STAT 410

Practice Problems 9

Fall 2020
A. Stepanov

1. Let X_1, X_2, X_3, X_4, X_5 be a random sample (i.i.d.) of size $n = 5$ from the distribution with probability density function

$$f(x) = 3x^2, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find $P(\max X_i < 0.87)$.

b) $E(\max X_i)$.

c) Find $P(\min X_i < 0.65)$.

2. Let X_1 and X_2 be independent random variables with probability density functions

$$f_1(x) = 3x^2, \quad 0 < x < 1, \quad \text{zero otherwise,} \qquad f_2(x) = 2(1-x), \quad 0 < x < 1, \quad \text{zero otherwise,}$$

respectively.

a) Let $Y_1 = \min(X_1, X_2)$. Find $P(Y_1 < 0.60)$.

b) Let $Y_2 = \max(X_1, X_2)$. Find $P(Y_2 < 0.60)$.

c) Let $Y_2 = \max(X_1, X_2)$. Find $E(Y_2)$.

3. Let X_1, X_2, X_3 represent the independent failure times in years of three components in parallel. The respective p.d.f.s are

$$f_1(x) = 3x^2,\ 0 < x < 1; \quad f_2(x) = 4x^3,\ 0 < x < 1; \quad f_3(x) = 6x^5,\ 0 < x < 1.$$

a) Let $Y_3 = \max(X_1, X_2, X_3)$. Find the probability $P(Y_3 > 0.98)$.

b) Find the expected lifetime of the system, $E(Y_3)$.

c) Let $Y_1 = \min(X_1, X_2, X_3)$. Find the probability $P(Y_1 < 0.50)$.

4. Let X_1, X_2, X_3 be i.i.d. with probability mass function

$$p(k) = \frac{k}{10}, \qquad k = 1, 2, 3, 4.$$

a) Find the probability mass function of $Y_3 = \max(X_1, X_2, X_3)$.

b) Find the probability mass function of $Y_1 = \min(X_1, X_2, X_3)$.

5. **2.6.3** (7th and 6th edition)

Let X_1, X_2, X_3, and X_4 be four independent random variables, each with pdf $f_X(x) = 3(1-x)^2$, $0 < x < 1$, zero elsewhere. If Y is the minimum of these four variables, find the cdf and the pdf of Y.

Hint: $\quad P(Y > y) = P(X_i > y,\ i = 1, \dots, 4)$.

6. **2.6.4** (7th and 6th edition)

A fair die is cast at random three independent times. Let the random variable X_i be equal to the number of spots that appear on the ith trial, $i = 1, 2, 3$. Let the random variable Y be equal to $\max(X_i)$. Find the cdf and the pmf of Y.

Hint: $\quad P(Y \le y) = P(X_i \le y,\ i = 1, 2, 3)$.

7. Let X and Y be independent Geometric random variables with the probabilities of "success" $1/4$ and $1/5$, respectively.

a) Find $P(X = Y)$.

b) Let $W = \min(X, Y)$. What is the probability distribution of W?

8. Four components are placed in a series (that is, the system fails with the failure of one of the components). The time in hours to failure of each component has the p.d.f.

$$f(x) = \frac{2x}{5^2} e^{-(x/5)^2}, \qquad 0 < x < \infty.$$

Since they are in a series, we are concerned about the minimum time Y to failure of the four. Assuming independence, find the probability $P(Y < 3)$.

b) Let W denote the maximum time to failure of the four components (in parallel). Find the probability $P(W < 7)$.

9. Three components are placed in a series (that is, the system fails with the failure of one of the components). The time in hours to failure of each component has the p.d.f.

$$f(x) = \frac{x}{5^2} e^{-(x/5)}, \qquad 0 < x < \infty.$$

Since they are in a series, we are concerned about the minimum time Y to failure of the three. Assuming independence, find the probability $P(Y < 3)$.

b) Let W denote the maximum time to failure of the three components (in parallel). Find the probability $P(W < 7)$.

10. Suppose two independent claims are made on two insured cars, where each claim has p.d.f.

$$f(x) = 5/x^6, \qquad x > 1,$$

in which the unit is \$1000.

a) Find the expected value of the smaller claim.
That is, let $Y_1 = \min(X_1, X_2)$. Find $E(Y_1)$.

b) Find the expected value of the larger claim.
That is, let $Y_2 = \max(X_1, X_2)$. Find $E(Y_2)$.

11 – 12. Dick and Jane have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Jane's arrival time by X, Dick's by Y, and suppose X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

11. a) Let T_1 denote the arrival time of the person who arrives first. Find the p.d.f. of T_1.

b) Let T_2 denote the arrival time of the person who arrives second. Find the p.d.f. of T_2.

c) What is the expected amount of time that the one who arrives first must wait for the person who arrives second?

