STAT 410

Examples for 11/16/2020 (2)

Fall 2020
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$X_1, X_2, \ldots, X_n$ i.i.d. p.d.f. or p.m.f. $f(x;\theta)$. $\quad H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$.

Likelihood Ratio:

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(\theta_0; x_1, x_2, \ldots, x_n)}{L(\theta_1; x_1, x_2, \ldots, x_n)}.$$

Neyman-Pearson Lemma:

$$C = \{ (x_1, x_2, \ldots, x_n) : \lambda(x_1, x_2, \ldots, x_n) \le k \}.$$

("Reject H_0 if $\lambda(x_1, x_2, \ldots, x_n) \le k$ ")

is the best (most powerful) rejection region.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with mean θ. That is, let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the p.d.f.

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad x > 0.$$

We wish to test $H_0: \theta = 0.5$ vs. $H_1: \theta > 0.5$.

a) If $n = 7$, find a uniformly most powerful rejection region with the significance level $\alpha = 0.05$.

Hint: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

Let $\theta > 0.5$.

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(H_0; x_1, x_2, \ldots, x_n)}{L(H_1; x_1, x_2, \ldots, x_n)} = \frac{L(0.5; x_1, x_2, \ldots, x_n)}{L(\theta; x_1, x_2, \ldots, x_n)}$$

$$= \frac{\prod_{i=1}^{n} 2 e^{-2x_i}}{\prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}} = (2\theta)^n \exp\left\{ \left(\frac{1-2\theta}{\theta} \right) \sum_{i=1}^{n} x_i \right\}.$$

Since $\theta > 0.5$, $1 - 2\theta < 0$, and $\lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad \sum_{i=1}^{n} x_i \ge c.$

If H_0 is true, $\sum_{i=1}^{7} X_i$ has a Gamma$(\alpha = 7, \theta = 1/2)$ distribution.

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{7} X_i \ge c \mid \theta = 1/2\right)$$

$$= P\left(4\sum_{i=1}^{7} X_i \ge 4c \mid \theta = 1/2\right) = P(\chi^2(14) \ge 4c).$$

$$\Rightarrow \quad 4c = \chi^2_{0.05}(14) = 23.68. \qquad \Rightarrow \quad c = \mathbf{5.92}.$$

Reject H_0 if $\sum_{i=1}^{7} x_i \ge \mathbf{5.92}$.

b) Find the power of the rejection rule from part (a) at $\theta = 0.75$.

$$\text{Power}(\theta = 0.75) = P(\text{Reject } H_0 \mid \theta = 0.75) = P\left(\sum_{i=1}^{7} X_i \ge 5.92 \mid \theta = 0.75\right) = \ldots$$

$$= P\left(\frac{2}{0.75}\sum_{i=1}^{7} X_i \ge \frac{2}{0.75}\cdot 5.92 \mid \theta = 0.75\right) = P\left(\chi^2(14) \ge \frac{2}{0.75}\cdot 5.92\right)$$

=CHISQ.DIST.RT(2*5.92/0.75,14) 0.326575

OR

$$= P\left(\text{Poisson}\left(\frac{5.92}{0.75}\right) \le 7 - 1\right) = P\left(\text{Poisson}\left(\frac{5.92}{0.75}\right) \le 6\right)$$

=POISSON.DIST(6,5.92/0.75,1) 0.326575

c) Find the significance level if the rejection rule is "Reject H_0 if $\sum_{i=1}^{7} X_i \ge 6$".

Hint: If T has a Gamma(α, θ) distribution, where α is an integer, then

$P(T \ge t) = P(Y \le \alpha - 1)$, where Y has a Poisson$\left(\frac{t}{\theta}\right)$ distribution.

$$\text{significance level} = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{7} X_i \ge 6 \mid \theta = 1/2\right)$$

$$= P(Y \le 7 - 1), \quad \text{where Y has a Poisson}\left(\frac{1}{1/2} \cdot 6 = 12\right) \text{ distribution}$$

using Cumulative Poisson Probabilities table:

$$= P(Y \le 6) = \mathbf{0.046}.$$

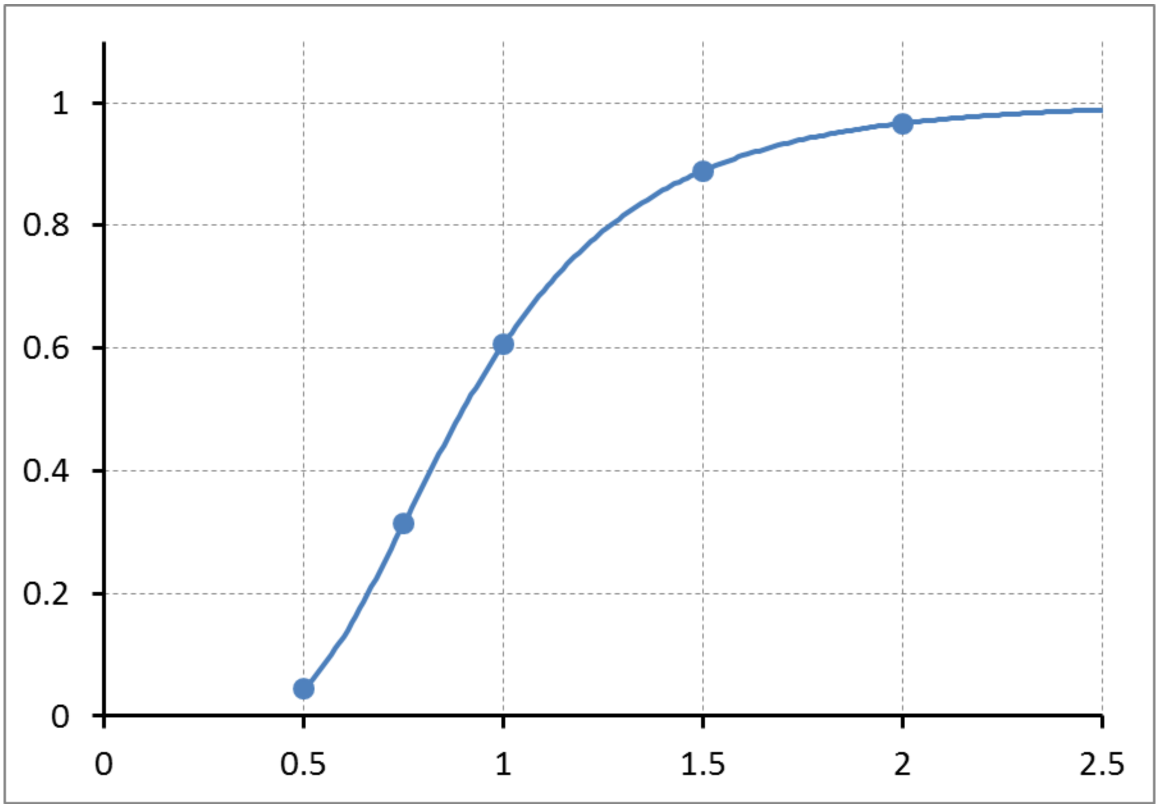
d) Find the power of the rejection rule is "Reject H_0 if $\sum_{i=1}^{7} X_i \ge 6$" at $\theta = 0.75$, $\theta = 1$, $\theta = 1.5$, $\theta = 2$.

$$\text{Power}(\theta) = P(\text{Reject } H_0 \mid \theta) = P\left(\sum_{i=1}^{7} X_i \ge 6 \mid \theta\right) = P(Y \le 7 - 1) = P(Y \le 6),$$

where Y has a Poisson$\left(\frac{1}{\theta} \cdot 6 = \frac{6}{\theta}\right)$ distribution.

Using Cumulative Poisson Probabilities table:

$\theta = 0.75$	$\frac{6}{\theta} = 8$	$=$ **0.313**.
$\theta = 1$	$\frac{6}{\theta} = 6$	$=$ **0.606**.
$\theta = 1.5$	$\frac{6}{\theta} = 4$	$=$ **0.889**.
$\theta = 2$	$\frac{6}{\theta} = 3$	$=$ **0.966**.



e) Suppose $\sum_{i=1}^{7} x_i = 6.5$. Find the p-value of this test.

p-value = $P\left(\sum_{i=1}^{7} X_i \text{ as extreme or more extreme than } \left(\sum_{i=1}^{7} x_i \right)_{\text{observed}} \middle| H_0 \text{ true} \right)$

$$= P\left(\sum_{i=1}^{7} X_i \ge 6.5 \mid \theta = {}^1\!/_2 \right) = P(Y \le 6)$$

where Y has a Poisson$\left(\frac{1}{1/2} \cdot 6.5 = 13 \right)$ distribution

$= \mathbf{0.026}$.