STAT 410 | **Examples for 09/23/2020 (Disc)** | Fall 2020
A. Stepanov

Examples for 09/09/2020 (Disc) & Examples for 09/16/2020 (Disc) (continued)

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let the joint probability density function of X and Y be defined by

$$f(x,y) = \frac{x+4y}{9}, \qquad 0<y<1, \quad y<x<3, \qquad \text{zero otherwise.}$$

l) Find the probability distribution of W = X + Y.

m) Let $\mathrm{V} = \mathrm{X} \cdot \mathrm{Y}$.

Find the joint probability density function of (V, Y), $f_{\mathrm{V,Y}}(v, y)$.

Sketch the support of (V, Y).

n) Find the probability distribution of $\mathrm{V} = \mathrm{X} \cdot \mathrm{Y}$.

o) Let $\mathrm{U} = \mathrm{X} / \mathrm{Y}$.

Find the joint probability density function of (X, U), $f_{\mathrm{X,U}}(x, u)$.

Sketch the support of (X, U).

p) Find the probability distribution of $\mathrm{U} = \mathrm{X} / \mathrm{Y}$.

2. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = \frac{1}{32}x^2 y, \qquad 0<x<4, \;\; 0<y<\sqrt{x}, \qquad \text{zero elsewhere.}$$

l) Let $U = Y$ and $V = X/Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u,v)$.

Sketch the support of (U, V).

m) Use part (l) to find the probability density function of $V = X/Y$, $f_V(v)$.

4. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{9}{2\,(2x+y)^3}, \qquad y>1, \;\; 0<x<y, \qquad \text{zero elsewhere.}$$

m) Find the probability distribution of $W = X + Y$.

n) Let $U = X/Y$ and $V = Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u,v)$.

Sketch the support of (U, V).

o) Use part (n) to find the p.d.f. of U, $f_U(u)$.

p) Let $V = X \cdot Y$ and $W = X$.

Find the joint probability density function of (V, W), $f_{V,W}(v,w)$.

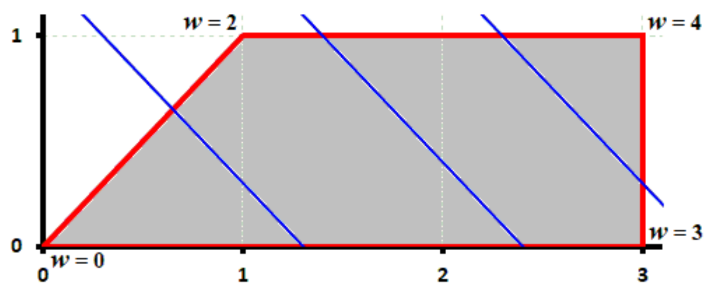
Sketch the support of (V, W).

Answers:

1. Let the joint probability density function of X and Y be defined by

$$f(x,y) = \frac{x+4y}{9}, \qquad 0 < y < 1, \quad y < x < 3, \qquad \text{zero otherwise.}$$

l) Find the probability distribution of W = X + Y.



$F_W(w) = P(X + Y \le w) = \ldots$

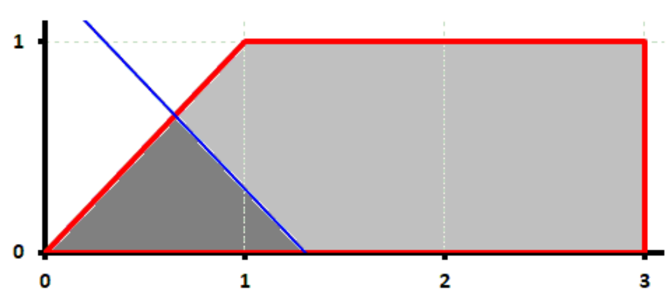
There are 3 cases:

$0 < w < 2, \quad 2 < w < 3, \quad 3 < w < 4.$

Technically, there are 5 cases: $w < 0, \quad 0 < w < 2, \quad 2 < w < 3, \quad 3 < w < 4, \quad w > 4,$

but $w < 0$ and $w > 4$ are boring.

Case 1: $0 \le w < 2.$



$x = y \;\;\&\;\; x + y = w$

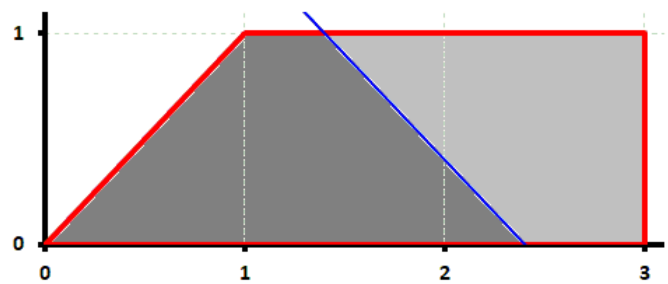
$\Rightarrow \quad x = \frac{w}{2}, \;\; y = \frac{w}{2}.$

$$F_W(w) = \int_0^{\frac{w}{2}} \left(\int_y^{w-y} \frac{x+4y}{9}\, dx \right) dy$$

$$F_W(w) = \int_0^{\frac{w}{2}} \left(\int_0^{x} \frac{x+4y}{9}\, dy \right) dx + \int_{\frac{w}{2}}^{w} \left(\int_0^{w-x} \frac{x+4y}{9}\, dy \right) dx$$

$$= \frac{w^3}{72} + \frac{w^3}{54} = \frac{7w^3}{216}.$$

Case 2: $2 \le w < 3$.

