STAT 410

Examples for 09/18/2020 (2)

Fall 2020
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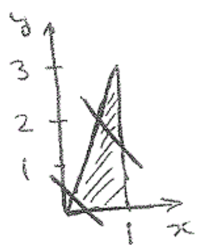
1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{4}{3}x^3 y, \qquad 0 < x < 1, \quad 0 < y < 3x, \qquad \text{zero otherwise.}$$

l) Let W = X + Y. Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.



Case 1. $0 < w < 1$.

Case 2. $1 < w < 4$.

$$f_W(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx.$$

$0 < x < 1$

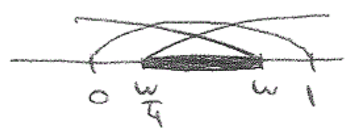
$0 < y \Rightarrow 0 < w - x$

$x < w$

$y < 3x \Rightarrow w - x < 3x$

$x > \frac{w}{4}$

Case 1. $0 < w < 1$.



$$f_W(w) = \int_{w/4}^{w} \frac{4}{3}x^3(w-x)\,dx = \left(\frac{wx^4}{3} - \frac{4x^5}{15}\right)\Bigg|_{w/4}^{w}$$

$$= \frac{w^5}{3} - \frac{4w^5}{15} - \frac{w^5}{3\cdot 256} + \frac{w^5}{15\cdot 256}$$

$$= \frac{21}{320}w^5, \qquad 0 < w < 1.$$

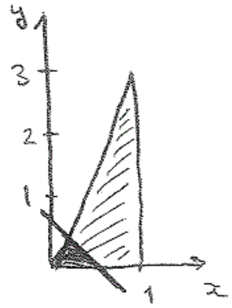
Case 2. $1 < w < 4$.

$$f_W(w) = \int_{w/4}^{1} \frac{4}{3}x^3(w-x)\,dx = \left(\frac{wx^4}{3} - \frac{4x^5}{15}\right)\Bigg|_{w/4}^{1}$$

$$= \frac{w}{3} - \frac{4}{15} - \frac{w^5}{960}, \qquad 1 < w < 4.$$

OR

Case 1.
$0 < w < 1.$

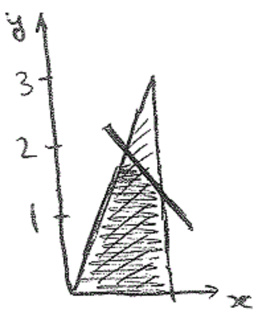


$$F_W(w) = \int_0^{3w/4} \left(\int_{y/3}^{w-y} \frac{4}{3} x^3 y \, dx \right) dy$$

$$\text{OR} \quad \int_0^{w/4} \left(\int_0^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{w/4}^{w} \left(\int_0^{w-x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$f_W(w) = F_W'(w).$$

Case 2.
$1 < w < 4.$



$$F_W(w) = 1 - \int_{\frac{w}{4}}^{1} \left(\int_{w-x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$\text{OR} \quad \int_0^{\frac{w}{4}} \left(\int_0^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{\frac{w}{4}}^{1} \left(\int_0^{w-x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$f_W(w) = F_W'(w).$$

$$F_W(w) = \begin{cases} \frac{7}{640} w^6, & 0 < w < 1 \\ \frac{1}{9} - \frac{4}{15} w + \frac{1}{6} w^2 - \frac{1}{5760} w^6, & 1 < w < 4 \end{cases}$$

2. Consider the following joint probability distribution $p(x, y)$ of two discrete random variables X and Y:

		x = 1	x = 2	$p_Y(y)$
y	1	0.14	0.06	0.20
	2	0.12	0.18	0.30
	3	0.14	0.36	0.50
	$p_X(x)$	0.40	0.60	1.00

Find the probability distribution of $W = X + Y$ and of $V = X \cdot Y$.

