STAT 410 | Fall 2020
A. Stepanov

Homework #4

(due Friday, September 25, by 5:00 p.m. CDT)

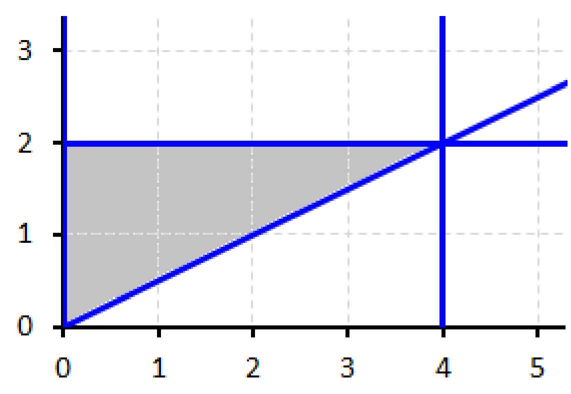
Please include your name (with your last name underlined), your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

5. Suppose that the joint probability density function for X, the time (in minutes) a weak-minded lazy Chegg enthusiast or a weak-minded lazy CourseHero worshiper would spend "trying to solve" a homework problem (which involves staring at the problem and hoping that a solution would appear), and Y, the time (in minutes) a weak-minded lazy Chegg enthusiast or a weak-minded lazy CourseHero worshiper would spend "actually try to solve" it (which involves looking through Compass2g for the same words that were used in the problem) before posting the problem on Chegg or CourseHero (because they care so much about their grades, obviously) and generously promising to "thumbs up if done neatly and correctly", is

$$f(x,y) = \frac{5}{128} x^2 (4-y), \qquad 0 \le x \le 4, \quad \frac{x}{2} \le y \le 2, \qquad \text{zero otherwise.}$$

Find $\operatorname{Var}(X \mid Y)$.



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{2y} \frac{5}{128} x^2 (4-y)\,dx$$

$$= \frac{5}{128 \cdot 3} x^3 (4-y) \Big|_{x=0}^{x=2y} = \frac{5 \cdot 8y^3}{128 \cdot 3}(4-y), \qquad 0 \le y \le 2.$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{3x^2}{8y^3}, \qquad 0 < x < 2y.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \int_0^{2y} x \cdot \frac{3x^2}{8y^3}\,dx = \frac{3x^4}{32y^3}\Bigg|_{x=0}^{x=2y} = \frac{3}{2}y = 1.5y.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}) = 1.5\,\mathrm{Y}.$$

$$\mathrm{E}(\mathrm{X}^2 \mid \mathrm{Y}=y) = \int_0^{2y} x^2 \cdot \frac{3x^2}{8y^3}\,dx = \frac{3x^5}{40y^3}\Bigg|_{x=0}^{x=2y} = \frac{12}{5}y^2 = 2.4y^2.$$

$$\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}=y) = \frac{12}{5}y^2 - \left(\frac{3}{2}y\right)^2 = 2.4y^2 - 2.25y^2 = \frac{3}{20}y^2 = 0.15y^2.$$

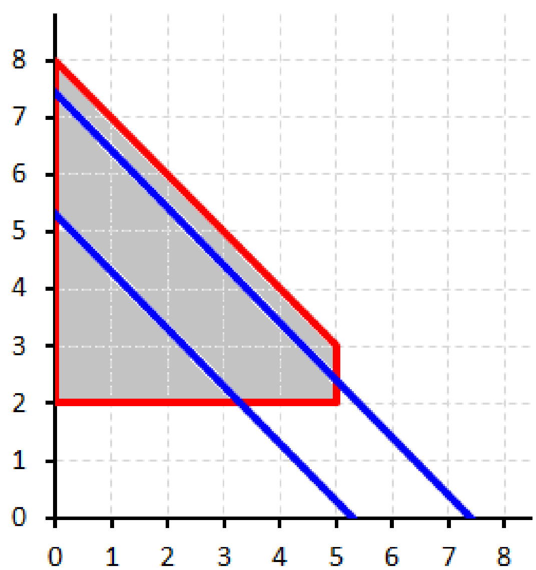
$$\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}) = \mathbf{\frac{3}{20}\,Y^2} = \mathbf{0.15\,Y^2}.$$

3. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Since the people become disobedient if they do not have food, the amount spent monthly on butter is always at least 2 million dollars. However, there is a law that prevents the government of Neverland spending more than 5 million dollars on guns in one month. In addition, the government of Neverland cannot afford to spend more than 8 million dollars per month on guns and butter. Suppose that the joint density function for (X, Y) is

$$f(x,y) = \frac{7x + 2y}{375}, \qquad x \ge 0, \quad y \ge 2, \quad x \le 5, \quad x + y \le 8, \qquad \text{zero otherwise.}$$

X – guns, Y – butter.

o) Find the probability distribution of $W = X + Y$, the total amount the government of Neverland spends on guns and butter in one month.
(Specify whether you answer is the p.d.f. or the c.d.f.
and include the support of the probability distribution.)



$$F_W(w) = P(X + Y \le w) = \ldots$$

There are 2 cases:

$2 < w < 7, \qquad 7 < w < 8.$

Technically, there are 4 cases:

$w < 2,$

$2 < w < 7, \qquad 7 < w < 8,$

$w > 8,$

but $w < 2$ and $w > 8$ are boring.

Two battle plans:

$$f_W(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx = \int_{-\infty}^{\infty} f(w-y, y)\, dy.$$

$$F_W(w) = P(X + Y \le w).$$

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx.$$

$$f(x, w-x) = \frac{5x + 2w}{375}.$$

$x \geq 0$

$y \geq 2 \quad \Rightarrow \quad w - x \geq 2 \quad \Rightarrow \quad x \leq w - 2$

$x \leq 5$

$x + y \leq 8 \quad \Rightarrow \quad x + w - x \leq 8 \quad \Rightarrow \quad w \leq 8$

$x \geq 0 \quad \& \quad x \leq w - 2, \quad x \leq 5.$

Case 1: $\quad 2 < w < 7. \qquad w - 2 < 5.$

$$f_W(w) = \int_0^{w-2} \frac{5x + 2w}{375}\, dx = \frac{\frac{5}{2}(w-2)^2 + 2w(w-2)}{375} = \frac{20 - 28w + 9w^2}{750},$$

$2 < w < 7.$

Case 2: $\quad 7 < w < 8. \qquad 5 < w - 2.$

$$f_W(w) = \int_0^{5} \frac{5x + 2w}{375}\, dx = \frac{\frac{125}{2} + 10w}{375} = \frac{25 + 4w}{150}, \qquad 7 < w < 8.$$

p.d.f.

$$f_W(w) = \begin{cases} \dfrac{20 - 28w + 9w^2}{750} & 2 < w < 7 \\ \dfrac{25 + 4w}{150} & 7 < w < 8 \\ 0 & \text{otherwise} \end{cases}$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy.$$

$$f(w-y, y) = \frac{7w - 5y}{375}.$$

$x \ge 0 \quad \Rightarrow \quad w - y \ge 0 \quad \Rightarrow \quad y \le w$

$y \ge 2$

$x \le 5 \quad \Rightarrow \quad w - y \le 5 \quad \Rightarrow \quad y \ge w - 5$

$x + y \le 8 \quad \Rightarrow \quad w - y + y \le 8 \quad \Rightarrow \quad w \le 8$

$y \ge 2, \;\; y \ge w - 5 \quad \& \quad y \le w.$

Case 1: $\quad 2 < w < 7. \qquad\qquad 2 > w - 5.$

$$f_W(w) = \int_2^w \frac{7w - 5y}{375}\, dx = \frac{7w(w-2) - \frac{5}{2}\left(w^2 - 4\right)}{375} = \frac{20 - 28w + 9w^2}{750},$$

