STAT 410 Fall 2020

Practice Problems 3

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1. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x, y) = C\,x^2 y^3, \quad 0 < x < 1, \quad 0 < y < \sqrt{x}, \quad \text{zero elsewhere.}$$

a) What must the value of C be so that $f_{X,Y}(x, y)$ is a valid joint p.d.f.?

b) Find $P(X + Y < 1)$.

c) Let $0 < a < 1$. Find $P(Y < aX)$.

d) Let $a > 1$. Find $P(Y < aX)$.

e) Let $0 < a < 1$. Find $P(XY < a)$.

f) Find $f_X(x)$.

g) Find $E(X)$.

h) Find $f_Y(y)$.

i) Find $E(Y)$.

j) Find $E(XY)$.

k) Find $\mathrm{Cov}(X, Y)$.

l) Are X and Y independent?

2. Let X and Y be two random variables with joint p.d.f.

$$f(x, y) = 64\,x \exp\{-4y\} = 64\,x e^{-4y}, \qquad 0 < x < y < \infty,$$

zero elsewhere.

a) Find $P(X^2 > Y)$.

b) Find the marginal p.d.f. $f_X(x)$ of X.

c) Find the marginal p.d.f. $f_Y(y)$ of Y.

d) Are X and Y independent? If not, find $\mathrm{Cov}(X, Y)$ and $\rho = \mathrm{Corr}(X, Y)$.

e) Let $a > 1$. Find $P(Y > aX)$.

f) Let $a > 0$. Find $P(X + Y < a)$.

3. Let X denote the number of times a photocopy machine will malfunction: 0, 1, or 2 times, on any given week. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x, y)$ is presented in the table below:

	x		
y	0	1	2
0	0.15	0.10	0.05
1	0.10	0.25	0.15
2	0	0.05	0.15

a) Find $P(Y > X)$.

b) Find $p_X(x)$, the marginal p.m.f. for the number of machine malfunctions.

c) Find $p_Y(y)$, the marginal p.m.f. for the number of times a technician is called.

d) Is the number of emergency calls independent of the number of machine malfunctions? If not, find $\text{Cov}(X, Y)$.

4. Suppose that the random variables X and Y have joint p.d.f. $f(x, y)$ given by

$$f(x, y) = C x^2 y, \quad 0 < x < y, \quad x + y < 2, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0 < x < y, \ x + y < 2 \}$.

b) What must the value of C be so that $f(x, y)$ is a valid joint p.d.f.?

c) Find $P(Y < 2X)$. d) Find $P(X + Y < 1)$.

e) Find the marginal probability density function for X.

f) Find the marginal probability density function for Y.

"Hint": Consider two cases: $0 < y < 1$ and $1 < y < 2$.

g) Find $E(X)$. h) Find $E(Y)$. i) Find $E(XY)$.

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y)=\begin{cases} x\,e^{-x(1+y)} & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal probability density function of X, $f_X(x)$.

b) Find the marginal probability density function of Y, $f_Y(y)$.

c) What is the probability that the lifetime of at least one component exceeds 1 year (when the manufacturer's warranty expires)?

6. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0<x<2, \quad 0<y<1, \qquad \text{zero otherwise.}$$

a) Find the probability $P(X>Y)$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

7. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{2}, \qquad x>0, \quad y>0, \quad 3x+y<3, \qquad \text{zero otherwise.}$$

a) Find the probability $P(X<Y)$.

b) Find the marginal probability density function of X, $f_X(x)$.

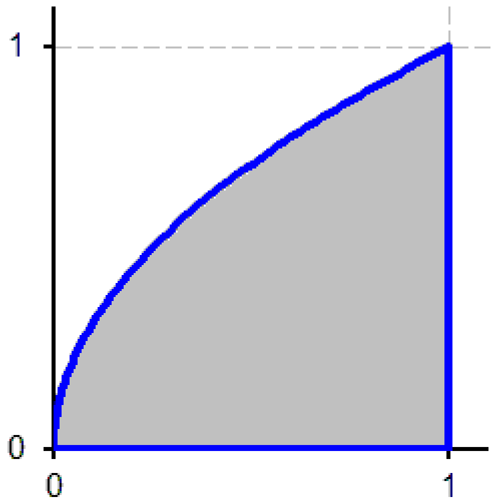
c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

1. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^2 y^3, \quad 0<x<1, \quad 0<y<\sqrt{x}, \qquad \text{zero elsewhere.}$$

a) What must the value of C be so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.?

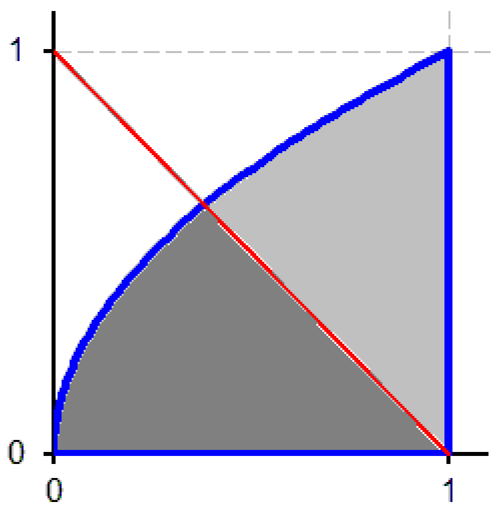


$$\int_0^1 \left(\int_0^{\sqrt{x}} C\,x^2 y^3\,dy \right) dx = \int_0^1 \frac{C}{4} x^4\,dx$$

$$= \frac{C}{20} = 1.$$

$$\Rightarrow \qquad C = \mathbf{20}.$$

b) Find $P(X+Y<1)$.



