Examples for 09/16/2020 (1)

0. **2.1.6** (7th and 6th edition)

Let $f(x,y) = e^{-x-y}$, $0 < x < \infty$, $0 < y < \infty$, zero elsewhere, be the pdf of X and Y. Then if $Z = X + Y$, compute $P(Z \le 0)$, $P(Z \le 6)$, and, more generally, $P(Z \le z)$, for $0 < z < \infty$. What is the pdf of Z?

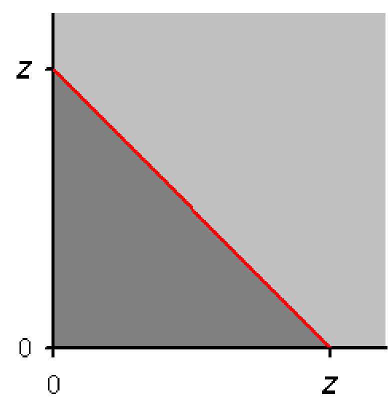
$$F_Z(z) = P(Z \le z) = P(Y \le z - X)$$

$$= \int_0^z \left(\int_0^{z-x} e^{-x-y}\, dy \right) dx$$

$$= \int_0^z e^{-x} \left(\int_0^{z-x} e^{-y}\, dy \right) dx$$

$$= \int_0^z e^{-x} \left(1 - e^{-z+x} \right) dx$$

$$= \int_0^z e^{-x}\, dx - \int_0^z e^{-z}\, dx = 1 - e^{-z} - z e^{-z}, \qquad z > 0.$$



$$f_Z(z) = F_Z'(z) = e^{-z} - e^{-z} + z e^{-z} = z e^{-z}, \qquad z > 0.$$

Another approach:

X and Y are two independent Exponential random variables with mean 1.

$$M_X(t) = \frac{1}{1-t}, \quad t < 1. \qquad\qquad M_Y(t) = \frac{1}{1-t}, \quad t < 1.$$

$$\Rightarrow \quad M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \left(\frac{1}{1-t} \right)^2, \quad t < 1.$$

$\Rightarrow$ Z = X + Y has a Gamma distribution with $\alpha = 2, \theta = 1$.

$$\Rightarrow \quad f_Z(z) = \frac{1}{\Gamma(2) \cdot 1^2} \cdot z^{2-1} \cdot e^{-z/1} = z e^{-z}, \qquad z > 0.$$

b) $W = 2X + Y;$

$$\mathrm{M}_{2\mathrm{X}+\mathrm{Y}}(t) = \mathrm{E}(\mathrm{e}^{2\mathrm{X}t+\mathrm{Y}t}) = \mathrm{M}_{\mathrm{X}}(2t)\cdot\mathrm{M}_{\mathrm{Y}}(t) = \left(\frac{1}{1-2t}\right)\cdot\left(\frac{1}{1-t}\right), \qquad t < {}^{1}\!/_{2}.$$

$$f_{2\mathrm{X}+\mathrm{Y}}(w) = ???$$

$$\mathrm{F}_{\mathrm{W}}(w) = \mathrm{P}(\mathrm{W} \le w) = \mathrm{P}(\mathrm{Y} \le w - 2\mathrm{X}) = \int_0^{w/2}\left(\int_0^{w-2x} e^{-x-y}\,dy\right)dx$$

$$= \int_0^{w/2} e^{-x}\left(-e^{-y}\right)\Big|_0^{w-2x} dx = \int_0^{w/2}\left(e^{-x} - e^{-w+x}\right)dx$$

$$= \left(-e^{-x} - e^{-w}e^{x}\right)\Big|_0^{w/2} = 1 + e^{-w} - 2e^{-w/2}, \qquad w > 0.$$

$$f_{\mathrm{W}}(w) = \mathrm{F}'_{\mathrm{W}}(w) = e^{-w/2} - e^{-w}, \qquad w > 0.$$

c) $V = \dfrac{Y}{X};$

$$\mathrm{F}_{\mathrm{V}}(v) = \mathrm{P}(\mathrm{V} \le v) = \mathrm{P}(\mathrm{Y} \le v\mathrm{X}) = \int_0^{\infty}\left(\int_{y/v}^{\infty}\left(e^{-x-y}\right)dx\right)dy$$

$$= \int_0^{\infty} e^{-y}\left(\int_{y/v}^{\infty}\left(e^{-x}\right)dx\right)dy = \int_0^{\infty} e^{-y}\left(e^{-y/v}\right)dy = \int_0^{\infty} e^{-(1+1/v)y}\,dy$$

$$= \frac{v}{1+v} = 1 - \frac{1}{1+v}, \qquad v > 0.$$

$$f_{\mathrm{V}}(v) = \frac{1}{(1+v)^2}, \qquad v > 0.$$

d) $U = \dfrac{X}{X+Y}$;

$$F_U(u) = P(U \le u) = P(X \le u(X+Y)) = P\left(Y \ge \frac{1-u}{u} X\right) = \frac{1}{1+\frac{1-u}{u}} = u,$$