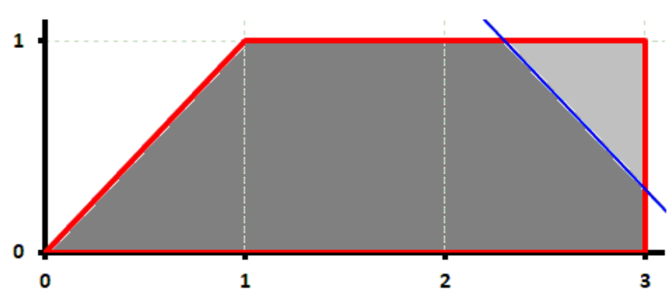


$$F_W(w) = \int_0^{1} \left(\int_y^{w-y} \frac{x+4y}{9}\, dx \right) dy \qquad F_W(w) = 1 - \int_0^{1} \left(\int_{w-y}^{3} \frac{x+4y}{9}\, dx \right) dy$$

$$F_W(w) = \int_0^{1} \left(\int_0^{x} \frac{x+4y}{9}\, dy \right) dx + \int_1^{w-1} \left(\int_0^{1} \frac{x+4y}{9}\, dy \right) dx + \int_{w-1}^{w} \left(\int_0^{w-x} \frac{x+4y}{9}\, dy \right) dx$$

$$= \frac{1}{9} + \frac{w^2+2w-8}{18} + \frac{3w+2}{54} = \frac{3w^2+9w-16}{54} = \frac{w^2}{18} + \frac{w}{6} - \frac{8}{27}.$$

Case 3: $3 \le w < 4$.



$$F_W(w) = 1 - \int_{w-1}^{3} \left(\int_{w-x}^{1} \frac{x+4y}{9}\, dy \right) dx$$

$$F_W(w) = 1 - \int_{w-3}^{1} \left(\int_{w-y}^{3} \frac{x+4y}{9}\, dx \right) dy$$

$$F_W(w) = \int_{0}^{1} \left(\int_{0}^{x} \frac{x+4y}{9}\, dy \right) dx + \int_{1}^{w-1} \left(\int_{0}^{1} \frac{x+4y}{9}\, dy \right) dx + \int_{w-1}^{3} \left(\int_{0}^{w-x} \frac{x+4y}{9}\, dy \right) dx$$

$$= \frac{1}{9} + \frac{w^2+2w-8}{18} + \frac{-5w^3+36w^2-78w+56}{54} = \frac{-5w^3+39w^2-72w+38}{54}.$$

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \dfrac{7w^3}{216} & 0 \le w < 2 \\ \dfrac{3w^2+9w-16}{54} & 2 \le w < 3 \\ \dfrac{-5w^3+39w^2-72w+38}{54} & 3 \le w < 4 \\ 1 & w \ge 4 \end{cases}$$

$$\frac{d}{dw}\left(\frac{7w^3}{216}\right) = \frac{7w^2}{72},$$

$$\frac{d}{dw}\left(\frac{3w^2+9w-16}{54}\right) = \frac{w}{9} + \frac{1}{6},$$

$$\frac{d}{dw}\left(\frac{-5w^3+39w^2-72w+38}{54}\right) = \frac{-5w^2+26w-24}{18}.$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$$f(x, w-x) = \frac{4w-3x}{9}.$$

$$0 < y \quad \Rightarrow \quad 0 < w - x \quad \Rightarrow \quad x < w$$

$$y < 1 \quad \Rightarrow \quad w - x < 1 \quad \Rightarrow \quad x > w - 1$$

$$y < x \quad \Rightarrow \quad w - x < x \quad \Rightarrow \quad x > \frac{w}{2}$$

$$x < 3$$

$$x > w - 1, \quad x > \frac{w}{2} \qquad \& \qquad x < w, \quad x < 3.$$

Case 1: $0 < w < 2$. $\qquad w - 1 < \frac{w}{2} \quad \& \quad w < 3.$

$$f_W(w) = \int_{\frac{w}{2}}^{w} \frac{4w-3x}{9}\, dx = \frac{7w^2}{72}, \qquad 0 < w < 2.$$

Case 2: $2 < w < 3$. $\qquad \frac{w}{2} < w - 1 \quad \& \quad w < 3.$

$$f_W(w) = \int_{w-1}^{w} \frac{4w-3x}{9}\, dx = \frac{2w+3}{18}, \qquad 2 < w < 3.$$

Case 3: $3 < w < 4$. $\qquad \frac{w}{2} < w - 1 \quad \& \quad 3 < w.$

$$f_W(w) = \int_{w-1}^{3} \frac{4w-3x}{9}\, dx = \frac{-5w^2 + 26w - 24}{18}, \qquad 3 < w < 4.$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$$f(w-y, y) = \frac{w+3y}{9}.$$

$0 < y < 1$

$y < x \quad \Rightarrow \quad y < w - y \quad \Rightarrow \quad y < \frac{w}{2}$

