STAT 410

Practice Problems 4

Fall 2020
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1. Let X and Y have the joint p.d.f.

$$f_{XY}(x, y) = 20\, x^2 y^3, \quad 0 < x < 1, \; 0 < y < \sqrt{x}, \quad \text{zero elsewhere.}$$

Recall (Practice Problems 3):

$$f_X(x) = 5\, x^4, \quad 0 < x < 1, \qquad\qquad f_Y(y) = \frac{20}{3} \cdot \left(y^3 - y^9 \right), \quad 0 < y < 1.$$

m) Find $f_{X|Y}(x|y)$.

n) Find $E(X \mid Y = y)$.

o) Find $f_{Y|X}(y|x)$.

p) Find $E(Y \mid X = x)$.

2. Let X and Y be two random variables with joint p.d.f.

$$f(x, y) = 64\, x \exp\{-4y\} = 64\, x e^{-4y}, \quad 0 < x < y < \infty, \quad \text{zero elsewhere.}$$

Recall (Practice Problems 3):

$$f_X(x) = 16\, x e^{-4x}, \quad 0 < x < \infty, \qquad\qquad f_Y(y) = 32\, y^2 e^{-4y}, \quad 0 < y < \infty.$$

g) Find $P(X > 2 \mid Y = 5)$.

h) Find $E(X \mid Y = y), \quad y > 0$.

i) Find $P(Y > 5 \mid X = 2)$.

j) Find $E(Y \mid X = x), \quad x > 0$.

2.5. Let the joint probability density function for (X, Y) be

$$f(x, y) = C\, x\, y, \qquad x > 0, \quad y > 0, \quad x^2 + (y+3)^2 < 25,$$

zero elsewhere.

a) Find the value of C so that $f(x, y)$ is a valid joint p.d.f.

b) Find $P(2X + Y > 2)$.

c) Find $P(X - 3Y > 0)$.

d) Find $P(X > 2 \mid Y = 1)$.

3. Let X denote the number of times a photocopy machine will malfunction: 0, 1, or 2 times, on any given week. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x, y)$ is presented in the table below:

	x			
y	0	1	2	$p_Y(y)$
0	0.15	0.10	0.05	0.30
1	0.10	0.25	0.15	0.50
2	0	0.05	0.15	0.20
$p_X(x)$	0.25	0.40	0.35	1.00

e) Construct the probability distribution of $E(Y \mid X)$.

4. Suppose that the random variables X and Y have joint p.d.f. $f(x, y)$ given by

$$f(x, y) = 6x^2 y, \quad 0 < x < y, \ x + y < 2, \qquad \text{zero elsewhere.}$$

Recall (Practice Problems 3):

$$f_X(x) = 12x^2(1-x), \quad 0 < x < 1. \qquad f_Y(y) = \begin{cases} 2y^4 & 0 < y < 1 \\ 2y(2-y)^3 & 1 < y < 2 \end{cases}$$

j) Find $f_{Y|X}(y|x)$.

k) Find $P(Y > 1.25 \mid X = 0.25)$.

l) Find $P(Y < 0.90 \mid X = 0.75)$.

m) Find $E(Y \mid X = x)$.

n) Find $f_{X|Y}(x|y)$.

o) Find $P(X < 0.20 \mid Y = 0.50)$.

p) Find $P(X > 0.60 \mid Y = 0.75)$.

q) Find $P(X > 0.40 \mid Y = 1.20)$.

r) Find $P(X < 0.10 \mid Y = 1.50)$.

s) Find $E(X \mid Y = y)$.

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y) = \begin{cases} x\,e^{-x(1+y)} & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall (Practice Problems 3):

$$f_X(x) = e^{-x}, \quad x \ge 0, \qquad\qquad f_Y(y) = \frac{1}{(1+y)^2}, \quad y \ge 0.$$

d) Find $f_{Y|X}(y|x)$.

e) Find $E(Y \mid X = x)$.

f) Find $P(Y > 0.8 \mid X = 0.5)$.

g) Find $P(Y < 1.5 \mid X = 0.6)$.

h) Find $f_{X|Y}(x|y)$.

i) Find $E(X \mid Y = y)$.

j) Find $P(X > 1 \mid Y = 2)$.

6. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0 < x < 2, \quad 0 < y < 1, \qquad \text{zero otherwise.}$$

Recall (Practice Problems 3):

$$f_X(x) = \frac{2x+1}{6}, \quad 0 < x < 2, \qquad\qquad f_Y(y) = \frac{2+2y}{3}, \quad 0 < y < 1.$$

e) Find $P(Y > 0.5 \mid X = 0.75)$.

f) Find $P(Y > 0.5 \mid X < 0.75)$.

g) Find $E(X \mid Y = y)$.

7. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{2}, \quad x > 0, \quad y > 0, \quad 3x + y < 3, \qquad \text{zero otherwise.}$$

Recall (Practice Problems 3):

$$f_X(x) = \frac{9}{4} - 3x + \frac{3}{4}x^2, \quad 0 < x < 1, \qquad f_Y(y) = \frac{1}{4} + \frac{1}{3}y - \frac{5}{36}y^2, \quad 0 < y < 3.$$

e) Find $P(Y > 0.8 \mid X = 0.5)$.

f) Find $P(Y > 0.8 \mid X < 0.5)$.

g) Find $E(Y \mid X = x)$.

Answers:

1. Let X and Y have the joint p.d.f.

$$f_{XY}(x,y) = 20\,x^2 y^3, \quad 0<x<1, \;\; 0<y<\sqrt{x}, \quad \text{zero elsewhere.}$$

Recall (Practice Problems 3):

$$f_X(x) = 5\,x^4, \quad 0<x<1, \qquad\qquad f_Y(y) = \frac{20}{3}\cdot\left(y^3 - y^9\right), \quad 0<y<1.$$

m) Find $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{3x^2}{1-y^6}, \qquad\qquad y^2<x<1.$$

n) Find $E(X \mid Y=y)$.

$$E(X \mid Y=y) = \int_{y^2}^{1} x\cdot\frac{3x^2}{1-y^6}\,dx = \frac{3}{4}\cdot\frac{1-y^8}{1-y^6}, \quad 0<y<1.$$

o) Find $f_{Y|X}(y|x)$.

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{4y^3}{x^2}, \quad 0<y<\sqrt{x}.$$

p) Find $E(Y \mid X=x)$.

$$E(Y \mid X=x) = \int_{0}^{\sqrt{x}} y\cdot\frac{4y^3}{x^2}\,dy = \frac{4}{5}\sqrt{x}, \quad 0<x<1.$$

2. Let X and Y be two random variables with joint p.d.f.

$$f(x, y) = 64\, x \exp\{-4y\} = 64\, x\, e^{-4y}, \qquad 0 < x < y < \infty, \qquad \text{zero elsewhere.}$$

Recall (Practice Problems 3):

$$f_X(x) = 16\, x\, e^{-4x}, \quad 0 < x < \infty, \qquad f_Y(y) = 32\, y^2\, e^{-4y}, \quad 0 < y < \infty.$$

g) Find $P(X > 2 \mid Y = 5)$.

$$f_{X|Y}(x|y) = \frac{64\, x\, e^{-4y}}{32\, y^2\, e^{-4y}} = \frac{2x}{y^2}, \qquad 0 < x < y.$$

$$P(X > 2 \mid Y = 5) = \int_2^5 \frac{2x}{25}\, dx = \frac{25 - 4}{25} = \frac{21}{25} = 0.84.$$

h) Find $E(X \mid Y = y)$, $\quad y > 0$.

$$E(X \mid Y = y) = \int_0^y x \cdot \frac{2x}{y^2}\, dx = \frac{2}{3}\, y.$$

i) Find $P(Y > 5 \mid X = 2)$.

$$f_{Y|X}(y|x) = \frac{64\, x\, e^{-4y}}{16\, x\, e^{-4x}} = 4\, e^{-4(y - x)}, \qquad x < y < \infty.$$

$$P(Y > 5 \mid X = 2) = \int_5^\infty 4 e^{-4(y-2)}\, dy = e^{-12}.$$

j) Find $E(Y \mid X = x)$, $\quad x > 0$.

$$E(Y \mid X = x) = \int_x^\infty y \cdot 4 e^{-4(y - x)}\, dy = x + \frac{1}{4}, \qquad x > 0.$$

