Quantum Computing Assignment 6 - Group 18

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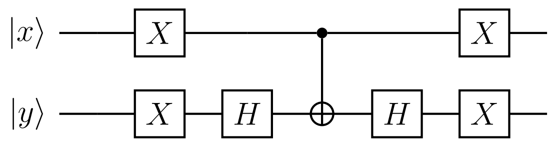
December 16, 2020

Exercise 6.1

(Two bit Quantum Search)

(a)



We're given the negated phase gate appearing in the Grover Operator, ie; $-2(|00\rangle\langle 00| - I)$
This term can be expanded into matrix form as follows :

$$-2\begin{pmatrix}1\\0\\0\\0\end{pmatrix}_{4\times 1} \times \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}_{1\times 4} - \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}_{4\times 4} = \begin{pmatrix}-1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}_{4\times 4} \quad \textbf{(A)}$$

Implementing the above circuit on some arbitrary states $|x\rangle$ & $|y\rangle$.
Consider $|x\rangle = 0$ & $|y\rangle = 0$:

$$|00\rangle \xrightarrow{X\otimes X} |11\rangle \xrightarrow{I\otimes H} |1\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \xrightarrow{\text{CNOT}} |0\rangle\left(\frac{|1\rangle - |0\rangle}{\sqrt{2}}\right) \xrightarrow{I\otimes H} -|11\rangle \xrightarrow{X\otimes X} -|00\rangle \quad \textbf{(B)}$$

Consider $|x\rangle = 0$ & $|y\rangle = 1$:

$$|01\rangle \xrightarrow{X\otimes X} |10\rangle \xrightarrow{I\otimes H} |1\rangle\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \xrightarrow{\text{CNOT}} |1\rangle\left(\frac{|1\rangle + |0\rangle}{\sqrt{2}}\right) \xrightarrow{I\otimes H} |10\rangle \xrightarrow{X\otimes X} |01\rangle \quad \textbf{(C)}$$

Consider $|x\rangle = 1$ & $|y\rangle = 0$:

$$|10\rangle \xrightarrow{X\otimes X} |01\rangle \xrightarrow{I\otimes H} |0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \xrightarrow{\text{CNOT}} |0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \xrightarrow{I\otimes H} |01\rangle \xrightarrow{X\otimes X} |10\rangle \quad \textbf{(D)}$$

Consider $|x\rangle = 1$ & $|y\rangle = 1$:

$$|11\rangle \xrightarrow{X\otimes X} |00\rangle \xrightarrow{I\otimes H} |0\rangle\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \xrightarrow{\text{CNOT}} |0\rangle\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \xrightarrow{I\otimes H} |00\rangle \xrightarrow{X\otimes X} |11\rangle \quad \textbf{(E)}$$

The four results **(B)**, **(C)**, **(D)**, & **(E)** are the column vectors of the result **(A)**

$\therefore$ The result of the above circuit is equivalent to the negated phase gate appearing in the Grover Operator

(b)

Given M = 1, N = 4, & $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{M}{N}}$

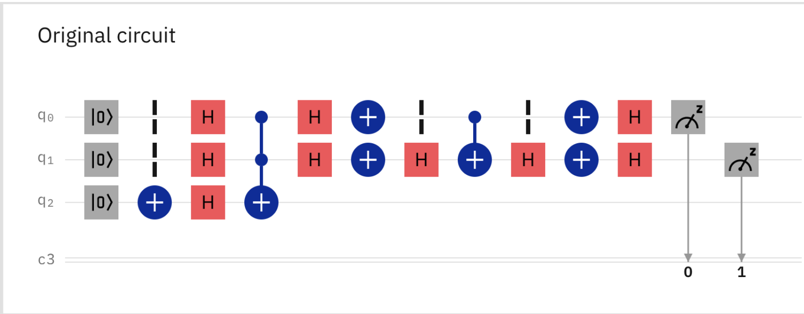
$\implies \frac{\theta}{2} = \sin^{-1}\left(\sqrt{\frac{1}{4}}\right) \implies \frac{\theta}{2} = \sin^{-1}\left(\pm\frac{1}{2}\right) \implies \frac{\theta}{2} = 30° \implies \theta = 60°$

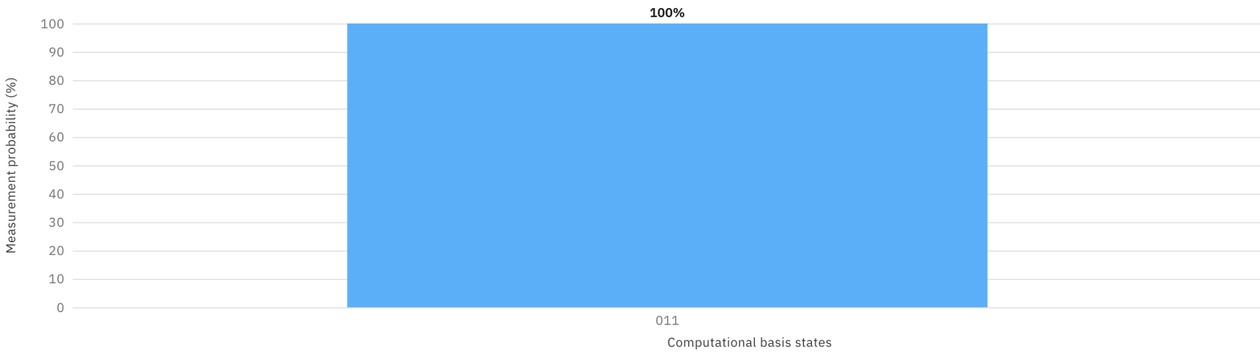
Given $G^k |\psi\rangle = \cos\left((\frac{1}{2}+k)\theta\right)|\alpha\rangle + \sin\left((\frac{1}{2}+k)\theta\right)|\beta\rangle$

Consider single application of Grover's Operator, i.e; k=1.