$y=\sqrt{x}$ and $y=1-x$

$x=y^2$ and $x=1-y$

$$\Rightarrow \qquad y = \frac{\sqrt{5}-1}{2}.$$

$$P(X+Y<1) = \int_0^{\frac{\sqrt{5}-1}{2}} \left(\int_{y^2}^{1-y} 20\,x^2 y^3\,dx \right) dy$$

$$= \int_0^{\frac{\sqrt{5}-1}{2}} \left(\frac{20}{3}(1-y)^3 y^3 - \frac{20}{3} y^9 \right) dy$$

$$= \int_0^{\frac{\sqrt{5}-1}{2}} \left(\frac{20}{3} y^3 - 20\,y^4 + 20\,y^5 - \frac{20}{3} y^6 - \frac{20}{3} y^9 \right) dy$$

$$= \left(\frac{5}{3} y^4 - 4y^5 + \frac{10}{3} y^6 - \frac{20}{21} y^7 - \frac{2}{3} y^{10} \right) \Bigg|_0^{\frac{\sqrt{5}-1}{2}} \approx 0.030022.$$

OR

$$y < \sqrt{x} \quad \text{and} \quad y = 1 - x \qquad \Rightarrow \qquad x = \left(\frac{\sqrt{5}-1}{2} \right)^2 = 1 - \frac{\sqrt{5}-1}{2} = \frac{3-\sqrt{5}}{2}.$$

$$\mathrm{P}(\mathrm{X} + \mathrm{Y} < 1) = 1 - \int_{\frac{3-\sqrt{5}}{2}}^{1} \left(\int_{1-x}^{\sqrt{x}} 20\, x^2\, y^3\, dy \right) dx$$

$$= 1 - \int_{\frac{3-\sqrt{5}}{2}}^{1} \left(5x^4 - 5x^2 (1-x)^4 \right) dx$$

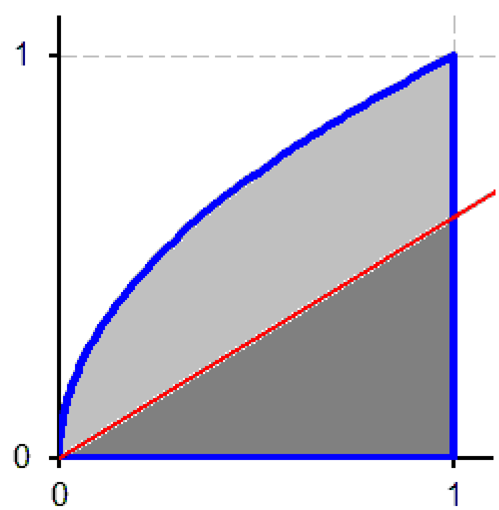
$$= 1 - \int_{\frac{3-\sqrt{5}}{2}}^{1} \left(-5x^2 + 20x^3 - 25x^4 + 20x^5 - 5x^6 \right) dy$$

$$= 1 - \left(-\frac{5}{3} x^3 + 5x^4 - 5x^5 + \frac{10}{3} x^6 - \frac{5}{7} x^7 \right) \Bigg|_{\frac{3-\sqrt{5}}{2}}^{1} \approx 0.030022.$$

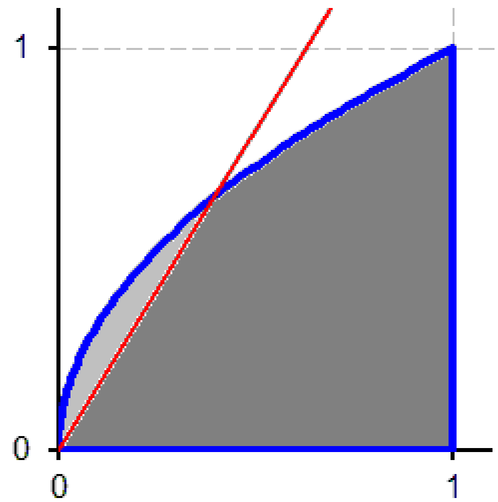
c) Let $0 < a < 1$. Find $\mathrm{P}(\mathrm{Y} < a\,\mathrm{X})$.

$$\mathrm{P}(\mathrm{Y} < a\,\mathrm{X}) = \int_0^1 \left(\int_0^{ax} 20\, x^2\, y^3\, dy \right) dx$$

$$= \int_0^1 5a^4 x^6\, dx = \frac{5}{7} a^4.$$



d) Let $a > 1$. Find $\mathrm{P}(\mathrm{Y} < a\,\mathrm{X})$.



