STAT 410 | **Examples for 10/28/2020 (1)** | Fall 2020
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Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. Let F_{X_n} and F_X be, respectively, the c.d.f.s of X_n and X. Let $C(F_X)$ denote the set of all points where F_X is continuous. We say that X_n **converges in distribution** to X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \qquad \text{for all } x \in C(F_X).$$

We denote this convergence by $$X_n \xrightarrow{D} X.$$

Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. We say that X_n **converges in probability** to X, if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \ge \varepsilon) = 0, \qquad \lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1.$$

We denote this convergence by $$X_n \xrightarrow{P} X.$$

Theorem 1 $\quad X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{D} X$

Theorem 2 $\quad X_n \xrightarrow{D} b,\ b$ – constant $\quad \Rightarrow \quad X_n \xrightarrow{P} b$

In general, $\quad X_n \xrightarrow{D} X \quad \not\Rightarrow \quad X_n \xrightarrow{P} X$

Example 1(a):

Let $\{X_n\}, X$ be i.i.d. with p.m.f. $P(X=-1)=\frac{1}{2}$, $P(X=1)=\frac{1}{2}$.

Then $X_n \xrightarrow{D} X$, since for all n $F_{X_n}(x)=F_X(x)=\begin{cases} 0 & x<-1 \\ 1/2 & -1\le x<1 \\ 1 & x\ge 1 \end{cases}$

However, for all n $P(|X_n-X|\ge 1)=\frac{1}{2}$, so X_n does not converge to X in probability.

Example 1(b):

Let X be a random variable with p.m.f. $P(X=-1)=\frac{1}{2}$, $P(X=1)=\frac{1}{2}$.

Let $X_n=(-1)^n X$, $n=1, 2, 3, \dots$. Then $X_n \xrightarrow{D} X$.

However, $P(|X_n-X|\ge 1)=1$ if n is odd, $P(|X_n-X|\ge 1)=0$ if n is even, so X_n does not converge to X in probability.

Example 1(c):

Let X be a random variable with p.m.f. $P(X=-1)=\frac{1}{2}$, $P(X=1)=\frac{1}{2}$.

Let $X_n=-X$, $n=1, 2, 3, \dots$. Then $X_n \xrightarrow{D} X$.

However, for all n $P(|X_n-X|\ge 1)=1$, so X_n does not converge to X in probability.

$X_n \xrightarrow{P} -X$.

Example 2(a): Suppose $U \sim \text{Uniform}(0, 1)$.

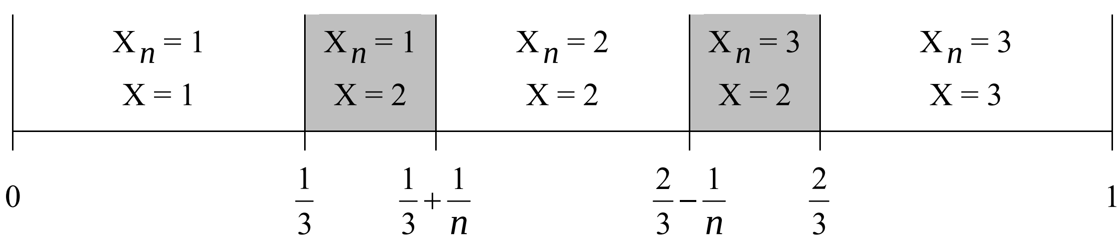
Let

$$X_n = \begin{cases} 1 & \text{if} \quad U \in \left(0, \frac{1}{3} + \frac{1}{n}\right) \\ 2 & \text{if} \quad U \in \left[\frac{1}{3} + \frac{1}{n}, \frac{2}{3} - \frac{1}{n}\right) \\ 3 & \text{if} \quad U \in \left[\frac{2}{3} - \frac{1}{n}, 1\right) \end{cases} \qquad X = \begin{cases} 1 & \text{if} \quad U \in \left(0, \frac{1}{3}\right) \\ 2 & \text{if} \quad U \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ 3 & \text{if} \quad U \in \left[\frac{2}{3}, 1\right) \end{cases}$$

Then $P(|X_n - X| \ge \varepsilon) = \dfrac{2}{n}$, $0 < \varepsilon \le 1$, $P(|X_n - X| \ge \varepsilon) = 0$, $\varepsilon > 1$.