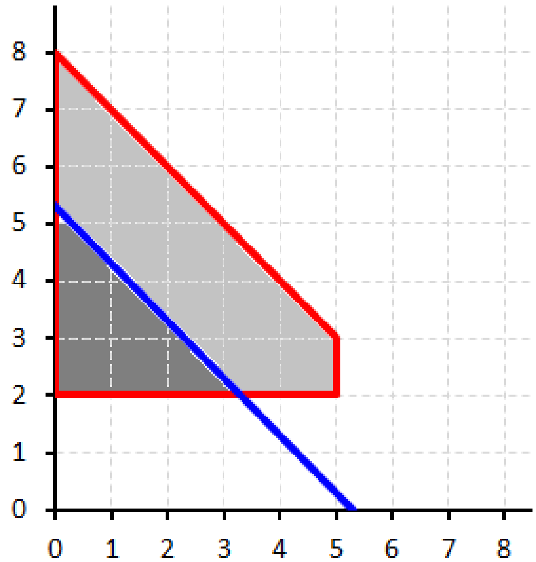
$$2 < w < 7.$$

Case 2: $\quad 7 < w < 8. \qquad\qquad w - 5 > 2.$

$$f_W(w) = \int_{w-5}^{w} \frac{7w - 5y}{375}\, dx = \frac{35w - \frac{5}{2}\left(w^2 - (w-5)^2\right)}{375} = \frac{25 + 4w}{150},$$

$$7 < w < 8.$$

OR



Case 1: $\quad 2 \le w < 7.$

$$F_W(w) = P(X + Y \le w) = \ldots$$

$$\ldots = \int_0^{w-2} \left(\int_2^{w-x} \frac{7x + 2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_2^{w} \left(\int_0^{w-y} \frac{7x + 2y}{375}\, dx \right) dy$$

OR

$$\ldots = 1 - \int_0^{w-2} \left(\int_{w-x}^{8-x} \frac{7x + 2y}{375}\, dy \right) dx - \int_{w-2}^{5} \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

OR

$$\ldots = 1 - \int_2^{3} \left(\int_{w-y}^{5} \frac{7x + 2y}{375}\, dx \right) dy - \int_3^{w} \left(\int_{w-y}^{8-y} \frac{7x + 2y}{375}\, dx \right) dy$$

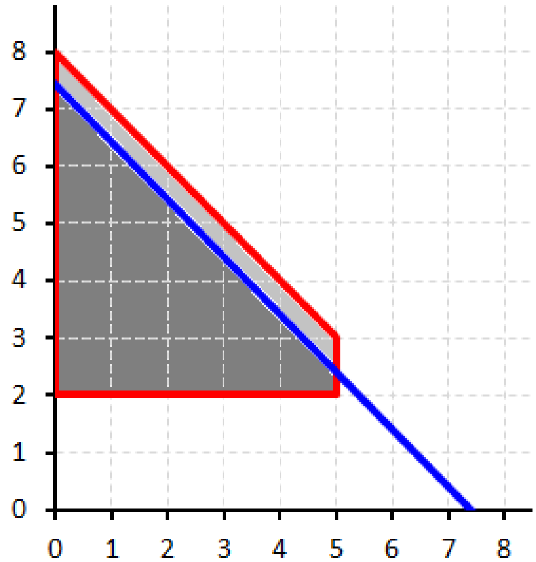
$$- \int_w^{8} \left(\int_0^{8-y} \frac{7x + 2y}{375}\, dx \right) dy$$

$$\int_0^{w-2} \left(\int_2^{w-x} \frac{7x + 2y}{375}\, dy \right) dx = \int_0^{w-2} \frac{7x(w-x-2) + (w-x)^2 - 4}{375}\, dx$$

$$= \int_0^{w-2} \frac{w^2 - 4 + 5wx - 14x - 6x^2}{375}\, dx$$

$$= \frac{2w^2(w-2) - 8(w-2) + 5w(w-2)^2 - 14(w-2)^2 - 4(w-2)^3}{750}$$

$$= \frac{3w^3 - 14w^2 + 20w - 8}{750}, \qquad 2 \le w < 7.$$



Case 2: $7 \le w < 8.$

$$F_W(w) = P(X+Y \le w) = \ldots$$

$$\ldots = \int_0^5 \left(\int_2^{w-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = 1 - \int_0^5 \left(\int_{w-x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_2^{w-5} \left(\int_0^5 \frac{7x+2y}{375}\, dx \right) dy + \int_{w-5}^{w} \left(\int_0^{w-y} \frac{7x+2y}{375}\, dx \right) dy$$

