STAT 410 — Fall 2020 — A. Stepanov

Examples for 09/23/2020 (1)

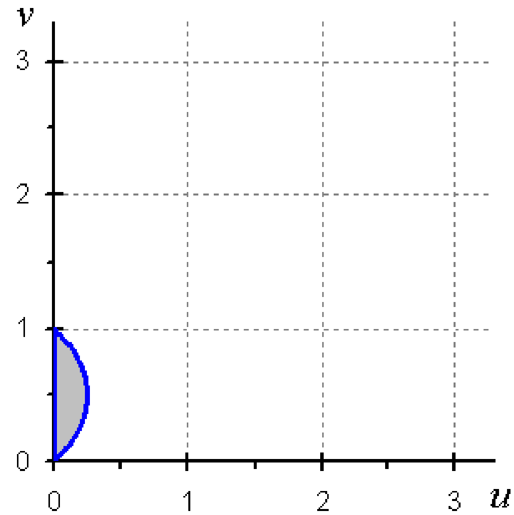
3. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = 60\,x^2 y, \qquad x > 0,\ \ y > 0,\ \ x + y < 1, \qquad \text{zero elsewhere.}$$

a) Let U = X Y and V = X.

Find the joint probability density function of (U, V), $f_{U,V}(u,v)$.

Sketch the support of (U, V).



$$X = V, \qquad Y = \frac{U}{V}.$$

$x > 0 \quad \Rightarrow \quad v > 0,$

$y > 0 \quad \Rightarrow \quad u > 0,$

$x + y < 1 \quad \Rightarrow \quad v + \dfrac{u}{v} < 1 \quad \Rightarrow \quad u < v - v^2.$

$$J = \begin{vmatrix} 0 & 1 \\ \dfrac{1}{v} & -\dfrac{u}{v^2} \end{vmatrix} = -\frac{1}{v}. \qquad\qquad |J| = \frac{1}{v}.$$

$$f_{U,V}(u,v) = f_{X,Y}\left(v, \frac{u}{v}\right) \cdot |J| = \left(60\,v^2\,\frac{u}{v}\right) \cdot \frac{1}{v} = 60\,u,$$

$$0 < v < 1, \qquad 0 < u < v - v^2,$$

$$f_{U,V}(u,v) = 0 \qquad \text{otherwise.}$$

b) Consider $U = X \times Y$. Use the answers to part (a) to find the p.d.f. of U, $f_U(u)$.

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv.$$

$u < v - v^2 \quad \Rightarrow \quad v_1 < v < v_2$, where

$$v_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - u}\,, \qquad v_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - u}\,.$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_{v_1}^{v_2} 60u\,dv = 60\,u\,(v_2 - v_1)$$

$$= 120u\sqrt{\frac{1}{4} - u} = 60\,u\,\sqrt{1 - 4u}\,, \qquad 0 < u < \frac{1}{4}.$$

4. **2.7.1** (7th and 6th edition)

Let X_1, X_2, X_3 be iid, each with the distribution having pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere. Show that

$$Y_1 = \frac{X_1}{X_1 + X_2}, \qquad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \qquad Y_3 = X_1 + X_2 + X_3$$

are mutually independent.

$$Y_1 = \frac{X_1}{X_1 + X_2}, \qquad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \qquad Y_3 = X_1 + X_2 + X_3$$

$$\Rightarrow \quad X_1 = Y_1 Y_2 Y_3$$

$$X_1 + X_2 = Y_2 Y_3 \quad \Rightarrow \quad X_2 = (1 - Y_1) Y_2 Y_3$$

$$X_3 = Y_3 - X_1 - X_2 = Y_3 - Y_2 Y_3 = (1 - Y_2) Y_3$$

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ -y_2 y_3 & (1-y_1) y_3 & (1-y_1) y_2 \\ 0 & -y_3 & (1-y_2) \end{vmatrix} = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= y_2 y_3^2$$

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = e^{-x_1} e^{-x_2} e^{-x_3}, \qquad x_1 > 0, \quad x_2 > 0, \quad x_3 > 0.$$

$$f_{Y_1, Y_2, Y_3}(y_1, y_2, y_3) = e^{-y_3} \times y_2 y_3^2 = y_2 y_3^2 e^{-y_3},$$

$$0 < y_1 < 1, \qquad 0 < y_2 < 1, \qquad 0 < y_3 < \infty.$$

$$f_{Y_1}(y_1) = 1, \qquad 0 < y_1 < 1, \qquad\qquad f_{Y_2}(y_2) = 2 y_2, \qquad 0 < y_2 < 1,$$

$$f_{Y_3}(y_3) = \frac{1}{2} y_3^2 e^{-y_3}, \qquad 0 < y_3 < \infty.$$

Y_1, Y_2, Y_3 are mutually independent.

