STAT 410 **Examples for 10/21/2020 (1)** Fall 2020
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1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a uniform distribution on the interval $(0, \theta)$.

$$f(x;\theta) = \begin{cases} \dfrac{1}{\theta} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases} \qquad F(x;\theta) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{\theta} & 0 < x < \theta \\ 1 & x > \theta \end{cases}$$

$$E(X) = \frac{\theta}{2} \qquad \operatorname{Var}(X) = \frac{\theta^2}{12}$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$E(X) = \frac{\theta}{2}. \qquad \Rightarrow \qquad \overline{X} = \frac{\tilde{\theta}}{2}. \qquad \Rightarrow \qquad \tilde{\theta} = 2\overline{X}.$$

b) Is $\tilde{\theta}$ unbiased for θ? That is, does $E(\tilde{\theta})$ equal θ?

$$E(\overline{X}) = E(X) = \frac{\theta}{2}. \qquad \Rightarrow \qquad E(\tilde{\theta}) = E(2\overline{X}) = 2E(\overline{X}) = \theta. \qquad \checkmark$$

$\tilde{\theta}$ is unbiased for θ.

c) Compute $\operatorname{Var}(\tilde{\theta})$.

$$\tilde{\theta} = 2\overline{X}. \qquad \operatorname{Var}(\tilde{\theta}) = \operatorname{Var}(2\overline{X}) = 4\operatorname{Var}(\overline{X}) = 4 \cdot \frac{\sigma^2}{n}.$$

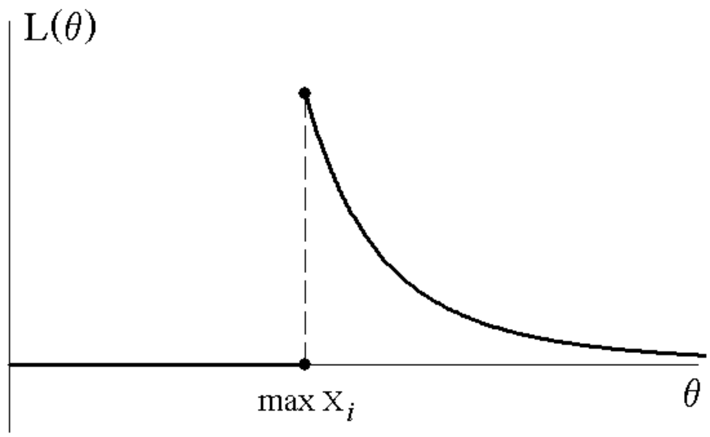
For Uniform $(0, \theta)$, $$\sigma^2 = \frac{\theta^2}{12}. \qquad \Rightarrow \qquad \operatorname{Var}(\tilde{\theta}) = \frac{\theta^2}{3 \cdot n}.$$

d) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

Likelihood function:

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) = \frac{1}{\theta^n}, \qquad \theta > \max X_i,$$

$$L(\theta) = 0, \qquad \theta < \max X_i.$$



Therefore, $\hat{\theta} = \max X_i$.

e) Is $\hat{\theta}$ unbiased for θ? That is, does $E(\hat{\theta})$ equal θ?

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \ldots \cdot P(X_n \le x) = \left(\frac{x}{\theta}\right)^n, \quad 0 < x < \theta.$$

$$f_{\max X_i}(x) = F'_{\max X_i}(x) = \frac{n \cdot x^{n-1}}{\theta^n}, \quad 0 < x < \theta.$$

$$E(\hat{\theta}) = \int_0^{\theta} x \cdot \frac{n \cdot x^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n} \cdot \int_0^{\theta} x^n \, dx = \frac{n}{\theta^n} \cdot \left(\frac{x^{n+1}}{n+1}\right)\Bigg|_0^{\theta} = \frac{n \cdot \theta}{n+1} \ne \theta.$$

$\hat{\theta}$ is NOT unbiased for θ.

f) What must c equal if $c\,\hat{\theta}$ is to be an unbiased estimator for θ?

$$\mathrm{E}\left(\frac{n+1}{n}\cdot\hat{\theta}\right)=\frac{n+1}{n}\cdot\mathrm{E}\left(\hat{\theta}\right)=\frac{n+1}{n}\cdot\frac{n\cdot\theta}{n+1}=\theta. \qquad c=\frac{n+1}{n}.$$

g) Compute $\mathrm{Var}(\hat{\theta})$ and $\mathrm{Var}\left(\frac{n+1}{n}\,\hat{\theta}\right)$.

