Examples for 09/16/2020 (2)

1. Consider two continuous random variables X and Y with joint p.d.f.

$f_{X,Y}(x,y) = 60\,x^2 y, \qquad x > 0, \ y > 0, \ x + y < 1, \qquad\qquad$ zero elsewhere.

Consider $W = X + Y$. Find the p.d.f. of W, $f_W(w)$.

$$F_W(w) = P(W \le w) = \int_0^w \left(\int_0^{w-x} 60\,x^2\,y\,dy \right) dx = \int_0^w 30\,x^2 (w-x)^2\,dx$$

$$= 10\,w^5 - 15\,w^5 + 6\,w^5 = w^5, \qquad 0 < w < 1.$$

$$f_W(w) = 5\,w^4, \qquad 0 < w < 1.$$

2. When a person applies for citizenship in Neverland, first he/she must wait X years for an interview, and then Y more years for the oath ceremony. Thus the total wait is $W = X + Y$ years. Suppose that X and Y are independent, the p.d.f. of X is

$$f_X(x) = {}^2\!/_{x^3}, \qquad x > 1, \qquad \text{zero otherwise,}$$

and Y has a Uniform distribution on interval $(0, 1)$.

Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

"Hint": Consider two cases: $1 < w < 2$ and $w > 2$.

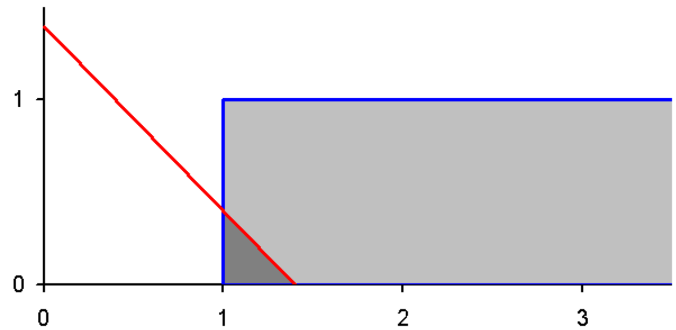
$f_X(x) = {}^2\!/_{x^3}, \qquad x > 1, \qquad$ zero otherwise,

$f_Y(y) = 1, \qquad 0 < y < 1, \qquad$ zero otherwise,

X and Y are independent.

$W = X + Y$.

Case 1: $1 < w < 2$.

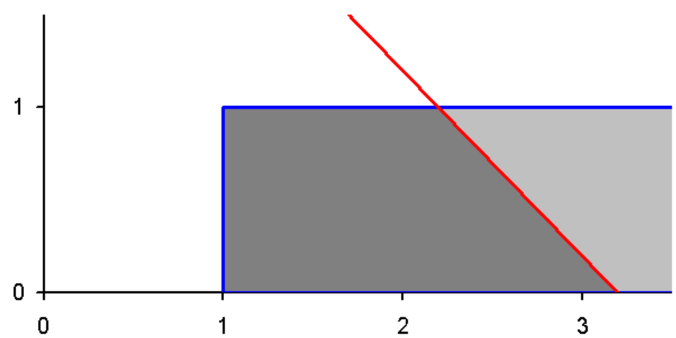


$$F_W(w) = P(X+Y \le w) = \int_1^w \left(\int_0^{w-x} \left(\frac{2}{x^3} \cdot 1 \right) dy \right) dx = \int_0^{w-1} \left(\int_1^{w-y} \left(\frac{2}{x^3} \cdot 1 \right) dx \right) dy$$

$$= \int_1^w \left(\frac{2w}{x^3} - \frac{2}{x^2} \right) dx = \left(-\frac{w}{x^2} + \frac{2}{x} \right) \Bigg|_1^w = \frac{1}{w} + w - 2, \qquad 1 < w < 2.$$

$$f_W(w) = 1 - \frac{1}{w^2}, \qquad 1 < w < 2.$$

Case 2: $w > 2$.



$$F_W(w) = P(X+Y \le w) = \int_0^1 \left(\int_1^{w-y} \left(\frac{2}{x^3} \cdot 1 \right) dx \right) dy = \int_0^1 \left(1 - \frac{1}{(w-y)^2} \right) dy$$

$$= \left(y - \frac{1}{w-y} \right) \Bigg|_0^1 = 1 - \frac{1}{w-1} + \frac{1}{w}, \qquad w > 2.$$

