STAT 410

Practice Problems 6

Fall 2020
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1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = 5\,e^{-2x-y}, \quad x > 0, \quad y > 3x, \qquad \text{zero elsewhere.}$$

a) Find $P(X > 0.8 \mid Y = 6)$.

b) Find $E(Y \mid X = x)$.

c) What is the probability distribution of $W = Y/X$?

2. Let X and Y be two random variables with joint p.d.f.

$$f(x,y) = 64\,x \exp\{-4y\} = 64\,x\,e^{-4y}, \qquad 0 < x < y < \infty,$$

zero elsewhere.

a) Find the p.d.f. $f_W(w)$ of $W = X + Y$.

b) Let $U = X$ and $V = X/Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u,v)$.

Sketch the support of (U, V).

c) Find the p.d.f. $f_V(v)$ of $V = X/Y$.

3. Suppose that the random variables X and Y have joint p.d.f. $f(x,y)$ given by

$$f(x,y) = 6\,x^2 y, \quad 0 < x < y, \quad x + y < 2, \qquad \text{zero elsewhere.}$$

a) Find the probability density function of $W = X + Y$, $f_W(w) = f_{X+Y}(w)$.

b) Find the probability density function of $V = X \times Y$, $f_V(v) = f_{X \times Y}(v)$.

c) Find the probability density function of $U = Y/X$, $f_U(u) = f_{Y/X}(u)$.

4. Let $\theta > 0$. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x, y) = \frac{\theta^2}{2\sqrt{y}} e^{-\theta x}, \qquad x > 0, \quad 0 < y < x^2.$$

a) Find $P(Y > 4 \mid X = 5)$.

b) Find $E(Y \mid X = x)$.

c) Find $P(X > 5 \mid Y = 4)$.

d) Find $E(X \mid Y = y)$.

5. Let $\theta > 0$. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x, y) = \frac{\theta^2}{2\sqrt{y}} e^{-\theta x}, \qquad x > 0, \quad 0 < y < x^2.$$

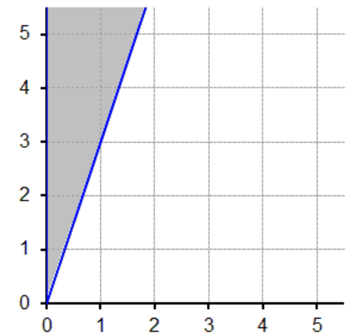
Let $U = X$ and $V = X/Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u, v)$.

Sketch the support of (U, V).

Answers:

1. Consider two continuous random variables X and Y with joint p.d.f.



$$f_{X,Y}(x,y) = 5\,e^{-2x-y}, \qquad x > 0, \quad y > 3x,$$

zero elsewhere.

a) Find $P(X > 0.8 \mid Y = 6)$.

$$f_Y(y) = \int_0^{y/3} 5\,e^{-2x-y}\,dx = \frac{5}{2}e^{-y}\left(1-e^{-2y/3}\right), \qquad 0 < y < \infty.$$

$$f_{X|Y}(x|y) = \frac{5\,e^{-2x-y}}{\frac{5}{2}e^{-y}\left(1-e^{-2y/3}\right)} = \frac{2\,e^{-2x}}{1-e^{-2y/3}}, \qquad 0 < x < \frac{y}{3}.$$

$$f_{X|Y}(x|6) = \frac{2\,e^{-2x}}{1-e^{-4}}, \qquad 0 < x < 2.$$

$$P(X > 0.8 \mid Y = 6) = \int_{0.8}^{2} \frac{2\,e^{-2x}}{1-e^{-4}}\,dx = \frac{e^{-1.6}-e^{-4}}{1-e^{-4}} = \frac{e^{2.4}-1}{e^{4}-1} \approx 0.187.$$

b) Find $E(Y \mid X = x)$.

