STAT 410

Examples for 10/23/2020 (3)

Fall 2020
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Examples for 10/19/2020 (4) & 10/21/2020 (2) (continued)

1. The Pareto probability distribution has many applications in economics, biology, and physics. Let $\beta > 0$ and $\delta > 0$ be the population parameters, and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x; \beta, \delta) = \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}, \qquad x > \beta, \qquad \text{zero otherwise.}$$

Suppose β is known.

Recall: a method of moments estimator of δ is $\tilde{\delta} = \dfrac{\overline{X}}{\overline{X} - \beta}$; $\qquad E(X) = \dfrac{\beta\,\delta}{\delta - 1}$;

the maximum likelihood estimator of δ is $\hat{\delta} = \dfrac{n}{\sum\limits_{i=1}^{n} \ln\left(\dfrac{X_i}{\beta}\right)} = \dfrac{n}{\sum\limits_{i=1}^{n} \ln X_i - n \cdot \ln \beta}$;

$W = \ln\left(\dfrac{X}{\beta}\right) = \ln X - \ln \beta$ has an Exponential$\left(\theta = \dfrac{1}{\delta}\right)$ distribution.

u) Show that $\tilde{\delta}$ is a consistent estimator of δ.
(NOT enough to say "because it is a method of moments estimator")

v) Show that $\hat{\delta}$ is a consistent estimator of δ.
(NOT enough to say "because it is the maximum likelihood estimator")

Suppose δ is known.

Recall: $E(X) = \dfrac{\beta\,\delta}{\delta - 1}$; $\quad F_X(x) = 1 - \dfrac{\beta^{\delta}}{x^{\delta}}, \quad x > \beta$;

a method of moments estimator of β is $\tilde{\beta} = \dfrac{\delta - 1}{\delta}\,\overline{X}$;

the maximum likelihood estimator of β is $\hat{\beta} = \min X_i$.

w) Show that $\tilde{\beta}$ is a consistent estimator of β.

(NOT enough to say "because it is a method of moments estimator")

x) Show that $\hat{\beta}$ is a consistent estimator of β.

(NOT enough to say "because it is the maximum likelihood estimator")

Answers:

1. The Pareto probability distribution has many applications in economics, biology, and physics. Let $\beta > 0$ and $\delta > 0$ be the population parameters, and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

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u) Show that $\tilde{\delta}$ is a consistent estimator of δ.
(NOT enough to say "because it is a method of moments estimator")

By WLLN, $\overline{X} \xrightarrow{P} \mu = E(X) = \dfrac{\beta\,\delta}{\delta - 1}$.

$X_n \xrightarrow{P} a$, g is continuous at a $\Rightarrow$ $g(X_n) \xrightarrow{P} g(a)$

$g(x) = \dfrac{x}{x - \beta}$ is continuous at μ.

$$\tilde{\delta} = \frac{\overline{X}}{\overline{X} - \beta} = g(\overline{X}) \xrightarrow{P} g(\mu) = \frac{\mu}{\mu - \beta} = \frac{\dfrac{\beta\,\delta}{\delta - 1}}{\dfrac{\beta\,\delta}{\delta - 1} - \beta} = \delta.$$

v) Show that $\hat{\delta}$ is a consistent estimator of δ.

(NOT enough to say "because it is the maximum likelihood estimator")

By WLLN, $\overline{W} \xrightarrow{P} E(W) = \dfrac{1}{\delta}$.

$$\hat{\delta} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right)} = \frac{1}{\overline{W}} \xrightarrow{P} \frac{1}{E(W)} = \delta.$$

OR

$$E(\ln X) = \int_{\beta}^{\infty} \ln x \cdot \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}\, dx = \delta \cdot \beta^{\delta} \cdot \int_{\beta}^{\infty} \ln x \cdot x^{-\delta-1}\, dx$$

$$= \delta \cdot \beta^{\delta} \cdot \left[\frac{1}{-\delta} \cdot \ln x \cdot x^{-\delta} - \frac{1}{(-\delta)^2} \cdot x^{-\delta} \right] \Bigg|_{\beta}^{\infty} = \ln\beta + \frac{1}{\delta}.$$

