STAT 410 Fall 2020

Homework #10

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(due Friday, November 13, by 5:00 p.m. CST)

No credit will be given without supporting work.

7. Let $\psi > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\psi) = \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall: $W = X^2$ has $\text{Gamma}\left(\alpha = \frac{1}{2}, \theta = \psi\right)$ distribution.

n) Find the Fisher information $I(\psi)$.

$$\ln f(x;\psi) = \ln 2 - \frac{1}{2}\cdot\ln\pi - \frac{1}{2}\cdot\ln\psi - \frac{1}{\psi}\cdot x^2.$$

$$\frac{\partial}{\partial\psi}\ln f(x;\psi) = -\frac{1}{2\psi} + \frac{1}{\psi^2}\cdot x^2. \qquad \frac{\partial^2}{\partial\psi^2}\ln f(x;\psi) = \frac{1}{2\psi^2} - \frac{2}{\psi^3}\cdot x^2.$$

$$\begin{aligned} I(\psi) &= \text{Var}\left[\frac{\partial}{\partial\psi}\ln f(X;\psi)\right] \\ &= \text{Var}\left[-\frac{1}{2\psi} + \frac{1}{\psi^2}\cdot X^2\right] \\ &= \frac{1}{\psi^4}\,\text{Var}(W) \\ &= \frac{1}{\psi^4}\,\alpha\,\theta^2 \\ &= \frac{1}{\psi^4}\,\frac{1}{2}\,\psi^2 = \frac{1}{2\psi^2}. \end{aligned}$$

$$\begin{aligned} I(\psi) &= -E\left[\frac{\partial^2}{\partial\psi^2}\ln f(X;\psi)\right] \\ &= -E\left[\frac{1}{2\psi^2} - \frac{2}{\psi^3}\cdot X^2\right] \\ &= -\frac{1}{2\psi^2} + \frac{2}{\psi^3}\,E(W) \\ &= -\frac{1}{2\psi^2} + \frac{2}{\psi^3}\,\alpha\,\theta \\ &= -\frac{1}{2\psi^2} + \frac{2}{\psi^3}\,\frac{1}{2}\,\psi = \frac{1}{2\psi^2}. \end{aligned}$$

Recall: $\hat{\psi} = \frac{2}{n} \sum_{i=1}^{n} X_i^2 = 2\,\overline{X^2} = 2\,\overline{W}$ is an unbiased estimator for ψ.

Recall: For large n, $\hat{\psi}$ is approximately $N\left(\psi, \frac{2\psi^2}{n} = \frac{1}{n\,I(\psi)}\right)$.

o) Is $\hat{\psi}$ an efficient estimator of ψ? If $\hat{\psi}$ is not efficient, find its efficiency.

① Find $\mathrm{Var}(\hat{\psi})$. ("Hint": $\mathrm{Var}(\overline{W}) = \frac{\sigma_W^2}{n}$, $\mathrm{Var}(a☺) = a^2\,\mathrm{Var}(☺)$.)

② Find the Rao-Cramér lower bound.

③ Is $\hat{\psi}$ an efficient estimator of ψ? Does $\mathrm{Var}(\hat{\psi})$ attain the R.C.L.B.?
If $\hat{\psi}$ is not efficient, find its efficiency.

$$\sigma_W^2 = \mathrm{Var}(W) = \alpha\,\theta^2 = \frac{1}{2}\psi^2.$$

$$\mathrm{Var}(\hat{\psi}) = \mathrm{Var}(2\,\overline{W}) = 4\,\mathrm{Var}(\overline{W}) = 4\,\frac{\sigma_W^2}{n} = \frac{2\psi^2}{n}.$$

$$\text{Rao-Cramer lower bound} = \frac{1}{n\,I(\psi)} = \frac{2\psi^2}{n}.$$

$\mathrm{Var}(\hat{\psi})$ DOES attain its Rao-Cramer lower bound.

$\Rightarrow$ $\hat{\psi}$ IS an efficient estimator of ψ.