w	$p_W(w)$
2	0.14
3	0.18
4	0.32
5	0.36

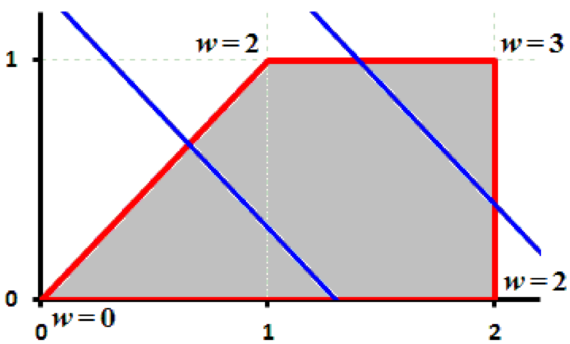
v	$p_V(v)$
1	0.14
2	0.18
3	0.14
4	0.18
6	0.36

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{12}{5} x y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

o) Find the probability distribution of $W = X + Y$.

$$F_W(w) = P(X + Y \le w) = \ldots$$



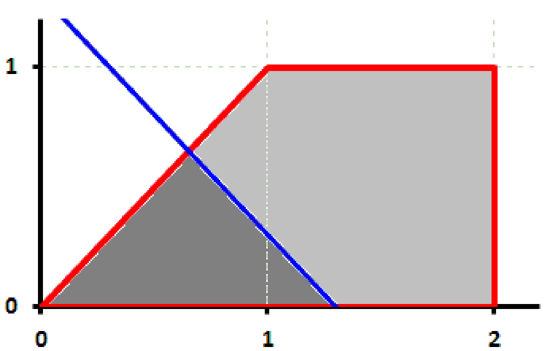
There are 2 cases: $0 < w < 2$, $2 < w < 3$.

Technically, there are 4 cases:

$w < 0$, $0 < w < 2$, $2 < w < 3$, $w > 3$,

but $w < 0$ and $w > 3$ are boring.

Case 0: $w < 0$. $F_W(w) = 0$.



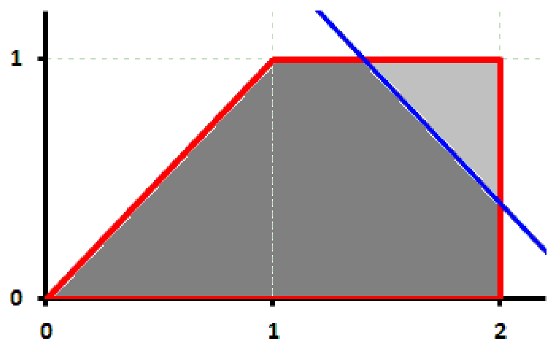
Case 1: $0 \le w < 2$.

$$F_W(w) = \int_0^{\frac{w}{2}} \left(\int_y^{w-y} \frac{12}{5} x y^3 \, dx \right) dy$$

$$= \int_0^{\frac{w}{2}} \frac{6}{5} \left((w-y)^2 - y^2 \right) y^3 \, dy$$

$$= \int_0^{\frac{w}{2}} \left(\frac{6\,w^2\,y^3}{5} - \frac{12\,w\,y^4}{5} \right) dy \;=\; \left(\frac{3\,w^2\,y^4}{10} - \frac{12\,w\,y^5}{25} \right) \Bigg|_0^{\frac{w}{2}}$$

$$= \frac{3\,w^6}{160} - \frac{3\,w^6}{200} \;=\; \frac{3\,w^6}{800}, \qquad\qquad 0 \le w < 2.$$