$0 < u < 1.$

$$\left[\quad P(Y \le vX) = 1 - \frac{1}{1+v} \qquad \Rightarrow \qquad P(Y \ge vX) = \frac{1}{1+v}, \quad \text{then } v = \frac{1-u}{u}. \quad \right]$$

$f_U(u) = 1, \qquad 0 < u < 1.$

U has a Uniform$(0, 1)$ distribution.

OR

$$U = \frac{X}{X+Y} = \frac{1}{1+\frac{Y}{X}} = \frac{1}{1+V} = g(V).$$

$$V = \frac{1}{U} - 1 = g^{-1}(U) \qquad \Rightarrow \qquad \frac{dv}{du} = -\frac{1}{u^2}.$$

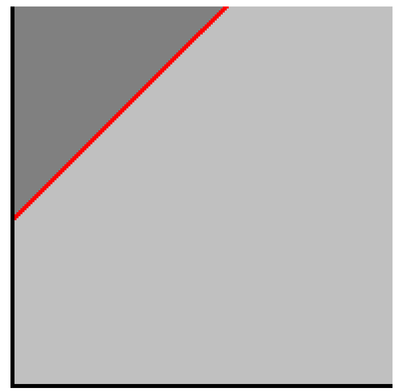
$$f_U(u) = f_V(g^{-1}(u)) \times \left| \frac{dv}{du} \right| = \frac{1}{\left(1+\frac{1}{u}-1\right)^2} \times \left| -\frac{1}{u^2} \right| = 1, \qquad 0 < u < 1.$$

U has a Uniform$(0, 1)$ distribution.

e) $T = X - Y$.

$$F_T(t) = P(T \le t) = P(X - Y \le t) = \ldots$$

Case 1: $t < 0$.



$$\ldots = P(Y \ge X - t)$$

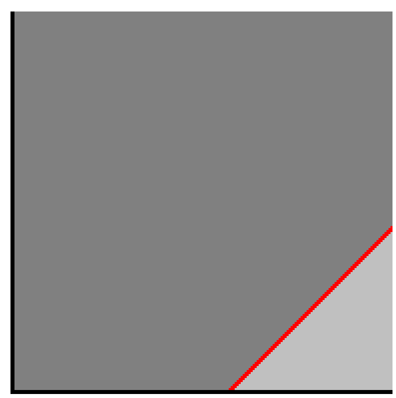
$$= \int_0^{\infty}\left(\int_{x-t}^{\infty}\left(e^{-x-y}\right)dy\right)dx$$

$$= \int_0^{\infty} e^{-2x+t}\,dx$$

$$= \frac{1}{2}e^{t}, \qquad t < 0.$$

$$f_T(t) = \frac{1}{2}e^{t}, \qquad t < 0.$$

Case 2: $t > 0$.



$$\ldots = 1 - P(X \ge Y + t)$$

$$= 1 - \int_0^{\infty}\left(\int_{y+t}^{\infty}\left(e^{-x-y}\right)dx\right)dy$$

$$= 1 - \int_0^{\infty} e^{-2y-t}\,dy$$

$$= 1 - \frac{1}{2}e^{-t}, \qquad t > 0.$$

$$f_T(t) = \frac{1}{2}e^{-t}, \qquad t > 0.$$

$$\Rightarrow \qquad f_T(t) = \frac{1}{2}e^{-|t|}, \qquad -\infty < t < \infty.$$

Another approach:

X and Y are two independent Exponential random variables with mean 1.

$$\mathrm{M}_{\mathrm{X}}(s) = \frac{1}{1-s}, \quad s < 1. \qquad\qquad \mathrm{M}_{\mathrm{Y}}(s) = \frac{1}{1-s}, \quad s < 1.$$

$$\Rightarrow \qquad \mathrm{M}_{\mathrm{X-Y}}(s) = \mathrm{M}_{\mathrm{X}}(s) \cdot \mathrm{M}_{\mathrm{Y}}(-s) = \frac{1}{1-s} \cdot \frac{1}{1+s} = \frac{1}{1-s^2}, \qquad -1 < s < 1.$$

$\Rightarrow$ (Recall Example 13 from STAT 400 Review (1))

$$f_{\mathrm{T}}(t) = \frac{1}{2}\, e^{-|t|}, \qquad -\infty < t < \infty.$$

(double exponential p.d.f.)