

Functions of One Random Variable

Example 1:

x	$p_X(x)$		$y = x^2$	$p_Y(y) = p_X(\sqrt{y})$
1	0.2	$Y = X^2$	1	0.2
2	0.4		4	0.4
3	0.3		9	0.3
4	0.1		16	0.1

Example 2:

x	$p_X(x)$		y	$p_Y(y)$
-2	0.2	$Y = X^2$	0	$p_X(0) = 0.4$
0	0.4		4	$p_X(-2) + p_X(2) = 0.5$
2	0.3		9	$p_X(3) = 0.1$
3	0.1			

Example 3:

$$X \sim \text{Poisson}(\lambda): \quad p_X(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, 3, 4, 5, 6, \dots$$

$$Y = X^2 \quad \Rightarrow \quad p_Y(y) = \frac{\lambda^{\sqrt{y}} \cdot e^{-\lambda}}{(\sqrt{y})!}, \quad y = 0, 1, 4, 9, 16, 25, 36, \dots$$

Let X be a continuous random variable.

Let $Y = g(X)$. What is the probability distribution of Y ?

Cumulative Distribution Function approach:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int_{\{x: g(x) \leq y\}} f_X(x) dx = \dots$$

Moment-Generating Function approach:

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{g(X) \cdot t}) = \int_{-\infty}^{\infty} e^{g(x) \cdot t} f_X(x) dx = \dots$$

1. Let U be a $\text{Uniform}(0, 1)$ random variable:

$$f_U(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases} \quad F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \leq u < 1 \\ 1 & u \geq 1 \end{cases}$$

Consider $Y = U^2$. What is the probability distribution of Y ?

$$0 < u < 1 \quad Y = U^2 \quad \Rightarrow \quad 0 < y < 1.$$

$$F_Y(y) = P(Y \leq y) = P(U^2 \leq y) = \dots$$

$$y < 0 \quad P(U^2 \leq y) = 0 \quad F_Y(y) = 0.$$

$$y \geq 1 \quad P(U^2 \leq y) = 1 \quad F_Y(y) = 1.$$

$$0 \leq y < 1 \quad P(U^2 \leq y) = P(U \leq \sqrt{y}) = F_U(\sqrt{y}) = \sqrt{y}.$$

$$\text{OR} \quad P(U^2 \leq y) = P(U \leq \sqrt{y}) = \int_0^{\sqrt{y}} 1 \, du = \sqrt{y}.$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Change-of-Variable Technique:

Theorem 1.7.1 X – continuous r.v. with p.d.f. $f_X(x)$.

$Y = g(X)$ $g(x)$ – one-to-one, differentiable

$$dx/dy = d[g^{-1}(y)]/dy$$

$$\Rightarrow \quad f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

1. Let U be a Uniform(0, 1) random variable.

Consider $Y = U^2$. What is the probability distribution of Y ?

$$f_U(u) = 1, \quad 0 < u < 1.$$

$$g(u) = u^2 \quad \quad \quad g^{-1}(y) = \sqrt{y} = y^{1/2} \quad \quad \quad du/dy = \frac{1}{2} y^{-1/2}$$

$$f_Y(y) = f_U(g^{-1}(y)) \left| \frac{du}{dy} \right| = (1) \left| \frac{1}{2} y^{-1/2} \right| = \frac{1}{2} y^{-1/2}, \quad 0 < y < 1.$$

1.5. Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} 6x^5 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the probability distribution of $Y = 1/X^2$.

Support of $X = \{ 0 < x < 1 \}$

$Y = 1/X^2 \Rightarrow$ Support of $Y = \{ y > 1 \}$

$$g(x) = 1/x^2 \quad g^{-1}(y) = \frac{1}{\sqrt{y}} = y^{-1/2} \quad dx/dy = -\frac{1}{2} y^{-3/2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (6y^{-5/2}) \left(\frac{1}{2} y^{-3/2} \right) = 3y^{-4} \quad y > 1.$$

OR

$$f_X(x) = \begin{cases} 6x^5 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \quad F_X(x) = \begin{cases} 0 & x < 0 \\ x^6 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(1/X^2 \leq y) = P(X \geq 1/\sqrt{y}) = 1 - F_X(1/\sqrt{y}) \\ &= 1 - y^{-3}, \quad y > 1. \end{aligned}$$

$$f_Y(y) = F'_Y(y) = 3y^{-4}, \quad y > 1.$$

2. Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

a) Find the probability distribution of $Y = \sqrt{X}$.

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases} \quad F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$0 < x < 1 \quad \Rightarrow \quad 0 < y < 1$$

$$y < 0 \quad F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = 0.$$

$$y \geq 0 \quad F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2).$$

$$0 \leq y < 1 \quad F_Y(y) = F_X(y^2) = y^4.$$

$$y \geq 1 \quad F_Y(y) = F_X(y^2) = 1.$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^4 & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases} \quad f_Y(y) = F'_Y(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

OR

$$g(x) = \sqrt{x} \quad g^{-1}(y) = y^2 \quad \frac{dx}{dy} = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (2y^2)(2y) = 4y^3, \quad 0 < y < 1.$$

b) Find the probability distribution of $W = \frac{1}{X+1}$.

$$0 < x < 1 \quad \Rightarrow \quad \frac{1}{2} < w < 1$$

$$\begin{aligned} F_W(w) &= P(W \leq w) = P\left(\frac{1}{X+1} \leq w\right) = P\left(X \geq \frac{1}{w} - 1\right) = 1 - F_X\left(\frac{1}{w} - 1\right) \\ &= 1 - \left(\frac{1}{w} - 1\right)^2 = \frac{2}{w} - \frac{1}{w^2}, \quad \frac{1}{2} < w < 1. \end{aligned}$$

$$f_W(w) = F'_W(w) = -\frac{2}{w^2} + \frac{2}{w^3} = \frac{2-2w}{w^3}, \quad \frac{1}{2} < w < 1.$$

OR

$$g(x) = \frac{1}{x+1} \quad g^{-1}(w) = \frac{1}{w} - 1 \quad \frac{dx}{dw} = -\frac{1}{w^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \left[2\left(\frac{1}{w} - 1\right) \right] \left(\frac{1}{w^2}\right) = \frac{2-2w}{w^3}, \quad \frac{1}{2} < w < 1.$$

3. Consider a continuous random variable X with the p.d.f. $f_X(x) = \frac{24}{x^4}$, $x > 2$.

a) Let $Y = 1/X$. Find the p.d.f. of Y , $f_Y(y)$.

Support of $X = \{x > 2\}$

$Y = 1/X \Rightarrow$ Support of $Y = \{0 < y < 1/2\}$

$$g(x) = 1/x \quad g^{-1}(y) = 1/y \quad dx/dy = -1/y^2$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (24y^4)(y^{-2}) = 24y^2, \quad 0 < y < 1/2.$$

OR

$$F_X(x) = 1 - \frac{8}{x^3}, \quad x > 2.$$

$$F_Y(y) = P(Y \leq y) = P(1/X \leq y) = P(X \geq 1/y) = 1 - F_X(1/y) = 8y^3, \\ 0 < y < 1/2.$$

$$f_Y(y) = 24y^2, \quad 0 < y < 1/2.$$

b) Find the probability distribution of $Y = 1/X^2$.

$$\text{Support of } X = \{x > 2\}$$

$$Y = 1/X^2 \Rightarrow \text{Support of } Y = \{0 < y < 1/4\}$$

$$g(x) = 1/x^2 \quad g^{-1}(y) = 1/\sqrt{y} = y^{-1/2} \quad dx/dy = -\frac{1}{2} y^{-3/2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (24y^2) \left(\frac{1}{2} y^{-3/2} \right) = 12y^{1/2} = 12\sqrt{y},$$

$$0 < y < 1/4.$$

OR

$$F_X(x) = 1 - \frac{8}{x^3}, \quad x > 2.$$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(1/X^2 \leq y) = P(X \geq 1/\sqrt{y}) \\ &= 1 - F_X(1/\sqrt{y}) = 8y^{3/2}, \quad 0 < y < 1/4. \end{aligned}$$

$$f_Y(y) = 12y^{1/2} = 12\sqrt{y}, \quad 0 < y < 1/4.$$