12. Let W denote the waiting time, the time that the person who arrives first must wait for the person who arrives second. Find the p.d.f. of W, $f_W(w)$.

Suggestion: Find $1 - F_W(w) = P(W > w) = P(|X - Y| > w), \ 0 \le w \le 1$.

13. Let X and Y be two independent random variables, with probability density functions $f_X(x)$ and $f_Y(y)$, respectively.

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the p.d.f. $f_W(w)$ of $W = X + Y$.

14. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y) = \begin{cases} x\,e^{-x(1+y)} & x \ge 0,\, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the probability density function of $\mathrm{U} = \min(\mathrm{X}, \mathrm{Y})$, $f_{\mathrm{U}}(u)$.

b) Find the probability density function of $\mathrm{V} = \max(\mathrm{X}, \mathrm{Y})$, $f_{\mathrm{V}}(v)$.

15. Let X_1 and X_2 be i.i.d. with the probability density function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathrm{Y} = \max(\mathrm{X}_1, \mathrm{X}_2)$. Find $\mathrm{E}(\mathrm{Y})$.

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Answers:

1. Let X_1, X_2, X_3, X_4, X_5 be a random sample (i.i.d.) of size $n = 5$ from the distribution with probability density function

$$f(x) = 3x^2, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find $P(\max X_i < 0.87)$.

$$P(\max X_i < 0.87) = [P(X < 0.87)]^5 = [(0.87)^3]^5 = 0.87^{15} \approx \mathbf{0.1238}.$$

b) $E(\max X_i)$.

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \ldots \cdot P(X_n \le x) = (F(x))^n$$

$$= (x^3)^5 = x^{15}, \qquad 0 < x < 1.$$

$$f_{\max X_i}(x) = 15x^{14}, \qquad 0 < x < 1.$$

$$E(\max X_i) = \int_0^1 x \cdot 15x^{14}\, dx = \mathbf{\frac{15}{16}}.$$

c) Find $P(\min X_i < 0.65)$.

$$P(\min X_i < 0.65) = 1 - P(\min X_i \ge 0.65) = 1 - [P(X \ge 0.65)]^5$$

$$= 1 - [1 - (0.65)^3]^5 \approx \mathbf{0.7992}.$$

2. Let X_1 and X_2 be independent random variables with probability density functions

$$f_1(x) = 3x^2, \quad 0 < x < 1, \qquad\qquad f_2(x) = 2(1-x), \quad 0 < x < 1,$$

zero otherwise, zero otherwise,

respectively.

a) Let $Y_1 = \min(X_1, X_2)$. Find $P(Y_1 < 0.60)$.

$$F_1(x) = P(X_1 \le x) = x^3, \qquad 0 < x < 1.$$

$$F_2(x) = P(X_2 \le x) = 2x - x^2, \qquad 0 < x < 1.$$

$$P(Y_1 < x) = 1 - P(\min(X_1, X_2) \ge x) = 1 - P(X_1 \ge x) \cdot P(X_2 \ge x)$$

$$= 1 - (1 - x^3) \cdot (1 - 2x + x^2) = 1 - (1 - x^3) \cdot (1 - x)^2, \qquad 0 < x < 1.$$

$$P(Y_1 < 0.60) = 1 - (1 - 0.6^3) \cdot (1 - 0.6)^2 = \mathbf{0.87456}.$$

b) Let $Y_2 = \max(X_1, X_2)$. Find $P(Y_2 < 0.60)$.

$$F_{Y_2}(x) = P(\max(X_1, X_2) \le x) = P(X_1 \le x) \cdot P(X_2 \le x)$$

$$= x^3 \cdot (2x - x^2) = 2x^4 - x^5, \qquad 0 < x < 1.$$

$$P(Y_2 < 0.60) = 2 \cdot 0.6^4 - 0.6^5 = \mathbf{0.18144}.$$

c) Let $Y_2 = \max(X_1, X_2)$. Find $E(Y_2)$.

$$f_{Y_2}(x) = 8x^3 - 5x^4, \qquad 0 < x < 1.$$

$$E(Y_2) = \int_0^1 x \cdot \left(8x^3 - 5x^4\right) dx = \frac{8}{5} - \frac{5}{6} = \mathbf{\frac{23}{30}}.$$

3. Let X_1, X_2, X_3 represent the independent failure times in years of three components in parallel. The respective p.d.f.s are

$$f_1(x) = 3x^2,\ 0 < x < 1; \quad f_2(x) = 4x^3,\ 0 < x < 1; \quad f_3(x) = 6x^5,\ 0 < x < 1.$$

a) Let $Y_3 = \max(X_1, X_2, X_3)$. Find the probability $P(Y_3 > 0.98)$.