$$f_W(w) = \frac{1}{(w-1)^2} - \frac{1}{w^2}, \qquad w > 2.$$

Case 3: $w < 1$. $\qquad F_W(w) = 0. \qquad f_W(w) = 0.$

2. Suppose that X and Y are independent, the p.d.f. of X is

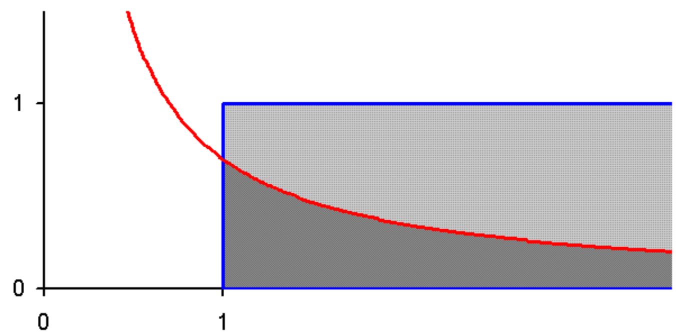
$$f_X(x) = {}^2/_{x^3}, \qquad x > 1, \qquad \text{zero otherwise},$$

and Y has a Uniform distribution on interval $(0, 1)$.

b) Let $V = X \times Y$. Find the p.d.f. of V, $f_V(v) = f_{X \times Y}(v)$.

"Hint": Consider two cases: $0 < v < 1$ and $v > 1$.

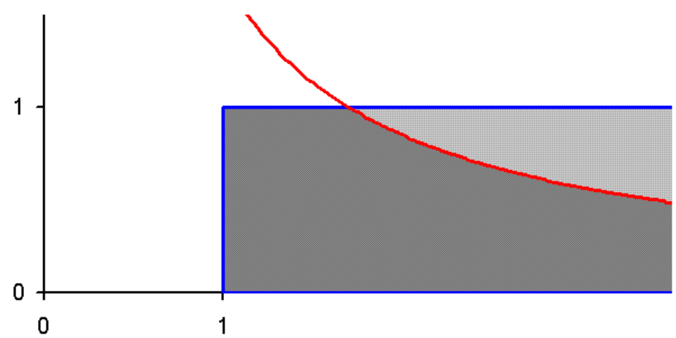
Case 1: $0 < v < 1$.



$$F_V(v) = P(XY \le v) = \int_1^{\infty} \left(\int_0^{v/x} \left(\frac{2}{x^3} \cdot 1 \right) dy \right) dx = \int_0^{v} \left(\int_1^{v/y} \left(\frac{2}{x^3} \cdot 1 \right) dx \right) dy = \frac{2}{3} v.$$

$$f_V(v) = \frac{2}{3}, \qquad 0 < v < 1.$$

Case 2: $v > 1$.



$$F_V(v) = P(XY \le v) = \int_0^{1} \left(\int_1^{v/y} \left(\frac{2}{x^3} \cdot 1 \right) dx \right) dy = 1 - \frac{1}{3v^2}.$$

$$f_V(v) = \frac{2}{3v^3}, \qquad v > 1.$$

Case 3: $v < 0$. $\qquad F_V(v) = 0. \qquad f_V(v) = 0.$

2. Suppose that X and Y are independent, the p.d.f. of X is

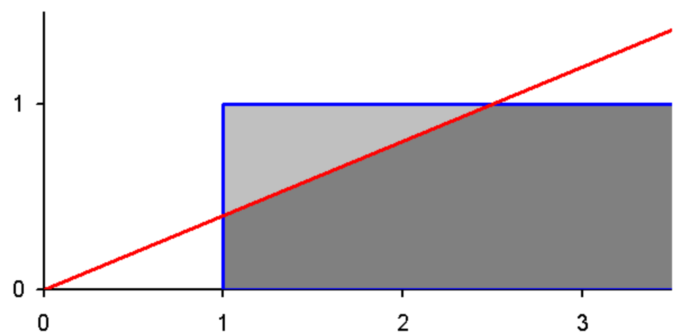
$$f_X(x) = {}^2/_{x^3}, \qquad x > 1, \qquad \text{zero otherwise,}$$

and Y has a Uniform distribution on interval $(0, 1)$.

c) Let $U = {}^Y/_X$. Find the c.d.f. and p.d.f. of U.

$x > 1, \ 0 < y < 1 \qquad \Rightarrow \qquad 0 < u < 1.$

$$F_U(u) = P({}^Y/_X \le u) = P(Y \le uX)$$



$$= 1 - \int_1^{1/u} \left(\int_{ux}^{1} \left(\frac{2}{x^3} \cdot 1 \right) dy \right) dx = 1 - \int_u^1 \left(\int_1^{y/u} \left(\frac{2}{x^3} \cdot 1 \right) dx \right) dy$$

$$= 1 - \int_1^{1/u} \left(\frac{2}{x^3} - \frac{2u}{x^2} \right) dx = 1 - \left(-\frac{1}{x^2} + \frac{2u}{x} \right) \Bigg|_1^{1/u} = 2u - u^2,$$

$0 < u < 1.$

$$f_U(u) = 2 - 2u, \qquad 0 < u < 1.$$

$F_U(u) = 0, \qquad f_U(u) = 0, \qquad u < 0.$

$F_U(u) = 1, \qquad f_U(u) = 0, \qquad u > 1.$