STAT 410 **Homework #1** Fall 2020
A. Stepanov

(due Friday, September 4, by 5:00 p.m. CDT)

Please include your name (with your last name underlined), your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

1. Grades on Fall 2020 STAT 410 Exam 1 were not very good*. Graphed, their distribution had a shape similar to the probability density function.

$$f_X(x) = \frac{\sqrt{x+6}}{C}, \qquad 3 \le x \le 75, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_3^{75} \frac{\sqrt{x+6}}{C}\,dx = \frac{2\,(x+6)^{1.5}}{3\,C}\Bigg|_3^{75} = \frac{2\,(729-27)}{3\,C} = \frac{468}{C}.$$

$\Rightarrow \quad C = \mathbf{468}.$

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(75) = 1$.

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \int_3^{x} \frac{\sqrt{u+6}}{468}\,du = \frac{(u+6)^{1.5}}{702}\Bigg|_3^{x}$$

$$= \frac{(x+6)^{1.5} - 27}{702} = \frac{(x+6)^{1.5}}{702} - \frac{1}{26}, \qquad 3 \le x < 75.$$

Obviously, $F_X(x) = 0, \quad x < 3, \qquad F_X(x) = 1, \quad x \ge 75.$

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* The probability distribution is fictional, the exam has not happened yet. Hopefully, the actual grades will be slightly better than these.

1. (continued)

As a way of "curving" the results, the instructor announced that he would replace each person's grade, X, with a new grade, $\mathrm{Y} = g(\mathrm{X})$, where $g(x) = 5\sqrt{2x+75}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

$3 \le x \le 75 \quad \Rightarrow \quad 81 \le 2x + 75 \le 225$

$\Rightarrow \quad 9 \le \sqrt{2x+75} \le 15 \quad \Rightarrow \quad 45 \le 5\sqrt{2x+75} \le 75.$

$\Rightarrow \quad \mathbf{45 \le y \le 75}.$

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \ldots$.

$$F_Y(y) = P(Y \le y) = P(5\sqrt{2X+75} \le y) = P\left(\sqrt{2X+75} \le \frac{y}{5}\right)$$

$$= P\left(2X + 75 \le \frac{y^2}{25}\right) = P\left(X \le \frac{y^2}{50} - 37.5\right) = F_X\left(\frac{y^2}{50} - 37.5\right)$$

$$= \frac{\left(\frac{y^2}{50} - 31.5\right)^{1.5} - 27}{702} = \frac{\left(\frac{y^2}{50} - 31.5\right)^{1.5}}{702} - \frac{1}{26}$$

$$= \frac{\left(2y^2 - 3150\right)^{1.5} - 27{,}000}{702{,}000} = \frac{\left(2y^2 - 3{,}150\right)^{1.5}}{702{,}000} - \frac{1}{26},$$

$45 \le y < 75.$

Obviously, $F_Y(y) = 0, \quad y < 45, \qquad F_Y(y) = 1, \quad y \ge 75.$

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = 5\sqrt{2x+75} \qquad\qquad x = \frac{y^2}{50} - 37.5 \qquad\qquad \frac{dx}{dy} = \frac{y}{25}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{\sqrt{\dfrac{y^2}{50} - 31.5}}{468} \cdot \left| \frac{y}{25} \right|$$

$$= \frac{y\sqrt{\dfrac{y^2}{50} - 31.5}}{11{,}700} = \frac{y\sqrt{2y^2 - 3{,}150}}{117{,}000}, \qquad\qquad 45 \le y \le 75.$$