Therefore, $X_n \xrightarrow{P} X$ $\left(\text{and } X_n \xrightarrow{D} X\right)$.

The same (one) random variable U was used to create the entire sequence of random variables $\{X_n\}$ and the limiting random variable X.



$\Rightarrow \quad P(X_n \text{ and } X \text{ are “not close”}) = \dfrac{2}{n} \to 0 \quad \text{as} \quad n \to \infty.$

(Same as Example 4 from Examples for 10/23/2020 (1))

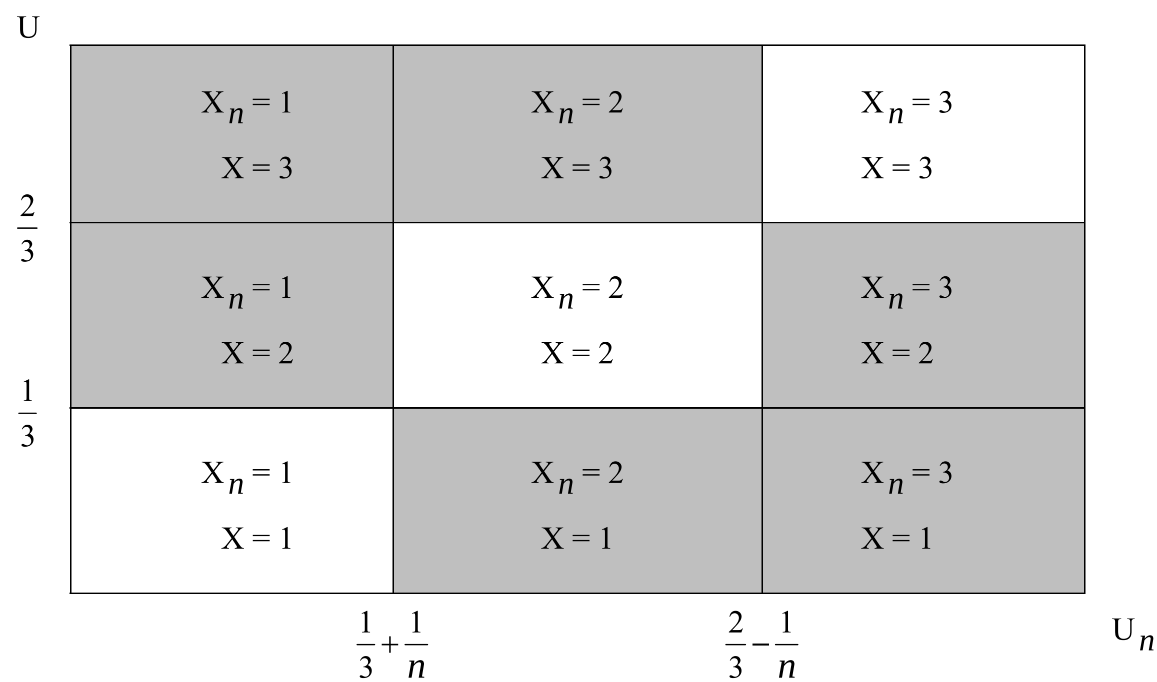
Example 2(b): Suppose $\{U_n\}, U$ are i.i.d. Uniform$(0, 1)$.

Let

$$X_n = \begin{cases} 1 & \text{if} \quad U_n \in \left(0, \frac{1}{3} + \frac{1}{n}\right) \\ 2 & \text{if} \quad U_n \in \left[\frac{1}{3} + \frac{1}{n}, \frac{2}{3} - \frac{1}{n}\right) \\ 3 & \text{if} \quad U_n \in \left[\frac{2}{3} - \frac{1}{n}, 1\right) \end{cases} \qquad X = \begin{cases} 1 & \text{if} \quad U \in \left(0, \frac{1}{3}\right) \\ 2 & \text{if} \quad U \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ 3 & \text{if} \quad U \in \left[\frac{2}{3}, 1\right) \end{cases}$$

Then $X_n \xrightarrow{D} X$, but not in probability, since $P(|X_n - X| \geq \varepsilon) \to \frac{2}{3}$, $0 < \varepsilon \leq 1$.

