Examples for 09/23/2020 (2)

Fact: Let X and Y be continuous random variables with joint p.d.f. $f(x, y)$. Then

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx$$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\,dy$$

Another Proof:

Let V = X and W = X + Y.

V = X $\Rightarrow$ X = V

W = X + Y $\Rightarrow$ Y = W – X = W – V

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

Then $f_{V,W}(v, w) = f_{X,Y}(v, w-v) \times 1 = f_{X,Y}(v, w-v)$.

Therefore, $f_W(w) = \int_{-\infty}^{\infty} f_{V,W}(v, w)\,dv = \int_{-\infty}^{\infty} f_{X,Y}(v, w-v)\,dv$.

~ 2.2.5

Fact: Let X and Y be independent continuous random variables. Then

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx$$

(convolution)

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y) \cdot f_Y(y)\,dy$$

0. Find the pdf of $W = X + Y$, where X and Y have the joint p.d.f.
$f_{X,Y}(x, y) = 2\,e^{-(x+y)}$, $0 < x < y < \infty$, zero elsewhere.

~ 2.2.7

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\,dx\,.$$

$0 < x$

$x < y \quad \Rightarrow \quad x < w - x \quad \Rightarrow \quad x < 0.5\,w$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\,dx = \int_{0}^{0.5\,w} 2\,e^{-w}\,dx = w\,e^{-w}, \quad w > 0.$$

OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_{X,Y}(w-y, y)\,dy\,.$$

$0 < x \quad \Rightarrow \quad 0 < w - y \quad \Rightarrow \quad y < w$

$x < y \quad \Rightarrow \quad w - y < y \quad \Rightarrow \quad y > 0.5\,w$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_{X,Y}(w-y, y)\,dy = \int_{0.5\,w}^{w} 2\,e^{-w}\,dy = w\,e^{-w}, \quad w > 0.$$

OR

$$F_{X+Y}(w) = \int_{0}^{w/2} \left(\int_{x}^{w-x} 2\,e^{-x-y}\,dy \right) dx = \int_{0}^{w/2} \left(-2\,e^{-x-y} \right) \Big|_{x}^{w-x} dx$$

$$= \int_{0}^{w/2} \left(2\,e^{-2x} - 2\,e^{-w} \right) dx = 1 - e^{-w} - w\,e^{-w}, \quad w > 0.$$

$$f_{X+Y}(w) = w\,e^{-w}, \quad w > 0.$$

OR

$$M_W(t) = E(e^{W \cdot t}) = E(e^{(X+Y) \cdot t}) = \int_0^{\infty} \left(\int_x^{\infty} e^{(x+y)t} \, 2 e^{-x-y} \, dy \right) dx$$

$$= \int_0^{\infty} \left(\int_x^{\infty} 2 e^{-(1-t)(x+y)} \, dy \right) dx = \int_0^{\infty} \left(\int_{2x}^{\infty} 2 e^{-(1-t)u} \, du \right) dx$$

$$= \int_0^{\infty} \frac{2}{1-t} e^{-2(1-t)x} \, dx = \frac{1}{(1-t)^2}, \qquad t < 1.$$

$\Rightarrow$ W has a Gamma distribution with $\alpha = 2$, $\theta = 1$.

$\Rightarrow$ $f_W(w) = w e^{-w}, \quad w > 0.$

OR

Recall:

Examples for 09/21/2020 (1):

1. Let X and Y have joint p.d.f. $f_{X,Y}(x, y) = 2 e^{-(x+y)}$, $0 < x < y < \infty$.

b) Find the joint p.d.f. $f_{W,Z}(w, z)$ of the variables

$W = X + Y$ and $Z = Y / X$.