$$F_{Y_3}(x) = F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \dots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \dots \cdot P(X_n \le x)$$

$$= x^3 \cdot x^4 \cdot x^6 = x^{13}, \qquad 0 < x < 1.$$

$$f_{Y_3}(x) = 13\,x^{12}, \quad 0 < x < 1.$$

$$P(Y_3 > 0.98) = 1 - F_{Y_3}(0.98) = 1 - 0.98^{13} \approx \mathbf{0.2310}.$$

b) Find the expected lifetime of the system, $E(Y_3)$.

$$E(Y_3) = \int_0^1 x \cdot 13\,x^{12}\,dx = \mathbf{\frac{13}{14}}.$$

c) Let $Y_1 = \min(X_1, X_2, X_3)$. Find the probability $P(Y_1 < 0.50)$.

$$1 - F_{Y_1}(x) = 1 - F_{\min X_i}(x) = P(\min X_i > x) = P(X_1 > x, X_2 > x, \dots, X_n > x)$$

$$= P(X_1 > x) \cdot P(X_2 > x) \cdot \dots \cdot P(X_n > x)$$

$$= (1 - x^3) \cdot (1 - x^4) \cdot (1 - x^6), \qquad 0 < x < 1.$$

$$P(Y_1 < 0.50) = 1 - (1 - 0.50^3) \cdot (1 - 0.50^4) \cdot (1 - 0.50^6) \approx \mathbf{0.1925}.$$

4. Let X_1, X_2, X_3 be i.i.d. with probability mass function

$$p(k) = \frac{k}{10}, \qquad k = 1, 2, 3, 4.$$

$F_X(1) = 0.1, \qquad F_X(2) = 0.3, \qquad F_X(3) = 0.6, \qquad F_X(4) = 1.0.$

a) Find the probability mass function of $Y_3 = \max(X_1, X_2, X_3)$.

$$F_{\max X_i}(x) = (F_X(x))^n = (F_X(x))^3.$$

$\Rightarrow \quad F_{\max X_i}(1) = 0.001, \qquad F_{\max X_i}(2) = 0.027,$

$F_{\max X_i}(3) = 0.216, \qquad F_{\max X_i}(4) = 1.000.$

$\Rightarrow \quad P(\max X_i = 1) = \mathbf{0.001}, \qquad P(\max X_i = 2) = \mathbf{0.026},$

$P(\max X_i = 3) = \mathbf{0.189}, \qquad P(\max X_i = 4) = \mathbf{0.784}.$

b) Find the probability mass function of $Y_1 = \min(X_1, X_2, X_3)$.

$$F_{\min X_i}(x) = 1 - (1 - F_X(x))^n = 1 - (1 - F_X(x))^3.$$

$\Rightarrow \quad F_{\min X_i}(1) = 0.271, \qquad F_{\min X_i}(2) = 0.657,$

$F_{\min X_i}(3) = 0.936, \qquad F_{\min X_i}(4) = 1.000.$

$\Rightarrow \quad P(\min X_i = 1) = \mathbf{0.271}, \qquad P(\min X_i = 2) = \mathbf{0.386},$

$P(\min X_i = 3) = \mathbf{0.279}, \qquad P(\min X_i = 4) = \mathbf{0.064}.$

5. **2.6.3** (7th and 6th edition)

Let X_1, X_2, X_3, and X_4 be four independent random variables, each with pdf $f_X(x) = 3(1-x)^2$, $0 < x < 1$, zero elsewhere. If Y is the minimum of these four variables, find the cdf and the pdf of Y.

Hint: $P(Y > y) = P(X_i > y,\ i = 1, \dots, 4)$.

$$f_X(x) = 3(1-x)^2, \qquad 0 < x < 1.$$

$$1 - F_X(x) = \int_x^1 3(1-y)^2\,dy = -(1-y)^3 \Big|_x^1 = (1-x)^3, \qquad 0 < x < 1.$$

$$1 - F_{\min X_i}(x) = P(\min X_i > x) = P(X_1 > x, X_2 > x, \dots, X_n > x)$$

$$= P(X_1 > x) \cdot P(X_2 > x) \cdot \dots \cdot P(X_n > x) = (1 - F_X(x))^n.$$

$$F_{\min X_i}(x) = 1 - (1 - F_X(x))^n.$$

$$f_{\min X_i}(x) = F'_{\min X_i}(x) = n \cdot (1 - F_X(x))^{n-1} \cdot f_X(x).$$

$$1 - F_Y(y) = P(Y > y) = P(X_1 > y, X_2 > y, X_3 > y, X_4 > y) = (1-y)^{12},$$

$$0 < y < 1.$$

c.d.f. $\quad F_Y(y) = 1 - (1-y)^{12}, \qquad 0 < y < 1.$

p.d.f. $\quad f_Y(y) = F'_Y(y) = 12(1-y)^{11}, \qquad 0 < y < 1.$

6. **2.6.4** (7th and 6th edition)

A fair die is cast at random three independent times. Let the random variable X_i be equal to the number of spots that appear on the ith trial, $i = 1, 2, 3$. Let the random variable Y be equal to $\max(X_i)$. Find the cdf and the pmf of Y.