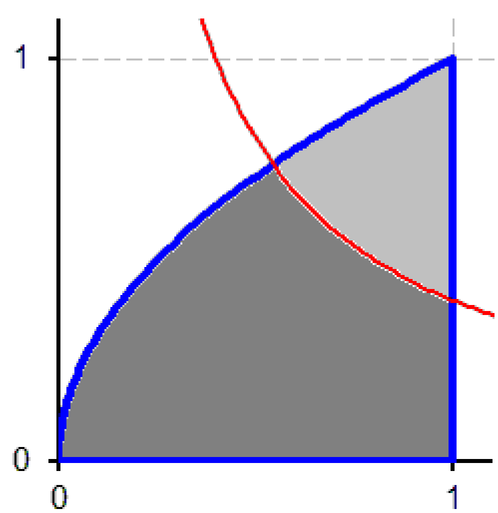
$y = \sqrt{x}$ and $y = a\,x$

$\Rightarrow \quad x = \dfrac{1}{a^2}, \qquad y = \dfrac{1}{a}.$

$$\mathrm{P}(\mathrm{Y} < a\,\mathrm{X}) = 1 - \int_0^{1/a} \left(\int_{y^2}^{y/a} 20\,x^2\,y^3\,dx \right) dy = 1 - \int_0^{1/a} \left(\frac{20\,y^6}{3\,a^3} - \frac{20}{3}\,y^9 \right) dy = 1 - \frac{2}{7\,a^{10}}.$$

$$\mathrm{P}(\mathrm{Y} < a\,\mathrm{X}) = 1 - \int_0^{1/a^2} \left(\int_{a\,x}^{\sqrt{x}} 20\,x^2\,y^3\,dy \right) dx = 1 - \int_0^{1/a^2} \left(5\,x^4 - 5\,a^4\,x^6 \right) dx = 1 - \frac{2}{7\,a^{10}}.$$

e) Let $0 < a < 1$. Find $\mathrm{P}(\mathrm{X\,Y} < a)$.



$y = \sqrt{x}$ and $y = \dfrac{a}{x}$

$\Rightarrow \quad x = a^{2/3}.$

$$\mathrm{P}(\mathrm{X\,Y} < a) = 1 - \int_{a^{2/3}}^{1} \left(\int_{a/x}^{\sqrt{x}} 20\,x^2\,y^3\,dy \right) dx = 1 - \int_{a^{2/3}}^{1} \left(5\,x^4 - 5\,\frac{a^4}{x^2} \right) dx$$

$$= 1 - \left(x^5 + 5\frac{a^4}{x} \right) \Bigg|_{a^{2/3}}^{1} = 6a^{10/3} - 5a^4.$$

f) Find $f_X(x)$.

$$f_X(x) = \int_0^{\sqrt{x}} 20x^2 y^3 \, dy = 5x^4, \qquad 0 < x < 1.$$

g) Find $E(X)$.

$$E(X) = \int_0^1 x \cdot 5x^4 \, dx = \mathbf{\frac{5}{6}}.$$

h) Find $f_Y(y)$.

$$f_Y(y) = \int_{y^2}^{1} 20x^2 y^3 \, dx = \frac{20}{3} \cdot \left(y^3 - y^9 \right), \qquad 0 < y < 1.$$

i) Find $E(Y)$.

$$E(Y) = \int_0^1 y \cdot \frac{20}{3}\left(y^3 - y^9 \right) dy = \int_0^1 \left(\frac{20}{3} y^4 - \frac{20}{3} y^{10} \right) dy = \frac{4}{3} - \frac{20}{33} = \mathbf{\frac{8}{11}}.$$

j) Find $E(XY)$.

$$E(XY) = \int_0^1 \left(\int_0^{\sqrt{x}} xy \cdot 20x^2 y^3 \, dy \right) dx = \int_0^1 4x^{11/2} \, dx = \mathbf{\frac{8}{13}}.$$

k) Find $\mathrm{Cov}(X, Y)$.

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{8}{13} - \frac{5}{6} \cdot \frac{8}{11} = \mathbf{\frac{8}{858}} \approx 0.009324.$$

l) Are X and Y independent?

$f(x,y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

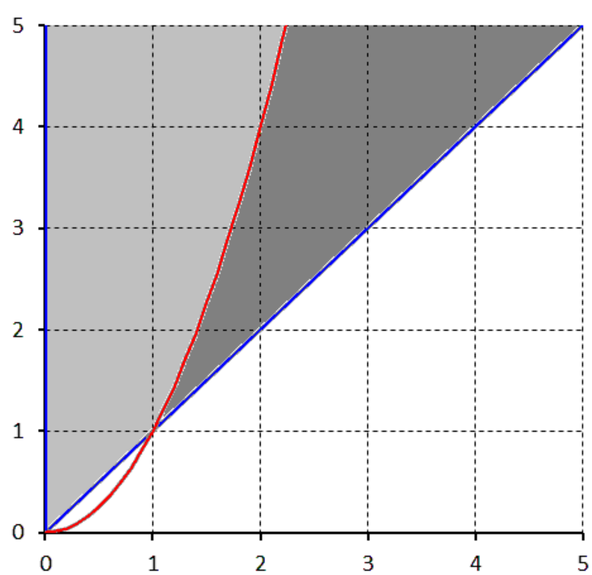
$\operatorname{Cov}(X, Y) \neq 0$. $\Rightarrow$ X and Y are **NOT independent**.

