STAT 410

Examples for 10/21/2020 (2)

Fall 2020
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Examples for 10/19/2020 (4) (continued)

1. The Pareto probability distribution has many applications in economics, biology, and physics. Let $\beta > 0$ and $\delta > 0$ be the population parameters, and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x; \beta, \delta) = \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}, \qquad x > \beta, \qquad \text{zero otherwise.}$$

Suppose δ is known.

l) Obtain the maximum likelihood estimator of β, $\hat{\beta}$.

m) Suppose $n = 5$, $\delta = 1.5$, and $x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$.
Find the maximum likelihood estimate of β.

n) Assume $n > \frac{1}{\delta}$. Is the maximum likelihood estimator $\hat{\beta}$ an unbiased estimator of β? If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\hat{\beta}$.

o) Assume $n > \frac{2}{\delta}$. Find $\operatorname{MSE}(\hat{\beta}) = (\operatorname{bias}(\hat{\beta}))^2 + \operatorname{Var}(\hat{\beta})$.

p) Assume $\delta > 1$. Obtain a method of moments estimator of β, $\tilde{\beta}$.

q) Suppose $n = 5$, $\delta = 1.5$, and $x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$.
Find a method of moments estimate of β.

r) Assume $\delta > 1$. Is the method of moments estimator of β, $\tilde{\beta}$, an unbiased estimator of β? If $\tilde{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\tilde{\beta}$.

s) Assume $\delta > 2$. Find $\operatorname{MSE}(\tilde{\beta}) = (\operatorname{bias}(\tilde{\beta}))^2 + \operatorname{Var}(\tilde{\beta})$.

t) Which estimator is "better", $\hat{\beta}$ or $\tilde{\beta}$? *Justify you decision.*

"Hint": $\quad \mathrm{E}(\overline{\mathrm{V}}) = \mu_{\mathrm{V}} = \mathrm{E}(\mathrm{V})$. $\qquad \operatorname{Var}(\overline{\mathrm{V}}) = \dfrac{\sigma_{\mathrm{V}}^2}{n} = \dfrac{\operatorname{Var}(\mathrm{V})}{n}$.

$\operatorname{Var}(\mathrm{V}) = \mathrm{E}(\mathrm{V}^2) - [\mathrm{E}(\mathrm{V})]^2$.

$\mathrm{E}(a☺) = a\,\mathrm{E}(☺)$. $\qquad \operatorname{Var}(a☺) = a^2 \operatorname{Var}(☺)$.

$F_{\max X_i}(x) = (F(x))^n$. $\qquad f_{\max X_i}(x) = n \cdot (F(x))^{n-1} \cdot f(x)$.

$F_{\min X_i}(x) = 1 - (1 - F(x))^n$. $\qquad f_{\min X_i}(x) = n \cdot (1 - F(x))^{n-1} \cdot f(x)$.

Answers:

1. The Pareto probability distribution has many applications in economics, biology, and physics. Let $\beta > 0$ and $\delta > 0$ be the population parameters, and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta,\delta) = \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}, \qquad x > \beta, \qquad \text{zero otherwise.}$$

Suppose δ is known.

l) Obtain the maximum likelihood estimator of β, $\hat{\beta}$.

$$L(\beta) = \prod_{i=1}^{n} f\left(x_i ; \beta, \delta\right) = \ldots$$

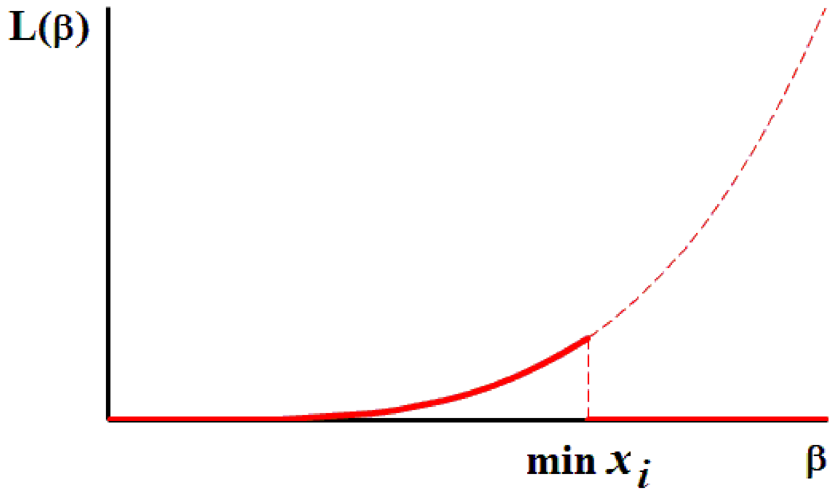
$$\ldots = \delta^{n} \cdot \beta^{n\delta} \cdot \left(\prod_{i=1}^{n} x_i\right)^{-(\delta+1)}, \qquad x_1 > \beta,\ x_2 > \beta,\ \ldots,\ x_n > \beta,$$

