STAT 410

Examples for 10/23/2020 (2)

Fall 2020
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Examples for 10/19/2020 (2) (continued)

1. Let $\beta > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

Recall: $W = \ln X$ has a $\text{Gamma}(\alpha = 2, \theta = \frac{1}{\beta})$ distribution. $(\lambda = \beta)$

i) Recall: The maximum likelihood estimator for β is $\hat{\beta} = \dfrac{2n}{\sum_{i=1}^{n} \ln X_i}$.

Is $\hat{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

j) Assume $\beta > 1$.

Recall: A method of moments estimator for β is $\tilde{\beta} = \dfrac{\sqrt{\overline{X}}}{\sqrt{\overline{X}} - 1}$.

Is $\tilde{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

Battleplan: ① WLLN, $\overline{\heartsuit} = \frac{1}{n} \cdot \sum_{i=1}^{n} \heartsuit_i \xrightarrow{P} E(\heartsuit)$.

② $\spadesuit \xrightarrow{P} a$, g is continuous at a $\Rightarrow$ $g(\spadesuit) \xrightarrow{P} g(a)$

Answers:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

Recall: $W = \ln X$ has a $\text{Gamma}(\alpha = 2, \theta = \frac{1}{\beta})$ distribution. $(\lambda = \beta)$

i) Recall: The maximum likelihood estimator for β is $\hat{\beta} = \dfrac{2n}{\sum_{i=1}^{n} \ln X_i}$.

Is $\hat{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

$$\hat{\beta} = \frac{2n}{\sum_{i=1}^{n} \ln X_i} = \frac{2}{\frac{1}{n}\cdot\sum_{i=1}^{n} \ln X_i} = \frac{2}{\frac{1}{n}\cdot\sum_{i=1}^{n} W_i} = \frac{2}{\overline{W}}.$$

By WLLN, $\overline{W} = \frac{1}{n}\cdot\sum_{i=1}^{n} W_i \xrightarrow{P} E(W) = \alpha\,\theta = \frac{2}{\beta}.$

$\spadesuit \xrightarrow{P} a$, g is continuous at a $\Rightarrow$ $g(\spadesuit) \xrightarrow{P} g(a)$

Since $g(x) = \frac{2}{x}$ is continuous at $\frac{2}{\beta}$,

$\hat{\beta} = g(\overline{W}) \xrightarrow{P} g(\frac{2}{\beta}) = \beta.$ $\qquad$ $\hat{\beta}$ is a consistent estimator of β.

j) Assume $\beta > 1$.

Recall: A method of moments estimator for β is $\tilde{\beta} = \dfrac{\sqrt{\overline{X}}}{\sqrt{\overline{X}} - 1}$.

Is $\tilde{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

By WLLN, $$\overline{X} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i \xrightarrow{P} \mu = \frac{\beta^2}{(\beta-1)^2}.$$

$$\spadesuit \xrightarrow{P} a, \; g \text{ is continuous at } a \;\Rightarrow\; g(\spadesuit) \xrightarrow{P} g(a)$$

Since $g(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 1}$ is continuous at $\dfrac{\beta^2}{(\beta-1)^2}$,

$$\tilde{\beta} = g(\overline{X}) \xrightarrow{P} g\left(\frac{\beta^2}{(\beta-1)^2}\right) = \beta.$$

$\tilde{\beta}$ is a consistent estimator of β.