Different (independent) random variables $\{U_n\}, U$ were used to create the sequence of random variables $\{X_n\}$ and the limiting random variable X.



$\Rightarrow \quad P(X_n \text{ and X are "not close"}) \to \frac{2}{3} \quad \text{as} \quad n \to \infty.$

= =

$$\left(1+\frac{a}{n^{\beta}}\right)^{n} \underset{n\to\infty}{\to} \begin{cases} e^{a} & \beta=1 \\ 1 & \beta>1 \\ \begin{matrix} \infty & a>0 \\ 0 & a<0 \end{matrix} & 0<\beta<1 \end{cases}$$

1. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{1}{\theta}\cdot x^{\frac{1}{\theta}-1}, \qquad 0<x<1, \qquad 0<\theta<\infty.$$

Let $Y_1 < Y_2 < \dots < Y_n$ denote the corresponding order statistics.

$$F_X(x) = x^{1/\theta}, \qquad 0<x<1.$$

a) For which values of β does $W_n = n^{\beta}(1-Y_n)$ converge in distribution? Find the limiting distribution of W_n.

$$F_{Y_n}(y) = P(Y_n \le y) = y^{n/\theta}, \qquad 0<y<1.$$

$$F_{W_n}(w) = P(n^{\beta}(1-Y_n) \le w) = P(Y_n \ge 1-\frac{w}{n^{\beta}})$$

$$= 1-\left(1-\frac{w}{n^{\beta}}\right)^{n/\theta}, \qquad 0<w<n^{\beta}.$$

If $\beta = 1$, $$F_{\infty}(w) = \lim_{n\to\infty} F_{W_n}(w) = 1 - e^{-w/\theta}, \qquad 0 < w < \infty,$$

The limiting distribution is an Exponential distribution with mean θ.

If $\beta < 1$, $$\lim_{n\to\infty} F_{W_n}(w) = 1, \qquad 0 < w < \infty,$$

Then $W_n \xrightarrow{D} 0$, and thus $W_n \xrightarrow{P} 0$.

If $\beta > 1$, $$\lim_{n\to\infty} F_{W_n}(w) = 0, \qquad 0 < w < \infty,$$

Then W_n does not have a limiting distribution.

b) For which values of γ does $V_n = n^{\gamma} Y_1$ converge in distribution?

Find the limiting distribution of V_n.

$$F_{Y_1}(y) = P(Y_1 \le y) = 1 - \left(1 - y^{1/\theta} \right)^n, \qquad 0 < y < 1.$$

$$F_{V_n}(v) = P(Y_1 \le \frac{v}{n^{\gamma}}) = 1 - \left(1 - \frac{v^{1/\theta}}{n^{\gamma/\theta}} \right)^n, \qquad 0 < v < n^{\gamma}.$$

If $\gamma = \theta$, $$F_{\infty}(v) = \lim_{n\to\infty} F_{V_n}(v) = 1 - e^{-v^{1/\theta}}, \qquad 0 < v < \infty.$$

The limiting distribution is a Weibull distribution.

If $\gamma < \theta$, $$\lim_{n\to\infty} F_{V_n}(v) = 1, \qquad 0 < v < \infty,$$

Then $V_n \xrightarrow{D} 0$, and thus $V_n \xrightarrow{P} 0$.

If $\gamma > \theta$, $$\lim_{n\to\infty} F_{V_n}(v) = 0, \qquad 0 < v < \infty,$$

Then V_n does not have a limiting distribution.

1¼. Let Y_{1n} denote the minimum (the first order statistic) of a random sample of size n from a distribution of the continuous type that has c.d.f. $F(x)$ and p.d.f. $f(x) = F'(x)$. Find the limiting distribution of $W_n = n\,F(Y_{1n})$.

$$F_{Y_{1n}}(x) = 1 - (1 - F(x))^n .$$

Since $W_n = n\,F(Y_{1n})$, $P(0 < W_n < n) = 1$.

Let $0 < w < n$.

$$F_{W_n}(w) = P(n\,F(Y_{1n}) \le w) = F_{Y_{1n}}\left(F^{-1}\left(\frac{w}{n}\right)\right)$$

$$= 1 - \left(1 - \frac{w}{n}\right)^n \to 1 - e^{-w} \qquad \text{as } n \to \infty.$$

$W_n \xrightarrow{D} W$. For the limiting distribution W of W_n,

$$F_W(w) = 1 - e^{-w},\ w > 0, \qquad f_W(w) = e^{-w},\ w > 0.$$

Therefore, W has Exponential distribution with mean 1.

