Examples for 11/06/2020 (7)

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \left(\theta^2 + \theta\right) x^{\theta-1}(1-x), \qquad 0 < x < 1, \qquad \theta > 0.$$

a) Obtain a method of moments estimator of θ, $\tilde{\theta}$.

b) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$.
Find a method of moments estimate of θ.

c) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

d) Is $\tilde{\theta}$ a consistent estimator of θ? *Justify your answer.*

e) Show that $\tilde{\theta}$ is asymptotically normally distributed (as $n \to \infty$).
Find the parameters.

f) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

That is, find $\hat{\theta} = \arg\max L(\theta) = \arg\max \ln L(\theta)$,

where $L(\theta) = \prod_{i=1}^{n} f\left(x_i ;\theta\right)$.

"Hint": ① $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;

② $\theta > 0$;

③ Since $0 < x < 1$, $\ln x < 0$.

g) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$.
Find the maximum likelihood estimate of θ.

h) Let $Y_1 < Y_2 < \ldots < Y_n$ denote the corresponding order statistics.

Find β so that $W_n = n^{\beta} Y_1$ converges in distribution.

Find the limiting distribution of W_n.

i) Find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for θ.

2. Let $\theta > 1$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{1}{x \ln \theta}, \qquad 1 < x < \theta.$$

a) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

b) Is $\hat{\theta}$ an unbiased estimator of θ?

c) Is $\hat{\theta}$ a consistent estimator of θ?

d) Obtain a method of moments estimate for θ, $\tilde{\theta}$.

Let $Y_1 < Y_2 < \ldots < Y_n$ denote the corresponding order statistics.

e) Let $Z_n = n \ln Y_1$. Find the limiting distribution of Z_n.

f) Let $W_n = n \ln \dfrac{\theta}{Y_n}$. Find the limiting distribution of W_n.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \left(\theta^2+\theta\right) x^{\theta-1}(1-x), \qquad 0<x<1, \qquad \theta>0.$$

a) Obtain a method of moments estimator of θ, $\tilde{\theta}$.

$$\mathrm{E}(X) = \int_0^1 x\cdot\left(\theta^2+\theta\right) x^{\theta-1}(1-x)\,dx = \left(\theta^2+\theta\right)\cdot\int_0^1 \left(x^{\theta}-x^{\theta+1}\right)dx$$

$$= \theta\cdot(\theta+1)\cdot\left(\frac{1}{\theta+1}x^{\theta+1} - \frac{1}{\theta+2}x^{\theta+2}\right)\Bigg|_0^1 = \frac{\theta\cdot(\theta+1)}{(\theta+1)\cdot(\theta+2)} = \frac{\theta}{\theta+2}.$$

OR

Beta distribution, $\alpha=\theta$, $\beta=2$. $\quad\Rightarrow\quad \mathrm{E}(X) = \dfrac{\theta}{\theta+2}.$

$$\frac{\tilde{\theta}}{\tilde{\theta}+2} = \overline{X} \qquad\qquad \tilde{\theta} = \overline{X}\cdot\left(\tilde{\theta}+2\right) \qquad\qquad \tilde{\theta} - \tilde{\theta}\,\overline{X} = 2\overline{X}$$

$$\Rightarrow \qquad \tilde{\theta} = \frac{2\overline{X}}{1-\overline{X}}, \qquad \text{where } \overline{X} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i\,.$$

b) Suppose $n=6$, and $x_1=0.3$, $x_2=0.5$, $x_3=0.6$, $x_4=0.65$, $x_5=0.75$, $x_6=0.8$. Find a method of moments estimate of θ.

$x_1=0.3$, $x_2=0.5$, $x_3=0.6$, $x_4=0.65$, $x_5=0.75$, $x_6=0.8$.

$$\overline{x} = 0.6. \qquad\qquad \tilde{\theta} = \frac{2\,\overline{x}}{1-\overline{x}} = \mathbf{3}.$$

c) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

Consider $g(x) = \dfrac{2x}{1-x}$. Then $g(\overline{X}) = \tilde{\theta}$, $g\left(\dfrac{\theta}{\theta+2}\right) = \theta$.

