STAT 410

Fall 2020
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Practice Problems 1

1. Let X have the pdf $f(x) = 4x^3$, for $0 < x < 1$, zero elsewhere.

a) **1.7.23** (7th and 6th edition)

Find the cdf and the pdf of $Y = -\ln X^4$.

b) Let $Y = e^X$. Find the probability distribution of Y.

c) Let $Y = X^2$. Find the probability distribution of Y.

d) Let $Y = \sqrt{X}$. Find the probability distribution of Y.

2. **1.7.22** (7th and 6th edition)

Let X have the uniform pdf $f_X(x) = \frac{1}{\pi}$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of $Y = \tan X$. This is the pdf of a **Cauchy distribution**.

3. **1.8.8** (7th edition) **1.8.10** (6th edition) + (a½)

Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the p.d.f. of X.

a) Compute $E(\frac{1}{X})$.

a½) Compute $E(X)$. Does $E(\frac{1}{X})$ equal $\frac{1}{E(X)}$?

b) Find the c.d.f. and the p.d.f. of $Y = \frac{1}{X}$.

c) Compute $E(Y)$ and compare this result with the answer obtained in part (a).

4. Let X be a random variable with probability density function

$$f_X(x) = \frac{1}{(1+x)^2}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = \frac{1}{4+X}$.

b) Find the probability distribution of $Y = e^{-X/2}$.

5. Let X be a random variable with probability density function

$$f_X(x) = -\ln x, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = -\ln X$.

b) Find the probability distribution of $Y = \sqrt{X}$.

c) Find the probability distribution of $Y = \frac{1}{\sqrt[3]{X}}$.

6. Let $\theta > 1$ and let X be a random variable with probability density function

$$f(x;\theta) = \frac{1}{x \ln \theta}, \qquad 1 < x < \theta.$$

a) Let $U = \ln X$. What is the probability distribution of U?

b) Let $a > 0$ and let $V = X^a$. What is the probability distribution of V?

c) Let $W = \frac{X}{\theta}$. What is the probability distribution of W?

d) Let $Y = \frac{1}{X}$. What is the probability distribution of Y?

7. Suppose a random variable X has the following probability density function:

$$f_X(x) = \begin{cases} x e^x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the moment-generating function of X, $M_X(t)$.

b) Let $Y = e^X$. Find the probability distribution of Y.

c) Find $\text{Var}(Y)$.

8. Let X be a random variable with probability density function

$$f_X(x) = \frac{x+1}{8}, \qquad -1 < x < 3, \qquad \text{zero otherwise.}$$

Find the probability distribution of $Y = X^2$.

9. Let X be a random variable with the probability density function

$$f_X(x) = \frac{11-2x}{50}, \qquad -2 < x < 3, \qquad \text{zero otherwise.}$$

Find the probability distribution of $Y = X^2$.

Answers:

1. Let X have the pdf $f(x) = 4x^3$, for $0 < x < 1$, zero elsewhere.

a) **1.7.23** (7th and 6th edition)

Find the cdf and the pdf of $Y = -\ln X^4$.

$F_X(x) = x^4, \qquad 0 < x < 1.$

$0 < x < 1 \qquad y = -4 \ln x \qquad \Rightarrow \qquad y > 0$

$F_Y(y) = P(Y \le y) = P(-4 \ln X \le y) = P(X \ge e^{-y/4}) = 1 - e^{-y}, \qquad y > 0.$

$\Rightarrow \quad f_Y(y) = F_Y'(y) = e^{-y}, \qquad y > 0.$

$\Rightarrow$ Y has Exponential distribution with mean 1.

OR

$y = g(x) = -4 \ln x \qquad \Rightarrow \qquad x = g^{-1}(y) = = e^{-y/4}$

$\Rightarrow \qquad {}^{dx}/_{dy} = -\frac{1}{4} e^{-y/4}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = 4\left(e^{-y/4}\right)^3 \times \left| -\frac{1}{4} e^{-y/4} \right| = e^{-y}, \qquad y > 0.$$

$\Rightarrow$ Y has Exponential distribution with mean 1.

