STAT 410 | Fall 2020
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Homework #12

(due Friday, December 11, by 5:00 p.m. CST)

No credit will be given without supporting work.

12. Consider two (discrete) probability mass functions:

x	1	2	3	4	5	6	7	8	9
$p_0(x)$	0	0.05	0.05	0.10	0.20	0.40	0.10	0.05	0.05
$p_1(x)$	0.08	0.08	0.10	0.25	0.05	0.08	0.16	0.20	0

You have only a single observation of X on which to base your choice between

$$H_0 : \text{X has p.m.f. } p_0(x) \quad \text{vs.} \quad H_1 : \text{X has p.m.f. } p_1(x).$$

a) (Warm-up) Consider the rejection region Reject H_0 if $x \in \{1, 2, 3\}$. Find …

(i) … the significance level α, (ii) … the power.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X \in \{1, 2, 3\} \mid H_0 \text{ is true})$$

$$= p_0(1) + p_0(2) + p_0(3) = 0 + 0.05 + 0.05 = \mathbf{0.10}.$$

$$\text{Power} = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(X \in \{1, 2, 3\} \mid H_0 \text{ is NOT true})$$

$$= p_1(1) + p_1(2) + p_1(3) = 0.08 + 0.08 + 0.10 = \mathbf{0.26}.$$

b) Find $\lambda(x) = \dfrac{L(H_0; x)}{L(H_1; x)} = \dfrac{p_0(x)}{p_1(x)}$ for $x = 1, 2, 3, \dots, 9$.

x	1	2	3	4	5	6	7	8	9
$p_0(x)$	0	0.05	0.05	0.10	0.20	0.40	0.10	0.05	0.05
$p_1(x)$	0.08	0.08	0.10	0.25	0.05	0.08	0.16	0.20	0
$\lambda(x)$	**0**	**0.625**	**0.5**	**0.4**	**4**	**5**	**0.625**	**0.25**	∞

most extreme least extreme

$\lambda(x)$	0	0.25	0.4	0.5	0.625	4	5	∞
x	1	8	4	3	2 & 7	5	6	9

Intuition on $\lambda(1) = 0$:

If I observe $x = 0$, then I would know with 100% certainty that H_1 is true. (since $x = 0$ cannot be observed if H_0 is true).

$x = 0$ is the possible value of X that has the most evidence against for H_0.

A rejection region "Reject H_0 if $x = 0$" would have had 0% significance level, yet positive power ($p_1(0) = 0.08$).

Intuition on $\lambda(9) = \infty$:

If I observe $x = 9$, then I immediately know with 100% certainty that H_0 is true (since $x = 9$ cannot be observed if H_1 is true).

$x = 9$ is the possible value of X that has the most support for H_0.

Since $p_1(9) = 0$ and $p_0(9) > 0$, we are "infinitely more likely" to observe $x = 9$ if H_0 is true than if H_1 is true.

c) Find the most powerful rejection region with significance level $\alpha = 0.20$.

"Hint": Reject H_0 if $x \in \{ ??? \}$.

"Hint": The Neyman-Pearson lemma "strongly suggests" that the values with the smallest $\lambda(x)$ are added to the rejection region first.

most extreme least extreme

$\lambda(x)$	0	0.25	0.4	0.5	0.625	4	5	∞
x	1	8	4	3	2 & 7	5	6	9
$p_0(x)$	0	0.05	0.10	0.05	…	…	…	…

$\mathrm{P}_0(\{1, 8, 4, 3\}) = 0 + 0.05 + 0.10 + 0.05 = 0.20.$

Reject H_0 if $x \in \mathbf{\{1, 8, 4, 3\}}$.

d) Find the power of the rejection region from part (c).

$\mathrm{P}_1(\{1, 8, 4, 3\}) = 0.08 + 0.20 + 0.25 + 0.10 = \mathbf{0.63}.$

IF $n = 2$, then you have $9 \cdot 9 = 81$ possible samples (x_1, x_2) to consider.

IF $n = 3$, then you have $9 \cdot 9 \cdot 9 = 729$ possible samples (x_1, x_2, x_3) to consider. **…**

$n = 5$ on Final Exam? (obviously, with different numbers, so you do not have time to prepare)

8. Let $\beta > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x \mid \beta) = \beta\,(1-x)^{\beta-1}, \qquad 0 < x < 1, \qquad \beta > 0, \qquad \text{zero otherwise.}$$

Recall that the maximum likelihood estimator for β is $\hat{\beta} = \dfrac{n}{\sum_{i=1}^{n}\left(-\ln(1-X_i)\right)}$.

Let the prior p.d.f. of β be $\text{Gamma}(\alpha, \theta)$. That is,

$$\pi(\beta) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,\beta^{\alpha-1}\,e^{-\beta/\theta}, \qquad \beta > 0.$$

s) Find the posterior distribution of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

HINT: $(1-x) = e^{-(-\ln(1-x))}$.

$$f(x_1, x_2, \ldots x_n, \beta) = \prod_{i=1}^{n}\beta\,(1-x_i)^{\beta-1} \times \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,\beta^{\alpha-1}\,e^{-\beta/\theta}$$

$$= \ldots\ \beta^{n+\alpha-1}\,e^{-\beta\left(\sum_{i=1}^{n}(-\ln(1-x_i)) + \frac{1}{\theta}\right)}.$$

$\Rightarrow$ the posterior distribution of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$,

is **Gamma** with New $\alpha = n + \alpha$ and

$$\text{New } \theta = \frac{1}{\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}}.$$

t) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Show that it is a weighted average of the maximum likelihood estimate $\hat{\beta}$ and the prior mean $\alpha\,\theta$. (What are the weights?)

