Examples for 11/11/2020 (2)

Fall 2020
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1. A certain STAT 410 instructor claims that the average time to complete a STAT 410 homework assignment is at most 150 minutes. Assume that the time to complete a STAT 410 homework assignment is approximately normally distributed with standard deviation $\sigma = 24$ minutes. We wish to test $H_0: \mu \le 150$ vs. $H_1: \mu > 150$.

a) A random sample of $n = 9$ students yields the sample mean of 162 minutes. Find the p-value for the test.

Test Statistic: $$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{162 - 150}{24 / \sqrt{9}} = \mathbf{1.50}.$$

$$\text{P - value} = P(Z \ge 1.50) = \mathbf{0.0668}.$$

p-value $> \alpha$ $\Rightarrow$ Do NOT Reject H_0.

p-value $< \alpha$ $\Rightarrow$ Reject H_0.

Since $0.0688 > 0.05$, Do NOT Reject H_0 at $\alpha = 0.05$.

Since $0.0688 < 0.10$, Reject H_0 at $\alpha = 0.10$.

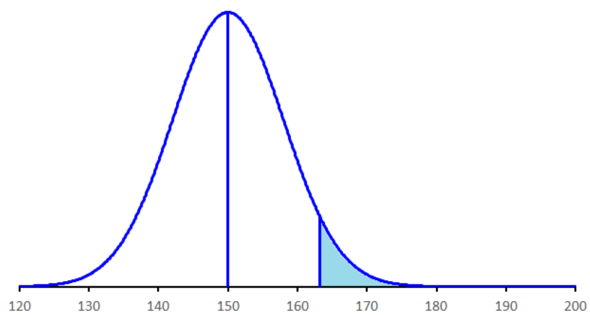
b) Find the Rejection Region for the test at $\alpha = 0.05$ if $n = 9$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if $n = 9$ and a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha .$$

$$Z = \frac{\overline{X} - 150}{24 / \sqrt{9}} > 1.645.$$

$$\overline{X} > 150 + 1.645 \cdot \frac{24}{\sqrt{9}} = \mathbf{163.16}.$$



c) Find the power of the test if the actual value of the average time to complete a STAT 410 homework assignment is (i) 159 minutes and (ii) 165 minutes, $n = 9$, and a 5% level of significance is used.

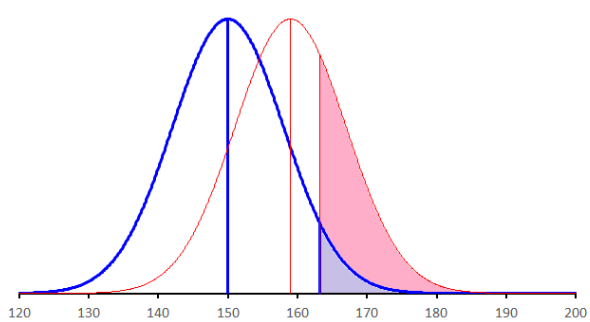
$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P(\overline{X} > 163.16 \mid \mu).$$

