STAT 410

Examples for 10/21/2020 (Disc)

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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0.$$

a) Obtain the maximum likelihood estimator of β, $\hat{\beta}$.

That is, find $\hat{\beta} = \arg\max L(\beta) = \arg\max \ln L(\beta)$, where $L(\beta) = \prod_{i=1}^{n} f(x_i;\beta)$.

Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Find the maximum likelihood estimate of β.

b) Show that $W = \sqrt{X}$ has a Gamma distribution. What are its parameters?

c) Suppose $n = 3$ and $\beta = 6.25$. Find the probability $P\left(\sum_{i=1}^{3}\sqrt{X_i} > 2\right)$.

d) Is the maximum likelihood estimator of β, $\hat{\beta}$, an unbiased estimator of β?

If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator $\hat{\hat{\beta}}$ of β based on $\hat{\beta}$.

e) Find $\text{MSE}(\hat{\beta}) = (\text{bias}(\hat{\beta}))^2 + \text{Var}(\hat{\beta})$.

f) Find a closed-form expression for $E(X^k)$ for $k > -2$.

Hint 1.1: $u = \beta\sqrt{x}$.

Hint 1.2: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du, \quad a > 0.$

Hint 2: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

g) Obtain a method of moments estimator of β, $\tilde{\beta}$.

That is, if $E(X) = h(\beta)$, solve $\overline{X} = h(\tilde{\beta})$ for $\tilde{\beta}$.

Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Find a method of moments estimate of β.

h) Is the method of moments estimator of β, $\tilde{\beta}$, an unbiased estimator of β?

If $\tilde{\beta}$ is not an unbiased estimator of β, does $\tilde{\beta}$ underestimate or overestimate β (on average)?

Answers:

1. Let $\beta > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0.$$

a) Obtain the maximum likelihood estimator of β, $\hat{\beta}$.

That is, find $\hat{\beta} = \arg\max \mathrm{L}(\beta) = \arg\max \ln \mathrm{L}(\beta)$, where $\mathrm{L}(\beta) = \prod_{i=1}^{n} f(x_i;\beta)$.

Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Find the maximum likelihood estimate of β.

$$\mathrm{L}(\beta) = \prod_{i=1}^{n}\left(\frac{\beta^4}{12}\, x_i\, e^{-\beta\sqrt{x_i}}\right) = \frac{\beta^{4n}}{12^n}\left(\prod_{i=1}^{n} x_i\right) e^{-\beta\sum_{i=1}^{n}\sqrt{x_i}}$$

$$\ln \mathrm{L}(\beta) = 4n\cdot\ln\beta - n\cdot\ln 12 + \sum_{i=1}^{n}\ln x_i - \beta\cdot\sum_{i=1}^{n}\sqrt{x_i}$$

$$\left(\ln \mathrm{L}(\beta)\right)' = \frac{4n}{\beta} - \sum_{i=1}^{n}\sqrt{x_i} = 0$$

$$\Rightarrow \qquad \hat{\beta} = \frac{4n}{\sum_{i=1}^{n}\sqrt{X_i}}.$$

$n = 3, \qquad x_1 = 0.25, \quad x_2 = 0.36, \quad x_3 = 0.81.$

$$\sum_{i=1}^{n} \sqrt{x_i} = 2. \qquad\qquad \hat{\beta} = \frac{12}{2} = \mathbf{6}.$$

b) Show that $W = \sqrt{X}$ has a Gamma distribution. What are its parameters?

$$W = \sqrt{X} = g(X) \qquad\qquad x = w^2 = g^{-1}(w) \qquad\qquad \frac{dx}{dw} = 2w$$

$$f_W(w) = f_X(g^{-1}(w)) \cdot \left| \frac{dx}{dw} \right| = \frac{\beta^4}{12} w^2 e^{-\beta w} \cdot 2w$$

$$= \frac{\beta^4}{6} w^3 e^{-\beta w} = \frac{\beta^4}{\Gamma(4)} w^{4-1} e^{-\beta w}, \qquad w > 0.$$

$\Rightarrow \quad W = \sqrt{X}$ has Gamma$\left(\alpha = 4,\ \theta = \frac{1}{\beta} \right)$ distribution.

c) Suppose $n = 3$ and $\beta = 6.25$. Find the probability $P\left(\sum_{i=1}^{3} \sqrt{X_i} > 2 \right)$.

