Examples for 09/28/2020 (1)

2. Three actuaries are independently hired to appraise the value of a company. The true value of the company is θ million dollars, and each actuary's estimate is uniformly distributed between $\theta - 2$ million dollars and $\theta + 3$ million dollars. Find the probability that the actual value of θ lies between the lowest and the highest estimate. That is, let X_1, X_2, X_3 be i.i.d. Uniform$(\theta - 2, \theta + 3)$. Find the probability $P(\min X_i < \theta < \max X_i)$. (from Actuarial Science Exam P)

$$P(\min X_i < \theta < \max X_i) = 1 - P(\min X_i > \theta) - P(\max X_i < \theta)$$

$$= 1 - [P(X > \theta)]^3 - [P(X < \theta)]^3$$

$$= 1 - \left(\frac{3}{5}\right)^3 - \left(\frac{2}{5}\right)^3 = 1 - \frac{27}{125} - \frac{8}{125}$$

$$= 1 - \frac{35}{125} = \frac{90}{125} = \mathbf{\frac{18}{25}} = \mathbf{0.72}.$$

OR

$$P(\min X_i < \theta < \max X_i) = P(\text{either 1 or 2 estimates below } \theta)$$

$$= \binom{3}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^2 + \binom{3}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^1$$

$$= \frac{54}{125} + \frac{36}{125} = \frac{90}{125} = \mathbf{\frac{18}{25}} = \mathbf{0.72}.$$

3. Let X_1 and X_2 represent the independent failure times in years of two components. The respective p.d.f.s are

$$f_1(x) = \frac{2}{9}x, \quad 0 < x < 3; \qquad f_2(x) = \frac{3}{16}\sqrt{x}, \quad 0 < x < 4.$$

a) Suppose the components are placed in a series. Find the c.d.f. and the p.d.f. of time $Y_1 = \min(X_1, X_2)$ to failure for the system.

$$F_1(x) = \frac{x^2}{9}, \quad 0 < x < 3; \qquad F_2(x) = \frac{x^{3/2}}{8}, \quad 0 < x < 4.$$

$$1 - F_{\min X_i}(x) = P(\min X_i > x) = P(X_1 > x, X_2 > x) = P(X_1 > x) \cdot P(X_2 > x)$$

$$= (1 - F_1(x)) \cdot (1 - F_2(x)) = \left(1 - \frac{x^2}{9}\right) \cdot \left(1 - \frac{x^{3/2}}{8}\right) = 1 - \frac{x^2}{9} - \frac{x^{3/2}}{8} + \frac{x^{7/2}}{72},$$

$$0 < x < 3.$$

$$F_{\min X_i}(x) = \frac{x^2}{9} + \frac{x^{3/2}}{8} - \frac{x^{7/2}}{72}, \qquad 0 < x < 3.$$

$$f_{\min X_i}(x) = \frac{2}{9}x + \frac{3}{16}\sqrt{x} - \frac{7}{144}x^{5/2}, \qquad 0 < x < 3.$$

b) Suppose the components are placed in parallel. Find the c.d.f. and the p.d.f. of time $Y_2 = \max(X_1, X_2)$ to failure for the system.

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x) = P(X_1 \le x) \cdot P(X_2 \le x)$$

$$= F_1(x) \cdot F_2(x) = \ldots$$

$0 < x < 3 \qquad \ldots = \dfrac{x^2}{9} \cdot \dfrac{x^{3/2}}{8} = \dfrac{x^{7/2}}{72}$

$3 < x < 4 \qquad \ldots = 1 \cdot \dfrac{x^{3/2}}{8} = \dfrac{x^{3/2}}{8}$

$$f_{\max X_i}(x) = \begin{cases} \dfrac{7}{144} x^{5/2} & 0 < x < 3 \\ \dfrac{3}{16} x^{1/2} & 3 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

3½. Let $X_1, X_2, \ldots, X_n$ be a random sample (i.i.d.) from Uniform$(0, a)$ probability distribution. Let $Y_k = k^{\text{th}}$ smallest of $X_1, X_2, \ldots, X_n$. Find $E(Y_k)$.

$$f_{Y_k}(x) = \frac{n!}{(k-1)! \cdot (n-k)!} \cdot \left(\frac{x}{a}\right)^{k-1} \cdot \left(1 - \frac{x}{a}\right)^{n-k} \cdot \frac{1}{a}, \qquad 0 < x < a.$$

$$E(Y_k) = \int_0^a x \cdot \frac{n!}{(k-1)! \cdot (n-k)!} \cdot \left(\frac{x}{a}\right)^{k-1} \cdot \left(1 - \frac{x}{a}\right)^{n-k} \cdot \frac{1}{a}\, dx$$

$$= \frac{k}{n+1} \cdot a \cdot \int_0^1 \frac{\Gamma(n+2)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} \cdot u^k \cdot (1-u)^{n-k}\, du = \frac{k}{n+1} \cdot a,$$

since $\dfrac{\Gamma(n+2)}{\Gamma(k+1) \cdot \Gamma(n-k+1)} \cdot u^k \cdot (1-u)^{n-k}$, $0 < u < 1$, is the p.d.f. of a Beta distribution with $\alpha = k+1$, $\beta = n-k+1$.

