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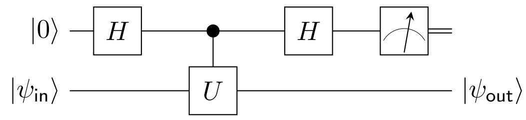
Tutorial 3 (Measuring an operator[1])
Suppose U is a single qubit operator with eigenvalues ± 1, so that U is both Hermitian and unitary, i.e., it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. Show that this is implemented by the following quantum circuit:



This exercise requires the concept of an orthogonal projection: a square matrix $P \in \mathbb{C}^{n \times n}$ is called an *orthogonal projection matrix* if P is Hermitian ($P^\dagger = P$) and $P^2 = P$, i.e., applying P a second time does not change the result any more. Note that a graphical projection is a special case of this abstract definition.

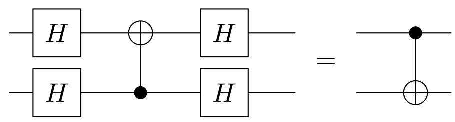
Exercise 3.1 (Basis transformation and measurement)

(a) Verify that the qubit states $|+\rangle$, $|-\rangle$ defined as $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ are orthonormal, i.e., that $\langle +|+\rangle = 1$, $\langle -|-\rangle = 1$, $\langle +|-\rangle = 0$.

(b) Compute the probabilities when measuring $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ with respect to the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ given by $|u_1\rangle = \frac{3}{5}|0\rangle + i\frac{4}{5}|1\rangle$ and $|u_2\rangle = \frac{4}{5}|0\rangle - i\frac{3}{5}|1\rangle$.

Hint: You can obtain the coefficients of $|\psi\rangle$ with respect to these basis states by computing the inner products $\langle u_j|\psi\rangle$ for $j = 1, 2$.

(c) The role of the control and target qubit of a CNOT gate can be reversed by switching to a different basis! First show that



with H the Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Use this identity to derive the following relations:

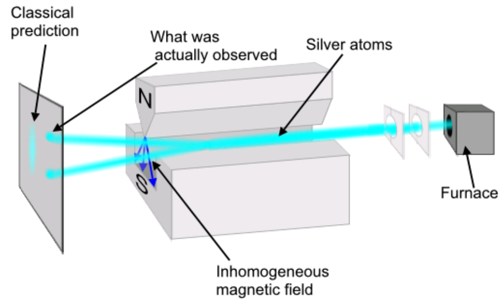
$$
\begin{aligned}
|+\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|+\rangle \\
|-\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|+\rangle \\
|+\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|-\rangle \\
|-\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|-\rangle
\end{aligned}
$$

with $|\pm\rangle$ as defined above. In other words, with respect to the $|\pm\rangle$ basis, the second qubit assumes the role of the control and the first qubit the role of the target.

Hint: Use that $H\,|+\rangle = |0\rangle$ and $H\,|-\rangle = |1\rangle$, and conversely $H\,|0\rangle = |+\rangle$ and $H\,|1\rangle = |-\rangle$.

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), exercise 4.34

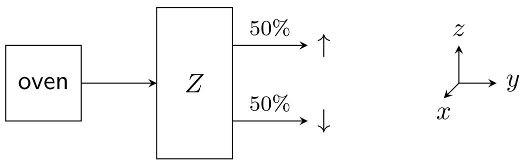
Exercise 3.2 (The Stern-Gerlach experiment)
The Stern-Gerlach experiment is a fundamental experiment in the history of quantum mechanics, leading to the insight that electrons have an intrinsic, quantized spin degree of freedom. Otto Stern conceived the experiment in 1921, and conducted it together with Walther Gerlach in 1922.
The setup is illustrated on the right. An oven (furnace) sends a beam of hot atoms through an inhomogeneous magnetic field, which causes the atoms to be deflected; the atoms are finally detected on a screen. The original experiment was conducted with silver atoms, but for our purpose it is simpler to discuss an analogous experiment with hydrogen atoms, which was performed in 1927.



https://commons.wikimedia.org/wiki/File:Stern-Gerlach_experiment.PNG

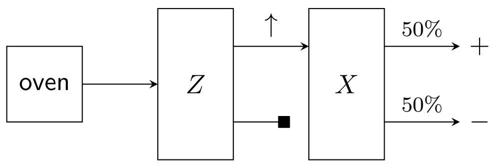
Based on classical physics, the electron orbiting around the proton in a hydrogen atom can be regarded as small magnetic dipole. One would then expect a continuous distribution of deflection angles, since the dipole axes are oriented randomly in space. Quantum mechanics predicts zero magnetic dipole moment for the hydrogen atom, and correspondingly the beam should not be deflected at all. Instead, a splitting into two beams was observed in the experiment.

We use the following schematic to summarize the Stern-Gerlach experiment:

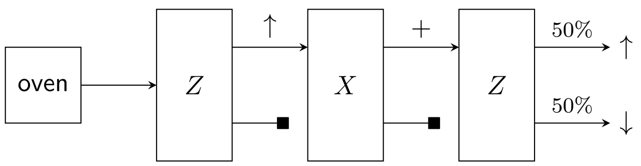


The coordinate system is chosen such that the beam propagates in y-direction. The inhomogeneous magnetic field (which we take to be oriented along the z-direction) splits the beam into two parts, one deflected up and the other down. Based on this description, one could hypothesize that each electron carries a classical bit of information, which specifies whether the atom goes up or down.

Now suppose we block the lower beam and send the upper beam through another inhomogeneous magnetic field, which is oriented along the x-direction. Classically, a dipole pointing in z-direction has zero moment in the x-direction, so one might expect that the final output is a single peak. Instead, experimentally one finds again two peaks, which we label $+$ and $-$:



Thus maybe each electron carries two classical bits of information, for selecting $\uparrow$ or $\downarrow$ and $+$ or $-$? If this was the case, and the electrons retained this information, then sending one beam of the previous output through another z-oriented field should result in a single beam deflected upwards. Instead, again two beams of equal intensity are observed:



Without any knowledge of quantum mechanics, it appears indeed challenging to invent a model explaining these observations.

Conversely, in the following we investigate the predictions of quantum mechanics when identifying the electronic spin as qubit, with $|0\rangle$ assigned to $\uparrow$ and $|1\rangle$ assigned to $\downarrow$. The inhomogeneous magnetic fields oriented along z- and x-direction measure the spin with respect to the eigenstates of the Pauli-Z and Pauli-X matrices, respectively.

(a) Compute the eigenvalues and normalized eigenvectors of the Pauli-X, Y and Z matrices.

(b) Calculate the probabilities when measuring $|0\rangle$ with respect to the eigenvectors of X, and compare your results with the second schematic above.

(c) Explain the experimental observations of the third schematic setup. What would happen when orienting the last magnetic field along the x-direction instead of the z-direction?