2.5 Independent Random Variables

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

$x \setminus y$	0	1	2	
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
	0.40	0.40	0.20	

Recall: A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

a) Are events $\{X = 1\}$ and $\{Y = 1\}$ independent?

$$P(X = 1 \cap Y = 1) = p(1, 1) = 0.10 = 0.25 \times 0.40 = P(X = 1) \times P(Y = 1).$$

$\{X = 1\}$ and $\{Y = 1\}$ are **independent**.

Def Random variables X and Y are **independent** if and only if

discrete $\quad p(x, y) = p_X(x) \cdot p_Y(y) \quad$ for all x, y.

continuous $\quad f(x, y) = f_X(x) \cdot f_Y(y) \quad$ for all x, y.

$F(x, y) = P(X \le x, Y \le y)$. $\qquad f(x, y) = \partial^2 F(x, y) / \partial x \, \partial y$.

Def Random variables X and Y are **independent** if and only if

$$F(x, y) = F_X(x) \cdot F_Y(y) \quad \text{for all } x, y.$$

b) Are random variables X and Y independent?

$$p(1, 0) = 0.15 \ne 0.25 \times 0.40 = p_X(1) \times p_Y(0).$$

X and Y are **NOT independent**.

2. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall:
$$f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$$
$$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1.$$

Are random variables X and Y independent?

The support of (X, Y) is not a rectangle.

X and Y are **NOT independent**.

OR

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$,

X and Y are **NOT independent**.

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$\begin{aligned}
f_1(x) &= \int_0^1 (x+y)\,dy \\
&= \left[xy + \frac{1}{2}y^2\right]_0^1 = x + \frac{1}{2}, \qquad 0 \le x \le 1\ ; \\
f_2(y) &= \int_0^1 (x+y)\,dx = y + \frac{1}{2}, \qquad 0 \le y \le 1; \\
f(x,y) &= x + y \neq \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = f_1(x) f_2(y).
\end{aligned}$$

X and Y are **NOT independent**.

4. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 12\,x(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$f_{\mathrm{X}}(x)=\int_0^{\infty} 12\,x(1-x)e^{-2y}\,dy = 6\,x(1-x),\quad 0<x<1.$$

$$f_{\mathrm{Y}}(y)=\int_0^{1} 12\,x(1-x)e^{-2y}\,dx = 2\,e^{-2y},\quad y>0.$$

Since $f(x,y)=f_{\mathrm{X}}(x)\cdot f_{\mathrm{Y}}(y)$ for all x, y, X and Y are **independent**.

If random variables X and Y are independent, then

$$\mathrm{E}(g(\mathrm{X})\cdot h(\mathrm{Y})) = \mathrm{E}(g(\mathrm{X}))\cdot \mathrm{E}(h(\mathrm{Y})).$$

$\Rightarrow$ $$\mathrm{M}_{\mathrm{X,Y}}(t_1,t_2) = \mathrm{M}_{\mathrm{X,Y}}(t_1,0)\cdot \mathrm{M}_{\mathrm{X,Y}}(0,t_2) = \mathrm{M}_{\mathrm{X}}(t_1)\cdot \mathrm{M}_{\mathrm{Y}}(t_2).$$

5. Suppose the probability density functions of T_1 and T_2 are

$$f_{\mathrm{T}_1}(x)=\alpha\,e^{-\alpha x},\quad x>0,\qquad f_{\mathrm{T}_2}(y)=\beta\,e^{-\beta y},\quad y>0,$$

respectively. Suppose T_1 and T_2 are independent. Find $\mathrm{P}(2\,\mathrm{T}_1>\mathrm{T}_2)$.

$$\mathrm{P}(2\,\mathrm{T}_1>\mathrm{T}_2)=\int_0^{\infty}\left(\int_{y/2}^{\infty}\left(\alpha\beta\,e^{-\alpha x-\beta y}\right)dx\right)dy=\int_0^{\infty}\beta e^{-\beta y}\left(\int_{y/2}^{\infty}\left(\alpha\,e^{-\alpha x}\right)dx\right)dy$$

$$=\int_0^{\infty}\beta e^{-\beta y}\left(e^{-\alpha y/2}\right)dy=\int_0^{\infty}\beta e^{-(\alpha/2+\beta)y}\,dy=\frac{\mathbf{2\,\beta}}{\boldsymbol{\alpha}+\mathbf{2\,\beta}}.$$