$x < 3 \quad \Rightarrow \quad w - y < 3 \quad \Rightarrow \quad y > w - 3$

$y > 0, \quad y > w - 3 \qquad \& \qquad y < 1, \quad y < \frac{w}{2}.$

Case 1: $\quad 0 < w < 2. \qquad w - 3 < 0 \quad \& \quad \frac{w}{2} < 1.$

$$f_W(w) = \int_0^{\frac{w}{2}} \frac{w+3y}{9}\, dy = \frac{7w^2}{72}, \qquad 0 < w < 2.$$

Case 2: $\quad 2 < w < 3. \qquad w - 3 < 0 \quad \& \quad 1 < \frac{w}{2}.$

$$f_W(w) = \int_0^1 \frac{w+3y}{9}\, dy = \frac{w}{9} + \frac{1}{6}, \qquad 2 < w < 3.$$

Case 3: $\quad 3 < w < 4. \qquad 0 < w - 3 \quad \& \quad 1 < \frac{w}{2}.$

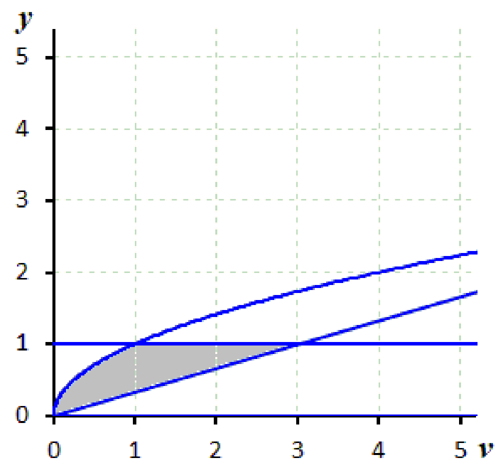
$$f_W(w) = \int_{w-3}^1 \frac{w+3y}{9}\, dy = \frac{-5w^2 + 26w - 24}{18}, \qquad 3 < w < 4.$$

m) Let $V = X \cdot Y$.

Find the joint probability density function of (V, Y), $f_{V,Y}(v, y)$.

Sketch the support of (V, Y).

$X = V/Y, \qquad Y = Y.$



$0 < y < 1,$

$y < x \quad \Rightarrow \quad y < \dfrac{v}{y} \quad \Rightarrow \quad v > y^2,$

$x < 3 \quad \Rightarrow \quad \dfrac{v}{y} < 3 \quad \Rightarrow \quad v < 3y.$

$$J = \begin{vmatrix} \dfrac{1}{y} & -\dfrac{v}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}.$$

$$f_{V,Y}(v,y) = f_{X,Y}\left(\frac{v}{y}, y\right) \times |J| = \frac{\frac{v}{y} + 4y}{9} \times \frac{1}{y} = \frac{v + 4y^2}{9y^2} = \frac{v}{9y^2} + \frac{4}{9},$$

$0 < y < 1, \quad y^2 < v < 3y.$

or $\quad 0 < v < 3, \quad \dfrac{v}{3} < y < \min(1, \sqrt{v}).$

n) Find the probability distribution of $V = X \cdot Y$.

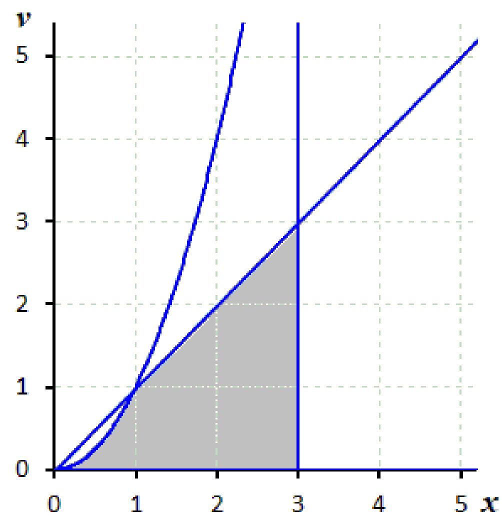
From part (m):

$0 < v < 1,$
$$\int_{\frac{v}{3}}^{\sqrt{v}} \left(\frac{v}{9y^2} + \frac{4}{9} \right) dy = \left(-\frac{v}{9y} + \frac{4y}{9} \right) \Bigg|_{y=\frac{v}{3}}^{y=\sqrt{v}} = \frac{1}{3} + \frac{\sqrt{v}}{3} - \frac{4v}{27}.$$

$1 < v < 3,$
$$\int_{\frac{v}{3}}^{1} \left(\frac{v}{9y^2} + \frac{4}{9} \right) dy = \left(-\frac{v}{9y} + \frac{4y}{9} \right) \Bigg|_{y=\frac{v}{3}}^{y=1} = \frac{7}{9} - \frac{7v}{27}.$$

OR

$X = X, \qquad Y = V / X.$



$0 < y < 1 \quad \Rightarrow \quad 0 < \dfrac{v}{x} < 1 \quad \Rightarrow \quad 0 < v < x,$

$y < x \quad \Rightarrow \quad \dfrac{v}{x} < x \quad \Rightarrow \quad v < x^2,$

