

STAT 425

Analysis of Factor Level Means

Question:

What happens when the F -test leads to the conclusion that the factor level means μ_i differ?

- Analysis of the factor level means of interest using *estimation* techniques.
- Statistical *tests* concerning the factor level means of interest.

Inference for Factor Level Means

- ① A single factor level mean.
- ② A difference between two factor level means.
- ③ A contrast among factor level means.
- ④ A linear combination of factor level means.

Single Factor Level Mean

- Estimation of μ_i : $\hat{\mu}_i = \bar{y}_i$.
- Distribution of $\hat{\mu}_i$: $E(\hat{\mu}_i) = \mu_i$, $Var(\hat{\mu}_i) = \frac{\sigma^2}{n_i}$.

The estimated variance of \bar{y}_i is $s_{\bar{y}_i}^2 = \frac{1}{n_i} \cdot \frac{RSS}{n-r}$.

- Under the H_0

$\frac{\bar{y}_i - \mu_i}{s_{\bar{y}_i}}$ is distributed as T_{n-r} .

- Confidence Interval for μ_i :

$$\mu_i \in \bar{y}_i \pm T_{n-r}(\alpha/2) s_{\bar{y}_i}.$$

- In order to obtain confidence intervals for the factor level means in R, we fit the *means model* and we use the `confint` command:

```
# CI for single Factor Level Mean  
gl=lm(coag~diet-1)  
confint(gl)
```

```
##           2.5 %    97.5 %  
## dietA 58.53185 63.46815  
## dietB 63.98477 68.01523  
## dietC 65.98477 70.01523  
## dietD 59.25476 62.74524
```

Difference between Two Factor Level Means

The difference between two factor level means (**pairwise comparison**) is defined as

$$D = \mu_i - \mu_{i'}$$

- Estimation of D : $\hat{D} = \bar{y}_{i\cdot} - \bar{y}_{i'\cdot}$.
- Distribution of \hat{D} : $E(\hat{D}) = \mu_i - \mu_{i'}$, $Var(\hat{D}) = \sigma^2 \left(\frac{1}{n_i} + \frac{1}{n_{i'}} \right)$.

The estimated variance of \hat{D} is

$$s_{\hat{D}}^2 = \frac{RSS}{n - r} \cdot \left(\frac{1}{n_i} + \frac{1}{n_{i'}} \right).$$

- Under the H_0

$$\frac{\hat{D} - D}{s_{\hat{D}}} \text{ is distributed as } T_{n-r}$$

- Confidence Interval for D : $D \in \hat{D} \pm T_{n-r}(\alpha/2) s_{\hat{D}}$
- Hypothesis Test for D :

$$\left\{ \begin{array}{l} H_0 : \mu_i = \mu_{i'} \\ H_\alpha : \mu_i \neq \mu_{i'} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} H_0 : \mu_i - \mu_{i'} = D = 0 \\ H_\alpha : \mu_i - \mu_{i'} \neq 0 \end{array} \right.$$

The test statistic is $t = \frac{\hat{D}}{s_{\hat{D}}} \sim T_{n-r}$.

Contrast of Factor Level Means

A contrast is a comparison involving two or more level means:

$$L = \sum_{i=1}^r c_i \mu_i, \quad \text{where } \sum_{i=1}^r c_i = 0.$$

- Estimation of L : $\hat{L} = \sum_{i=1}^r c_i \bar{y}_i$.
- Distribution of \hat{L} : $E(\hat{L}) = \sum_{i=1}^r c_i \mu_i$, $Var(\hat{L}) = \sigma^2 \sum_{i=1}^r \frac{c_i^2}{n_i}$

The estimated variance of \hat{L} is

$$s_{\hat{L}}^2 = \frac{RSS}{n - r} \cdot \sum_{i=1}^r \frac{c_i^2}{n_i}$$

- Under the H_0

$$\frac{\hat{L} - L}{s_{\hat{L}}} \text{ is distributed as } T_{n-r}$$

- Confidence Interval for L : $L \in \hat{L} \pm T_{n-r}(\alpha/2) s_{\hat{L}}$
- Hypothesis Testing for L :

$$\begin{cases} H_0 : L = 0 \\ H_\alpha : L \neq 0 \end{cases}$$

The test statistic is $t = \frac{\hat{L}}{s_{\hat{L}}} \sim T_{n-r}$.

Linear Combination of Factor Level Means

$$L = \sum_{i=1}^r c_i \mu_i, \quad \text{no restrictions on } c_i's$$

- Point estimator and estimated variance same as before.
- Single Degree of Freedom Tests

$$\begin{cases} H_0 : L = c \\ H_\alpha : L \neq c \end{cases}$$

Under the H_0 the test statistic here is

$$F = t^2 = \left(\frac{\hat{L} - c}{s_{\hat{L}}} \right)^2 \sim F_{1, n-r}$$

Limitations of Inference Procedures

- The confidence coefficient $1 - \alpha$ for the estimation procedures described is a statement confidence coefficient and applies only to a particular estimate, not to a series of estimates.
- Similarly the specified Type I error rate α applies only to a particular test and not to a series of tests.

