Examples for 11/11/2020 (1)

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2. Suppose $n = 49$ observations are taken from a normal distribution where $\sigma = 8.0$ for the purpose of testing $H_0 : \mu = 60$ versus $H_1 : \mu > 60$.

a) What is the significance level associated with the rejection region "Reject H_0 if $\overline{x} > 62$"?

significance level $= \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\overline{X} > 62 \mid \mu = 60)$

$$= P\left(Z > \frac{62 - 60}{8 / \sqrt{49}} \right) = P(Z > 1.75) = \mathbf{0.0401}.$$

b) Find the power of the rejection region in part (a) if the true mean is $\mu_1 = 61$ and if $\mu_1 = 62$.

$\text{Power}(\mu = 61) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 62 \mid \mu = 61)$

$$= P\left(Z > \frac{62 - 61}{8 / \sqrt{49}} \right) = P(Z > 0.875) = \mathbf{0.1908}.$$

$\text{Power}(\mu = 62) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 62 \mid \mu = 62) = \mathbf{0.50}.$

c) Find the "best" rejection region with the significance level $\alpha = 0.05$.

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha \qquad \Rightarrow \qquad \overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \qquad \overline{X} > 60 + 1.645 \frac{8}{\sqrt{49}} \qquad \Rightarrow \qquad \overline{X} > \mathbf{61.88}$$

d) Find the power of the test if the true mean is $\mu_1 = 61$ at the $\alpha = 0.05$ level of significance.

$$\text{Power}(\mu = 61) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 61.88 \mid \mu = 61)$$

$$= P\left(Z > \frac{61.88 - 61}{8/\sqrt{49}} \right) = P(Z > 0.77) = \mathbf{0.2206}.$$

e) Repeat part (d) for the case when the true value of the mean is $\mu_1 = 62$ and $\mu_1 = 63$.

$$\text{Power}(\mu = 62) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 61.88 \mid \mu = 62)$$

$$= P\left(Z > \frac{61.88 - 62}{8/\sqrt{49}} \right) = P(Z > -0.105) = \mathbf{0.5418}.$$

$$\text{Power}(\mu = 63) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 61.88 \mid \mu = 63)$$

$$= P\left(Z > \frac{61.88 - 63}{8/\sqrt{49}} \right) = P(Z > -0.98) = \mathbf{0.8365}.$$

f) Repeat parts (c) – (e) using a larger sample size of $n = 100$.

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha \quad \Rightarrow \quad \overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \overline{X} > 60 + 1.645 \frac{8}{\sqrt{100}} \quad \Rightarrow \quad \overline{X} > \mathbf{61.316}$$

$$\text{Power}(\mu = 61) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\overline{X} > 61.316 \mid \mu = 61)$$

$$= P\left(Z > \frac{61.316 - 61}{8/\sqrt{100}} \right) = P(Z > 0.395) = \mathbf{0.3464}.$$

$$\text{Power}(\mu = 62) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\bar{X} > 61.316 \mid \mu = 62)$$

$$= P\left(Z > \frac{61.316 - 62}{8/\sqrt{100}}\right) = P(Z > -0.855) = \mathbf{0.8037}.$$

$$\text{Power}(\mu = 63) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\bar{X} > 61.316 \mid \mu = 63)$$

$$= P\left(Z > \frac{61.316 - 63}{8/\sqrt{100}}\right) = P(Z > -2.105) = \mathbf{0.98235}.$$

g) Repeat parts (c) – (e) at the $\alpha = 0.10$ level of significance.

