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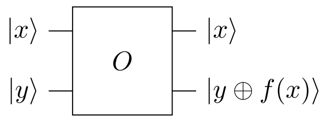
Tutorial 5 (Quantum search / Grover's algorithm)
Classically, searching through a search space of N unordered elements takes $\mathcal{O}(N)$ operations. The following quantum search algorithm, also known as Grover's algorithm, speeds this up to $\mathcal{O}(\sqrt{N})$ operations. For convenience, we assume that $N = 2^n$, so the index of one element in the search space can be stored in n bits, and that there are exactly M solutions. We represent the search problem by an indicator function f which is able to recognize the element(s) we are looking for, i.e., f takes as input an integer between 0 and $N-1$, and outputs $f(x) = 1$ if x is a solution and $f(x) = 0$ otherwise.

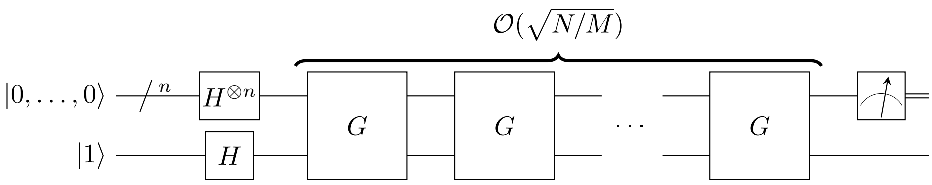
We will require an *oracle* O which performs the operation shown on the right (same as U_f in exercise 5.2). The so-called oracle qubit $|y\rangle$ is flipped precisely if $f(x) = 1$. In the following, we will initially set $|y\rangle = H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Since $|0\rangle$ and $|1\rangle$ are interchanged if x is a solution, the action of the oracle is



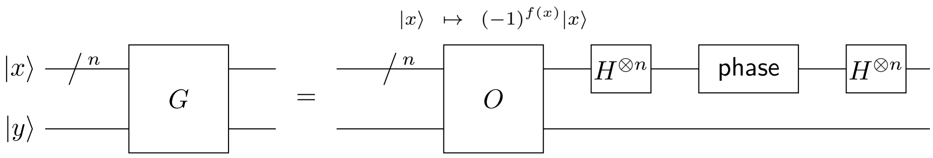
$$|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \overset{O}{\mapsto} (-1)^{f(x)} |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Conceptually, only the effective operation $|x\rangle \overset{O}{\mapsto} (-1)^{f(x)}|x\rangle$ is important, while the oracle qubit can be regarded as constant.

The quantum search algorithm consists of repeated applications of the so-called *Grover operator* G:



G is defined by the following circuit:



Here the phase gate maps $|0\rangle \mapsto |0\rangle$, $|x\rangle \mapsto -|x\rangle$ for $x > 0$, which is precisely the reflection operation $2|0\rangle\langle 0| - I$. We will repeatedly encounter the equal superposition state defined as

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle.$$

(a) Show that the Hadamard-phase-Hadamard operation appearing in G is a reflection about $|\psi\rangle$:

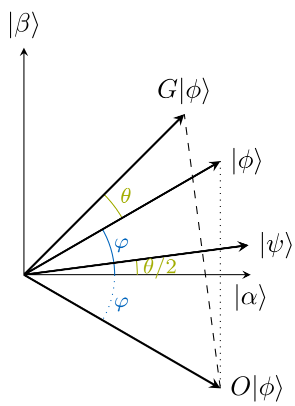
$$H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} = 2|\psi\rangle\langle\psi| - I.$$

Thus the Grover operator can be written as $G = (2|\psi\rangle\langle\psi| - I)O$.

(b) Define the orthonormal states

$$|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum_{f(x)=0} |x\rangle, \qquad |\beta\rangle = \frac{1}{\sqrt{M}} \sum_{f(x)=1} |x\rangle$$

and the angle θ via $\sin(\frac{\theta}{2}) = \sqrt{M/N}$, such that $|\psi\rangle = \cos(\frac{\theta}{2})|\alpha\rangle + \sin(\frac{\theta}{2})|\beta\rangle$. Show that the action of O in the plane spanned by $|\alpha\rangle$ and $|\beta\rangle$ is a reflection about $|\alpha\rangle$, and of G a rotation by θ, as illustrated on the right. In the picture, the input is $|\phi\rangle = \cos(\varphi)|\alpha\rangle + \sin(\varphi)|\beta\rangle$.