$$\ldots = 0, \qquad \text{otherwise.}$$

$$\ln L(\beta) = n \cdot \ln \delta + n\,\delta \cdot \ln \beta - (\delta + 1) \cdot \sum_{i=1}^{n} \ln x_i .$$

$$\frac{d}{d\beta} \ln L(\beta) = \frac{n\,\delta}{\beta} = 0 \qquad ???$$

$$x_1 > \beta,\ x_2 > \beta,\ \dots\ ,\ x_n > \beta \qquad \Leftrightarrow \qquad \beta < \min x_i .$$



$$\Rightarrow \qquad \hat{\beta} = \mathbf{\min X_i} .$$

m) Suppose $n = 5$, $\delta = 1.5$, and $x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$.
Find the maximum likelihood estimate of β.

$x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$. $\qquad \hat{\beta} = \min x_i = \mathbf{3.9}$.

n) Assume $n > \dfrac{1}{\delta}$. Is the maximum likelihood estimator $\hat{\beta}$ an unbiased estimator of β? If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\hat{\beta}$.

$$F_X(x) = P(X \le x) = \int_\beta^x \frac{\delta \cdot \beta^\delta}{u^{\delta+1}}\, du = -\frac{\beta^\delta}{u^\delta}\Bigg|_\beta^x = 1 - \frac{\beta^\delta}{x^\delta}, \qquad x > \beta.$$

$$F_{\min X_i}(x) = 1 - \left(1 - F_X(x)\right)^n = 1 - \left(\frac{\beta^\delta}{x^\delta}\right)^n = 1 - \frac{\beta^{\delta n}}{x^{\delta n}}, \qquad x > \beta.$$

$$f_{\min X_i}(x) = \frac{\delta\, n\, \beta^{\delta n}}{x^{\delta n + 1}}, \qquad x > \beta.$$

$$E(\hat{\beta}) = E(\min X_i) = \int_\beta^\infty x \cdot \frac{\delta\, n\, \beta^{\delta n}}{x^{\delta n + 1}}\, dx = \delta\, n\, \beta^{\delta n} \cdot \int_\beta^\infty x^{-\delta n}\, dx$$

$$= \frac{\delta n}{\delta n - 1}\, \beta \ne \beta.$$

$\hat{\beta}$ is NOT an unbiased estimator of β.

Consider $\quad \hat{\hat{\beta}} = \dfrac{\delta n - 1}{\delta n}\, \hat{\beta} = \dfrac{\delta n - 1}{\delta n}\, \min X_i.$

$$E(\hat{\hat{\beta}}) = \frac{\delta n - 1}{\delta n}\, E(\hat{\beta}) = \beta.$$

$\hat{\hat{\beta}}$ is an unbiased estimator of β.

o) Assume $n > \frac{2}{\delta}$. Find $\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta})$.

$$\mathrm{E}(\hat{\beta}^2) = \mathrm{E}[(\min \mathrm{X}_i)^2] = \int_{\beta}^{\infty} x^2 \cdot \frac{\delta\, n\, \beta^{\delta n}}{x^{\delta n + 1}}\, dx = \frac{\delta\, n}{\delta\, n - 2}\, \beta^2.$$