Rejection Region: Reject H_0 if

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha \quad \Rightarrow \quad \bar{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \bar{X} > 60 + 1.282 \frac{8}{\sqrt{49}} \quad \Rightarrow \quad \bar{X} > \mathbf{61.465}$$

$$\text{Power}(\mu = 61) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\bar{X} > 61.465 \mid \mu = 61)$$

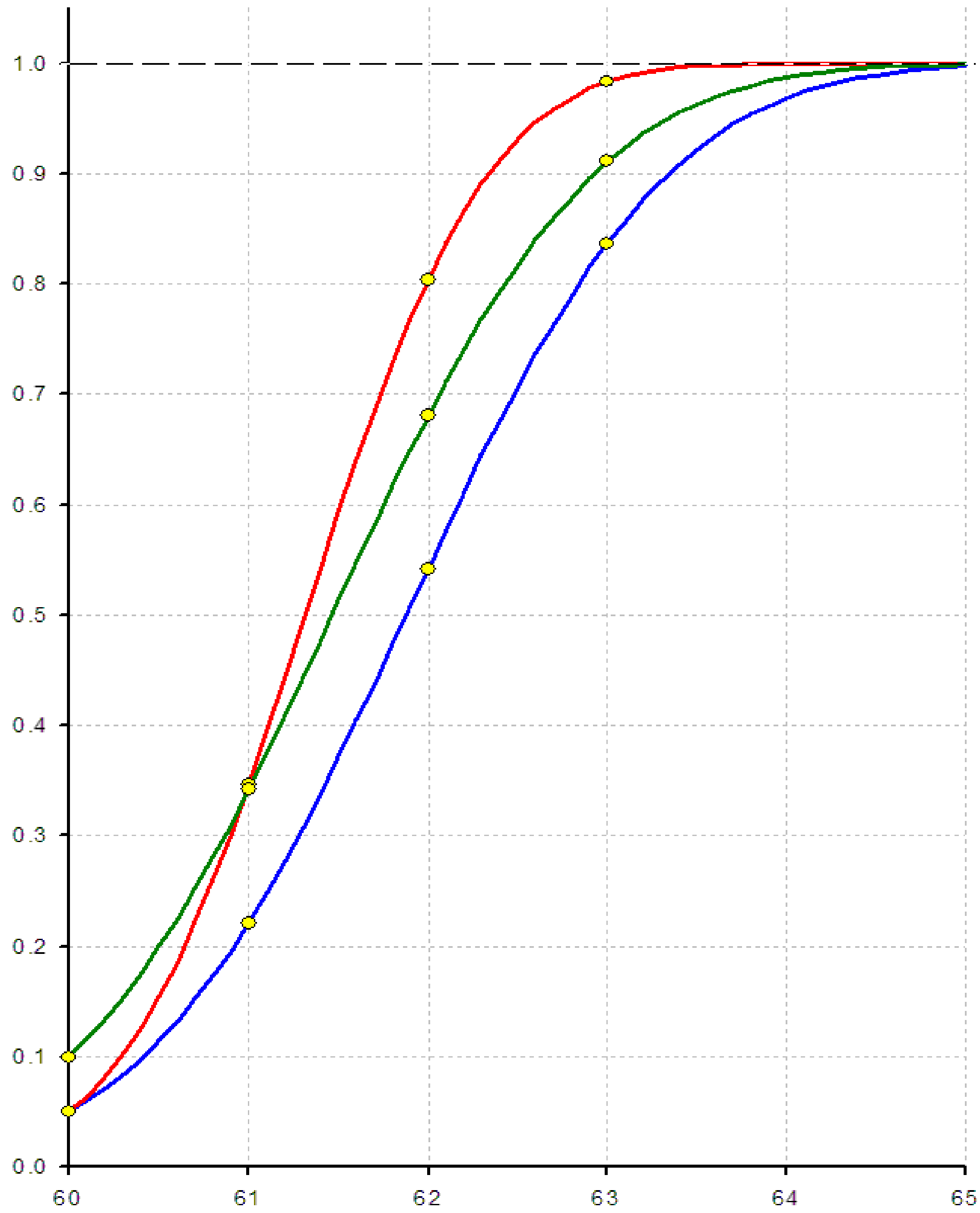
$$= P\left(Z > \frac{61.465 - 61}{8/\sqrt{49}}\right) = P(Z > 0.407) = \mathbf{0.3409}.$$

