STAT 410 **Examples for 10/12/2020 (1)** Fall 2020
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Gamma Distribution:

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad 0 \le x < \infty \qquad\qquad f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad 0 \le x < \infty$$

$$\mathrm{E}(\mathrm{X}) = \alpha\,\theta \qquad\qquad \mathrm{E}(\mathrm{X}) = \alpha/\lambda,$$

$$\mathrm{Var}(\mathrm{X}) = \alpha\,\theta^2 \qquad\qquad \mathrm{Var}(\mathrm{X}) = \alpha/\lambda^2$$

If T_{α} has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$\mathrm{F}_{\mathrm{T}_{\alpha}}(t) = \mathrm{P}(\mathrm{T}_{\alpha} \le t) = \mathrm{P}(\mathrm{X}_t \ge \alpha),$$

$$\mathrm{P}(\mathrm{T}_{\alpha} > t) = \mathrm{P}(\mathrm{X}_t \le \alpha - 1),$$

where X_t has a Poisson$\left(\lambda t = \frac{t}{\theta}\right)$ distribution.

1. Alex is told that he needs to take bus #5 to the train station. He misunderstands the directions and decides to wait for the fifth bus. Suppose that the buses arrive to the bus stop according to Poisson process with the average rate of one bus per 20 minutes.

X_t = number of buses in t hours. Poisson(λt)

T_k = arrival time of the kth bus. Gamma, $\alpha = k$.

one bus per 20 minutes $\Rightarrow$ $\lambda = 3$.

a) Find the probability that Alex would have to wait longer than 1 hour for the fifth bus to arrive.

$\mathrm{P}(\mathrm{T}_5 > 1) = \mathrm{P}(\mathrm{X}_1 \le 4) = \mathrm{P}(\text{Poisson}(3) \le 4) =$ **0.815**.

OR

$$P(T_5 > 1) = \int_1^{\infty} \frac{3^5}{\Gamma(5)} t^{5-1} e^{-3t}\, dt = \int_1^{\infty} \frac{3^5}{4!} t^4 e^{-3t}\, dt = \ldots$$

b) Find the probability that the fifth bus arrives during the third hour.

$$P(2 < T_5 < 3) = P(T_5 > 2) - P(T_5 > 3) = P(X_2 \le 4) - P(X_3 \le 4)$$
$$= P(\text{Poisson}(6) \le 4) - P(\text{Poisson}(9) \le 4) = 0.285 - 0.055 = \mathbf{0.230}.$$

OR

$$P(2 < T_5 < 3) = \int_2^{3} \frac{3^5}{\Gamma(5)} t^{5-1} e^{-3t}\, dt = \int_2^{3} \frac{3^5}{4!} t^4 e^{-3t}\, dt = \ldots$$

c) Find the probability that the fifth bus arrives during the second hour.

$$P(1 < T_5 < 2) = P(T_5 > 1) - P(T_5 > 2) = P(X_1 \le 4) - P(X_2 \le 4)$$
$$= P(\text{Poisson}(3) \le 4) - P(\text{Poisson}(6) \le 4) = 0.815 - 0.285 = \mathbf{0.530}.$$

OR

$$P(1 < T_5 < 2) = \int_1^{2} \frac{3^5}{\Gamma(5)} t^{5-1} e^{-3t}\, dt = \int_1^{2} \frac{3^5}{4!} t^4 e^{-3t}\, dt = \ldots$$

2. a) Mistakes that Alex makes in class occur according to Poisson process with the average rate of one mistake per 10 minutes. Find the probability that the third mistake Alex makes occurs during the last 15 minutes of a 50-minute class.

Notations: X_t = number of server failures in t days.

T_k = time of the kth server failure.