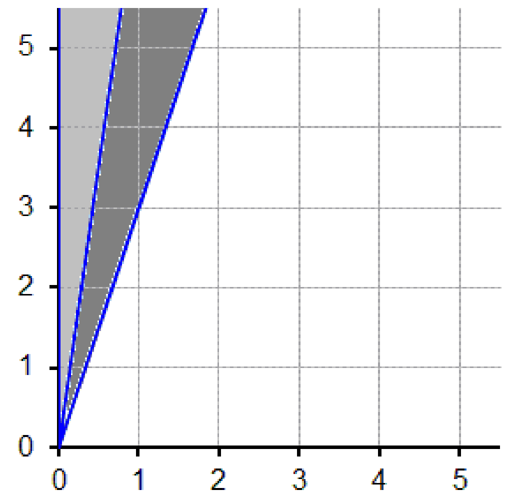
$$f_X(x) = \int_{3x}^{\infty} 5\,e^{-2x-y}\,dy = 5\,e^{-5x}, \qquad 0 < x < \infty.$$

$$f_{Y|X}(y|x) = \frac{5\,e^{-2x-y}}{5\,e^{-5x}} = e^{-y+3x}, \qquad 3x < y < \infty.$$

$$E(Y \mid X = x) = \int_{3x}^{\infty} y \cdot e^{-y+3x}\,dy = 3x + 1, \qquad x > 0.$$

c) What is the probability distribution of $W = {}^{Y}/_{X}$?

$y > 3x \quad \Rightarrow \quad w > 3$



$$F_W(w) = P\left({}^{Y}/_{X} \le w\right)$$

$$= P(Y \le wX)$$

$$= 1 - \int_0^{\infty}\int_{wx}^{\infty} 5e^{-2x-y}\,dy\,dx$$

$$= 1 - \int_0^{\infty} 5e^{-2x-wx}\,dx$$

$$1 - \frac{5}{2+w}, \qquad w > 3.$$

$$f_W(w) = F_W'(w) = \frac{5}{(2+w)^2}, \qquad w > 3.$$

OR $$F_W(w) = \int_0^{\infty}\int_{3x}^{wx} 5e^{-2x-y}\,dy\,dx = \ldots$$

OR

Let $V = X$, $W = {}^{Y}/_{X}$. Then $X = V$, $Y = VW$.

$x > 0 \quad \Rightarrow \quad v > 0$

$y > 3x \quad \Rightarrow \quad w > 3$

$$J = \begin{vmatrix} 1 & 0 \\ w & v \end{vmatrix} = v.$$

$$f_{V,W}(v,w) = f_{X,Y}(v, vw)\,|v| = 5\,v\,e^{-2v-wv}.$$

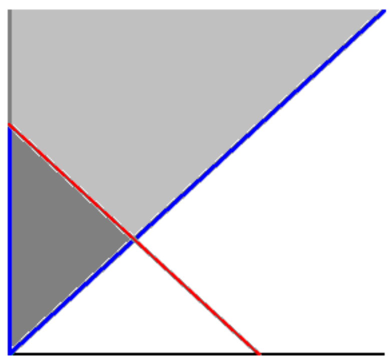
$$f_W(w) = \int_{-\infty}^{\infty} f_{V,W}(v,w)\,dv = \int_0^{\infty} 5\,v\,e^{-2v-wv}\,dv = \frac{5}{(2+w)^2}, \qquad w > 3.$$

2. Let X and Y be two random variables with joint p.d.f.

$$f(x,y) = 64\,x\exp\{-4y\} = 64\,x\,e^{-4y}, \qquad 0 < x < y < \infty,$$

zero elsewhere.

a) Find the p.d.f. $f_W(w)$ of $W = X + Y$.



$$F_W(w) = P(X + Y \le w) = \int_0^{w/2} \int_x^{w-x} 64\,x\,e^{-4y}\,dy\,dx$$

$$= \int_1^{w/2} \left(16\,x\,e^{-4x} - 16\,x\,e^{-4w}\,e^{4x}\right) dx$$

$$= \left(-4\,x\,e^{-4x} - e^{-4x} - 4\,x\,e^{-4w}\,e^{4x} + e^{-4w}\,e^{4x}\right)\Bigg|_0^{w/2}$$

$$= 1 - e^{-4w} - 4\,w\,e^{-2w}, \qquad w > 0.$$

$$f_W(w) = F'_W(w) = = 4\,e^{-4w} - 4\,e^{-2w} + 8\,w\,e^{-2w}, \qquad w > 0.$$

OR

$$f_W(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx.$$

$0 < x$

$x < y \qquad \Rightarrow \qquad x < w - x \qquad \Rightarrow \qquad x < \frac{w}{2}$

