STAT 410

Homework #3

Fall 2020
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(due Friday, September 18, by 5:00 p.m. CDT)

Please include your name (with your last name underlined),
your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

4. A fair 6-sided die is rolled.

If the outcome is 1 or 2, then two (2) fair coins are tossed.

If the outcome is 6 or 5 or 4 or 3, then three (3) fair coins are tossed.

Let X be the number of coins tossed. Let Y be the number of **Tails** observed.

a) Construct the joint probability distribution of (X, Y).

"Hint": There are 2 possible values for X, 4 possible values for Y.

You have $p_X(x)$ and $p_{Y|X}(y|x)$.

Write the values of the joint p.m.f. $p(x,y)$ in a 2×4 rectangular array.

$$p_X(2) = \frac{2}{6} = \frac{1}{3}. \qquad\qquad p_X(3) = \frac{4}{6} = \frac{2}{3}.$$

$(Y \mid X = x)$ has a Binomial$\left(n = x, p = \frac{1}{2}\right)$ distribution.

$$p_{Y|X}(0|2) = \frac{1}{4}. \quad p_{Y|X}(1|2) = \frac{2}{4}. \quad p_{Y|X}(2|2) = \frac{1}{4}.$$

$$p_{Y|X}(0|3) = \frac{1}{8}. \quad p_{Y|X}(1|3) = \frac{3}{8}. \quad p_{Y|X}(2|3) = \frac{3}{8}. \quad p_{Y|X}(3|3) = \frac{1}{8}.$$

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)}. \qquad \Rightarrow \qquad p(x,y) = p_X(x) \cdot p_{Y|X}(y|x).$$

		y				$p_X(x)$
		0	1	2	3	
x	2	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	0	$\frac{1}{3}$
	3	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{2}{3}$
	$p_Y(y)$	$\frac{2}{12}$	$\frac{5}{12}$	$\frac{4}{12}$	$\frac{1}{12}$	1

b) Construct the probability distribution of $E(X \mid Y)$.

"Hint": There are 4 possible values for Y. … Meh, no one reads the hints anyway…

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}.$$

$(X \mid Y = 0)$

x	prob.	$x \cdot$ prob.
2	$\frac{1/12}{2/12} = 0.50$	1.00
3	$\frac{1/12}{2/12} = 0.50$	1.50
	1.00	2.50

$(X \mid Y = 1)$

x	prob.	$x \cdot$ prob.
2	$\frac{2/12}{5/12} = 0.40$	0.80
3	$\frac{3/12}{5/12} = 0.60$	1.80
	1.00	2.60

$(X \mid Y = 2)$

x	prob.	$x \cdot$ prob.
2	$\frac{1/12}{4/12} = 0.25$	0.50
3	$\frac{3/12}{4/12} = 0.75$	2.25
	1.00	2.75

$(X \mid Y = 3)$

x	prob.	$x \cdot$ prob.
2	$\frac{0}{1/12} = 0.00$	0.00
3	$\frac{1/12}{1/12} = 1.00$	3.00
	1.00	3.00

$E(X \mid Y)$ is a random variable, a function of random variable Y.

If $Y = y$, then $E(X \mid Y) = E(X \mid Y = y)$.

$E(X \mid Y)$:

y	$E(X \mid Y = y)$	prob.
0	**2.50**	$\mathbf{\frac{2}{12}}$
1	**2.60**	$\mathbf{\frac{5}{12}}$
2	**2.75**	$\mathbf{\frac{4}{12}}$
3	**3.00**	$\mathbf{\frac{1}{12}}$
		1

3. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Since the people become disobedient if they do not have food, the amount spent monthly on butter is always at least 2 million dollars. However, there is a law that prevents the government of Neverland spending more than 5 million dollars on guns in one month. In addition, the government of Neverland cannot afford to spend more than 8 million dollars per month on guns and butter. Suppose that the joint density function for (X, Y) is

$$f(x,y) = \frac{7x + 2y}{375}, \qquad x \ge 0,\ \ y \ge 2,\ \ x \le 5,\ \ x + y \le 8, \qquad \text{zero otherwise.}$$

X – guns, Y – butter.

