Quantum Computing Assignment 1 - Group 18

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Exercise 1.1

(Bloch sphere and single qubit quantum gates)

(a) $|\psi\rangle = e^{i\gamma}[\cos(\frac{\theta}{2})|o\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle]$

$|\psi\rangle = i[\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}i|1\rangle]$

$e^{i\gamma} = i$ This implies $\gamma = \frac{\pi}{2}$

$\cos(\frac{\theta}{2}) = \frac{1}{2}$ This implies $\theta = \frac{2\pi}{3}$

$e^{i\phi}\sin(\frac{\theta}{2}) = \frac{\sqrt{3}}{2}i$ This implies $\phi = \frac{\pi}{2}$

(b) Rotation operation about Pauli X Matrix is given by

$$R_X(\theta) := e^{\frac{-i\theta X}{2}} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X$$

Based on the above

$$R_x(\frac{2\pi}{3}) = \cos\frac{\pi}{3}I - i\sin\frac{\pi}{3}X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}_{2\times2} - \frac{\sqrt{3}i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2\times2} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}_{2\times2} \quad \textbf{(A)}$$

Given $|\phi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2}i \end{pmatrix}_{2\times1}$ **(B)**

(A) times **(B)** gives, $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}_{2\times2} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2}i \end{pmatrix}_{2\times1} = \frac{1}{4}\begin{pmatrix} 1 & 3 \\ -\sqrt{3}i & \sqrt{3}i \end{pmatrix}_{2\times1}$

(c) Since the Hadamard gate (H) is a Unitary Matrix It can be represented as follows

$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_z(\delta)$

Simplifying the above equations through Matrix Multiplication we get the following

$U = \begin{pmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\gamma/2 & -e^{i(\alpha-\beta/2+\delta/2)}\sin\gamma/2 \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)}\cos\gamma/2 \end{pmatrix}$

The Hadamard gate is defined as

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Since H and U represent the same matrix, we equate element wise and arrive at

$$e^{i(\alpha-\beta/2-\delta/2)}\cos\gamma/2 = \frac{1}{\sqrt{2}} \tag{1}$$
$$-e^{i(\alpha-\beta/2+\delta/2)}\sin\gamma/2 = \frac{1}{\sqrt{2}} \tag{2}$$
$$e^{i(\alpha+\beta/2-\delta/2)}\sin\gamma/2 = \frac{1}{\sqrt{2}} \tag{3}$$
$$e^{i(\alpha+\beta/2+\delta/2)}\cos\gamma/2 = -\frac{1}{\sqrt{2}} \tag{4}$$

From the above equations it can be inferred that $\gamma = \pi/2$

Simplifying (1)-(4) we arrive at

$$e^{i(\alpha-\beta/2-\delta/2)} = 1 \tag{5}$$
$$e^{i(\alpha-\beta/2+\delta/2)} = -1 \tag{6}$$
$$e^{i(\alpha+\beta/2-\delta/2)} = 1 \tag{7}$$
$$e^{i(\alpha+\beta/2+\delta/2)} = -1 \tag{8}$$

From (5),(7) using $e^{i0} = 1$

$\alpha - \beta/2 - \delta/2 = 0$
$\alpha + \beta/2 - \delta/2 = 0$

From (6),(8) using $e^{i\pi} = -1$

$\alpha - \beta/2 + \delta/2 = \pi$
$\alpha + \beta/2 + \delta/2 = \pi$

On solving the above equations we get

$\alpha = \pi/2, \beta = 0, \delta = \pi$

$\therefore H = e^{i\pi/2} R_x(0) R_y(\pi/2) R_z(\pi)$

Exercise 1.2

(Multiple qubits and tensor products)

(a) Y $= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ Z $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Y $\otimes Z = \begin{pmatrix} y_{11}Z & y_{12}Z \\ y_{21}Z & y_{22}Z \end{pmatrix}$

$$= \begin{pmatrix} y_{11}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} & y_{12}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \\ y_{21}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} & y_{22}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} & -i\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \\ i\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} & 0\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

(b) Code to Compute Kronecker Product of Pauli Y and Z

```
import numpy as np

# Pauli Y
Y = np.array([[0, -np.complex(0,1)], [np.complex(0,1), 0]])
# Pauli Z
Z = np.array([[1, 0], [0, -1]])

# Computing the Kronecker product
kron_YZ = np.kron(Y, Z)
print(kron_YZ)
```

Output

```
[[ 0.+0.j   0.+0.j   0.-1.j   0.-0.j]
 [ 0.+0.j  -0.+0.j   0.-0.j   0.+1.j]
 [ 0.+1.j   0.+0.j   0.+0.j   0.+0.j]
 [ 0.+0.j  -0.-1.j   0.+0.j  -0.+0.j]]
```

(c) Evaluating $(Y \otimes Z)(|v\rangle \otimes |w\rangle)$

$$Given, \quad v = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}_{2\times1} \quad and \quad \mathrm{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times1}$$

Performing Kronecker Product according to Eqn(3),

$$(v \otimes w) = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}_{2\times1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times1} = \begin{pmatrix} 0 \\ \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}_{4\times1}$$

From Q1(a) we know,

$$(Y \otimes Z) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}_{4\times 4}$$

$$(Y \otimes Z)(v \otimes w) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} 0 \\ \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}_{2\times 1} = \begin{pmatrix} 0 \\ \frac{4}{5}i \\ 0 \\ -\frac{3}{5}i \end{pmatrix}_{4\times 1} \qquad \textbf{(A)}$$

Evaluating $(Y|v\rangle) \otimes (Z|w\rangle)$

$$Y|v\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2\times 2} \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}_{2\times 1} = \begin{pmatrix} -\frac{4}{5}i \\ \frac{3}{5}i \end{pmatrix}_{2\times 1}$$

$$Z|w\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{2\times 2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2\times 1} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{2\times 1}$$

$$(Y|v\rangle) \otimes (Z|w\rangle) = \begin{pmatrix} -\frac{4}{5}i \\ \frac{3}{5}i \end{pmatrix}_{2\times 1} \otimes \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{2\times 1} = \begin{pmatrix} 0 \\ \frac{4}{5}i \\ 0 \\ -\frac{3}{5}i \end{pmatrix}_{4\times 1} \qquad \textbf{(B)}$$

*As **(A)** & **(B)** are equal. Hence the results are in agreement.*

(d) Let $|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ and $|w\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$

$|v\rangle \otimes |w\rangle = (ac, ad, bc, bd)^T$

$ac = 1/\sqrt{2}, bd = 1, ad = 0, bc = 1/\sqrt{2}$

On solving we get $a/b = 0, ac = 1/\sqrt{2}$

This implies a=0. But this is not possible since $ac = 1/\sqrt{2}$.

Thus there does not exist any value of a,b,c,d