$$P(\overline{X} > 163.16 \mid \mu = 159)$$

$$= P\left(Z > \frac{163.16 - 159}{24 / \sqrt{9}} \right)$$

$$= P(Z > 0.52)$$

$$= \mathbf{0.3015}.$$

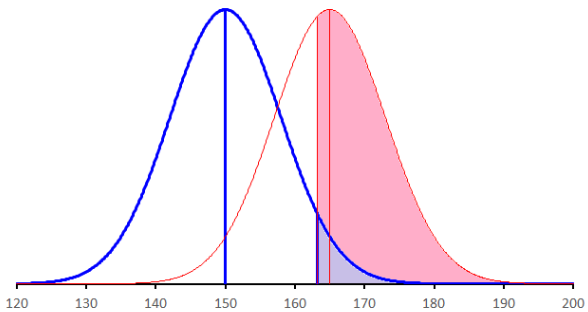


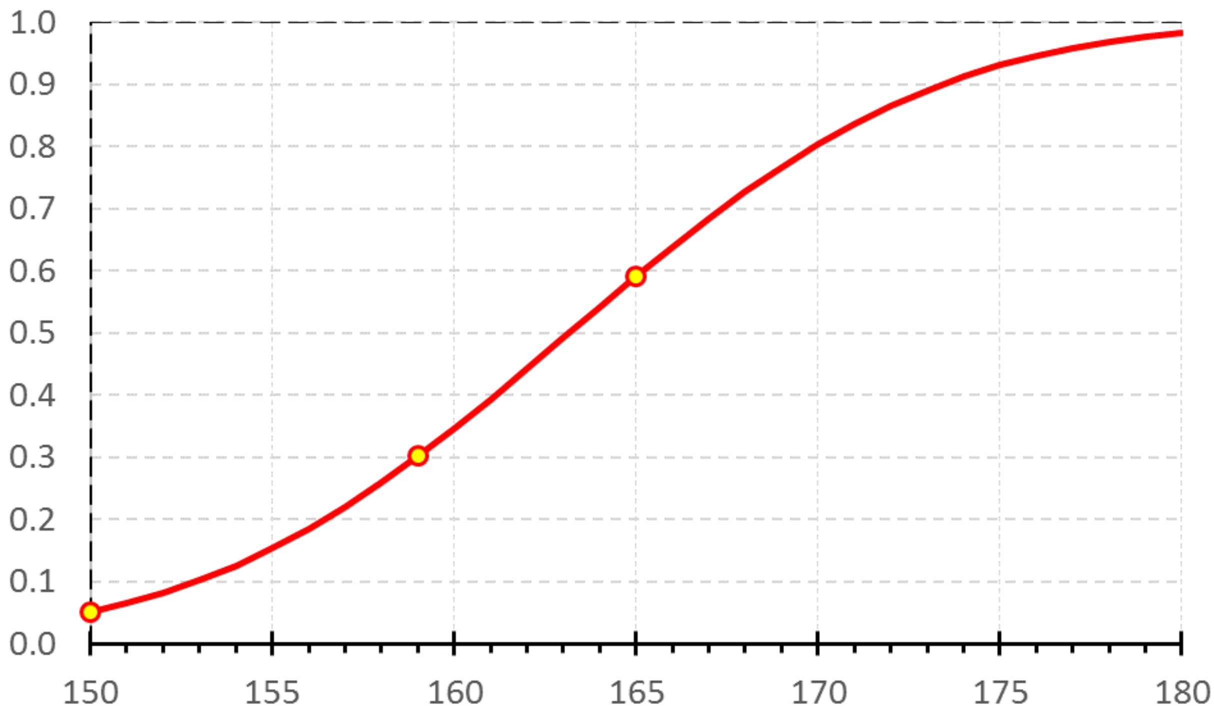
$$P(\overline{X} > 163.16 \mid \mu = 165)$$

$$= P\left(Z > \frac{163.16 - 165}{24/\sqrt{9}}\right)$$

$$= P(Z > -0.23)$$

$$= \mathbf{0.5910}.$$





d) Find the Rejection Region for the test at $\alpha = 0.05$ if $n = 25$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if $n = 25$ and a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha. \qquad Z = \frac{\overline{X} - 150}{24 / \sqrt{25}} > 1.645.$$

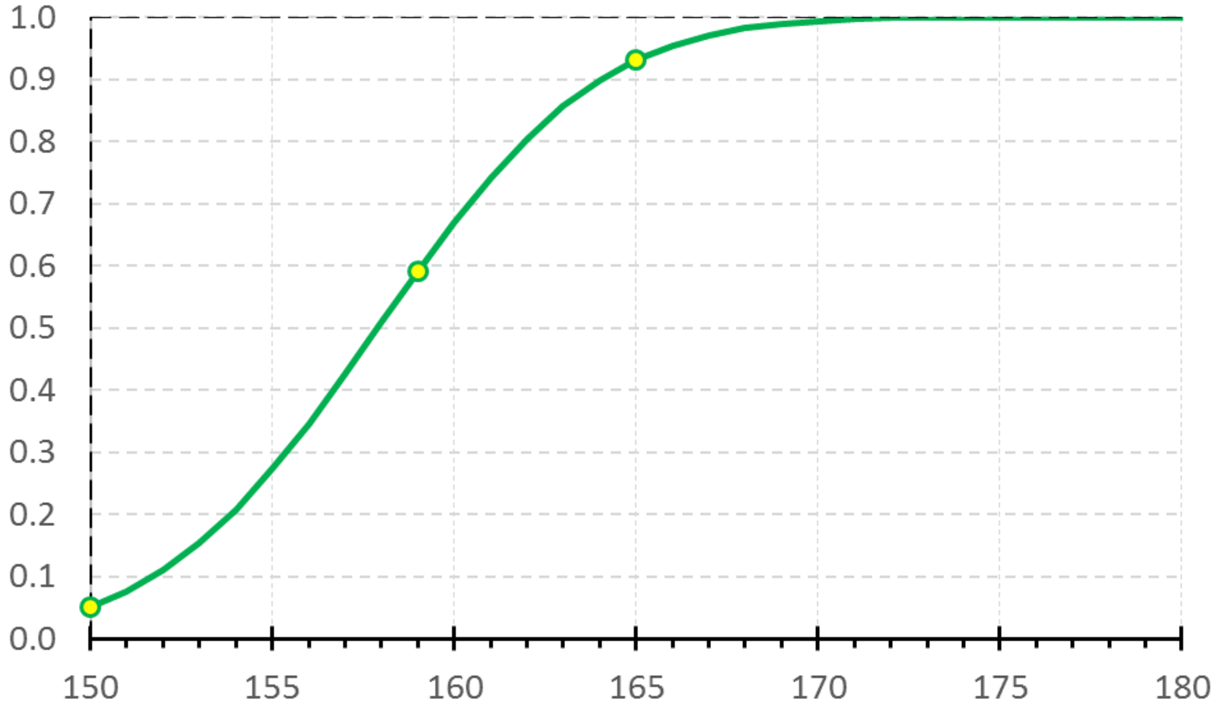
$$\overline{X} > 150 + 1.645 \cdot \frac{24}{\sqrt{25}} = \mathbf{157.896}.$$