6. Let X and Y be two independent random variables, X has a Geometric distribution with the probability of "success" $p = {}^1/_3$, Y has a Poisson distribution with mean 3. That is,

$$p_X(x) = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1}, \qquad x = 1, 2, 3, \ldots,$$

$$p_Y(y) = \frac{3^y e^{-3}}{y!}, \quad y = 0, 1, 2, 3, \ldots.$$

a) Find $P(X = Y)$.

$$P(X = Y) = \sum_{k=1}^{\infty} p_X(k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{k-1} \cdot \frac{3^k e^{-3}}{k!}$$

$$= e^{-3} \cdot \sum_{k=1}^{\infty} \frac{2^{k-1}}{k!} = \frac{e^{-3}}{2} \cdot \left[\sum_{k=0}^{\infty} \frac{2^k}{k!} - 1 \right] = \frac{e^{-3}}{2} \cdot \left[e^2 - 1 \right]$$

$$= \frac{e^{-1} - e^{-3}}{2} \approx 0.159.$$

b) Find $P(X = 2Y)$.

$$P(X = 2Y) = \sum_{k=1}^{\infty} p_X(2k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{2k-1} \cdot \frac{3^k e^{-3}}{k!}$$

$$= \frac{e^{-3}}{2} \cdot \sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k \cdot \frac{1}{k!} = \frac{e^{-3}}{2} \cdot \left[e^{4/3} - 1 \right] \approx 0.069544.$$

c) Find $P(X > Y)$.

$$P(X > Y) = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1} \cdot \frac{3^y e^{-3}}{y!} = \sum_{y=0}^{\infty} \left(\frac{2}{3}\right)^y \cdot \frac{3^y e^{-3}}{y!}$$

$$= e^{-3} \cdot \sum_{y=0}^{\infty} \frac{2^y}{y!} = e^{-1} \approx 0.368.$$

2.4 Covariance and Correlation Coefficient

Covariance of X and Y

$$\sigma_{XY} = \mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

(a) $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$;

(b) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$;

(c) $\mathrm{Cov}(a X + b, Y) = a \,\mathrm{Cov}(X, Y)$;

(d) $\mathrm{Cov}(X + Y, W) = \mathrm{Cov}(X, W) + \mathrm{Cov}(Y, W)$.

$$\begin{aligned}\mathrm{Cov}(a X + b Y, c X + d Y) \\ &= a c \,\mathrm{Var}(X) + (a d + b c)\,\mathrm{Cov}(X, Y) + b d \,\mathrm{Var}(Y).\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(a X + b Y) &= \mathrm{Cov}(a X + b Y, a X + b Y) \\ &= a^2 \,\mathrm{Var}(X) + 2 a b \,\mathrm{Cov}(X, Y) + b^2 \,\mathrm{Var}(Y).\end{aligned}$$

0. Find in terms of σ_X^2, σ_Y^2, and σ_{XY}:

a) $\mathrm{Cov}(2X + 3Y, X - 2Y)$,

$$\mathrm{Cov}(2X + 3Y, X - 2Y) = 2\,\mathrm{Var}(X) - \mathrm{Cov}(X, Y) - 6\,\mathrm{Var}(Y).$$

b) $\mathrm{Var}(2X + 3Y)$,

$$\begin{aligned}\mathrm{Var}(2X + 3Y) &= \mathrm{Cov}(2X + 3Y, 2X + 3Y) \\ &= 4\,\mathrm{Var}(X) + 12\,\mathrm{Cov}(X, Y) + 9\,\mathrm{Var}(Y).\end{aligned}$$

c) $\mathrm{Var}(X - 2Y)$.

$$\begin{aligned}\mathrm{Var}(X - 2Y) &= \mathrm{Cov}(X - 2Y, X - 2Y) \\ &= \mathrm{Var}(X) - 4\,\mathrm{Cov}(X, Y) + 4\,\mathrm{Var}(Y).\end{aligned}$$

$$E(aX+bY) = aE(X)+bE(Y),$$
$$\mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}(X)+2ab\,\mathrm{Cov}(X,Y)+b^2\,\mathrm{Var}(Y).$$

If $X_1, X_2, \dots, X_n$ are n random variables and $a_0, a_1, a_2, \dots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$ has mean

$$E(U) = a_0 + a_1 E(X_1) + a_2 E(X_2) + \dots + a_n E(X_n)$$

and variance

$$\mathrm{Var}(U) = \sum_{i=1}^{n} a_i^2\,\mathrm{Var}\left(X_i\right) + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} a_i a_j\,\mathrm{Cov}\left(X_i, X_j\right)$$

$$= \sum_{i=1}^{n} a_i^2\,\mathrm{Var}\left(X_i\right) + 2\sum_{i<j}\sum a_i a_j\,\mathrm{Cov}\left(X_i, X_j\right)$$

