Examples for 09/18/2020 (1)

Fact: Let X and Y be continuous random variables with joint p.d.f. $f(x, y)$. Then

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

Proof:

$$F_{X+Y}(w) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{w-x} f(x, y)\, dy \right) dx.$$

let $u = y + x$, then $du = dy$, $y = u - x$,

$-\infty \to -\infty$, $w - x \to w$

$$F_{X+Y}(w) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{w} f(x, u-x)\, du \right) dx = \int_{-\infty}^{w} \left(\int_{-\infty}^{\infty} f(x, u-x)\, dx \right) du.$$

$$\Rightarrow \qquad f_{X+Y}(w) = F'_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx \qquad (\text{by FTC}).$$

Fact: Let X and Y be independent continuous random variables. Then

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx$$

(convolution)

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y) \cdot f_Y(y)\, dy$$

0. a) Let X and Y be two independent Exponential random variables with mean 1. Find the probability distribution of $Z = X + Y$. That is, find $f_Z(z) = f_{X+Y}(z)$.

$$f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(w-x) = \begin{cases} e^{-w+x} & w-x > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{-w+x} & x < w \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $w > 0$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx$$

$$= \int_{-\infty}^{0} 0 \cdot e^{-w+x}\, dx + \int_{0}^{w} e^{-x} \cdot e^{-w+x}\, dx + \int_{w}^{\infty} e^{-x} \cdot 0\, dx$$

$$= \int_{0}^{w} e^{-x} \cdot e^{-w+x}\, dx = e^{-w} \cdot \int_{0}^{w} dx = w e^{-w}, \qquad w > 0.$$

Case 2: $w < 0$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx = \int_{-\infty}^{\infty} 0\, dx = 0, \qquad w < 0.$$

½. Let X be an Exponential random variables with mean 1. Suppose the p.d.f. of Y is $f_Y(y) = 2y,\ 0 < y < 1,$ zero elsewhere. Assume that X and Y are independent. Find the p.d.f. of $W = X + Y$, $f_W(w) = f_{X+Y}(w)$.

$$f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y) \cdot f_Y(y)\,dy$$

$$f_X(w-y) = \begin{cases} e^{-(w-y)} & w-y > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{y-w} & y < w \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $0 < w < 1$. $\quad f_W(w) = \int_0^w e^{y-w} \cdot 2y\,dy = 2(e^{-w} - 1 + w).$

Case 2: $w > 1$. $\quad f_W(w) = \int_0^1 e^{y-w} \cdot 2y\,dy = 2e^{-w}.$

OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx$$

$$f_Y(w-x) = \begin{cases} 2(w-x) & 0 < w-x < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2(w-x) & w-1 < x < w \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $0 < w < 1$. $\quad f_W(w) = \int_0^w e^{-x} \cdot 2(w-x)\,dx = \ldots$

Case 2: $w > 1$. $\quad f_W(w) = \int_{w-1}^{w} e^{-x} \cdot 2(w-x)\,dx = \ldots \quad$ since $w - 1 > 0$

1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} 60\,x^2\,y & x>0,\ y>0,\ x+y<1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Consider W = X + Y. Find the p.d.f. of W, $f_W(w)$.

$$f_W(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx = \ldots$$

$x > 0$

$y > 0 \quad \Rightarrow \quad w - x > 0 \quad \Rightarrow \quad x < w$

$x + y < 1 \quad \Rightarrow \quad x + (w - x) < 1 \quad \Rightarrow \quad w < 1$

$$\ldots = \int_0^w 60\,x^2\,(w-x)\,dx = 20\,w^4 - 15\,w^4 = 5\,w^4, \qquad 0 < w < 1.$$

OR

$$f_W(w) = \int_{-\infty}^{\infty} f(w-y, y)\,dy = \ldots$$

$x > 0 \quad \Rightarrow \quad w - y > 0 \quad \Rightarrow \quad y < w$

$y > 0$

$x + y < 1 \quad \Rightarrow \quad (w - y) + y < 1 \quad \Rightarrow \quad w < 1$

$$\ldots = \int_0^w 60\,(w-y)^2\,y\,dy = 30\,w^4 - 40\,w^4 + 15\,w^4 = 5\,w^4, \qquad 0 < w < 1.$$

2. a) When a person applies for citizenship in Neverland, first he/she must wait X years for an interview, and then Y more years for the oath ceremony. Thus the total wait is W = X + Y years. Suppose that X and Y are independent, the p.d.f. of X is

$$f_X(x) = 2/x^3, \qquad x > 1, \qquad \text{zero otherwise,}$$

and Y has a Uniform distribution on interval $(0, 1)$.

Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

Hint: Consider two cases: $1 < w < 2$ and $w > 2$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx.$$

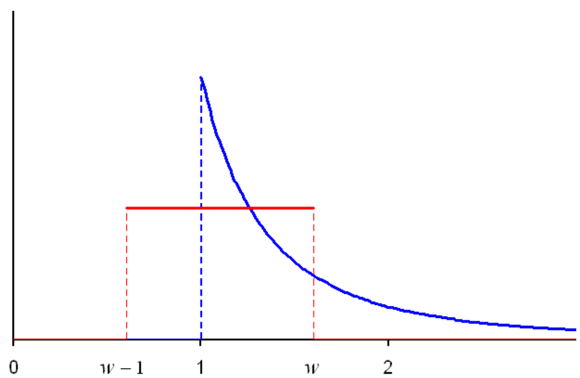
$$f_X(x) = 2/x^3, \qquad x > 1, \qquad \text{zero otherwise.}$$

$$f_Y(w-x) = 1, \qquad 0 < w-x < 1 \quad \text{OR} \quad w-1 < x < w, \qquad \text{zero otherwise.}$$

Case 1: $1 < w < 2$.

$\Rightarrow \qquad 0 < w-1 < 1.$

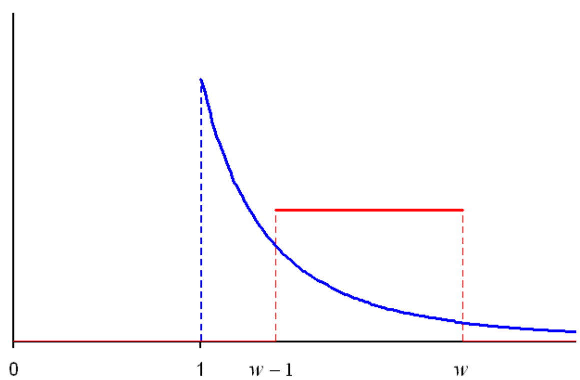
$$f_W(w) = \int_1^w \frac{2}{x^3} \cdot 1\,dx = 1 - \frac{1}{w^2}.$$



Case 2: $w > 2$.

$\Rightarrow \qquad w-1 > 1.$

$$f_W(w) = \int_{w-1}^{w} \frac{2}{x^3} \cdot 1\,dx = \frac{1}{(w-1)^2} - \frac{1}{w^2}.$$



Case 3: $w < 1$. $\qquad f_W(w) = 0.$

3. Let X and Y be two independent Poisson random variables with mean λ_1 and λ_2, respectively. Let $W = X + Y$.

a) What is the probability distribution of W?

$$P(W = n) = \sum_{k=0}^{n} P(X = k) \cdot P(Y = n-k) = \sum_{k=0}^{n} \frac{\lambda_1^k \cdot e^{-\lambda_1}}{k!} \cdot \frac{\lambda_2^{n-k} \cdot e^{-\lambda_2}}{(n-k)!}$$

$$= \frac{(\lambda_1 + \lambda_2)^n \cdot e^{-(\lambda_1 + \lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k! \cdot (n-k)!} \cdot \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$$

$$= \frac{(\lambda_1 + \lambda_2)^n \cdot e^{-(\lambda_1 + \lambda_2)}}{n!}.$$

Therefore, W is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

OR

$$M_W(t) = M_X(t) \cdot M_Y(t) = e^{\lambda_1 (e^t - 1)} \cdot e^{\lambda_2 (e^t - 1)} = e^{(\lambda_1 + \lambda_2)(e^t - 1)}.$$

Therefore, W is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

b) What is the conditional distribution of X given $W = n$?

$$P(X = k \mid W = n) = \frac{P(X = k \cap W = n)}{P(W = n)} = \frac{P(X = k \cap Y = n-k)}{P(W = n)}$$

$$= \frac{\dfrac{\lambda_1^k \cdot e^{-\lambda_1}}{k!} \cdot \dfrac{\lambda_2^{n-k} \cdot e^{-\lambda_2}}{(n-k)!}}{\dfrac{(\lambda_1 + \lambda_2)^n \cdot e^{-(\lambda_1 + \lambda_2)}}{n!}}$$

$$= \frac{n!}{k! \cdot (n-k)!} \cdot \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}.$$

$\Rightarrow$ $X \mid W = n$ has a Binomial distribution, $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

