STAT 410
Fall 2020

Conflict Final Exam

Friday, December 18, 7:00 – 10:00 pm CST

There are 8 problems on the exam, most of them have multiple parts. The point value of each question is shown in parentheses before the question. The total number of points for the exam is 150.

Make sure that you include everything you wish to submit, and that the submission process has completed. You do not need to include the question statements with your work. However, please label you work clearly. Neatness and organization are appreciated. Please put your final answers at the end of your work and mark them clearly.

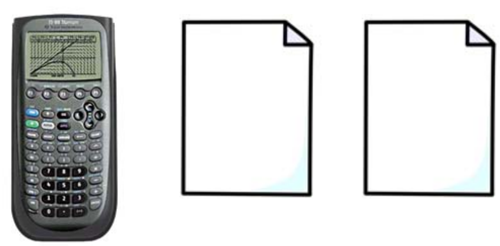
If the answer is a function, specify what kind of function it is (p.d.f., p.m.f., c.d.f., or m.g.f.), and **its support must be included**.

Be sure to show all your work; your partial credit might depend on it.

No credit will be given without supporting work.

The exam is closed book and closed notes.

You are allowed to use a calculator and
two 8½" x 11" sheets (both sides) with notes.



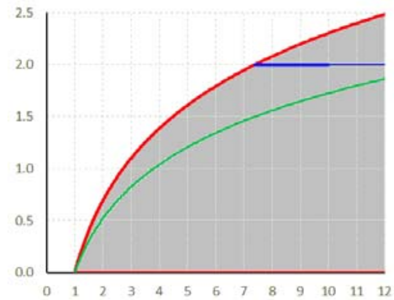
You are allowed to use
https://www.wolframalpha.com/calculators/integral-calculator/
https://www.symbolab.com/solver/definite-integral-calculator
https://www.integral-calculator.com/
https://www.desmos.com/calculator

You are allowed to use
R, RStudio, and Microsoft Excel.

1. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{8y^2}{x^3}, \qquad x > 1, \quad 0 < y < \ln x,$$

zero otherwise.



a) (9) Find $E(Y \mid X = x)$.

$$f_X(x) = \int_0^{\ln x} \frac{8y^2}{x^3}\,dy = \frac{8(\ln x)^3}{3x^3}, \qquad x > 1.$$

$$f_{Y|X}(y \mid x) = \frac{\dfrac{8y^2}{x^3}}{\dfrac{8(\ln x)^3}{3x^3}} = \frac{3y^2}{(\ln x)^3}, \qquad 0 < y < \ln x.$$

$$E(Y \mid X = x) = \int_0^{\ln x} y \cdot \frac{3y^2}{(\ln x)^3}\,dy = \frac{3}{4}\ln x, \qquad x > 1.$$

b) (9) Find $P(X < 10 \mid Y = 2)$.

$$f_Y(y) = \int_{e^y}^{\infty} \frac{8y^2}{x^3}\,dx = 4y^2 e^{-2y}, \qquad y > 0.$$

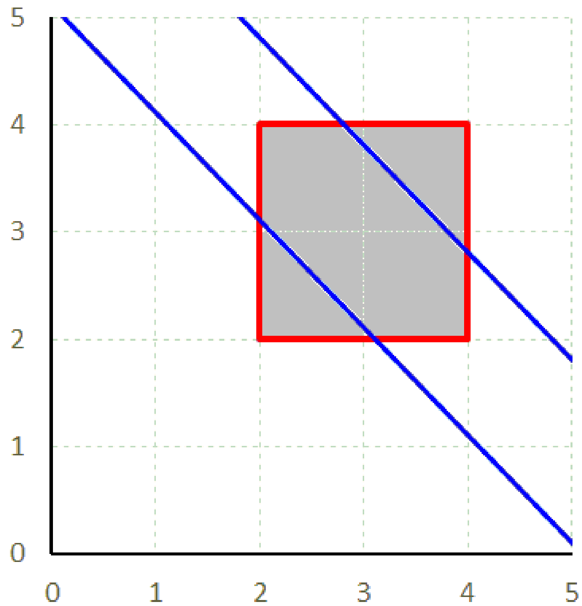
$$f_{X|Y}(x \mid y) = \frac{\dfrac{8y^2}{x^3}}{4y^2 e^{-2y}} = \frac{2e^{2y}}{x^3}, \qquad x > e^y.$$

$$P(X < 10 \mid Y = 2) = \int_{e^2}^{10} \frac{2e^4}{x^3}\,dx = 1 - \frac{e^4}{100} \approx 0.454018.$$

2. Suppose that the p.d.f. of X is $f_X(x) = \dfrac{x}{6}$, $2 < x < 4$, zero otherwise,

Y has a Uniform distribution on interval $(2, 4)$, and X and Y are independent.

a) (16) Find the probability distribution of $W = X + Y$.



$$F_W(w) = P(X + Y \le w) = \ldots$$

There are 2 cases:

$4 < w < 6$, $6 < w < 8$.

