STAT 410 **Examples for 09/09/2020 (2)** Fall 2020
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2. Consider the following joint probability distribution $p(x, y)$ of two discrete random variables X and Y:

		x = 1	x = 2	$p_{\mathrm{Y}}(y)$
y	1	0.14	0.06	0.20
	2	0.12	0.18	0.30
	3	0.14	0.36	0.50
	$p_{\mathrm{X}}(x)$	0.40	0.60	1.00

Recall STAT 400: $\quad \mathrm{P}(\mathrm{A} \mid \mathrm{B}) = \dfrac{\mathrm{P}(\mathrm{A} \cap \mathrm{B})}{\mathrm{P}(\mathrm{B})}$, provided $\mathrm{P}(\mathrm{B}) > 0$.

$$\Rightarrow \quad \mathrm{P}(\mathrm{Y} = y \mid \mathrm{X} = x) = \frac{\mathrm{P}(\mathrm{X} = x, \mathrm{Y} = y)}{\mathrm{P}(\mathrm{X} = x)}, \quad \text{provided } \mathrm{P}(\mathrm{X} = x) > 0.$$

Def Conditional probability mass function of Y given $\mathrm{X} = x$ is defined by

$$p_{\mathrm{Y}|\mathrm{X}}(y|x) = \frac{p(x,y)}{p_{\mathrm{X}}(x)}.$$

For a fixed x,

- $p_{\mathrm{Y}|\mathrm{X}}(y|x) \geq 0 \quad \forall\, y.$
- $\displaystyle\sum_y p_{\mathrm{Y}|\mathrm{X}}(y|x) = \sum_y \frac{p(x,y)}{p_{\mathrm{X}}(x)} = \frac{1}{p_{\mathrm{X}}(x)} \sum_y p(x,y) = \frac{1}{p_{\mathrm{X}}(x)}\, p_{\mathrm{X}}(x) = 1.$

Similarly, $\quad \mathrm{P}(\mathrm{X} = x \mid \mathrm{Y} = y) = \dfrac{\mathrm{P}(\mathrm{X} = x, \mathrm{Y} = y)}{\mathrm{P}(\mathrm{Y} = y)}$, provided $\mathrm{P}(\mathrm{Y} = y) > 0$.

Def Conditional probability mass function of X given $\mathrm{Y} = y$ is defined by

$$p_{\mathrm{X}|\mathrm{Y}}(x|y) = \frac{p(x,y)}{p_{\mathrm{Y}}(y)}.$$

For a fixed y,

- $p_{X|Y}(x|y) \geq 0 \qquad \forall\, x.$
- $\sum\limits_x p_{X|Y}(x|y) = \sum\limits_x \dfrac{p(x,y)}{p_Y(y)} = \dfrac{1}{p_Y(y)} \sum\limits_x p(x,y) = \dfrac{1}{p_Y(y)} p_Y(y) = 1.$

Y given X = 1

y	$p_{Y\|X}(y\|1)$
1	0.14 / 0.40 = 0.35
2	0.12 / 0.40 = 0.30
3	0.14 / 0.40 = 0.35
	1.00

Y given X = 2

y	$p_{Y\|X}(y\|2)$
1	0.06 / 0.60 = 0.10
2	0.18 / 0.60 = 0.30
3	0.36 / 0.60 = 0.60
	1.00

For example, $P(Y \geq 2 \mid X = 1) = 0.65.$ $\qquad P(Y \leq 2 \mid X = 2) = 0.40.$

Def $\quad E(Y|X=x) = \sum\limits_y y\, P(Y=y|X=x) = \sum\limits_y y\, p_{Y|X}(y|x).$

Y given X = 1

y	$p_{Y\|X}(y\|1)$	
1	0.35	0.35
2	0.30	0.60
3	0.35	1.05
	1.00	2.00

$E(Y|X=1) = 2.00$

Y given X = 2

y	$p_{Y\|X}(y\|2)$	
1	0.10	0.10
2	0.30	0.60
3	0.60	1.80
	1.00	2.50

$E(Y|X=2) = 2.50$

Denote by $\mathrm{E}(\mathrm{Y}|\mathrm{X})$ that function of the random variable X whose value at $\mathrm{X}=x$ is $\mathrm{E}(\mathrm{Y}|\mathrm{X}=x)$. Note that $\mathrm{E}(\mathrm{Y}|\mathrm{X})$ is itself a random variable, it depends on the (random) value of X that occurs.