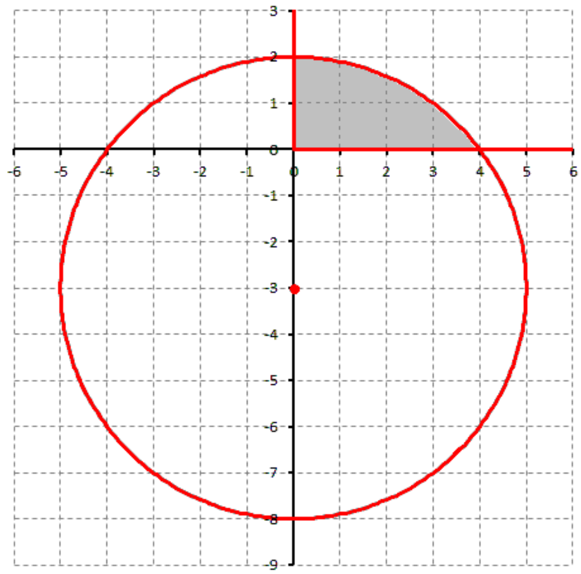
2.5. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y,$$

$$x > 0, \qquad y > 0,$$

$$x^2 + (y+3)^2 < 25,$$

zero elsewhere.



a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

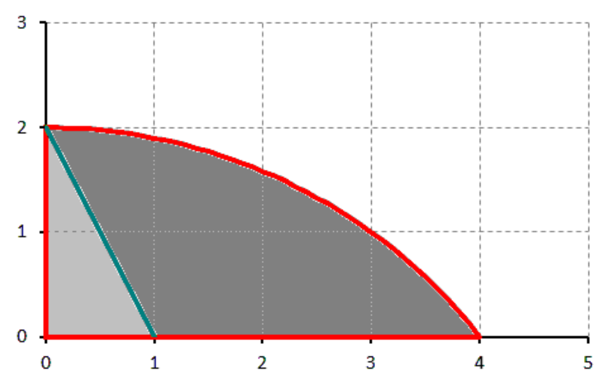
Must have $$1 = \int_0^2 \left[\int_0^{\sqrt{25-(y+3)^2}} C\,x\,y\,dx \right] dy = \int_0^2 \frac{C}{2}\,y\left[25-(y+3)^2\right]dy$$

$$= \frac{C}{2}\int_0^2 \left[16\,y - 6\,y^2 - y^3\right]dy$$

$$= \frac{C}{2}\left[8\,y^2 - 2\,y^3 - \frac{1}{4}\,y^4\right]\Bigg|_0^2 = 6\,C.$$

$$\Rightarrow \qquad C = \mathbf{\frac{1}{6}}.$$

b) Find $P(2X + Y > 2)$.



$$1 - \int_0^1 \left(\int_0^{2-2x} \frac{1}{6}\,x\,y\,dy \right) dx$$

$$= 1 - \int_0^1 \frac{1}{12}(2-2x)^2\,x\,dx$$

$$= 1 - \int_0^1 \frac{1}{3}\left(x - 2x^2 + x^3\right)dx = 1 - \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1 - \frac{1}{36} = \mathbf{\frac{35}{36}}.$$

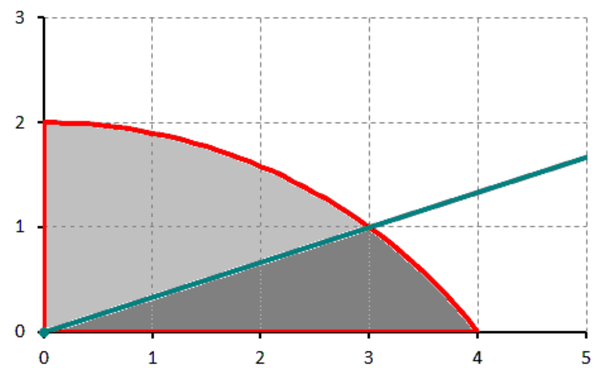
OR

$$1 - \int_0^2 \left(\int_0^{\frac{2-y}{2}} \frac{1}{6}\, x\, y\, dx \right) dy = \ldots \qquad\qquad \int_0^2 \left(\int_{\frac{2-y}{2}}^{\sqrt{25-(y+3)^2}} \frac{1}{6}\, x\, y\, dx \right) dy = \ldots$$