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\,\sigma_Y} = \frac{\mathrm{Cov}(X,\,Y)}{\sqrt{\mathrm{Var}(X)}\cdot\sqrt{\mathrm{Var}(Y)}} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\cdot\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

(a) $-1 \le \rho_{XY} \le 1$;

(b) ρ_{XY} is either $+1$ or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$E(g(X)\cdot h(Y)) = E(g(X))\cdot E(h(Y)).$$

$$\Rightarrow \qquad \mathrm{Cov}(X,Y) = \sigma_{XY} = 0, \qquad \mathrm{Corr}(X,Y) = \rho_{XY} = 0.$$

If $X_1, X_2, \dots, X_n$ are n independent random variables and $a_0, a_1, a_2, \dots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$ has variance

$$\mathrm{Var}(U) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) + \dots + a_n^2\,\mathrm{Var}(X_n)$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$E(X) = 1.75$,

$E(Y) = 0.8$,

$E(XY) = 1.5$.

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

$$\text{Cov}(X, Y) = \sigma_{XY} = 1.5 - 1.75 \cdot 0.8 = \mathbf{0.10}.$$

$$E(X^2) = 1 \times 0.25 + 4 \times 0.75 = 3.25.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 3.25 - 1.75^2 = 0.1875.$$

$$E(Y^2) = 0 \times 0.40 + 1 \times 0.40 + 4 \times 0.20 = 1.2.$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1.2 - 0.8^2 = 0.56.$$

$$\text{Corr}(X, Y) = \rho_{XY} = \frac{0.10}{\sqrt{0.1875} \cdot \sqrt{0.56}} \approx \mathbf{0.3086}.$$

2. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60\, x^2\, y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall: $f_X(x) = 30\, x^2 (1 - x)^2, \quad 0 < x < 1,$ $\quad E(X) = \frac{1}{2},$

$f_Y(y) = 20\, y (1 - y)^3, \quad 0 < y < 1,$ $\quad E(Y) = \frac{1}{3},$ $\quad E(XY) = \frac{1}{7}.$

$$\text{Cov}(X, Y) = \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{3} = -\frac{\mathbf{1}}{\mathbf{42}}. \qquad \text{Var}(X) = \frac{9}{252}. \qquad \text{Var}(Y) = \frac{8}{252}.$$

$$\rho_{XY} = \frac{-1/42}{\sqrt{9/252} \cdot \sqrt{8/252}} = -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = -\frac{\sqrt{\mathbf{2}}}{\mathbf{2}} \approx \mathbf{-0.7071}.$$

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Cov}(X,Y) = \sigma_{XY}$ and $\mathrm{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = x+\frac{1}{2},\ 0<x<1.$ $f_Y(y) = y+\frac{1}{2},\ 0<y<1.$

$$\mu_X = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \left[\frac{1}{3}x^3+\frac{1}{4}x^2\right]_0^1 = \frac{7}{12};$$

$$\mu_Y = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \frac{7}{12};$$

$$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \left[\frac{1}{4}x^4+\frac{1}{6}x^3\right]_0^1 = \frac{5}{12},$$

$$\sigma_X^2 = E(X^2)-\mu_X^2 = \frac{5}{12}-\left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Similarly, $\sigma_Y^2 = \frac{11}{144}$.

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 \left(x^2y+xy^2\right)dx\,dy = \int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \frac{1}{3}.$$

$$\mathrm{Cov}(X,Y) = \frac{1}{3}-\frac{7}{12}\cdot\frac{7}{12} = -\mathbf{\frac{1}{144}}. \qquad \rho_{XY} = \frac{-1/144}{\sqrt{11/144}\cdot\sqrt{11/144}} = -\mathbf{\frac{1}{11}}.$$

4. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 12\,x(1-x)e^{-2y} & 0\le x\le 1, y\ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{Cov}(X,Y) = \sigma_{XY}$ and $\mathrm{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 6\,x(1-x),\ 0<x<1.$ $f_Y(y) = 2e^{-2y},\ y>0.$

Since X and Y are independent,

$\mathrm{Cov}(X,Y) = \sigma_{XY} = \mathbf{0},$ $\mathrm{Corr}(X,Y) = \rho_{XY} = \mathbf{0}.$