8. Let $\beta > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\beta) = \beta\,(1-x)^{\beta-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

Recall: $W = -\ln(1-X)$ has an $\text{Exponential}\left(\theta = \frac{1}{\beta}\right)$

$= \text{Gamma}\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

$$\Rightarrow \qquad Y = \sum_{i=1}^{n}\big(-\ln(1-X_i)\big) = \sum_{i=1}^{n} W_i \ \text{ has a } \text{Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right) \text{ distribution.}$$

n) Find the Fisher information $I(\beta)$.

$$\ln f(x;\beta) = \ln\beta + (\beta-1)\cdot\ln(1-x).$$

$$\frac{\partial}{\partial\beta}\ln f(x;\beta) = \frac{1}{\beta} + \ln(1-x). \qquad\qquad \frac{\partial^2}{\partial\beta^2}\ln f(x;\beta) = -\frac{1}{\beta^2}.$$

$$I(\beta) = \text{Var}\left[\frac{\partial}{\partial\beta}\ln f(X;\beta)\right] = \text{Var}\left[\frac{1}{\beta} + \ln(1-X)\right] = \text{Var}(W) = \alpha\,\theta^2 = \frac{1}{\beta^2}.$$

$$I(\beta) = -E\left[\frac{\partial^2}{\partial\beta^2}\ln f(X;\beta)\right] = -E\left[-\frac{1}{\beta^2}\right] = \frac{1}{\beta^2}.$$

Recall: $\hat{\hat{\beta}} = \dfrac{n-1}{\sum\limits_{i=1}^{n} \left(-\ln\left(1-X_i\right)\right)} = \dfrac{n-1}{Y}$ is an unbiased estimator for β.

o) Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

① Find $\operatorname{Var}(\hat{\hat{\beta}})$. (“Hint”: Recall **8** (f) of Homework #8.)

② Find the Rao-Cramér lower bound.

③ Is $\hat{\hat{\beta}}$ an efficient estimator of β? Does $\operatorname{Var}(\hat{\hat{\beta}})$ attain the R.C.L.B.?

If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

Recall (**8** (f) of Homework #8.):

$$\operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{(n-1)(n-2)} - \left(\frac{\beta}{n-1}\right)^2 = \frac{\beta^2}{(n-1)^2\,(n-2)}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \operatorname{Var}\left(\frac{n-1}{Y}\right) = (n-1)^2 \operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{n-2}.$$

$$\text{Rao-Cramer lower bound} = \frac{1}{n \cdot I(\beta)} = \frac{\beta^2}{n}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \frac{\beta^2}{n-2} > \frac{\beta^2}{n}.$$

$\operatorname{Var}(\hat{\hat{\beta}})$ does NOT attain its Rao-Cramer lower bound.

$\Rightarrow$ $\hat{\hat{\beta}}$ is NOT an efficient estimator of β,

its efficiency $= \dfrac{n-2}{n} \to 1$ as $n \to \infty$.

$\Rightarrow$ $\hat{\hat{\beta}}$ is an asymptotically efficient estimator of β.

9. Hagrid is concerned that, with Lord Voldemort's threat gone, the unicorns in the Forbidden Forest have gained extra weight, which would make it more difficult for them to elude future threats. The average weight for adult unicorns is supposed to be 1,200 pounds. Assume that the weight of adult unicorns in the Forbidden Forest is approximately normally distributed with the standard deviation $\sigma = 40$ pounds. Hagrid plans to obtain a random sample of $n = 49$ adult unicorns in order to test $H_0: \mu \le 1{,}200$ versus $H_0: \mu > 1{,}200$, where μ is the overall average weight of adult unicorns in the Forbidden Forest.

a) The average of unicorns in the sample was 1,212 pounds. Find the p-value of the test.