e) Find the power of the test if the actual value of the average time to complete a STAT 410 homework assignment is (i) 159 minutes and (ii) 165 minutes, $n = 25$, and a 5% level of significance is used.

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P(\overline{X} > 157.896 \mid \mu).$$

$$P(\overline{X} > 157.896 \mid \mu = 159) = P\left(Z > \frac{157.896 - 159}{24 / \sqrt{25}}\right) = P(Z > -0.23) = \mathbf{0.5910}.$$

$$P(\overline{X} > 157.896 \mid \mu = 165) = P\left(Z > \frac{157.896 - 165}{24 / \sqrt{25}}\right) = P(Z > -1.48) = \mathbf{09306}.$$



f) Find the Rejection Region for the test at $\alpha = 0.05$ if $n = 49$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if $n = 49$ and a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha. \qquad\qquad Z = \frac{\overline{X} - 150}{24/\sqrt{49}} > 1.645.$$

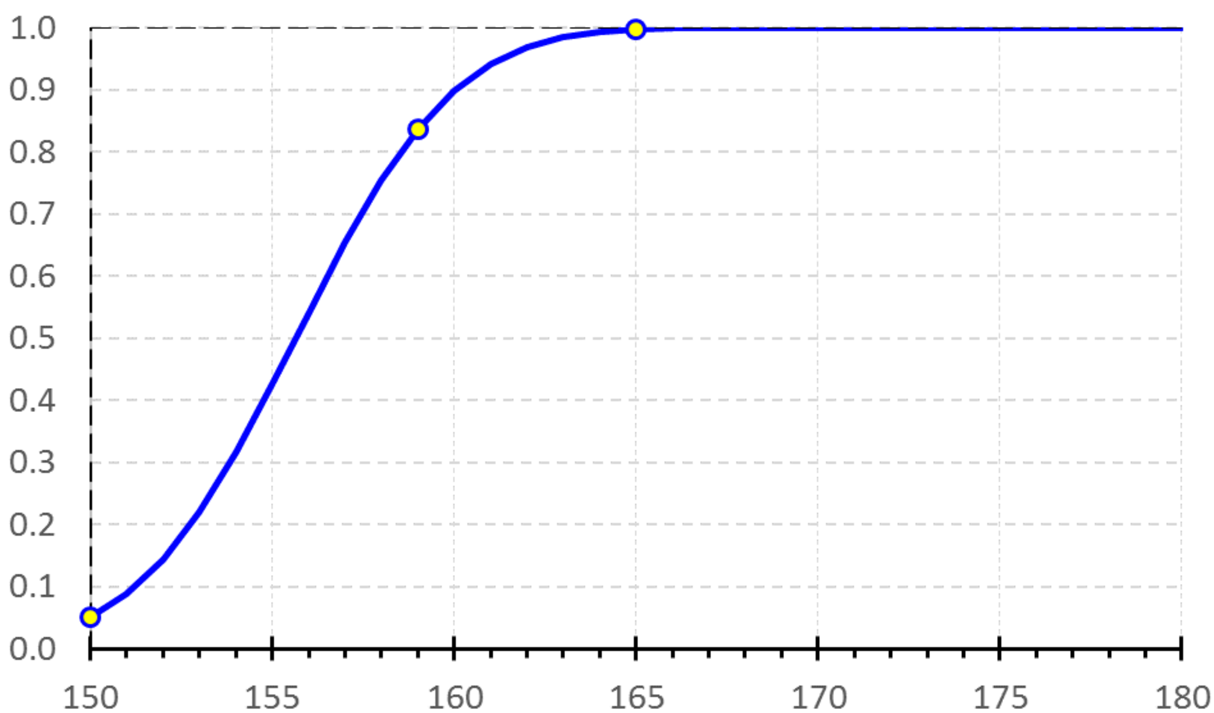
$$\overline{X} > 150 + 1.645 \cdot \frac{24}{\sqrt{49}} = \mathbf{155.64}.$$