2. Let X and Y be two random variables with joint p.d.f.

$$f(x,y) = 64\,x \exp\{-4y\} = 64\,x e^{-4y}, \qquad 0 < x < y < \infty,$$

zero elsewhere.

a) Find $P(X^2 > Y)$.



$$P(X^2 > Y) = \int_1^{\infty} \int_x^{x^2} 64\,x e^{-4y}\,dy\,dx$$

$$= \int_1^{\infty} 16\,x e^{-4x}\,dx - \int_1^{\infty} 16\,x e^{-4x^2}\,dx$$

$$u = 4x^2 \qquad du = 8x\,dx$$

$$= \left(-4\,x e^{-4x} - e^{-4x}\right)\Big|_1^{\infty} - \int_4^{\infty} 2e^{-u}\,du$$

$$= 4e^{-4} + e^{-4} - 2e^{-4} = \mathbf{3\,e^{-4}} \approx 0.055.$$

b) Find the marginal p.d.f. $f_X(x)$ of X.

$$f_X(x) = \int_x^{\infty} 64\, x\, e^{-4y}\, dy = 16\, x\, e^{-4x}, \qquad 0 < x < \infty.$$

X has a Gamma distribution with $\alpha = 2,\ \lambda = 4$.

c) Find the marginal p.d.f. $f_Y(y)$ of Y.

$$f_Y(y) = \int_0^{y} 64\, x\, e^{-4y}\, dx = 32\, y^2\, e^{-4y}, \qquad 0 < y < \infty.$$

Y has a Gamma distribution with $\alpha = 3,\ \lambda = 4$.

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$ and $\rho = \operatorname{Corr}(X, Y)$.

$f(x, y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

X has a Gamma distribution with $\alpha = 2,\ \lambda = 4$. $\qquad E(X) = \frac{1}{2}, \quad \operatorname{Var}(X) = \frac{1}{8}$.

Y has a Gamma distribution with $\alpha = 3,\ \lambda = 4$. $\qquad E(Y) = \frac{3}{4}, \quad \operatorname{Var}(Y) = \frac{3}{16}$.

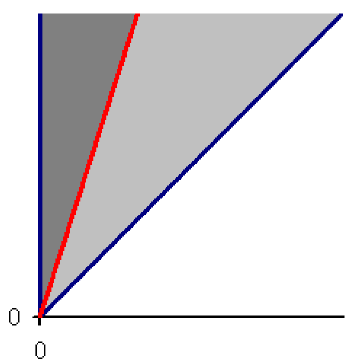
$$E(XY) = \int_0^{\infty}\int_0^{y} xy \cdot 64\, x\, e^{-4y}\, dx\, dy = \int_0^{\infty} \frac{64}{3}\, y^4\, e^{-4y}\, dy = \int_0^{\infty} \frac{4^3}{3}\, y^4\, e^{-4y}\, dy$$

$$= \frac{8}{4^2} \cdot \int_0^{\infty} \frac{4^5}{24}\, y^4\, e^{-4y}\, dy = \frac{1}{2} \cdot \int_0^{\infty} \frac{4^5}{\Gamma(5)}\, y^{5-1}\, e^{-4y}\, dy = \frac{1}{2}.$$

$$\operatorname{Cov}(X, Y) = \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{4} = \mathbf{\frac{1}{8}} = \mathbf{0.125}.$$

$$\rho = \operatorname{Corr}(X, Y) = \frac{1/8}{\sqrt{1/8} \cdot \sqrt{3/16}} = \frac{\sqrt{\mathbf{2}}}{\sqrt{\mathbf{3}}} \approx 0.8165.$$

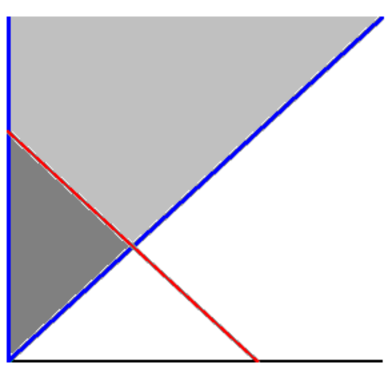
e) Let $a > 1$. Find $P(Y > aX)$.



$$P(Y > aX) = \int_0^{\infty} \int_{ax}^{\infty} 64\, x\, e^{-4y}\, dy\, dx$$

$$= \int_0^{\infty} 16\, x\, e^{-4ax}\, dx = \frac{1}{a^2}.$$

f) Let $a > 0$. Find $P(X + Y < a)$.



$$P(X + Y < a) = \int_0^{a/2} \int_x^{a-x} 64\, x\, e^{-4y}\, dy\, dx$$

$$= \int_1^{a/2} \left(16\, x\, e^{-4x} - 16\, x\, e^{-4a}\, e^{4x}\right) dx$$

$$= \left(-4\, x\, e^{-4x} - e^{-4x} - 4\, x\, e^{-4a}\, e^{4x} + e^{-4a}\, e^{4x}\right) \Big|_0^{a/2}$$

$$= 1 - e^{-4a} - 4\, a\, e^{-2a}.$$

3. Let X denote the number of times a photocopy machine will malfunction: 0, 1, or 2 times, on any given week. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x, y)$ is presented in the table below:

	x			
y	0	1	2	$p_Y(y)$
0	0.15	0.10	0.05	**0.30**
1	0.10	0.25	0.15	**0.50**
2	0	0.05	0.15	**0.20**
$p_X(x)$	**0.25**	**0.40**	**0.35**	1.00

a) Find $P(Y > X)$.