• • •

$$f_{W,Z}(w, z) = 2 e^{-w} \times \frac{w}{(1+z)^2}, \qquad w > 0, \quad z > 1.$$

$$\Rightarrow \quad f_W(w) = \int_{-\infty}^{\infty} f_{W,Z}(w, z) dz = \int_1^{\infty} \frac{2 w e^{-w}}{(1+z)^2} dz = w e^{-w}, \quad w > 0.$$

1. When a person applies for citizenship in Neverland, first he/she must wait X years for an interview, and then Y more years for the oath ceremony. Thus the total wait is W = X + Y years. Suppose that X and Y are independent, the p.d.f. of X is

$$f_X(x) = {}^{2}/_{x^3}, \qquad x > 1, \qquad \text{zero otherwise,}$$

and Y has a Uniform distribution on interval $(0, 1)$.

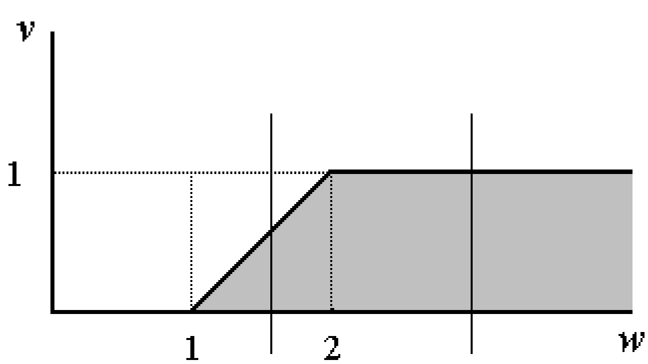
Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

Hint: Consider two cases: $1 < w < 2$ and $w > 2$.

$$\begin{aligned} V &= Y & &\Rightarrow & X &= W - V \\ W &= X + Y & &\Rightarrow & Y &= V \end{aligned} \qquad J = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$f_{V,W}(v, w) = f_{X,Y}(w-v, v) \times 1 = \frac{2}{(w-v)^3}.$$

$$\begin{aligned} x &> 1 & &\Rightarrow & w - v &> 1 & &\Rightarrow & w &> v + 1 \\ 0 &< y < 1 & &\Rightarrow & 0 &< v < 1 \end{aligned}$$



$1 < w < 2$
$$f_W(w) = \int_0^{w-1} \frac{2}{(w-v)^3}\, dv = \left(\frac{1}{(w-v)^2} \right) \Bigg|_0^{w-1} = 1 - \frac{1}{w^2}.$$

$w > 2$
$$f_W(w) = \int_0^{1} \frac{2}{(w-v)^3}\, dv = \left(\frac{1}{(w-v)^2} \right) \Bigg|_0^{1} = \frac{1}{(w-1)^2} - \frac{1}{w^2}.$$

OR

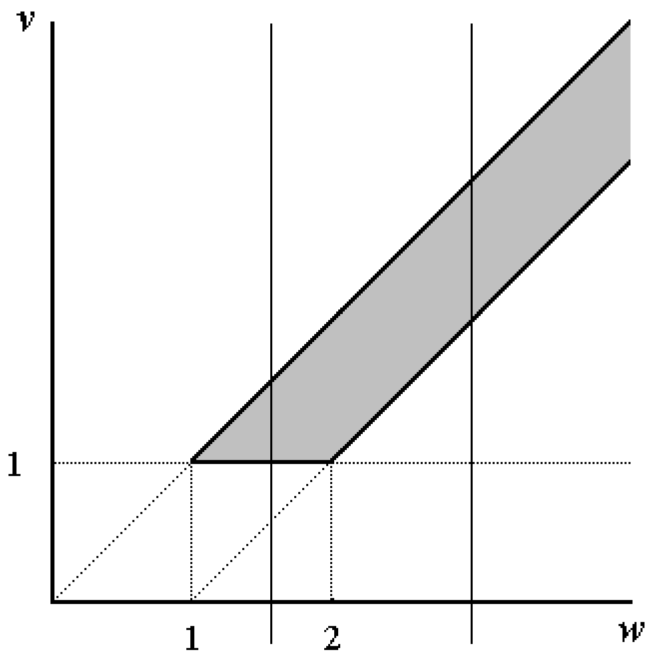
$$V = X \quad \Rightarrow \quad X = V$$
$$W = X + Y \quad \Rightarrow \quad Y = W - V$$

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$f_{V,W}(v, w) = f_{X,Y}(v, w-v) \times 1 = \frac{2}{v^3}.$$

$x > 1 \quad \Rightarrow \quad v > 1$

$0 < y < 1 \quad \Rightarrow \quad 0 < w - v < 1 \qquad v < w < v + 1$



$1 < w < 2$
$$f_W(w) = \int_1^w \frac{2}{v^3}\,dv = \left(-\frac{1}{v^2} \right)\Bigg|_1^w = 1 - \frac{1}{w^2}.$$

$w > 2$
$$f_W(w) = \int_{w-1}^{w} \frac{2}{v^3}\,dv = \left(-\frac{1}{v^2} \right)\Bigg|_{w-1}^{w} = \frac{1}{(w-1)^2} - \frac{1}{w^2}.$$

<u>Fact</u>: Let X and Y be continuous random variables with joint p.d.f. $f(x, y)$.

Let $Z = XY$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \frac{1}{|x|} \, dx$$

<u>Proof</u>: Let $W = X$, $Z = XY$.

Then $X = W$, $Y = {}^{Z}/_{W}$.

$$J = \begin{vmatrix} 1 & 0 \\ -\frac{z}{w^2} & \frac{1}{w} \end{vmatrix} = \frac{1}{w}.$$

Then $f_{W,Z}(w, z) = f_{X,Y}\left(w, \frac{z}{w}\right) \frac{1}{|w|}$.

Therefore, $f_Z(z) = \int_{-\infty}^{\infty} f_{WZ}(w, z)\, dw = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{1}{|w|}\, dw$.

2. Suppose that X and Y are independent, the p.d.f. of X is

$$f_X(x) = {}^{2}/_{x^3}, \qquad x > 1, \qquad \text{zero otherwise,}$$

and Y has a Uniform distribution on interval $(0, 1)$.

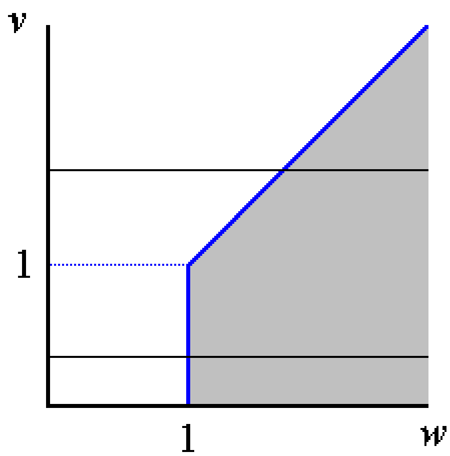
Let $V = X \times Y$. Find the p.d.f. of V, $f_V(v) = f_{X \times Y}(v)$.

Hint: Consider two cases: $0 < v < 1$ and $v > 1$.

Let $W = X$, $V = XY$. Then $X = W$, $Y = {}^{V}/_{W}$.

$x > 1 \quad \Rightarrow \quad w > 1$.

$0 < y < 1 \quad \Rightarrow \quad 0 < {}^{v}/_{w} < 1 \quad \Rightarrow \quad 0 < v < w$.



$$f_V(v) = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{v}{w}\right) \frac{1}{|w|}\, dw = \ldots$$

$$\ldots = \int_{v}^{\infty} \left(\frac{2}{w^3} \cdot 1 \right) \cdot \frac{1}{w}\, dw = \frac{2}{3\, v^3}, \qquad v > 1.$$

$$\ldots = \int_{1}^{\infty} \left(\frac{2}{w^3} \cdot 1 \right) \cdot \frac{1}{w}\, dw = \frac{2}{3}, \qquad 0 < v < 1.$$

<u>Fact</u>: Let X and Y be continuous random variables with joint p.d.f. $f(x, y)$.

Let $Z = X/Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f(yz, y)\, |y|\, dy$$

<u>Proof</u>: Let $W = Y$, $Z = X/Y$.

Then $X = WZ$, $Y = W$.

$$J = \begin{vmatrix} z & w \\ 1 & 0 \end{vmatrix} = -w.$$

Then $f_{W,Z}(w, z) = f_{X,Y}(wz, w)\, |w|$.

Therefore, $f_Z(z) = \int_{-\infty}^{\infty} f_{W,Z}(w, z)\, dw = \int_{-\infty}^{\infty} f_{X,Y}(wz, w)\, |w|\, dw.$

3. Suppose that X and Y are independent, the p.d.f. of X is

$$f_X(x) = {}^{2}\!/_{x^3}, \qquad x > 1, \qquad \text{zero otherwise,}$$

and Y has a Uniform distribution on interval $(0, 1)$.

Let $U = {}^{Y}\!/_{X}$. Find the p.d.f. of U, $f_U(u)$.

Let $X = X$, $U = {}^{Y}\!/_{X}$. Then $X = X$, $Y = UX$.

$$J = \begin{vmatrix} 1 & 0 \\ u & x \end{vmatrix} = x.$$

Then $f_{X,U}(x,u) = f_{X,Y}(x,ux)\,|x|$.

Therefore, $f_U(u) = \int\limits_{-\infty}^{\infty} f_{X,U}(x,u)\,dx = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,ux)\,|x|\,dx$.

$x > 1 \qquad \Rightarrow \qquad x > 1$

$0 < y < 1 \qquad \Rightarrow \qquad 0 < ux < 1 \qquad \Rightarrow \qquad 0 < x < \dfrac{1}{u}$

Need $x > 1$ & $0 < x < \dfrac{1}{u}$.

$\Rightarrow \qquad 0 < u < 1 \qquad$ (otherwise $\dfrac{1}{u} < 1$).

$$f_U(u) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,ux)\,|x|\,dx = \int\limits_{1}^{\frac{1}{u}} \frac{2}{x^3}\cdot 1\cdot |x|\,dx = \int\limits_{1}^{\frac{1}{u}} \frac{2}{x^2}\,dx$$

$$= -\frac{2}{x}\bigg|_{1}^{1/u} = 2 - 2u, \qquad 0 < u < 1.$$

OR

Let $\quad Y = Y, \quad U = {}^{Y}\!/_{X}.$ $\qquad$ Then $\quad X = {}^{Y}\!/_{U}, \quad Y = Y.$

$$J = \begin{vmatrix} -\dfrac{y}{u^2} & \dfrac{1}{u} \\ 0 & 1 \end{vmatrix} = -\frac{y}{u^2}.$$

Then $\quad f_{Y,U}(y,u) = f_{X,Y}\left(\dfrac{y}{u}, y\right)\left|-\dfrac{y}{u^2}\right|.$

Therefore, $\quad f_U(u) = \displaystyle\int_{-\infty}^{\infty} f_{Y,U}(y,u)\,dy = \int_{-\infty}^{\infty} f_{X,Y}\left(\frac{y}{u}, y\right)\left|-\frac{y}{u^2}\right| dy.$

$x > 1 \qquad \Rightarrow \qquad \dfrac{y}{u} > 1 \qquad\qquad y > u$

$0 < y < 1 \qquad \Rightarrow \qquad 0 < y < 1$

Need $\qquad y > u \quad \& \quad 0 < y < 1.$

$\Rightarrow \qquad 0 < u < 1.$

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}\left(\frac{y}{u}, y\right)\left|-\frac{y}{u^2}\right| dy = \int_{u}^{1} \frac{2}{(\,y/u\,)^3}\cdot 1\cdot\left|-\frac{y}{u^2}\right| dy$$

$$= \int_{u}^{1} \frac{2u}{y^2}\,dy = -\frac{2u}{y}\,\Bigg|_{u}^{1} = 2 - 2u, \qquad\qquad 0 < u < 1.$$