STAT 410 | **Examples for 10/30/2020 (2)** | Fall 2020
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**1.** Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{1}{\theta} \cdot x^{\frac{1}{\theta}-1}, \qquad 0 < x < 1, \qquad 0 < \theta < \infty.$$

a) Recall that $W = -\ln X$ has an Exponential distribution with mean $\theta$.
Suggest a confidence interval for $\theta$ with $(1-\alpha)\,100\,\%$ confidence level.

$-\sum\limits_{i=1}^{n} \ln X_i = \sum\limits_{i=1}^{n} W_i$ has Gamma$(\alpha = n, \theta)$ distribution.

If Y has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where $\alpha$ is an integer, then $2Y/\theta = 2\lambda Y$ has a chi-square distribution with $2\alpha$ degrees of freedom.

$\frac{2}{\theta} \sum\limits_{i=1}^{n} W_i$ has a $\chi^2(2\alpha = 2n)$ distribution.

$$\Rightarrow \quad P\left( \chi^2_{1-\alpha/2}(2n) < \frac{2}{\theta} \sum_{i=1}^{n} W_i < \chi^2_{\alpha/2}(2n) \right) = 1-\alpha.$$

$$\Rightarrow \quad P\left( \frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} W_i} < \frac{1}{\theta} < \frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} W_i} \right) = 1-\alpha.$$

$$\Rightarrow \quad P\left( \frac{2\sum_{i=1}^{n} W_i}{\chi^2_{1-\alpha/2}(2n)} > \theta > \frac{2\sum_{i=1}^{n} W_i}{\chi^2_{\alpha/2}(2n)} \right) = 1-\alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for $\theta$

$$\left( \frac{2\sum\limits_{i=1}^{n} w_i}{\chi^2_{1-\alpha/2}(2n)},\ \frac{2\sum\limits_{i=1}^{n} w_i}{\chi^2_{\alpha/2}(2n)} \right).$$

b) Suppose $n = 3$, and $x_1 = 0.2$, $x_2 = 0.3$, $x_3 = 0.5$.

Use part (a) to construct a 90% confidence interval for $\theta$.

$\chi^2_{0.95}(6) = 1.635$, $\qquad\qquad \chi^2_{0.05}(6) = 12.59$.

$$\sum_{i=1}^{n} w_i = -\ln 0.2 - \ln 0.3 - \ln 0.5 = -\ln 0.03 \approx 3.50656.$$

$$\left( \frac{2\sum\limits_{i=1}^{n} w_i}{\chi^2_{1-\alpha/2}(2n)},\ \frac{2\sum\limits_{i=1}^{n} w_i}{\chi^2_{\alpha/2}(2n)} \right) = \left( \frac{2\cdot 3.50656}{12.59},\ \frac{2\cdot 3.50656}{1.635} \right)$$

$$= \mathbf{(0.557,\, 4.289)}.$$

Recall: $$\hat{\theta} = -\frac{1}{n}\cdot\sum_{i=1}^{n} \ln x_i = -\frac{1}{3}\cdot\ln 0.03 \approx 1.16885.$$

**2.** Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

Recall: $\quad Y = \sum_{i=1}^{n} X_i^2$ has Gamma( $\alpha = 2n$, "usual $\theta$" $= \frac{1}{\lambda}$ ) distribution.

a) Suggest a confidence interval for $\lambda$ with $(1-\alpha)\,100\,\%$ confidence level.

If Y has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where $\alpha$ is an integer, then $2Y/\theta = 2\lambda Y$ has a chi-square distribution with $2\alpha$ degrees of freedom.

$2\lambda \sum_{i=1}^{n} X_i^2$ has a $\chi^2(2\alpha = 4n)$ distribution.

$$\Rightarrow \qquad P\left(\chi^2_{1-\alpha/2}(4n) < 2\lambda \sum_{i=1}^{n} X_i^2 < \chi^2_{\alpha/2}(4n)\right) = 1-\alpha.$$

$$\Rightarrow \qquad P\left(\frac{\chi^2_{1-\alpha/2}(4n)}{2\sum_{i=1}^{n} X_i^2} < \lambda < \frac{\chi^2_{\alpha/2}(4n)}{2\sum_{i=1}^{n} X_i^2}\right) = 1-\alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for $\lambda$

$$\left(\frac{\chi^2_{1-\alpha/2}(4n)}{2\sum_{i=1}^{n} x_i^2}, \frac{\chi^2_{\alpha/2}(4n)}{2\sum_{i=1}^{n} x_i^2}\right).$$

b) Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\sum_{i=1}^{n} x_i^2 = 40.$

(i) Use part (a) to construct a 90% confidence interval for $\lambda$.

$\chi^2_{0.95}(20) = 10.85,$ $\chi^2_{0.05}(20) = 31.41.$

$$\left( \frac{\chi^2_{1-\alpha/2}(4n)}{2\sum_{i=1}^{n} x_i^2}, \frac{\chi^2_{\alpha/2}(4n)}{2\sum_{i=1}^{n} x_i^2} \right) = \left( \frac{10.85}{2 \cdot 40}, \frac{31.41}{2 \cdot 40} \right) \approx \mathbf{(\,0.1356\,,\,0.3926\,)}.$$

(ii) Use part (a) to construct a 95% confidence interval for $\lambda$.

$\chi^2_{0.975}(20) = 9.591,$ $\chi^2_{0.025}(20) = 34.17.$

$$\left( \frac{\chi^2_{1-\alpha/2}(4n)}{2\sum_{i=1}^{n} x_i^2}, \frac{\chi^2_{\alpha/2}(4n)}{2\sum_{i=1}^{n} x_i^2} \right) = \left( \frac{9.591}{2 \cdot 40}, \frac{34.17}{2 \cdot 40} \right) \approx \mathbf{(\,0.120\,,\,0.427\,)}.$$

Recall: $$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} x_i^2} = 0.25.$$