Test Statistic: $$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{1{,}212 - 1{,}200}{40 / \sqrt{49}} = \mathbf{2.10}.$$

$$\text{P - value} = P(Z \ge 2.10) = \mathbf{0.0179}.$$

b) Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha. \qquad Z = \frac{\overline{X} - 1{,}200}{40 / \sqrt{49}} > 1.645.$$

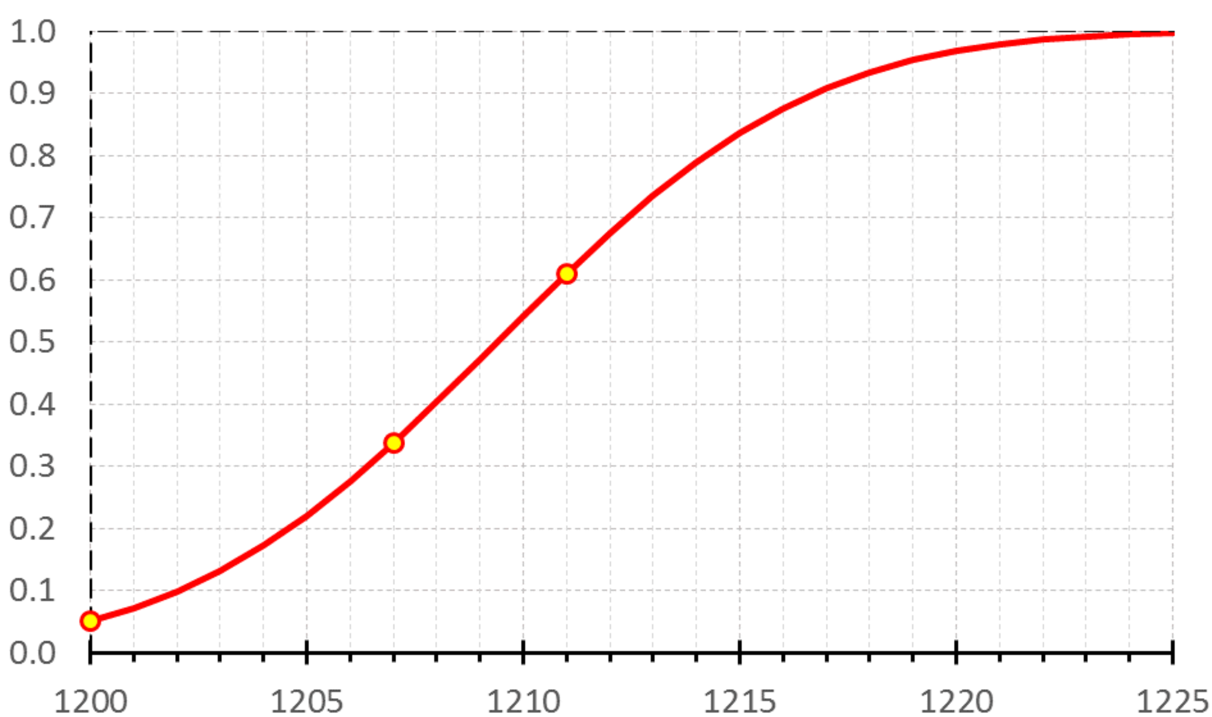
$$\overline{X} > 1{,}200 + 1.645 \cdot \frac{40}{\sqrt{49}} = \mathbf{1{,}209.4}.$$

c) Find the power of the test if the true value of the average weight of adult unicorns in the Forbidden Forest is (i) 1,207 pounds and (ii) 1,211 pounds, and a 5% level of significance is used.

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 1{,}209.4 \mid \mu).$$

$$P(\overline{X} > 1{,}209.4 \mid \mu = 1{,}207) = P\left(Z > \frac{1{,}209.4 - 1{,}207}{40/\sqrt{49}} \right) = P(Z > 0.42) = \mathbf{0.3372}.$$

$$P(\overline{X} > 1{,}209.4 \mid \mu = 1{,}211) = P\left(Z > \frac{1{,}209.4 - 1{,}211}{40/\sqrt{49}} \right) = P(Z > -0.28) = \mathbf{0.6103}.$$



10. 9. (continued)

We wish to test $H_0: \mu = 1{,}200$ versus $H_0: \mu \neq 1{,}200$.

a) The average of unicorns in the sample was 1,212 pounds. Find the p-value of the test.

Test Statistic: $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \dfrac{1{,}212 - 1{,}200}{40 / \sqrt{49}} = \mathbf{2.10}$.