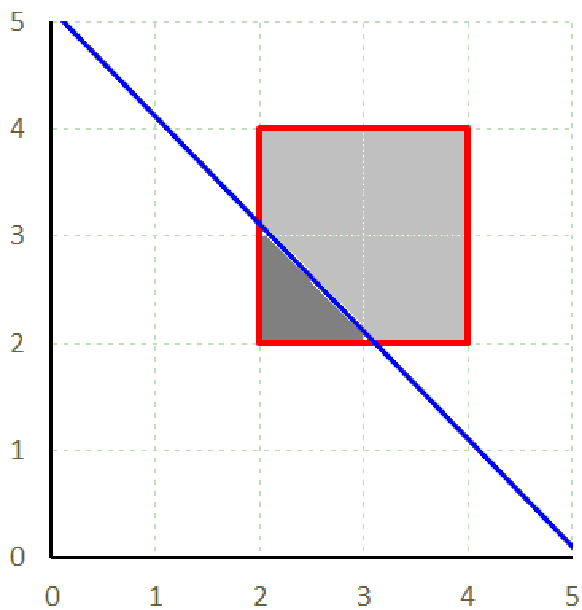
Technically, there are 4 cases:

$w < 4$,

$4 < w < 6$, $6 < w < 8$,

$w > 8$,

but $w < 4$ and $w > 8$ are boring.



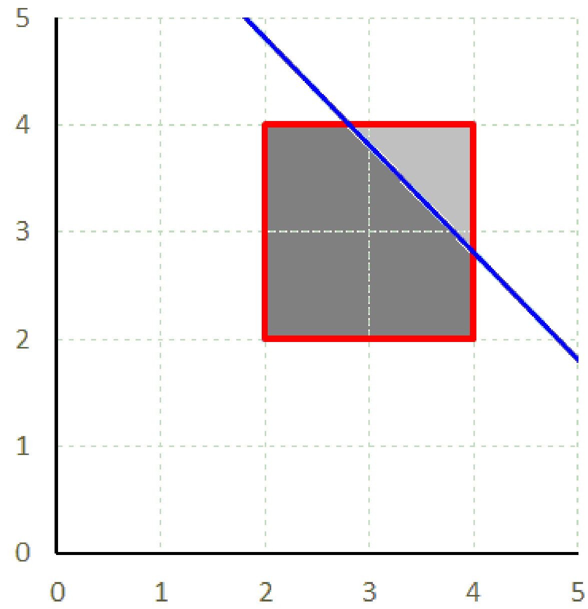
Case 1: $4 < w < 6$.

$$F_W(w) = \int_2^{w-2} \left(\int_2^{w-x} \frac{x}{6} \cdot \frac{1}{2} \, dy \right) dx$$

$$= \frac{w^3 - 6w^2 + 32}{72}$$

$$= \frac{(w-4)^2 (w+2)}{72},$$

$4 < w < 6$.



Case 2: $6 < w < 8$.

$$F_W(w) = 1 - \int_{w-4}^{4} \left(\int_{w-x}^{4} \frac{x}{6} \cdot \frac{1}{2}\, dy \right) dx$$

$$= 1 - \frac{w^3 - 12w^2 + 256}{72}$$

$$= 1 - \frac{(w-8)^2 (w+4)}{72},$$

$6 < w < 8$.

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$$f(x, w-x) = \frac{x}{6} \cdot \frac{1}{2}.$$

$2 < x < 4$

$2 < y < 4 \quad \Rightarrow \quad 2 < w - x < 4 \quad \Rightarrow \quad w - 4 < x < w - 2$

$x > 2 \quad \& \quad x > w - 4 \qquad\qquad x < 4 \quad \& \quad x < w - 2$

Case 1: $4 < w < 6$. Then $w - 4 < 2 < w - 2 < 4$.

$$f_W(w) = \int_{2}^{w-2} \frac{x}{6} \cdot \frac{1}{2}\, dx = \frac{(w-2)^2 - 4}{24} = \frac{w^2 - 4w}{24} = \frac{w(w-4)}{24}, \quad 4 < w < 6.$$

Case 2: $6 < w < 8$. Then $2 < w - 4 < 4 < w - 2$.

$$f_W(w) = \int_{w-4}^{4} \frac{x}{6} \cdot \frac{1}{2}\, dx = \frac{16 - (w-4)^2}{24} = \frac{8w - w^2}{24} = \frac{w(8-w)}{24}, \quad 6 < w < 8.$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$$f(w-y, y) = \frac{w-y}{6} \cdot \frac{1}{2}.$$