Hint: $P(Y \le y) = P(X_i \le y,\ i = 1, 2, 3)$.

$p_X(x) = {}^1\!/_6$, $\quad x = 1, 2, 3, 4, 5, 6.$

$F_X(x) = {}^x\!/_6$, $\quad x = 0, 1, 2, 3, 4, 5, 6.$

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \dots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \dots \cdot P(X_n \le x) = (F_X(x))^n.$$

c.d.f. $$F_Y(y) = P(Y \le y) = P(X_1 \le y, X_2 \le y, X_3 \le y) = \left(\frac{y}{6}\right)^3 = \frac{y^3}{6^3},$$

$y = 0, 1, 2, 3, 4, 5, 6.$

p.m.f. $$p_Y(y) = P(Y \le y) = F_Y(y) - F_Y(y-1) = \frac{y^3 - (y-1)^3}{6^3},$$

$y = 1, 2, 3, 4, 5, 6.$

y	$F_Y(y)$
1	$^1/_{216}$
2	$^8/_{216}$
3	$^{27}/_{216}$
4	$^{64}/_{216}$
5	$^{125}/_{216}$
6	$^{216}/_{216}$

y	$p_Y(y)$
1	$^1/_{216}$
2	$^7/_{216}$
3	$^{19}/_{216}$
4	$^{37}/_{216}$
5	$^{61}/_{216}$
6	$^{91}/_{216}$

7. Let X and Y be independent Geometric random variables with the probabilities of "success" $^1/_4$ and $^1/_5$, respectively.

$$p_X(x) = \left(\frac{1}{4}\right) \cdot \left(\frac{3}{4}\right)^{x-1}, \quad x = 1, 2, 3, \dots,$$

$$p_Y(y) = \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^{y-1}, \quad y = 1, 2, 3, \dots.$$

a) Find $P(X = Y)$.

$$P(X = Y) = \sum_{k=1}^{\infty} p_X(k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right) \cdot \left(\frac{3}{4}\right)^{k-1} \cdot \left(\frac{1}{5}\right) \cdot \left(\frac{4}{5}\right)^{k-1}$$

$$= \left(\frac{1}{4} \cdot \frac{1}{5}\right) \cdot \sum_{k=1}^{\infty} \left(\frac{3}{4} \cdot \frac{4}{5}\right)^{k-1} = \left(\frac{1}{20}\right) \cdot \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1} = \left(\frac{1}{20}\right) \cdot \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^{k}$$

$$= \left(\frac{1}{20}\right) \cdot \frac{1}{1 - \frac{3}{5}} = \mathbf{\frac{1}{8}}.$$

b) Let $W = \min(X, Y)$. What is the probability distribution of W?

$$P(X > x) = \left(\frac{3}{4}\right)^{x}, \quad x = 0, 1, 2, 3, \dots, \qquad P(Y > y) = \left(\frac{4}{5}\right)^{y}, \quad y = 0, 1, 2, 3, \dots.$$

$$P(W > w) = P(X > w \cap Y > w) = P(X > w) \cdot P(Y > w)$$

$$= \left(\frac{3}{4}\right)^{w} \cdot \left(\frac{4}{5}\right)^{w} = \left(\frac{3}{5}\right)^{w}, \qquad w = 0, 1, 2, 3, \dots.$$

$$F_W(w) = 1 - P(W > w) = 1 - \left(\frac{3}{5}\right)^{w}, \qquad w = 0, 1, 2, 3, \dots.$$

$$p_W(w) = F_W(w) - F_W(w-1) = \left(\frac{2}{5}\right) \cdot \left(\frac{3}{5}\right)^{w-1}, \qquad w = 1, 2, 3, \ldots .$$

W is a Geometric random variable with the probability of "success" $2/5$.

OR

Geometric distribution describes the number of independent attempts needed to get the first "success".

$\{ W = w \} = \{$ either X or Y (or both) have the first "success" on attempt # w $\}$

$= \{ (w-1)$ "failures" for both X and Y AND then "success" for either X or Y (or both) $\}$

$$P(W = w) = \left(\frac{3}{4}\right)^{w-1} \cdot \left(\frac{4}{5}\right)^{w-1} \cdot \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{4} \cdot \frac{1}{5}\right)$$

$$= \left(\frac{3}{5}\right)^{w-1} \cdot \left(\frac{8}{20}\right) = \left(\frac{3}{5}\right)^{w-1} \cdot \left(\frac{2}{5}\right), \qquad w = 1, 2, 3, \ldots .$$

W is a Geometric random variable with the probability of "success" $2/5$.

