STAT 410 | **Examples for 09/14/2020 (1)** | Fall 2020
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2.3 Conditional Distributions and Expectations. (continued)

Def $\quad \mathrm{Var}(X\,|\,Y) = E[(X - E(X\,|\,Y))^2\,|\,Y] = E(X^2\,|\,Y) - [E(X\,|\,Y)]^2$

Theorem $\quad E(E(X\,|\,Y)) = E(X)$

$\mathrm{Var}(E(X\,|\,Y)) \le \mathrm{Var}(X)$

Furthermore, $\quad \mathrm{Var}(X) = \mathrm{Var}(E(X\,|\,Y)) + E[\mathrm{Var}(X\,|\,Y)]$

$\mathrm{Var}(X)$	$=$	$\mathrm{Var}(E(X\,|\,Y))$	$+$	$E[\mathrm{Var}(X\,|\,Y)]$
		Portion of the distribution of X that is related to Y		Portion of the distribution of X that is independent of Y

If X is a function of Y, then

$E(X\,|\,Y) = X$ and $\mathrm{Var}(E(X\,|\,Y)) = \mathrm{Var}(X)$,

$\mathrm{Var}(X\,|\,Y) = 0$ and $E[\mathrm{Var}(X\,|\,Y)] = 0$.

If X and Y are independent, then

$E(X\,|\,Y) = E(X)$ and $\mathrm{Var}(E(X\,|\,Y)) = 0$ since $E(X)$ is a constant,

$\mathrm{Var}(X\,|\,Y) = \mathrm{Var}(X)$ and $E[\mathrm{Var}(X\,|\,Y)] = \mathrm{Var}(X)$.

2. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$

$$E(X) = \frac{1}{2}, \qquad \operatorname{Var}(X) = \frac{9}{252}, \qquad E(X^2) = \frac{72}{252} = \frac{2}{7};$$

$$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1,$$

$$E(Y) = \frac{1}{3}, \qquad \operatorname{Var}(Y) = \frac{8}{252}, \qquad E(Y^2) = \frac{36}{252} = \frac{1}{7};$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2\,y}{(1-x)^2}, \qquad 0 < y < 1-x, \qquad 0 < x < 1.$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{3\,x^2}{(1-y)^3}, \qquad 0 < x < 1-y, \qquad 0 < y < 1.$$

$$E(Y|X=x) = \int_0^{1-x} y \cdot \frac{2\,y}{(1-x)^2}\,dy = \frac{2}{3}(1-x), \qquad 0 < x < 1.$$

$$E(Y|X) = \frac{2}{3}(1-X). \qquad E(E(Y|X)) = \frac{1}{3} = E(Y).$$

$$E[(E(Y|X))^2] = \frac{4}{9}\left(1-2\,E(X)+E\left(X^2\right)\right) = \frac{4}{9}\left(1-2\cdot\frac{1}{2}+\frac{2}{7}\right) = \frac{8}{63}.$$

$$\operatorname{Var}(E(Y|X)) = \frac{8}{63}-\left(\frac{1}{3}\right)^2 = \frac{1}{63}.$$

OR

$$\operatorname{Var}(E(Y|X)) = \operatorname{Var}\left(\frac{2}{3}(1-X)\right) = \frac{4}{9}\operatorname{Var}(X) = \frac{4}{9}\cdot\frac{9}{252} = \frac{4}{252} = \frac{1}{63}.$$

$$\operatorname{Var}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X})) = \frac{1}{63} = \frac{4}{252} < \frac{8}{252} = \operatorname{Var}(\mathrm{Y}).$$

$$\mathrm{E}(\mathrm{Y}^2 \mid \mathrm{X} = x) = \int_0^{1-x} y^2 \cdot \frac{2y}{(1-x)^2}\, dy = \frac{1}{2}(1-x)^2, \quad 0 < x < 1.$$

$$\begin{aligned}\operatorname{Var}(\mathrm{Y} \mid \mathrm{X} = x) &= \mathrm{E}(\mathrm{Y}^2 \mid \mathrm{X} = x) - [\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)]^2 \\ &= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2 = \frac{1}{18}(1-x)^2, \qquad 0 < x < 1.\end{aligned}$$

$$\operatorname{Var}(\mathrm{Y} \mid \mathrm{X}) = \frac{1}{18}(1-\mathrm{X})^2.$$

$$\mathrm{E}[\operatorname{Var}(\mathrm{Y} \mid \mathrm{X})] = \frac{1}{18}\left(1 - 2\,\mathrm{E}(\mathrm{X}) + \mathrm{E}\left(\mathrm{X}^2\right)\right) = \frac{1}{18}\left(1 - 2 \cdot \frac{1}{2} + \frac{2}{7}\right) = \frac{1}{63}.$$

$$\operatorname{Var}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X})) + \mathrm{E}[\operatorname{Var}(\mathrm{Y} \mid \mathrm{X})] = \frac{1}{63} + \frac{1}{63} = \frac{8}{252} = \operatorname{Var}(\mathrm{Y}).$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y) = \int_0^{1-y} x \cdot \frac{3x^2}{(1-y)^3}\,dx = \frac{3}{4}(1-y), \qquad 0 < y < 1.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}) = \frac{3}{4}(1-\mathrm{Y}).$$

$$\mathrm{E}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = \frac{3}{4}(1-\mathrm{E}(\mathrm{Y})) = \frac{3}{4}\left(1-\frac{1}{3}\right) = \frac{1}{2} = \mathrm{E}(\mathrm{X}).$$

$$\mathrm{Var}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = \mathrm{Var}\left(\frac{3}{4}(1-\mathrm{Y})\right) = \frac{9}{16}\mathrm{Var}(\mathrm{Y}) = \frac{9}{16} \cdot \frac{8}{252} = \frac{1}{56}.$$

