STAT 410

Examples for 12/07/2020 (1)

Fall 2020
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$f(x;\theta) = f(x \mid \theta)$ – p.d.f. (or p.m.f.) of x for given θ.

$\pi(\theta)$ – prior distribution of θ.

$f(x,\theta) = f(x \mid \theta) \times \pi(\theta)$ – joint p.d.f. of x and θ.

$f(x)$ – marginal p.d.f. of x.

$\pi(\theta \mid x) = \dfrac{f(x,\theta)}{f(x)} = \dfrac{f(x|\theta) \times \pi(\theta)}{f(x)}$ – posterior distribution of θ, given x.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0, \qquad \lambda > 0.$$

Let the prior p.d.f. of λ be $\text{Gamma}(\alpha, \theta)$.

Recall: The maximum likelihood estimator of λ is $\hat{\lambda} = \dfrac{2n}{\sum_{i=1}^{n} X_i^2}$.

a) Find the posterior distribution of λ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$f(x_1, x_2, \ldots x_n \mid \lambda) = f(x_1;\lambda)\, f(x_2;\lambda)\, \ldots\, f(x_n;\lambda)$$

$$= \prod_{i=1}^{n} \left(2\lambda^2 x_i^3 e^{-\lambda x_i^2} \right)$$

$$= 2^n \lambda^{2n} \left(\prod_{i=1}^{n} x_i^3 \right) e^{-\lambda \sum_{i=1}^{n} x_i^2}.$$

$$f(x_1, x_2, \ldots x_n, \lambda) = f(x_1, x_2, \ldots x_n \mid \lambda) \times \pi(\lambda)$$

$$= 2^n \lambda^{2n} \left(\prod_{i=1}^{n} x_i^3 \right) e^{-\lambda \sum_{i=1}^{n} x_i^2} \times \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} \lambda^{\alpha-1} e^{-\lambda/\theta}$$

$$= \ldots \lambda^{2n+\alpha-1} e^{-\lambda \left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta} \right)}.$$

$\Rightarrow$ the posterior distribution of λ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$,

is **Gamma** with New $\alpha = 2n + \alpha$ and New $\theta = \dfrac{1}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}}$.

b) Find the conditional mean of λ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Show that it is a weighted average of the maximum likelihood estimate $\hat{\lambda}$ and the prior mean $\alpha\,\theta$.

(conditional mean of λ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$)

$$= (\text{New } \alpha) \times (\text{New } \theta) = \frac{2n+\alpha}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}}.$$

$$\frac{2n+\alpha}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}} = \frac{2n}{\sum_{i=1}^{n} x_i^2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}} + \alpha\,\theta \cdot \frac{\frac{1}{\theta}}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}}$$

$$= \hat{\lambda} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}} + \alpha\,\theta \cdot \frac{\frac{1}{\theta}}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}}.$$

c) Use part (a) to construct a $(1-\gamma)\,100\,\%$ credible interval for λ, given that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

That is, construct an interval estimate for λ with posterior probability $(1-\gamma)$.

$$2\left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}\right)(\lambda \mid x_1, x_2, \ldots, x_n) \text{ has a } \chi^2(4n+2\alpha) \text{ distribution.}$$

$$P\left(\chi^2_{1-\gamma/2}(4n+2\alpha) < 2\left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}\right)(\lambda \mid x_1, x_2, \ldots, x_n) < \chi^2_{\gamma/2}(4n+2\alpha)\right) = 1-\gamma.$$

$$P\left(\frac{\chi^2_{1-\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}\right)} < (\lambda \mid x_1, x_2, \ldots, x_n) < \frac{\chi^2_{\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}\right)}\right) = 1-\gamma.$$

$$\Rightarrow \quad \left(\frac{\chi^2_{1-\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}\right)}, \frac{\chi^2_{\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}\right)}\right)$$

is a $(1-\gamma)\,100\%$ credible interval for λ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

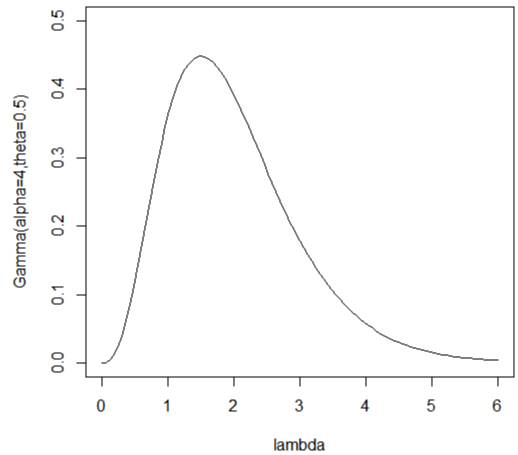
Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\quad \sum_{i=1}^{n} x_i^2 = 40.$

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} x_i^2} = 0.25.$$

Let $\alpha = 4$, $\theta = 0.50$.

prior mean $= \alpha\,\theta = 2$.



$$\left(\text{conditional mean of } \lambda, \text{ given } X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right) = \frac{2n + \alpha}{\sum_{i=1}^{n} x_i^2 + \frac{1}{\theta}} = \mathbf{\frac{1}{3}}.$$

95% credible interval for λ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$:

28 degrees of freedom $\left(\frac{15.31}{84}, \frac{44.46}{84}\right) = \mathbf{(0.182, 0.529)}$.

2. ~ **Example** **6.8-2** **9.2-2** 7.2-2 (STAT 400 textbook)

Let X have a Binomial(n, p) distribution. Let p have a prior p.d.f. which is Beta with parameters α and β.

Recall: the maximum likelihood estimator of p is $\hat{p} = \dfrac{X}{n}$.

a) Find the posterior distribution of p, given $X = x$.

$$f(x \mid p) = \binom{n}{x} p^{x} (1-p)^{n-x}.$$

$$f(x, p) = f(x \mid p) \times \pi(p)$$

$$= \binom{n}{x} p^{x} (1-p)^{n-x} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}$$

$$= \ldots\ p^{x+\alpha-1} (1-p)^{n-x+\beta-1}.$$

$\Rightarrow$ the posterior distribution of p, given $X = x$,

is **Beta** with New $\alpha = x + \alpha$ and New $\beta = n - x + \beta$.

b) Find the conditional mean of p, given $X = x$. Show that it is a weighted average of the maximum likelihood estimate $\hat{p}$ and the prior mean.

$\Rightarrow$ (conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$)

$$= \frac{(\text{New }\alpha)}{(\text{New }\alpha) + (\text{New }\beta)} = \frac{x+\alpha}{n+\alpha+\beta}.$$

$$\frac{x+\alpha}{n+\alpha+\beta} = \frac{x}{n} \cdot \frac{n}{n+\alpha+\beta} + \frac{\alpha}{\alpha+\beta} \cdot \frac{\alpha+\beta}{n+\alpha+\beta}.$$

$$= \hat{p} \cdot \frac{n}{n+\alpha+\beta} + (\text{prior mean}) \cdot \frac{\alpha+\beta}{n+\alpha+\beta}.$$