$$\mathrm{E}\left(\hat{\theta}^{\,2}\right)=\int_0^{\theta} x^2\cdot\frac{n\cdot x^{\,n-1}}{\theta^{\,n}}\,dx=\frac{n}{\theta^{\,n}}\cdot\int_0^{\theta} x^{\,n+1}\,dx=\frac{n}{\theta^{\,n}}\cdot\left(\frac{x^{\,n+2}}{n+2}\right)\Bigg|_0^{\theta}=\frac{n\cdot\theta^2}{n+2}.$$

$$\mathrm{Var}\left(\hat{\theta}\right)=\mathrm{E}\left(\hat{\theta}^{\,2}\right)-\left[\mathrm{E}\left(\hat{\theta}\right)\right]^2=\frac{n\cdot\theta^2}{n+2}-\left(\frac{n\cdot\theta}{n+1}\right)^2=\frac{n\cdot\theta^2}{(n+2)\cdot(n+1)^2}.$$

$$\mathrm{Var}\left(\frac{n+1}{n}\hat{\theta}\right)=\left(\frac{n+1}{n}\right)^2\cdot\mathrm{Var}\left(\hat{\theta}\right)=\frac{\theta^2}{(n+2)\cdot n}.$$

Def Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators for θ. $\hat{\theta}_1$ is said to be **more efficient** than $\hat{\theta}_2$ if $\mathrm{Var}(\hat{\theta}_1)<\mathrm{Var}(\hat{\theta}_2)$.

The **relative efficiency** of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is $\mathrm{Var}(\hat{\theta}_2)/\mathrm{Var}(\hat{\theta}_1)$.

h) Which estimator for θ is more efficient, $\tilde{\theta}$ or $\frac{n+1}{n}\,\hat{\theta}$? What is the relative efficiency of $\frac{n+1}{n}\,\hat{\theta}$ with respect to $\tilde{\theta}$?

Since $\frac{\theta^2}{(n+2)\cdot n}<\frac{\theta^2}{3\cdot n}$ for $n>1$, $\frac{n+1}{n}\hat{\theta}$ is more efficient than $\tilde{\theta}$.

Relative efficiency of $\frac{n+1}{n}\hat{\theta}$ with respect to $\tilde{\theta}$ $=\frac{n+2}{3}$.

For an estimator $\hat{\theta}$ of θ, define the **Mean Squared Error** of $\hat{\theta}$ by

$$\text{MSE}(\hat{\theta}) = \text{E}\left[(\hat{\theta} - \theta)^2\right].$$

$$\text{E}\left[(\hat{\theta} - \theta)^2\right] = \left(\text{E}(\hat{\theta}) - \theta\right)^2 + \text{Var}(\hat{\theta}) = \left(\text{bias}(\hat{\theta})\right)^2 + \text{Var}(\hat{\theta}).$$

i) Find $\text{MSE}(\tilde{\theta})$.

$\text{bias}(\tilde{\theta}) = 0$ and $\text{Var}\left(\tilde{\theta}\right) = \dfrac{\theta^2}{3n}$.

$$\Rightarrow \quad \text{MSE}(\tilde{\theta}) = \text{E}\left[(\tilde{\theta} - \theta)^2\right] = 0 + \frac{\theta^2}{3n} = \frac{\theta^2}{3n} \to 0 \quad \text{as } n \to \infty.$$

j) Find $\text{MSE}(\hat{\theta})$.

$\text{bias}(\hat{\theta}) = \dfrac{n \cdot \theta}{n+1} - \theta = -\dfrac{\theta}{n+1}$ and $\text{Var}(\hat{\theta}) = \dfrac{n\theta^2}{(n+1)^2(n+2)}$.

$$\Rightarrow \quad \text{MSE}(\hat{\theta}) = \text{E}\left[(\hat{\theta} - \theta)^2\right] = \left(-\frac{\theta}{n+1}\right)^2 + \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$= \frac{2\theta^2}{(n+1)(n+2)} \to 0 \quad \text{as } n \to \infty.$$

k) Which estimator is “better”, $\tilde{\theta}$ or $\hat{\theta}$?

$\text{MSE}(\tilde{\theta}) = \dfrac{\theta^2}{3n}$. $\qquad\qquad$ $\text{MSE}(\hat{\theta}) = \dfrac{2\theta^2}{(n+1)(n+2)}$.