$2 < x < 4 \quad \Rightarrow \quad 2 < w - y < 4 \quad \Rightarrow \quad w - 4 < y < w - 2$

$2 < y < 4$

$y > 2 \quad \& \quad y > w - 4 \qquad\qquad y < 4 \quad \& \quad y < w - 2$

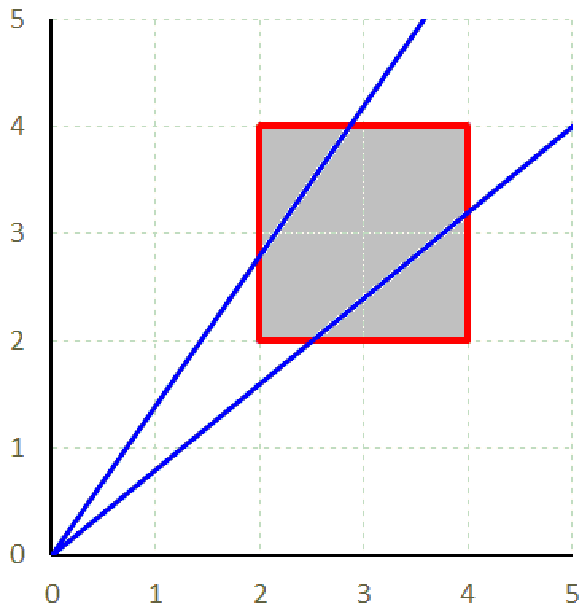
Case 1: $4 < w < 6.$ Then $w - 4 < 2 < w - 2 < 4.$

$$f_W(w) = \int_2^{w-2} \frac{w-y}{6} \cdot \frac{1}{2}\, dy = \frac{w^2 - 4w}{24} = \frac{w\,(w-4)}{24}, \qquad 4 < w < 6.$$

Case 2: $6 < w < 8.$ Then $2 < w - 4 < 4 < w - 2.$

$$f_W(w) = \int_{w-4}^{4} \frac{w-y}{6} \cdot \frac{1}{2}\, dy = \frac{8w - w^2}{24} = \frac{w\,(8-w)}{24}, \qquad 6 < w < 8.$$

b) (16) Find the probability distribution of $U = {}^{Y}\!/_{X}$.



$$F_U(u) = P\left({}^{Y}\!/_{X} \le u\right) = P(Y \le u X) = \ldots$$

There are 2 cases:

$\frac{1}{2} < u < 1, \qquad 1 < u < 2.$

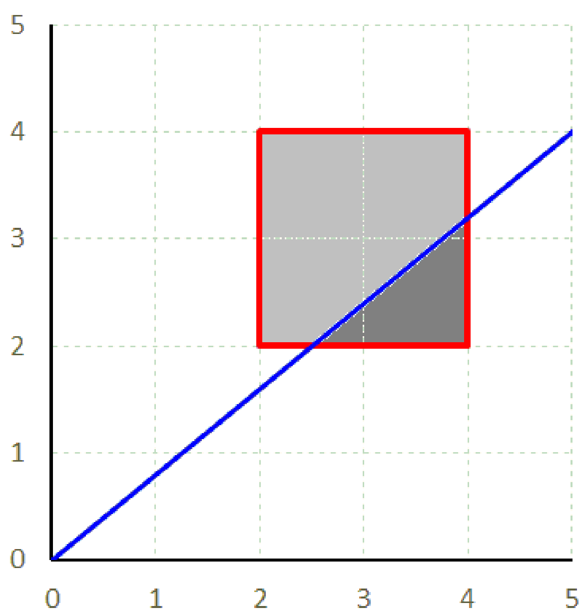
Technically, there are 4 cases:

$u < \frac{1}{2},$

$\frac{1}{2} < u < 1, \qquad 1 < u < 2,$

$u > 2,$

but $u < \frac{1}{2}$ and $u > 2$ are boring.