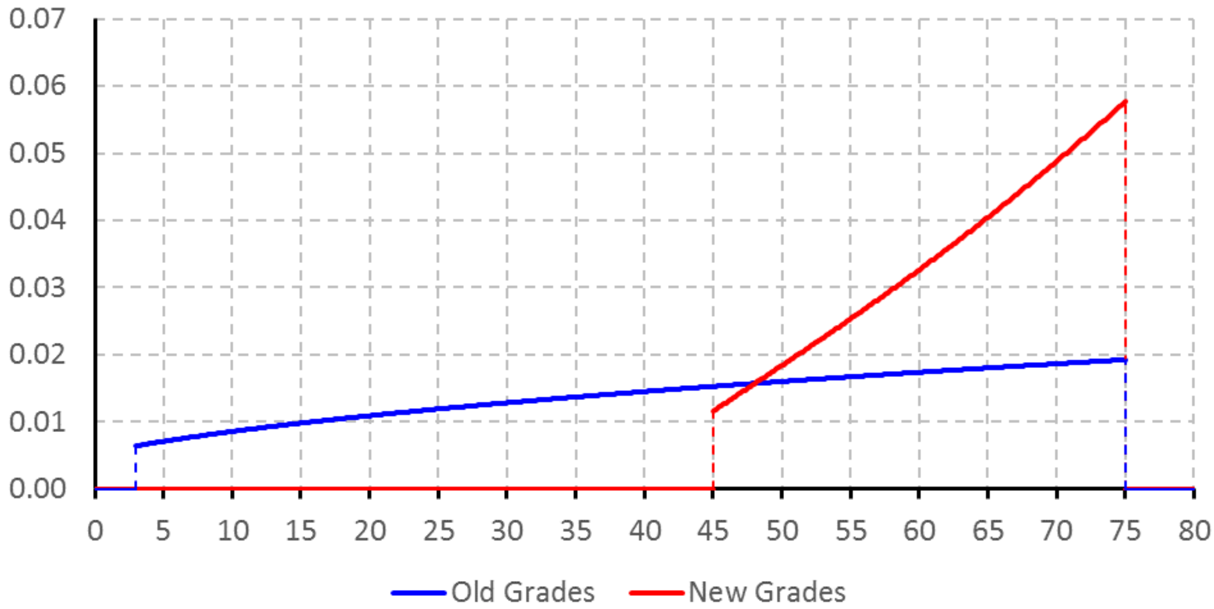
Indeed, $$\frac{d}{dy}\left(\frac{\left(2y^2 - 3150\right)^{1.5} - 27{,}000}{702{,}000} \right) = \frac{y\sqrt{2y^2 - 3{,}150}}{117{,}000}. \qquad ☺$$

For fun:

$\dfrac{45}{75} = 60\%.$ Everyone passed! ☺

$E(X) = \dfrac{2877}{65} \approx 44.26154.$ 59.0154% ☹

$E(Y) \approx 63.43215.$ 84.5762% ☺



Partially inspired by

https://www.reddit.com/r/UIUC/comments/djhcoa/when_you_see_your_physics_325_grade_precurve_and/

A child returns home from school (whatever that means).

Child: Mom, I got an A in zoology today.

Mom: That is great, Sweetie! What did you do?

Child: I said that penguins have three legs.

Mom: But, Sweetie, penguins have only two legs.

Child: Yeah, I know now. But everyone else in class said that penguins have four legs, so my answer was the closest to the right one.

From Fall 2020 STAT 410 Syllabus:

Grades are not curved or adjusted. This is not to dishearten students, but to let them know that their grade is based on individual effort and not on comparative effort.

2. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{3-x}{8}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

Consider $Y = g(X) = \dfrac{9}{X^2}$. Find the probability distribution of Y.

$-2 \le x \le 4$	$-1 \le x < 0$	$0 < x \le 3$
$y = \dfrac{9}{x^2}$	$9 \le y < \infty$	$\infty > y \ge 1$

$y < 1$ $$F_Y(y) = P(Y \le y) = P\left(\frac{9}{X^2} \le y\right) = P\left(X^2 \ge \frac{9}{y}\right) = 0.$$

$y \ge 1$ $$F_Y(y) = P(Y \le y) = P\left(\frac{9}{X^2} \le y\right) = P\left(X^2 \ge \frac{9}{y}\right)$$

$$= P\left(X \le -\frac{3}{\sqrt{y}}\right) + P\left(X \ge \frac{3}{\sqrt{y}}\right)$$

$$= F_X\left(-\frac{3}{\sqrt{y}}\right) + 1 - F_X\left(\frac{3}{\sqrt{y}}\right).$$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-1}^{x} \frac{3-u}{8}\,du = \left.\frac{6u - u^2}{16}\right|_{-1}^{x}$$

