STAT 410

Fall 2020

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# Examples for 08/28/2020 (3)

**1.** The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0$, $\delta > 0$. Consider the probability density function

$$f_X(x) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Find the probability distribution of $W = X^{\delta}$.

a) Determine the probability distribution of W by finding the c.d.f. of W, $F_W(w)$.

i) Find the c.d.f. of X, $F_X(x) = P(X \le x)$.

"Hint" 1: $u$-substitution: $u = \heartsuit^{\delta}$.

"Hint" 2: There is no such thing as a negative cumulative distribution function.

"Hint" 3: Should be $F_X(0) = 0$, $F_X(\infty) = 1$.

ii) Find the c.d.f. of W, $F_W(w) = P(W \le w) = P(X^{\delta} \le w)$.

iii) What is the name of the probability distribution of W? What are its parameters?

b) Determine the probability distribution of W by finding the p.d.f. of W, $f_W(w)$, using the change-of-variable technique.

i) Find the p.d.f. of W, $f_W(w) = f_X(g^{-1}(w))\left|\dfrac{dx}{dw}\right|$.

ii) What is the name of the probability distribution of W? What are its parameters?

c) Determine the probability distribution of W by finding the m.g.f. of W, $M_W(t)$.

i) Find the m.g.f. of W, $M_W(t) = E(e^{tW}) = E(e^{tX^\delta})$.

"Hint" 1: $u$-substitution: $u = ♥^{\delta}$.

"Hint" 2: Must have $t < \beta$ for the integral to converge.

ii) What is the name of the probability distribution of W? What are its parameters?

**2.** Let X have an exponential distribution with $\theta = 1$; that is, the p.d.f. of X is $f(x) = e^{-x}$, $0 < x < \infty$. Let T be defined by $T = \ln X$.

a) Show that the p.d.f. of T is

$$f_T(t) = e^t e^{-e^t}, \quad -\infty < t < \infty,$$

which is the p.d.f. of an **extreme value distribution**.

b) Let W be defined by $T = \alpha + \beta \ln W$, where $-\infty < \alpha < \infty$ and $\beta > 0$. Show that W has a Weibull distribution.

## Answers:

**1.** The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0$, $\delta > 0$. Consider the probability density function

$$f_X(x) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Find the probability distribution of $W = X^{\delta}$.

a) Determine the probability distribution of W by finding the c.d.f. of W, $F_W(w)$.

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"Hint" 1: $u$-substitution: $u = ♥^{\delta}$.

"Hint" 2: There is no such thing as a negative cumulative distribution function.

"Hint" 3: Should be $F_X(0) = 0$, $F_X(\infty) = 1$.

ii) Find the c.d.f. of W, $F_W(w) = P(W \le w) = P(X^{\delta} \le w)$.

iii) What is the name of the probability distribution of W? What are its parameters?

$$F_X(x) = \int_0^x \beta\,\delta\,y^{\delta-1}\,e^{-\beta y^{\delta}}\,dy \qquad u = y^{\delta} \qquad du = \delta\,y^{\delta-1}\,dy$$

$$= \int_0^{x^{\delta}} \beta\,e^{-\beta u}\,du = -e^{-\beta u}\Big|_0^{x^{\delta}} = 1 - e^{-\beta x^{\delta}}, \qquad x > 0.$$

$x > 0 \quad \Rightarrow \quad w > 0.$

$$F_W(w) = P(W \le w) = P(X^\delta \le w) = P(X \le w^{1/\delta})$$

$$= F_X(w^{1/\delta}) = 1 - e^{-\beta w}, \qquad w > 0.$$

$$f_W(w) = \beta e^{-\beta w}, \qquad w > 0.$$

W has an Exponential$\left(\theta = \frac{1}{\beta}\right)$ distribution.

b) Determine the probability distribution of W by finding the p.d.f. of W, $f_W(w)$, using the change-of-variable technique.

i) Find the p.d.f. of W, $f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right|$.

ii) What is the name of the probability distribution of W? What are its parameters?

$W = X^\delta \qquad X = g^{-1}(w) = W^{1/\delta} \qquad \frac{dx}{dw} = \frac{1}{\delta} w^{(1-\delta)/\delta}$

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right| = \beta \delta w^{(\delta-1)/\delta} e^{-\beta w} \cdot \frac{1}{\delta} w^{(1-\delta)/\delta} = \beta e^{-\beta w},$$

$$w > 0.$$

W has an Exponential$\left(\theta = \frac{1}{\beta}\right)$ distribution.

c) Determine the probability distribution of W by finding the m.g.f. of W, $M_W(t)$.

i) Find the m.g.f. of W, $M_W(t) = E(e^{tW}) = E(e^{tX^\delta})$.

"Hint" 1: $u$-substitution: $u = ♥^\delta$.

"Hint" 2: Must have $t < \beta$ for the integral to converge.

ii) What is the name of the probability distribution of W? What are its parameters?

$$M_W(t) = E(e^{tW}) = E(e^{tX^\delta}) = \int_0^\infty e^{tx^\delta} \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^\delta}\,dx$$

$$= \int_0^\infty \beta\,e^{-(\beta-t)x^\delta}\,\delta\,x^{\delta-1}\,dx$$

Indeed, must have $t < \beta$ for the integral to converge.

$u = x^\delta \qquad du = \delta\,x^{\delta-1}\,dx$

$$= \int_0^\infty \beta\,e^{-(\beta-t)u}\,du = \frac{\beta}{\beta-t} = \frac{1}{1-\frac{1}{\beta}t}, \qquad t < \beta.$$

W has an Exponential$\left(\theta = \frac{1}{\beta}\right)$ distribution.

**2.** Let X have an exponential distribution with $\theta = 1$; that is, the p.d.f. of X is $f(x) = e^{-x}$, $0 < x < \infty$. Let T be defined by $T = \ln X$.

a) Show that the p.d.f. of T is

$$f_T(t) = e^t e^{-e^t}, \quad -\infty < t < \infty,$$

which is the p.d.f. of an **extreme value distribution**.

$t = \ln(x) \qquad x = e^t \qquad \dfrac{dx}{dt} = e^t$

$x > 0 \quad \Rightarrow \quad -\infty < t < \infty$

$$f_T(t) = f_X(e^t)\left|\frac{dx}{dt}\right| = e^t e^{-e^t}, \quad -\infty < t < \infty.$$

OR

$$F_X(x) = 1 - e^{-x}, \quad x > 0.$$

$$F_T(t) = P(T \le t) = P(X \le e^t) = F_X(e^t) = 1 - e^{-e^t}, \quad -\infty < t < \infty.$$

$$f_T(t) = e^t e^{-e^t}, \quad -\infty < t < \infty.$$

b) Let W be defined by $T = \alpha + \beta \ln W$, where $-\infty < \alpha < \infty$ and $\beta > 0$. Show that W has a Weibull distribution.

$t = \alpha + \beta \ln w \qquad \dfrac{dt}{dw} = \dfrac{\beta}{w}$

$$f_W(t) = f_T(\alpha + \beta \ln w)\left|\frac{dt}{dw}\right| = e^{\alpha + \beta \ln w} e^{-e^{\alpha + \beta \ln w}} \cdot \frac{\beta}{w}$$

$$= e^{\alpha} \beta w^{\beta - 1} e^{-e^{\alpha} w^{\beta}}, \quad w > 0.$$