Recall:

$$f_X(x) = \frac{60 + 26x - 6x^2}{375} = \frac{2(5+3x)(6-x)}{375}, \qquad 0 \le x \le 5.$$

$$f_Y(y) = \begin{cases} \dfrac{35+4y}{150} & 2 \le y \le 3 \\[2ex] \dfrac{448 - 80y + 3y^2}{750} = \dfrac{(56-3y)(8-y)}{750} & 3 \le y \le 8 \end{cases}$$

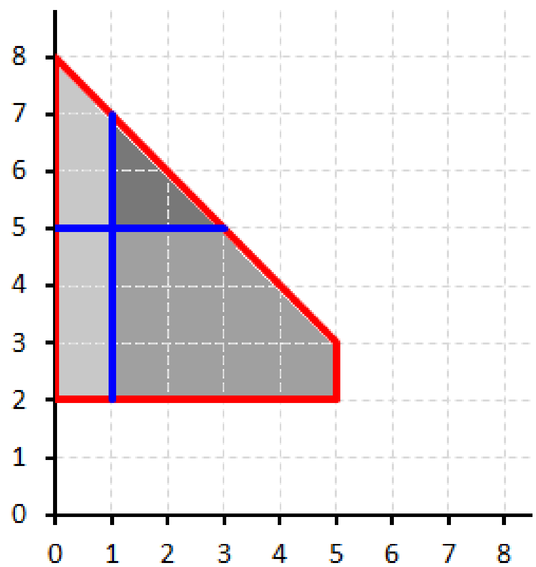
i) Suppose that 1 million dollars is spent on guns in a given month. What is the probability that more than 5 million dollars is spent on butter during this month? That is, find $P(Y > 5 \mid X = 1)$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{7x + 2y}{60 + 26x - 6x^2}, \qquad 2 < y < 8 - x.$$

$$f_{Y|X}(y|1) = \frac{7 + 2y}{80}, \qquad 2 < y < 7.$$

$$P(Y > 5 \mid X = 1) = \int_5^7 \frac{7 + 2y}{80}\, dy = \left.\frac{7y + y^2}{80}\right|_{y=5}^{y=7} = \frac{38}{80} = \mathbf{\frac{19}{40}} = \mathbf{0.475}.$$

j) Suppose that more than 1 million dollars is spent on guns in a given month. What is the probability that more than 5 million dollars is spent on butter during this month? That is, find $P(Y > 5 \mid X > 1)$.



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

provided $P(B) > 0$.

$$P(B) = P(X > 1)$$

$$= \int_1^5 \frac{60 + 26x - 6x^2}{375}\, dx$$

$$= \left. \frac{60x + 13x^2 - 2x^3}{375} \right|_{x=1}^{x=5}$$

$$= \frac{304}{375} \approx 0.810667.$$

$$P(A \cap B) = P(Y > 5 \cap X > 1) = \int_1^3 \left(\int_5^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$= \int_1^3 \left(\left. \frac{7xy + y^2}{375} \right) \right|_{y=5}^{y=8-x} dx = \int_1^3 \frac{7x(8-x) + (8-x)^2 - 35x - 25}{375}\, dx$$

$$= \int_1^3 \frac{39 + 5x - 6x^2}{375}\, dx = \left. \frac{39x + 2.5x^2 - 2x^3}{375} \right|_{x=1}^{x=3} = \frac{46}{375} \approx 0.122667.$$

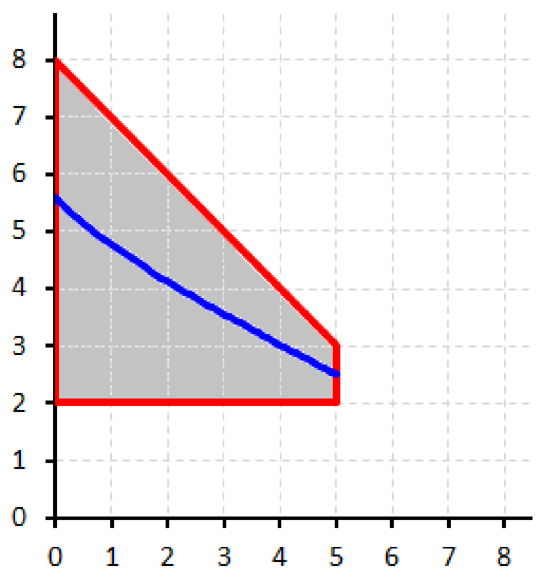
$$P(Y > 5 \mid X > 1) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{46}{375}}{\frac{304}{375}} = \frac{46}{304} = \mathbf{\frac{23}{152}} \approx 0.151316.$$

k) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$, the expected amount spent on butter during a month when x million dollars is spent on guns.