$x < 3.$

$$J = \begin{vmatrix} 1 & 0 \\ -\dfrac{v}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{1}{x}.$$

$$f_{X,V}(x,v) = f_{X,Y}(x, \frac{v}{x}) \times |J| = \frac{x+4\frac{v}{x}}{9} \times \frac{1}{x} = \frac{x^2+4v}{9x^2} = \frac{1}{9}+\frac{4v}{9x^2},$$

$$0 < v < 3, \quad \max(\sqrt{v}, v) < x < 3.$$

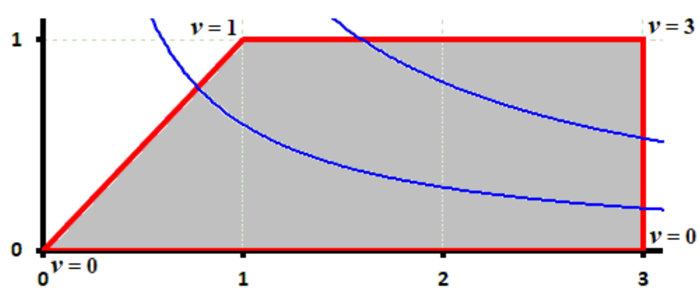
$0 < v < 1,$

$$\int_{\sqrt{v}}^{3} \left(\frac{1}{9}+\frac{4v}{9x^2}\right) dx = \left(\frac{x}{9}-\frac{4v}{9x}\right)\Bigg|_{x=\sqrt{v}}^{x=3} = \frac{1}{3}+\frac{\sqrt{v}}{3}-\frac{4v}{27}.$$

$1 < v < 3,$

$$\int_{v}^{3} \left(\frac{1}{9}+\frac{4v}{9x^2}\right) dx = \left(\frac{x}{9}-\frac{4v}{9x}\right)\Bigg|_{x=v}^{x=3} = \frac{7}{9}-\frac{7v}{27}.$$

OR



$$F_V(v) = P(X \cdot Y \le v)$$

$$= P\left(Y \le \frac{v}{X}\right) = \ldots$$

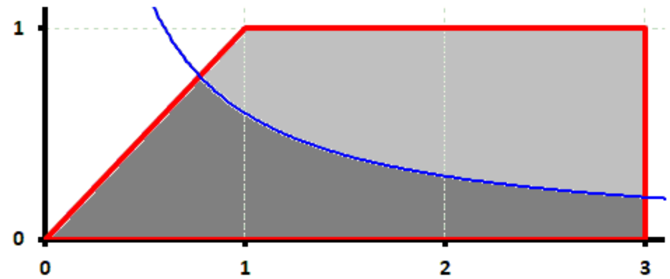
There are 2 cases:

$0 < v < 1, \quad 1 < v < 3.$

Technically, there are 4 cases: $v < 0,$ $0 < v < 1,$ $1 < v < 3,$ $v > 3,$

but $v < 0$ and $v > 3$ are boring.

Case 1: $0 \le v < 1$.



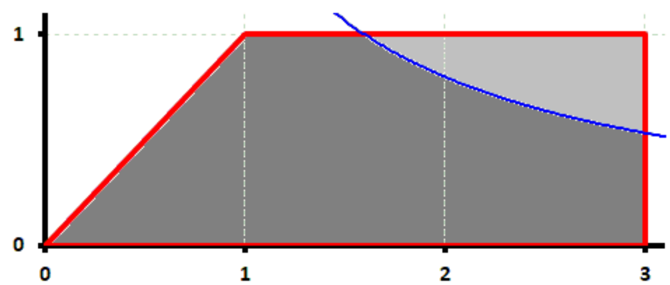
$x = y$ & $x \cdot y = v$

$\Rightarrow \quad x = \sqrt{v}, \; y = \sqrt{v}$.

$$F_V(v) = \int_0^{\sqrt{v}} \left(\int_0^x \frac{x+4y}{9}\,dy \right) dx + \int_{\sqrt{v}}^3 \left(\int_0^{\frac{v}{x}} \frac{x+4y}{9}\,dy \right) dx$$

$$= \frac{v^{3/2}}{9} + \left(\frac{v}{3} - \frac{2v^2}{27} + \frac{v^{3/2}}{9} \right) = \frac{v}{3} - \frac{2v^2}{27} + \frac{2v^{3/2}}{9}.$$

Case 2: $1 \le v < 3$.



$$F_V(v) = 1 - \int_v^3 \left(\int_{\frac{v}{x}}^1 \frac{x+4y}{9}\,dy \right) dx \qquad F_V(v) = 1 - \int_{\frac{v}{3}}^1 \left(\int_{\frac{v}{y}}^3 \frac{x+4y}{9}\,dx \right) dy$$

$$F_V(v) = \int_0^1 \left(\int_0^x \frac{x+4y}{9}\,dy \right) dx + \int_1^v \left(\int_0^1 \frac{x+4y}{9}\,dy \right) dx + \int_v^3 \left(\int_0^{\frac{v}{x}} \frac{x+4y}{9}\,dy \right) dx$$

$$= \frac{1}{9} + \frac{v^2+4v-5}{18} + \frac{15v-5v^2}{27} = \frac{-9+42v-7v^2}{54} = -\frac{1}{6} + \frac{7v}{9} - \frac{7v^2}{54}.$$

$$F_V(v) = \begin{cases} 0 & v < 0 \\ \dfrac{v}{3} - \dfrac{2v^2}{27} + \dfrac{2v^{3/2}}{9} & 0 \le v < 1 \\ -\dfrac{1}{6} + \dfrac{7v}{9} - \dfrac{7v^2}{54} & 1 \le v < 3 \\ 1 & v \ge 3 \end{cases}$$