$$\mathrm{Var}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = \frac{1}{56} = \frac{4.5}{252} < \frac{9}{252} = \mathrm{Var}(\mathrm{X}).$$

$$\mathrm{E}(\mathrm{X}^2 \mid \mathrm{Y} = y) = \int_0^{1-y} x^2 \cdot \frac{3x^2}{(1-y)^3}\,dx = \frac{3}{5}(1-y)^2, \quad 0 < y < 1.$$

$$\begin{aligned}\mathrm{Var}(\mathrm{X} \mid \mathrm{Y} = y) &= \mathrm{E}(\mathrm{X}^2 \mid \mathrm{Y} = y) - [\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y)]^2 \\ &= \frac{3}{5}(1-y)^2 - \frac{9}{16}(1-y)^2 = \frac{3}{80}(1-y)^2, \qquad 0 < y < 1.\end{aligned}$$

$$\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}) = \frac{3}{80}(1-\mathrm{Y})^2.$$

$$\mathrm{E}[\mathrm{Var}(\mathrm{X} \mid \mathrm{Y})] = \frac{3}{80}\left(1 - 2\,\mathrm{E}(\mathrm{Y}) + \mathrm{E}\left(\mathrm{Y}^2\right)\right) = \frac{3}{80}\left(1 - 2 \cdot \frac{1}{3} + \frac{1}{7}\right) = \frac{1}{56}.$$

$$\mathrm{Var}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) + \mathrm{E}[\mathrm{Var}(\mathrm{X} \mid \mathrm{Y})] = \frac{1}{56} + \frac{1}{56} = \frac{9}{252} = \mathrm{Var}(\mathrm{X}).$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	

x	$p_{X\|Y}(x\|0)$
1	$0.15/0.40 = 0.375$
2	$0.25/0.40 = 0.625$

x	$p_{X\|Y}(x\|1)$
1	$0.10/0.40 = 0.25$
2	$0.30/0.40 = 0.75$

x	$p_{X\|Y}(x\|2)$
1	$0.00/0.20 = 0.00$
2	$0.20/0.20 = 1.00$

$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=0) = 1.625$ $\quad$ $\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=1) = 1.75$ $\quad$ $\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=2) = 2$

$\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}=0) = 0.234375$ $\quad$ $\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}=1) = 0.1875$ $\quad$ $\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}=2) = 0$

Def $\quad \mathrm{Var}(\mathrm{X} \mid \mathrm{Y})) = \mathrm{E}[(\mathrm{X}-\mathrm{E}(\mathrm{X} \mid \mathrm{Y}))^2 \mid \mathrm{Y}] = \mathrm{E}(\mathrm{X}^2 \mid \mathrm{Y}) - [\mathrm{E}(\mathrm{X} \mid \mathrm{Y})]^2$

y	$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y)$	$p_Y(y)$
0	1.625	0.40
1	1.75	0.40
2	2	0.20

$\mathrm{Var}(\mathrm{X} \mid \mathrm{Y}=y)$	$p_Y(y)$
0.234375	0.40
0.1875	0.40
0	0.20

$\mathrm{E}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = 1.75 = \mathrm{E}(\mathrm{X})$ $\qquad$ $\mathrm{E}(\mathrm{Var}(\mathrm{X} \mid \mathrm{Y})) = 0.16875$

$\mathrm{Var}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = 0.01875 \quad < \quad 0.1875 = \mathrm{Var}(\mathrm{X}).$

$\mathrm{Var}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) + \mathrm{E}(\mathrm{Var}(\mathrm{X} \mid \mathrm{Y})) = 0.01875 + 0.16875 = 0.1875 = \mathrm{Var}(\mathrm{X}).$

y	$p_{Y\|X}(y\|1)$
0	$^{0.15}/_{0.25} = 0.60$
1	$^{0.10}/_{0.25} = 0.40$
2	$^{0.00}/_{0.25} = 0.00$

y	$p_{Y\|X}(y\|2)$
0	$^{0.25}/_{0.75} = {}^{5}/_{15}$
1	$^{0.30}/_{0.75} = {}^{6}/_{15}$
2	$^{0.20}/_{0.75} = {}^{4}/_{15}$

$\mathrm{E}(Y \mid X=1) = 0.4 = {}^{6}/_{15}$

$\mathrm{E}(Y \mid X=2) = {}^{14}/_{15}$

$\mathrm{Var}(Y \mid X=1) = 0.24 = {}^{54}/_{225}$

$\mathrm{Var}(Y \mid X=2) = {}^{134}/_{225}$

x	$\mathrm{E}(Y\|X=x)$	$p_X(x)$
1	$^{6}/_{15}$	0.25
2	$^{14}/_{15}$	0.75

$\mathrm{Var}(Y\|X=x)$	$p_X(x)$
$^{54}/_{225}$	0.25
$^{134}/_{225}$	0.75

$\mathrm{E}(\mathrm{E}(Y \mid X)) = {}^{12}/_{15} = 0.80 = \mathrm{E}(Y)$

$\mathrm{E}(\mathrm{Var}(Y \mid X)) = {}^{38}/_{75}$

$\mathrm{Var}(\mathrm{E}(Y \mid X)) = {}^{4}/_{75} \approx 0.053333 \quad < \quad 0.56 = \mathrm{Var}(Y)$

$\mathrm{Var}(Y) = 0.56 = {}^{4}/_{75} + {}^{38}/_{75} = \mathrm{Var}(\mathrm{E}(Y \mid X)) + \mathrm{E}(\mathrm{Var}(Y \mid X))$