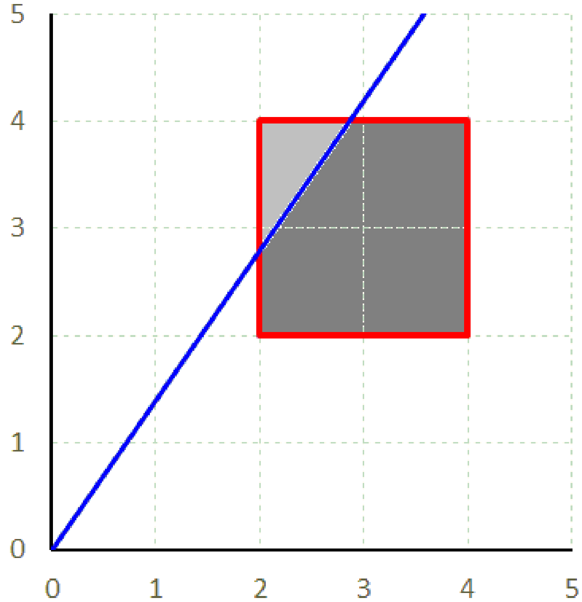


Case 1: $\frac{1}{2} < u < 1.$

$$F_U(u) = \int_{2/u}^{4} \left(\int_{2}^{ux} \frac{x}{6} \cdot \frac{1}{2}\, dy \right) dx$$

$$= \frac{16u^3 - 12u^2 + 1}{9u^2}$$

$$= \frac{(2u-1)^2\,(4u+1)}{9u^2}, \qquad \frac{1}{2} < u < 1.$$



Case 2: $1 < u < 2$.

$$F_U(u) = 1 - \int_2^{\frac{4}{u}} \left(\int_{ux}^{4} \frac{x}{6} \cdot \frac{1}{2}\, dy \right) dx$$

$$= 1 - \frac{2u^3 - 6u^2 + 8}{9u^2}$$

$$= 1 - \frac{2(u-2)^2 (u+1)}{9u^2},$$

$1 < u < 2$.

OR

Let $X = X, \quad U = {}^{Y}\!/_{X}$.

Then $X = X, \quad Y = U\,X$.

$$J = \begin{vmatrix} 1 & 0 \\ u & x \end{vmatrix} = x.$$

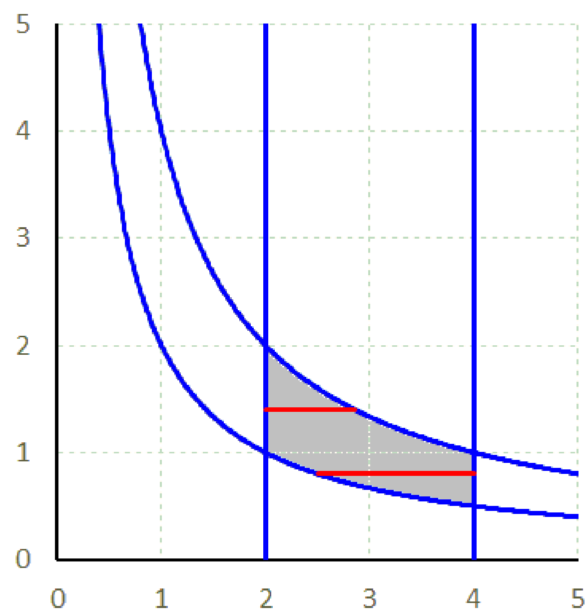
$$f_{X,U}(x,u) = f_{X,Y}(x, u\,x)\,|x|$$

$$= \frac{x}{6} \cdot \frac{1}{2} \cdot |x| = \frac{x^2}{12}.$$

$2 < x < 4$

$2 < y < 4 \quad \Rightarrow \quad 2 < u\,x < 4$

$\Rightarrow \quad \frac{2}{u} < x < \frac{4}{u}$



Case 1: $\frac{1}{2} < u < 1.$

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,U}(x,u)\, dx = \int_{\frac{2}{u}}^{4} \frac{x^2}{12}\, dx = \frac{4^3 - \left(\frac{2}{u}\right)^3}{36} = \frac{16u^3 - 2}{9u^3},$$

$$\frac{1}{2} < u < 1.$$

Case 2: $1 < u < 2.$

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,U}(x,u)\, dx = \int_{2}^{\frac{4}{u}} \frac{x^2}{12}\, dx = \frac{\left(\frac{4}{u}\right)^3 - 2^3}{36} = \frac{16 - 2u^3}{9u^3},$$

$$1 < u < 2.$$

3. (8) Let X_1, X_2, X_3, X_4 be a random sample of size $n = 4$ from a probability distribution with the probability density function

$$f_X(x) = \frac{1}{4\sqrt{x}}, \qquad 0 < x < 4, \qquad \text{zero elsewhere.}$$

Find $P(\min X_i < 0.4)$.

$$F_X(x) = P(X \le x) = \int_0^x \frac{1}{4\sqrt{u}}\,du = \left(\frac{\sqrt{u}}{2}\right)\Bigg|_0^x = \frac{\sqrt{x}}{2}, \qquad 0 \le x < \beta.$$

$$P(\min X_i < 0.4) = 1 - P(\min X_i \ge 0.4) = 1 - [P(X \ge 0.4)]^4$$

$$= 1 - [1 - F_X(0.4)]^4 = 1 - \left[1 - \frac{\sqrt{0.4}}{2}\right]^4 \approx \mathbf{0.7814}.$$

4. (8) Let $\theta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a discrete distribution with probability mass function

$$P(X_i = 1) = \frac{\theta}{4+\theta}, \quad P(X_i = 2) = \frac{3}{4+\theta}, \quad P(X_i = 4) = \frac{1}{4+\theta}, \qquad \theta > 0.$$

Construct a consistent estimator of θ based on $\sum_{i=1}^{n} \ln X_i$.