OR

$$\int_0^1 \left(\int_{2-2x}^{-3+\sqrt{25-x^2}} \frac{1}{6}\, x\, y\, dy \right) dx + \int_1^4 \left(\int_0^{-3+\sqrt{25-x^2}} \frac{1}{6}\, x\, y\, dy \right) dx = \ldots$$

c) Find $P(X - 3Y > 0)$.



$$P(X - 3Y > 0) = P(X > 3Y)$$

$$= \int_0^1 \left[\int_{3y}^{\sqrt{25-(y+3)^2}} \frac{1}{6}\, x\, y\, dx \right] dy$$

$$= \int_0^1 \frac{1}{12}\, y \left[25 - (y+3)^2 - 9y^2 \right] dy = \int_0^1 \frac{1}{12}\, y \left[16 - 6y - 10y^2 \right] dy$$

$$= \int_0^1 \left[\frac{4}{3} y - \frac{1}{2} y^2 - \frac{5}{6} y^3 \right] dy = \frac{2}{3} - \frac{1}{6} - \frac{5}{24} = \mathbf{\frac{7}{24}} \approx 0.2916667.$$

d) Find $P(X > 2 \mid Y = 1)$.

$$f_Y(y) = \int_0^{\sqrt{25-(y+3)^2}} \frac{1}{6}\, x\, y\, dx = \frac{1}{12}\, y \left[25 - (y+3)^2 \right] = \frac{1}{12}\, y \left[16 - 6y - y^2 \right]$$

$$= \frac{1}{12}\, y\, (8+y)(2-y), \qquad 0 < y < 2.$$

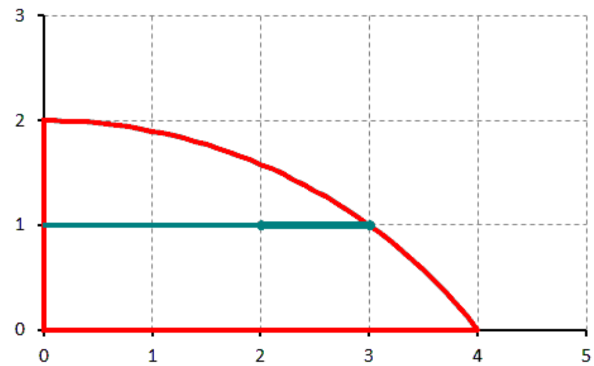
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2x}{25-(y+3)^2} = \frac{2x}{16-6y-y^2} = \frac{2x}{(8+y)(2-y)},$$

$$0 < x < \sqrt{25-(y+3)^2} = \sqrt{16-6y-y^2} = \sqrt{(8+y)(2-y)}.$$

$y = 1 \qquad \Rightarrow \qquad 0 < x < \sqrt{9} = 3.$

$$\mathrm{P}(X > 2 \mid Y = 1) = \int_2^3 \frac{2x}{9}\,dx$$

$$= \frac{3^2 - 2^2}{9} = \mathbf{\frac{5}{9}}.$$



3. Let X denote the number of times a photocopy machine will malfunction: 0, 1, or 2 times, on any given week. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x, y)$ is presented in the table below:

	x			$p_Y(y)$
y	0	1	2	
0	0.15	0.10	0.05	0.30
1	0.10	0.25	0.15	0.50
2	0	0.05	0.15	0.20
$p_X(x)$	0.25	0.40	0.35	1.00

e) Construct the probability distribution of $E(Y|X)$.

y	$p_{Y\|X}(y\|0)$
0	$^{3}/_{5}$
1	$^{2}/_{5}$
2	0

y	$p_{Y\|X}(y\|0)$
0	$^{2}/_{8}$
1	$^{5}/_{8}$
2	$^{1}/_{8}$

y	$p_{Y\|X}(y\|0)$
0	$^{1}/_{7}$
1	$^{3}/_{7}$
2	$^{3}/_{7}$

$E(Y|X=0) = {}^{2}/_{5}$ $\qquad E(Y|X=1) = {}^{7}/_{8}$ $\qquad E(Y|X=2) = {}^{9}/_{7}$

$E(Y|X)$:

x	$E(Y\|X=x)$	$p_X(x)$
0	$\mathbf{^{2}/_{5}}$	**0.25**
1	$\mathbf{^{7}/_{8}}$	**0.40**
2	$\mathbf{^{9}/_{7}}$	**0.35**