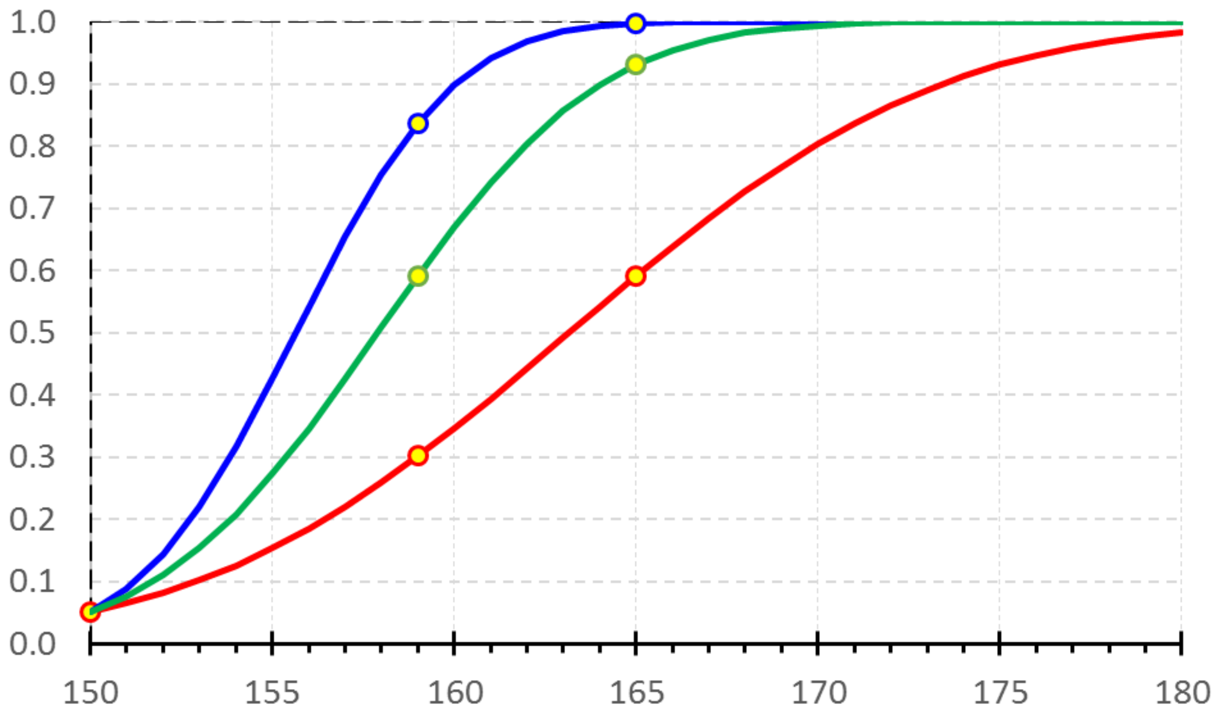
g) Find the power of the test if the actual value of the average time to complete a STAT 410 homework assignment is (i) 159 minutes and (ii) 165 minutes, $n = 49$, and a 5% level of significance is used.

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P(\overline{X} > 155.64 \mid \mu).$$

$$P(\overline{X} > 155.64 \mid \mu = 159) = P\left(Z > \frac{155.64 - 159}{24/\sqrt{49}}\right) = P(Z > -0.98) = \mathbf{0.8365}.$$

$$P(\overline{X} > 155.64 \mid \mu = 165) = P\left(Z > \frac{155.64 - 165}{24/\sqrt{49}}\right) = P(Z > -2.73) = \mathbf{0.9968}.$$





Right – tailed test. $\qquad H_0 : \mu \le 150 \quad \text{vs.} \quad H_1 : \mu > 150.$

Rejection Region:

$$\frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha . \qquad\qquad \overline{X} > \mu_0 + z_\alpha \cdot \frac{\sigma}{\sqrt{n}} .$$

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P\left(\overline{X} > \mu_0 + z_\alpha \cdot \frac{\sigma}{\sqrt{n}} \mid \mu\right).$$

$$= 1 - \Phi\left(\frac{\mu_0 - \mu}{\sigma / \sqrt{n}} + z_\alpha\right).$$

h) What is the minimum sample size required if we want to have the power of at least 0.90 at $\mu = 159$ minutes, if $n = 9$, and a 5% level of significance is used?