Since $\theta = \Omega(\sqrt{M/N})$, on the order of $\sqrt{N/M}$ applications of G are sufficient to rotate the initial state $H^{\otimes n}|0\rangle = |\psi\rangle$ close to $|\beta\rangle$. A subsequent observation (measurement) will then yield one of the solutions encoded in $|\beta\rangle$ with high probability.

Exercise 5.1 (Bell inequality)

(a) Let $\{|a\rangle, |b\rangle\}$ be an orthonormal basis of $\mathbb{C}^2$, and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ chosen such that

$$|0\rangle = \alpha|a\rangle + \beta|b\rangle,$$
$$|1\rangle = \gamma|a\rangle + \delta|b\rangle.$$

Verify the relation

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma)\frac{|ab\rangle - |ba\rangle}{\sqrt{2}}.$$

(b) In the quantum experiment violating the Bell inequality, Charlie prepares the "spin singlet" quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and sends the first qubit to Alice and the second to Bob. Alice then measures the observable Q or R on her qubit and Bob the observable S or T on his qubit, defined as

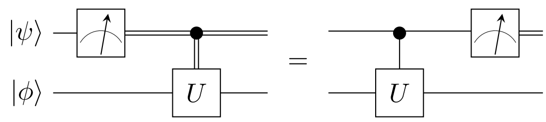
$$Q = Z, \qquad S = \frac{-Z - X}{\sqrt{2}},$$
$$R = X, \qquad T = \frac{Z - X}{\sqrt{2}}.$$

Verify the following average values (which violate the Bell inequality):

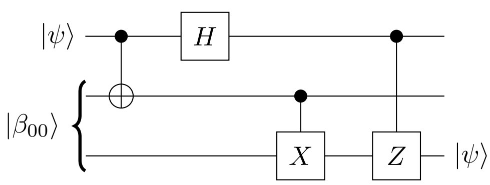
$$\langle\psi|Q \otimes S|\psi\rangle = \frac{1}{\sqrt{2}}, \quad \langle\psi|R \otimes S|\psi\rangle = \frac{1}{\sqrt{2}}, \quad \langle\psi|R \otimes T|\psi\rangle = \frac{1}{\sqrt{2}}, \quad \langle\psi|Q \otimes T|\psi\rangle = -\frac{1}{\sqrt{2}}.$$

Exercise 5.2 (Quantum teleportation circuit using IBM Q and Qiskit)

(a) By the *principle of deferred measurement*, measurement operations can always be moved to the end of the circuit, and classically controlled operations by conditional quantum operations. Verify this statement for the following controlled-U circuit, by inserting $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and computing the intermediate states:



(b) For the purpose of simulating the quantum teleportation circuit, we can exploit the principle of deferred measurement to omit the measurements altogether and rewrite the circuit in the following modified form:[1]



Construct this circuit in the IBM Q Circuit Composer. Insert a rotation operation at the beginning to prepare the initial qubit as $|\psi\rangle = R_y(\frac{\pi}{3})|0\rangle$. To check that the circuit works as intended, first compute the amplitudes α and β of the representation $R_y(\frac{\pi}{3})|0\rangle = \alpha|0\rangle + \beta|1\rangle$. Now insert a measurement operation at the end of the bottom qubit line, run 1024 "shots" your of circuit and compare the resulting measurement histogram with the expected probabilities $|\alpha|^2$ and $|\beta|^2$.

Also print a picture of your circuit and the corresponding `OPENQASM` code (shown in the Circuit Editor).

Hint: You can use the gates from exercise 3.1(b) to prepare the initial entangled pair $|\beta_{00}\rangle$.

(c) Construct the circuit in (b) using Qiskit, and execute the circuit via Aer's statevector_simulator. Print your code together with the final state vector.

Hint: See also https://quantum-computing.ibm.com/jupyter/tutorial/advanced/aer/1_aer_provider.ipynb.

[1] Note that the usual quantum teleportation protocol assumes that Alice and Bob are far from each other, such that conditional quantum operations between their qubits would be impractical.