STAT 410

Examples for 11/06/2020 (6)

Fall 2020
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1. Let $\delta > 2$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x; \delta) = (\delta^2 - 4)\, e^{-(\delta^2 - 4)x}, \qquad x > 0.$$

a) Find a sufficient statistic for δ.

b) Obtain a method of moments estimator $\tilde{\delta}$ for δ.

c) Obtain the maximum likelihood estimator $\hat{\delta}$ for δ.

d) Suppose $n = 9$ and $\sum_{i=1}^{n} x_i = 4$. Obtain the maximum likelihood estimate $\hat{\delta}$ for δ.

e) Suggest a $(1 - \alpha)\,100\,\%$ confidence interval for δ.

f) Suppose $n = 9$ and $\sum_{i=1}^{n} x_i = 4$. Construct a 90% confidence interval for δ.

Answers:

1. Let $\delta > 2$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\delta) = (\delta^2 - 4)\, e^{-(\delta^2-4)x}, \qquad x > 0.$$

$X_1, X_2, \ldots, X_n$ are i.i.d. Exponential$\left(\theta = \dfrac{1}{\delta^2 - 4}\right)$. $\qquad (\lambda = \delta^2 - 4)$

a) Find a sufficient statistic for δ.

$$\prod_{i=1}^{n} f(x_i;\delta) = \prod_{i=1}^{n} \left(\delta^2 - 4\right) e^{-\left(\delta^2-4\right) x_i} = \left(\delta^2 - 4\right)^n e^{-\left(\delta^2-4\right)\sum_{i=1}^n x_i}.$$

By Factorization Theorem, $\displaystyle\sum_{i=1}^{n} X_i$ is a sufficient statistic for δ.

[$\Rightarrow$ $\overline{X}$ is also a sufficient statistic for λ.]

b) Obtain a method of moments estimator $\tilde{\delta}$ for δ.

$$E(X) = \theta = \frac{1}{\delta^2 - 4}. \qquad \overline{X} = \frac{1}{\delta^2 - 4}.$$

$$\Rightarrow \qquad \tilde{\delta} = \sqrt{\frac{1}{\overline{X}} + 4}.$$

c) Obtain the maximum likelihood estimator $\hat{\delta}$ for δ.

$$L(\delta) = \prod_{i=1}^{n} f(x_i;\delta) = \left(\delta^2 - 4\right)^n e^{-\left(\delta^2 - 4\right)\sum_{i=1}^{n} x_i}.$$

$$\ln L(\delta) = n \ln\left(\delta^2 - 4\right) - \left(\delta^2 - 4\right)\sum_{i=1}^{n} x_i.$$

$$\frac{d}{d\delta} \ln L(\delta) = \frac{n}{\delta^2 - 4}\, 2\delta - 2\delta \sum_{i=1}^{n} x_i = 0.$$

$$\hat{\delta} = \sqrt{\frac{n}{\sum_{i=1}^{n} X_i} + 4} = \sqrt{\frac{1}{\overline{X}} + 4}.$$

d) Suppose $n = 9$ and $\sum_{i=1}^{n} x_i = 4$. Obtain the maximum likelihood estimate $\hat{\delta}$ for δ.

$$\hat{\delta} = \sqrt{\frac{n}{\sum_{i=1}^{n} X_i} + 4} = \sqrt{\frac{9}{4} + 4} = \mathbf{2.5}.$$

e) Suggest a $(1 - \alpha)\,100\,\%$ confidence interval for δ.

$\sum_{i=1}^{n} X_i$ has a Gamma$\left(\alpha = n,\ \theta = \frac{1}{\delta^2 - 4}\right)$ distribution.

$$\frac{2\,T_\alpha}{\theta} = 2\lambda T_\alpha \text{ has a } \chi^2(2\alpha) \text{ distribution.}$$

$$\Rightarrow \quad 2\left(\delta^2 - 4\right)\sum_{i=1}^{n} X_i \text{ has a } \chi^2(2\alpha = 2n) \text{ distribution.}$$

$$\Rightarrow \quad P\Big(\chi^2_{1-\alpha/2}(2n) < 2\left(\delta^2 - 4\right)\sum_{i=1}^{n} X_i < \chi^2_{\alpha/2}(2n)\Big) = 1-\alpha.$$

$$\Rightarrow \quad P\Bigg(\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i} < \delta^2 - 4 < \frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i}\Bigg) = 1-\alpha.$$

$$\Rightarrow \quad P\Bigg(\sqrt{\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i} + 4} < \delta < \sqrt{\frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i} + 4}\Bigg) = 1-\alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for δ:

$$\left(\sqrt{\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i} + 4}\ ,\ \sqrt{\frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i} + 4}\right).$$

f) Suppose $n = 9$ and $\sum_{i=1}^{n} x_i = 4$. Construct a 90% confidence interval for δ.

$$\chi^2_{0.95}(18) = 9.390, \qquad \chi^2_{0.05}(18) = 28.87.$$

$$\left(\sqrt{\frac{9.390}{2\cdot 4} + 4}\ ,\ \sqrt{\frac{28.87}{2\cdot 4} + 4}\right). \qquad \mathbf{(2.2746, 2.7584)}.$$