}{\| M_m |\psi\rangle \|}$$

$$\implies \frac{M_0 |+\rangle}{\| M_0 |+\rangle \|} = \frac{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}{\left\| \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \right\|} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$$

$$\implies \frac{M_1 |+\rangle}{\| M_1 |+\rangle \|} = \frac{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}{\left\| \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \right\|} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$$

In Schematic Diagram 3: Orienting the last magnetic field along the x-direction from z-direction. $|\psi\rangle = |+\rangle$ (Given)

$$M_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \& \quad M_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$p(+) = \langle +| M_+^\dagger M_+ |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 = 100\%$$

$$p(-) = \langle +| M_-^\dagger M_- |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0\%$$

State after measurement corresponds to

$$\implies \frac{M_+ |+\rangle}{\| M_+ |+\rangle \|} = \frac{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\left\| \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$