OR

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{-4 \ln X \cdot t}) = E(X^{-4t}) = \int_0^1 \left(x^{-4t} \cdot 4x^3\right) dx$$

$$= \int_0^1 4x^{3-4t}\, dx = \frac{4}{4-4t} = \frac{1}{1-t}, \qquad t < 1.$$

$\Rightarrow$ Y has Exponential distribution with mean 1.

b) Let $Y = e^X$. Find the probability distribution of Y.

$0 < x < 1 \qquad y = e^x \qquad \Rightarrow \qquad 1 < y < e.$

$y = g(x) = e^x \qquad \Rightarrow \qquad x = g^{-1}(y) = \ln y$

$\Rightarrow \qquad dx/dy = \dfrac{1}{y}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = 4(\ln y)^3 \times \left| \frac{1}{y} \right| = \frac{4}{y}(\ln y)^3, \qquad 1 < y < e.$$

OR

$F_X(x) = x^4, \qquad 0 < x < 1.$

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = (\ln y)^4, \qquad 1 < y < e.$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = 4(\ln y)^3 \times \frac{1}{y} = \frac{4}{y}(\ln y)^3, \qquad 1 < y < e.$$

c) Let $Y = X^2$. Find the probability distribution of Y.

$$F_X(x) = x^4, \qquad 0 < x < 1. \qquad\qquad 0 < x < 1 \quad y = x^2 \quad \Rightarrow \quad 0 < y < 1.$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}\,) = F_X(\sqrt{y}\,) = y^2, \quad 0 < y < 1.$$

$$\Rightarrow \qquad f_Y(y) = F_Y'(y) = 2y, \qquad 0 < y < 1.$$

OR

$$g(x) = x^2 \qquad\qquad g^{-1}(y) = \sqrt{y} = y^{1/2} \qquad\qquad {}^{dx}/_{dy} = \frac{1}{2}\,y^{-1/2}$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = (4y^{3/2})\left|\frac{1}{2}\,y^{-1/2}\right| = 2y, \qquad 0 < y < 1.$$

d) Let $Y = \sqrt{X}$. Find the probability distribution of Y.

$$F_X(x) = x^4, \qquad 0 < x < 1. \qquad\qquad 0 < x < 1 \quad y = \sqrt{x} \quad \Rightarrow \quad 0 < y < 1.$$

$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = F_X(y^2) = y^8, \qquad 0 < y < 1.$$

$$\Rightarrow \qquad f_Y(y) = F_Y'(y) = 8y^7, \qquad 0 < y < 1.$$

OR

$$g(x) = \sqrt{x} \qquad\qquad g^{-1}(y) = y^2 \qquad\qquad {}^{dx}/_{dy} = 2y$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = = (4y^6)\left|2y\right| = 8y^7, \qquad 0 < y < 1.$$

2. **1.7.22** (7th and 6th edition)

Let X have the uniform pdf $f_X(x) = \frac{1}{\pi}$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of Y = tan X. This is the pdf of a **Cauchy distribution**.

$$f_X(x) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < -\frac{\pi}{2} \\ \frac{x}{\pi} + \frac{1}{2} & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 1 & x \ge \frac{\pi}{2} \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(\tan X \le y) = P(X \le \arctan(y)) = \frac{1}{\pi}\arctan(y) + \frac{1}{2}, \quad -\infty < y < \infty.$$

$$f_Y(y) = \frac{1}{\pi\left(1+y^2\right)}, \qquad -\infty < y < \infty. \qquad \text{(Standard) Cauchy distribution.}$$

OR

$$g(x) = \tan x \qquad g^{-1}(y) = \arctan(y) \qquad {}^{dx}/_{dy} = \frac{1}{1+y^2}$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = \left(\frac{1}{\pi}\right)\left(\frac{1}{1+y^2}\right) = \frac{1}{\pi\left(1+y^2\right)}, \qquad -\infty < y < \infty.$$

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\pi\left(1+u^2\right)}du = \frac{1}{\pi}\arctan(y) + \frac{1}{2}, \qquad -\infty < y < \infty.$$

3. **1.8.8** (7th edition) **1.8.10** (6th edition) + (a½)

Let $f(x) = 2x,\ 0 < x < 1$, zero elsewhere, be the p.d.f. of X.

a) Compute $E(\frac{1}{X})$.

$$E(\frac{1}{X}) = \int_0^1 \frac{1}{x} \cdot 2x\,dx = \int_0^1 2\,dx = \mathbf{2}.$$

a½) Compute $E(X)$. Does $E(\frac{1}{X})$ equal $\frac{1}{E(X)}$?

$$E(X) = \int_0^1 x \cdot 2x\,dx = \int_0^1 2x^2\,dx = \mathbf{\frac{2}{3}}. \qquad E(\frac{1}{X}) \neq \frac{1}{E(X)}.$$

b) Find the c.d.f. and the p.d.f. of $Y = \frac{1}{X}$.

