STAT 410 **Examples for 09/14/2020 (2)** Fall 2020
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Def $\mathrm{Var}(X|Y) = E[(X - E(X|Y))^2|Y] = E(X^2|Y) - [E(X|Y)]^2$

1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{4}{3}x^3 y, \qquad 0<x<1, \quad 0<y<3x, \qquad \text{zero otherwise.}$$

Recall: $f_X(x) = 6x^5, \qquad 0<x<1.$

$$f_Y(y) = \frac{1}{3}y - \frac{1}{243}y^5, \qquad 0<y<3.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{9x^2}, \qquad 0<y<3x, \quad 0<x<1.$$

$$E(Y|X=x) = \int_0^{3x} y \cdot \frac{2y}{9x^2}\,dy = \frac{2}{9x^2}\cdot\frac{(3x)^3}{3} = 2x, \quad 0<x<1.$$

$$E(Y|X) = 2X.$$

$$E(Y^2|X=x) = \int_0^{3x} y^2 \cdot \frac{2y}{9x^2}\,dy = \frac{2}{9x^2}\cdot\frac{(3x)^4}{4} = \frac{9}{2}x^2, \quad 0<x<1.$$

$$E(Y^2|X) = \frac{9}{2}X.$$

$$\mathrm{Var}(Y|X=x) = \frac{9}{2}x^2 - (2x)^2 = \frac{1}{2}x^2, \quad 0<x<1.$$

$$\mathrm{Var}(Y|X) = \frac{1}{2}X^2.$$

$$\mathrm{E}(\mathrm{X}) = \int_0^1 x \cdot 6x^5 \, dx = \frac{6}{7}.$$

$$\mathrm{E}(\mathrm{X}^2) = \int_0^1 x^2 \cdot 6x^5 \, dx = \frac{6}{8} = \frac{3}{4}.$$

$$\mathrm{Var}(\mathrm{X}) = \frac{3}{4} - \left(\frac{6}{7}\right)^2 = \frac{3}{196}.$$

$$\mathrm{E}[\mathrm{Var}(\mathrm{Y}|\mathrm{X})] = \mathrm{E}[\frac{1}{2}\mathrm{X}^2] = \frac{1}{2}\mathrm{E}(\mathrm{X}^2) = \frac{3}{8}.$$

$$\mathrm{Var}[\mathrm{E}(\mathrm{Y}|\mathrm{X})] = \mathrm{Var}[2\mathrm{X}] = 4\,\mathrm{Var}(\mathrm{X}) = \frac{3}{49}.$$

$$\mathrm{E}(\mathrm{Y}) = \int_0^3 y \cdot \left(\frac{1}{3}y - \frac{1}{243}y^5\right) dy = \left(\frac{1}{3\cdot 3}y^3 - \frac{1}{7\cdot 243}y^7\right)\Big|_0^3 = \frac{12}{7}.$$

$$\mathrm{E}(\mathrm{Y}^2) = \int_0^3 y^2 \cdot \left(\frac{1}{3}y - \frac{1}{243}y^5\right) dy = \left(\frac{1}{4\cdot 3}y^4 - \frac{1}{8\cdot 243}y^8\right)\Big|_0^3 = \frac{27}{8}.$$

$$\mathrm{Var}(\mathrm{Y}) = \frac{27}{8} - \left(\frac{12}{7}\right)^2 = \frac{171}{392}.$$

$$\mathrm{Var}(\mathrm{Y}) = \mathrm{Var}[\mathrm{E}(\mathrm{Y}|\mathrm{X})] + \mathrm{E}[\mathrm{Var}(\mathrm{X}|\mathrm{Y})]$$

$$\frac{171}{392} = \frac{3}{49} + \frac{3}{8} = \frac{24+147}{392}.$$ ☺

2. Consider the following joint probability distribution $p(x, y)$ of two discrete random variables X and Y:

		x		
		1	2	$p_{\mathrm{Y}}(y)$
	1	0.14	0.06	0.20
y	2	0.12	0.18	0.30
	3	0.14	0.36	0.50
	$p_{\mathrm{X}}(x)$	0.40	0.60	1.00

Y given X = 1

y	$p_{\mathrm{Y\mid X}}(y \mid 1)$
1	0.14 / 0.40 = 0.35
2	0.12 / 0.40 = 0.30
3	0.14 / 0.40 = 0.35
	1.00

Y given X = 2

y	$p_{\mathrm{Y\mid X}}(y \mid 2)$
1	0.06 / 0.60 = 0.10
2	0.18 / 0.60 = 0.30
3	0.36 / 0.60 = 0.60
	1.00