4. Let X_1 and X_2 be be two independent χ^2 random variables with m and n degrees of freedom, respectively. Find the probability distribution of $W = X_1 + X_2$.

$$f_1(x_1) = \frac{1}{\Gamma(m/2)2^{m/2}} x_1^{m/2-1} e^{-x_1/2}, \qquad x_1 > 0,$$

$$f_2(x_2) = \frac{1}{\Gamma(n/2)2^{n/2}} x_2^{n/2-1} e^{-x_2/2}, \qquad x_2 > 0.$$

$$f_W(w) = \int_{-\infty}^{\infty} f_1(x) \cdot f_2(w-x)\,dx$$

$$= \int_0^w \frac{1}{\Gamma(m/2)2^{m/2}} x^{m/2-1} e^{-x/2} \cdot \frac{1}{\Gamma(n/2)2^{n/2}} (w-x)^{n/2-1} e^{-(w-x)/2}\,dx$$

$$= \frac{e^{-w/2}}{\Gamma((m+n)/2)2^{(m+n)/2}} \int_0^w \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \cdot x^{m/2-1} \cdot (w-x)^{n/2-1}\,dx$$

let $x = wy$, then $dx = w\,dy$,

$0 \to 0, \quad w \to 1$

$$= \frac{w^{(m+n)/2-1} \cdot e^{-w/2}}{\Gamma((m+n)/2)2^{(m+n)/2}} \int_0^1 \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \cdot y^{m/2-1} \cdot (1-y)^{n/2-1}\,dy$$

$$= \frac{1}{\Gamma((m+n)/2)2^{(m+n)/2}} \cdot w^{(m+n)/2-1} \cdot e^{-w/2},$$

since $\dfrac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \cdot y^{m/2-1} \cdot (1-y)^{n/2-1}$, $0 < y < 1$, is the p.d.f. of a Beta distribution with $\alpha = m/2$, $\beta = n/2$.

$\Rightarrow$ W has a $\chi^2(m+n)$ distribution.

If random variables X and Y are independent, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

$$M_1(t) = \frac{1}{(1-2t)^{m/2}}, \quad t < 1/2, \qquad M_2(t) = \frac{1}{(1-2t)^{n/2}}, \quad t < 1/2.$$

$$M_W(t) = M_1(t) \cdot M_2(t) = \frac{1}{(1-2t)^{(m+n)/2}}, \quad t < 1/2.$$

$\Rightarrow$ W has a $\chi^2(m+n)$ distribution.

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If X and Y are independent,

X is Bernoulli(p), Y is Bernoulli(p) $\Rightarrow$ X + Y is Binomial$(n=2, p)$;

X is Binomial(n_1, p), Y is Binomial(n_2, p) $\Rightarrow$ X + Y is Binomial(n_1+n_2, p);

X is Geometric(p), Y is Geometric(p) $\Rightarrow$ X + Y is Neg. Binomial$(r=2, p)$;

X is Neg. Binomial(r_1, p), Y is Neg. Binomial(r_2, p)

$\Rightarrow$ X + Y is Neg. Binomial(r_1+r_2, p);

X is Poisson(λ_1), Y is Poisson(λ_2) $\Rightarrow$ X + Y is Poisson$(\lambda_1+\lambda_2)$;

X is Exponential(θ), Y is Exponential(θ) $\Rightarrow$ X + Y is Gamma$(\alpha=2, \theta)$;

X is $\chi^2(r_1)$, Y is $\chi^2(r_2)$ $\Rightarrow$ X + Y is $\chi^2(r_1+r_2)$;

X is Gamma(α_1, θ), Y is Gamma(α_2, θ) $\Rightarrow$ X + Y is Gamma$(\alpha_1+\alpha_2, \theta)$;

X is Normal(μ_1, σ_1^2), Y is Normal(μ_2, σ_2^2)

$\Rightarrow$ X + Y is Normal$(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

5. Let X and Y be independent random variables, each geometrically distributed with the probability of "success" p, $0 < p < 1$. That is,

$$p_X(k) = p_Y(k) = p \cdot (1-p)^{k-1}, \qquad k = 1, 2, 3, \dots,$$

a) Find $P(X + Y = n)$, $n = 2, 3, 4, \dots$.

$$P(X+Y=n) = \sum_{k=1}^{n-1} P(X=k) \cdot P(Y=n-k)$$

$$= \sum_{k=1}^{n-1} p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{n-k-1} = \sum_{k=1}^{n-1} p^2 \cdot (1-p)^{n-2}$$

$$= (n-1) \cdot p^2 \cdot (1-p)^{n-2}, \qquad n = 2, 3, 4, \dots.$$

If X and Y both have Geometric(p) distribution and are independent, then X + Y has Negative Binomial distribution with $r = 2$.