Case 2: $\quad 2 \le w < 3.$

$$\mathrm{F_W}(w) \;=\; 1 - \int_{w-2}^{1} \left(\int_{w-y}^{2} \frac{12}{5}\,x\,y^3\,dx \right) dy$$

$$= 1 - \int_{w-2}^{1} \frac{6}{5} \left(2^2 - (w-y)^2 \right) y^3\,dy$$

$$= 1 - \int_{w-2}^{1} \left(\frac{24\,y^3}{5} - \frac{6\,w^2\,y^3}{5} + \frac{12\,w\,y^4}{5} - \frac{6\,y^5}{5} \right) dy$$

$$= 1 - \left(\frac{6\,y^4}{5} - \frac{3\,w^2\,y^4}{10} + \frac{12\,w\,y^5}{25} - \frac{y^6}{5} \right) \Bigg|_{w-2}^{1}$$

$$= \left(\frac{15\,w^2 - 24\,w}{50} \right) + \left(\frac{6}{5} - \frac{3\,w^2}{10} + \frac{12\,w\,(w-2)}{25} - \frac{(w-2)^2}{5} \right) (w-2)^4$$

$$= \left(\frac{15\,w^2 - 24\,w}{50} \right) + \left(\frac{20 - 8\,w - w^2}{50} \right) (w-2)^4$$

$$= \frac{320 - 792\,w + 735\,w^2 - 320\,w^3 + 60\,w^4 - w^6}{50}, \qquad\qquad 2 \le w < 3.$$

Case 3: $\quad w \ge 3. \qquad\qquad \mathrm{F_W}(w) \;=\; 1.$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$$f(x, w-x) = \frac{12}{5}\, x\, (w-x)^3.$$

$0 < y < 1$ $\quad$ $0 < w - x < 1$ $\quad$ $w - 1 < x < w$

$y < x$ $\quad$ $w - x < x$ $\quad$ $x > \dfrac{w}{2}$

$x < 2$

$x > w - 1 \quad \& \quad x > \dfrac{w}{2}$ $\qquad$ $x < w \quad \& \quad x < 2$

Case 1: $\quad 0 < w < 2.$ $\qquad$ $w - 1 < \dfrac{w}{2} \quad \& \quad w < 2.$

$$f_W(w) = \int_{\frac{w}{2}}^{w} \frac{12}{5}\, x\, (w-x)^3\, dx = \ldots$$

Case 2: $\quad 2 < w < 3.$ $\qquad$ $\dfrac{w}{2} < w - 1 \quad \& \quad 2 < w.$

$$f_W(w) = \int_{w-1}^{2} \frac{12}{5}\, x\, (w-x)^3\, dx = \ldots$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$$f(w-y, y) = \frac{12}{5}\,(w-y)\,y^3.$$

$0 < y < 1$

$y < x$	$y < w - y$	$y < \frac{w}{2}$
$x < 2$	$w - y < 2$	$y > w - 2$

$y > 0 \quad \& \quad y > w - 2$ $\qquad\qquad$ $y < 1 \quad \& \quad y < \frac{w}{2}$

Case 1: $\quad 0 < w < 2. \qquad w - 2 < 0 \quad \& \quad \frac{w}{2} < 1.$

$$f_W(w) = \int_0^{\frac{w}{2}} \frac{12}{5}\,(w-y)\,y^3\, dy = \int_0^{\frac{w}{2}} \left(\frac{12\,w\,y^3}{5} - \frac{12\,y^4}{5} \right) dy$$

$$= \left(\frac{3\,w\,y^4}{5} - \frac{12\,y^5}{25} \right) \Bigg|_0^{\frac{w}{2}} = \frac{3\,w^5}{80} - \frac{3\,w^5}{200} = \frac{9\,w^5}{400}, \qquad 0 < w < 2.$$

Case 2: $\quad 2 < w < 3. \qquad 0 < w - 2 \quad \& \quad 1 < \frac{w}{2}.$

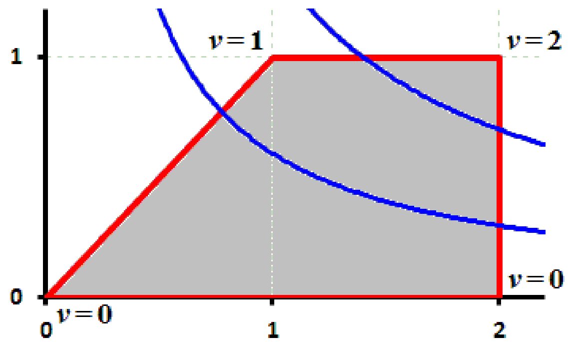
$$f_W(w) = \int_{w-2}^{1} \frac{12}{5}\,(w-y)\,y^3\, dy = \int_{w-2}^{1} \left(\frac{12\,w\,y^3}{5} - \frac{12\,y^4}{5} \right) dy$$