$P(Y > X) = p_{X,Y}(0, 1) + p_{X,Y}(0, 2) + p_{X,Y}(1, 2) = 0.10 + 0 + 0.05 = \mathbf{0.15}$.

b) Find $p_X(x)$, the marginal p.m.f. for the number of machine malfunctions.

c) Find $p_Y(y)$, the marginal p.m.f. for the number of times a technician is called.

d) Is the number of emergency calls independent of the number of machine malfunctions? If not, find $\text{Cov}(X, Y)$.

$p_{X,Y}(0, 0) = 0.15 \neq 0.075 = 0.25 \times 0.30 = p_X(0) \times p_Y(0)$.

X and Y are **NOT independent**.

$E(X) = 0 \times 0.25 + 1 \times 0.40 + 2 \times 0.35 = 1.10$.

$E(Y) = 0 \times 0.30 + 1 \times 0.50 + 2 \times 0.20 = 0.90$.

$E(XY) = 0.25 + 0.30 + 0.10 + 0.60 = 1.25$.

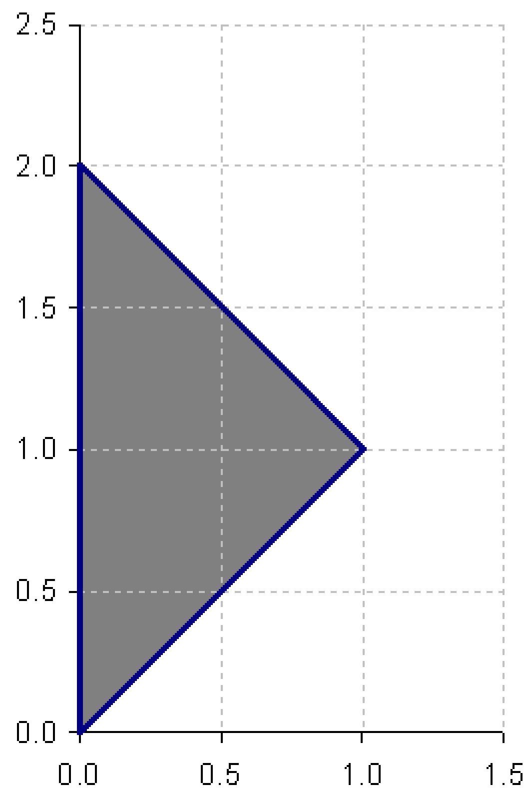
$\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = 1.25 - 1.10 \times 0.90 = \mathbf{0.26}$.

4. Suppose that the random variables X and Y have joint p.d.f. $f(x,y)$ given by

$$f(x,y) = C x^2 y, \quad 0 < x < y, \quad x + y < 2, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch

$$\{\, 0 < x < y, \quad x + y < 2 \,\}.$$



b) What must the value of C be so that $f(x,y)$ is a valid joint p.d.f.?

Must have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

$$\int_0^1 \left(\int_x^{2-x} C x^2 y \, dy \right) dx$$

$$= \int_0^1 \left(\frac{C}{2} x^2 y^2 \right) \Bigg|_{y = x}^{y = 2 - x} dx$$

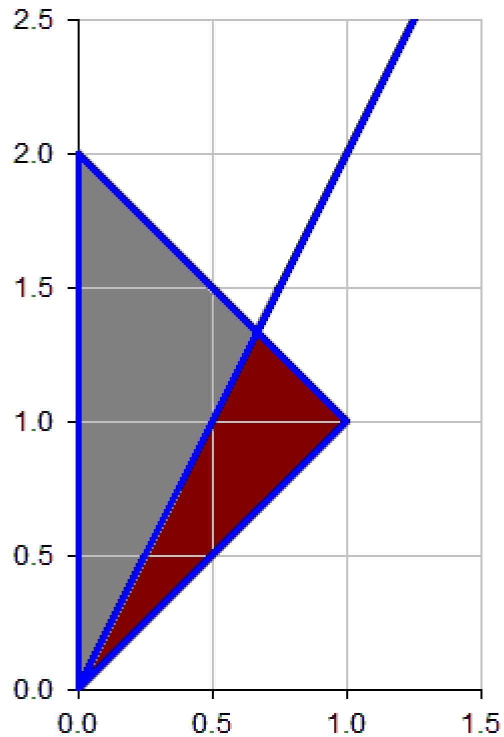
$$= \int_0^1 \left(\frac{C}{2} x^2 \left[(2 - x)^2 - x^2 \right] \right) dx$$

$$= \int_0^1 \left(2 C x^2 - 2 C x^3 \right) dx$$

$$= \left(\frac{2C}{3} x^3 - \frac{C}{2} x^4 \right) \Bigg|_0^1 = \frac{C}{6} = 1.$$

$\Rightarrow \quad C = \mathbf{6}.$

c) Find $P(Y < 2X)$.