OR

$$\ldots = 1 - \int_{w-5}^{3} \left(\int_{w-y}^{5} \frac{7x+2y}{375}\, dx \right) dy - \int_3^{w} \left(\int_{w-y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$- \int_w^8 \left(\int_0^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$\int_0^5 \left(\int_2^{w-x} \frac{7x+2y}{375}\, dy \right) dx = \int_0^5 \frac{7x(w-x-2) + (w-x)^2 - 4}{375}\, dx$$

$$= \int_0^5 \frac{w^2 - 4 + 5wx - 14x - 6x^2}{375}\, dx$$

$$= \frac{10w^2 - 40 + 125w - 350 - 500}{750} = \frac{2w^2 + 25w - 178}{150},$$

$7 \le w < 8.$

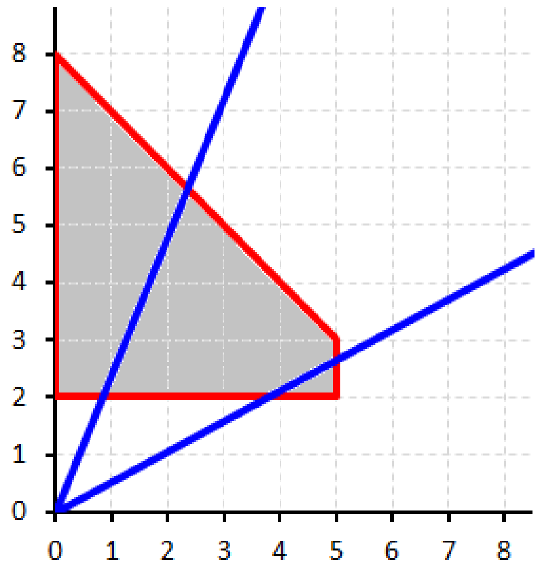
c.d.f. $$F_W(w) = \begin{cases} 0 & w < 2 \\ \dfrac{3w^3 - 14w^2 + 20w - 8}{750} & 2 \le w < 7 \\ \dfrac{2w^2 + 25w - 178}{150} & 7 \le w < 8 \\ 1 & w \ge 8 \end{cases}$$

p.d.f. $$f_W(w) = \begin{cases} \dfrac{9w^2 - 28w + 20}{750} & 2 < w < 7 \\ \dfrac{4w + 25}{150} & 7 < w < 8 \\ 0 & \text{otherwise} \end{cases}$$

p) Let $U = \frac{Y}{X} = \frac{\text{butter}}{\text{guns}}$. Determine the range of possible values of U.

Set up the integral(s) for the c.d.f. of U, $F_U(u)$.

*You do NOT have to evaluate the integral(s). **



$$F_U(u) = P\left(\frac{Y}{X} \le u\right) = P(Y \le uX) = \ldots$$

There are 2 cases:

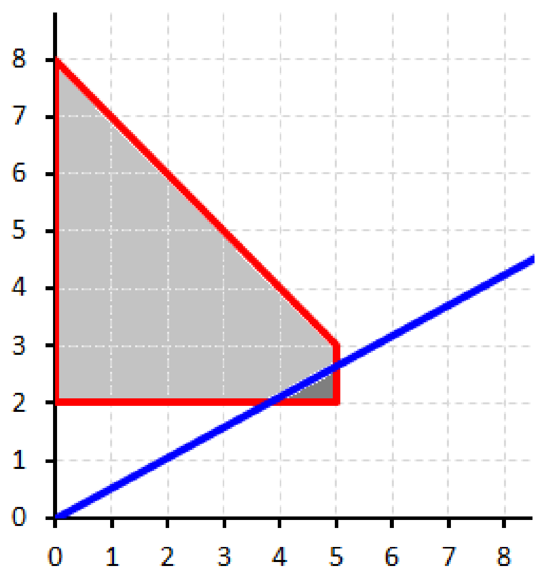
$0.4 < u < 0.6$, $u > 0.6$.