By WLLN, $$\frac{\sum_{i=1}^{n} \ln X_i}{n} \overset{P}{\to} E(\ln X).$$

$$E(\ln X) = 0 \times \frac{\theta}{4+\theta} + \ln 2 \times \frac{3}{4+\theta} + \ln 4 \times \frac{1}{4+\theta} = \frac{5 \ln 2}{4+\theta} = \frac{\ln 32}{4+\theta}.$$

$$\Rightarrow \qquad \hat{\theta} = \frac{n \ln 32}{\sum_{i=1}^{n} \ln X_i} - 4 = \frac{5n \ln 2}{\sum_{i=1}^{n} \ln X_i} - 4 \overset{P}{\to} \frac{\ln 32}{\frac{\ln 32}{4+\theta}} - 4 = \theta.$$

5. Consider a random sample $X_1, X_2, \ldots, X_n$ from a Geometric distribution for which p is the probability of success. That is,

$$f(x \mid p) = (1-p)^{x-1} p, \qquad x = 1, 2, 3, 4, 5, \ldots .$$

Recall: the maximum likelihood estimator of p is $\hat{p} = \dfrac{1}{\overline{X}}$.

Let p have a prior p.d.f. which is Beta with parameters α and β.

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, p^{\alpha-1} (1-p)^{\beta-1}, \qquad 0 < p < 1.$$

Recall: $E[\text{Beta}(\alpha, \beta)] = \dfrac{\alpha}{\alpha+\beta}$.

a) (7) Find the posterior distribution of p, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$f(x_1, x_2, \ldots x_n \mid p) = f_X(x_1; p) \cdot f_X(x_2; p) \cdot \ldots \cdot f_X(x_n; p)$$

$$= \prod_{i=1}^{n} (1-p)^{x_i - 1}\, p = (1-p)^{\sum_{i=1}^{n} x_i - n}\, p^{n}.$$

$$f(x_1, x_2, \ldots x_n, p) = f(x_1, x_2, \ldots x_n \mid p) \times \pi(p)$$

$$= (1-p)^{\sum_{i=1}^{n} x_i - n}\, p^{n} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, p^{\alpha-1} (1-p)^{\beta-1}$$

$$= \ldots\, p^{n+\alpha-1} (1-p)^{\sum_{i=1}^{n} x_i - n + \beta - 1}.$$

$\Rightarrow$ the posterior distribution of p, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$,

is **Beta** with New $\alpha = n + \alpha$ and New $\beta = \sum_{i=1}^{n} x_i - n + \beta$.

b) (6) Find the conditional mean of p, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Show that it is a weighted average of the maximum likelihood estimate $\hat{p}$ and the prior mean.

$\left(\text{conditional mean of } p, \text{ given } X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right)$

$$= \frac{(\text{New } \alpha)}{(\text{New } \alpha) + (\text{New } \beta)} = \frac{n + \alpha}{\sum_{i=1}^{n} x_i + \alpha + \beta}.$$

$$\frac{n + \alpha}{\sum_{i=1}^{n} x_i + \alpha + \beta} = \frac{n}{\sum_{i=1}^{n} x_i} \cdot \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i + \alpha + \beta} + \frac{\alpha}{\alpha + \beta} \cdot \frac{\alpha + \beta}{\sum_{i=1}^{n} x_i + \alpha + \beta}$$

$$= \hat{p} \cdot \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i + \alpha + \beta} + (\text{prior mean}) \cdot \frac{\alpha + \beta}{\sum_{i=1}^{n} x_i + \alpha + \beta}.$$

6. Suppose a random sample of size $n = 9$ is taken from a normal distribution with $\sigma = 12$ for the purpose of testing $H_0: \mu = 70$ vs. $H_1: \mu < 70$ at a 5% level of significance.

a) (3) What is the p-value of this test if the observed value of the sample mean is $\bar{x} = 64.2$?

Test Statistic: $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \dfrac{64.2 - 70}{12 / \sqrt{9}} = -1.45.$

Left-tailed. P-value $= P(Z \le -1.45) = \mathbf{0.0735}$.

b) (6) What is the power of this test if $\mu = 66.66$?