$x + y = 2 \quad \& \quad y = 2x$

$\Rightarrow \quad x = \frac{2}{3}, \; y = \frac{4}{3}.$

$$1 - \int_0^{2/3} \left(\int_{2x}^{2-x} 6x^2 y \, dy \right) dx$$

$$= 1 - \int_0^{2/3} \left(3x^2 y^2 \right) \Big|_{y=2x}^{y=2-x} dx$$

$$= 1 - \int_0^{2/3} \left(3x^2 \left[(2-x)^2 - 4x^2 \right] \right) dx$$

$$= 1 - \int_0^{2/3} \left(12x^2 - 12x^3 - 9x^4 \right) dx$$

$$= 1 - \left(4x^3 - 3x^4 - \frac{9}{5} x^5 \right) \Big|_0^{2/3} = 1 - 4\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^4 + \frac{9}{5}\left(\frac{2}{3}\right)^5 = \mathbf{\frac{87}{135}}.$$

OR

$$\int_0^{2/3} \left(\int_x^{2x} 6x^2 y \, dy \right) dx + \int_{2/3}^{1} \left(\int_x^{2-x} 6x^2 y \, dy \right) dx = \ldots$$

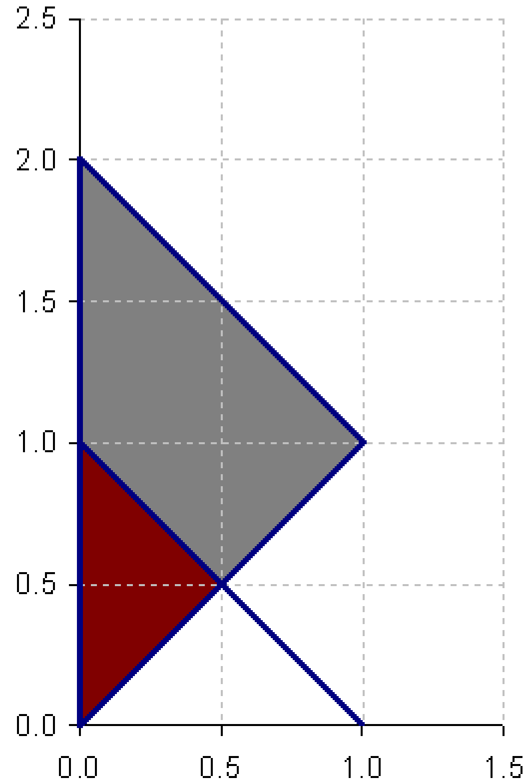
OR

$$\int_0^{1} \left(\int_{y/2}^{y} 6x^2 y \, dx \right) dy + \int_1^{4/3} \left(\int_{y/2}^{2-y} 6x^2 y \, dx \right) dy = \ldots$$

OR

$$1 - \int_0^{4/3} \left(\int_0^{y/2} 6x^2 y \, dx \right) dy - \int_{4/3}^{2} \left(\int_0^{2-y} 6x^2 y \, dx \right) dy = \ldots$$

d) Find $P(X+Y<1)$.



$$\int_0^{0.5}\left(\int_x^{1-x} 6x^2 y\,dy\right)dx$$

$$= \int_0^{0.5} \left(3x^2 y^2\right)\Big|_{y=x}^{y=1-x} dx$$

$$= \int_0^{0.5} \left(3x^2\left[(1-x)^2 - x^2\right]\right)dx$$

$$= \int_0^{0.5} \left(3x^2 - 6x^3\right)dx$$

$$= \left(x^3 - \frac{3}{2}x^4\right)\Big|_0^{0.5}$$

$$= \left(\frac{1}{2}\right)^3 - \frac{3}{2}\left(\frac{1}{2}\right)^4$$

$$= \frac{1}{8} - \frac{3}{32} = \mathbf{\frac{1}{32}} = \mathbf{0.03125}.$$

e) Find the marginal probability density function for X.

First, X can only take values in $(0,1)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{2-x} 6x^2 y\,dy = \left(3x^2 y^2\right)\Big|_{y=x}^{y=2-x}$$

$$= 3x^2\left\{(2-x)^2 - x^2\right\} = 12x^2 - 12x^3 = 12x^2(1-x), \quad 0<x<1.$$

f) Find the marginal probability density function for Y.

"Hint": Consider two cases: $0 < y < 1$ and $1 < y < 2$.