Technically, there are 3 cases:

$u < 0.4$,

$0.4 < u < 0.6$, $u > 0.6$.

but $u < 0.4$ is boring.



Case 1: $0.4 < u < 0.6$.

$$F_U(u) = P\left(\frac{Y}{X} \le u\right) = P(Y \le uX) = \ldots$$

$$\ldots = \int_{2/u}^{5} \left(\int_{2}^{ux} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_{2}^{5u} \left(\int_{y/u}^{5} \frac{7x+2y}{375}\, dx \right) dy$$

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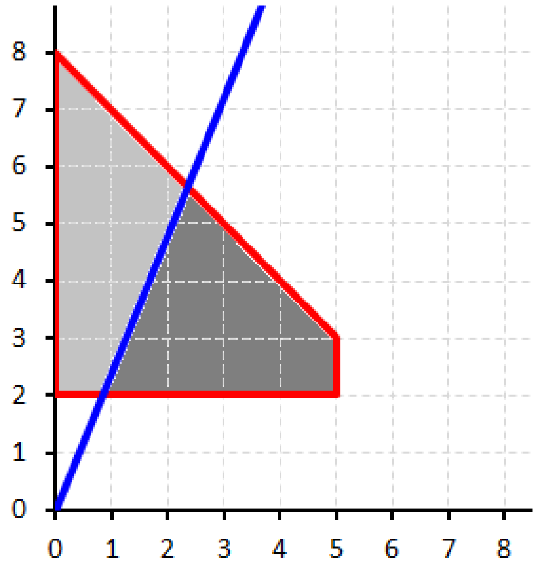
* Obviously, you would have to evaluate all integrals on an exam.



Case 2: $u > 0.6$.

$$F_U(u) = P\left(\frac{Y}{X} \le u\right) = P(Y \le u X) = \ldots$$

$x + y = 8$ & $y = u x$

$\Rightarrow \quad x + u x = 8$

$\Rightarrow \quad x = \dfrac{8}{1+u}, \quad y = \dfrac{8u}{1+u}.$

$$\ldots = \int_{2/u}^{\frac{8}{1+u}} \left(\int_{2}^{ux} \frac{7x+2y}{375}\, dy \right) dx + \int_{\frac{8}{1+u}}^{5} \left(\int_{2}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = 1 - \int_{0}^{2/u} \left(\int_{2}^{8-x} \frac{7x+2y}{375}\, dy \right) dx - \int_{2/u}^{\frac{8}{1+u}} \left(\int_{ux}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_{2}^{3} \left(\int_{y/u}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{3}^{\frac{8u}{1+u}} \left(\int_{y/u}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

OR

$$\ldots = 1 - \int_{2}^{\frac{8u}{1+u}} \left(\int_{0}^{y/u} \frac{7x+2y}{375}\, dx \right) dy - \int_{\frac{8u}{1+u}}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

q) Let $V = X \cdot Y$.

Set up the integral(s) for the c.d.f. of V, $F_V(v)$.

You do NOT have to evaluate the integral(s). *

"Hint": $0 < v < 16$.

$$F_V(v) = P(X \cdot Y \le v) = \dots$$

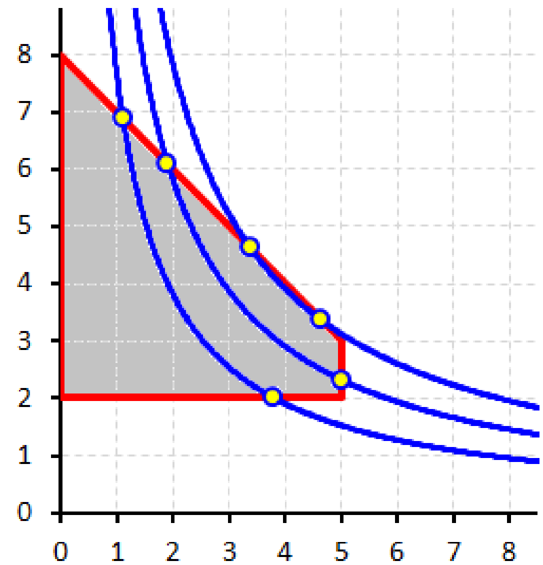
While $0 < v < 10$ & $10 < v < 15$ are predictable, where does $15 < v < 16$ come from?