5. Let X_1 and X_2 have independent Gamma distributions with parameters α, θ and β, θ, respectively. Consider

$$Y_1 = \frac{X_1}{X_1 + X_2} \quad \text{and} \quad Y_2 = X_1 + X_2.$$

Show that Y_1 has a Beta distribution with parameters α and β, Y_2 has a Gamma distribution with parameters $\alpha + \beta$ and θ, and Y_1 and Y_2 are independent.

$$X_1 = Y_1 Y_2 \qquad X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2$$

$$0 < y_1 < 1, \qquad y_2 > 0.$$

$$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = y_2$$

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}(y_1 y_2)^{\alpha-1} e^{-(y_1 y_2)/\theta} \cdot \frac{1}{\Gamma(\beta)\theta^{\beta}}(y_2 - y_1 y_2)^{\beta-1} e^{-(y_2 - y_1 y_2)/\theta} \cdot y_2$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(y_1)^{\alpha-1}(1-y_1)^{\beta-1} \cdot \frac{1}{\Gamma(\alpha+\beta)\theta^{\alpha+\beta}}(y_2)^{\alpha+\beta-1} e^{-y_2/\theta},$$

$$0 < y_1 < 1, \quad y_2 > 0.$$

$\Rightarrow$ Y_1 has a Beta distribution with parameters α and β,

Y_2 has a Gamma distribution with parameters $\alpha + \beta$ and θ,

Y_1 and Y_2 are independent.

6. Let X_1 and X_2 be independent random variables, each with p.d.f.

$$f(x) = e^{-x}, \qquad 0 < x < \infty.$$

Let $Y_1 = X_1 - X_2$, $\quad Y_2 = X_1 + X_2$.

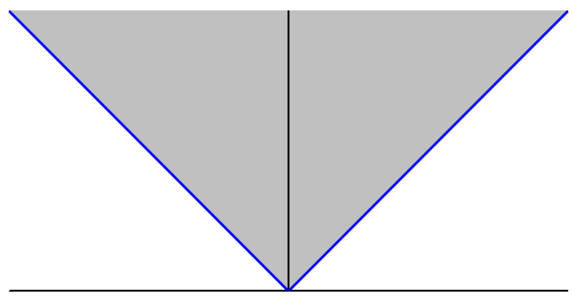
a) Find the joint pd.f. of Y_1 and Y_2.

$$X_1 = \frac{Y_1 + Y_2}{2}, \qquad X_2 = \frac{Y_2 - Y_1}{2}. \qquad J = \frac{1}{2}.$$

$$f_{Y_1,Y_2}(y_1, y_2) = e^{-x_1 - x_2}, \qquad x_1 > 0, \quad x_2 > 0.$$

$x_1 > 0 \qquad y_2 > -y_1$

$x_2 > 0 \qquad y_2 > y_1$



$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2}\, e^{-y_2},$$

$$0 < y_2 < \infty, \quad -y_2 < y_1 < y_2.$$

OR

$$y_2 > |y_1|$$

b) Find the marginal p.d.f.'s of Y_1 and Y_2.

$$f_{Y_1}(y_1) = \frac{1}{2}\, e^{-|y_1|}, \qquad -\infty < y_1 < \infty. \qquad (\text{double exponential})$$

$$f_{Y_2}(y_2) = y_2\, e^{-y_2}, \qquad 0 < y_2 < \infty. \qquad (\text{Gamma}, \ \alpha = 2, \ \theta = 1)$$

c) Are Y_1 and Y_2 independent?

Y_1 and Y_2 are **NOT independent**.

d) Find $f_{Y_1|Y_2}(y_1 \mid y_2)$.

$$f_{Y_1|Y_2}(y_1 \mid y_2) = \frac{1}{2\,y_2}, \qquad -y_2 < y_1 < y_2, \qquad 0 < y_2 < \infty.$$

$Y_1 \mid Y_2 = y_2$ is $\text{Uniform}(-y_2, y_2)$.

e) Find $f_{Y_2|Y_1}(y_2 \mid y_1)$.

$$f_{Y_2|Y_1}(y_2 \mid y_1) = e^{|y_1| - y_2}, \qquad y_2 > |y_1|, \qquad -\infty < y_1 < \infty.$$