By WLLN, $\dfrac{1}{n} \sum_{i=1}^{n} \ln X_i = \overline{\ln X} \xrightarrow{P} E(\ln X)$.

$$\hat{\delta} = \frac{n}{\sum_{i=1}^{n} \ln X_i - n \cdot \ln\beta} = \frac{1}{\overline{\ln X} - \ln\beta} \xrightarrow{P} \frac{1}{E(\ln X) - \ln\beta} = \delta.$$

OR

$$\mathrm{MSE}(\hat{\delta}) = \frac{(n+2)\,\delta^2}{(n-2)\,(n-1)} \to 0 \quad \text{as } n \to \infty.$$

$$\Rightarrow \quad \hat{\delta} \xrightarrow{P} \delta.$$

Suppose δ is known.

Recall: $\quad \mathrm{E}(\mathrm{X}) = \dfrac{\beta\,\delta}{\delta-1}; \qquad \mathrm{F}_{\mathrm{X}}(x) = 1-\dfrac{\beta^{\delta}}{x^{\delta}}, \quad x>\beta;$

a method of moments estimator of β is $\tilde{\beta} = \dfrac{\delta-1}{\delta}\,\overline{\mathrm{X}}$;

the maximum likelihood estimator of β is $\hat{\beta} = \min \mathrm{X}_i$.

w) Show that $\tilde{\beta}$ is a consistent estimator of β.

(NOT enough to say "because it is a method of moments estimator")

By WLLN, $\quad \overline{\mathrm{X}} \xrightarrow{P} \mu = \mathrm{E}(\mathrm{X}) = \dfrac{\beta\,\delta}{\delta-1}.$

$$\mathrm{X}_n \xrightarrow{P} \mathrm{X}, \ a = \text{const} \ \Rightarrow \ a\,\mathrm{X}_n \xrightarrow{P} a\,\mathrm{X}$$

$$\tilde{\beta} = \frac{\delta-1}{\delta}\,\overline{\mathrm{X}} \xrightarrow{P} \frac{\delta-1}{\delta}\cdot\frac{\beta\,\delta}{\delta-1} = \beta.$$

OR

$$\mathrm{MSE}(\tilde{\beta}) = \frac{\beta^2}{(\delta-2)\,\delta\,n} \to 0 \quad \text{as } n\to\infty.$$

$$\Rightarrow \qquad \tilde{\beta} \xrightarrow{P} \beta.$$

x) Show that $\hat{\beta}$ is a consistent estimator of β.

(NOT enough to say "because it is the maximum likelihood estimator")

$$F_{\min X_i}(x) = 1-\left(1-F_X(x)\right)^n = 1-\left(\frac{\beta^\delta}{x^\delta}\right)^n = 1-\frac{\beta^{\delta n}}{x^{\delta n}}, \qquad x > \beta.$$

Let $\varepsilon > 0$. Then

$$P\left(\left|\hat{\beta}-\beta\right| \ge \varepsilon\right) = P(\min X_i \le \beta-\varepsilon) + P(\min X_i \ge \beta+\varepsilon)$$

$$= 0 + P(\min X_i \ge \beta+\varepsilon)$$

$$= 1 - F_{\min X_i}(\beta+\varepsilon) = \left(\frac{\beta}{\beta+\varepsilon}\right)^{\delta n} \to 0 \qquad \text{as } n \to \infty.$$

$$\Rightarrow \qquad \hat{\beta} \xrightarrow{P} \beta.$$

OR

$$\text{MSE}(\hat{\beta}) = \frac{2\beta^2}{(\delta n-2)(\delta n-1)} \to 0 \quad \text{as } n \to \infty.$$

$$\Rightarrow \qquad \tilde{\beta} \xrightarrow{P} \beta.$$