See **4.4.9** (7th edition) **5.2.9** (6th edition)

4. Let X_1, X_2, X_3 be a random sample (i.i.d.) of size $n = 3$ from a Normal distribution with mean $\mu = 23$ and standard deviation $\sigma = 5$.

Let $Y_k = k^{\text{th}}$ smallest of X_1, X_2, X_3, $\quad k = 1, 2, 3$.

a) Find $P(Y_3 < 25) = P(\max X_i < 25)$.

$$P(\max X_i < 25) = [P(X < 25)]^3 = [P(Z < 0.40)]^3 = [0.6554]^3 \approx \mathbf{0.2815}.$$

b) Find $P(Y_3 > 30) = P(\max X_i > 30)$.

$$P(\max X_i > 30) = 1 - P(\max X_i < 30) = 1 - [P(X < 30)]^3$$
$$= 1 - [P(Z < 1.40)]^3 = 1 - [0.9192]^3 \approx \mathbf{0.2233}.$$

c) Find $P(Y_1 > 20) = P(\min X_i > 20)$.

$$P(\min X_i > 20) = [P(X > 20)]^3 = [P(Z > -0.60)]^3 = [0.7257]^3 \approx \mathbf{0.3822}.$$

d) Find $P(Y_1 < 15) = P(\min X_i < 15)$.

$$P(\min X_i < 15) = 1 - P(\min X_i > 15) = 1 - [P(X > 15)]^3$$
$$= 1 - [P(Z > -1.60)]^3 = 1 - [0.9452]^3 \approx \mathbf{0.1556}.$$

e) Find $P(Y_2 < 25)$.

$$P(Y_2 < 25) = P(\text{either 2 or 3 observations less than 25})$$
$$= {}_3C_2\,(0.6554)^2\,(0.3446)^1 + {}_3C_3\,(0.6554)^3\,(0.3446)^0 \approx \mathbf{0.7256}.$$

f) Find $P(Y_2 > 20)$.

$$P(Y_2 > 20) = P(\text{either 0 or 1 observations less than 20})$$
$$= {}_3C_0\,(0.2743)^0\,(0.7257)^3 + {}_3C_1\,(0.2743)^1\,(0.7257)^2 \approx \mathbf{0.8156}.$$

5. Consider a system consisting of three independent components. Suppose the lifetimes of these components follow Exponential distribution with mean 200 hours, 250 hours, and 500 hours, respectively.

a) Suppose that the system works only if at least one component is functional (e.g., parallel connection). Find the probability that the system still works after 300 hours. That is, find the probability $P(\max(X_1, X_2, X_3) > 300)$.

$$P(\max(X_1, X_2, X_3) > 300) = 1 - P(\max(X_1, X_2, X_3) \le 300)$$
$$= 1 - P(X_1 \le 300,\ X_2 \le 300,\ X_3 \le 300)$$
$$= 1 - \left(1 - e^{-300/200}\right) \bullet \left(1 - e^{-300/250}\right) \bullet \left(1 - e^{-300/500}\right) \approx 0.755.$$

b) Suppose that the system works only if all three components are functional (e.g., series connection). Find the probability that the system still works after 300 hours. That is, find the probability $P(\min(X_1, X_2, X_3) > 300)$.

$$P(\min(X_1, X_2, X_3) > 300) = P(X_1 > 300,\ X_2 > 300,\ X_3 > 300)$$
$$= e^{-300/200} \bullet e^{-300/250} \bullet e^{-300/500} = \boldsymbol{e^{-3.3}} \approx 0.0369.$$

6. Let $T_1, T_2, \ldots, T_k$ be independent Exponential random variables. Suppose $E(T_i) = \frac{1}{\lambda_i}$, $i = 1, 2, \ldots, k$. That is, $f_{T_i}(t) = \lambda_i e^{-\lambda_i t}$, $t > 0$, $i = 1, 2, \ldots, k$.

Denote $T_{\min} = \min(T_1, T_2, \ldots, T_k)$.

a) Show that $T_{\min}$ also has an Exponential distribution. What is the mean of $T_{\min}$?

"Hint": Consider $P(T_{\min} > t) = P(T_1 > t \text{ AND } T_2 > t \text{ AND } \ldots \text{ AND } T_k > t)$.