1½. **5.2.3** (7th edition) **4.3.3** (6th edition)

Let Y_n denote the maximum (the last order statistic) of a random sample of size n from a distribution of the continuous type that has c.d.f. $F(x)$ and p.d.f. $f(x) = F'(x)$. Find the limiting distribution of $Z_n = n(1 - F(Y_n))$.

$$F_{Y_n}(x) = (F(x))^n .$$

Since $Z_n = n(1 - F(Y_n))$, $P(0 < Z_n < n) = 1$.

Let $0 < z < n$.

$$F_{Z_n}(z) = P(n(1 - F(Y_n)) \le z) = P(F(Y_n) \ge 1 - \frac{z}{n})$$

$$= 1 - F_{Y_n}\left(F^{-1}\left(1 - \frac{z}{n}\right)\right) = 1 - \left(1 - \frac{z}{n}\right)^n \to 1 - e^{-z} \quad \text{as } n \to \infty.$$

For the limiting distribution Z of Z_n,

$F_Z(z) = 1 - e^{-z}, \qquad z > 0, \qquad\qquad f_Z(z) = e^{-z}, \qquad z > 0.$

Therefore, the limiting distribution of Z_n is Exponential with mean 1.

(Exponential distribution with mean 1 is same as Gamma distribution with $\alpha = 1$, $\beta = 1$.)

1¾. 5.2.4 (7th edition) **4.3.4** (6th edition)

Let Y_2 denote the second smallest item of a random sample of size n from a distribution of the continuous type that has c.d.f. $F(x)$ and p.d.f. $f(x) = F'(x)$. Find the limiting distribution of $W_n = n\,F(Y_2)$.

$$F_{Y_2}(x) = \sum_{i=2}^{n} \binom{n}{i} \cdot (F(x))^i \cdot (1 - F(x))^{n-i}$$

$$= 1 - (1 - F(x))^n - n \cdot (F(x)) \cdot (1 - F(x))^{n-1}.$$

Since $W_n = n\,F(Y_2)$, $P(0 < W_n < n) = 1$.

Let $0 < w < n$.

$$F_{W_n}(w) = P(n\,F(Y_2) \le w) = F_{Y_2}\left(F^{-1}\left(\frac{w}{n}\right)\right)$$

$$= 1 - \left(1 - \frac{w}{n}\right)^n - n \cdot \left(\frac{w}{n}\right) \cdot \left(1 - \frac{w}{n}\right)^{n-1} \to 1 - e^{-w} - w\,e^{-w}$$

as $n \to \infty$.

For the limiting distribution W of W_n,

$F_W(w) = 1 - e^{-w} - w\,e^{-w}, \qquad w > 0, \qquad\qquad f_W(w) = w\,e^{-w}, \qquad w > 0.$

Therefore, W has Gamma distribution with $\alpha = 2$, $\beta = 1$.

Theorem 3 $\quad X_n \xrightarrow{D} X$, g is continuous on the support of X

$$\Rightarrow \quad g(X_n) \xrightarrow{D} g(X)$$

Theorem 4 $\quad X_n \xrightarrow{D} X$, $\; Y_n \xrightarrow{P} 0 \quad \Rightarrow \quad X_n + Y_n \xrightarrow{D} X$

Theorem 5 $\quad$ Slutsky's Theorem

$X_n \xrightarrow{D} X$, $\; A_n \xrightarrow{P} a$, $\; B_n \xrightarrow{P} b$

$$\Rightarrow \quad A_n + B_n \cdot X_n \xrightarrow{D} a + b \cdot X$$

Theorem 6 $\quad M_{X_n}(t) \to M_X(t)$ for $|t| < h \quad \Rightarrow \quad X_n \xrightarrow{D} X$

2. Let X_n be Binomial$(n, \, p_n = \lambda / n)$. Find the limiting distribution of X_n.

Let X_n be Binomial$(n, p_n = \lambda / n)$. Then

$$M_{X_n}(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t\right)^n \to e^{\lambda(e^t - 1)} \text{ as } n \to \infty.$$

$M_X(t) = e^{\lambda(e^t - 1)}$, where X has a Poisson(λ) distribution.