Bonferroni Correction

When? The family of interest is a **particular set of pairwise comparisons, contrasts, or linear combinations** that is specified by the user.

- Suppose m is the number of statements in the family.
- In order to control the family wise error rate to be α , we need to reduce the error rate for each individual comparison to be α/m .
- That is we need to increase the significance level from $(1 - \alpha)$ to $(1 - \alpha/m)$.
- Not applicable when m is large, since the CIs would be too wide due to the increase of the significant level.

- In R, we can obtain the p -values for the Bonferroni corrections for pairwise differences using the `pairwise.t.test` command.

```
# Bonferroni correction: this test outputs the p-values for the corresponding differences.  
pairwise.t.test(coag, diet, p.adjust.method = "bonferroni")
```

```
##  
## Pairwise comparisons using t tests with pooled SD  
##  
## data: coag and diet  
##  
## A B C  
## B 0.02282 - -  
## C 0.00108 0.95266 -  
## D 1.00000 0.00518 0.00014  
##  
## P value adjustment method: bonferroni
```

Tukey's Paired Comparison Procedures

When? the family of interest is a set of **all pairwise comparisons** of factor level means, i.e. it consists of estimates of all pairs

$$D = \mu_i - \mu_{i'}.$$

A confidence interval is given by

$$D \in \hat{D} + \frac{q(\alpha/2; r, n - r)}{\sqrt{2}} s(\hat{D}),$$

where $q(\alpha/2; r, n - r)$ refers to the $\alpha/2$ upper quantile of the **studentized range** for r means and $n - r$ degrees of freedom.

The coverage probability is exact when the sample sizes in each group are identical and is approximate otherwise.

Remark: The **studentized range** refers to the distribution of

$$\max_{i \neq j} \sqrt{n}(\bar{y}_i - \bar{y}_j) / \hat{\sigma}$$

where \bar{y}_i and \bar{y}_j are sample means from independent samples of size n from normal distributions with common means and variance σ^2 .

- To obtain Tukey family CIs for all pairwise comparisons in R, we use the TukeyHSD command.

```
# Tukey Simultaneous 95% CI for all mean differences  
TukeyHSD(aov(coag~diet), data=coagulation)
```

```
## Tukey multiple comparisons of means  
## 95% family-wise confidence level  
##  
## Fit: aov(formula = coag ~ diet)  
##  
## $diet  
##      diff      lwr      upr      p adj  
## B-A      5    0.5932529  9.406747 0.0228300  
## C-A      7    2.5932529 11.406747 0.0013858  
## D-A      0   -4.2789880  4.278988 1.0000000  
## C-B      2   -1.9415144  5.941514 0.4988550  
## D-B     -5   -8.7981383 -1.201862 0.0075558  
## D-C     -7  -10.7981383 -3.201862 0.0002854
```


Scheffé's Method for Contrasts

When? The family of interest is the set of **contrasts** among the factor level means:

$$L = \sum c_i \mu_i, \text{ where } \sum c_i = 0$$

An confidence interval is given by

$$L \in \hat{L} + (r - 1) F_{r-1, n-r}(\alpha) s_{\hat{L}}$$

- To obtain Scheffé family CIs for all pairwise comparisons in R, we use the `ScheffeTest` in the *DescTools* library.

```
g2aov=aov(coag~diet)
```

```
# If you want all the pairwise comparisons with Scheffe's method:  
ScheffeTest(g2aov)
```

```
##  
## Posthoc multiple comparisons of means: Scheffe Test  
## 95% family-wise confidence level  
##  
## $diet  
##      diff      lwr.ci      upr.ci      pval  
## B-A      5      0.342883      9.657117      0.03233 *  
## C-A      7      2.342883     11.657117      0.00210 **  
## D-A      0     -4.418129      4.418129      1.00000  
## C-B      2     -2.165452      6.165452      0.55494  
## D-B     -5     -8.896424     -1.103576      0.00876 **  
## D-C     -7    -10.896424     -3.103576      0.00031 ***  
##  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- If we want to obtain intervals for specific contrasts, such as

$$L_1 = \mu_A - \frac{1}{2}\mu_B - \frac{1}{2}\mu_C \text{ and } L_2 = \mu_B - \frac{1}{2}\mu_C - \frac{1}{2}\mu_D$$

then we can specify this in the contrasts argument as follows:

```
ScheffeTest(g2aov, contrasts=matrix(c(1,-0.5,-0.5,0,  
                                     0,1,-0.5,-0.5), ncol=2))
```

```
##  
## Posthoc multiple comparisons of means: Scheffe Test  
## 95% family-wise confidence level  
##  
## $diet  
##      diff      lwr.ci      upr.ci    pval  
## A-B,C -6.0 -10.165452 -1.834548 0.0032 **  
## B-C,D  1.5 -2.031434  5.031434 0.6482  
##  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```