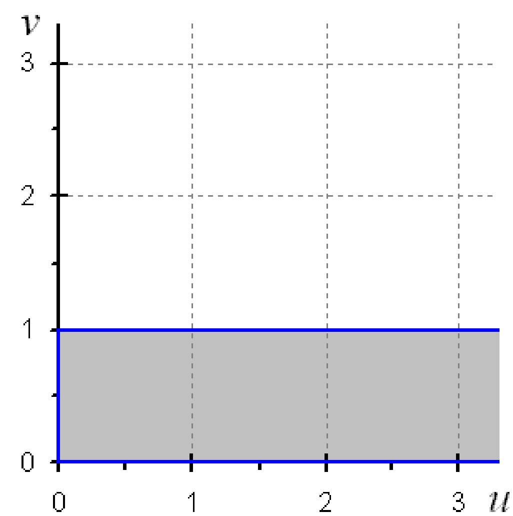
$$f_W(w) = \int_0^{w/2} 64\,x\,e^{-4w+4x}\,dx = e^{-4w}\left[16\,x\,e^{4x} - 4\,e^{4x}\right]\Big|_0^{w/2}$$

$$= 4\,e^{-4w} - 4\,e^{-2w} + 8\,w\,e^{-2w}, \qquad w > 0.$$

b) Let $U = X$ and $V = X/Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u, v)$.

Sketch the support of (U, V).



$X = U$

$Y = U/V$

$0 < x \quad \Rightarrow \quad u > 0$

$x < y \quad \Rightarrow \quad u < u/v \quad \Rightarrow \quad v < 1$

$$J = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v^2}.$$

$$f_{U,V}(u, v) = f_{X,Y}\left(u, \frac{u}{v}\right) \cdot |J| = 64\,u\,e^{-4u/v} \cdot \frac{u}{v^2} = 64\,\frac{u^2}{v^2}\,e^{-4u/v},$$

$$u > 0, \quad 0 < v < 1.$$

c) Find the p.d.f. $f_V(v)$ of $V = X/Y$.

$$0 < x < y < \infty \qquad \Rightarrow \qquad 0 < v < 1.$$

Part (b):

$U = X$ and $V = X/Y$.

$$f_{U,V}(u,v) = 64\frac{u^2}{v^2}e^{-4u/v}, \qquad u > 0, \quad 0 < v < 1.$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \int_0^{\infty} 64\frac{u^2}{v^2}e^{-4u/v}\,du$$

$$= 2v \cdot \int_0^{\infty} \frac{4^3}{\Gamma(3)v^3} u^{3-1} e^{-4u/v}\,du = 2v, \qquad 0 < v < 1.$$

OR

$$F_V(v) = P\left(X/Y \le v\right) = P\left(Y \ge X/v\right) = \int_0^{\infty}\int_{x/v}^{\infty} 64\,x\,e^{-4y}\,dy\,dx$$

$$= \int_0^{\infty} 16\,x\,e^{-4x/v}\,dx = v^2, \qquad 0 < v < 1.$$

$$f_V(v) = F_V'(v) = 2v, \qquad 0 < v < 1.$$

3. Suppose that the random variables X and Y have joint p.d.f. $f(x,y)$ given by

$$f(x,y) = 6x^2 y, \quad 0 < x < y, \ x + y < 2, \qquad \text{zero elsewhere.}$$

a) Find the probability density function of $W = X + Y$, $f_W(w) = f_{X+Y}(w)$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$0 < x$ | $x < y$, $x < w - x$, $x < \frac{w}{2}$ | $x + y < 2$, $x + (w - x) < 2$, $w < 2$

$$f_{X+Y}(w) = \int_0^{w/2} 6x^2 (w - x)\, dx$$

$$= \int_0^{w/2} (6wx^2 - 6x^3)\, dx$$

$$= \left(2wx^3 - \frac{3}{2}x^4\right)\Big|_0^{w/2} = \frac{1}{4}w^4 - \frac{3}{32}w^4$$

$$= \frac{5}{32}w^4, \qquad 0 < w < 2.$$

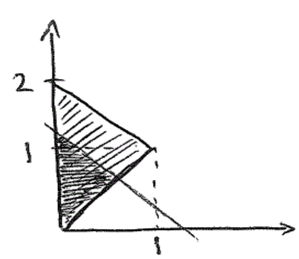
OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$0 < x$, $0 < w - y$, $y < w$ | $x < y$, $w - y < y$, $y > \frac{w}{2}$ | $x + y < 2$, $(w - y) + y < 2$, $w < 2$

$$f_{X+Y}(w) = \int_{w/2}^{w} 6(w-y)^2 y\, dy = \ldots$$

OR



$$F_{X+Y}(w) = \int_0^{w/2} \left(\int_x^{w-x} 6x^2 y\, dy \right) dx = \ldots$$

$$f_{X+Y}(w) = F'_{X+Y}(w) = \ldots$$

b) Find the probability density function of $V = X \times Y$, $f_V(v) = f_{X \times Y}(v)$.