8. Four components are placed in a series (that is, the system fails with the failure of one of the components). The time in hours to failure of each component has the p.d.f.

$$f(x) = \frac{2x}{5^2} e^{-(x/5)^2}, \qquad 0 < x < \infty.$$

Since they are in a series, we are concerned about the minimum time Y to failure of the four. Assuming independence, find the probability $P(Y < 3)$.

$$F_X(x) = 1 - e^{-(x/5)^2}, \qquad 0 < x < \infty.$$

$$1 - F_{\min X_i}(x) = P(\min X_i > x) = P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

$$= P(X_1 > x) \cdot P(X_2 > x) \cdot \ldots \cdot P(X_n > x) = (1 - F_X(x))^n$$

$$= \left[e^{-(x/5)^2} \right]^4 = e^{-4(x/5)^2}, \qquad 0 < x < \infty.$$

$$F_Y(x) = 1 - e^{-4(x/5)^2}, \qquad 0 < x < \infty.$$

$$P(Y < 3) = 1 - e^{-4(3/5)^2} = 1 - e^{-1.44} \approx 0.7631.$$

b) Let W denote the maximum time to failure of the four components (in parallel). Find the probability $P(W < 7)$.

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \ldots \cdot P(X_n \le x) = (F(x))^n$$

$$= \left[1 - e^{-(x/5)^2} \right]^4, \qquad 0 < x < \infty.$$

$$P(W < 7) = \left[1 - e^{-(7/5)^2} \right]^4 \approx 0.5448.$$

9. Three components are placed in a series (that is, the system fails with the failure of one of the components). The time in hours to failure of each component has the p.d.f.

$$f(x) = \frac{x}{5^2} e^{-(x/5)}, \qquad 0 < x < \infty.$$

Since they are in a series, we are concerned about the minimum time Y to failure of the three. Assuming independence, find the probability $P(Y < 3)$.

$$F_X(x) = 1 - e^{-(x/5)} - \frac{x}{5} e^{-(x/5)}, \qquad 0 < x < \infty.$$

$$1 - F_{\min X_i}(x) = P(\min X_i > x) = P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

$$= P(X_1 > x) \cdot P(X_2 > x) \cdot \ldots \cdot P(X_n > x) = (1 - F_X(x))^n$$

$$= \left[e^{-(x/5)} + \frac{x}{5} e^{-(x/5)} \right]^3, \qquad 0 < x < \infty.$$

$$F_Y(x) = 1 - \left[e^{-(x/5)} + \frac{x}{5} e^{-(x/5)} \right]^3, \qquad 0 < x < \infty.$$

$$P(Y < 3) = 1 - \left[\frac{8}{5} e^{-(3/5)} \right]^3 = 1 - \left[1.6\, e^{-0.6} \right]^3 \approx 0.3229.$$

b) Let W denote the maximum time to failure of the three components (in parallel). Find the probability $P(W < 7)$.

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \ldots \cdot P(X_n \le x) = (F(x))^n$$

$$= \left[1 - e^{-(x/5)} - \frac{x}{5} e^{-(x/5)} \right]^3, \qquad 0 < x < \infty.$$

$$P(W < 7) = \left[1 - \frac{12}{5} e^{-(7/5)} \right]^3 \approx 0.0680.$$

10. Suppose two independent claims are made on two insured cars, where each claim has p.d.f.

$$f(x) = 5/x^6, \qquad x > 1,$$

in which the unit is \$1000.

a) Find the expected value of the smaller claim.
That is, let $Y_1 = \min(X_1, X_2)$. Find $E(Y_1)$.

$$F(x) = \int_1^x \frac{5}{y^6}\,dy = -\frac{1}{y^5}\Bigg|_1^x = 1 - \frac{1}{x^5}, \qquad x > 1.$$

$$f_{\min X_i}(x) = n \cdot (1 - F(x))^{n-1} \cdot f(x) = 2 \cdot \left(\frac{1}{x^5}\right)^{2-1} \cdot \frac{5}{x^6} = \frac{10}{x^{11}}, \qquad x > 1.$$

$$E(Y_1) = \int_1^\infty x \cdot \frac{10}{x^{11}}\,dx = \int_1^\infty \frac{10}{x^{10}}\,dx = \mathbf{\frac{10}{9}}.$$

b) Find the expected value of the larger claim.
That is, let $Y_2 = \max(X_1, X_2)$. Find $E(Y_2)$.

$$f_{\max X_i}(x) = n \cdot (F(x))^{n-1} \cdot f(x) = 2 \cdot \left(1 - \frac{1}{x^5}\right)^{2-1} \cdot \frac{5}{x^6} = \frac{10}{x^6} - \frac{10}{x^{11}}, \qquad x > 1.$$

$$E(Y_2) = \int_1^\infty x \cdot \left(\frac{10}{x^6} - \frac{10}{x^{11}}\right) dx = \int_1^\infty \left(\frac{10}{x^5} - \frac{10}{x^{10}}\right) dx = \frac{10}{4} - \frac{10}{9} = \mathbf{\frac{25}{18}}.$$

OR

$$E(Y_1 + Y_2) = E(X_1 + X_2) = 2 \times E(X) = 2 \times \frac{5}{4} = \frac{5}{2}.$$

$$E(Y_1) = \frac{10}{9}. \qquad \Rightarrow \qquad E(Y_2) = \frac{5}{2} - \frac{10}{9} = \mathbf{\frac{25}{18}}.$$

11 – 12. Dick and Jane have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Jane's arrival time by X, Dick's by Y, and suppose X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

11. a) Let T_1 denote the arrival time of the person who arrives first. Find the p.d.f. of T_1.