4. Suppose that the random variables X and Y have joint p.d.f. $f(x, y)$ given by

$$f(x,y) = 6x^2 y, \quad 0 < x < y, \; x + y < 2, \qquad \text{zero elsewhere.}$$

Recall (Practice Problems 3):

$$f_X(x) = 12x^2(1-x), \; 0 < x < 1. \qquad f_Y(y) = \begin{cases} 2y^4 & 0 < y < 1 \\ 2y(2-y)^3 & 1 < y < 2 \end{cases}$$

j) Find $f_{Y|X}(y|x)$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{y}{2-2x}, \qquad x < y < 2 - x, \qquad 0 < x < 1.$$

k) Find $P(Y > 1.25 \mid X = 0.25)$.

$$f_{Y|X}(y|0.25) = \frac{y}{1.5} = \frac{2y}{3}, \qquad 0.25 < y < 1.75.$$

$$P(Y > 1.25 \mid X = 0.25) = \int_{1.25}^{1.75} \frac{2y}{3}\, dy = \mathbf{0.50}.$$

l) Find $P(Y < 0.90 \mid X = 0.75)$.

$$f_{Y|X}(y|0.75) = \frac{y}{0.5} = 2y, \qquad 0.75 < y < 1.25.$$

$$P(Y < 0.90 \mid X = 0.75) = \int_{0.75}^{0.90} 2y\, dy = \mathbf{0.2475}.$$

m) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$.

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x) = \int_{x}^{2-x} \frac{y^2}{2-2x}\,dy = \frac{(2-x)^3 - x^3}{6-6x} = \frac{8 - 12x + 6x^2 - 2x^3}{6-6x},$$

$$= \frac{4 - 2x + x^2}{3}, \qquad 0 < x < 1.$$

n) Find $f_{\mathrm{X}|\mathrm{Y}}(x|y)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x|y) = \frac{f(x,y)}{f_{\mathrm{Y}}(y)} = \begin{cases} \dfrac{3x^2}{y^3}, & 0 < x < y, \qquad 0 < y < 1 \\[2ex] \dfrac{3x^2}{(2-y)^3}, & 0 < x < 2-y, \qquad 1 < y < 2 \end{cases}$$

o) Find $\mathrm{P}(\mathrm{X} < 0.20 \mid \mathrm{Y} = 0.50)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x|0.50) = \frac{3x^2}{0.125} = 24x^2, \qquad 0 < x < 0.50.$$

$$\mathrm{P}(\mathrm{X} < 0.20 \mid \mathrm{Y} = 0.50) = \int_{0}^{0.20} 24x^2\,dx = \mathbf{0.064}.$$

p) Find $\mathrm{P}(\mathrm{X} > 0.60 \mid \mathrm{Y} = 0.75)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x|0.75) = \frac{3x^2}{0.421875} = \frac{64x^2}{9}, \qquad 0 < x < 0.75.$$

$$\mathrm{P}(\mathrm{X} > 0.60 \mid \mathrm{Y} = 0.75) = \int_{0.60}^{0.75} \frac{64x^2}{9}\,dx = \mathbf{0.488}.$$

q) Find $\mathrm{P}(\mathrm{X} < 0.40 \mid \mathrm{Y} = 1.20)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x \mid 1.20) = \frac{3x^2}{0.512} = \frac{375x^2}{64}, \qquad 0 < x < 0.80.$$

$$\mathrm{P}(\mathrm{X} < 0.40 \mid \mathrm{Y} = 1.20) = \int_0^{0.40} \frac{375x^2}{64}\,dx = \mathbf{0.125}.$$

r) Find $\mathrm{P}(\mathrm{X} > 0.10 \mid \mathrm{Y} = 1.50)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x \mid 1.50) = \frac{3x^2}{0.125} = 24x^2, \qquad 0 < x < 0.50.$$

$$\mathrm{P}(\mathrm{X} > 0.10 \mid \mathrm{Y} = 1.50) = \int_{0.10}^{0.50} 24x^2\,dx = \mathbf{0.992}.$$

s) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y)$.