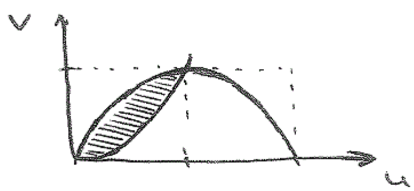
Let $U = X$, $V = X \cdot Y$.

$X = U$ $\quad Y = \frac{V}{U}$

$x > 0 \qquad u > 0$

$y > x \qquad \frac{v}{u} > u \qquad v > u^2$

$x + y > 2 \qquad u + \frac{v}{u} > 2 \qquad v < 2u - u^2$



$$J = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}$$

$$f_{U,V}(u,v) = 6\,u^2\left(\frac{v}{u}\right) \cdot \left|\frac{1}{u}\right| = 6v, \qquad 0 < u < 1,\quad u^2 < v < 2u - u^2$$

$u^2 < v \qquad u < \sqrt{v}$

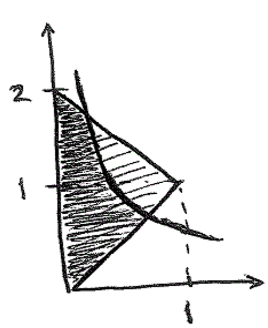
$v < 2u - u^2 \qquad u^2 - 2u + v < 0$

$1 - \sqrt{1-v} < u < 1 + \sqrt{1-v}$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du$$

$$= \int_{1-\sqrt{1-v}}^{\sqrt{v}} 6v\,du = 6v\left(\sqrt{v} + \sqrt{1-v} - 1\right), \qquad 0 < v < 1.$$

OR



$$F_V(v) = \int_0^{1-\sqrt{1-v}} \left(\int_x^{2-x} 6x^2 y\,dy \right) dx + \int_{1-\sqrt{1-v}}^{\sqrt{v}} \left(\int_x^{v/x} 6x^2 y\,dy \right) dx = \ldots$$

$$f_V(v) = F_V'(v) = \ldots$$

c) Find the probability density function of $U = {}^{Y}/_{X}$, $f_U(u) = f_{Y/X}(u)$.

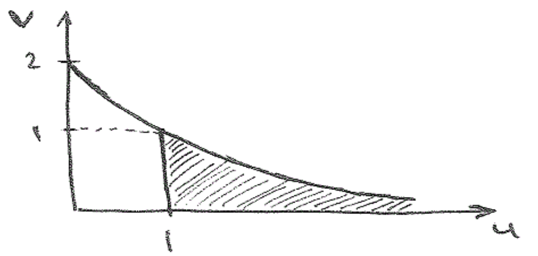
Let $U = \frac{Y}{X}$, $V = X$.

$X = V$ $\quad$ $Y = UV$

$x > 0$ $\quad$ $v > 0$

$y > x$ $\quad$ $uv > v$ $\quad$ $u > 1$

$x + y < 2$ $\quad$ $v + uv < 2$ $\quad$ $v < \frac{2}{1+u}$



$$J = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v$$

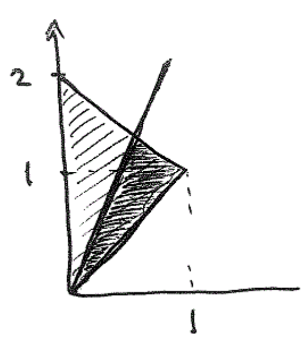
$$f_{U,V}(u,v) = 6v^2(uv)\cdot|-v| = 6uv^4, \quad u > 1, \quad 0 < v < \frac{2}{1+u}.$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv$$

$$= \int_0^{\frac{2}{1+u}} 6uv^4\,dv = \frac{6}{5}u\left(\frac{2}{1+u}\right)^5$$

$$= \frac{192u}{5(1+u)^5}, \quad u > 1.$$

OR



$$F_U(u) = 1 - \int_0^{\frac{2}{1+u}} \left(\int_{ux}^{2-x} 6x^2y\,dy \right) dx = \ldots$$

$$f_U(u) = F_U'(u) = \ldots$$

4. Let $\theta > 0$. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{\theta^2}{2\sqrt{y}}\, e^{-\theta x}, \qquad x > 0, \quad 0 < y < x^2.$$

a) Find $P(Y > 4 \mid X = 5)$.