Rejection Region:

$$\frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha . \qquad \overline{X} > 150 + 1.645 \cdot \frac{24}{\sqrt{n}} .$$

Want $\quad P\left(\overline{X} > 150 + 1.645 \cdot \frac{24}{\sqrt{n}} \;\middle|\; \mu = 159 \right) \ge 0.90.$

$$P\left(\overline{X} > 150 + 1.645 \cdot \frac{24}{\sqrt{n}} \;\middle|\; \mu = 159 \right) = P\left(Z > \frac{150 - 159}{24 / \sqrt{n}} + 1.645 \right)$$

$$= P\left(Z > -\frac{3\sqrt{n}}{8} + 1.645 \right)$$

$P(Z > -1.282) = 0.90, \qquad -\frac{3\sqrt{n}}{8} + 1.645 \le -1.282 .$

$\Rightarrow \quad \sqrt{n} \ge \frac{8}{3} \cdot (1.282 + 1.645) \approx 7.805333.$

$\Rightarrow \quad n \ge 7.805333^2 \approx 60.923. \qquad$ Round up. $\qquad n \ge \mathbf{61}.$

i) Find the Rejection Region for the test at $\alpha = 0.01$ if $n = 9$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if $n = 9$ and a 1% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha. \qquad\qquad Z = \frac{\overline{X} - 150}{24 / \sqrt{9}} > 2.326.$$

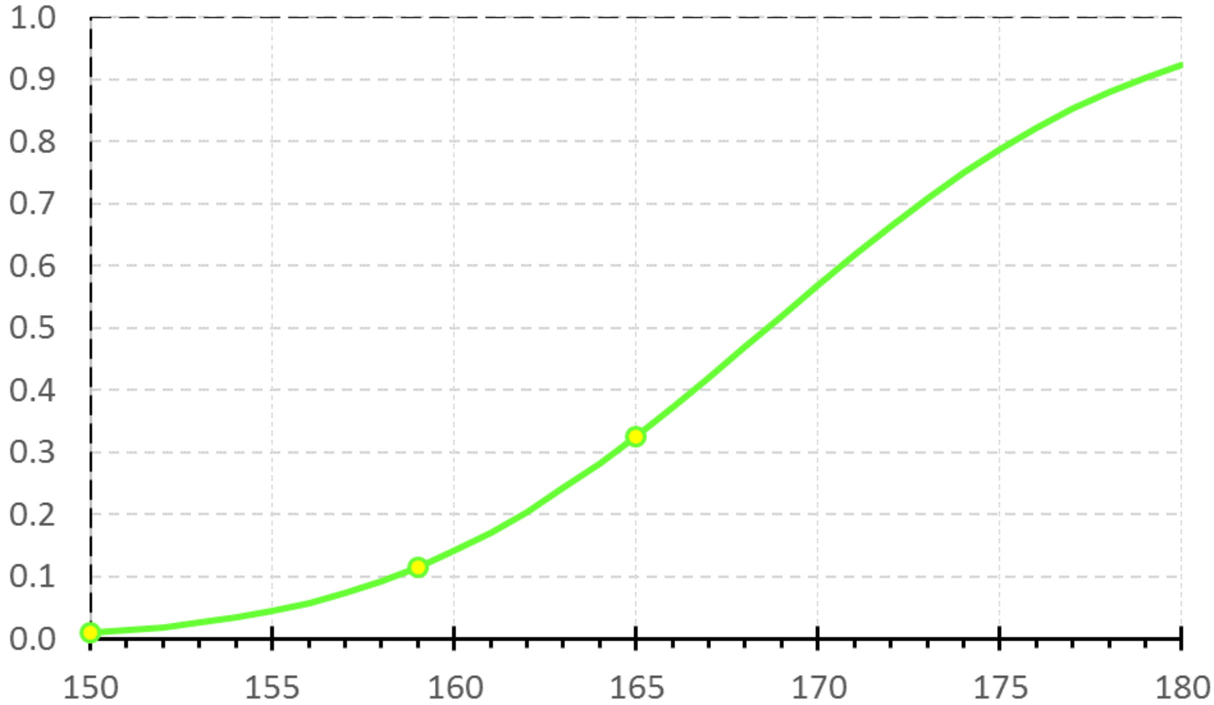
$$\overline{X} > 150 + 2.326 \cdot \frac{24}{\sqrt{9}} = \mathbf{168.608}.$$