Also $g''(x) = \dfrac{4}{(1-x)^3} > 0$ for $0 < x < 1$, i.e., $g(x)$ is strictly convex.

By Jensen's Inequality,

$$E(\tilde{\theta}) = E[g(\overline{X})] > g(E(\overline{X})) = g(\mu_X) = g\left(\frac{\theta}{\theta+2}\right) = \theta.$$

Therefore, $\tilde{\theta}$ is NOT an unbiased estimator of θ.

d) Is $\tilde{\theta}$ a consistent estimator of θ? *Justify your answer.*

By WLLN, $\overline{X} \overset{P}{\rightarrow} E(X) = \dfrac{\theta}{\theta+2}$.

Consider $g(x) = \dfrac{2x}{1-x}$. Then $g(x)$ is continuous at $\dfrac{\theta}{\theta+2}$.

$g(\overline{X}) = \tilde{\theta}$ $\qquad$ $g\left(\dfrac{\theta}{\theta+2}\right) = \theta$.

$X_n \overset{P}{\rightarrow} a$, g is continuous at a $\Rightarrow$ $g(X_n) \overset{P}{\rightarrow} g(a)$

$\Rightarrow$ $\tilde{\theta} \overset{P}{\rightarrow} \theta$. $\qquad$ $\tilde{\theta}$ is a consistent estimator of θ.

e) Show that $\tilde{\theta}$ is asymptotically normally distributed $(\text{as } n \to \infty)$.
Find the parameters.

Beta distribution, $\alpha = \theta, \quad \beta = 2.$ $\Rightarrow$ $\operatorname{Var}(X) = \dfrac{2\theta}{(\theta+3)(\theta+2)^2}.$

OR

$$\mathrm{E}(X^2) = \int_0^1 x^2 \cdot \left(\theta^2 + \theta\right) x^{\theta-1}(1-x)\,dx = \left(\theta^2+\theta\right)\cdot\int_0^1 \left(x^{\theta+1} - x^{\theta+2}\right)dx$$

$$= \theta\cdot(\theta+1)\cdot\left(\frac{1}{\theta+2}x^{\theta+2} - \frac{1}{\theta+3}x^{\theta+3}\right)\Bigg|_0^1 = \frac{\theta\cdot(\theta+1)}{(\theta+2)\cdot(\theta+3)}.$$

$$\operatorname{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{\theta\cdot(\theta+1)}{(\theta+2)\cdot(\theta+3)} - \left(\frac{\theta}{\theta+2}\right)^2 = \frac{2\theta}{(\theta+3)(\theta+2)^2}.$$

By CLT, $\sqrt{n}\left(\overline{X} - \mu\right)$ is approx. $N(0, \sigma^2)$ for large n.

$g(x) = \dfrac{2x}{1-x}.$ $\qquad g'(x) = \dfrac{2}{(1-x)^2}.$

$g(\overline{X}) = \tilde{\theta}$ $\qquad g\left(\dfrac{\theta}{\theta+2}\right) = \theta.$ $\qquad g'\left(\dfrac{\theta}{\theta+2}\right) = \dfrac{(\theta+2)^2}{2}.$

By the Δ-method, $\sqrt{n}\left(g\left(\overline{X}\right) - g(\mu)\right) = \sqrt{n}\left(\tilde{\theta} - \theta\right)$ is approx.

$$N\left(0, \left(\frac{(\theta+2)^2}{2}\right)^2 \frac{2\theta}{(\theta+3)\,(\theta+2)^2}\right) = N\left(0, \frac{\theta(\theta+2)^2}{2(\theta+3)}\right) \text{ for large } n.$$

For large n, $\tilde{\theta}$ is approximately $N\left(\theta, \dfrac{\theta(\theta+2)^2}{2(\theta+3)n}\right).$

f) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

That is, find $\hat{\theta} = \arg\max \mathrm{L}(\theta) = \arg\max \ln \mathrm{L}(\theta)$,

where $\mathrm{L}(\theta) = \prod_{i=1}^{n} f(x_i ;\theta)$.