$0 < x < 1 \qquad Y = {}^1/_X \qquad \Rightarrow \qquad y > 1.$

$g(x) = {}^1/_x \qquad g^{-1}(y) = {}^1/_y = y^{-1} \qquad {}^{dx}/_{dy} = -y^{-2}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (2y^{-1})(y^{-2}) = 2y^{-3}, \qquad y > 1.$$

OR

$$F_Y(y) = P(Y \le y) = P({}^1/_X \le y) = P(X \ge {}^1/_y) = 1 - P(X < {}^1/_y)$$
$$= 1 - F_X({}^1/_y) = 1 - {}^1/_{y^2}, \qquad y > 1.$$

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ 1 - {}^1/_{y^2} & y \ge 1 \end{cases} \qquad f_Y(y) = \begin{cases} 0 & y < 1 \\ {}^2/_{y^3} & y \ge 1 \end{cases}$$

c) Compute $E(Y)$ and compare this result with the answer obtained in part (a).

$$E(Y) = \int_1^\infty y \cdot \frac{2}{y^3}\,dy = \int_1^\infty \frac{2}{y^2}\,dy = \left(-\frac{2}{y}\right)\Bigg|_1^\infty = \mathbf{2}.$$

Same ☺.

4. Let X be a random variable with probability density function

$$f_X(x) = \frac{1}{(1+x)^2}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = \dfrac{1}{4+X}$.

$$y = \frac{1}{4+x} \qquad x > 0 \qquad \Rightarrow \qquad 0 < y < \frac{1}{4}.$$

$$y = \frac{1}{4+x} \qquad x = \frac{1}{y} - 4 = g^{-1}(y) \qquad \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{1}{\left(1 + \frac{1}{y} - 4\right)^2} \times \left| -\frac{1}{y^2} \right| = \frac{1}{(1-3y)^2},$$

$$0 < y < \frac{1}{4}.$$

OR

$$F_X(x) = P(X \le x) = 1 - \frac{1}{1+x}, \qquad x > 0.$$

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{4+X} \le y\right) = P\left(X \ge \frac{1}{y} - 4\right)$$

$$= 1 - F_X\left(\frac{1}{y} - 4\right) = \frac{1}{1 + \frac{1}{y} - 4} = \frac{y}{1-3y}, \qquad 0 < y < \frac{1}{4}.$$

$$f_Y(y) = F_Y'(y) = \frac{(1-3y) - y(-3)}{(1-3y)^2} = \frac{1}{(1-3y)^2}, \qquad 0 < y < \frac{1}{4}.$$

b) Find the probability distribution of $\mathrm{Y} = e^{-\mathrm{X}/2}$.

$$y = e^{-x/2} \qquad x > 0 \qquad \Rightarrow \qquad 0 < y < 1.$$

$$y = e^{-x/2} \qquad x = -2\ln y = g^{-1}(y) \qquad \frac{dx}{dy} = -\frac{2}{y}$$

$$f_{\mathrm{Y}}(y) = f_{\mathrm{X}}(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{1}{(1-2\ln y)^2} \times \left| -\frac{2}{y} \right| = \frac{2}{y(1-2\ln y)^2},$$

$$0 < y < 1.$$

OR

$$\mathrm{F}_{\mathrm{X}}(x) = \mathrm{P}(\mathrm{X} \le x) = 1 - \frac{1}{1+x}, \qquad x > 0.$$

$$\mathrm{F}_{\mathrm{Y}}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}(e^{-\mathrm{X}/2} \le y) = \mathrm{P}(\mathrm{X} \ge -2\ln y)$$

$$= 1 - \mathrm{F}_{\mathrm{X}}(-2\ln y) = \frac{1}{1-2\ln y}, \qquad 0 < y < 1.$$

$$f_{\mathrm{Y}}(y) = \mathrm{F}_{\mathrm{Y}}'(y) = -\frac{1}{(1-2\ln y)^2} \cdot \left(-\frac{2}{y} \right) = \frac{2}{y(1-2\ln y)^2}, \qquad 0 < y < 1.$$

5. Let X be a random variable with probability density function

$$f_X(x) = -\ln x, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = -\ln X$.