$\sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has Gamma$\left(\alpha = 4n,\ \theta = \frac{1}{\beta} \right)$ distribution.

$$P\left(\sum_{i=1}^{3} \sqrt{X_i} > 2 \right) = P\left(\text{Gamma}\left(\alpha = 12, \theta = \frac{1}{6.25} \right) > 2 \right) = P(T_{12} > 2)$$

$$= P(\text{Poisson}(6.25 \cdot 2) \le 12 - 1) = P(\text{Poisson}(12.5) \le 11) = \mathbf{0.406}.$$

OR

$$P(T_{12} > 2) = \int_2^\infty \frac{6.25^{12}}{\Gamma(12)} t^{12-1} e^{-6.25t}\, dt = \int_2^\infty \frac{6.25^{12}}{11!} t^{11} e^{-6.25t}\, dt = \ldots$$

OR

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T_\alpha/\theta = 2\lambda T_\alpha$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$2\lambda T_{12} = 2\beta T_{12} = 12.5\, T_{12} \sim \chi^2(2 \cdot 12) = \chi^2(24).$$

$$P(T_{12} > 2) = P(12.5\, T_{12} > 12.5 \cdot 2) = P(\chi^2(24) > 25) = \mathbf{0.405761}.$$

d) Is the maximum likelihood estimator of β, $\hat{\beta}$, an unbiased estimator of β?
If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator $\hat{\hat{\beta}}$ of β based on $\hat{\beta}$.

$$Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i \quad \text{has Gamma}(\alpha = 4n,\ \theta = \frac{1}{\beta}) \text{ distribution.}$$

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\, \Gamma(\alpha + m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha + m)}{\lambda^m\, \Gamma(\alpha)}, \qquad m > -\alpha.$$

$$E\left(\frac{1}{Y}\right) = E(Y^{-1}) = \frac{\Gamma(\alpha - 1)}{\lambda^{-1}\, \Gamma(\alpha)} = \frac{\lambda}{\alpha - 1} = \frac{\beta}{4n - 1}.$$

$$\mathrm{E}(\hat{\beta}) = \mathrm{E}\left(\frac{4n}{\mathrm{Y}}\right) = \frac{4n}{4n-1}\cdot\beta = \beta+\frac{\beta}{4n-1} \neq \beta.$$

$\hat{\beta}$ is NOT an unbiased estimator of β. $\hat{\beta}$ is an asymptotically unbiased estimator of β.

$$\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta})-\beta = \frac{\beta}{4n-1}.$$

Consider $\quad \hat{\hat{\beta}} = \dfrac{4n-1}{4n}\cdot\hat{\beta} = \dfrac{4n-1}{\sum_{i=1}^{n}\sqrt{\mathrm{X}_i}}.$

Then $\quad \mathrm{E}(\hat{\hat{\beta}}) = \dfrac{4n-1}{4n}\cdot \mathrm{E}(\hat{\beta}) = \beta.$

$\hat{\hat{\beta}}$ is an unbiased estimator of β.

e) Find $\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2+\mathrm{Var}(\hat{\beta})$.

$$\mathrm{E}\left(\frac{1}{\mathrm{Y}^2}\right) = \mathrm{E}(\mathrm{Y}^{-2}) = \frac{\Gamma(\alpha-2)}{\lambda^{-2}\,\Gamma(\alpha)} = \frac{\lambda^2}{(\alpha-2)(\alpha-1)} = \frac{\beta^2}{(4n-2)(4n-1)}.$$

$$\mathrm{Var}(\hat{\beta}) = 16n^2\,\mathrm{Var}\left(\frac{1}{\mathrm{Y}}\right) = 16n^2\left[\frac{\beta^2}{(4n-2)(4n-1)}-\frac{\beta^2}{(4n-1)^2}\right]$$

$$= \frac{16n^2\beta^2}{(4n-2)(4n-1)^2}.$$

$$\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2+\mathrm{Var}(\hat{\beta}) = \left(\frac{\beta}{4n-1}\right)^2+\frac{16n^2\beta^2}{(4n-2)(4n-1)^2}$$

$$= \frac{\left(16n^2+4n-2\right)\beta^2}{(4n-2)(4n-1)^2} = \frac{(4n+2)\beta^2}{(4n-2)(4n-1)}.$$

f) Find a closed-form expression for $E(X^k)$ for $k > -2$.

Hint 1.1: $u = \beta\sqrt{x}$. Hint 1.2: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du, \quad a > 0.$

Hint 2: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m \Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m \Gamma(\alpha)}, \qquad m > -\alpha.$$

$\Rightarrow$ $W = \sqrt{X}$ has Gamma$(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution.

$$E(X^k) = E(W^{2k}) = \frac{\Gamma(\alpha+m)}{\lambda^m \Gamma(\alpha)} = \frac{\Gamma(2k+4)}{\beta^{2k}\Gamma(4)} = \frac{\Gamma(2k+4)}{3!\cdot\beta^{2k}} = \frac{\Gamma(2k+4)}{6\beta^{2k}}.$$