$x \cdot y = v$ & $x + y = 8$

$$\Rightarrow \quad x + \frac{v}{x} = 8$$

$$\Rightarrow \quad x^2 - 8x + v = 0$$

$$\Rightarrow \quad x = 4 \pm \sqrt{16 - v}\,.$$



Technically, there are 5 cases:

$v < 0$,

$0 < v < 10$, $10 < v < 15$, $15 < v < 16$,

$v > 16$,

but $v < 0$ and $v > 16$ are boring.

Case 1: $0 < v < 10$.

$$\dots = \int_0^{4-\sqrt{16-v}} \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx + \int_{4-\sqrt{16-v}}^{v/2} \left(\int_2^{v/x} \frac{7x + 2y}{375}\, dy \right) dx$$

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* Obviously, you would have to evaluate all integrals on an exam.

OR

$$\ldots = 1 - \int_{4-\sqrt{16-v}}^{v/2} \left(\int_{u/x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx - \int_{v/2}^{5} \left(\int_{2}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_{2}^{4+\sqrt{16-v}} \left(\int_{0}^{v/y} \frac{7x+2y}{375}\, dx \right) dy + \int_{4+\sqrt{16-v}}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

OR

$$\ldots = 1 - \int_{2}^{3} \left(\int_{v/y}^{5} \frac{7x+2y}{375}\, dx \right) dy - \int_{3}^{4+\sqrt{16-v}} \left(\int_{v/y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

Case 2: $10 < v < 15$.

$$\ldots = \int_{0}^{4-\sqrt{16-v}} \left(\int_{2}^{8-x} \frac{7x+2y}{375}\, dy \right) dx + \int_{4-\sqrt{16-v}}^{5} \left(\int_{2}^{v/x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = 1 - \int_{4-\sqrt{16-v}}^{5} \left(\int_{v/x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_{2}^{v/5} \left(\int_{0}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{v/5}^{4+\sqrt{16-v}} \left(\int_{0}^{v/y} \frac{7x+2y}{375}\, dx \right) dy$$

$$+ \int_{4+\sqrt{16-v}}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

OR

$$\ldots = 1 - \int_{v/5}^{3} \left(\int_{v/y}^{5} \frac{7x+2y}{375}\, dx \right) dy - \int_{3}^{4+\sqrt{16-v}} \left(\int_{v/y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

Case 2: $15 < v < 16$.

$$\ldots = \int_{0}^{4-\sqrt{16-v}} \left(\int_{2}^{8-x} \frac{7x+2y}{375}\, dy \right) dx + \int_{4-\sqrt{16-v}}^{4+\sqrt{16-v}} \left(\int_{2}^{v/x} \frac{7x+2y}{375}\, dy \right) dx$$

$$+ \int_{4+\sqrt{16-v}}^{5} \left(\int_{2}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = 1 - \int_{4-\sqrt{16-v}}^{4+\sqrt{16-v}} \left(\int_{v/x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

OR

$$\ldots = \int_{2}^{3} \left(\int_{0}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{3}^{4-\sqrt{16-v}} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$\int_{4-\sqrt{16-v}}^{4+\sqrt{16-v}} \left(\int_{0}^{v/y} \frac{7x+2y}{375}\, dx \right) dy + \int_{4+\sqrt{16-v}}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

OR

$$\ldots = 1 - \int_{4-\sqrt{16-v}}^{4+\sqrt{16-v}} \left(\int_{v/y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

These integrals do not look inviting… ☹