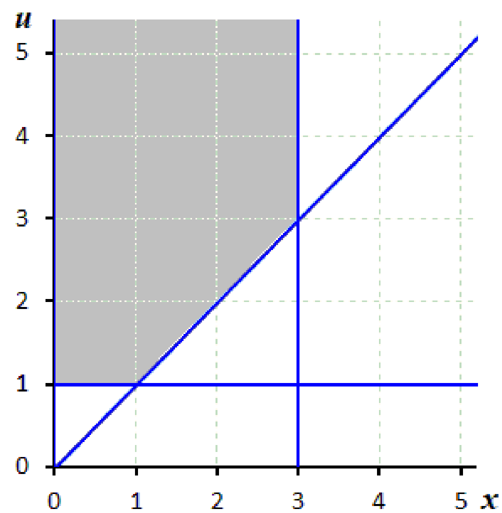
Indeed, $$\frac{d}{dv}\left(\frac{v}{3} - \frac{2v^2}{27} + \frac{2v^{3/2}}{9}\right) = \frac{1}{3} - \frac{4v}{27} + \frac{\sqrt{v}}{3},$$

$$\frac{d}{dv}\left(-\frac{1}{6} + \frac{7v}{9} - \frac{7v^2}{54}\right) = \frac{7}{9} - \frac{7v}{27}.$$

o) Let $U = {}^{X}/_{Y}$.

Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.

Sketch the support of (X, U).



$X = X, \qquad Y = X/U.$

$0 < y < 1 \quad \Rightarrow \quad 0 < \dfrac{x}{u} < 1 \quad \Rightarrow \quad 0 < x < u,$

$y < x \quad \Rightarrow \quad \dfrac{x}{u} < x \quad \Rightarrow \quad u > 1,$

$x < 3.$

$$J = \begin{vmatrix} 1 & 0 \\ \dfrac{1}{u} & -\dfrac{x}{u^2} \end{vmatrix} = -\frac{x}{u^2}.$$

$$f_{X,U}(x, u) = f_{X,Y}\left(x, \frac{x}{u}\right) \times |J| = \frac{x + 4\dfrac{x}{u}}{9} \times \frac{x}{u^2} = \frac{u+4}{9u^3} x^2,$$

$0 < x < 3, \quad u > \max(1, x).$

or $\quad u > 1, \quad 0 < x < \min(3, u).$

p) Find the probability distribution of $U = X/Y$.

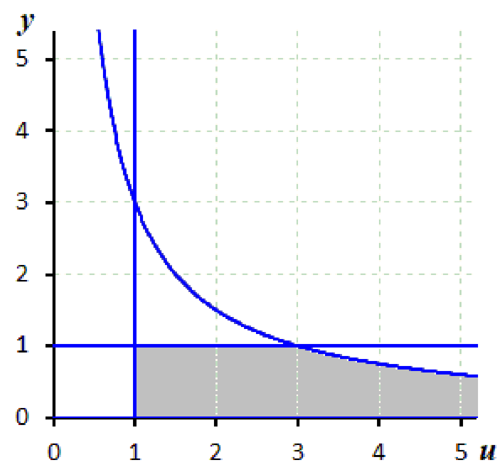
From part (o):

$1 < u < 3,$ $$\int_0^u \frac{u+4}{9u^3} x^2\,dx = \frac{u+4}{27u^3} x^3 \Big|_{x=0}^{x=u} = \frac{u+4}{27}.$$

$u > 3,$ $$\int_0^3 \frac{u+4}{9u^3} x^2\,dx = \frac{u+4}{27u^3} x^3 \Big|_{x=0}^{x=3} = \frac{u+4}{u^3}.$$

OR

$X = U\,Y, \qquad Y = Y.$



$0 < y < 1,$

$y < x \quad \Rightarrow \quad y < u\,y \quad \Rightarrow \quad u > 1,$

$x < 3 \quad \Rightarrow \quad u\,y < 3.$

$$J = \begin{vmatrix} y & u \\ 0 & 1 \end{vmatrix} = y.$$

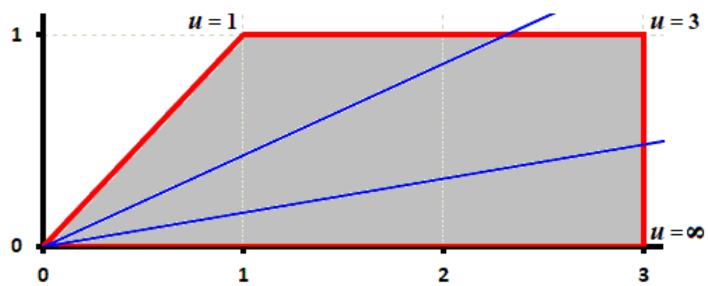
$$f_{U,Y}(u, y) = f_{X,Y}(u\,y, y) \times |J| = \frac{u\,y + 4y}{9} \times y = \frac{u+4}{9} y^2,$$

$$u > 1, \quad 0 < y < \min\left(1, \frac{3}{u}\right).$$

$1 < u < 3,\qquad \displaystyle\int_0^1 \frac{u+4}{9}\,y^2\,dy = \frac{u+4}{27}\,y^3\,\Big|_{y=0}^{y=1} = \frac{u+4}{27}.$