$\Rightarrow \quad X_n \xrightarrow{D} X \qquad$ (Poisson approximation to Binomial distribution).

3. Let X_n be $\chi^2(n)$.

a) Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

Let X_n be $\chi^2(n)$. Let $Y_n = X_n/n$. Then

$$M_{Y_n}(t) = E(e^{tX_n/n}) = M_{X_n}(t/n) = \frac{1}{\left(1-2\frac{t}{n}\right)^{n/2}} \to e^t \text{ as } n \to \infty.$$

$M_X(t) = e^t$, where $P(X=1)=1$.

$$\Rightarrow \quad Y_n \xrightarrow{D} 1. \qquad \Rightarrow \quad Y_n \xrightarrow{P} 1.$$

b) Let $Z_n = \left(X_n - n\right)/\sqrt{2n}$. Find the limiting distribution of Z_n.

$$M_{Z_n}(t) = e^{-t\sqrt{n/2}}\, M_{X_n}(t/\sqrt{2n}) = e^{-t\sqrt{n/2}} \cdot \frac{1}{\left(1-2\frac{t}{\sqrt{2n}}\right)^{n/2}}$$

$$= \left(e^{t\sqrt{\frac{2}{n}}} - t\sqrt{\frac{2}{n}}\, e^{t\sqrt{\frac{2}{n}}}\right)^{-n/2}, \qquad t < \sqrt{\frac{n}{2}}.$$

$$e^{t\sqrt{\frac{2}{n}}} = 1 + t\sqrt{\frac{2}{n}} + \frac{t^2}{n} + o\left(\frac{1}{n}\right).$$

$$M_{Z_n}(t) = \left(1 - \frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)^{-n/2}, \qquad t < \sqrt{\frac{n}{2}}.$$

As $n \to \infty$, $\qquad M_{Z_n}(t) \to \exp\left\{\frac{t^2}{2}\right\} = M_Z(t),$

where Z has Standard Normal N(0, 1) distribution.

$$\Rightarrow \qquad Z_n \xrightarrow{D} Z, \qquad Z \sim N(0, 1).$$

c) **5.2.9** (7th edition) **4.3.9** (6th edition)

Let X be $\chi^2(50)$. Approximate $P(40 < X < 60)$.

$\chi^2(r = 50)$ can be approximated by $N(r, 2r) = N(50, 100)$.

$$P(40 < X < 60) \approx P\left(\frac{40-50}{10} < Z < \frac{60-50}{10}\right) = P(-1.00 < Z < 1.00)$$

$$= 0.8413 - 0.1587 = \mathbf{0.6826}.$$

```
> pchisq(60,50)
[1] 0.842758
> pchisq(40,50)
[1] 0.1567726
> pchisq(60,50)-pchisq(40,50)
[1] 0.6859854
```

=CHISQ.DIST(60,50,1)
0.842758
=CHISQ.DIST(40,50,1)
0.156773
=CHISQ.DIST(60,50,1)-CHISQ.DIST(40,50,1)
0.685985

=CHISQ.DIST.RT(40,50)
0.843227
=CHISQ.DIST.RT(60,50)
0.157242

=CHISQ.DIST.RT(40,50)-CHISQ.DIST.RT(60,50)
0.685985

4. a) **5.2.11** (7th edition) **4.3.11** (6th edition)

$Z_n \sim \text{Poisson}(n) \qquad Y_n = (Z_n - n)/\sqrt{n}$

$M_{Z_n}(t) = e^{n(e^t - 1)}.$

$$M_{Y_n}(t) = E\left(e^{t(Z_n - n)/\sqrt{n}}\right) = e^{-t\sqrt{n}}\,E\left(e^{tZ_n/\sqrt{n}}\right)$$

$$= e^{-t\sqrt{n}}\,M_{Z_n}\left(\frac{t}{\sqrt{n}}\right) = \exp\left\{-t\sqrt{n} + n\cdot\left(e^{t/\sqrt{n}} - 1\right)\right\}.$$

$$e^{t/\sqrt{n}} = 1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2n} + o\left(\frac{1}{n}\right).$$

$$M_{Y_n}(t) = \exp\left\{-t\sqrt{n} + n\cdot\left(\frac{t}{\sqrt{n}} + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)\right\} = \exp\left\{\frac{t^2}{2} + o(1)\right\}.$$

As $n \to \infty$, $\qquad M_{Y_n}(t) \to \exp\left\{\frac{t^2}{2}\right\} = M_Z(t),$

where Z has Standard Normal $N(0, 1)$ distribution.

