Quantum Computing Assignment 11 - Group 18

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Exercise 11.1

(**Fidelity of the Amplite Damping Channel**)

(a) The Operator-Sum representation of Amplitute Damping is given by

$$\varepsilon_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

$$F(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)) = tr\Big(\sqrt{\langle\psi| \sum_k E_k |\psi\rangle \langle\psi| E_k^\dagger |\psi\rangle}\Big)$$

Consider $|\psi\rangle = \begin{pmatrix}\alpha & \beta\end{pmatrix}^T$, where $E_0 = \begin{pmatrix}1 & 0 \\ 0 & \sqrt{1-\gamma}\end{pmatrix}$ & $E_1 = \begin{pmatrix}0 & \sqrt{\gamma} \\ 0 & 0\end{pmatrix}$

$\implies F(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)) = tr\left(\sqrt{\underbrace{\langle\psi| E_0 |\psi\rangle}_{\text{A}} \underbrace{\langle\psi| E_0^\dagger |\psi\rangle}_{\text{B}} + \underbrace{\langle\psi| E_1 |\psi\rangle}_{\text{C}} \underbrace{\langle\psi| E_1^\dagger |\psi\rangle}_{\text{D}}}\right)$

$\implies A = \begin{pmatrix}\alpha & \beta\end{pmatrix} \begin{pmatrix}1 & 0 \\ 0 & \sqrt{1-\gamma}\end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\sqrt{1-\gamma}\end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\alpha^2 + \beta^2\sqrt{1-\gamma}\end{pmatrix}$

$\implies B = \begin{pmatrix}\alpha & \beta\end{pmatrix} \begin{pmatrix}1 & 0 \\ 0 & \sqrt{1-\gamma}\end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\sqrt{1-\gamma}\end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\alpha^2 + \beta^2\sqrt{1-\gamma}\end{pmatrix}$

$\implies C = \begin{pmatrix}\alpha & \beta\end{pmatrix} \begin{pmatrix}0 & \sqrt{\gamma} \\ 0 & 0\end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \alpha\beta\sqrt{\gamma}$ & $D = \begin{pmatrix}\alpha & \beta\end{pmatrix} \begin{pmatrix}0 & 0 \\ \sqrt{\gamma} & 0\end{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \alpha\beta\sqrt{\gamma}$

Since $A = B$ & $C = D$

$$\implies tr\left(\sqrt{\langle\psi| E_0 |\psi\rangle \langle\psi| E_0^\dagger |\psi\rangle + \langle\psi| E_1 |\psi\rangle \langle\psi| E_1^\dagger |\psi\rangle}\right) = tr\Big(\sqrt{\sum_k |\langle\psi| E_k |\psi\rangle|^2}\Big)$$

Solving $F\big(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)\big)$ using A, B, C, & D

$\implies F\big(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)\big) = tr\left(\sqrt{\Big(\alpha^2 + \beta^2\sqrt{1-\gamma}\Big)^2 + \Big(\alpha\beta\sqrt{\gamma}\Big)^2}\right)$

$\implies F\big(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)\big) = tr\left(\sqrt{\Big(\alpha^2 + \beta^2\Big)^2 - \beta^2\Big((\alpha^2+\beta^2)\gamma\Big)}\right)$

Consider $|\alpha|^2 = \cos^2(\frac{\theta}{2})$ & $|\beta|^2 = \sin^2(\frac{\theta}{2})$

$\implies F\big(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)\big) = tr\left(\sqrt{1 - \sin^2(\frac{\theta}{2})\gamma}\right)$

To minimize $F\big(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)\big)$, $\sin^2(\frac{\theta}{2})\gamma$ should take the maximum value

$\implies F_{min}\big(|\psi\rangle, \varepsilon_{AD}(|\psi\rangle\langle\psi|)\big) = tr\left(\sqrt{1-\gamma}\right) = \sqrt{1-\gamma}$ $\quad\big(\because \sin^2(\frac{\theta}{2}) = 1, \theta = n\pi\ \forall\ \text{n}\ \epsilon\ \mathbb{Z} - \{0\}\big)$

Exercise 11.2

(Pauli Group & Check Matrix)

(a) Let us verify for 3 cases

$$\alpha = 1, \beta = 2$$
$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\sigma_1\sigma_2$$

$$\alpha = 1, \beta = 3$$
$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\sigma_1\sigma_3$$

$$\alpha = 2, \beta = 3$$
$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\sigma_3\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\sigma_2\sigma_3$$

