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Tutorial 1 (Pauli matrices and rotation operators)
The Pauli matrices, which are often used in the study of quantum computation, are defined as

$$\sigma_1 \equiv X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 \equiv Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $\{I, \sigma_1, \sigma_2, \sigma_3\}$ are linearly independent and therefore form a basis for any complex 2×2 matrix.

(b) Verify the following commutation relations (here $[A, B] = A\,B - B\,A$ denotes the *commutator* of two matrices):

$$[\sigma_1, \sigma_2] = 2i\sigma_3,$$
$$[\sigma_2, \sigma_3] = 2i\sigma_1,$$
$$[\sigma_3, \sigma_1] = 2i\sigma_2.$$

(c) Let x be a real number and $A \in \mathbb{C}^{n\times n}$ $(n \in \mathbb{N})$ such that $A^2 = I$ (self-inverse). Show that

$$\exp(iAx) = \cos(x)I + i\sin(x)A.$$

The matrix exponential is defined by its power series expansion, i.e.,

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

(d) Let $A \in \mathbb{C}^{n\times n}$ $(n \in \mathbb{N})$ be Hermitian. Show that $\exp(iA)$ is unitary.

A matrix A is called Hermitian if $A^\dagger = A$, with $A^\dagger = (A^*)^T$ denoting the conjugate transpose of A. It is called unitary if $A^\dagger A = I$.

(e) Given a real unit vector $\vec{v} \in \mathbb{R}^3$, show that the linear combination $\vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$ is Hermitian and unitary (i.e. self-inverse).

For a real unit vector $\vec{v} \in \mathbb{R}^3$, we define the rotation by an angle θ about the $\vec{v}$ axis as

$$R_{\vec{v}}(\theta) = \exp(-i\theta\, \vec{v} \cdot \vec{\sigma}/2).$$

Combining (c), (d) and (e) implies that $R_{\vec{v}}(\theta)$ is unitary, and leads to the representation

$$R_{\vec{v}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(\vec{v} \cdot \vec{\sigma}).$$

(f) Compute $R_{\vec{v}}(\theta)$ for the special case $\vec{v} = (0, 0, 1)$, and interpret its action on the Bloch sphere representation of a qubit.

(g) Show that an arbitrary single qubit unitary operator U can be written in the form

$$U = \mathrm{e}^{i\alpha} R_{\vec{v}}(\theta)$$

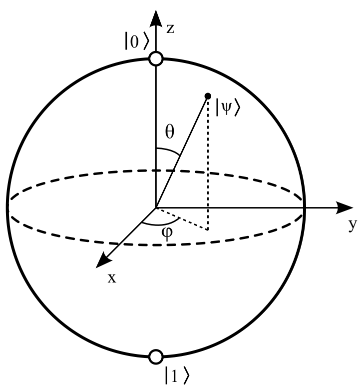
for some real number α. For example, the Hadamard gate equals

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \mathrm{e}^{i\pi/2} R_{\frac{1}{\sqrt{2}}(1,0,1)}(\pi).$$

Exercise 1.1 (Bloch sphere and single qubit quantum gates)
An arbitrary single qubit quantum state can be parametrized as

$$|\psi\rangle = \mathrm{e}^{i\gamma} \left(\cos\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{i\varphi} \sin\frac{\theta}{2}\,|1\rangle \right)$$

where θ, φ and γ are real numbers, which can be chosen such that $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. The angles θ and φ define the Bloch sphere representation of $|\psi\rangle$, as shown on the right.



https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg

For a real unit vector $\vec{v} \in \mathbb{R}^3$, recall that the rotation by an angle ω about the $\vec{v}$ axis is given by

$$R_{\vec{v}}(\omega) = \exp(-i\omega\, \vec{v} \cdot \vec{\sigma}/2) = \cos(\omega/2)I - i\sin(\omega/2)(\vec{v} \cdot \vec{\sigma}),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector. The rotations R_x, R_y, R_z about the standard axes correspond to the special cases $\vec{v} = (1, 0, 0)$, $\vec{v} = (0, 1, 0)$ and $\vec{v} = (0, 0, 1)$, respectively.

The *Z-Y decomposition* theorem states the following: given any unitary 2×2 matrix U, there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

(a) Find the Bloch angles θ and φ of $|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ (Hint: $\cos(\frac{\pi}{3}) = \frac{1}{2}$), and the corresponding Bloch vector

$$\vec{r} = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)).$$

(b) Compute $R_x(\frac{2\pi}{3})|\psi\rangle$ for the state $|\psi\rangle$ defined in (a), and visualize this operation on the Bloch sphere.

(c) Find the Z-Y decomposition of the Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Exercise 1.2 (Multiple qubits and tensor products)
We have learned that a single qubit is a superposition of the computational basis states $|0\rangle$ and $|1\rangle$. For two qubits, this generalizes to

$$|00\rangle, \; |01\rangle, \; |10\rangle, \; |11\rangle \tag{1}$$

as basis states, which we identify with the unit vectors

$$|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, \quad |01\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, \quad |10\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \quad |11\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T. \tag{2}$$

Accordingly, an arbitrary two-qubit state can be represented as

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

with $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11} \in \mathbb{C}$ such that $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$.

From a mathematical perspective, the two-qubit vector space is the tensor product $V \otimes V$, with $V = \mathbb{C}^2$ the single qubit space. The basis states in Eq. (1) are then interpreted as tensor products of basis states of V, e.g., $|01\rangle = |0\rangle \otimes |1\rangle$. Accordingly, one introduces the following vector notation for the tensor product:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix} \quad \text{for all } v_1, v_2, w_1, w_2 \in \mathbb{C}. \tag{3}$$

You should convince yourself that this is consistent with Eq. (2). Note that Eq. (3) can be canonically generalized to higher dimensions.

Given linear operators A and B, one can define $A \otimes B$ by its action on tensor products of states:

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = (A|v\rangle) \otimes (B|w\rangle),$$

which uniquely specifies $A \otimes B$ together with linearity. Using matrix notation, this leads to the *Kronecker product* of two matrices $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp \times nq}, \tag{4}$$

with a_{ij} the entries of A, and the terms $a_{ij}B$ denoting $p \times q$ submatrices which are proportional to B, with overall prefactor a_{ij}.

(a) Compute the Kronecker product $Y \otimes Z$ (with Y and Z denoting the corresponding Pauli matrices).

(b) Compute $Y \otimes Z$ again using Python/NumPy or (alternatively) Julia.

Hint: the imaginary unit is `1j` in Python and `im` in Julia. Both languages offer a `kron` command for calculating the Kronecker product defined in Eq. (4). Please hand in your code and the output of your program with the rest of your homework.

(c) Given $|v\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ and $|w\rangle = |1\rangle$, evaluate $(Y \otimes Z)(|v\rangle \otimes |w\rangle)$ by first computing $|v\rangle \otimes |w\rangle$ according to Eq. (3). Then check that your result agrees with $(Y|v\rangle) \otimes (Z|w\rangle)$.

(d) Show that the so-called Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

cannot be written as tensor product of the form $|v\rangle \otimes |w\rangle$ with $|v\rangle, |w\rangle \in \mathbb{C}^2$.