(conditional mean of β given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$)

$$= (\text{New } \alpha) \times (\text{New } \theta) = \frac{n+\alpha}{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right) + \frac{1}{\theta}}$$

$$= \frac{n}{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right)} \cdot \frac{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right)}{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right) + \frac{1}{\theta}} + \alpha\,\theta \cdot \frac{\frac{1}{\theta}}{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right) + \frac{1}{\theta}}$$

$$= (\text{MLE}) \cdot \frac{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right)}{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right) + \frac{1}{\theta}} + \begin{pmatrix}\text{prior}\\ \text{mean}\end{pmatrix} \cdot \frac{\frac{1}{\theta}}{\sum_{i=1}^{n}\left(-\ln\left(1-x_i\right)\right) + \frac{1}{\theta}}.$$

u) Use part (s) to construct a $(1-\gamma)100\%$ credible interval for β, given $X_1 = x_1$, $X_2 = x_2, \dots, X_n = x_n$.

$(\beta \mid x_1, x_2, \dots, x_n)$ has a

$$\text{Gamma}\left(\text{New } \alpha = n + \alpha,\ \text{New } \theta = \frac{1}{\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}}\right) \text{ distribution.}$$

$\dfrac{2}{\text{New } \theta}(\beta \mid x_1, x_2, \dots, x_n)$ has a $\chi^2(2 \times \text{New } \theta)$ distribution.

$2\left(\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \dfrac{1}{\theta}\right)(\beta \mid x_1, x_2, \dots, x_n)$ has a $\chi^2(2n + 2\alpha)$ distribution.

$$P\left(\chi^2_{1-\gamma/2}(2n+2\alpha) < 2\left(\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}\right)(\beta \mid x_1, x_2, \dots, x_n) < \chi^2_{\gamma/2}(2n+2\alpha)\right) = 1-\gamma.$$

$$P\left(\frac{\chi^2_{1-\gamma/2}(2n+2\alpha)}{2\left(\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}\right)} < (\beta \mid x_1, x_2, \dots, x_n) < \frac{\chi^2_{\gamma/2}(2n+2\alpha)}{2\left(\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}\right)}\right) = 1-\gamma.$$

$$\left(\frac{\chi^2_{1-\gamma/2}(2n+2\alpha)}{2\left(\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}\right)},\ \frac{\chi^2_{\gamma/2}(2n+2\alpha)}{2\left(\sum_{i=1}^{n}\left(-\ln(1-x_i)\right) + \frac{1}{\theta}\right)}\right)$$

is a $(1-\gamma)100\%$ credible interval for β, given $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$.

v) Suppose $n = 3$, and $x_1 = 0.31$, $x_2 = 0.77$, $x_3 = 0.93$.

Let $\alpha = 2$, $\theta = 1.20$.

$$n = 3, \qquad \sum_{i=1}^{n} \left(-\ln\left(1 - x_i\right) \right) = -\ln 0.69 - \ln 0.23 - \ln 0.07 \approx 4.5.$$

(i) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$\frac{n + \alpha}{\sum_{i=1}^{n} \left(-\ln\left(1 - x_i\right) \right) + \frac{1}{\theta}} \approx \frac{3 + 2}{4.5 + \frac{1}{1.20}} = \mathbf{0.9375}.$$

Last chance for Alex to be annoying:

$$\hat{\beta} \approx \frac{3}{4.5} = \frac{2}{3}. \qquad \text{Prior mean} = \alpha\,\theta = 2.4.$$

$$\frac{\sum_{i=1}^{n} \left(-\ln\left(1 - x_i\right) \right)}{\sum_{i=1}^{n} \left(-\ln\left(1 - x_i\right) \right) + \frac{1}{\theta}} \approx \frac{4.5}{4.5 + \frac{1}{1.2}} = 0.84375,$$

$$\frac{\frac{1}{\theta}}{\sum_{i=1}^{n} \left(-\ln\left(1 - x_i\right) \right) + \frac{1}{\theta}} \approx \frac{\frac{1}{1.2}}{4.5 + \frac{1}{1.2}} = 0.15625.$$

$$\frac{2}{3} \cdot 0.84375 + 2.4 \cdot 0.15625 = \mathbf{0.9375}.$$

(ii) Construct a 95% credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$\chi^2_{0.975}(10) = 3.247, \qquad \chi^2_{0.025}(10) = 20.48.$$

$$\left(\frac{3.247}{2 \cdot \left(4.5 + \frac{1}{1.2} \right)}, \frac{20.48}{2 \cdot \left(4.5 + \frac{1}{1.2} \right)} \right) \qquad \mathbf{(0.3044,\ 1.92)}$$