STAT 410 Fall 2020

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Examples for 11/30/2020 (1)

$\mathrm{H}_0 : \theta \in \Theta_0$ vs. $\mathrm{H}_1 : \theta \in \Theta_1$ $\Theta_0 \cap \Theta_1 = \varnothing$.

Reject H_0 if $\Lambda^* = \dfrac{\max\limits_{\theta \in \Theta_0} \mathrm{L}(\theta)}{\max\limits_{\theta \in \Theta_1} \mathrm{L}(\theta)} \le k \quad \Leftrightarrow \quad \Lambda = \dfrac{\max\limits_{\theta \in \Theta_0} \mathrm{L}(\theta)}{\max\limits_{\theta \in \Theta_0 \cup \Theta_1} \mathrm{L}(\theta)} \le k$

since $\Lambda = \min(\Lambda^*, 1)$

$\mathrm{H}_0 : \theta = \theta_0$ vs. $\mathrm{H}_1 : \theta \ne \theta_0$ Reject H_0 if $\Lambda = \dfrac{\mathrm{L}(\theta_0)}{\mathrm{L}(\hat{\theta})} \le k$

(Likelihood Ratio Test)

Example 1:

Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample of size n from an Exponential distribution with mean $1/\lambda$. That is,

$$f_{\mathrm{X}}(x) = \lambda e^{-\lambda x}, \qquad x > 0.$$

$\mathrm{H}_0 : \lambda = \lambda_0$ vs. $\mathrm{H}_1 : \lambda \ne \lambda_0$

$$\mathrm{L}(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n \exp\left\{ -\lambda \sum_{i=1}^{n} x_i \right\}. \qquad \hat{\lambda} = \frac{1}{\overline{\mathrm{X}}}.$$

$$\Lambda = \frac{\mathrm{L}(\lambda_0)}{\mathrm{L}(\hat{\lambda})} = \frac{\lambda_0^n \exp\left\{ -\lambda_0 \sum_{i=1}^{n} x_i \right\}}{\left(\frac{1}{\overline{x}} \right)^n \exp\left\{ -\left(\frac{1}{\overline{x}} \right) \sum_{i=1}^{n} x_i \right\}} = e^n \lambda_0^n \left(\overline{x} \right)^n \exp\left\{ -n \lambda_0 \overline{x} \right\}.$$

$$\Lambda \le k \qquad \Leftrightarrow \qquad \overline{x} \exp\left\{ -\lambda_0 \overline{x} \right\} \le c$$

Reject H_0 if $\overline{x} \exp\left\{ -\lambda_0 \overline{x} \right\} \le c$.

Example 2:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$ distribution $(\sigma^2 \text{ known})$.

$H_0 : \mu = \mu_0 \quad$ vs. $\quad H_1 : \mu \neq \mu_0$

$\hat{\mu} = \overline{X}$.

$$\Lambda = \frac{L(\mu_0)}{L(\hat{\mu})} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - \mu_0)^2 \right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - \overline{x})^2 \right\}}$$

$$= \exp\left\{ \frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (x_i - \overline{x})^2 - \sum_{i=1}^{n} (x_i - \mu_0)^2 \right] \right\}$$

$$= \exp\left\{ -\frac{n(\overline{x} - \mu_0)^2}{2\sigma^2} \right\}.$$

$$\Lambda \le k \quad \Leftrightarrow \quad \frac{n(\overline{x} - \mu_0)^2}{\sigma^2} \ge k_1 \quad \Leftrightarrow \quad \left| \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}} \right| \ge c$$

Reject H_0 if $\left| \dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}} \right| \ge c.$

Recall: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}}$ has $N(0, 1)$ distribution. $\Rightarrow \quad c = z_{\alpha/2}.$

Example 3:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$ distribution $(\sigma^2 \text{ unknown})$.