OR

$$E(X^k) = \int_0^\infty x^k \cdot \frac{\beta^4}{12} x e^{-\beta\sqrt{x}}\, dx \qquad u = \beta\sqrt{x} \qquad du = \frac{\beta}{2\sqrt{x}}\, dx$$

$$x = \left(\frac{u}{\beta}\right)^2 \qquad dx = \frac{2u}{\beta^2}\, du$$

$$= \int_0^\infty \left(\frac{u}{\beta}\right)^{2k} \cdot \frac{\beta^4}{12}\left(\frac{u}{\beta}\right)^2 e^{-u} \cdot \frac{2u}{\beta^2}\, du$$

$$= \frac{1}{6\beta^{2k}} \int_0^\infty u^{2k+3} e^{-u}\, du = \frac{1}{6\beta^{2k}} \int_0^\infty u^{(2k+4)-1} e^{-u}\, du = \frac{\Gamma(2k+4)}{6\beta^{2k}}.$$

OR

$$E(X^k) = \int_0^\infty x^k \cdot \frac{\beta^4}{12} x e^{-\beta\sqrt{x}} dx \qquad u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}} dx$$

$$x = u^2 \qquad dx = 2u\,du$$

$$= \int_0^\infty u^{2k} \cdot \frac{\beta^4}{12} u^2 e^{-\beta u} \cdot 2u\,du$$

$$= \frac{\Gamma(2k+4)}{6\beta^{2k}} \int_0^\infty \frac{\beta^{2k+4}}{\Gamma(2k+4)} u^{2k+3} e^{-\beta u} du = \frac{\Gamma(2k+4)}{6\beta^{2k}},$$

since $\frac{\beta^{2k+4}}{\Gamma(2k+4)} u^{2k+3} e^{-\beta u}$ is the probability density function of

Gamma($\alpha = 2k+4,\ \theta = \frac{1}{\beta}$) distribution.

g) Obtain a method of moments estimator of β, $\tilde{\beta}$.

That is, if $E(X) = h(\beta)$, solve $\overline{X} = h(\tilde{\beta})$ for $\tilde{\beta}$.

Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Find a method of moments estimate of β.

$$E(X) = E(X^1) = \frac{\Gamma(6)}{6\beta^2} = \frac{5!}{6\beta^2} = \frac{20}{\beta^2}.$$

$$\overline{X} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i = \frac{20}{\tilde{\beta}^2} \qquad \Rightarrow \qquad \tilde{\beta} = \sqrt{\frac{20}{\overline{X}}} = \sqrt{\frac{20 \cdot n}{\sum_{i=1}^{n} X_i}}.$$

$n = 3$, $\quad x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

$$\sum_{i=1}^{n} x_i = 1.42. \qquad \bar{x} = \frac{1.42}{3}. \qquad \tilde{\beta} = \sqrt{\frac{20 \cdot 3}{1.42}} \approx \mathbf{6.5}.$$

OR

$$\mathrm{E}(\sqrt{\mathrm{X}}) = \mathrm{E}(\mathrm{X}^{1/2}) = \frac{\Gamma(5)}{6\,\beta^{1}} = \frac{4!}{6\,\beta} = \frac{4}{\beta}.$$

$$\overline{\sqrt{\mathrm{X}}} = \frac{1}{n} \cdot \sum_{i=1}^{n} \sqrt{\mathrm{X}_i} = \frac{4}{\tilde{\beta}} \qquad \Rightarrow \qquad \tilde{\beta} = \frac{4}{\overline{\sqrt{\mathrm{X}}}} = \frac{4n}{\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}}.$$

$n = 3$, $\quad x_1 = 0.25$, $\quad x_2 = 0.36$, $\quad x_3 = 0.81$.

$$\sum_{i=1}^{n} \sqrt{x_i} = 2. \qquad \overline{\sqrt{x}} = \frac{2}{3}. \qquad \tilde{\beta} = \frac{4}{2/3} = \frac{12}{2} = \mathbf{6}.$$

h) Is the method of moments estimator of β, $\tilde{\beta}$, an unbiased estimator of β?
If $\tilde{\beta}$ is not an unbiased estimator of β, does $\tilde{\beta}$ underestimate or overestimate β (on average)?

$$\mathrm{E}(\overline{\mathrm{X}}) = \mu = \mathrm{E}(\mathrm{X}^{1}) = \frac{\Gamma(6)}{6\,\beta^{2}} = \frac{20}{\beta^{2}}.$$

Consider $g(x) = \sqrt{\dfrac{20}{x}}$. $\qquad g''(x) = \sqrt{\dfrac{45}{4\,x^{5}}}$. $\qquad g(x)$ is convex.

By Jensen's Inequality,

$$\mathrm{E}(\tilde{\beta}) = \mathrm{E}[g(\overline{\mathrm{X}})] > g(\mathrm{E}(\overline{\mathrm{X}})) = g\left(\frac{20}{\beta^{2}}\right) = \beta.$$

$\tilde{\beta}$ is NOT an unbiased estimator of β.

$\tilde{\beta}$ overestimates β (on average).