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y) = \begin{cases} \displaystyle\int_0^{y} \frac{3x^3}{y^3}\,dx & 0 < y < 1 \\ \displaystyle\int_0^{2-y} \frac{3x^3}{(2-y)^3}\,dx & 1 < y < 2 \end{cases}$$

$$= \begin{cases} \dfrac{3}{4}y & 0 < y < 1 \\ \dfrac{3}{4}(2-y) & 1 < y < 2 \end{cases}$$

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y)=\begin{cases} x\,e^{-x(1+y)} & x\ge 0,\ y\ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall (Practice Problems 3):

$$f_X(x) = e^{-x}, \quad x \ge 0, \qquad\qquad f_Y(y) = \frac{1}{(1+y)^2}, \quad y \ge 0.$$

d) Find $f_{Y|X}(y|x)$.

$$f_{Y|X}(y|x) = x\,e^{-xy}, \quad y \ge 0.$$

Exponential with mean $\theta = \dfrac{1}{x}$.

e) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \theta = \frac{1}{x}.$$

f) Find $P(Y > 0.8 \mid X = 0.5)$.

$$P(Y > 0.8 \mid X = 0.5) = e^{-0.5 \cdot 0.8} = \boldsymbol{e^{-0.4}} \approx 0.67032.$$

g) Find $P(Y < 1.5 \mid X = 0.6)$.

$$P(Y < 1.5 \mid X = 0.6) = 1 - e^{-0.6 \cdot 1.5} = \mathbf{1 - \boldsymbol{e^{-0.9}}} \approx 0.59343.$$

h) Find $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = (1+y)^2\, x\, e^{-x(1+y)}, \quad x \ge 0.$$

Gamma with $\alpha = 2$ and $\theta = \dfrac{1}{1+y}$.

i) Find $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \alpha\,\theta = \frac{2}{1+y}.$$

j) Find $P(X > 1 \mid Y = 2)$.

$$P(X > 1 \mid Y = 2) = \int_1^{\infty} 9\,x\,e^{-3x}\,dx = \left(-3\,x\,e^{-3x} - e^{-3x}\right)\Big|_1^{\infty}$$

$$= \mathbf{4\,e^{-3}} \approx 0.19915.$$

OR

$$P(X > 1 \mid Y = 2) = P(\text{Gamma}(\alpha = 2, \theta = \tfrac{1}{3}) > 1) = P(\text{Poisson}(3) \le 2 - 1)$$

$$= \frac{3^0\,e^{-3}}{0!} + \frac{3^1\,e^{-3}}{1!} = \mathbf{4\,e^{-3}} \approx 0.19915.$$

6. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0 < x < 2, \quad 0 < y < 1, \qquad \text{zero otherwise.}$$

Recall (Practice Problems 3):

$$f_X(x) = \frac{2x+1}{6}, \quad 0 < x < 2, \qquad f_Y(y) = \frac{2+2y}{3}, \quad 0 < y < 1.$$

e) Find $P(Y > 0.5 \mid X = 0.75)$.

$$f_X(x) = \int_0^1 \frac{x+y}{3}\,dy = \left(\frac{xy}{3} + \frac{y^2}{6}\right)\Bigg|_0^1 = \frac{2x+1}{6}, \qquad 0 < x < 2.$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{x+y}{3}}{\frac{2x+1}{6}} = \frac{2x+2y}{2x+1}, \qquad 0 < y < 1.$$

$$f_{Y|X}(y|0.75) = \frac{1.5+2y}{2.5}, \qquad 0 < y < 1.$$

$$P(Y > 0.5 \mid X = 0.75) = \int_{0.5}^{1} \frac{1.5+2y}{2.5}\,dy = \mathbf{0.6}.$$

f) Find $P(Y > 0.5 \mid X < 0.75)$.

Def $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.

$$P(B) = P(X < 0.75) = \int_0^{0.75} \frac{2x+1}{6}\,dx = \frac{x^2+x}{6}\Bigg|_0^{0.75} = \frac{21}{96} = \frac{14}{64}.$$

$$\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{Y} > 0.5 \cap \mathrm{X} < 0.75) = \int_0^{0.75} \left(\int_{0.5}^{1} \frac{x+y}{3}\, dy \right) dx$$

$$= \int_0^{0.75} \left(\frac{xy}{3} + \frac{y^2}{6} \right) \Bigg|_{0.5}^{1} dx = \int_0^{0.75} \left(\frac{x}{6} + \frac{1}{8} \right) dx = \left(\frac{x^2}{12} + \frac{x}{8} \right) \Bigg|_0^{0.75} = \frac{9}{64}.$$

$$\mathrm{P}(\mathrm{Y} > 0.5 \mid \mathrm{X} < 0.75) = \frac{9/64}{14/64} = \mathbf{\frac{9}{14}} \approx 0.642857.$$

g) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y)$.