$$f_X(x) = \int_0^{x^2} \frac{\theta^2}{2\sqrt{y}}\, e^{-\theta x}\, dy = \theta^2\, x e^{-\theta x}, \qquad 0 < x < \infty.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{2x\sqrt{y}}, \qquad 0 < y < x^2, \quad 0 < x < \infty.$$

$$P(Y > 4 \mid X = 5) = \int_4^{25} \frac{1}{10\sqrt{y}}\, dy = \frac{\sqrt{y}}{5}\Bigg|_4^{25} = \frac{3}{5} = 0.60.$$

b) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \int_0^{x^2} y \cdot \frac{1}{2x\sqrt{y}}\, dy = \frac{x^2}{3}, \qquad 0 < x < \infty.$$

c) Find $P(X > 5 \mid Y = 4)$.

$$f_Y(y) = \int_{\sqrt{y}}^{\infty} \frac{\theta^2}{2\sqrt{y}}\, e^{-\theta x}\, dx = \frac{\theta}{2\sqrt{y}}\, e^{-\theta\sqrt{y}}, \qquad 0 < y < \infty.$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \theta\, e^{-\theta x + \theta\sqrt{y}}, \qquad \sqrt{y} < x < \infty, \qquad 0 < y < \infty.$$

$$P(X > 5 \mid Y = 4) = \int_5^{\infty} \theta\, e^{-\theta x + 2\theta}\, dx = e^{-3\theta}.$$

d) Find $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \int_{\sqrt{y}}^{\infty} x \cdot \theta\, e^{-\theta x + \theta\sqrt{y}}\, dx = \sqrt{y} + \frac{1}{\theta}, \qquad 0 < y < \infty.$$

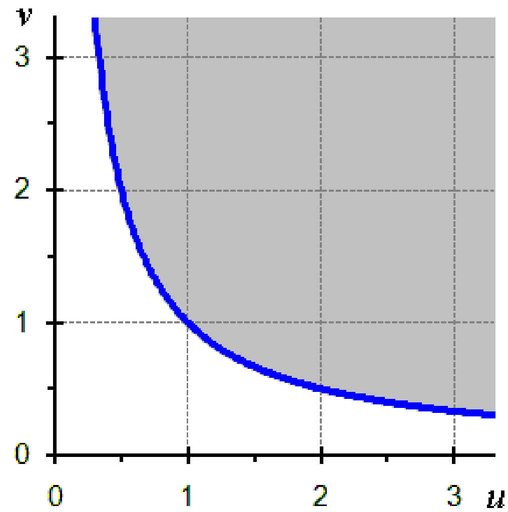
5. Let $\theta > 0$. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{\theta^2}{2\sqrt{y}}\, e^{-\theta x}, \qquad x > 0, \quad 0 < y < x^2.$$

Let $U = X$ and $V = {}^{X}\!/_{Y}$.

Find the joint probability density function of (U, V), $f_{U,V}(u,v)$.

Sketch the support of (U, V).



$X = U$

$Y = {}^{U}\!/_{V}$

$x > 0 \quad \Rightarrow \quad u > 0$

$0 < y \quad \Rightarrow \quad v > 0$

$y < x^2 \quad \Rightarrow \quad \frac{u}{v} < u^2 \quad \Rightarrow \quad u\,v > 1$

$$J = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v^2}. \qquad |J| = \frac{u}{v^2}.$$

$$f_{U,V}(u,v) = \frac{\theta^2}{2\sqrt{\frac{u}{v}}}\, e^{-\theta u} \times \frac{u}{v^2} = \frac{\theta^2 \sqrt{u}}{2\sqrt{v^3}}\, e^{-\theta u}, \qquad u > 0, \; v > 0, \; u\,v > 1$$