"Hint": ① $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;

② $\theta > 0$;

③ Since $0 < x < 1$, $\ln x < 0$.

$$\mathrm{L}(\theta) = \prod_{i=1}^{n} f(x_i ;\theta) = f(x_1;\theta)\, f(x_2;\theta)\, \ldots\, f(x_n;\theta)$$

$$= \left(\theta^2 + \theta\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta - 1} \prod_{i=1}^{n} (1 - x_i).$$

$$\ln \mathrm{L}(\theta) = n \ln(\theta^2 + \theta) + (\theta - 1) \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln(1 - x_i)$$

$$(\ln \mathrm{L}(\theta))' = \frac{2n\theta + n}{\theta^2 + \theta} + \sum_{i=1}^{n} \ln x_i = 0.$$

$$\Rightarrow \qquad \Sigma\, \hat{\theta}^2 + (2n + \Sigma)\, \hat{\theta} + n = 0, \qquad \text{where } \Sigma = \sum_{i=1}^{n} \ln \mathrm{X}_i .$$

$$\Rightarrow \qquad \hat{\theta} = \frac{-2n - \Sigma \pm \sqrt{(2n + \Sigma)^2 - 4\Sigma n}}{2\Sigma} = \frac{2n + \Sigma \pm \sqrt{4n^2 + \Sigma^2}}{-2\Sigma}.$$

Since $0 < x < 1$, $\ln x < 0$. $\Rightarrow$ $\Sigma < 0$.

$$\Rightarrow \qquad (2n + \Sigma)^2 = 4n^2 + 4n\Sigma + \Sigma^2 < 4n^2 + \Sigma^2.$$

$\Rightarrow \qquad |2n+\Sigma| < \sqrt{4n^2+\Sigma^2}$.

$\Rightarrow \qquad$ Since $\theta > 0$, $\qquad \hat{\theta} = \dfrac{2n+\Sigma + \sqrt{4n^2+\Sigma^2}}{-2\Sigma}$,

where $\Sigma = \sum_{i=1}^{n} \ln X_i$.

g) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$.
Find the maximum likelihood estimate of θ.

$\Sigma = \sum_{i=1}^{n} \ln X_i \approx -3.349554. \qquad \hat{\theta} \approx \mathbf{3.151}$.

h) Let $Y_1 < Y_2 < \ldots < Y_n$ denote the corresponding order statistics.
Find β so that $W_n = n^{\beta} Y_1$ converges in distribution.
Find the limiting distribution of W_n.

$$F_X(x) = \int_0^x \left(\theta^2+\theta\right) y^{\theta-1}(1-y)\,dy = \theta\cdot(\theta+1)\cdot\int_0^x \left(y^{\theta-1}-y^{\theta}\right)dy$$

$$= (\theta+1)x^{\theta} - \theta x^{\theta+1}, \qquad 0 < x < 1.$$

$$F_{Y_1}(x) = P(\min X_i \le x) = 1-(1-F(x))^n = 1-\left(1-(\theta+1)x^{\theta}+\theta x^{\theta+1}\right)^n,$$

$0 < x < 1$.

$$F_{W_n}(w) = P(W_n \le w) = P\left(Y_1 \le \frac{w}{n^{\beta}}\right)$$

$$= 1 - \left(1 - (\theta+1)\cdot\frac{w^{\theta}}{n^{\beta\theta}} + \theta\cdot\frac{w^{\theta+1}}{n^{\beta(\theta+1)}}\right)^{n}, \qquad 0 < w < n^{\beta}.$$

If $\beta = \frac{1}{\theta}$, $\quad F_{W_n}(w) = 1 - \left(1 - (\theta+1)\cdot\frac{w^{\theta}}{n} + \theta\cdot\frac{w^{\theta+1}}{n^{(\theta+1)/\theta}}\right)^{n},$

$0 < w < n^{\beta}.$

$$F_{\infty}(w) = \lim_{n\to\infty} F_{W_n}(w) = 1 - e^{-(\theta+1)w^{\theta}}, \qquad w > 0,$$

since $\frac{\theta+1}{\theta} > 1$.