$$\text{Power}(\mu = 62) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\bar{X} > 61.465 \mid \mu = 62)$$

$$= P\left(Z > \frac{61.465 - 62}{8/\sqrt{49}}\right) = P(Z > -0.468) = \mathbf{0.6808}.$$

$$\text{Power}(\mu = 63) = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(\bar{X} > 61.465 \mid \mu = 63)$$

$$= P\left(Z > \frac{61.465 - 63}{8/\sqrt{49}}\right) = P(Z > -1.343) = \mathbf{0.9099}.$$


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
60
61
62
63
64
65

2. (continued)

Suppose that the sample mean is $\overline{x} = 61.6$ for a random sample of size $n = 49$.

h) Find the p-value of the appropriate test.

The observed value of the test statistic is $z = \dfrac{61.6 - 60}{8 / \sqrt{49}} = \mathbf{1.40}$.

Right – tailed test.

P-value $= \mathrm{P}(\mathrm{Z} \ge 1.40) = \mathbf{0.0808}$.

i) State your decision $(\text{Reject } \mathrm{H}_0 \text{ or Do NOT Reject } \mathrm{H}_0)$ for $\alpha = 0.05$.

P-value $= 0.0808 > 0.05 = \alpha$ **Do NOT Reject $\mathbf{H_0}$** at $\alpha = 0.05$.

OR

$z = 1.40 < 1.645 = z_{0.05}$ **Do NOT Reject $\mathbf{H_0}$** at $\alpha = 0.05$.

OR

$\overline{x} = 61.6 < 61.88$ **Do NOT Reject $\mathbf{H_0}$** at $\alpha = 0.05$.

3. Suppose $n = 49$ observations are taken from a normal distribution where $\sigma = 8.0$ for the purpose of testing $\mathrm{H}_0: \mu = 60$ versus $\mathrm{H}_1: \mu \ne 60$.

a) What is the significance level associated with the rejection region "Reject H_0 if $\overline{x} < 58$ or $\overline{x} > 62$"?

significance level $= \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true})$

$$= \mathrm{P}(\overline{\mathrm{X}} < 58 \mid \mu = 60) + \mathrm{P}(\overline{\mathrm{X}} > 62 \mid \mu = 60)$$

$$= \mathrm{P}\left(\mathrm{Z} < \frac{58 - 60}{8/\sqrt{49}}\right) + \mathrm{P}\left(\mathrm{Z} > \frac{62 - 60}{8/\sqrt{49}}\right)$$

$$= P(Z < -1.75) + P(Z > 1.75)$$

$$= 0.0401 + 0.0401 = \mathbf{0.0802}.$$

b) Find the “best” rejection region with the significance level $\alpha = 0.05$.

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} < -z_{\alpha/2} \quad \text{or} \quad Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} > z_{\alpha/2}$$

$$\Rightarrow \quad \overline{X} < \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \overline{X} > \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \overline{X} < 60 - 1.96 \frac{8}{\sqrt{49}} \quad \text{or} \quad \overline{X} > 60 + 1.96 \frac{8}{\sqrt{49}}$$

$$\Rightarrow \quad \overline{X} < \mathbf{57.76} \quad \text{or} \quad \overline{X} > \mathbf{62.24}$$

c) What is the power of ths test when $\mu = 61$ if the significance level is $\alpha = 0.05$?