$\Rightarrow \qquad Y_n \xrightarrow{D} Z, \qquad Z \sim N(0, 1).$

b) **5.2.14** (7th edition) **4.3.14** (6th edition)

$X_1, X_2, \ldots, X_n$ are i.i.d. Poisson(1) $\qquad Y_n = \sqrt{n}\left(\overline{X}_n - 1\right)$

a) $M_{X_1}(t) = e^{(e^t - 1)}$.

$$M_{Y_n}(t) = E\left(e^{t\sqrt{n}\left(\overline{X}_n - 1\right)}\right) = e^{-t\sqrt{n}}\,E\left(e^{t\left(X_1 + X_2 + \ldots + X_n\right)/\sqrt{n}}\right)$$

$$= e^{-t\sqrt{n}}\left(M_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n = \exp\left\{-t\sqrt{n} + n\cdot\left(e^{t/\sqrt{n}} - 1\right)\right\}.$$

b) $e^{t/\sqrt{n}} = 1 + \dfrac{t}{\sqrt{n}} + \dfrac{t^2}{2n} + o\left(\dfrac{1}{n}\right)$.

$$M_{Y_n}(t) = \exp\left\{-t\sqrt{n} + n\cdot\left(\frac{t}{\sqrt{n}} + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)\right\} = \exp\left\{\frac{t^2}{2} + o(1)\right\}.$$

As $n \to \infty$, $\qquad M_{Y_n}(t) \to \exp\left\{\dfrac{t^2}{2}\right\} = M_Z(t)$,

where Z has Standard Normal $N(0, 1)$ distribution.

$\Rightarrow \qquad Y_n \xrightarrow{D} Z, \qquad Z \sim N(0, 1)$.

(a) and (b) really are the same since

$$X_1 + X_2 + \ldots + X_n = \text{Poisson}(1) + \text{Poisson}(1) + \ldots + \text{Poisson}(1) = \text{Poisson}(n) = Z_n.$$

Population: mean μ, variance σ^2, standard deviation σ.

Random Sample: $X_1, X_2, \dots, X_n$ are i.i.d.

The **sample mean** $\overline{X} = \dfrac{X_1 + X_2 + \dots + X_n}{n}$.

Then $$M_{\overline{X}}(t) = E\left(e^{t\overline{X}}\right) = E\left(e^{t(X_1 + X_2 + \dots + X_n)/n}\right) = \left(M_X\left(\frac{t}{n}\right)\right)^n.$$

Example 1: Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a $\mathbf{N}\left(\mu, \sigma^2\right)$, distribution.

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}. \qquad M_{\overline{X}}(t) = \left(M_X\left(\frac{t}{n}\right)\right)^n = e^{\mu t + \sigma^2 t^2/2n}.$$

Then $\overline{X}$ has a $\mathbf{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ distribution.

Example 2: Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a Gamma(α, θ) distribution. That is,

$$f_X(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad x > 0.$$

$$M_X(t) = (1 - \theta t)^{-\alpha}, \qquad t < 1/\theta.$$

$$M_{\overline{X}}(t) = \left(M_X\left(\frac{t}{n}\right)\right)^n = \left(1 - \frac{\theta t}{n}\right)^{-n\alpha}, \qquad t < n/\theta.$$

$\Rightarrow$ $\overline{X}$ has a Gamma distribution with $\alpha_{\overline{X}} = n\alpha$, $\theta_{\overline{X}} = \theta/n$.

Central Limit Theorem

$X_1, X_2, \dots, X_n$ are i.i.d. with mean μ and variance σ^2.

$$\sqrt{n}\left(\overline{X} - \mu\right)/\sigma = \left(\sum_{i=1}^n X_i - n\mu\right)\Big/\sqrt{n}\,\sigma \xrightarrow{D} Z, \qquad Z \sim N(0, 1).$$