$$= \left(\frac{3\,w\,y^4}{5} - \frac{12\,y^5}{25} \right) \Bigg|_{w-2}^{1} = \frac{15\,w - 12}{25} - \frac{3\,w + 24}{25}\,(w-2)^4$$

$$= \frac{-396 + 735\,w - 480\,w^2 + 120\,w^3 - 3\,w^5}{25}, \qquad 2 < w < 3.$$

p) Find the c.d.f. of $V = X \cdot Y$, $F_V(v)$.

$$F_V(v) = P(X \cdot Y \le v) = P\left(Y \le \frac{v}{X}\right) = \ldots$$



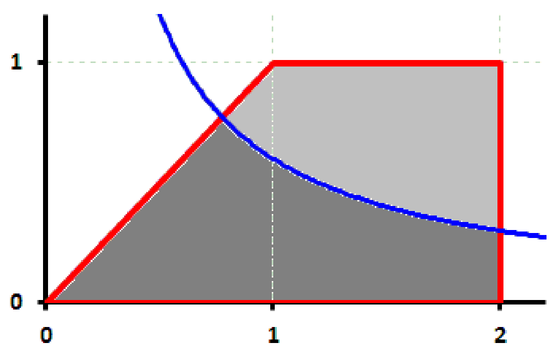
There are 2 cases: $0 < v < 1$, $1 < v < 2$.

Technically, there are 4 cases:

$v < 0$, $0 < v < 1$, $1 < v < 2$, $v > 2$,

but $v < 0$ and $v > 2$ are boring.

<u>Case 0</u>: $v < 0$. $F_V(v) = 0$.

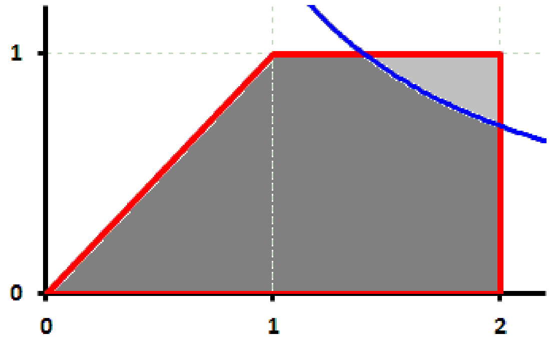


<u>Case 1</u>: $0 \le v < 1$.

$$F_V(v) = \int_0^{\sqrt{v}} \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_{\sqrt{v}}^2 \left(\int_0^{v/x} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= \int_0^{\sqrt{v}} \frac{3}{5} x^5 \, dx + \int_{\sqrt{v}}^2 \frac{3}{5} \frac{v^4}{x^3} \, dx = \frac{1}{10} x^6 \Big|_0^{\sqrt{v}} + \left(-\frac{3}{10} \frac{v^4}{x^2} \right) \Big|_{\sqrt{v}}^2$$