$\mathrm{E}(\mathrm{Y}|\mathrm{X})$:

	values	probabilities
x	$\mathrm{E}(\mathrm{Y}\vert\mathrm{X}=x)$	$p_{\mathrm{X}}(x)$
1	2.00	0.40
2	2.50	0.60
		1.00

$$\mathrm{E}[\mathrm{E}(\mathrm{Y}|\mathrm{X})] = 2.00\cdot 0.40 + 2.50\cdot 0.60 = 2.30 = \mathrm{E}(\mathrm{Y}).$$

<u>Fact</u>: $\mathrm{E}[\mathrm{E}(\mathrm{Y}|\mathrm{X})] = \mathrm{E}(\mathrm{Y})$.

<u>Proof</u>: $\mathrm{E}(\mathrm{Y}|\mathrm{X})$ is a function of X.

If $\mathrm{X}=x$, then $\mathrm{E}(\mathrm{Y}|\mathrm{X})=\mathrm{E}(\mathrm{Y}|\mathrm{X}=x)$.

Recall: $$\mathrm{E}[g(\mathrm{X})] = \sum_{\text{all } x} g(x)\cdot p_{\mathrm{X}}(x).$$

$$\begin{aligned}
\mathrm{E}[\mathrm{E}(\mathrm{Y}|\mathrm{X})] &= \sum_{\text{all } x} \mathrm{E}(\mathrm{Y}|\mathrm{X}=x)\cdot p_{\mathrm{X}}(x) \\
&= \sum_{\text{all } x}\left[\sum_{\text{all } y} y\cdot p_{\mathrm{Y}|\mathrm{X}}(y|x)\right]\cdot p_{\mathrm{X}}(x) \\
&= \sum_{\text{all } x}\left[\sum_{\text{all } y} y\cdot \frac{p(x,y)}{p_{\mathrm{X}}(x)}\right]\cdot p_{\mathrm{X}}(x) \\
&= \sum_{\text{all } x}\sum_{\text{all } y} y\cdot p(x,y) \\
&= \mathrm{E}(\mathrm{Y}).
\end{aligned}$$

X given Y = 1

x	$p_{X\|Y}(x\|1)$
1	$\frac{0.14}{0.20} = 0.70$
2	$\frac{0.06}{0.20} = 0.30$
	1.00

$E(X \mid Y=1) = 1.30$

X given Y = 2

x	$p_{X\|Y}(x\|2)$
1	$\frac{0.12}{0.30} = 0.40$
2	$\frac{0.18}{0.30} = 0.60$
	1.00

$E(X \mid Y=2) = 1.60$

X given Y = 3

x	$p_{X\|Y}(x\|3)$
1	$\frac{0.14}{0.50} = 0.28$
2	$\frac{0.36}{0.50} = 0.72$
	1.00

$E(X \mid Y=3) = 1.72$

$E(X \mid Y)$:

y	values $E(X \mid Y=y)$	probabilities $p_Y(y)$
1	1.30	0.20
2	1.60	0.30
3	1.72	0.50
		1.00

Def Conditional probability density function of Y given $\mathrm{X} = x$ is defined by

$$f_{\mathrm{Y}|\mathrm{X}}(y|x) = \frac{f(x,y)}{f_{\mathrm{X}}(x)}.$$

For a fixed x,

- $f_{\mathrm{Y}|\mathrm{X}}(y|x) \geq 0 \qquad \forall\, y.$

- $$\int_{-\infty}^{\infty} f_{\mathrm{Y}|\mathrm{X}}(y|x)\, dy = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_{\mathrm{X}}(x)}\, dy = \frac{1}{f_{\mathrm{X}}(x)} \int_{-\infty}^{\infty} f(x,y)\, dy$$
  $$= \frac{1}{f_{\mathrm{X}}(x)} f_{\mathrm{X}}(x) = 1.$$