Reject H_0 if $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} < -z_\alpha.$ $Z = \dfrac{\bar{X} - 70}{12 / \sqrt{9}} < -1.645.$

$$\bar{X} < 70 - 1.645 \cdot \frac{12}{\sqrt{9}} = 63.42.$$

$$\text{Power}(\mu = 66.66) = P(\text{Reject } H_0 \mid \mu = 66.66) = P(\bar{X} < 63.42 \mid \mu = 66.66)$$

$$= P\left(Z < \frac{63.42 - 66.66}{12 / \sqrt{9}}\right) = P(Z < -0.81) = \mathbf{0.2090}.$$

7. Let X have a Binomial distribution with the number of trials $n = 25$ and with probability of "success" p. We wish to test $\mathrm{H}_0 : p = 0.30$ vs. $\mathrm{H}_1 : p > 0.30$.

a) (5) Find the best rejection region with the significance level α closest to 0.10.

Decision rule: Reject H_0 if $\mathrm{X} \ge c$. Want $\mathrm{P}(\text{Type I error}) \approx 0.10$.

$$\mathrm{P}(\text{Type I error}) = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ true}) = \mathrm{P}(\mathrm{X} \ge c \mid p = 0.30)$$
$$= 1 - \mathrm{CDF}(c - 1 \mid p = 0.30).$$

Want $(1 - \mathrm{CDF}(c - 1 \mid p = 0.30)) \approx 0.10$, $\quad \mathrm{CDF}(c - 1 \mid p = 0.30) \approx 0.90$.

$\mathrm{CDF}(10 \mid p = 0.30) = 0.902$, $\quad c - 1 = 10$, $\quad c = 11$.

Decision rule: Reject H_0 if $\mathbf{X \ge 11}$.

b) (4) Find the power of the test from part (a) if $p = 0.40$.

$$\text{Power}(0.40) = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid p = 0.40) = \mathrm{P}(\mathrm{X} \ge 11 \mid p = 0.40)$$
$$= 1 - \mathrm{P}(\mathrm{X} \le 10 \mid p = 0.40) = 1 - 0.586 = \mathbf{0.414}.$$

c) (4) Suppose we observe $x = 9$. Find the p-value of the test.

$$\text{p-value} = \mathrm{P}(\text{value of X as extreme or more extreme than } \mathrm{X} = 9 \mid \mathrm{H}_0 \text{ true})$$
$$= \mathrm{P}(\mathrm{X} \ge 9 \mid p = 0.30) = 1 - \mathrm{CDF}(8 \mid p = 0.30) = 1 - 0.677 = \mathbf{0.323}.$$

8. Let $\zeta > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\zeta) = \zeta^3 x^5 e^{-\zeta x^2}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) (7) Show that $W = X^2$ follows a Gamma distribution.
What are the parameters α and θ for this Gamma distribution?
No credit will be given without proper justification.

Let $W = X^2$ $\qquad X = \sqrt{W} = v(W)$ $\qquad v'(w) = \dfrac{1}{2\sqrt{w}}$

$$f_W(w) = f_X(v(w)) \cdot |v'(w)| = \zeta^3 w^{5/2} e^{-\zeta w} \cdot \frac{1}{2\sqrt{w}} = \frac{\zeta^3}{2} w^2 e^{-\zeta w}$$

$$= \frac{\zeta^3}{\Gamma(3)} w^{3-1} e^{-\zeta w}, \qquad w > 0.$$

$\Rightarrow$ W has a Gamma$(\alpha = 3,\ \theta = \frac{1}{\zeta})$ distribution.

b) (2) Find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for ζ.

$$f(x_1, x_2, \ldots x_n; \zeta) = f(x_1;\zeta)\, f(x_2;\zeta)\, \ldots\, f(x_n;\zeta)$$

$$= \left[\zeta^{3n} e^{-\zeta \sum_{i=1}^n x_i^2} \right] \left(\prod_{i=1}^n x_i^5 \right).$$

By Factorization Theorem, $Y = \sum_{i=1}^n X_i^2$ is a sufficient statistic for ζ.

OR

$$f(x;\zeta) = \exp\left\{ -\zeta\cdot x^2 + 3\ln\zeta + 5\ln x \right\}. \qquad \Rightarrow \qquad \mathrm{K}(x) = x^2.$$

$$\Rightarrow \qquad \mathrm{Y} = \sum_{i=1}^{n} \mathrm{K}\left(\mathrm{X}_i\right) = \sum_{i=1}^{n} \mathrm{X}_i^2 \text{ is a sufficient statistic for } \zeta.$$

c) (9) (i) Obtain the maximum likelihood estimator of ζ, $\hat{\zeta}$.