T_1 = time the first person arrives = $\min(X, Y)$,

$$1 - F_{T_1}(t) = P(T_1 > t) = P(X > t, Y > t) = P(X > t) \times P(Y > t)$$

$$= (1 - t^3) \times (1 - t^2) = 1 - t^2 - t^3 + t^5, \qquad 0 < t < 1.$$

$$F_{T_1}(t) = t^2 + t^3 - t^5, \qquad 0 < t < 1.$$

$$f_{T_1}(t) = 2t + 3t^2 - 5t^4, \qquad 0 < t < 1.$$

b) Let T_2 denote the arrival time of the person who arrives second. Find the p.d.f. of T_2.

T_2 = time the second person arrives = $\max(X, Y)$,

$$F_{T_2}(t) = P(T_2 \le t) = P(X \le t, Y \le t) = P(X \le t) \times P(Y \le t)$$

$$= t^3 \times t^2 = t^5, \qquad 0 < t < 1.$$

$$f_{T_2}(t) = 5t^4, \qquad 0 < t < 1.$$

c) What is the expected amount of time that the one who arrives first must wait for the person who arrives second?

waiting time $= T_2 - T_1$.

$E(T_1) = \frac{7}{12}$, $E(T_2) = \frac{5}{6}$,

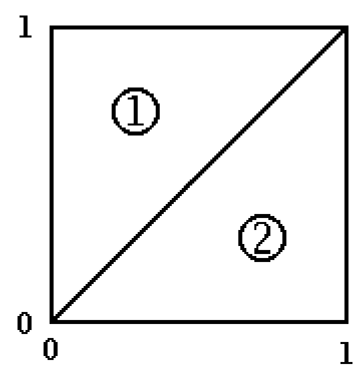
$E(\text{waiting time}) = E(T_2) - E(T_1) = \mathbf{\frac{1}{4}}$ hour $=$ **15** minutes.

OR

$f(x, y) = 6\,x^2 y,\qquad 0 \le x \le 1,\quad 0 \le y \le 1.$

① $y > x$ Jane is waiting for Dick
waiting time $= y - x$

② $x > y$ Dick is waiting for Jane
waiting time $= x - y$



$$\int_0^1 \left(\int_0^y (y-x)\cdot 6\,x^2 y\,dx \right) dy + \int_0^1 \left(\int_0^x (x-y)\cdot 6\,x^2 y\,dy \right) dx$$

$$= \frac{1}{12} + \frac{1}{6} = \mathbf{\frac{1}{4}}$$ hour $=$ **15** minutes.

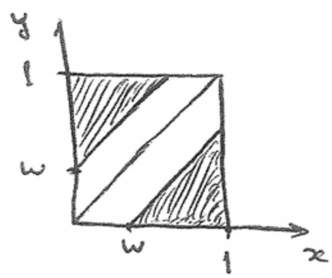
12. Let W denote the waiting time, the time that the person who arrives first must wait for the person who arrives second. Find the p.d.f. of W, $f_W(w)$.

Suggestion: Find $1 - F_W(w) = P(W > w) = P(|X - Y| > w), \quad 0 \le w \le 1.$

$$f(x,y) = 6x^2 y, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

$$W = |X - Y|.$$

$$1 - F_W(w) = P(|X - Y| > w)$$



$$= \int_w^1 \left(\int_0^{y-w} 6x^2 y \, dx \right) dy + \int_w^1 \left(\int_0^{x-w} 6x^2 y \, dy \right) dx$$

$$= \int_w^1 2y(y-w)^3 \, dy + \int_w^1 3x^2 (x-w)^2 \, dx$$

$$= \frac{(1-w)^4 (4+w)}{10} + \frac{(1-w)^3 (w^2 + 3w + 6)}{10}$$

$$= (1-w)^3, \qquad 0 \le w \le 1.$$

$$F_W(w) = 1 - (1-w)^3, \qquad 0 \le w \le 1.$$

$$f_W(w) = F_W'(w) = 3(1-w)^2, \qquad 0 \le w \le 1.$$

13. Let X and Y be two independent random variables, with probability density functions $f_X(x)$ and $f_Y(y)$, respectively.