P - value $=$ 2 tails $= 2 \times P(Z \geq 2.10) = 2 \times 0.0179 = \mathbf{0.0358}$.

b) Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} < -z_{\alpha/2} \quad \text{or} \quad Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_{\alpha/2}$$

$$\Rightarrow \quad \overline{X} < \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \overline{X} > \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \overline{X} < 1{,}200 - 1.96 \cdot \frac{40}{\sqrt{49}} \quad \text{or} \quad \overline{X} > 1{,}200 + 1.96 \cdot \frac{40}{\sqrt{49}}$$

$$\Rightarrow \quad \overline{X} < \mathbf{1{,}188.8} \quad \text{or} \quad \overline{X} > \mathbf{1{,}211.2}$$

c) Find the power of the test if the true value of the average weight of adult unicorns in the Forbidden Forest is (i) 1,194 pounds and (ii) 1,204 pounds, and a 5% level of significance is used.

$$\text{Power}(\mu) = P(\text{Reject } H_0 \mid H_0 \text{ is false})$$

$$= P(\overline{X} < 1{,}188.8 \mid \mu) + P(\overline{X} > 1{,}211.2 \mid \mu).$$

$$\text{Power}(\mu = 1{,}194) = P\left(Z < \frac{1{,}188.8 - 1{,}194}{40/\sqrt{49}}\right) + P\left(Z > \frac{1{,}211.2 - 1{,}194}{40/\sqrt{49}}\right)$$

$$= P(Z < -0.91) + P(Z > 3.01) = 0.1814 + 0.0013 = \mathbf{0.1827}.$$

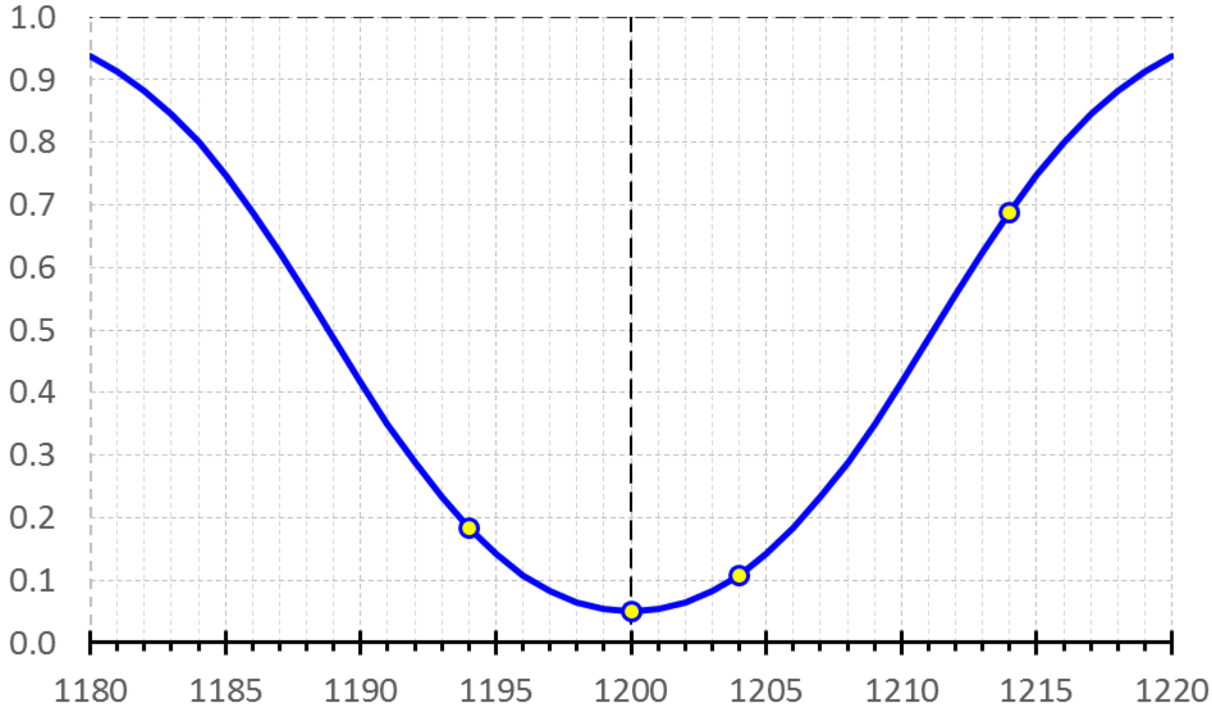
$$\text{Power}(\mu = 1{,}204) = P\left(Z < \frac{1{,}188.8 - 1{,}204}{40/\sqrt{49}}\right) + P\left(Z > \frac{1{,}211.2 - 1{,}204}{40/\sqrt{49}}\right)$$

