Quantum Computing Assignment 7 - Group 18

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Exercise 7.1

(**Schmidt Decomposition & Schrödinger Equation**)

(a) Part 1

$$|\psi\rangle = \frac{1}{\sqrt{2}} * (|00\rangle + |11\rangle)$$

$$|\psi\rangle = \sum A_{ij}|ij\rangle$$

$$A = \frac{1}{\sqrt{2}} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$svd(A) = \frac{1}{\sqrt{2}} * \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Schmidt decomposition is

$$|\psi\rangle = \sum \lambda_\alpha * |\psi_{A,\alpha}\rangle * |\psi_{B,\alpha}\rangle$$

$$= \begin{bmatrix} |\psi_{A,0}\rangle |\psi_{A,1}\rangle \end{bmatrix} * \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} * \begin{bmatrix} |\psi_{A,0}\rangle |\psi_{A,1}\rangle \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -|0\rangle & -|1\rangle \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} -|0\rangle \\ -|1\rangle \end{bmatrix}$$

Part 2

$$|\psi\rangle = \frac{1}{2} * (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$A = \frac{1}{2} * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$svd(A) = \frac{1}{\sqrt{2}} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} * \frac{1}{\sqrt{2}} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Schmidtt decomposition is

$$|\psi\rangle = \sum \lambda_\alpha * |\psi_{A,\alpha}\rangle * |\psi_{B,\alpha}\rangle$$

$$= \begin{bmatrix} |\psi_{A,0}\rangle |\psi_{A,1}\rangle \end{bmatrix} * \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} * \begin{bmatrix} |\psi_{A,0}\rangle |\psi_{A,1}\rangle \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} |0\rangle + |1\rangle & |0\rangle - |1\rangle \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} |0\rangle + |1\rangle \\ |0\rangle - |1\rangle \end{bmatrix}$$

(b)

Given $U_t = e^{-iHt}$, as $\hbar = 1$ & H $= \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$
Since H is Hermitian, i.e; $H^2 = I$
We can use Eq. 2.20, i.e; $e^{iAx} = \cos(x)I + i\sin(x)A$ to represent U_t as follows:

$$U_t = e^{-iHt} \implies U_t = \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin(t) \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

$$\implies \begin{pmatrix} cos(t) & 0 \\ 0 & cos(t) \end{pmatrix} - \begin{pmatrix} i\sin(t)\omega_1 & 0 \\ 0 & i\sin(t)\omega_2 \end{pmatrix}$$

$$\implies U_t = \begin{pmatrix} \cos(t) - i\sin(t)\omega_1 & 0 \\ 0 & \cos(t) - i\sin(t)\omega_2 \end{pmatrix} \quad \textbf{(A)}$$

Using **(A)** we can solve for $|\psi(t)\rangle$ given $|\psi(0)\rangle$.

(i) $|\psi(0)\rangle = |0\rangle$

$$\implies |\psi(t)\rangle = \begin{pmatrix} \cos(t) - i\sin(t)\omega_1 & 0 \\ 0 & \cos(t) - i\sin(t)\omega_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\implies |\psi(t)\rangle = \begin{pmatrix} \cos(t) - i\sin(t)\omega_1 \end{pmatrix} |0\rangle$$

(ii) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big)$

$\implies |\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(t) - i\sin(t)\omega_1 & 0 \\ 0 & \cos(t) - i\sin(t)\omega_2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$\implies |\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big(\Big(\cos(t) - i\sin(t)\omega_1\Big)|0\rangle + \Big(\cos(t) - i\sin(t)\omega_2\Big)|1\rangle\Big)$

(c) Given H=$\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix} + \epsilon\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

This can be re-written as:

$$H = \bar{w}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \triangle\,\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \epsilon\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \equiv \bar{\omega}I + \sqrt{\triangle\,\omega^2 + \epsilon^2}(\vec{v}.\vec{\sigma})$$

Where $\bar{\omega}$ is assumed to be $\frac{\omega_1+\omega_2}{2}$ & $\triangle\,\omega = \frac{\omega_1-\omega_2}{2}$