1 min $\quad \theta = 10, \quad \lambda = \frac{1}{10} = 0.10.$

$$P(35 < T_3 < 50) = P(T_3 > 35) - P(T_3 > 50) = P(X_{35} \le 2) - P(X_{50} \le 2) = \ldots$$

5 min $\quad \theta = 2, \quad \lambda = \frac{1}{2} = 0.50.$

$$P(7 < T_3 < 10) = P(T_3 > 7) - P(T_3 > 10) = P(X_7 \le 2) - P(X_{10} \le 2) = \ldots$$

10 min $\quad \theta = 1, \quad \lambda = 1.$

$$P(3.5 < T_3 < 5) = P(T_3 > 3.5) - P(T_3 > 5) = P(X_{3.5} \le 2) - P(X_5 \le 2) = \ldots$$

$$\ldots = P(\text{Poisson}(3.5) \le 2) - P(\text{Poisson}(5.0) \le 2) = 0.3208 - 0.1247 = \mathbf{0.1961}.$$

OR

1 min $$P(35 < T_3 < 50) = \int_{35}^{50} \frac{0.10^3}{\Gamma(3)} t^{3-1} e^{-0.10t}\, dt = \int_{35}^{50} \frac{1}{2{,}000} t^2 e^{-0.10t}\, dt = \ldots$$

5 min $$P(7 < T_3 < 10) = \int_{7}^{10} \frac{0.50^3}{\Gamma(3)} t^{3-1} e^{-0.50t}\, dt = \int_{7}^{10} \frac{1}{16} t^2 e^{-0.50t}\, dt = \ldots$$

10 min $$P(3.5 < T_3 < 5) = \int_{3.5}^{5.0} \frac{1}{\Gamma(3)} t^{3-1} e^{-t}\, dt = \int_{3.5}^{5.0} \frac{1}{2} t^2 e^{-t}\, dt = \ldots$$

b) Students ask questions in class according to Poisson process with the average rate of one question per 20 minutes. Find the probability that the third question is asked during the last 10 minutes of a 50-minute class.

1 min $\quad \theta = 20, \quad \lambda = \dfrac{1}{20} = 0.05.$

$$P(40 < T_3 < 50) = P(T_3 > 40) - P(T_3 > 50) = P(X_{40} \le 2) - P(X_{50} \le 2) = \ldots$$

5 min $\quad \theta = 4, \quad \lambda = \dfrac{1}{4} = 0.25.$

$$P(8 < T_3 < 10) = P(T_3 > 8) - P(T_3 > 10) = P(X_8 \le 2) - P(X_{10} \le 2) = \ldots$$

10 min $\quad \theta = 2, \quad \lambda = \dfrac{1}{2} = 0.50.$

$$P(4 < T_3 < 5) = P(T_3 > 4) - P(T_3 > 5) = P(X_4 \le 2) - P(X_5 \le 2) = \ldots$$

20 min $\quad \theta = 1, \quad \lambda = 1.$

$$P(2 < T_3 < 2.5) = P(T_3 > 2) - P(T_3 > 2.5) = P(X_2 \le 2) - P(X_{2.5} \le 2) = \ldots$$

$$\ldots = P(\text{Poisson}(2.0) \le 2) - P(\text{Poisson}(2.5) \le 2) = 0.6767 - 0.5438 = \mathbf{0.1329}.$$

OR

1 min $$P(40 < T_3 < 50) = \int_{40}^{50} \frac{0.05^3}{\Gamma(3)}\, t^{3-1} e^{-0.05t}\, dt = \int_{40}^{50} \frac{1}{16{,}000}\, t^2 e^{-0.05t}\, dt = \ldots$$

5 min $$P(8 < T_3 < 10) = \int_{8}^{10} \frac{0.25^3}{\Gamma(3)}\, t^{3-1} e^{-0.25t}\, dt = \int_{8}^{10} \frac{1}{128}\, t^2 e^{-0.25t}\, dt = \ldots$$

10 min $$P(4 < T_3 < 5) = \int_{4}^{5} \frac{0.50^3}{\Gamma(3)}\, t^{3-1} e^{-0.50t}\, dt = \int_{4}^{5} \frac{1}{16}\, t^2 e^{-0.50t}\, dt = \ldots$$

20 min $$P(2 < T_3 < 2.5) = \int_{2.0}^{2.5} \frac{1}{\Gamma(3)}\, t^{3-1} e^{-t}\, dt = \int_{2.0}^{2.5} \frac{1}{2}\, t^2 e^{-t}\, dt = \ldots$$