(b)

$$g = X_1, g' = Z_1$$
$$[g, g'] = X_1.Z_1$$
$$[g', g] = Z_1.X_1 = -X_1.Z_1 = AntiCommute$$

$$g = X_1, g' = Z_3$$
$$[g, g'] = (X_1 \otimes I \otimes I).(I \otimes I \otimes Z_3) = X \otimes I \otimes Z$$
$$[g', g] = (I \otimes I \otimes Z_3).(X_1 \otimes I \otimes I) = X \otimes I \otimes Z = Commute$$

$$g = X_1X_2X_3, g' = Y_2Z_3$$
$$[g, g'] = (X_1 \otimes X_2 \otimes X_3).(I \otimes Y_2 \otimes Z_3) = X \otimes XY \otimes XZ$$
$$[g', g] = (I \otimes Y_2 \otimes Z_3).(X_1 \otimes X_2 \otimes X_3) = X \otimes YX \otimes ZX = X \otimes -XY \otimes -XZ = X \otimes XY \otimes XZ$$
$$= Commute$$

(c)

$$g_1 = X_1Z_2Z_3X_4$$
$$g_2 = X_2Z_3Z_4X_5$$
$$g_3 = X_1X_3Z_4Z_5$$
$$g_4 = Z_1X_3X_4Z_5$$

$$\text{Check Matrix} = \begin{pmatrix} r(g_1) \\ r(g_2) \\ r(g_3) \\ r(g_4) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Given $r(X) = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $r(Y) = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $r(Z) = \begin{pmatrix} 0 & 1 \end{pmatrix}$, & $r(I) = \begin{pmatrix} 0 & 0 \end{pmatrix}$,

1. Consider $g = I$ & $g' = I$
$I \times I = I$ & $I + I = I$ (Following Modulo 2 Addition)

$$\text{Hence } r(I) + r(I) = r(I \times I)$$

2. Consider $g = X$ & $g' = X$
$X \times X = I$ & $r(X) + r(X) = \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$

$$\text{Hence } r(X) + r(X) = r(X \times X)$$

Similarly $Y \times Y = I$ & $Z \times Z = I$
Hence $r(Y) + r(Y) = r(Y \times Y)$ & $r(Z) + r(Z) = r(Z \times Z)$

3. Consider $g = X$ & $g' = Y$
$X \times Y = iZ = Z$ (Absorbing Prefactor Term)
$r(X) + r(Y) = \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} = r(Z)$

$$\text{Hence } r(X) + r(Y) = r(X \times Y)$$

4. Consider $g = Y$ & $g' = Z$
$Y \times Z = iX = X$ (Absorbing Prefactor Term)
$r(Y) + r(Z) = \begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} = r(X)$

$$\text{Hence } r(Y) + r(Z) = r(Y \times Z)$$

5. Consider $g = X$ & $g' = Z$
$X \times Z = -iY = Y$ (Absorbing Prefactor Term)
$r(X) + r(Z) = \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} = r(Y)$

$$\text{Hence } r(X) + r(Z) = r(X \times Z)$$

From the above results we can conclude $r(g) + r(g') = r(gg') \ \forall \ g, g' \ \epsilon \ G_n$

(e) <u>Case 1</u>: Consider $g = g'$

Let $g = g' = I \implies \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \equiv 0 \bmod 2$

Let $g = g' = X \implies \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \equiv 0 \bmod 2$

Let $g = g' = Y \implies \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \equiv 0 \bmod 2$

Let $g = g' = Z \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \equiv 0 \bmod 2$

Case 2: Consider $g \neq g'$

Let $g = X$ & $g' = Y \implies \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 1 \equiv 1 \bmod 2$

Let $g = X$ & $g' = Z \implies \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \equiv 1 \bmod 2$

Let $g = Y$ & $g' = Z \implies \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \equiv 1 \bmod 2$

Let $g = Y$ & $g' = X \implies \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1 \equiv 1 \bmod 2$

Let $g = Z$ & $g' = X \implies \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1 \equiv 1 \bmod 2$

Let $g = Z$ & $g' = Y \implies \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 1 \equiv 1 \bmod 2$

Verifying results for "Case 2"

$XY = \begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$, while $YX = \begin{pmatrix}-i & 0\\0 & i\end{pmatrix} \implies [X, Y] \neq 0$

$XZ = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$, while $ZX = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} \implies [X, Z] \neq 0$

$YZ = \begin{pmatrix}0 & i\\i & 0\end{pmatrix}$, while $ZY = \begin{pmatrix}0 & -i\\-i & 0\end{pmatrix} \implies [Y, Z] \neq 0$