OR

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \left[\frac{p e^{-t}}{1 - (1-p) e^{-t}} \right]^2, \qquad t < -\ln(1-p).$$

b) Find $P(X = k \mid X + Y = n)$, $k = 1, 2, 3, \dots, n-1$, $n = 2, 3, 4, \dots$.

$$P(X = k \mid X+Y=n) = \frac{P(X=k \cap X+Y=n)}{P(X+Y=n)} = \frac{P(X=k \cap Y=n-k)}{P(X+Y=n)}$$

$$= \frac{p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{n-k-1}}{(n-1) \cdot p^2 \cdot (1-p)^{n-2}} = \frac{1}{n-1}, \qquad k = 1, 2, 3, \dots, n-1.$$

$\Rightarrow$ $X \mid X + Y = n$ has a Uniform distribution on integers $1, 2, 3, \dots, n-1$.

c) Find $P(X > Y)$. [Hint: First, find $P(X = Y)$.]

$$P(X = Y) = \sum_{k=1}^{\infty} p_X(k) \cdot p_Y(k) = \sum_{k=1}^{\infty} p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{k-1}$$

$$= p^2 \cdot \sum_{k=1}^{\infty} \left[(1-p)^2 \right]^{k-1} = p^2 \cdot \sum_{n=0}^{\infty} \left[(1-p)^2 \right]^{n}$$

$$= \frac{p^2}{1-(1-p)^2} = \frac{p}{2-p}.$$

$$P(X > Y) + P(X = Y) + P(X < Y) = 1.$$

Since $P(X > Y) = P(X < Y)$,

$$P(X > Y) = \frac{1}{2} \cdot \left(1 - P(X = Y)\right) = \frac{1}{2} \cdot \left(1 - \frac{p}{2-p}\right) = \frac{1-p}{2-p}.$$

OR

$$P(X > Y) = \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} p \cdot (1-p)^{x-1} \cdot p \cdot (1-p)^{y-1}$$

$$= \sum_{y=1}^{\infty} p^2 \cdot (1-p)^{y-1} \cdot \sum_{x=y+1}^{\infty} (1-p)^{x-1}$$

$$= \sum_{y=1}^{\infty} p^2 \cdot (1-p)^{y-1} \cdot \frac{(1-p)^y}{1-(1-p)} = \sum_{y=1}^{\infty} p \cdot (1-p)^{2y-1}$$

$$= p \cdot (1-p) \cdot \sum_{n=0}^{\infty} \left[(1-p)^2 \right]^{n} = \frac{p \cdot (1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

d) Consider the discrete random variable $Q = \dfrac{X}{Y}$.

Find $E(X)$, $E(\frac{1}{Y})$, $E(Q)$.

[Hint: $\ln(1 - z) = -\sum_{k=1}^{\infty} \dfrac{z^k}{k}$ for $-1 < z < 1$.]

$$E(X) = \frac{1}{p}, \quad \text{since X has a Geometric}(p) \text{ distribution.}$$

$$E\left(\frac{1}{Y}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \cdot p \cdot (1-p)^{k-1} = \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} \frac{(1-p)^k}{k}$$

$$= -\ln(1-(1-p)) \cdot \frac{p}{1-p} = -\ln(p) \cdot \frac{p}{1-p}.$$

Since X and Y are independent,

$$E(Q) = E(X) \times E\left(\frac{1}{Y}\right) = \frac{-\ln(p)}{1-p}.$$

e) For any positive, irreducible fraction $\dfrac{a}{b}$, find $P(Q = \dfrac{a}{b})$.

$$P\left(Q = \frac{a}{b}\right) = \sum_{k=1}^{\infty} p_X(ka) \cdot p_Y(kb)$$

$$= \sum_{k=1}^{\infty} p \cdot (1-p)^{ka-1} \cdot p \cdot (1-p)^{kb-1}$$

$$= \left(\frac{p}{1-p}\right)^2 \cdot \sum_{k=1}^{\infty} \left[(1-p)^{a+b}\right]^k$$

$$= \frac{p^2}{(1-p)^2} \cdot \frac{(1-p)^{a+b}}{1-(1-p)^{a+b}}.$$

6. Suppose we have two 4-sided dice. Suppose that for the first die (X),

$$p_{\mathrm{X}}(1) = {}^{1}\!/_{10}, \quad p_{\mathrm{X}}(2) = {}^{2}\!/_{10}, \quad p_{\mathrm{X}}(3) = {}^{3}\!/_{10}, \quad p_{\mathrm{X}}(4) = {}^{4}\!/_{10}.$$