$u > 3,\qquad \displaystyle\int_0^{\frac{3}{u}} \frac{u+4}{9}\,y^2\,dy = \frac{u+4}{27}\,y^3\,\Big|_{y=0}^{y=\frac{3}{u}} = \frac{u+4}{u^3}.$

OR



$$F_U(u) = P\left(\frac{X}{Y} \le u\right)$$

$$= P\left(Y \ge \frac{X}{u}\right) = \ldots$$

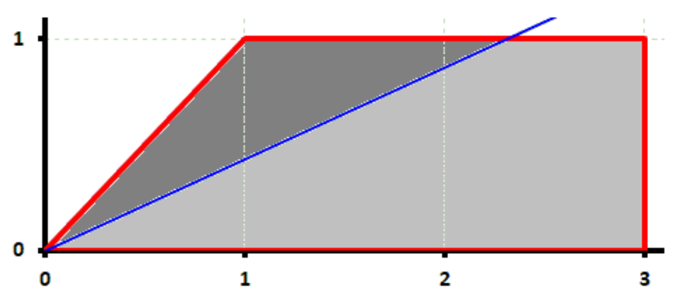
There are 2 cases:

$1 < u < 3,\quad u > 3.$

Technically, there are 3 cases: $u < 1,\quad 1 < u < 3,\quad u > 3,$

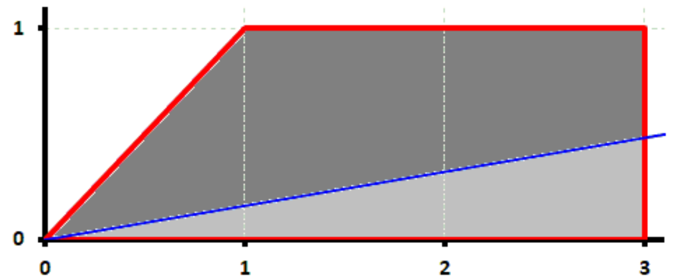
but $u < 1$ are boring.

Case 1: $1 \le u < 3.$



$$F_U(u) = \int_0^1 \left(\int_y^{uy} \frac{x+4y}{9}\,dx \right) dy = \int_0^1 \left(\frac{u^2y^2+8uy^2}{18} - \frac{y^2}{2} \right) dy = \frac{u^2+8u-9}{54}.$$

Case 2: $u \geq 3$.



$$F_U(u) = 1 - \int_0^3 \left(\int_0^{\frac{x}{u}} \frac{x+4y}{9}\, dy \right) dx = 1 - \int_0^3 \left(\frac{\frac{x^2}{u} + 2\frac{x^2}{u^2}}{9} \right) dx = 1 - \frac{1}{u} - \frac{2}{u^2}.$$

$$F_U(u) = \begin{cases} 0 & u < 1 \\ \dfrac{u^2 + 8u - 9}{54} & 1 \leq u < 3 \\ 1 - \dfrac{1}{u} - \dfrac{2}{u^2} & u \geq 3 \end{cases}$$

Indeed,

$$\frac{d}{du}\left(\frac{u^2 + 8u - 9}{54} \right) = \frac{u+4}{27},$$

$$\frac{d}{du}\left(1 - \frac{1}{u} - \frac{2}{u^2} \right) = \frac{1}{u^2} + \frac{4}{u^3} = \frac{u+4}{u^3}.$$

2. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = \frac{1}{32}x^2 y, \qquad 0 < x < 4, \quad 0 < y < \sqrt{x}, \qquad \text{zero elsewhere.}$$

l) Let $U = Y$ and $V = X/Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u,v)$.

Sketch the support of (U, V).

$x = u\,v \qquad y = u$

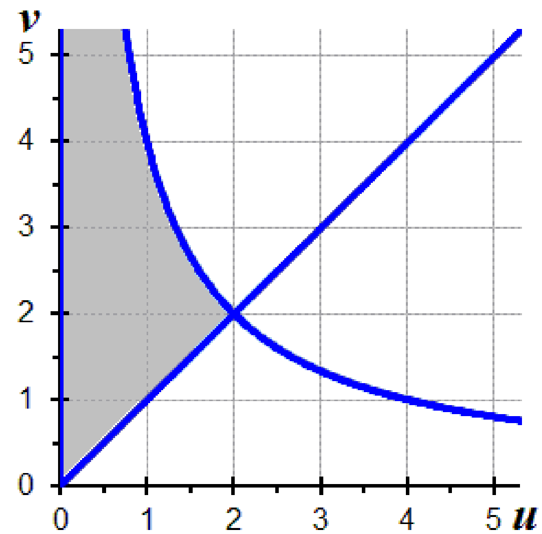
$0 < x < 4 \quad \Rightarrow \quad 0 < u\,v < 4$

$0 < y \quad \Rightarrow \quad 0 < u$

$y < \sqrt{x} \quad \Rightarrow \quad u < \sqrt{u\,v}$

$\Rightarrow \quad u < v$

$$J = \begin{vmatrix} v & u \\ 1 & 0 \end{vmatrix} = -u.$$



$$f_{U,V}(u,v) = f_{X,Y}(u\,v, u)\,|-u| = \frac{1}{32}(u\,v)^2 u \cdot u = \frac{1}{32}u^4 v^2,$$

$$0 < u < v, \qquad 0 < u\,v < 4.$$

m) Use part (l) to find the probability density function of $V = {}^{X}/_{Y}$, $f_V(v)$.