If $\beta < \frac{1}{\theta}$, $\quad F_{\infty}(w) = \lim_{n\to\infty} F_{W_n}(w) = 1, \qquad w > 0,$

since $\beta\theta < 1$.

Then $W_n \overset{D}{\to} 0$, and thus $W_n \overset{P}{\to} 0$.

If $\beta > \frac{1}{\theta}$, $\quad F_{\infty}(w) = \lim_{n\to\infty} F_{W_n}(w) = 0, \qquad w > 0.$

since $1 < \beta\theta < \beta(\theta+1)$.

Then W_n does not have a limiting distribution.

"Goldilocks" $\beta = \frac{1}{\theta}$.

Limiting distribution: $\quad F_{\infty}(w) = 1 - e^{-(\theta+1)w^{\theta}}, \qquad w > 0,$

$$f_{\infty}(w) = \left(\theta^{2} + \theta\right)\cdot w^{\theta-1}\cdot e^{-(\theta+1)w^{\theta}}, \qquad w > 0,$$

Weibull distribution.

i) Find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for θ.

$$f(x_1, x_2, \ldots x_n; \theta) = f(x_1; \theta)\, f(x_2; \theta)\, \ldots\, f(x_n; \theta)$$

$$= \left(\theta^2 + \theta\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta - 1} \prod_{i=1}^{n} \left(1 - x_i\right).$$

By Factorization Theorem, $Y = \prod_{i=1}^{n} X_i$ is a sufficient statistic for θ.

OR

$$f(x; \lambda) = \exp\left\{(\theta - 1) \cdot \ln x + \ln\left(\theta^2 + \theta\right) + \ln(1 - x)\right\}. \qquad \Rightarrow \qquad K(x) = \ln x.$$

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} K\left(X_i\right) = \sum_{i=1}^{n} \ln X_i \text{ is a sufficient statistic for } \lambda.$$

2. Let $\theta > 1$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

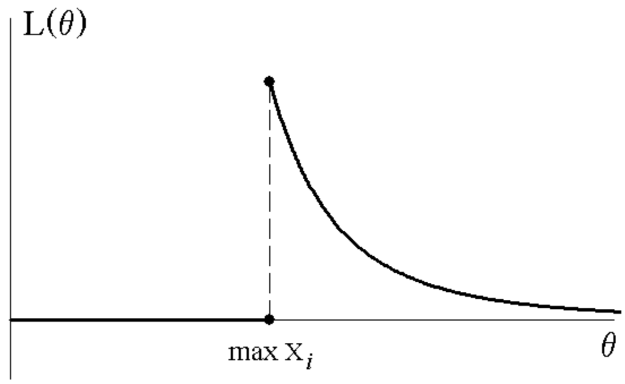
$$f(x;\theta) = \frac{1}{x \ln\theta}, \qquad 1 < x < \theta.$$

a) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

Likelihood function:

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{X_i \ln\theta} \right) = \frac{1}{(\ln\theta)^n} \cdot \prod_{i=1}^{n} \frac{1}{X_i}, \qquad \theta > \max X_i,$$

$$L(\theta) = 0, \qquad \theta < \max X_i.$$



Therefore,

$$\hat{\theta} = \max X_i.$$

b) Is $\hat{\theta}$ an unbiased estimator of θ?

Since $P(\max X_i < \theta) = 1$, $\qquad E(\max X_i) < \theta$.

$\Rightarrow \qquad \hat{\theta}$ is NOT an unbiased estimator for θ.

c) Is $\hat{\theta}$ a consistent estimator of θ?