j) Find the power of the test if the actual value of the average time to complete a STAT 410 homework assignment is (i) 159 minutes and (ii) 165 minutes, $n = 9$, and a 1% level of significance is used.

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P(\overline{X} > 168.608 \mid \mu).$$

$$P(\overline{X} > 168.608 \mid \mu = 159) = P\left(Z > \frac{168.608 - 159}{24 / \sqrt{9}}\right) = P(Z > 1.201) = \mathbf{0.1149}.$$

$$P(\overline{X} > 168.608 \mid \mu = 165) = P\left(Z > \frac{168.608 - 165}{24 / \sqrt{9}}\right) = P(Z > 0.451) = \mathbf{0.3260}.$$



k) Find the Rejection Region for the test at $\alpha = 0.10$ if $n = 9$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if $n = 9$ and a 10% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha}. \qquad\qquad Z = \frac{\overline{X} - 150}{24/\sqrt{9}} > 1.282.$$

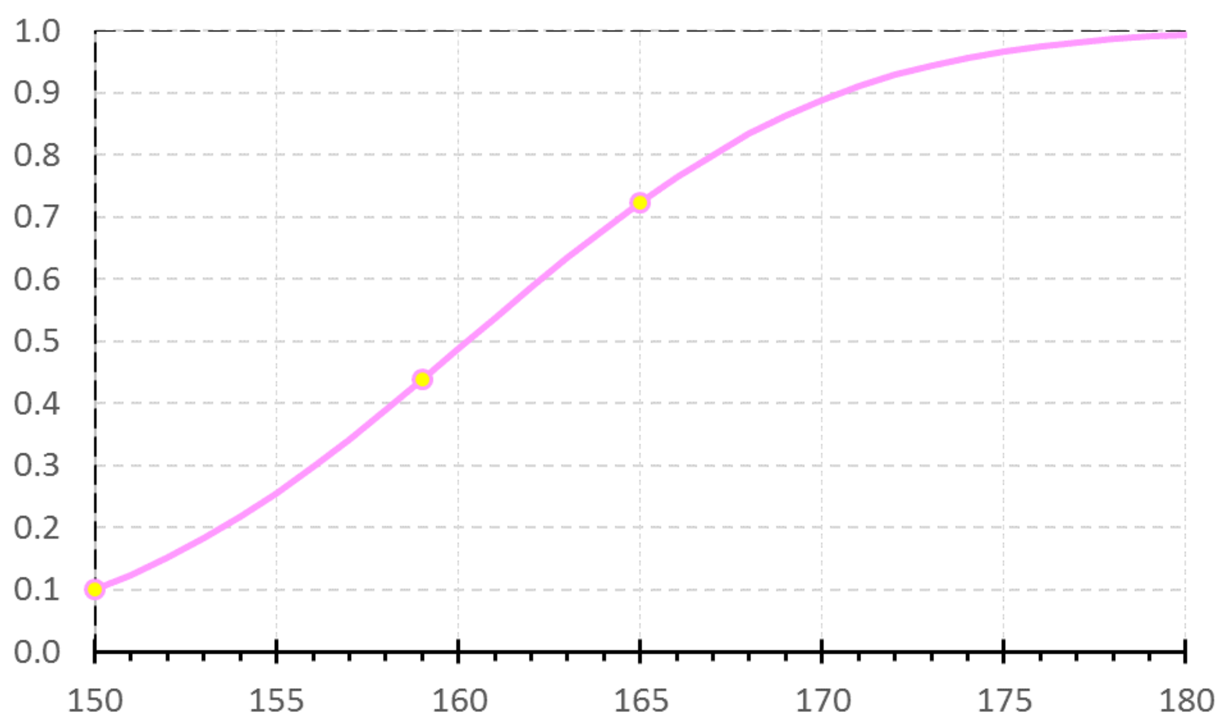
$$\overline{X} > 150 + 1.282 \cdot \frac{24}{\sqrt{9}} = \mathbf{160.256}.$$