Def $\mathrm{E}(\mathrm{Y}\,|\,\mathrm{X} = x) = \int_{-\infty}^{\infty} y \cdot f_{\mathrm{Y}|\mathrm{X}}(y|x)\, dy.$

Def Conditional probability density function of X given $\mathrm{Y} = y$ is defined by

$$f_{\mathrm{X}|\mathrm{Y}}(x|y) = \frac{f(x,y)}{f_{\mathrm{Y}}(y)}.$$

For a fixed y,

- $f_{\mathrm{X}|\mathrm{Y}}(x|y) \geq 0 \qquad \forall\, x.$

- $$\int_{-\infty}^{\infty} f_{\mathrm{X}|\mathrm{Y}}(x|y)\, dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_{\mathrm{Y}}(y)}\, dx = \frac{1}{f_{\mathrm{Y}}(y)} \int_{-\infty}^{\infty} f(x,y)\, dx$$
  $$= \frac{1}{f_{\mathrm{Y}}(y)} f_{\mathrm{Y}}(y) = 1.$$

Def $\mathrm{E}(\mathrm{X}\,|\,\mathrm{Y} = y) = \int_{-\infty}^{\infty} x \cdot f_{\mathrm{X}|\mathrm{Y}}(x|y)\, dx.$

1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{4}{3} x^3 y, \qquad 0 < x < 1, \quad 0 < y < 3x, \qquad \text{zero otherwise.}$$

Recall: $$f_X(x) = 6x^5, \qquad 0 < x < 1.$$

$$f_Y(y) = \frac{1}{3}y - \frac{1}{243}y^5, \qquad 0 < y < 3.$$

h) Find $P(Y > 1 \mid X = \frac{2}{3})$.

$$f_X(x) = \int_0^{3x} \frac{4}{3} x^3 y \, dy = 6x^5, \quad 0 < x < 1.$$

$$f_{Y|X}(y|x) = \frac{\frac{4}{3} x^3 y}{6x^5} = \frac{2y}{9x^2}, \quad 0 < y < 3x.$$

$$f_{Y|X}(y|\tfrac{2}{3}) = \frac{y}{2}, \quad 0 < y < 2.$$

$$P(Y > 1 \mid X = \tfrac{2}{3}) = \int_1^2 \frac{y}{2} \, dy = \frac{y^2}{4} \Big|_1^2 = \frac{3}{4}.$$

i) Find $E(Y \mid X = x)$.

$$E(Y|X=x) = \int_0^{3x} y \cdot \frac{2y}{9x^2} \, dy = \frac{2}{3} \frac{(3x)^3}{9x^2} = 2x,$$

$$0 < x < 1.$$

j) Find $P\left(X < \frac{2}{3} \mid Y = 1\right)$.

$$f_Y(y) = \int_{y/3}^{1} \frac{4}{3} x^3 y \, dx = \left(\frac{1}{3} x^4 y\right)_{x=\frac{y}{3}}^{x=1}$$

$$= \frac{1}{3} y - \frac{1}{243} y^5, \qquad 0 < y < 3.$$

$$f_{X|Y}(x|y) = \frac{\frac{4}{3} x^3 y}{\frac{1}{3} y - \frac{1}{243} y^5} = \frac{4x^3}{1 - \frac{y^4}{81}}, \qquad \frac{y}{3} < x < 1.$$

$$f_{X|Y}(x|1) = \frac{81}{20} x^3, \qquad \frac{1}{3} < x < 1.$$

$$P\left(X < \tfrac{2}{3} \mid Y = 1\right) = \int_{1/3}^{2/3} \frac{81}{20} x^3 \, dx = \frac{81}{80} x^4 \Big|_{1/3}^{2/3} = \frac{15}{80} = \frac{3}{16}.$$

k) Find $E(X \mid Y = y)$.

$$E(X|Y=y) = \int_{y/3}^{1} x \cdot \frac{4x^3}{1 - \frac{y^4}{81}} \, dx = \frac{4}{5} \cdot \frac{1 - \frac{y^5}{243}}{1 - \frac{y^4}{81}} = \frac{4}{15} \cdot \frac{243 - y^5}{81 - y^4},$$

$$0 < y < 3.$$