$$= \frac{7 + 6x - x^2}{16}, \qquad -1 \le x < 3.$$

Obviously, $F_X(x) = 0, \quad x < -1, \qquad F_X(x) = 1, \quad x \ge 3.$

Case 1: $1 \le y < 9$ $\qquad 3 \ge \dfrac{3}{\sqrt{y}} > 1.$

$$F_Y(y) = F_X\left(-\frac{3}{\sqrt{y}}\right) + 1 - F_X\left(\frac{3}{\sqrt{y}}\right) = 0 + 1 - \frac{7 + \frac{18}{\sqrt{y}} - \frac{9}{y}}{16}$$

$$= \frac{9 - \frac{18}{\sqrt{y}} + \frac{9}{y}}{16} = \frac{9y - 18\sqrt{y} + 9}{16y} = \frac{9\left(\sqrt{y}-1\right)^2}{16y}, \qquad 1 \le y < 9.$$

Case 2: $9 \le y < \infty$ $\qquad 1 \ge \dfrac{3}{\sqrt{y}} > 0.$

$$F_Y(y) = F_X\left(-\frac{3}{\sqrt{y}}\right) + 1 - F_X\left(\frac{3}{\sqrt{y}}\right) = \frac{7 - \frac{18}{\sqrt{y}} - \frac{9}{y}}{16} + 1 - \frac{7 + \frac{18}{\sqrt{y}} - \frac{9}{y}}{16}$$

$$= 1 - \frac{9}{4\sqrt{y}}, \qquad 9 \le y < \infty.$$

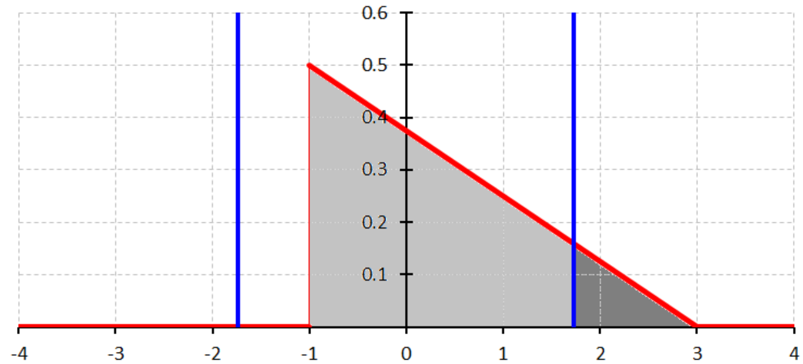
c.d.f. $$F_Y(y) = \begin{cases} 0 & y < 1 \\ \dfrac{9y - 18\sqrt{y} + 9}{16y} & 1 \le y < 9 \\ 1 - \dfrac{9}{4\sqrt{y}} & y \ge 9 \end{cases}$$

p.d.f. $$f_Y(y) = \begin{cases} 0 & y < 1 \\ \dfrac{9}{16y^{1.5}} - \dfrac{9}{16y^2} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases} = \begin{cases} 0 & y < 1 \\ \dfrac{9\left(\sqrt{y}-1\right)}{16y^2} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases}$$

OR

Case 1: $1 \le y < 9$

$$\Rightarrow \quad 3 \ge \frac{3}{\sqrt{y}} > 1.$$



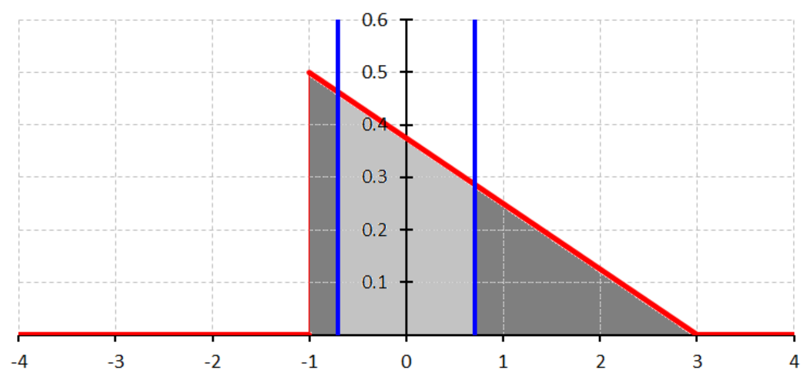
$$F_Y(y) = P(X \le -\frac{3}{\sqrt{y}}) + P(X \ge \frac{3}{\sqrt{y}})$$

$$= 0 + \int_{\frac{3}{\sqrt{y}}}^{3} \frac{3-x}{8}\,dx = \frac{6x - x^2}{16}\Bigg|_{\frac{3}{\sqrt{y}}}^{3} = \frac{9 - \frac{18}{\sqrt{y}} + \frac{9}{y}}{16}$$

$$= \frac{9y - 18\sqrt{y} + 9}{16y} = \frac{9\left(\sqrt{y}-1\right)^2}{16y}, \qquad 1 \le y < 9.$$