$\implies G|\psi\rangle = \cos\left(\frac{3}{2}\theta\right)|\alpha\rangle + \sin\left(\frac{3}{2}\theta\right)|\beta\rangle \implies G|\psi\rangle = \cos(90°)|\alpha\rangle + \sin(90°)|\beta\rangle$
$\implies G|\psi\rangle = |\beta\rangle$

(c)





Exercise 6.2

(Bloch Sphere for mixed-state qubits)

(a)

A density matrix is hermitian, thus

$$\rho = \begin{bmatrix} a & b* \\ b* & d \end{bmatrix} a = \frac{1+v_3}{2}, d = \frac{1-v_3}{2}, b = \frac{v_1 - i*v_2}{2}$$

$$\rho = \frac{1}{2} * \begin{bmatrix} 1+v_3 & v_1 - i*v_2 \\ v_1 + i*v_2 & 1-v_3 \end{bmatrix} = \frac{I + r.\sigma}{2}$$

Computing eigenvalues of this matrix, we get

$$det = (\frac{1}{2} * \begin{vmatrix} v_3 + 1 - \lambda & v_1 - i * v_2 \\ v1 + i * v_2 & -v_3 + 1 - \lambda \end{vmatrix})$$

$$\frac{1}{2} * [(1-\lambda)^2 - (v_3)^2 - v_1^2 - v_2^2] = 0$$

$$\lambda = 1 \pm \sqrt{v_1^2 + v_2^2 + v_3^3}$$

$$\lambda = 1 \pm |r|$$

Since ρ is Positive, we get

$$1 - |r| >= 0$$

$$|r| <= 1$$

Thus an arbitary density matrix ρ in a mixed state can be written in the form

$$\rho = \frac{I + r.\sigma}{2}$$

(b)

Since state ρ is pure, we get

$$tr[\rho^2] = 1$$

$$tr[\frac{I + r.\sigma}{2} * \frac{I + r.\sigma}{2}] = 1$$

$$tr[\frac{1}{4} * \begin{bmatrix} 1 + v_3 & v_1 - i * v_2 \\ v_1 + i * v_2 & 1 - v_3 \end{bmatrix} * \begin{bmatrix} 1 + v_3 & v_1 - i * v_2 \\ v_1 + i * v_2 & 1 - v_3 \end{bmatrix}] = 1$$

$$\frac{1}{4} * [(1 + v_3)^2 + 2(v_1^2 + v_2^2) + (1 - v_3)^2] = 1$$

$$2 + 2(v_1^2 + v_2^2 + v_3 * 2) = 4$$
$$|r|^2 = 1$$

$$|r| = 1$$

Thus a state is pure if and only if $|r| = 1$

(c)

For a pure state $|\psi\rangle$ we know that the density opertor is given by $\rho = |\psi\rangle \langle\psi|$.
Consider $|\psi\rangle = e^{i\gamma}\Big(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \Big)$.
We compute $|\psi\rangle\langle\psi|$ as follows:

$$|\psi\rangle\langle\psi| = e^{i\gamma}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \times e^{i\gamma}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\implies |\psi\rangle\langle\psi| = e^{2i\gamma}\begin{pmatrix} cos^2\left(\frac{\theta}{2}\right) & e^{-i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\implies |\psi\rangle\langle\psi| = \begin{pmatrix} 1+\cos(\theta) & e^{-i\varphi}\sin(\theta) \\ e^{i\varphi}\sin(\theta) & 1-\cos(\theta) \end{pmatrix} \qquad \textbf{(A)}$$

The phase term is inconsequential hence removed.

We applied the following trigonometric properties to obtain **(A)**.

$\sin(\theta) = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$, $\quad 1 + cos(\theta) = \cos^2(\frac{\theta}{2})$, and $\quad 1-\cos(\theta) = \sin^2(\frac{\theta}{2})$

Consider $\rho = \frac{1}{2}\Big(1 + r\sigma\Big) \implies \rho = \begin{pmatrix} 1+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1-r_3 \end{pmatrix}$ $\qquad$ **(B)**

Comparing **(A)** & **(B)** $\implies r = \begin{pmatrix} sin(\theta)cos(\varphi) \\ sin(\theta)sin(\varphi) \\ cos(\theta) \end{pmatrix}$

$|r| = \sqrt{\sin^2(\theta)\cos^2(\varphi) + \sin^2(\theta)\sin^2(\varphi) + cos^2(\theta)} = 1$

Hence r coincides with $|\psi\rangle$ on the Bloch Sphere.