First, Y can only take values in $(0, 2)$.

$$f_{\mathrm{Y}}(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} \int_0^y 6x^2 y\,dx & 0 < y < 1 \\ \int_0^{2-y} 6x^2 y\,dx & 1 < y < 2 \end{cases}$$

$$= \begin{cases} \left(2x^3 y\right)\Big|_{x=0}^{x=y} & 0 < y < 1 \\ \left(2x^3 y\right)\Big|_{x=0}^{x=2-y} & 1 < y < 2 \end{cases}$$

$$= \begin{cases} 2y^4 & 0 < y < 1 \\ 2y(2-y)^3 & 1 < y < 2 \end{cases}$$

g) Find $\mathrm{E}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f_{\mathrm{X}}(x)\,dx = \int_0^1 x \cdot 12x^2(1-x)\,dx = \mathbf{0.60}.$$

h) Find $\mathrm{E}(\mathrm{Y})$.

$$\mathrm{E}(\mathrm{Y}) = \int_{-\infty}^{\infty} y \cdot f_{\mathrm{Y}}(y)\,dy = \int_0^1 y \cdot 2y^4\,dy + \int_1^2 y \cdot 2y(2-y)^3\,dx = \frac{1}{3} + \frac{11}{15} = \mathbf{\frac{16}{15}}.$$

i) Find $\mathrm{E}(\mathrm{XY})$.

$$\mathrm{E}(\mathrm{XY}) = \int_0^1 \left(\int_x^{2-x} xy \cdot 6x^2 y\,dy \right) dx = \ldots = \mathbf{\frac{22}{35}}.$$

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y)=\begin{cases} x\,e^{-x(1+y)} & x\ge 0,\ y\ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^\infty x\,e^{-x(1+y)}\,dy = x\,e^{-x}\int_0^\infty e^{-xy}\,dy = e^{-x}, \qquad x\ge 0.$$

b) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^\infty x\,e^{-x(1+y)}\,dx = \frac{1}{(1+y)^2}, \qquad y\ge 0.$$

c) What is the probability that the lifetime of at least one component exceeds 1 year (when the manufacturer's warranty expires)?

$$\mathrm{P}(\mathrm{X}>1\cup \mathrm{Y}>1) = 1-\mathrm{P}(\mathrm{X}\le 1\cap \mathrm{Y}\le 1) = 1-\int_0^1\left(\int_0^1 x\,e^{-x(1+y)}\,dy\right)dx$$

$$= 1-\int_0^1 x\,e^{-x}\left(\int_0^1 e^{-xy}\,dy\right)dx = 1-\int_0^1 x\,e^{-x}\left(\frac{1}{x}-\frac{1}{x}e^{-x}\right)dx$$

$$= 1-\int_0^1\left(e^{-x}-e^{-2x}\right)dx = 1-\left(-e^{-x}+\frac{1}{2}e^{-2x}\right)\Big|_0^1$$

$$= 1-\left(-e^{-1}+\frac{1}{2}e^{-2}\right)+\left(-1+\frac{1}{2}\right) = \frac{1}{2}+e^{-1}-\frac{1}{2}e^{-2} \approx 0.800212.$$

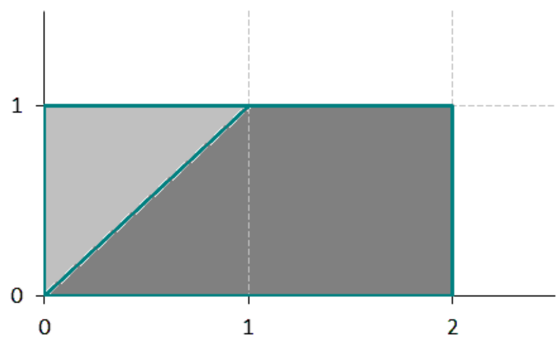
OR

$$\mathrm{P}(\mathrm{X}>1\cup \mathrm{Y}>1) = \mathrm{P}(\mathrm{X}>1) + \mathrm{P}(\mathrm{Y}>1) - \mathrm{P}(\mathrm{X}>1\cap \mathrm{Y}>1) = \ldots$$

6. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{3}, \qquad 0<x<2, \quad 0<y<1, \qquad \text{zero otherwise.}$$

a) Find the probability $P(X>Y)$.



$$P(X>Y) = 1-\int_0^1\left(\int_0^y \frac{x+y}{3}\,dx\right)dy$$

$$= 1-\int_0^1\left(\frac{y^2}{6}+\frac{y^2}{3}\right)dy$$

$$= 1-\int_0^1 \frac{y^2}{2}\,dy = 1-\frac{1}{6} = \mathbf{\frac{5}{6}}.$$

OR $$P(X>Y) = \int_0^1\left(\int_y^2 \frac{x+y}{3}\,dx\right)dy = \ldots$$

OR $$P(X>Y) = \int_0^1\left(\int_0^x \frac{x+y}{3}\,dy\right)dx+\int_1^2\left(\int_0^1 \frac{x+y}{3}\,dy\right)dx = \ldots$$

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^1 \frac{x+y}{3}\,dy = \left(\frac{xy}{3}+\frac{y^2}{6}\right)\Bigg|_0^1 = \frac{2x+1}{6}, \qquad 0<x<2.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^2 \frac{x+y}{3}\,dx = \left(\frac{x^2}{6}+\frac{xy}{3}\right)\Bigg|_0^2 = \frac{2+2y}{3}, \qquad 0<y<1.$$

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are **NOT independent**.