(ii) Suppose $n = 5$, and $x_1 = 0.5$, $x_2 = 0.6$, $x_3 = 0.7$, $x_4 = 0.9$, $x_5 = 1.7$.

Find the maximum likelihood estimate of ζ, $\hat{\zeta}$.

$$\mathrm{L}(\zeta) = \prod_{i=1}^{n} \left(\zeta^3 \, x_i^5 \, e^{-\zeta x_i^2} \right).$$

$$\ln \mathrm{L}(\zeta) = 3n\cdot\ln\zeta + \sum_{i=1}^{n} \ln\left(x_i^5\right) - \zeta\cdot\sum_{i=1}^{n} x_i^2 .$$

$$\left(\ln \mathrm{L}(\zeta)\right)' = \frac{3n}{\zeta} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\zeta} = \frac{3n}{\sum_{i=1}^{n} \mathrm{X}_i^2}.$$

$$x_1 = 0.5,\ x_2 = 0.6,\ x_3 = 0.7,\ x_4 = 0.9,\ x_5 = 1.7. \qquad \sum_{i=1}^{5} x_i^2 = 4.8.$$

$$\hat{\zeta} = \frac{3n}{\sum_{i=1}^{n} x_i^2} = \frac{3\cdot 5}{4.8} = \mathbf{3.125}.$$

d) (7) Is the maximum likelihood estimator $\hat{\zeta}$ an unbiased estimator of ζ?

If $\hat{\zeta}$ is not an unbiased estimator of ζ, construct an unbiased estimator of ζ from $\hat{\zeta}$.

$Y = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} X_i^2$ has a $\text{Gamma}\left(\alpha = 3n,\ \theta = \frac{1}{\zeta}\right)$ distribution.

$$\hat{\zeta} = \frac{3n}{\sum_{i=1}^{n} X_i^2} = \frac{3n}{Y}.$$

If T_α has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\, \Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\, \Gamma(\alpha)}, \qquad m > -\alpha.$$

$$E\left(\frac{1}{Y}\right) = E\left(Y^{-1}\right) = \frac{\Gamma(3n-1)}{\zeta^{-1}\, \Gamma(3n)} = \frac{\zeta}{3n-1}.$$

$$E(\hat{\zeta}) = E\left(\frac{3n}{Y}\right) = 3n\, E\left(\frac{1}{Y}\right) = 3n \cdot \frac{\zeta}{3n-1} = \frac{3n}{3n-1} \cdot \zeta = \zeta + \frac{\zeta}{3n-1} \neq \zeta.$$

$\hat{\zeta}$ is NOT an unbiased estimator of ζ. $\quad \text{bias}(\hat{\zeta}) = E(\hat{\zeta}) - \zeta = \dfrac{\zeta}{3n-1}.$

Consider $\hat{\hat{\zeta}} = \dfrac{3n-1}{3n} \cdot \hat{\zeta} = \dfrac{3n-1}{\sum_{i=1}^{n} X_i^2}.$

Then $E(\hat{\hat{\zeta}}) = \dfrac{3n-1}{3n} \cdot E(\hat{\zeta}) = \zeta.$ $\quad \hat{\hat{\zeta}}$ is an unbiased estimator of ζ.

e) (9) Suppose $n = 5$, and $x_1 = 0.5,\ x_2 = 0.6,\ x_3 = 0.7,\ x_4 = 0.9,\ x_5 = 1.7$.

Construct a 90% confidence interval for ζ.

$$Y = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} X_i^2$$ has a Gamma$\left(\alpha = 3n,\ \theta = \frac{1}{\zeta} \right)$ distribution.

$$2\zeta \sum_{i=1}^{n} X_i^2$$ has a $\chi^2(2\alpha = 6n)$ distribution.

$$\Rightarrow \quad P\left(\chi^2_{1-\alpha/2}(6n) < 2\zeta \sum_{i=1}^{n} X_i^2 < \chi^2_{\alpha/2}(6n) \right) = 1 - \alpha.$$

$$\Rightarrow \quad P\left(\frac{\chi^2_{1-\alpha/2}(6n)}{2\sum_{i=1}^{n} X_i^2} < \zeta < \frac{\chi^2_{\alpha/2}(6n)}{2\sum_{i=1}^{n} X_i^2} \right) = 1 - \alpha.$$

A $(1 - \alpha)\,100\,\%$ confidence interval for ζ:

$$\left(\frac{\chi^2_{1-\alpha/2}(6n)}{2\sum_{i=1}^{n} x_i^2},\ \frac{\chi^2_{\alpha/2}(6n)}{2\sum_{i=1}^{n} x_i^2} \right).$$

$x_1 = 0.5,\ x_2 = 0.6,\ x_3 = 0.7,\ x_4 = 0.9,\ x_5 = 1.7.$ $\qquad \sum_{i=1}^{5} x_i^2 = 4.8.$

