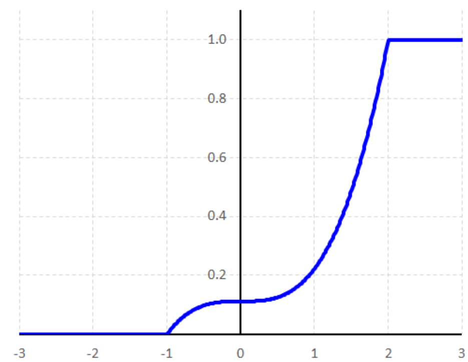
Examples for 09/02/2020 (2)

0. Find $\mathrm{E}(\mathrm{X})$ and $\mathrm{Var}(\mathrm{X})$, if random variable X has the following c.d.f.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3+1}{9} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



X is a continuous random variable with the p.d.f.

$$f_X(x) = F_X'(x) = \frac{x^2}{3}, \qquad -1 < x < 2, \qquad \text{zero otherwise.}$$

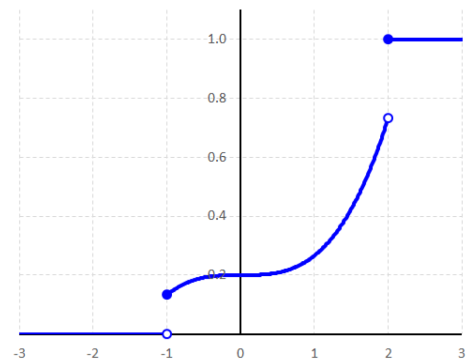
$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{-1}^{2} \frac{x^3}{3}\, dx = \frac{x^4}{12}\Bigg|_{-1}^{2} = \frac{16-1}{12} = \mathbf{\frac{5}{4}} = \mathbf{1.25}.$$

$$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_{-1}^{2} \frac{x^4}{3}\, dx = \frac{x^5}{15}\Bigg|_{-1}^{2} = \frac{32+1}{15} = 2.2.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = 2.2 - 1.25^2 = \mathbf{0.6375}.$$

1. Find $E(X)$ and $Var(X)$, if random variable X has the following c.d.f.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3+3}{15} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



X is a mixed random variable.

Discrete:

$F_X(x)$ "jumps" at $x = -1$ from 0 to $\frac{2}{15}$, $p(-1) = \frac{2}{15}$,

and $x = 2$ from $\frac{11}{15}$ to 1, $p(2) = \frac{4}{15}$.

Continuous:

$$f_X(x) = F_X'(x) = \frac{x^2}{5}, \qquad -1 < x < 2, \qquad \text{zero otherwise.}$$

$$E(X) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= -1 \cdot \frac{2}{15} + 2 \cdot \frac{4}{15} + \int_{-1}^{2} x \cdot \frac{x^2}{5}\, dx = \frac{6}{15} + \frac{x^4}{20}\Bigg|_{-1}^{2}$$

$$= \frac{2}{5} + \frac{16-1}{20} = \mathbf{\frac{23}{20}} = \mathbf{1.15}.$$

$$E(X^2) = \sum_{\text{all } x} x^2 \cdot p(x) + \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= (-1)^2 \cdot \frac{2}{15} + (2)^2 \cdot \frac{4}{15} + \int_{-1}^{2} x^2 \cdot \frac{x^2}{5}\, dx = \frac{18}{15} + \frac{x^5}{25}\Bigg|_{-1}^{2}$$

$$= 1.2 + \frac{32+1}{25} = 2.52.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2.52 - 1.15^2 = \mathbf{1.1975}.$$

2. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$p(2) = 0.08$, $p(4) = c$, zero otherwise,

and the p.d.f. of the continuous portion of the probability distribution

$f(x) = \frac{x^3}{100}$, $2 \le x \le 4$, zero elsewhere.

a) Find the value of c that would make this a valid probability distribution.

$$1 = \sum_{\text{all } x} p(x) + \int_{-\infty}^{\infty} f(x)\, dx$$

$$= [0.08 + c] + \int_{2}^{4} \frac{x^3}{100}\, dx = 0.08 + c + \frac{x^4}{400}\Bigg|_{2}^{4}$$

$$= 0.08 + c + \frac{256-16}{400} = 0.08 + c + 0.60 = c + 0.68.$$

$\Rightarrow$ $c = \mathbf{0.32}$.

b) Find $\mu_X = E(X)$.

$$E(X) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= 2 \cdot 0.08 + 4 \cdot 0.32 + \int_{2}^{4} x \cdot \frac{x^3}{100}\, dx = 0.16 + 1.28 + \left. \frac{x^5}{500} \right|_{2}^{4}$$

$$= 1.44 + \frac{1024 - 32}{500} = \mathbf{3.424}.$$

c) Find $\sigma_X^2 = \text{Var}(X)$.

$$E(X^2) = \sum_{\text{all } x} x^2 \cdot p(x) + \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= (2)^2 \cdot 0.08 + (4)^2 \cdot 0.32 + \int_{2}^{4} x^2 \cdot \frac{x^3}{100}\, dx = 0.32 + 5.12 + \left. \frac{x^6}{600} \right|_{2}^{4}$$

$$= 5.44 + \frac{4096 - 64}{600} = 12.16.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 12.16 - 3.424^2 = \mathbf{0.436224}.$$