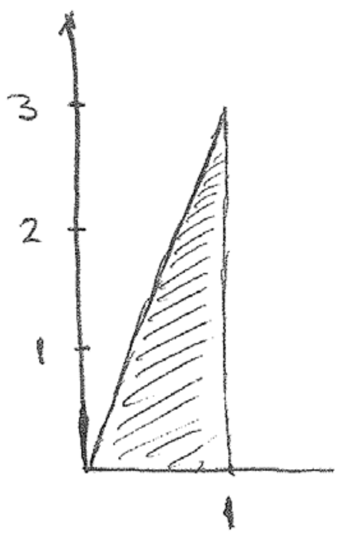
STAT 410 | Fall 2020 | A. Stepanov

Examples for 09/04/2020 (1)

1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^3\,y, \qquad 0 < x < 1, \quad 0 < y < 3x, \qquad \text{zero otherwise.}$$

a) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.



Must have

$$1 = \int_0^1 \left(\int_0^{3x} C\,x^3 y\,dy \right) dx$$

$$= \int_0^1 \frac{C}{2}\,x^3 (3x)^2\,dx$$

$$= \frac{9C}{2} \int_0^1 x^5\,dx = \frac{3C}{4}.$$

$$\Rightarrow \quad C = \frac{4}{3}.$$

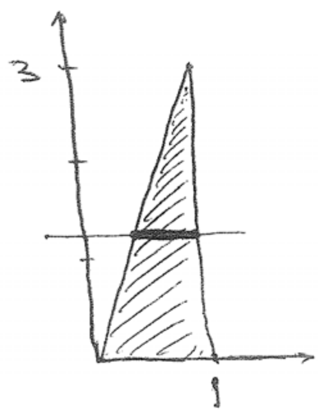
b) Find the marginal probability density function of X, $f_X(x)$.

Be sure to include its support.

$$f_X(x) = \int_0^{3x} \frac{4}{3}\,x^3 y\,dy = 6x^5, \qquad 0 < x < 1.$$

c) Find the marginal probability density function of Y, $f_{\mathrm{Y}}(y)$.

Be sure to include its support.



$$f_Y(y) = \int_{y/3}^{1} \frac{4}{3} x^3 y \, dx$$

$$= \left(\frac{1}{3} x^4 y \right)_{x = \frac{y}{3}}^{x=1}$$

$$= \frac{1}{3} y - \frac{1}{243} y^5, \qquad 0 < y < 3.$$

d) Are X and Y independent? If not, find $\mathrm{Cov}(\mathrm{X}, \mathrm{Y})$.

$f_{\mathrm{X,Y}}(x, y) \neq f_{\mathrm{X}}(x) \cdot f_{\mathrm{Y}}(y)$. $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

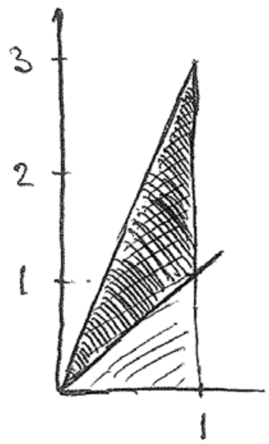
$$\mathrm{E}(\mathrm{X}) = \int_0^1 x \cdot 6x^5 \, dx = \frac{6}{7}.$$

$$\mathrm{E}(\mathrm{Y}) = \int_0^3 y \cdot \left(\frac{1}{3} y - \frac{1}{243} y^5 \right) dy = \left(\frac{1}{9} y^3 - \frac{1}{1701} y^7 \right) \Big|_0^3 = 3 - \frac{9}{7} = \frac{12}{7}.$$

$$\mathrm{E}(\mathrm{XY}) = \int_0^1 \left(\int_0^{3x} xy \cdot \frac{4}{3} x^3 y \, dy \right) dx = \int_0^1 \left(\int_0^{3x} \frac{4}{3} x^4 y^2 \, dy \right) dx = \int_0^1 12 x^7 \, dx = \frac{3}{2}.$$

$$\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X}) \cdot \mathrm{E}(\mathrm{Y}) = \frac{3}{2} - \frac{6}{7} \times \frac{12}{7} = \mathbf{\frac{3}{98}}.$$

e) Find $P(X < Y)$.



$$\int_0^1 \left(\int_x^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$1 - \int_0^1 \left(\int_0^x \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$1 - \int_0^1 \left(\int_y^1 \frac{4}{3} x^3 y \, dx \right) dy$$

OR

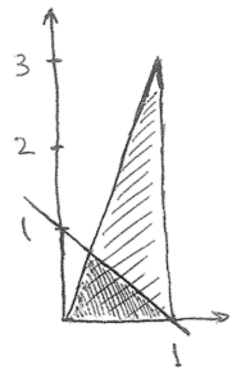
$$\int_0^1 \left(\int_{y/3}^{y} \frac{4}{3} x^3 y \, dx \right) dy + \int_1^3 \left(\int_{y/3}^{1} \frac{4}{3} x^3 y \, dx \right) dy$$

$$\int_0^1 \left(\int_x^{3x} \frac{4}{3} x^3 y \, dy \right) dx = \int_0^1 \frac{16}{3} x^5 \, dx = \mathbf{\frac{8}{9}}.$$

$$1 - \int_0^1 \left(\int_0^x \frac{4}{3} x^3 y \, dy \right) dx = 1 - \int_0^1 \frac{2}{3} x^5 \, dx = 1 - \frac{1}{9} = \mathbf{\frac{8}{9}}.$$

$$1 - \int_0^1 \left(\int_y^1 \frac{4}{3} x^3 y \, dx \right) dy = 1 - \int_0^1 \frac{1}{3} \left(y - y^5 \right) dy = 1 - \frac{1}{6} + \frac{1}{18} = \mathbf{\frac{8}{9}}.$$

f) Find $P(X+Y<1)$.