$$\text{Power}(\mu = 61) = P(\text{Reject } H_0 \mid H_0 \text{ is false})$$

$$= P(\overline{X} < 57.76 \mid \mu = 61) + P(\overline{X} > 62.24 \mid \mu = 61)$$

$$= P\left(Z < \frac{57.76 - 61}{8 / \sqrt{49}}\right) + P\left(Z > \frac{62.24 - 61}{8 / \sqrt{49}}\right)$$

$$= P(Z < -2.835) + P(Z > 1.085)$$

$$= 0.0023 + 0.1390 = \mathbf{0.1413}.$$

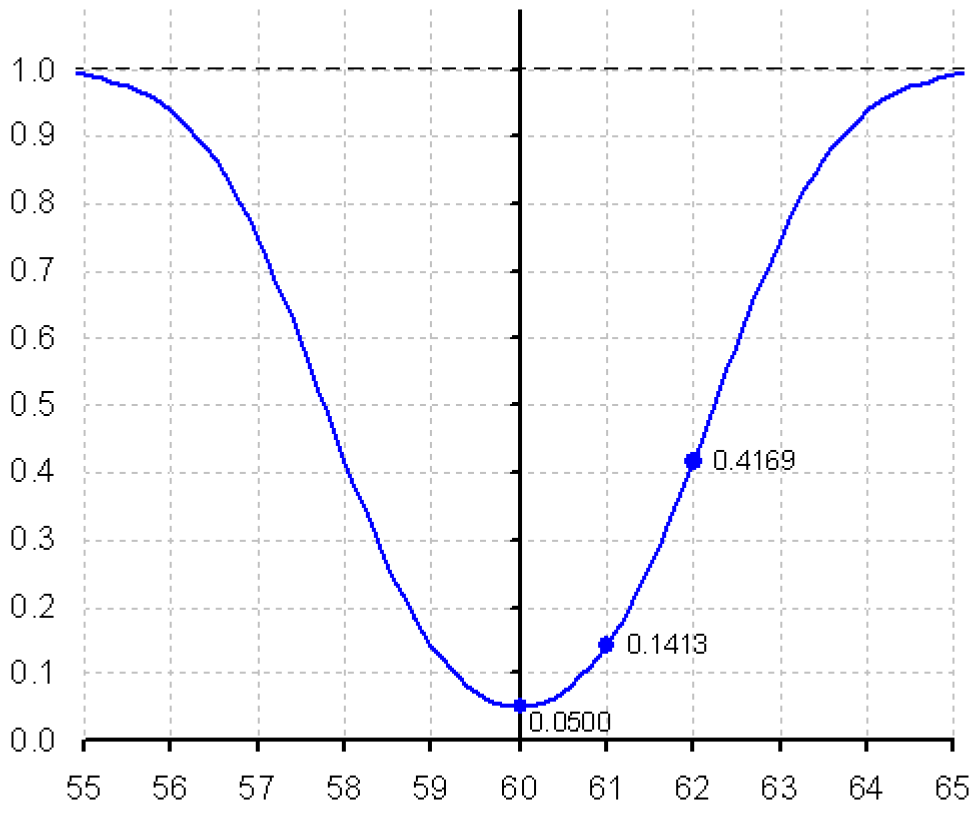
$$\text{Power}(\mu = 62) = P(\text{Reject } H_0 \mid H_0 \text{ is false})$$

$$= P(\overline{X} < 57.76 \mid \mu = 62) + P(\overline{X} > 62.24 \mid \mu = 62)$$

$$= P\left(Z < \frac{57.76 - 62}{8/\sqrt{49}} \right) + P\left(Z > \frac{62.24 - 62}{8/\sqrt{49}} \right)$$

$$= P(\,Z < -3.71\,) + P(\,Z > 0.21\,)$$

$$= 0.0001 + 0.4168 \;=\; \mathbf{0.4169}.$$



d) What is the p-value of the test if the observed value of the sample mean is $\overline{x} = 61.6$?

The observed value of the test statistic is $z = \dfrac{61.6 - 60}{8/\sqrt{49}} = \mathbf{1.40}$.

2 – tailed test.

P-value $= 2 \times P(Z \geq 1.40) = 2 \times 0.0808 = \mathbf{0.1616}$.