$0 < v < 2$,

$$f_V(v) = \int_0^v \frac{1}{32} u^4 v^2 \, du = \frac{1}{160} u^5 v^2 \Big|_{u=0}^{u=v} = \frac{1}{160} v^7.$$

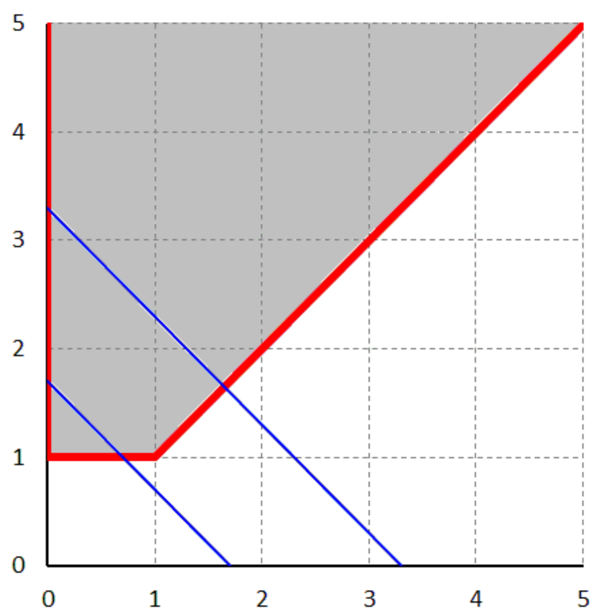
$v > 2$,

$$f_V(v) = \int_0^{\frac{4}{v}} \frac{1}{32} u^4 v^2 \, du = \frac{1}{160} u^5 v^2 \Big|_{u=0}^{u=\frac{4}{v}} = \frac{32}{5 v^3}.$$

4. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{9}{2\,(\,2\,x + y\,)^{3}}, \qquad y > 1, \quad 0 < x < y, \qquad \text{zero elsewhere.}$$

m) Find the probability distribution of $\mathrm{W} = \mathrm{X} + \mathrm{Y}$.



$$\mathrm{F}_{\mathrm{W}}(w) = \mathrm{P}\,(\,\mathrm{X} + \mathrm{Y} \le w\,) = \ldots$$

There are 2 cases:

$1 < w < 2, \qquad w > 2.$

Technically, there are 3 cases:

$w < 1, \qquad 1 < w < 2, \qquad w > 2,$

but $w < 1$ is boring.

See Examples for 09/09/2020 (Disc) Problem 4 parts (e) and (f) for the CDF approach.

OR

$$f_W(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx = \ldots$$

$$f_{X,Y}(x,y) = \frac{9}{2(2x+y)^3}, \qquad y > 1, \qquad x < y, \qquad x > 0.$$

$$f_{X,Y}(x, w-x) = \frac{9}{2(x+w)^3}, \qquad w-x > 1, \qquad x < w-x, \qquad x > 0.$$

$$x < w-1 \qquad x < w/2.$$

Need $\qquad x > 0 \qquad x < w-1 \;\;\&\;\; x < w/2.$

Case 1: $\qquad 1 < w < 2$

$$x < w-1 \;\;\&\;\; x < w/2 \qquad \Rightarrow \qquad x < w-1$$

$$f_W(w) = \int_0^{w-1} \frac{9}{2(x+w)^3}\,dx = -\frac{9}{4(x+w)^2}\Bigg|_0^{w-1} = \frac{9}{4w^2} - \frac{9}{4(2w-1)^2}, \qquad 1 < w < 2.$$

Case 1: $\qquad w > 2$

$$x < w-1 \;\;\&\;\; x < w/2 \qquad \Rightarrow \qquad x < w/2$$

$$f_W(w) = \int_0^{w/2} \frac{9}{2(x+w)^3}\,dx = -\frac{9}{4(x+w)^2}\Bigg|_0^{w/2} = \frac{9}{4w^2} - \frac{1}{w^2} = \frac{5}{4w^2}, \qquad w > 2.$$

OR

$$f_W(w) = \int_{-\infty}^{\infty} f(w-y, y)\,dy = \ldots$$

$$f_{X,Y}(x,y) = \frac{9}{2(2x+y)^3}, \qquad y > 1, \qquad x > 0, \qquad x < y.$$

$$f_{X,Y}(w-y,y) = \frac{9}{2(2w-y)^3}, \qquad y > 1, \qquad w-y > 0, \qquad w-y < y.$$

$$y < w \qquad y > w/2.$$

Need $\quad y > 1 \;\;\&\;\; y > w/2 \qquad y < w.$

Case 1: $\quad 1 < w < 2$

$$y > 1 \;\;\&\;\; y > w/2 \qquad \Rightarrow \qquad y > 1$$

$$f_W(w) = \int_1^w \frac{9}{2(2w-y)^3}\,dy = \left.\frac{9}{4(2w-y)^2}\right|_0^w = \frac{9}{4w^2} - \frac{9}{4(2w-1)^2}, \qquad 1 < w < 2.$$

Case 1: $\quad w > 2$

$$y > 1 \;\;\&\;\; y > w/2 \qquad \Rightarrow \qquad y > w/2$$

$$f_W(w) = \int_{w/2}^w \frac{9}{2(2w-y)^3}\,dy = \left.\frac{9}{4(2w-y)^2}\right|_{w/2}^w = \frac{9}{4w^2} - \frac{1}{w^2} = \frac{5}{4w^2}, \qquad w > 2.$$

n) Let $U = {}^{X}/_{Y}$ and $V = Y$.