$H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$

$$\hat{\mu} = \overline{X}, \qquad \hat{\sigma}^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X} \right)^2 .$$

Under H_0, $\qquad \hat{\mu}_0 = \mu_0, \qquad \hat{\sigma}_0^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} \left(X_i - \mu_0 \right)^2 .$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\hat{\sigma}_0} \exp\left\{ -\frac{1}{2\hat{\sigma}_0^2} (x_i - \hat{\mu}_0)^2 \right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\hat{\sigma}} \exp\left\{ -\frac{1}{2\hat{\sigma}^2} (x_i - \hat{\mu})^2 \right\}}$$

$$= \left(\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{\sum_{i=1}^{n} (x_i - \mu_0)^2} \right)^{n/2} = \left(\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n(\overline{x} - \mu_0)^2 + \sum_{i=1}^{n} (x_i - \overline{x})^2} \right)^{n/2} .$$

$$\Lambda \le k \quad \Leftrightarrow \quad \frac{n(\overline{x} - \mu_0)^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2} \ge k_1 \quad \Leftrightarrow \quad \left| \frac{\overline{x} - \mu_0}{s / \sqrt{n}} \right| \ge c$$

Reject H_0 if $\left| \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}} \right| \ge c.$

Recall: $\dfrac{\overline{X} - \mu}{S / \sqrt{n}}$ has $t(n-1)$ distribution. $\Rightarrow \quad c = t_{\alpha/2}(n-1).$

Example 4:

Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size n from a distribution with a p.d.f. $f(x;\theta) = {}^1/_\theta$, for $0 \le x \le \theta$, zero elsewhere, where $\theta > 0$.

$H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\left(\frac{1}{\theta_0}\right)^n I\{Y_n < \theta_0\}}{\left(\frac{1}{Y_n}\right)^n} = \left(\frac{Y_n}{\theta_0}\right)^n I\{Y_n < \theta_0\}.$$

$\Lambda \le k \quad \Leftrightarrow \quad Y_n \le c$ or $Y_n \ge \theta_0$

Reject H_0 if $Y_n \le c$ or $Y_n \ge \theta_0$.

Example 5: **8.2.2** (7th and 6th edition)

Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with a p.d.f. $f(x;\theta) = {}^1/_\theta$, for $0 \le x \le \theta$, zero elsewhere, where $0 < \theta$. Let the observed value of Y_4 be y_4. The hypothesis $H_0: \theta = 1$ is rejected and $H_1: \theta \ne 1$ is accepted if either $y_4 \le 1/2$ or $y_4 \ge 1$. Find and sketch the power function $K(\theta)$, $0 < \theta$, of the test.

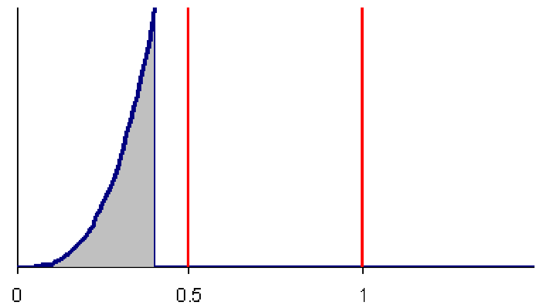
Hint: Consider three cases: $0 < \theta < 1/2$, $1/2 < \theta < 1$, and $\theta > 1$.

$$F_{Y_4}(y) = \frac{y^4}{\theta^4}, \qquad 0 < y < \theta. \qquad\qquad f_{Y_4}(y) = \frac{4y^3}{\theta^4}, \qquad 0 < y < \theta.$$

Reject H_0 if $y_4 \le {}^1/_2$ or $y_4 \ge 1$.

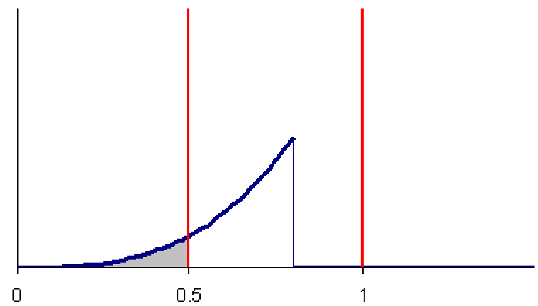
Case 1: $\theta < {}^1\!/_2$.

$\text{Power}(\theta) = 1.$



Case 2: ${}^1\!/_2 < \theta < 1$.

$$\text{Power}(\theta) = F_{Y_4}({}^1\!/_2) = \frac{1}{16\theta^4}.$$



Case 3: $\theta > 1$.

$$\text{Power}(\theta) = F_{Y_4}({}^1\!/_2) + [1 - F_{Y_4}(1)]$$

$$= \frac{1}{16\theta^4} + \left[1 - \frac{1}{\theta^4}\right] = 1 - \frac{15}{16\theta^4}.$$