$$= \frac{2}{5} v^3 - \frac{3}{40} v^4, \qquad 0 \le v < 1.$$



Case 2: $1 \le v < 2.$

$$F_V(v) = 1 - \int_v^2 \left(\int_{v/x}^1 \frac{12}{5} x y^3 \, dy \right) dx$$

$$= 1 - \int_v^2 \left(\frac{3}{5} x - \frac{3}{5} \frac{v^4}{x^3} \right) dx$$

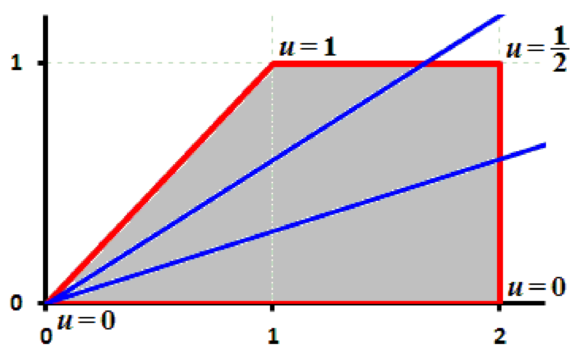
$$= 1 - \left(\frac{3}{10} x^2 + \frac{3}{10} \frac{v^4}{x^2} \right) \Bigg|_v^2 = -\frac{1}{5} + \frac{3}{5} v^2 - \frac{3}{40} v^4,$$

$1 \le v < 2.$

Case 3: $v \ge 2.$ $F_V(v) = 1.$

q) Find the c.d.f. of $U = \dfrac{Y}{X}$, $F_U(u)$.

$$F_U(u) = P\left(\frac{Y}{X} \le u\right) = P(Y \le u \cdot X) = \ldots$$



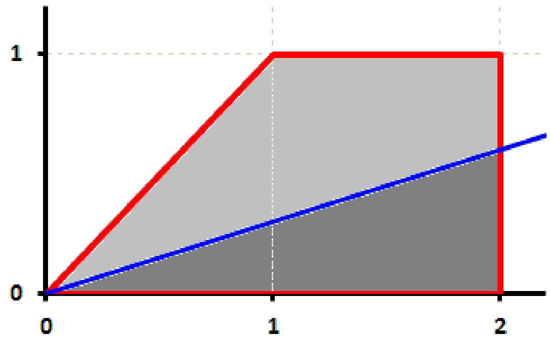
There are 2 cases: $0 < u < \frac{1}{2}$, $\frac{1}{2} < u < 1.$

Technically, there are 4 cases:

$u < 0$, $0 < u < \frac{1}{2}$, $\frac{1}{2} < u < 1$, $u > 1$,

but $u < 0$ and $u > 1$ are boring.

Case 0: $u < 0.$ $\qquad F_U(u) = 0.$

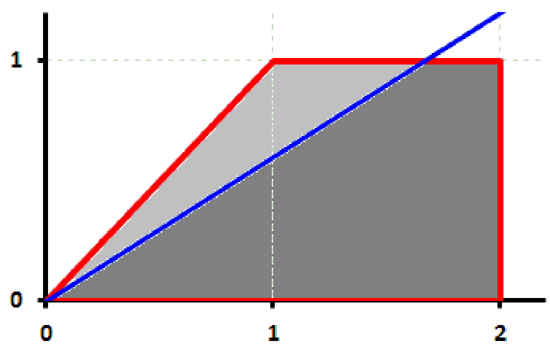


Case 1: $0 \le u < \frac{1}{2}.$

$$F_U(u) = \int_0^2 \left(\int_0^{ux} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= \int_0^2 \frac{3}{5} u^4 x^5 \, dx$$

$$= \frac{1}{10} u^4 x^6 \Big|_0^2 = \frac{32}{5} u^4, \qquad 0 \le u < \frac{1}{2}.$$



Case 2: $\frac{1}{2} \le u < 1.$

$$F_U(u) = 1 - \int_0^1 \left(\int_y^{y/u} \frac{12}{5} x y^3 \, dx \right) dy$$

$$= 1 - \int_0^1 \left(\frac{6}{5} \frac{y^5}{u^2} - \frac{6}{5} y^5 \right) dy$$

$$= 1 - \left(\frac{1}{5} \frac{y^6}{u^2} - \frac{1}{5} y^6 \right) \Big|_0^1 = \frac{6}{5} - \frac{1}{5u^2}, \qquad \frac{1}{2} \le u < 1.$$

Case 3: $u \ge 1.$ $\qquad F_U(u) = 1.$