Note that even though $\tilde{\theta} = 2\overline{\text{X}}$ is unbiased for θ

and $\hat{\theta} = \max \text{X}_i$ is not unbiased for θ,

$\text{MSE}(\hat{\theta}) \ll \text{MSE}(\tilde{\theta})$ for large n. $\qquad\qquad$ $\hat{\theta}$ is “better”.

$$\mathrm{MSE}(c\,\hat{\theta}) = \mathrm{E}\left[(c\,\hat{\theta}-\theta)^2\right] = c^2\,\mathrm{E}(\hat{\theta}^2) - 2\,c\,\mathrm{E}(\hat{\theta})\,\theta + \theta^2.$$

$$c_{\min} = \frac{\mathrm{E}(\hat{\theta})\cdot\theta}{\mathrm{E}(\hat{\theta}^2)}.$$

$\Rightarrow$ An estimator could be improved by multiplying it by a constant if $c_{\min}$ does NOT depend on θ.

For $\tilde{\theta} = 2\,\overline{\mathrm{X}}$,

$$c_{\min} = \frac{\mathrm{E}(2\,\overline{\mathrm{X}})\cdot\theta}{\mathrm{Var}(2\,\overline{\mathrm{X}})+\left[\mathrm{E}(2\,\overline{\mathrm{X}})\right]^2} = \frac{\theta^2}{\frac{\theta^2}{3\,n}+\theta^2} = \frac{3\,n}{3\,n+1}.$$

$$\tilde{\tilde{\theta}} = \frac{6\,n}{3\,n+1}\,\overline{\mathrm{X}}.$$

$$\mathrm{bias}(\tilde{\tilde{\theta}}) = \frac{3\,n\,\theta}{3\,n+1} - \theta = -\frac{\theta}{3\,n+1} \quad \text{and} \quad \mathrm{Var}\left(\tilde{\tilde{\theta}}\right) = \frac{3\,n\,\theta^2}{(3\,n+1)^2}.$$

$$\Rightarrow \quad \mathrm{MSE}(\tilde{\tilde{\theta}}) = \frac{\theta^2}{(3\,n+1)^2} + \frac{3\,n\,\theta^2}{(3\,n+1)^2} = \frac{\theta^2}{3\,n+1}.$$

For $\hat{\theta} = \max \mathrm{X}_i$,

$$c_{\min} = \frac{\mathrm{E}(\hat{\theta})\cdot\theta}{\mathrm{E}(\hat{\theta}^2)} = \frac{\frac{n\cdot\theta^2}{n+1}}{\frac{n\cdot\theta^2}{n+2}} = \frac{n+2}{n+1}.$$

$$\hat{\hat{\hat{\theta}}} = \frac{n+2}{n+1}\max \mathrm{X}_i.$$

$$\mathrm{bias}(\hat{\hat{\hat{\theta}}}) = \frac{(n+2)\,n\,\theta}{(n+1)^2} - \theta = -\frac{\theta}{(n+1)^2} \quad \text{and} \quad \mathrm{Var}(\hat{\hat{\hat{\theta}}}) = \frac{(n+2)\,n\,\theta^2}{(n+1)^4}.$$

$$\Rightarrow \quad \mathrm{MSE}(\hat{\hat{\hat{\theta}}}) = \frac{\theta^2}{(n+1)^4} + \frac{(n+2)\,n\,\theta^2}{(n+1)^4} = \frac{\theta^2}{(n+1)^2}.$$

Indeed,

$$\mathrm{MSE}(\tilde{\tilde{\theta}}) = \frac{\theta^2}{3n+1} < \frac{\theta^2}{3n} = \mathrm{MSE}(\tilde{\theta}).$$

$$\mathrm{MSE}(\hat{\hat{\hat{\theta}}}) = \frac{\theta^2}{(n+1)^2} = \frac{\theta^2}{n^2+2n+1} < \frac{\theta^2}{n^2+2n} = \mathrm{MSE}(\hat{\hat{\theta}}),$$

$$\text{where } \hat{\hat{\theta}} = \frac{n+1}{n}\hat{\theta} = \frac{n+1}{n}\max \mathrm{X}_i .$$

$$\mathrm{MSE}(\hat{\hat{\hat{\theta}}}) = \frac{\theta^2}{(n+1)^2} < \frac{2\theta^2}{(n+1)(n+2)} = \mathrm{MSE}(\hat{\theta}).$$

More on the Method of Moments:

For $\mathrm{U}(0,\theta)$, $\quad \mathrm{E}(\mathrm{X}^k) = \dfrac{\theta^k}{k+1}, \quad k > -1.$

$$\Rightarrow \qquad \tilde{\theta}_k = \left((k+1)\overline{\mathrm{X}^k}\right)^{1/k}, \quad \text{where } \overline{\mathrm{X}^k} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{X}_i^k .$$

For example, $\qquad \mathrm{E}(\mathrm{X}^2) = \dfrac{\theta^2}{3}, \qquad \tilde{\theta}_2 = \sqrt{3\overline{\mathrm{X}^2}}\,.$