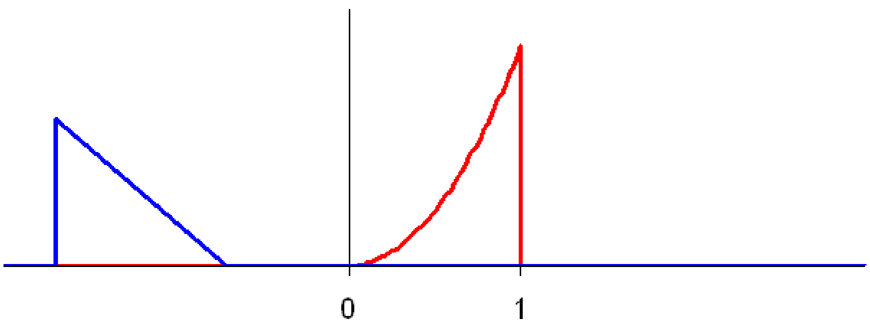
$$f_X(x)=\begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y)=\begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the p.d.f. $f_W(w)$ of $W = X + Y$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx.$$

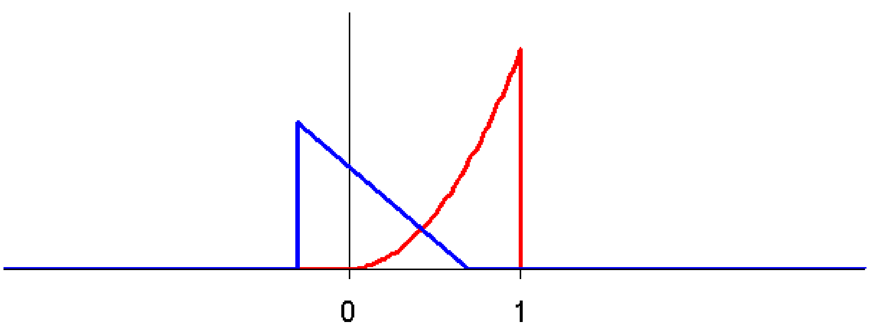
$$f_Y(w-x) = \begin{cases} 2(w-x) & \text{if } 0 \le w-x \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2(w-x) & \text{if } w-1 \le x \le w \\ 0 & \text{otherwise} \end{cases}$$

Case 1. $w < 0$.



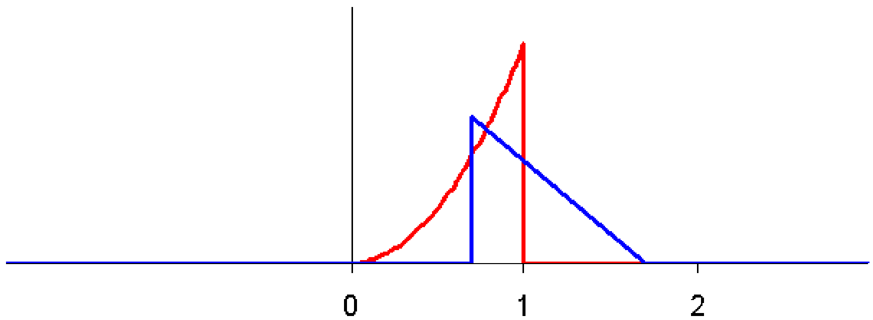
$$f_{X+Y}(w) = 0.$$

Case 2. $0 < w < 1$.



$$f_{X+Y}(w) = \int_0^w 3x^2\,2(w-x)\,dx = \left(2x^3 w - \frac{3}{2}x^4\right)\Bigg|_{x=0}^{x=w} = \frac{1}{2}w^4.$$

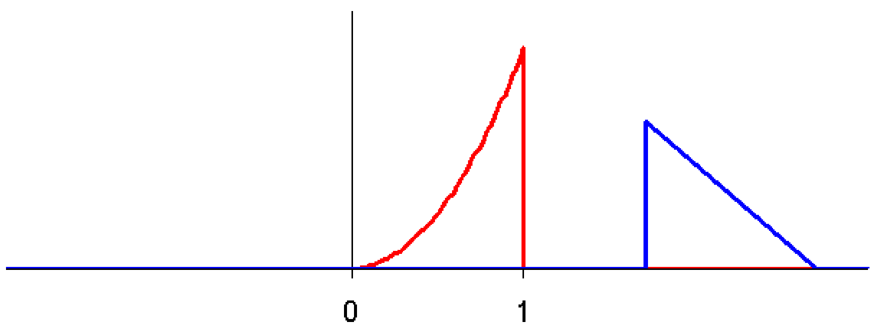
Case 3. $1 < w < 2$.



$$f_{X+Y}(w) = \int_{w-1}^{1} 3x^2\, 2(w-x)\,dx = \left(2x^3 w - \frac{3}{2}x^4 \right)\Bigg|_{x=w-1}^{x=1}$$