If X has Exponential distribution with mean θ, $\qquad f_X(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0,$

$$P(X > t) = \int_t^\infty \frac{1}{\theta} e^{-x/\theta}\, dx = \left(-e^{-x/\theta} \right) \Big|_t^\infty = e^{-t/\theta}, \qquad t > 0.$$

Since $T_1, T_2, \ldots, T_k$ are independent,

$$\begin{aligned} P(T_{\min} > t) &= P(T_1 > t \text{ AND } T_2 > t \text{ AND } \ldots \text{ AND } T_k > t) \\ &= P(T_1 > t) \times P(T_2 > t) \times \ldots \times P(T_k > t) \\ &= e^{-\lambda_1 t} \times e^{-\lambda_2 t} \times \ldots \times e^{-\lambda_k t} \\ &= e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_k) t}, \qquad t > 0. \end{aligned}$$

$$F_{T_{\min}}(t) = P(T_{\min} \le t) = 1 - P(T_{\min} > t) = 1 - e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_k) t}, \qquad t > 0.$$

$$f_{T_{\min}}(t) = (\lambda_1 + \lambda_2 + \ldots + \lambda_k)\, e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_k) t}, \qquad t > 0.$$

$\Rightarrow \quad T_{\min}$ has an Exponential distribution with mean $\dfrac{1}{\lambda_1 + \lambda_2 + \ldots + \lambda_k}$.

b) Find $P(T_1 = T_{min}) = P(T_1$ is the smallest of $T_1, T_2, \dots, T_k)$

$= P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k)$.

Since $T_1, T_2, \dots, T_k$ are independent, their joint probability density function is

$$f(t_1, t_2, \dots, t_k) = \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \dots \lambda_k e^{-\lambda_k t_k},$$

$$t_1 > 0, \quad t_2 > 0, \dots, \quad t_k > 0.$$

$$P(T_1 = T_{min}) = P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k)$$

$$= \int_0^\infty \left(\int_{t_1}^\infty \dots \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \dots \lambda_k e^{-\lambda_k t_k} \, dt_2 \dots dt_k \right) dt_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} \left(\int_{t_1}^\infty \lambda_2 e^{-\lambda_2 t_2} \, dt_2 \right) \dots \left(\int_{t_1}^\infty \lambda_k e^{-\lambda_k t_k} \, dt_k \right) dt_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \dots e^{-\lambda_k t_1} \, dt_1$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \dots + \lambda_k}.$$

$\{T_1 > t\}, \{T_2 > t\}, \dots, \{T_k > t\}$ are independent

(t is a (non-random) real-valued variable).

$\{T_1 < T_2\}, \dots, \{T_1 < T_k\}$ are **NOT** independent, since they all involve T_1.

For Exponential distribution, $\text{mean} = \dfrac{1}{\text{rate}}$, $\text{rate} = \dfrac{1}{\text{mean}}$, $\theta = \dfrac{1}{\lambda}$, $\lambda = \dfrac{1}{\theta}$.

7. Suppose $X_1, X_2, \dots, X_n$ are independent random variables, and X_i has Geometric distribution with probability of "success" p_i, $i = 1, 2, \dots, n$. Let $Y = \min X_i$. What is the probability distribution of Y?

Let X be a random variable with a Geometric distribution with probability of "success" p. Then

$$P(X > y) = \sum_{k=y+1}^{\infty} p \cdot (1-p)^{k-1} = (1-p)^y, \qquad y = 0, 1, 2, 3, \dots .$$

Let y be a positive integer.

$$P(Y > y) = P(X_1 > y) \cdot P(X_2 > y) \cdot \dots \cdot P(X_n > y)$$

$$= (1-p_1)^y \cdot (1-p_2)^y \cdot \dots \cdot (1-p_n)^y = \left(\prod_{i=1}^{n} (1-p_i) \right)^y,$$

$$y = 0, 1, 2, 3, \dots .$$

$$p_Y(y) = P(Y = y) = P(Y > y-1) - P(Y > y)$$

$$= \left(1 - \prod_{i=1}^{n} (1-p_i) \right) \cdot \left(\prod_{i=1}^{n} (1-p_i) \right)^{y-1}, \qquad y = 1, 2, 3, \dots .$$

Y has a Geometric distribution with probability of "success"

$$p = 1 - \prod_{i=1}^{n} (1-p_i).$$

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Theorem Let $Y_1 < Y_2 < \dots < Y_n$ be the order statistics of a random sample $X_1, X_2, \dots, X_n$ from a continuous distribution with p.d.f. $f(x)$ and support (a, b). Then the joint p.d.f. of $Y_1, Y_2, \dots, Y_n$ is given by

$$g(y_1, y_2, \dots, y_n) = n!\, f(y_1) f(y_2) \dots f(y_n), \qquad a < y_1 < y_2 < \dots < y_n < b,$$

$$= 0 \quad \text{elsewhere.}$$

(Since there are $n!$ ways to reorder y_i's to get x_i's.)