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{1}^{x} \frac{1}{y \ln\theta}\,dy = \frac{\ln x}{\ln\theta}, \qquad 1 < x < \theta.$$

$$F_{\max X_i}(x) = [F_X(x)]^n = \left[\frac{\ln x}{\ln\theta}\right]^n, \qquad 1 < x < \theta.$$

Let $\varepsilon > 0$. $\qquad P(\hat\theta \ge \theta + \varepsilon) = 0.$

If $\varepsilon \ge \theta - 1$, $\qquad P(\hat\theta \le \theta - \varepsilon) = 0.$

If $0 < \varepsilon < \theta - 1$,

$$P(\hat\theta \le \theta - \varepsilon) = F_{\max X_i}(\theta - \varepsilon) = \left[\frac{\ln(\theta - \varepsilon)}{\ln\theta}\right]^n \to 0 \quad \text{as } n \to \infty.$$

$$\Rightarrow \quad P(|\hat\theta - \theta| \ge \varepsilon) \to 0 \quad \text{as } n \to \infty, \quad \text{and} \quad \hat\theta \xrightarrow{P} \theta.$$

d) Obtain a method of moments estimate for θ, $\tilde\theta$.

$$E(X) = \int_1^\theta x \cdot \frac{1}{x\ln\theta}\,dx = \frac{\theta - 1}{\ln\theta}.$$

$\overline{X} = \dfrac{\tilde\theta - 1}{\ln\tilde\theta}$ CANNOT be solved algebraically for $\tilde\theta$.

$\dfrac{\tilde\theta - 1}{\ln\tilde\theta}$	$\tilde\theta$
1.5	2.144033
2.0	3.512862
2.5	5.046970
3.0	6.711441
3.5	8.483382
4.0	10.346652
4.5	12.289269
5.0	14.301995

Let $Y_1 < Y_2 < \ldots < Y_n$ denote the corresponding order statistics.

e) Let $Z_n = n \ln Y_1$. Find the limiting distribution of Z_n.

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{1}^{x} \frac{1}{y \ln\theta}\,dy = \frac{\ln x}{\ln\theta}, \qquad 1 < x < \theta.$$

$$F_{Y_1}(x) = F_{\min X_i}(x) = 1 - [1 - F_X(x)]^n = 1 - \left[1 - \frac{\ln x}{\ln\theta}\right]^n, \qquad 1 < x < \theta.$$

$$F_{Z_n}(z) = P(Y_1 \le e^{z/n}) = 1 - \left(1 - \frac{z}{n \ln\theta}\right)^n, \qquad 0 < z < n \ln\theta.$$

$$F_\infty(z) = \lim_{n\to\infty} F_{Z_n}(z) = 1 - e^{-z/\ln\theta}, \qquad 0 < z < \infty.$$

$Z_n \xrightarrow{D}$ Exponential distribution with mean $\ln\theta$.

f) Let $W_n = n \ln \dfrac{\theta}{Y_n}$. Find the limiting distribution of W_n.

$$F_X(x) = \frac{\ln x}{\ln\theta}, \qquad 1 < x < \theta.$$

$$F_{Y_n}(x) = F_{\max X_i}(x) = [F_X(x)]^n = \left[\frac{\ln x}{\ln\theta}\right]^n, \qquad 1 < x < \theta.$$

$$F_{W_n}(w) = P(Y_n \ge \theta\, e^{-w/n}) = 1 - F_{Y_n}(\theta\, e^{-w/n}) = 1 - \left(1 - \frac{w}{n \ln\theta}\right)^n,$$

$$0 < w < n \ln\theta.$$

$$F_\infty(w) = \lim_{n\to\infty} F_{W_n}(w) = 1 - e^{-w/\ln\theta}, \qquad 0 < w < \infty.$$

$W_n \xrightarrow{D}$ Exponential distribution with mean $\ln\theta$.