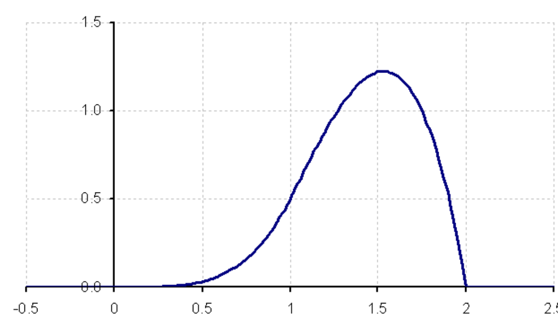
$$= 2w - \frac{3}{2} - 2(w-1)^3 w + \frac{3}{2}(w-1)^4 = -\frac{1}{2}w^4 + 3w^2 - 2w.$$

Case 4. $w > 2$.



$$f_{X+Y}(w) = 0.$$

$$f_{X+Y}(w) = \begin{cases} \frac{1}{2}w^4 & 0 < w < 1 \\ -\frac{1}{2}w^4 + 3w^2 - 2w & 1 < w < 2 \\ 0 & \text{otherwise} \end{cases}$$



14. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y) = \begin{cases} x\,e^{-x(1+y)} & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the probability density function of $U = \min(X, Y)$, $f_U(u)$.

$$P(\min(X,Y) > u) = P(X > u, Y > u)$$

$$= \int_u^\infty \int_u^\infty x e^{-x} e^{-xy}\, dy\, dx$$

$$= \int_u^\infty x e^{-x} \left(-\frac{1}{x} e^{-xy} \right) \Big|_u^\infty dx$$

$$= \int_u^\infty e^{-(1+u)x}\, dx = -\frac{1}{1+u} e^{-(1+u)x} \Big|_u^\infty$$

$$= \frac{1}{1+u} e^{-(1+u)u} = 1 - F_U(u), \quad u > 0.$$

$$F_U(u) = 1 - \frac{1}{1+u} e^{-(1+u)u}, \quad u > 0$$

$$f_U(u) = F_U'(u) = \frac{1}{(1+u)^2} e^{-(1+u)u} + \frac{2u+1}{1+u} e^{-(1+u)u}$$

$$= \frac{2u^2 + 3u + 2}{(u+1)^2} e^{-(1+u)u}, \quad u > 0.$$

b) Find the probability density function of $V = \max(X, Y)$, $f_V(v)$.

$$F_V(v) = P(\max(X,Y) \le v) = P(X \le v, Y \le v)$$

$$= \int_0^v \int_0^v x e^{-x} e^{-xy} \, dy \, dx$$

$$= \int_0^v \left(-e^{-x} e^{-xy}\right)\Big|_0^v \, dx$$

$$= \int_0^v \left(e^{-x} - e^{-(1+v)x}\right) dx$$

$$= 1 - e^{-v} - \frac{1}{v+1} + \frac{1}{v+1} e^{-(1+v)v}, \quad v > 0.$$

$$f_V(v) = e^{-v} + \frac{1}{(v+1)^2} - \frac{2v^2+3v+2}{(v+1)^2} e^{-(1+v)v}, \quad v > 0.$$

15. Let X_1 and X_2 be i.i.d. with the probability density function

$$f(x)=\begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y=\max(X_1, X_2)$. Find $E(Y)$.

$$F(x)=\frac{x^2}{2}, \qquad 0<x<1.$$

$$F(x)=\frac{1}{2}+\int_1^x (2-y)\,dy = -\frac{x^2}{2}+2x-1, \qquad 1<x<2.$$

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a continuous distribution with cumulative distribution function $F(x)$ and probability density function $f(x)$, then

$$f_{\max X_i}(x) = n\cdot(F(x))^{n-1}\cdot f(x).$$

$$f_Y(x)=2\cdot\left(\frac{x^2}{2}\right)\cdot x = x^3, \qquad 0<x<1.$$

$$f_Y(x)=2\cdot\left(-\frac{x^2}{2}+2x-1\right)\cdot(2-x)$$

$$= x^3-6x^2+10x-4, \qquad 1<x<2.$$

$$E(Y)=\int_0^1 x\cdot x^3dx+\int_1^2 x\cdot\left(x^3-6x^2+10x-4\right)dx$$

$$=\left[\frac{x^5}{5}\right]_0^1+\left[\frac{x^5}{5}-\frac{6x^4}{4}+\frac{10x^3}{3}-\frac{4x^2}{2}\right]_1^2$$

$$=\frac{1}{5}+\left[\frac{32}{5}-24+\frac{80}{3}-8\right]-\left[\frac{1}{5}-\frac{3}{2}+\frac{10}{3}-2\right] = \mathbf{\frac{37}{30}}.$$