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = 1) = 1 \cdot 0.35 + 2 \cdot 0.30 + 3 \cdot 0.35 = 2.00.$$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = 2) = 1 \cdot 0.10 + 2 \cdot 0.30 + 3 \cdot 0.60 = 2.50.$$

$$\mathrm{E}(\mathrm{Y}^2 \mid \mathrm{X} = 1) = 1 \cdot 0.35 + 4 \cdot 0.30 + 9 \cdot 0.35 = 4.70.$$

$$\mathrm{E}(\mathrm{Y}^2 \mid \mathrm{X} = 2) = 1 \cdot 0.10 + 4 \cdot 0.30 + 9 \cdot 0.60 = 6.70.$$

$$\mathrm{Var}(\mathrm{Y} \mid \mathrm{X} = 1) = 4.70 - 2.00^2 = 0.70.$$

$$\mathrm{Var}(\mathrm{Y} \mid \mathrm{X} = 2) = 6.70 - 2.50^2 = 0.45.$$

$\mathrm{Var}(\mathrm{Y} \mid \mathrm{X})$:

	values	probabilities
x	$\mathrm{Var}(\mathrm{Y} \mid \mathrm{X} = x)$	$p_{\mathrm{X}}(x)$
1	0.70	0.40
2	0.45	0.60
		1.00

$$\mathrm{E}[\,\mathrm{Var}(\mathrm{Y}\,|\,\mathrm{X})\,] \;=\; 0.70 \cdot 0.40 + 0.45 \cdot 0.60 \;=\; 0.55.$$

Recall:

$\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X})$:

	values	probabilities
x	$\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X}=x)$	$p_{\mathrm{X}}(x)$
1	2.00	0.40
2	2.50	0.60
		1.00

$$\mathrm{E}[\,\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X})\,] \;=\; 2.00 \cdot 0.40 + 2.50 \cdot 0.60 \;=\; 2.30.$$

$$\mathrm{E}[\,\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X})^2\,] \;=\; 2.00^2 \cdot 0.40 + 2.50^2 \cdot 0.60 \;=\; 5.35.$$

$$\mathrm{Var}[\,\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X})\,] \;=\; 5.35 - 2.30^2 \;=\; 0.06.$$

$$\mathrm{E}(\mathrm{Y}) \;=\; 1 \cdot 0.20 + 2 \cdot 0.30 + 3 \cdot 0.50 \;=\; 2.30.$$

$$\mathrm{E}(\mathrm{Y}^2) \;=\; 1^2 \cdot 0.20 + 2^2 \cdot 0.30 + 3^2 \cdot 0.50 \;=\; 5.90.$$

$$\mathrm{Var}(\mathrm{Y}) \;=\; 5.90 - 2.30^2 \;=\; 0.61.$$

$$\mathrm{Var}(\mathrm{Y}) \;=\; \mathrm{Var}[\,\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X})\,] + \mathrm{E}[\,\mathrm{Var}(\mathrm{Y}\,|\,\mathrm{X})\,]$$

$0.61 \;=\; 0.06 \;+\; 0.55.$ ☺

X given $Y = 1$

x	$p_{X\|Y}(x\|1)$
1	$\frac{0.14}{0.20} = 0.70$
2	$\frac{0.06}{0.20} = 0.30$
	1.00

X given $Y = 2$

x	$p_{X\|Y}(x\|2)$
1	$\frac{0.12}{0.30} = 0.40$
2	$\frac{0.18}{0.30} = 0.60$
	1.00

X given $Y = 3$

x	$p_{X\|Y}(x\|3)$
1	$\frac{0.14}{0.50} = 0.28$
2	$\frac{0.36}{0.50} = 0.72$
	1.00

$E(X|Y=1) = 1.30$ $E(X|Y=2) = 1.60$ $E(X|Y=3) = 1.72$

$\mathrm{Var}(X|Y=1) = 0.21$ $\mathrm{Var}(X|Y=2) = 0.24$ $\mathrm{Var}(X|Y=3) = 0.2016$

$\mathrm{Var}(X|Y)$:

	values	probabilities
y	$E(X\|Y=y)$	$p_Y(y)$
1	0.21	0.20
2	0.24	0.30
3	0.2016	0.50
		1.00

$E[\mathrm{Var}(X|Y)] = 0.21 \cdot 0.20 + 0.24 \cdot 0.30 + 0.2016 \cdot 0.50 = 0.2148.$

Recall:

$E(X|Y)$:

	values	probabilities
y	$E(X\|Y=y)$	$p_Y(y)$
1	1.30	0.20
2	1.60	0.30
3	1.72	0.50
		1.00

$$E[E(X|Y)] = 1.30 \cdot 0.20 + 1.60 \cdot 0.30 + 1.72 \cdot 0.50 = 1.60.$$

$$E[E(X|Y)^2] = 1.30^2 \cdot 0.20 + 1.60^2 \cdot 0.30 + 1.72^2 \cdot 0.50 = 2.5852.$$

$$\mathrm{Var}[E(X|Y)] = 2.5852 - 1.60^2 = 0.0252.$$

$$E(X) = 1 \cdot 0.40 + 2 \cdot 0.60 = 1.60.$$

$$E(X^2) = 1^2 \cdot 0.40 + 2^2 \cdot 0.60 = 2.80.$$

$$\mathrm{Var}(X) = 2.80 - 1.60^2 = 0.24.$$

$$\mathrm{Var}(X) = \mathrm{Var}[E(X|Y)] + E[\mathrm{Var}(X|Y)]$$

$0.24 = 0.0252 + 0.2148.$ ☺