Find the joint probability density function of (U, V), $f_{U,V}(u, v)$.

Sketch the support of (U, V).

$$U = \frac{X}{Y} \qquad V = Y \qquad \Rightarrow \qquad X = UV \qquad Y = V$$

$y > 1 \quad \Rightarrow \quad v > 1$

$x > 0 \quad \Rightarrow \quad uv > 0$

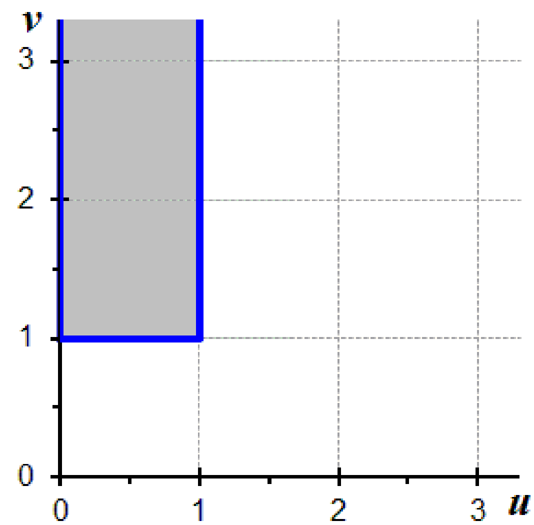
$\Rightarrow \quad u > 0$

$(\text{since } v > 1)$

$x < y \quad \Rightarrow \quad uv < v$

$\Rightarrow \quad u < 1$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$



$$f_{U,V}(u,v) = f_{X,Y}(uv, v) \cdot |J| = \frac{9}{2(2uv+v)^3} \cdot v = \frac{9}{2(2u+1)^3 v^2},$$

$$0 < u < 1, \qquad v > 1.$$

$$f_{U,V}(u,v) = 0 \qquad \text{otherwise.}$$

o) Use part (n) to find the p.d.f. of U, $f_U(u)$.

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_{1}^{\infty} \frac{9}{2(2u+1)^3 v^2}\,dv = \frac{9}{2(2u+1)^3}, \qquad 0 < u < 1.$$

To double-check: Recall Examples for 09/09/2020 (Disc) Problem 4 part (g):

Let $U = {}^{Y}/_{X}$.

$$F_U(u) = \frac{9}{8} - \frac{9}{8(2u+1)^2}, \qquad 0 < u < 1.$$

$$f_U(u) = F_U'(u) = \frac{9}{2(2u+1)^3}, \quad 0 < u < 1.$$

p) Let $V = X \cdot Y$ and $W = X$.

Find the joint probability density function of (V, W), $f_{V,W}(v, w)$.

Sketch the support of (V, W).

$x = w \qquad y = \dfrac{v}{w}$

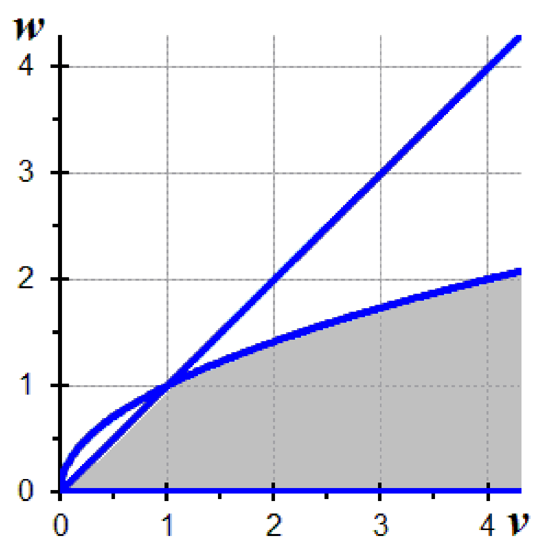
$y > 1 \quad \Rightarrow \quad \dfrac{v}{w} > 1$

$\Rightarrow \quad v > w$

$0 < x \quad \Rightarrow \quad 0 < w$

$x < y \quad \Rightarrow \quad w < \dfrac{v}{w}$

$\Rightarrow \quad v > w^2$



$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{w} & -\frac{v}{w^2} \end{vmatrix} = -\frac{1}{w}.$$

$$f_{V,W}(v,w) = f_{X,Y}\left(w, \frac{v}{w}\right)\left|-\frac{1}{w}\right| = \frac{9}{2\left(2w + \frac{v}{w}\right)^3} \cdot \frac{1}{w} = \frac{9w^2}{2\left(2w^2 + v\right)^3},$$

$$0 < w < v, \qquad v > w^2.$$