Suppose also that for the second die (Y),

$$p_{\mathrm{Y}}(1) = {}^{1}\!/_{30}, \quad p_{\mathrm{Y}}(2) = {}^{4}\!/_{30}, \quad p_{\mathrm{Y}}(3) = {}^{9}\!/_{30}, \quad p_{\mathrm{Y}}(4) = {}^{16}\!/_{30}.$$

Find the probability distribution of $\mathrm{U} = \mathrm{X} + \mathrm{Y}$.

X \ Y	1 $^{1}\!/_{30}$	2 $^{4}\!/_{30}$	3 $^{9}\!/_{30}$	4 $^{16}\!/_{30}$
1 $^{1}\!/_{10}$	$(1,1)$ 2 $^{1}\!/_{300}$	$(1,2)$ 3 $^{4}\!/_{300}$	$(1,3)$ 4 $^{9}\!/_{300}$	$(1,4)$ 5 $^{16}\!/_{300}$
2 $^{2}\!/_{10}$	$(2,1)$ 3 $^{2}\!/_{300}$	$(2,2)$ 4 $^{8}\!/_{300}$	$(2,3)$ 5 $^{18}\!/_{300}$	$(2,4)$ 6 $^{32}\!/_{300}$
3 $^{3}\!/_{10}$	$(3,1)$ 4 $^{3}\!/_{300}$	$(3,2)$ 5 $^{12}\!/_{300}$	$(3,3)$ 6 $^{27}\!/_{300}$	$(3,4)$ 7 $^{48}\!/_{300}$
4 $^{4}\!/_{10}$	$(4,1)$ 5 $^{4}\!/_{300}$	$(4,2)$ 6 $^{16}\!/_{300}$	$(4,3)$ 7 $^{36}\!/_{300}$	$(4,4)$ 8 $^{64}\!/_{300}$

u	$p(u)$
2	$^{1}\!/_{300}$
3	$^{6}\!/_{300}$
4	$^{20}\!/_{300}$
5	$^{50}\!/_{300}$
6	$^{75}\!/_{300}$
7	$^{84}\!/_{300}$
8	$^{64}\!/_{300}$

OR

$$\mathrm{M}_{\mathrm{U}}(t) = \mathrm{M}_{\mathrm{X}}(t) \cdot \mathrm{M}_{\mathrm{Y}}(t)$$

$$= \left(e^{t}\frac{1}{10} + e^{2t}\frac{2}{10} + e^{3t}\frac{3}{10} + e^{4t}\frac{4}{10} \right) \cdot \left(e^{t}\frac{1}{30} + e^{2t}\frac{4}{30} + e^{3t}\frac{9}{30} + e^{4t}\frac{16}{30} \right) = \ldots$$

7. Suppose X and Y are two independent discrete random variables with the following probability distributions:

$$p_X(1) = 0.2, \quad p_X(2) = 0.4, \quad p_X(3) = 0.3, \quad p_X(4) = 0.1,$$

$$p_Y(1) = 0.3, \quad p_Y(3) = 0.5, \quad p_Y(5) = 0.2.$$

Find the probability distribution of $W = X + Y$.

$$M_W(t) = M_X(t) \cdot M_Y(t)$$

$$= \left(0.2\,e^{t} + 0.4\,e^{2t} + 0.3\,e^{3t} + 0.1\,e^{4t}\right) \cdot \left(0.3\,e^{t} + 0.5\,e^{3t} + 0.2\,e^{5t}\right)$$

$$= 0.06\,e^{2t} + 0.12\,e^{3t} + 0.19\,e^{4t} + 0.23\,e^{5t} + 0.19\,e^{6t} + 0.13\,e^{7t} + 0.06\,e^{8t} + 0.02\,e^{9t}.$$

$$\Rightarrow \qquad p_W(2) = 0.06, \quad p_W(3) = 0.12, \quad p_W(4) = 0.19, \quad p_W(5) = 0.23,$$

$$p_W(6) = 0.19, \quad p_W(7) = 0.13, \quad p_W(8) = 0.06, \quad p_W(9) = 0.02.$$