Examples for 11/16/2020 (1)

$X_1, X_2, \dots, X_n$ i.i.d. p.d.f. or p.m.f. $f(x;\theta)$. $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$.

Likelihood Ratio:

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(\theta_0; x_1, x_2, \dots, x_n)}{L(\theta_1; x_1, x_2, \dots, x_n)}.$$

Neyman-Pearson Lemma:

$$C = \{ (x_1, x_2, \dots, x_n) : \lambda(x_1, x_2, \dots, x_n) \le k \}.$$

("Reject H_0 if $\lambda(x_1, x_2, \dots, x_n) \le k$ ")

is the best (most powerful) rejection region.

1. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a $N(\mu, \sigma^2)$ distribution (σ^2 is known). Use the likelihood ratio to find the best rejection region for the test $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(\mu_0; x_1, x_2, \dots, x_n)}{L(\mu_1; x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - \mu_0)^2 \right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - \mu_1)^2 \right\}}$$

$$= \exp\left\{ \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[(x_i - \mu_1)^2 - (x_i - \mu_0)^2 \right] \right\}$$

$$= \exp\left\{ \frac{n\left(\mu_1^2 - \mu_0^2\right)}{2\sigma^2} + \frac{n(\mu_0 - \mu_1)}{\sigma^2} \cdot \overline{x} \right\}$$

$$\lambda(x_1, x_2, \dots, x_n) \le k \quad \Leftrightarrow \quad (\mu_0 - \mu_1) \cdot \overline{x} \le k_1$$

$$\Leftrightarrow \quad \begin{cases} \overline{x} \ge c & \text{if } \mu_1 > \mu_0 \\ \overline{x} \le c & \text{if } \mu_1 < \mu_0 \end{cases}$$

2. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a Poisson distribution with mean λ. That is,

$$P(X_1 = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}, \qquad k = 0, 1, 2, 3, \dots .$$

Consider the test $H_0: \lambda = \lambda_0$ vs. $H_1: \lambda = \lambda_1$.

Show that the best rejection region is given by $\{(x_1, x_2, \dots, x_n): \sum_{i=1}^n x_i \ge c\}$ if $\lambda_1 > \lambda_0$, and by $\{(x_1, x_2, \dots, x_n): \sum_{i=1}^n x_i \le c\}$ if $\lambda_1 < \lambda_0$.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(\lambda_0;\, x_1, x_2, \dots, x_n)}{L(\lambda_1;\, x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^n \frac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!}}{\prod_{i=1}^n \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!}}.$$

$$= \left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^n x_i} e^{n(\lambda_1 - \lambda_0)}.$$

$$\lambda(x_1, x_2, \dots, x_n) \le k \qquad \Leftrightarrow \qquad \begin{cases} \sum_{i=1}^n x_i \ge c & \text{if} \quad \lambda_1 > \lambda_0 \\ \sum_{i=1}^n x_i \le c & \text{if} \quad \lambda_1 < \lambda_0 \end{cases}$$

3. Let X have a Binomial distribution with the number of trials n and with p either p_0 or p_1. We wish to test $H_0: p = p_0$ vs. $H_1: p = p_1$.
Show that the best rejection region is given by "Reject H_0 if $X \ge c$" if $p_1 > p_0$, and by" "Reject H_0 if $X \le c$" if $p_1 < p_0$.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(p_0; x)}{L(p_1; x)} = \frac{\binom{n}{x} p_0{}^x (1-p_0)^{n-x}}{\binom{n}{x} p_1{}^x (1-p_1)^{n-x}} = \left(\frac{\frac{p_0}{1-p_0}}{\frac{p_1}{1-p_1}}\right)^x \left(\frac{1-p_0}{1-p_1}\right)^n.$$

Since $\dfrac{p}{1-p}$, $0 < p < 1$, is strictly increasing,

$$\lambda(x_1, x_2, \dots, x_n) \le k \qquad \Leftrightarrow \qquad \begin{cases} x \ge c & \text{if} \quad p_1 > p_0 \\ x \le c & \text{if} \quad p_1 < p_0 \end{cases}$$