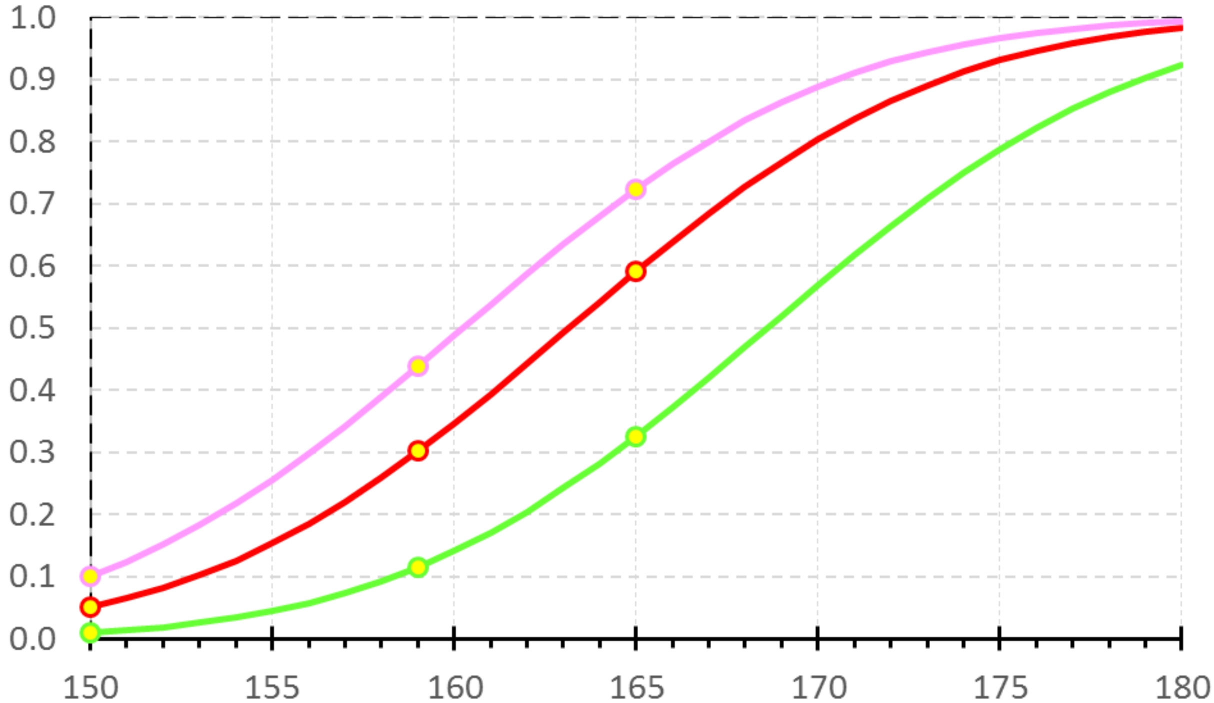
l) Find the power of the test if the actual value of the average time to complete a STAT 410 homework assignment is (i) 159 minutes and (ii) 165 minutes, $n = 9$, and a 10% level of significance is used.

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P(\overline{X} > 160.256 \mid \mu).$$

$$P(\overline{X} > 160.256 \mid \mu = 159) = P\left(Z > \frac{160.256 - 159}{24/\sqrt{9}}\right) = P(Z > 0.157) = \mathbf{0.4376}.$$

$$P(\overline{X} > 160.256 \mid \mu = 165) = P\left(Z > \frac{160.256 - 165}{24/\sqrt{9}}\right) = P(Z > -0.593) = \mathbf{0.7234}.$$




1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
150
155
160
165
170
175
180

2. Assume that the population is approximately normally distributed with standard deviation $\sigma = 24$ minutes. We wish to test $H_0: \mu = 150$ vs. $H_1: \mu \neq 150$.

a) A random sample of size $n = 9$ yields the sample mean of 162. Find the p-value for the test.

Test Statistic: $$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{162 - 150}{24 / \sqrt{9}} = \mathbf{1.50}.$$

$$\text{P - value} = \text{2 tails} = 2 \times P(Z \geq 1.50) = 2 \times 0.0668 = \mathbf{0.1336}.$$

b) Find the Rejection Region for the test at $\alpha = 0.05$ if $n = 9$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if $n = 9$ and a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} < -z_{\alpha/2} \quad \text{or} \quad Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_{\alpha/2}$$

$$\Rightarrow \quad \overline{X} < \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \overline{X} > \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \overline{X} < 150 - 1.96 \cdot \frac{24}{\sqrt{9}} \quad \text{or} \quad \overline{X} > 150 + 1.96 \cdot \frac{24}{\sqrt{9}}$$

$$\Rightarrow \quad \overline{X} < \mathbf{134.32} \quad \text{or} \quad \overline{X} > \mathbf{165.68}$$

c) Find the power of the test if the actual value of the population mean is 146 and 160, $n = 9$, and a 5% level of significance is used.