$\implies Ht = \bar{w}tI + \triangle\,\omega tZ + \epsilon tX \implies e^{-iHt} = e^{-i\bar{\omega}tI - i\triangle\omega tZ - i\epsilon tX} \approx \underbrace{e^{-i(\bar{w}t)I}}_{\textbf{(A)}}\cdot\underbrace{e^{-i(\triangle\omega t)Z}}_{\textbf{(B)}}\cdot\underbrace{e^{-i(\epsilon t)X}}_{\textbf{(C)}}$

Using Eq. 2.20 in **(A)** : $\begin{pmatrix}\cos(\bar{w}t) & 0\\ 0 & \cos(\bar{w}t)\end{pmatrix} - \begin{pmatrix}i\sin(\bar{w}t) & 0\\ 0 & i\sin(\bar{w}t)\end{pmatrix} = \begin{pmatrix}e^{-i(\bar{w}t)} & 0\\ 0 & e^{-i(\bar{w}t)}\end{pmatrix}$

Using Eq. 2.20 in **(B)** : $\begin{pmatrix}\cos(\triangle\,\omega t) - i\sin(\triangle\,\omega t) & 0\\ 0 & \cos(\triangle\,\omega t) + i\sin(\triangle\,\omega t)\end{pmatrix} = \begin{pmatrix}e^{-i(\triangle\omega t)} & 0\\ 0 & e^{i(\triangle\omega t)}\end{pmatrix}$

Using Eq. 2.20 in **(C)**: $\begin{pmatrix}\cos(\epsilon t) & -i\sin(\epsilon t)\\ -i\sin(\epsilon t) & \cos(\epsilon t)\end{pmatrix}$

$\implies U_t = e^{-iHt} \approx \textbf{(A)(B)(C)}$

(A)(B) : $\begin{pmatrix}e^{-i(\bar{w}t)} & 0\\ 0 & e^{-i(\bar{w}t)}\end{pmatrix} \times \begin{pmatrix}e^{-i(\triangle\omega t)} & 0\\ 0 & e^{i(\triangle\omega t)}\end{pmatrix} = \begin{pmatrix}e^{-i(\bar{\omega}+\triangle\omega)t} & 0\\ 0 & e^{i(\triangle\omega-\bar{\omega})t}\end{pmatrix} = \begin{pmatrix}e^{-i(\omega_1)t} & 0\\ 0 & e^{-i(\omega_2)t}\end{pmatrix}$

$U_t = \begin{pmatrix}e^{-i(\omega_1)t} & 0\\ 0 & e^{-i(\omega_2)t}\end{pmatrix} \times \begin{pmatrix}\cos(\epsilon t) & -i\sin(\epsilon t)\\ -i\sin(\epsilon t) & \cos(\epsilon t)\end{pmatrix} = \begin{pmatrix}e^{-i(\omega_1)t}\cos(\epsilon t) & -ie^{-i(\omega_1)t}\sin(\epsilon t)\\ -ie^{-i(\omega_2)t}\sin(\epsilon t) & e^{-i(\omega_2)t}\cos(\epsilon t)\end{pmatrix}$

$|\psi(t)\rangle = U_t|0\rangle \implies |\psi(t)\rangle = \begin{pmatrix}e^{-i(\omega_1)t}\cos(\epsilon t)\\ -ie^{-i(\omega_2)t}\sin(\epsilon t)\end{pmatrix}$

$$\therefore \langle 1|\psi(t)\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}e^{-i(\omega_1)t}\cos(\epsilon t)\\ -ie^{-i(\omega_2)t}\sin(\epsilon t)\end{pmatrix} = -ie^{-i(\omega_2)t}\sin(\epsilon t)$$

(d)

$$|\psi(t)\rangle = U_t|\psi(0)\rangle$$

$$\rho = \sum_j p_j|\psi_j(t)\rangle\langle\psi_j(t)| = \sum_j p_j U_t|\psi(0)\rangle\langle\psi(0)|U_t^\dagger = U_t A U_t^\dagger$$

$$U_t = e^{\frac{-iHt}{h}}$$

$$\frac{dU_t}{dt} = \frac{-iH}{h}U_t$$

$$\frac{d\rho}{d_t} = (\frac{-iHU_t}{h}) * AU_t^\dagger + U_t^\dagger A(\frac{iHU_t^\dagger}{h})$$

$$= \frac{-i}{h}[H\rho - \rho H]$$

$$ih\frac{d\rho(t)}{d_t} = [H\rho - \rho H] = [H, \rho(t)]$$

Thus the derivation of Von Neumann equation is proved.

(Python/Numpy implementation of the partial trace)

(a)

In [39]:
```python
# ANSWER 7.2
# a)

import numpy as np
def partial_trace(rho,dimA,dimB):
    reshaped_dm = rho.reshape([dimA,dimB,dimA,dimB])
    # compute the partial trace
    rho_A= np.einsum('ijik->jk', reshaped_dm)
    rho_B = np.einsum('jiki->jk', reshaped_dm)
    a = (rho_A,rho_B)
    return a
```

In [40]:
```python
dimA=4
dimB=8
rho = np.random.randn(dimA*dimB,dimA*dimB)
partial_trace(rho,dimA,dimB)
```

Out[40]:
```
(array([[ 1.4201677 ,  2.39504506, -2.2483172 ,  2.10753928,  2.22660272,
         -0.23774537,  3.38391736, -0.12099788],
        [-0.07790885,  3.0741709 , -2.12218315, -0.96384127,  3.82175562,
         -4.2751483 ,  4.63409173,  1.07739852],
        [-0.43215101,  1.88827875,  1.51959742,  0.70667535,  1.10531461,
          1.57533666,  0.67035551,  4.23955212],
        [ 3.02463034,  2.97591351,  1.12200783,  0.33641079,  0.58748035,
          0.69172361, -2.75042606, -0.84158349],
        [ 1.51968918,  1.12140566,  4.87596295,  0.11029747,  1.18433153,
          3.03211712,  1.33704395,  0.84814695],
        [-1.88134385, -0.49163687, -6.78719296, -3.6769222 , -0.5633687 ,
          0.65842035, -4.53086354,  0.97387264],
        [-0.82960378, -2.51742567, -0.21867956, -3.36792719,  1.11556678,
          3.45606045, -0.68361483,  3.60800759],
        [-0.78327494,  2.02959473,  2.21159498, -0.4650718 , -0.95604822,
          0.17083995,  2.68733761,  1.86004572]]),
 array([[ 2.55597097,  5.6823249 , -1.77814963, -4.2631279 ],
        [-5.22056583,  4.04964453,  1.63559599, -3.48303142],
        [ 2.57096543,  2.19484942,  2.94299248,  3.3851535 ],
        [-5.73840189, -0.36136427, -1.98193132, -0.17907841]]))
```

(b)

In [41]:
```python
# Answer b)
psi = np.array([1 ,0 ,0, 1])/np.sqrt(2)
rho1=np.outer(psi,psi.conj())
print(rho1)
partial_trace(rho1,2,2)
```

```
[[0.5 0.  0.  0.5]
 [0.  0.  0.  0. ]
 [0.  0.  0.  0. ]
 [0.5 0.  0.  0.5]]
```

Out[41]:
```
(array([[0.5, 0. ],
        [0. , 0.5]]), array([[0.5, 0. ],
        [0. , 0.5]]))
```

(c)

In [48]:
```
# Answer c)
def construct_random_density_matrix(d):
    """
    Construct a complex random density matrix of dimension d x d.
    """
    # ensure that rho is positive semidefinite
    A = (np.random.randn(d, d) + 1j*np.random.randn(d, d))/np.sqrt(2)
    rho = A @ A.conj().T
    # normalization
    rho /= np.trace(rho)
    return rho

def construct_random_operator(d):
    """
    Construct a complex random Hermitian matrix of dimension d x d.
    """
    # ensure that M is Hermitian
    A = (np.random.randn(d, d) + 1j*np.random.randn(d, d))/np.sqrt(2)
    M = 0.5*(A + A.conj().T)
    return M
```

In [84]:
```
rho3 = construct_random_density_matrix(4)
M= construct_random_operator(2)
rho_A,rho_B=partial_trace(rho3,2,2)
LHS=np.trace(rho_B*M)
RHS=np.trace(np.kron(M,np.identity(2))*rho3)
print(LHS,RHS)
#ma=np.mean((np.square(np.subtract(LHS,RHS))))
print(RHS-LHS)

np.isclose(LHS,RHS,rtol=1e-05, atol=1e-06)
```

```
(-0.2859130294523684+6.635801494981807e-20j) (-0.2859130294523684+6.635801494981807e-20j)
0j
```

Out[84]:
```
True
```