$$f_{\mathrm{Y}}(y) = \int_0^2 \frac{x+y}{3}\, dx = \left(\frac{x^2}{6} + \frac{xy}{3} \right) \Bigg|_0^2 = \frac{2+2y}{3}, \qquad 0 < y < 1.$$

$$f_{\mathrm{X}|\mathrm{Y}}(x|y) = \frac{f_{\mathrm{X,Y}}(x,y)}{f_{\mathrm{Y}}(y)} = \frac{\frac{x+y}{3}}{\frac{2+2y}{3}} = \frac{x+y}{2+2y}, \qquad 0 < x < 2.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y) = \int_0^2 x \cdot \frac{x+y}{2+2y}\, dx = \frac{4+3y}{3+3y}, \qquad 0 < y < 1.$$

7. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{2}, \quad x > 0, \quad y > 0, \quad 3x + y < 3, \qquad \text{zero otherwise.}$$

Recall (Practice Problems 3):

$$f_X(x) = \frac{9}{4} - 3x + \frac{3}{4}x^2, \quad 0 < x < 1,$$

$$f_Y(y) = \frac{1}{4} + \frac{1}{3}y - \frac{5}{36}y^2, \quad 0 < y < 3.$$

e) Find $P(Y > 0.8 \mid X = 0.5)$.

$$f_X(x) = \int_0^{3-3x} \frac{x+y}{2}\,dy = \frac{9}{4} - 3x + \frac{3}{4}x^2, \qquad 0 < x < 1.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{x+y}{2}}{\frac{9}{4} - 3x + \frac{3}{4}x^2}, \qquad 0 < y < 3 - 3x, \qquad 0 < x < 1.$$

$$f_{Y|X}(y|0.5) = \frac{0.50 + y}{1.875}, \qquad 0 < y < 1.50.$$

$$P(Y > 0.8 \mid X = 0.5) = \int_{0.8}^{1.5} \frac{0.50 + y}{1.875}\,dy = \left.\frac{y + y^2}{3.75}\right|_{0.8}^{1.5} = \frac{2.31}{3.75} = \mathbf{\frac{77}{125}} = \mathbf{0.616}.$$

f) Find $P(Y > 0.8 \mid X < 0.5)$.

Def $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.

$$P(B) = P(X < 0.5) = \int_0^{0.5} \left(\frac{9}{4} - 3x + \frac{3}{4}x^2\right) dx = \left.\frac{9}{4}x - \frac{3}{2}x^2 + \frac{1}{4}x^3\right|_0^{0.5} = \frac{25}{32}.$$

$$\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{Y} > 0.8 \cap \mathrm{X} < 0.5) = \int_0^{0.5} \left(\int_{0.8}^{3-3x} \frac{x+y}{2}\, dy \right) dx$$

$$= \int_0^{0.5} \frac{209 - 340x + 75x^2}{100}\, dx = \frac{521}{800}.$$

$$\mathrm{P}(\mathrm{Y} > 0.8 \,|\, \mathrm{X} < 0.5) = \frac{\frac{521}{800}}{\frac{25}{32}} = \mathbf{\frac{521}{625}} = \mathbf{0.8336}.$$

g) Find $\mathrm{E}(\mathrm{Y} \,|\, \mathrm{X} = x)$.

$$\mathrm{E}(\mathrm{Y} \,|\, \mathrm{X} = x) = \int_0^{3-3x} y \cdot \frac{\frac{x+y}{2}}{\frac{9}{4} - 3x + \frac{3}{4}x^2}\, dy = \int_0^{3-3x} \frac{2xy + 2y^2}{9 - 12x + 3x^2}\, dy$$

$$= \frac{x(3-3x)^2 + \frac{2}{3}(3-3x)^3}{9 - 12x + 3x^2} = \frac{18 - 45x + 36x^2 - 9x^3}{9 - 12x + 3x^2}$$

$$= \frac{6 - 15x + 12x^2 - 3x^3}{3 - 4x + x^2} = \frac{6 - 9x + 3x^2}{3 - x} = \frac{3(1-x)(2-x)}{3-x},$$

$0 < x < 1$.