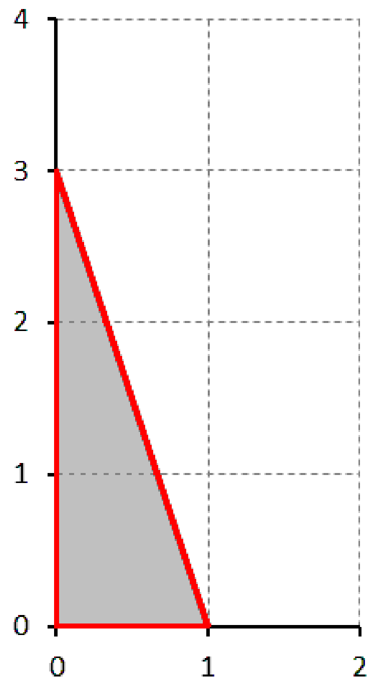
$$\text{E}(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 x \cdot \frac{2x+1}{6}\,dx = \left(\frac{x^3}{9} + \frac{x^2}{12} \right) \Bigg|_0^2 = \frac{11}{9}.$$

$$\text{E}(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot \frac{2+2y}{3}\,dy = \left(\frac{y^2}{3} + \frac{y^3}{9} \right) \Bigg|_0^1 = \frac{5}{9}.$$

$$\text{E}(XY) = \int_0^2 \left(\int_0^1 x\,y \cdot \frac{x+y}{3}\,dy \right) dx = \int_0^2 \left(\frac{x^2}{6} + \frac{x}{9} \right) dx = \left(\frac{x^3}{18} + \frac{x^2}{18} \right) \Bigg|_0^2 = \frac{2}{3}.$$

$$\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X) \times \text{E}(Y) = \frac{2}{3} - \frac{11}{9} \cdot \frac{5}{9} = -\mathbf{\frac{1}{81}} \approx -0.012345679.$$

7. Let the joint probability density function for (X, Y) be

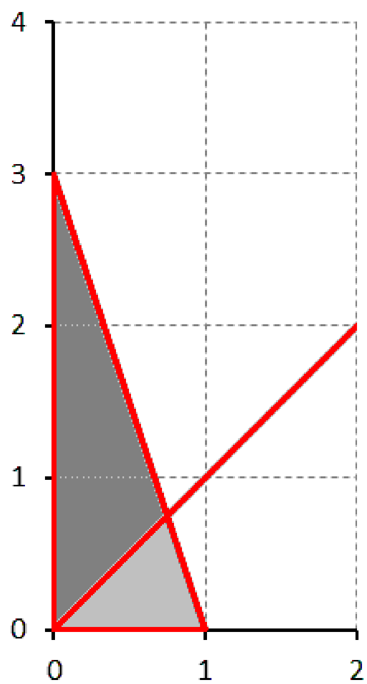


$$f(x,y) = \frac{x+y}{2}, \qquad x > 0, \quad y > 0,$$

$$3x + y < 3,$$

zero otherwise.

a) Find the probability $P(X < Y)$.



intersection point:

$y = x$ and $x + 3y = 3$

$x = \frac{3}{4}$ and $y = \frac{3}{4}$

$$P(X < Y) = \int_0^{3/4} \left(\int_x^{3-3x} \frac{x+y}{2}\, dy \right) dx$$

$$= \int_0^{3/4} \left(\frac{9}{4} - 3x \right) dx = \mathbf{\frac{27}{32}}.$$

OR

$$P(X < Y) = 1 - \int_0^{3/4} \left(\int_y^{1-(y/3)} \frac{x+y}{2}\, dx \right) dy = 1 - \int_0^{3/4} \left(\frac{1}{4} + \frac{1}{3} y - \frac{8}{9} y^2 \right) dy = \mathbf{\frac{27}{32}}.$$

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{3-3x} \frac{x+y}{2}\, dy = \frac{9}{4} - 3x + \frac{3}{4} x^2, \qquad 0 < x < 1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^{1-(y/3)} \frac{x+y}{2}\,dx = \frac{1}{4} + \frac{1}{3}y - \frac{5}{36}y^2, \qquad 0 < y < 3.$$

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{X,Y}(x,y) \neq f_X(x) \times f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot \left(\frac{9}{4} - 3x + \frac{3}{4}x^2\right) dx$$

$$= \int_0^1 \left(\frac{9}{4}x - 3x^2 + \frac{3}{4}x^3\right) dx = \frac{9}{8} - 1 + \frac{3}{16} = \frac{5}{16}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^3 y \cdot \left(\frac{1}{4} + \frac{1}{3}y - \frac{5}{36}y^2\right) dy$$

$$= \int_0^3 \left(\frac{1}{4}y + \frac{1}{3}y^2 - \frac{5}{36}y^3\right) dy = \frac{9}{8} + 3 - \frac{405}{144} = \frac{21}{16}.$$

$$E(XY) = \int_0^1 \left(\int_0^{3-3x} xy \cdot \frac{x+y}{2}\,dy \right) dx = \int_0^1 \left(\frac{x^2}{4}(3-3x)^2 + \frac{x}{6}(3-3x)^3 \right) dx$$

$$= = \int_0^1 \left(\frac{9}{2}x - \frac{45}{4}x^2 + 9x^3 - \frac{9}{4}x^4\right) dx = \frac{9}{4} - \frac{15}{4} + \frac{9}{4} - \frac{9}{20} = \frac{6}{20} = \frac{3}{10}.$$

$$\operatorname{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \frac{3}{10} - \frac{5}{16} \times \frac{21}{16} = -\mathbf{\frac{141}{1280}} \approx -0.11016.$$