$y = 3x$

$y = 1-x$

$\Rightarrow x = \frac{1}{4}$

$y = \frac{3}{4}$

$$\int_0^{3/4} \left(\int_{y/3}^{1-y} \frac{4}{3} x^3 y \, dx \right) dy$$

OR

$$1 - \int_{1/4}^{1} \left(\int_{1-x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$\int_0^{1/4} \left(\int_0^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{1/4}^{1} \left(\int_0^{1-x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$1 - \int_{1/4}^{1} \left(\int_{1-x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx = 1 - \int_{1/4}^{1} \frac{2}{3} x^3 \left(8x^2 + 2x - 1 \right) dx$$

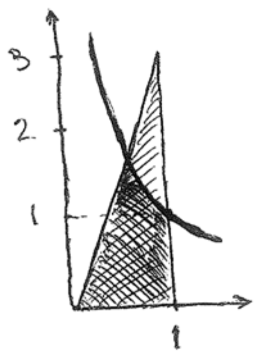
$$= 1 - \left(\frac{8}{9} x^6 + \frac{4}{15} x^5 - \frac{1}{6} x^4 \right) \Bigg|_{1/4}^{1} = \mathbf{\frac{7}{640}} = \mathbf{0.0109375}.$$

$$\int_0^{1/4} \left(\int_0^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{1/4}^{1} \left(\int_0^{1-x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$= \int_0^{1/4} 6x^5 \, dx + \int_{1/4}^{1} \frac{2}{3} x^3 \left(1 - 2x + x^2 \right) dx$$

$$= \frac{1}{4096} + \left(\frac{1}{6} x^4 - \frac{4}{15} x^5 + \frac{1}{9} x^6 \right) \Bigg|_{1/4}^{1} = \mathbf{\frac{7}{640}} = \mathbf{0.0109375}.$$

g) Find $P(X \cdot Y < 1)$.



$$1 - \int_{1/\sqrt{3}}^{1} \left(\int_{1/x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$\int_{0}^{1/\sqrt{3}} \left(\int_{0}^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{1/\sqrt{3}}^{1} \left(\int_{0}^{1/x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$\int_{0}^{1} \left(\int_{y/3}^{1} \frac{4}{3} x^3 y \, dx \right) dy + \int_{1}^{\sqrt{3}} \left(\int_{y/3}^{1/y} \frac{4}{3} x^3 y \, dx \right) dy$$

$y = 3x$

$x \cdot y = 1$

$\Rightarrow x = \frac{1}{\sqrt{3}}$

$y = \sqrt{3}$

$$1 - \int_{1/\sqrt{3}}^{1} \left(\int_{1/x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx = 1 - \int_{1/\sqrt{3}}^{1} \frac{2}{3} x^3 \left(9x^2 - \frac{1}{x^2} \right) dx$$

$$= 1 - \int_{1/\sqrt{3}}^{1} \left(6x^5 - \frac{2x}{3} \right) dx = 1 - \left(x^6 - \frac{x^2}{3} \right) \Bigg|_{1/\sqrt{3}}^{1}$$

$$= \mathbf{\frac{7}{27}} \approx 0.25926.$$

$$\int_{0}^{1/\sqrt{3}} \left(\int_{0}^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{1/\sqrt{3}}^{1} \left(\int_{0}^{1/x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$= \int_{0}^{1/\sqrt{3}} 6x^5 \, dx + \int_{1/\sqrt{3}}^{1} \frac{2x}{3} dx = \frac{1}{27} + \left(\frac{1}{3} - \frac{1}{9} \right) = \mathbf{\frac{7}{27}} \approx 0.25926.$$

2. Consider the following joint probability distribution $p(x, y)$ of two discrete random variables X and Y:

		x = 1	2	$p_Y(y)$
y	1	0.14	0.06	0.20
	2	0.12	0.18	0.30
	3	0.14	0.36	0.50
	$p_X(x)$	0.40	0.60	1.00

a) Find $P(X + Y = 4)$.

$$P(X + Y = 4) = p(1, 3) + p(2, 2)$$
$$= 0.14 + 0.18 = \mathbf{0.32}.$$

b) Find $P(X > Y)$.

$$P(Y > X) = p(1, 2) + p(1, 3) + p(2, 3) = 0.12 + 0.14 + 0.36 = \mathbf{0.62}.$$

c) Find $p_X(x)$, the marginal p.m.f. for X. ↑

d) Find $p_Y(y)$, the marginal p.m.f. for Y. ↑

e) Find $E(X)$, $E(Y)$, $E(X+Y)$, $E(X \cdot Y)$, $Cov(X, Y)$.

$$E(X) = 1 \times 0.40 + 2 \times 0.60 = \mathbf{1.60}.$$

$$E(Y) = 1 \times 0.20 + 2 \times 0.30 + 3 \times 0.50 = \mathbf{2.30}.$$

$$E(X+Y) = E(X) + E(Y) = 1.60 + 2.30 = \mathbf{3.90}.$$

$$E(X \cdot Y) = 1 \times 0.14 + 2 \times 0.06 + 2 \times 0.12 + 4 \times 0.18 + 3 \times 0.14 + 6 \times 0.36 = \mathbf{3.80}.$$

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y) = 3.80 - 1.60 \cdot 2.30 = \mathbf{0.12}.$$