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}(\min \mathrm{X}_i) = \frac{\delta\, n}{\delta\, n - 2}\, \beta^2 - \left(\frac{\delta\, n}{\delta\, n - 1}\, \beta \right)^2 = \frac{\beta^2\, \delta\, n}{(\delta\, n - 2)\, (\delta\, n - 1)^2}.$$

$$\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta = \frac{\delta\, n}{\delta\, n - 1}\, \beta - \beta = \frac{\beta}{\delta\, n - 1}.$$

$$\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta}) = \left(\frac{\beta}{\delta\, n - 1} \right)^2 + \frac{\beta^2\, \delta\, n}{(\delta\, n - 2)\, (\delta\, n - 1)^2}$$

$$= \frac{2\, \beta^2}{(\delta\, n - 2)\, (\delta\, n - 1)}.$$

p) Assume $\delta > 1$. Obtain a method of moments estimator of β, $\tilde{\beta}$.

$$E(X) = \int_{\beta}^{\infty} x \cdot \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}\, dx = \delta \beta^{\delta} \cdot \int_{\beta}^{\infty} x^{-\delta}\, dx = \frac{\beta \delta}{\delta - 1}.$$

$$\overline{X} = \frac{\beta \delta}{\delta - 1}. \qquad \Rightarrow \qquad \tilde{\beta} = \frac{\delta - 1}{\delta} \overline{X}.$$

q) Suppose $n = 5$, $\delta = 1.5$, and $x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$.
Find a method of moments estimate of β.

$n = 5$, $\beta = 3$, $x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$.

$$\sum_{i=1}^{n} x_i = 38.1. \qquad \overline{x} = 7.62. \qquad \tilde{\beta} = \frac{1.5 - 1}{1.5} \cdot 7.62 = \mathbf{2.54}.$$

r) Assume $\delta > 1$. Is the method of moments estimator of β, $\tilde{\beta}$, an unbiased estimator of β? If $\tilde{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\tilde{\beta}$.

$$E(\tilde{\beta}) = \frac{\delta - 1}{\delta} E\left(\overline{X}\right) = \frac{\delta - 1}{\delta} \mu = \frac{\delta - 1}{\delta} \cdot \frac{\beta \delta}{\delta - 1} = \beta.$$

$\tilde{\beta}$ is an unbiased estimator of β.

s) Assume $\delta > 2$. Find $\text{MSE}(\tilde{\beta}) = (\text{bias}(\tilde{\beta}))^2 + \text{Var}(\tilde{\beta})$.

$$\text{E}(\text{X}^2) = \int_{\beta}^{\infty} x^2 \cdot \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}\, dx = \frac{\beta^2 \delta}{\delta - 2}.$$

$$\sigma^2 = \text{Var}(\text{X}) = \text{E}(\text{X}^2) - [\text{E}(\text{X})]^2 = \frac{\beta^2 \delta}{\delta - 2} - \left(\frac{\beta \delta}{\delta - 1}\right)^2 = \frac{\beta^2 \delta}{(\delta - 2)(\delta - 1)^2}.$$

$$\text{Var}(\tilde{\beta}) = \left(\frac{\delta - 1}{\delta}\right)^2 \text{Var}\left(\overline{\text{X}}\right) = \left(\frac{\delta - 1}{\delta}\right)^2 \cdot \frac{\sigma^2}{n} = \frac{\beta^2}{(\delta - 2)\,\delta n}.$$

Since $\tilde{\beta}$ is an unbiased estimator of β, $\text{bias}(\tilde{\beta}) = 0$,

and $$\text{MSE}(\tilde{\beta}) = \text{Var}(\tilde{\beta}) = \frac{\beta^2}{(\delta - 2)\,\delta n}.$$

t) Which estimator is "better", $\hat{\beta}$ or $\tilde{\beta}$? *Justify you decision.*

$$\text{MSE}(\hat{\beta}) = \frac{2\beta^2}{(\delta n - 2)(\delta n - 1)} \sim \frac{const}{n^2}.$$

$$\text{MSE}(\tilde{\beta}) = \frac{\beta^2}{(\delta - 2)\,\delta n} \sim \frac{const}{n}.$$

$\text{MSE}(\hat{\beta})$ decreases faster than $\text{MSE}(\tilde{\beta})$ as n increases.

$\text{MSE}(\hat{\beta})$ will be smaller than $\text{MSE}(\tilde{\beta})$ for larger n.

$\hat{\beta}$ is a better estimator.