Case 2: $9 \le y < \infty$

$$\Rightarrow \quad 1 \ge \frac{3}{\sqrt{y}} > 0.$$



$$F_Y(y) = P(X \le -\frac{3}{\sqrt{y}}) + P(X \ge \frac{3}{\sqrt{y}})$$

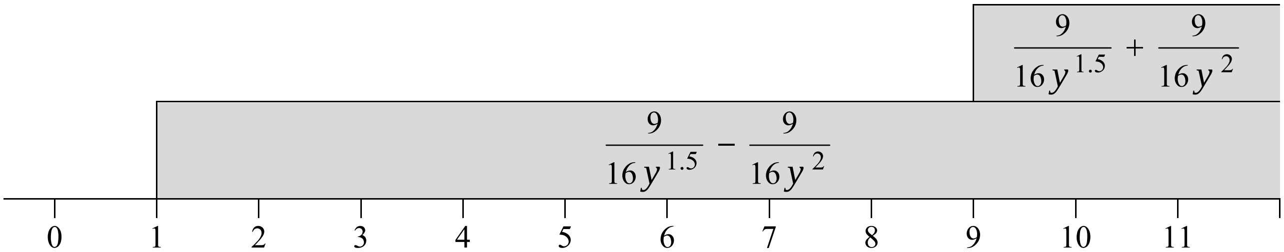
$$= \int_{-1}^{-\frac{3}{\sqrt{y}}} \frac{3-x}{8}\,dx + \int_{\frac{3}{\sqrt{y}}}^{3} \frac{3-x}{8}\,dx$$

$$= \left.\frac{6x-x^2}{16}\right|_{-1}^{-\frac{3}{\sqrt{y}}} + \left.\frac{6x-x^2}{16}\right|_{\frac{3}{\sqrt{y}}}^{3}$$

$$= \frac{-\frac{18}{\sqrt{y}} - \frac{9}{y} + 7}{16} + \frac{9 - \frac{18}{\sqrt{y}} + \frac{9}{y}}{16} = 1 - \frac{9}{4\sqrt{y}}, \qquad 9 \le y < \infty.$$

OR

$-1 \le x < 0$	$0 < x \le 3$
$Y = g(X) = \dfrac{9}{X^2}$	$Y = g(X) = \dfrac{9}{X^2}$
$x = -\dfrac{3}{\sqrt{y}} = g^{-1}(y)$	$x = \dfrac{3}{\sqrt{y}} = g^{-1}(y)$
$\dfrac{dx}{dy} = \dfrac{3}{2y^{1.5}}$	$\dfrac{dx}{dy} = -\dfrac{3}{2y^{1.5}}$
$9 \le y < \infty$	$\infty > y \ge 1$
$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$	$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$
$\dfrac{3 + \frac{3}{\sqrt{y}}}{8} \times \left\lvert \dfrac{3}{2y^{1.5}} \right\rvert = \dfrac{9}{16y^{1.5}} + \dfrac{9}{16y^2}$	$\dfrac{3 - \frac{3}{\sqrt{y}}}{8} \times \left\lvert -\dfrac{3}{2y^{1.5}} \right\rvert = \dfrac{9}{16y^{1.5}} - \dfrac{9}{16y^2}$



$$f_Y(y) = \frac{9}{16y^{1.5}} - \frac{9}{16y^{2}}, \qquad 1 < y < 9,$$

$$f_Y(y) = \frac{9}{8y^{1.5}}, \qquad 9 < y < \infty.$$

p.d.f.

$$f_Y(y) = \begin{cases} 0 & y < 1 \\ \dfrac{9}{16y^{1.5}} - \dfrac{9}{16y^{2}} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases} = \begin{cases} 0 & y < 1 \\ \dfrac{9\left(\sqrt{y}-1\right)}{16y^{2}} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases}$$