$\chi^2_{0.95}(30) = 18.49, \qquad \chi^2_{0.05}(30) = 43.77.$

$$\left(\frac{18.49}{2 \cdot 4.8},\ \frac{43.77}{2 \cdot 4.8} \right) \approx \mathbf{(1.926,\ 4.559)}.$$

8. (continued)

We wish to test $H_0 : \zeta = 2.5$ vs. $H_1 : \zeta > 2.5$.

f) (9) Suppose $n = 5$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

Let $\zeta > 2.5$.

$$\frac{L(2.5;\ x_1, x_2, \ldots, x_n)}{L(\zeta;\ x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} f(x_i;\ 2.5)}{\prod_{i=1}^{n} f(x_i;\ \zeta)} = \frac{\prod_{i=1}^{n} \left(2.5^3\, x_i^5\, e^{-2.5 x_i^2} \right)}{\prod_{i=1}^{n} \left(\zeta^3\, x_i^5\, e^{-\zeta x_i^2} \right)}$$

$$= \left(\frac{2.5}{\zeta} \right)^{3n} e^{(\zeta - 2.5) \sum_{i=1}^{n} x_i^2} \le k.$$

Since $\zeta > 2.5$,

$$\left(\frac{2.5}{\zeta} \right)^{3n} e^{(\zeta - 2.5) \sum_{i=1}^{n} x_i^2} \le k \qquad \Leftrightarrow \qquad \sum_{i=1}^{n} x_i^2 \le c.$$

$Y = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} X_i^2$ has a $\text{Gamma}(\alpha = 3n,\ \theta = \frac{1}{\zeta})$ distribution.

$2\zeta \sum_{i=1}^{n} X_i^2$ has a $\chi^2(2\alpha = 6n)$ distribution.

$$0.10 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{n} X_i^2 \le c \mid \zeta = 2.5 \right)$$

$$= P\left(2\zeta \sum_{i=1}^{5} X_i^2 \le 2\zeta c \mid \zeta = 2.5 \right) = P(\chi^2(30) \le 5c).$$

$\chi^2_{0.90}(30) = 20.60 = 5c. \qquad \Rightarrow \qquad c = \mathbf{4.12}.$

```
> qgamma(0.10,15,2.5)
[1] 4.119847
>
> qchisq(0.10,30)
[1] 20.59923
> qchisq(0.10,30)/5
[1] 4.119847
```

The uniformly most powerful rejection region is

$$\text{“Reject } H_0 \text{ if } \sum_{i=1}^{5} x_i^2 \le 4.12.\text{”}$$

g) (6) Suppose $n = 5$, and $x_1 = 0.5,\ x_2 = 0.6,\ x_3 = 0.7,\ x_4 = 0.9,\ x_5 = 1.7.$
Find the p-value of the test.

$x_1 = 0.5,\ x_2 = 0.6,\ x_3 = 0.7,\ x_4 = 0.9,\ x_5 = 1.7.$ $\qquad \sum_{i=1}^{5} x_i^2 = 4.8.$

p-value $= P(\ \ldots$ as extreme or more extreme than the observed $\ldots \mid H_0$ true $).$

$$\text{p-value} = P\left(\sum_{i=1}^{n} X_i^2 \le 4.8 \mid \zeta = 2.5\right) = P\left(\text{Gamma}\left(\alpha = 15,\ \theta = \frac{1}{2.5}\right) \le 4.8\right).$$

```
> pgamma(4.8,15,2.5)
[1] 0.2279755
```

$$\text{p-value} = P\left(\sum_{i=1}^{n} X_i^2 \le 4.8 \mid \zeta = 2.5\right) = P\left(\text{Gamma}\left(\alpha = 15,\ \theta = \frac{1}{2.5}\right) \le 4.8\right)$$

$$= P(\text{Poisson}(2.5 \cdot 4.8) \ge 15) = 1 - P(\text{Poisson}(12) \le 14) = 1 - 0.772 = \mathbf{0.228}.$$

```
> 1-ppois(14,12)
[1] 0.2279755
```

$$\text{p-value} = P\Big(\sum_{i=1}^{n} X_i^2 \le 4.8 \,\Big|\, \zeta = 2.5\Big) = P\Big(\text{Gamma}\big(\alpha = 15,\ \theta = \frac{1}{2.5}\big) \le 4.8\Big)$$

$$= P\big(\chi^2(2 \cdot 15) \le 2 \cdot 2.5 \cdot 4.8\big) = P\big(\chi^2(30) \le 24\big).$$

```
> pchisq(24,30)
[1] 0.2279755
```

p-value $\approx$ **0.228**.