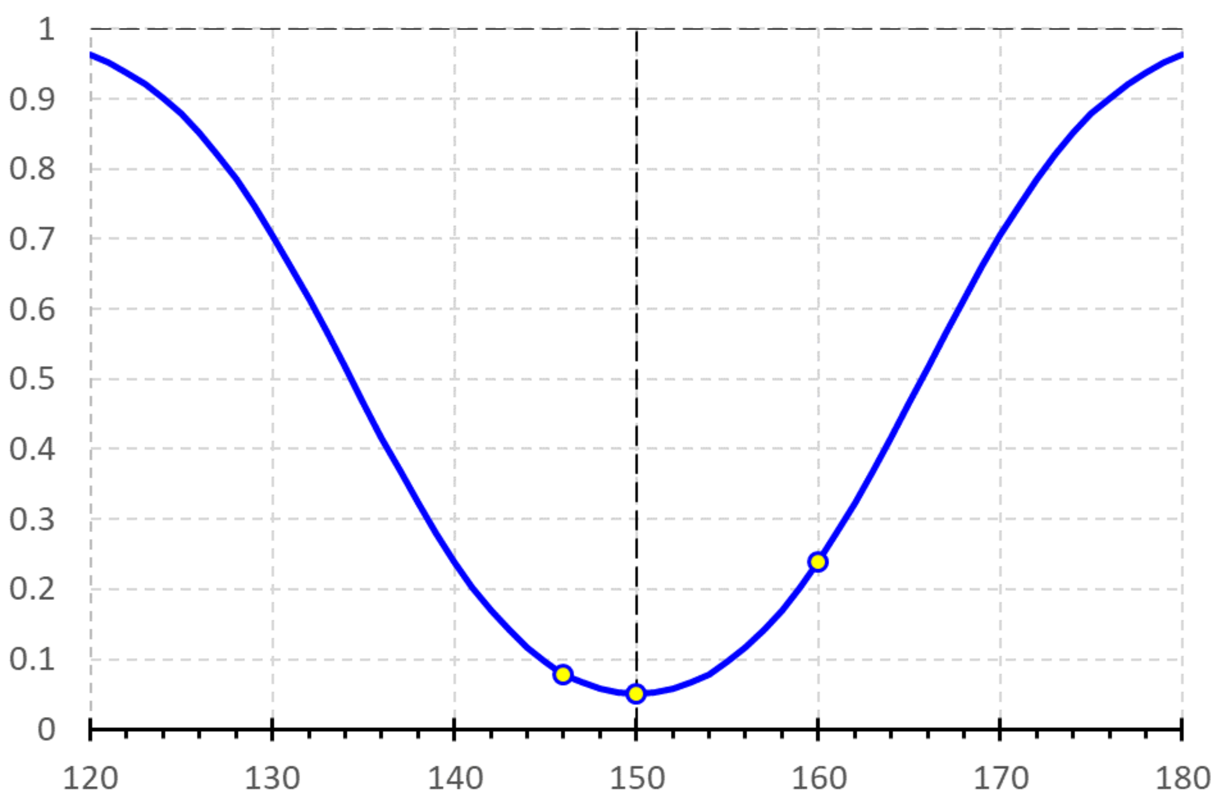
$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid \mu) = P(\overline{X} < 134.32 \mid \mu) + P(\overline{X} > 165.68 \mid \mu).$$

$$\text{Power}(\mu = 146) = P\left(Z < \frac{134.32 - 146}{24/\sqrt{9}}\right) + P\left(Z > \frac{165.68 - 146}{24/\sqrt{9}}\right)$$

$$= P(Z < -1.46) + P(Z > 2.46) = 0.0721 + 0.0069 = \mathbf{0.0790}.$$

$$\text{Power}(\mu = 160) = P\left(Z < \frac{134.32 - 160}{24/\sqrt{9}}\right) + P\left(Z > \frac{165.68 - 160}{24/\sqrt{9}}\right)$$

$$= P(Z < -3.21) + P(Z > 0.71) = 0.0007 + 0.2389 = \mathbf{0.2396}.$$



1 and **2**: