STAT 410 | Fall 2020

Practice Problems 13

A. Stepanov

1. The label on 1-gallon can of paint states that the amount of paint in the can is sufficient to paint at least 350 square feet (on average). Suppose the amount of coverage is approximately normally distributed, and the overall standard deviation of the amount of coverage is 32 square feet. A random sample of 16 cans yields the sample mean amount of coverage of 338 square feet. We wish to test

$$H_0 : \mu = 350 \quad \text{against} \quad H_1 : \mu < 350.$$

a) Find the p-value for the test.

b) Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

c) State your decision, Reject H_0 or Do NOT Reject H_0, at $\alpha = 0.05$.

d) Find the power of the test if the actual value of the mean coverage of all 1-gallon cans is 345, 341, 337, 333, and 329 square feet, if a 5% level of significance is used.

e) Sketch the graph of the power function.

2. Suppose the lifetime of a particular brand of light bulbs is normally distributed. A random sample of 27 light bulbs yields an average lifetime of 410 hours and a sample standard deviation of 67 hours.

a) Find the p-value of the test $H_0 : \mu = 432$ hours vs. $H_1 : \mu < 432$ hours.

b) Test $H_0 : \sigma = 56$ hours vs. $H_1 : \sigma > 56$ hours at a 5% level of significance.

c) Find the power of the test in part (b) if $\sigma = 84$ hours at $\alpha = 0.05$.

3. An advertisement for a particular brand of automobile states that it accelerates from 0 to 60 mph in an average of 5.0 seconds. Makers of a competing automobile feel that the true average number of seconds it takes to reach 60 mph from zero is above 5.0. Suppose the population standard deviation is believed to be 0.43 seconds. We wish to test $H_0: \mu \le 5.0$ vs. $H_1: \mu > 5.0$.

a) Fifty automobiles were tested (each automobile was tested a single time). The sample mean time to reach 60 mph from zero was 5.09 seconds. Find the p-value of this test.

b) For which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance will be used, and 50 automobiles will be tested (each automobile to be tested a single time)?

c) Find the power of this test if the true average time is 5.15 seconds, a 5% level of significance will be used, and 50 automobiles will be tested (each automobile to be tested a single time).

d) What is the minimum sample size required if we want to have the power of at least 0.90 at $\mu = 5.15$ for the test with a 5% level of significance.

4. **4.5.4** (7th edition) **5.5.4** (6th edition)

5. **4.5.8** (7th edition) **5.5.8** (6th edition)

6. **4.5.9** (7th edition) **5.5.9** (6th edition)

7. **4.5.12** (7th edition) **5.5.12** (6th edition)

4.5.4. Let X have a binomial distribution with the number of trials $n = 10$ and with p either $1/4$ or $1/2$. The simple hypothesis $H_0 : p = \frac{1}{2}$ is rejected, and the alternative simple hypothesis $H_1 : p = \frac{1}{4}$ is accepted, if the observed value of X_1, a random sample of size 1, is less than or equal to 3. Find the significance level and the power of the test.

4.5.8. Let us say the life of a tire in miles, say X, is normally distributed with mean θ and standard deviation 5000. Past experience indicates that $\theta = 30{,}000$. The manufacturer claims that the tires made by a new process have mean $\theta > 30{,}000$. It is possible that $\theta = 35{,}000$. Check his claim by testing $H_0 : \theta = 30{,}000$ against $H_1 : \theta > 30{,}000$. We observe n independent values of X, say $x_1, \ldots, x_n$, and we reject H_0 (thus accept H_1) if and only if $\overline{x} \geq c$. Determine n and c so that the power function $\gamma(\theta)$ of the test has the values $\gamma(30{,}000) = 0.01$ and $\gamma(35{,}000) = 0.98$.

4.5.9. Let X have a Poisson distribution with mean θ. Consider the simple hypothesis $H_0 : \theta = \frac{1}{2}$ and the alternative composite hypothesis $H_1 : \theta < \frac{1}{2}$. Thus $\Omega = \{\theta : 0 < \theta \leq \frac{1}{2}\}$. Let $X_1, \ldots, X_{12}$ denote a random sample of size 12 from this distribution. We reject H_0 if and only if the observed value of $Y = X_1 + \cdots + X_{12} \leq 2$. If $\gamma(\theta)$ is the power function of the test, find the powers $\gamma(\frac{1}{2})$, $\gamma(\frac{1}{3})$, $\gamma(\frac{1}{4})$, $\gamma(\frac{1}{6})$, and $\gamma(\frac{1}{12})$. Sketch the graph of $\gamma(\theta)$. What is the significance level of the test?

4.5.12. Let $X_1, X_2, \ldots, X_8$ be a random sample of size $n = 8$ from a Poisson distribution with mean μ. Reject the simple null hypothesis $H_0 : \mu = 0.5$ and accept $H_1 : \mu > 0.5$ if the observed sum $\sum_{i=1}^{8} x_i \geq 8$.

(a) Compute the significance level α of the test.

(b) Find the power function $\gamma(\mu)$ of the test as a sum of Poisson probabilities.

(c) Using Table I of Appendix C, determine $\gamma(0.75)$, $\gamma(1)$, and $\gamma(1.25)$.

Answers:

1. The label on 1-gallon can of paint states that the amount of paint in the can is sufficient to paint at least 350 square feet (on average). Suppose the amount of coverage is approximately normally distributed, and the overall standard deviation of the amount of coverage is 32 square feet. A random sample of 16 cans yields the sample mean amount of coverage of 338 square feet. We wish to test

$$H_0 : \mu = 350 \quad \text{against} \quad H_1 : \mu < 350.$$

a) Find the p-value for the test.

Test Statistic: $$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{338 - 350}{32 / \sqrt{16}} = \mathbf{-\,1.50}.$$

$$\text{P - value} = P(Z \le -1.50) = \Phi(-1.50) = \mathbf{0.0668}.$$

b) Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} < -z_\alpha. \qquad\qquad Z = \frac{\overline{X} - 350}{32 / \sqrt{16}} < -1.645.$$

$$\overline{X} < 350 - 1.645 \cdot \frac{32}{\sqrt{16}} = \mathbf{336.84}.$$

c) State your decision, Reject H_0 or Do NOT Reject H_0, at $\alpha = 0.05$.

$\overline{x} = 338 > 336.84.$ OR p-value $= 0.0668 > 0.05 = \alpha.$

Do NOT Reject H_0 at $\alpha = 0.05$.

d) Find the power of the test if the actual value of the mean coverage of all 1-gallon cans is 345, 341, 337, 333, and 329 square feet, if a 5% level of significance is used.

$$P(\overline{X} < 336.84 \mid \mu = 345) = P\left(Z < \frac{336.84 - 345}{32/\sqrt{16}}\right) = P(Z < -1.02) = \mathbf{0.1539}.$$

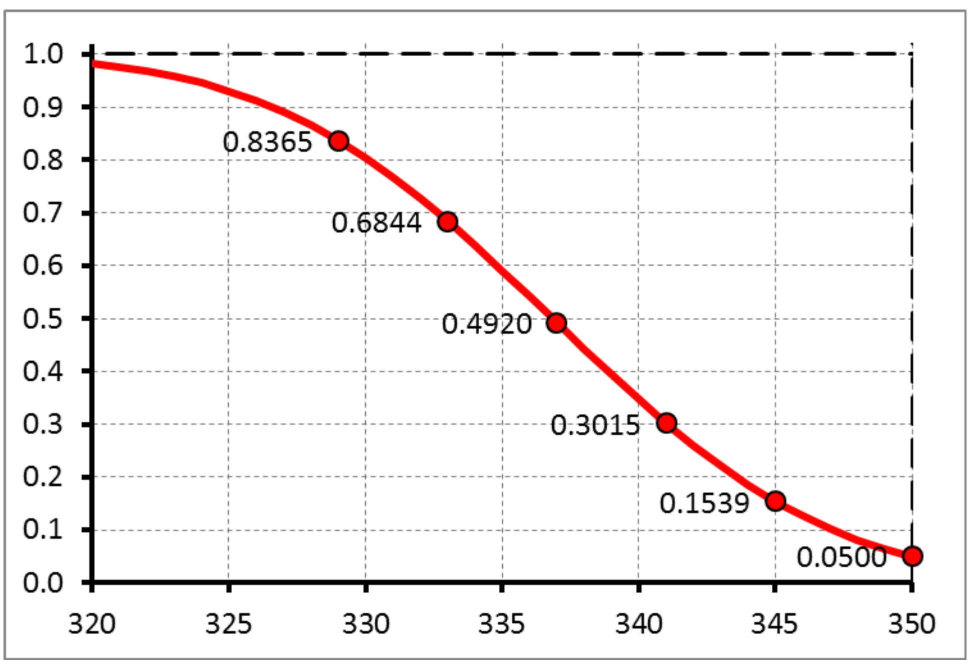
$$P(\overline{X} < 336.84 \mid \mu = 341) = P\left(Z < \frac{336.84 - 341}{32/\sqrt{16}}\right) = P(Z < -0.52) = \mathbf{0.3015}.$$

$$P(\overline{X} < 336.84 \mid \mu = 337) = P\left(Z < \frac{336.84 - 337}{32/\sqrt{16}}\right) = P(Z < -0.02) = \mathbf{0.4920}.$$

$$P(\overline{X} < 336.84 \mid \mu = 333) = P\left(Z < \frac{336.84 - 333}{32/\sqrt{16}}\right) = P(Z < 0.48) = \mathbf{0.6844}.$$

$$P(\overline{X} < 336.84 \mid \mu = 329) = P\left(Z < \frac{336.84 - 329}{32/\sqrt{16}}\right) = P(Z < 0.98) = \mathbf{0.8365}.$$

e) Sketch the graph of the power function.



2. Suppose the lifetime of a particular brand of light bulbs is normally distributed. A random sample of 27 light bulbs yields an average lifetime of 410 hours and a sample standard deviation of 67 hours.

a) Find the p-value of the test $H_0 : \mu = 432$ hours vs. $H_1 : \mu < 432$ hours.

Test Statistic: σ is unknown.

$$T = \frac{\overline{X} - \mu_0}{s / \sqrt{n}} = \frac{410 - 432}{67 / \sqrt{27}} = \mathbf{-1.7062}.$$

$n - 1 = 26$ degrees of freedom

Left – tailed test.

p-value $=$ (area of the left tail) $= P(T(26) \le -1.7062) \approx \mathbf{0.05}$.

b) Test $H_0 : \sigma = 56$ hours vs. $H_1 : \sigma > 56$ hours at a 5% level of significance.

Test Statistic: $$\chi^2 = \frac{(n-1) \cdot s^2}{\sigma_0^2} = \frac{(27-1) \cdot 67^2}{56^2} = \mathbf{37.2175}.$$

Rejection Region: Right – tailed.

Reject H_0 if $\chi^2 > \chi^2_\alpha$ $\quad$ $n - 1 = 26$ degrees of freedom.

$\alpha = 0.05$ $\quad$ $\chi^2_{0.05}(26) = 38.88$.

Reject H_0 if $\chi^2 > 38.88$.

The value of the test statistic **does not** fall into the Rejection Region.

Do NOT Reject H_0 at $\alpha = 0.05$.

c) Find the power of the test in part (b) if $\sigma = 84$ hours at $\alpha = 0.05$.

Test Statistic: $$\chi^2 = \frac{(n-1)\cdot s^2}{\sigma_0^2} = \frac{(27-1)\cdot s^2}{56^2}.$$

Reject H_0 if $\chi^2 > \chi^2_\alpha(n-1) = \chi^2_{0.05}(26) = 38.88$.

$$\frac{(27-1)\cdot s^2}{56^2} > 38.88 \qquad \Leftrightarrow \qquad s^2 > 4689.526, \quad s > \mathbf{68.48}.$$

$$\text{Power} = P(\text{Reject } H_0 \mid H_0 \text{ is not true}) = P(S^2 > 4689.526 \mid \sigma = 84)$$

$$= P\left(\frac{(n-1)\cdot S^2}{\sigma^2} > \frac{(27-1)\cdot 4689.526}{84^2} \mid \sigma = 84\right)$$

$$= P(\chi^2(26) > 17.28) \approx \mathbf{0.90}.$$

3. An advertisement for a particular brand of automobile states that it accelerates from 0 to 60 mph in an average of 5.0 seconds. Makers of a competing automobile feel that the true average number of seconds it takes to reach 60 mph from zero is above 5.0. Suppose the population standard deviation is believed to be 0.43 seconds. We wish to test $H_0: \mu \le 5.0$ vs. $H_1: \mu > 5.0$.

a) Fifty automobiles were tested (each automobile was tested a single time). The sample mean time to reach 60 mph from zero was 5.09 seconds. Find the p-value of this test.

Test Statistic: σ is known. $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \dfrac{5.09 - 5.0}{0.43 / \sqrt{50}} = \mathbf{1.48}$.

Right – tailed test. p-value = (area of the right tail) = $P(Z > 1.48) = \mathbf{0.0694}$.

b) For which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance will be used, and 50 automobiles will be tested (each automobile to be tested a single time)?

$\sigma = 0.43$. $n = 50$. $\alpha = 0.05$.

σ is known. Test Statistic: $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \dfrac{\overline{X} - 5.0}{0.43 / \sqrt{50}}$.

Rejection Region: Right - tailed. Reject H_0 if $Z > z_\alpha = 1.645$.

$$\overline{X} > 5.0 + 1.645 \cdot \frac{0.43}{\sqrt{50}} = \mathbf{5.1}.$$

c) Find the power of this test if the true average time is 5.15 seconds, a 5% level of significance will be used, and 50 automobiles will be tested (each automobile to be tested a single time).

Power = $P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(\overline{X} > 5.1 \mid \mu = 5.15)$

$$= P\left(Z > \frac{5.1 - 5.15}{0.43 / \sqrt{50}}\right) = P(Z > -0.82) = \mathbf{0.7939}.$$

d) What is the minimum sample size required if we want to have the power of at least 0.90 at $\mu = 5.15$ for the test with a 5% level of significance.

Rejection Region: Reject H_0 if $\overline{X} > 5.0 + 1.645 \cdot \frac{0.43}{\sqrt{n}}$.

$$\text{Power}(5.15) = P(\text{Reject } H_0 \mid \mu = 5.15) = P\left(\overline{X} > 5.0 + 1.645 \cdot \frac{0.43}{\sqrt{n}} \mid \mu = 5.15\right)$$

$$= P\left(Z > \frac{5.0 + 1.645 \cdot \frac{0.43}{\sqrt{n}} - 5.15}{0.43 / \sqrt{n}}\right) = P\left(Z > \frac{-0.15\sqrt{n}}{0.43} + 1.645\right).$$

$P(Z > -1.282) = 0.90$.

$$\Rightarrow \quad \frac{-0.15\sqrt{n}}{0.43} + 1.645 \le -1.282. \qquad \Rightarrow \quad \frac{-0.15\sqrt{n}}{0.43} \le -2.927.$$

$$\Rightarrow \quad \sqrt{n} \ge 8.39073\overline{3}. \qquad \Rightarrow \quad n \ge 70.4044. \qquad \boldsymbol{n \ge 71}.$$

4. **4.5.4** (7th edition) **5.5.4** (6th edition)

$H_0 : p = 1/2$ $\qquad$ $H_1 : p = 1/4$

Significance level $= \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X \le 3 \mid p = 1/2)$

$$= \sum_{k=0}^{3} {}_{10}C_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{10-k} = \mathbf{0.171875}.$$

Power $= P(\text{Reject } H_0 \mid H_0 \text{ is not true}) = P(X \le 3 \mid p = 1/4)$

$$= \sum_{k=0}^{3} {}_{10}C_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{10-k} = \mathbf{0.775875}.$$

5. **4.5.8** (7th edition) **5.5.8** (6th edition)

$H_0 : \theta = \mu = 30{,}000$ vs. $H_1 : \theta = \mu > 30{,}000$ Right - tailed.

$\gamma(30{,}000) = \alpha = 0.01.$ $z_{0.01} = 2.326.$

Reject H_0 if $\bar{x} > \mu_0 + z_\alpha \dfrac{\sigma}{\sqrt{n}} = 30{,}000 + 2.326 \dfrac{5{,}000}{\sqrt{n}}$ – critical value.

$$\gamma(35{,}000) = P(\text{Reject } H_0 \mid \mu = 35{,}000)$$

$$= P\left(\overline{X} > 30{,}000 + 2.326 \frac{5{,}000}{\sqrt{n}} \mid \mu = 35{,}000\right)$$

$$= P\left(Z > \frac{30{,}000 + 2.326 \frac{5{,}000}{\sqrt{n}} - 35{,}000}{\frac{5{,}000}{\sqrt{n}}} \right)$$

$$= P\left(Z > 2.326 - \sqrt{n}\right) = 0.98.$$

$P\left(Z > -2.054\right) = 0.98.$

$\Rightarrow$ $2.326 - \sqrt{n} = -2.054.$ $\Rightarrow$ $\sqrt{n} = 4.38.$

$n = 19.1844.$

Round up. $n = \mathbf{20}.$ $c = \mathbf{32{,}600.547}.$

OR

$z_{0.01} = 2.33.$ $z_{0.98} = -2.05.$ (a bit rougher rounding)

$2.33 - \sqrt{n} = -2.05.$ $\sqrt{n} = 4.38.$ $n = 19.1844.$

Round up. $n = \mathbf{20}.$ $c = \mathbf{32{,}605.019}.$

6. **4.5.9** (7th edition) **5.5.9** (6th edition)

$H_0 : \theta = {}^1\!/_2 \qquad H_1 : \theta < {}^1\!/_2$

Reject H_0 if $X_1 + \ldots + X_{12} \le 2$

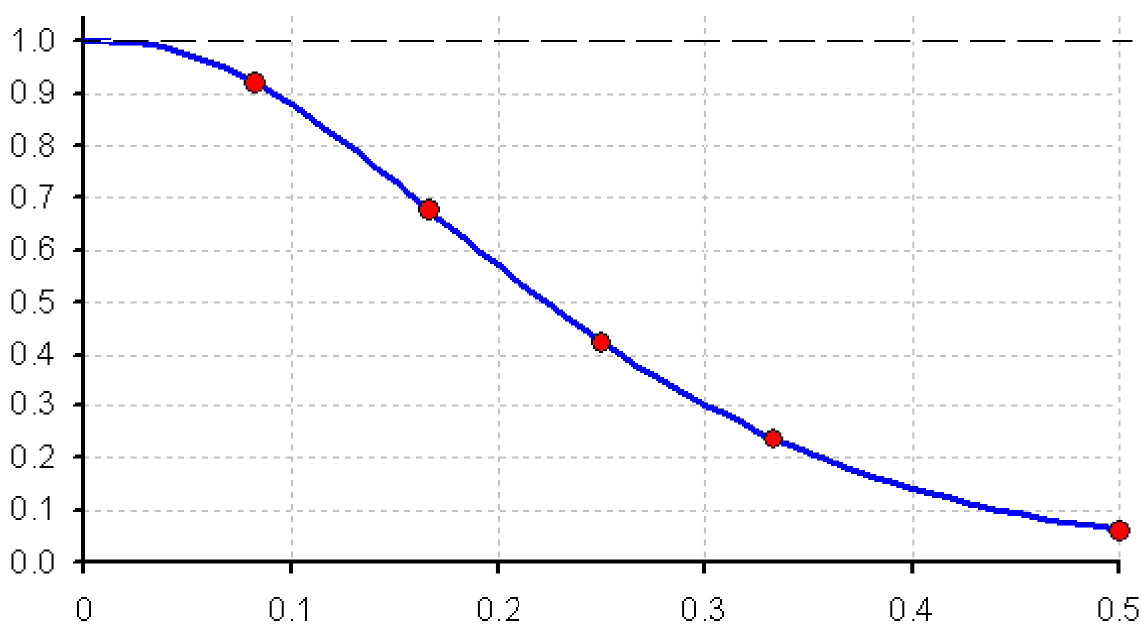
$X_1 + \ldots + X_{12}$ has Poisson distribution with mean $12\,\theta$.

Power: $\gamma(\theta) = P(\text{Poisson}(12\,\theta) \le 2) = \sum_{k=0}^{2} \frac{(12\theta)^k e^{-12\theta}}{k!}$

$\gamma(\frac{1}{2}) = P(\text{Poisson}(6) \le 2) = \mathbf{0.062}$ = significance level = α

$\gamma(\frac{1}{3}) = P(\text{Poisson}(4) \le 2) = \mathbf{0.238}$ $\qquad \gamma(\frac{1}{4}) = P(\text{Poisson}(3) \le 2) = \mathbf{0.423}$

$\gamma(\frac{1}{6}) = P(\text{Poisson}(2) \le 2) = \mathbf{0.677}$ $\qquad \gamma(\frac{1}{12}) = P(\text{Poisson}(1) \le 2) = \mathbf{0.920}$



7. **4.5.12** (7th edition) **5.5.12** (6th edition)

$H_0 : \mu = 0.5$ $H_1 : \mu > 0.5$

Reject H_0 if $X_1 + \ldots + X_8 \ge 8$

$X_1 + \ldots + X_8$ has Poisson distribution with mean 8μ.

a) significance level $= \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\text{Poisson}(4) \ge 8)$

$= 1 - P(\text{Poisson}(4) \le 7) = 1 - 0.949 = \mathbf{0.051}$

b) $\gamma(\mu) = P(\text{Poisson}(8\mu) \ge 8) = \sum_{k=8}^{\infty} \frac{(8\mu)^k e^{-8\mu}}{k!} = 1 - \sum_{k=0}^{7} \frac{(8\mu)^k e^{-8\mu}}{k!}.$

c) $\gamma(0.75) = P(\text{Poisson}(6) \ge 8) = 1 - 0.744 = \mathbf{0.256}$

$\gamma(1.00) = P(\text{Poisson}(8) \ge 8) = 1 - 0.453 = \mathbf{0.547}$

$\gamma(1.25) = P(\text{Poisson}(10) \ge 8) = 1 - 0.220 = \mathbf{0.780}$