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x) = \int_2^{8-x} y \cdot \frac{7x + 2y}{60 + 26x - 6x^2}\, dy = \left. \frac{\frac{7}{2} x y^2 + \frac{2}{3} y^3}{60 + 26x - 6x^2} \right|_{y=2}^{y=8-x}$$

$$= \frac{\frac{7}{2} x (8-x)^2 + \frac{2}{3}(8-x)^3 - 14x - \frac{16}{3}}{60 + 26x - 6x^2}$$

$$= \frac{2016 + 492x - 240x^2 + 17x^3}{6\left(60 + 26x - 6x^2\right)} = \frac{336 + 138x - 17x^2}{12(5 + 3x)}, \qquad 0 \le x \le 5.$$



l) Suppose that 2.5 million dollars is spent on butter in a given month. What is the probability that more than 2 million dollars is spent on guns during this month? That is, find $P(X > 2 \mid Y = 2.5)$.

For $2 < y < 3$, $\quad f_Y(y) = \dfrac{35 + 4y}{150}$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{14x + 4y}{175 + 20y}, \qquad 0 < x < 5.$$

$$f_{X|Y}(x|2.5) = \frac{14x + 10}{225}, \qquad 0 < x < 5.$$

$$P(X > 2 \mid Y = 2.5) = \int_2^5 \frac{14x + 10}{225}\, dx = \left. \frac{7x^2 + 10x}{225} \right|_{x=2}^{x=5}$$

$$= \frac{177}{225} = \mathbf{\frac{59}{75}} \approx 0.786667.$$

m) Suppose that 5 million dollars is spent on butter in a given month. What is the probability that more than 2 million dollars is spent on guns during this month? That is, find $P(X > 2 \mid Y = 5)$.

For $3 < y < 8$, $\quad f_Y(y) = \dfrac{448 - 80y + 3y^2}{750}$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{14x + 4y}{448 - 80y + 3y^2}, \qquad 0 < x < 8 - y.$$

$$f_{X|Y}(x|5) = \frac{14x + 20}{123}, \qquad 0 < x < 3.$$

$$P(X > 2 \mid Y = 5) = \int_2^3 \frac{14x + 20}{123}\, dx = \left. \frac{7x^2 + 20x}{123} \right|_{x=2}^{x=3} = \mathbf{\frac{55}{123}} \approx 0.447154.$$

n) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y)$, the expected amount spent on guns during a month when y million dollars is spent on butter.

For $2<y<3$, $\qquad f_{\mathrm{X}|\mathrm{Y}}(x|y) = \dfrac{14x+4y}{175+20y}, \qquad 0<x<5.$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \int_0^5 x \cdot \frac{14x+4y}{175+20y}\, dx = \left.\frac{\frac{14}{3}x^3+2x^2y}{175+20y}\right|_{x=0}^{x=5} = \frac{350+30y}{105+12y},$$

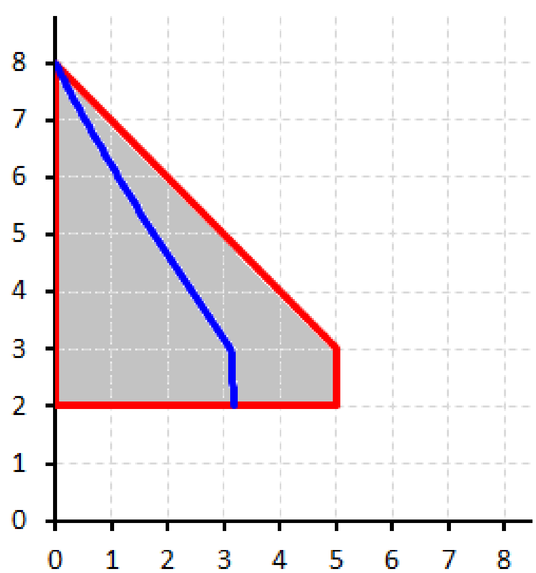
$2<y<3.$

For $3<y<8$, $\qquad f_{\mathrm{X}|\mathrm{Y}}(x|y) = \dfrac{14x+4y}{448-80y+3y^2}, \qquad 0<x<8-y.$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \int_0^{8-y} x \cdot \frac{14x+4y}{448-80y+3y^2}\, dx = = \left.\frac{\frac{14}{3}x^3+2x^2y}{448-80y+3y^2}\right|_{x=0}^{x=8-y}$$

$$= \frac{14(8-y)^3+6(8-y)^2y}{3(56-3y)(8-y)} = \frac{14(8-y)^2+6(8-y)y}{3(56-3y)}$$

$$= \frac{896-176y+8y^2}{3(56-3y)} = \frac{8(14-y)(8-y)}{3(56-3y)}, \qquad 3<y<8.$$



$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \begin{cases} \dfrac{350+30y}{105+12y} & 2<y<3 \\[2ex] \dfrac{896-176y+8y^2}{168-9y} & 3<y<8 \end{cases}$$