$$= P(Z < -2.66) + P(Z > 1.26) = 0.0039 + 0.1038 = \mathbf{0.1077}.$$

For fun:

$$\text{Power}(\mu = 1{,}214) = P\left(Z < \frac{1{,}188.8 - 1{,}214}{40/\sqrt{49}}\right) + P\left(Z > \frac{1{,}211.2 - 1{,}214}{40/\sqrt{49}}\right)$$

$$= P(Z < -4.41) + P(Z > -0.49) = 0.0000 + 0.6879 = \mathbf{0.6879}.$$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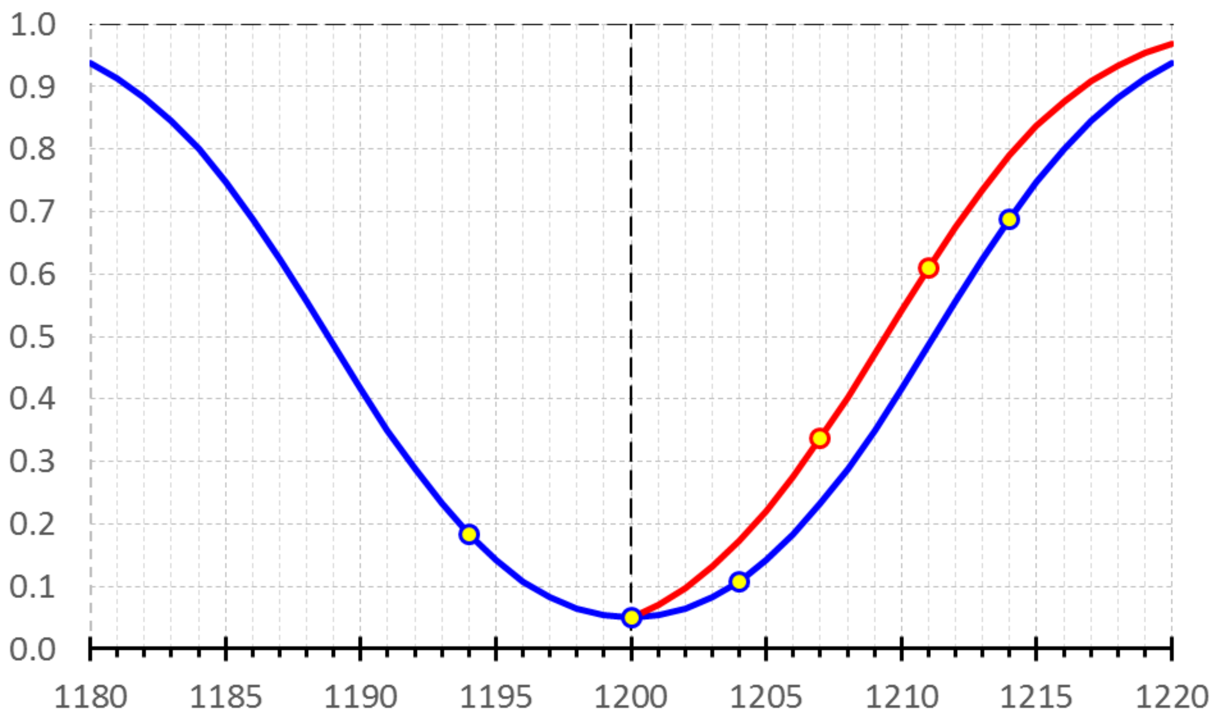
9. and **10.**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