$y = -\ln x \qquad 0 < x < 1 \qquad \Rightarrow \qquad 0 < y < \infty.$

$y = -\ln x \qquad x = e^{-y} \qquad \dfrac{dx}{dy} = -e^{-y}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = y \times \left| -e^{-y} \right| = y\,e^{-y}, \qquad 0 < y < \infty.$$

Y has a Gamma distribution with $\alpha = 2$ and $\theta = 1$.

b) Find the probability distribution of $Y = \sqrt{X}$.

$y = \sqrt{x} \qquad 0 < x < 1 \qquad \Rightarrow \qquad 0 < y < 1.$

$y = \sqrt{x} \qquad x = y^2 \qquad \dfrac{dx}{dy} = 2y$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = -2\ln y \times \left| 2y \right| = -4y\ln y, \qquad 0 < y < 1.$$

c) Find the probability distribution of $Y = \dfrac{1}{\sqrt[3]{X}}$.

$y = \dfrac{1}{\sqrt[3]{x}} \qquad 0 < x < 1 \qquad \Rightarrow \qquad 1 < y < \infty.$

$y = \dfrac{1}{\sqrt[3]{x}} \qquad x = y^{-3} \qquad \dfrac{dx}{dy} = -3y^{-4}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = 3\ln y \times \left| -3y^{-4} \right| = 9y^{-4}\ln y = \frac{9\ln y}{y^4},$$

$$1 < y < \infty.$$

6. Let $\theta > 1$ and let X be a random variable with probability density function

$$f(x;\theta) = \frac{1}{x\ln\theta}, \qquad 1 < x < \theta.$$

a) Let $\mathrm{U} = \ln \mathrm{X}$. What is the probability distribution of U?

$1 < x < \theta \qquad u = \ln x \qquad \Rightarrow \qquad 0 < u < \ln\theta$

$x = e^u \qquad {}^{dx}/_{du} = e^u$

$$f_{\mathrm{U}}(u) = f_{\mathrm{X}}(g^{-1}(u))\left|\frac{dx}{du}\right| = \frac{1}{e^u \ln\theta}\cdot e^u = \frac{1}{\ln\theta}, \qquad 0 < u < \ln\theta.$$

Uniform on $(0, \ln\theta)$.

b) Let $a > 0$ and let $\mathrm{V} = \mathrm{X}^a$. What is the probability distribution of V?

$1 < x < \theta \qquad v = x^a \qquad \Rightarrow \qquad 1 < v < \theta^a$

$x = v^{1/a} \qquad {}^{dx}/_{dv} = \frac{1}{a} v^{(1/a)-1}$

$$f_{\mathrm{V}}(v) = f_{\mathrm{X}}(g^{-1}(v))\left|\frac{dx}{dv}\right| = \frac{1}{v^{(1/a)}\ln\theta}\cdot\frac{1}{a}v^{(1/a)-1} = \frac{1}{a\,v\ln\theta} = \frac{1}{v\ln\theta^a},$$

$$1 < v < \theta^a.$$

c) Let $W = \dfrac{X}{\theta}$. What is the probability distribution of W?

$$1 < x < \theta \qquad w = \frac{x}{\theta} \qquad \Rightarrow \qquad \frac{1}{\theta} < w < 1$$

$$x = \theta w \qquad {}^{dx}\!/_{dw} = \theta$$

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{1}{\theta\, w \ln\theta} \cdot \theta = \frac{1}{w \ln\theta}, \qquad \frac{1}{\theta} < w < 1.$$

d) Let $Y = \dfrac{1}{X}$. What is the probability distribution of Y?

$$1 < x < \theta \qquad y = \frac{1}{x} \qquad \Rightarrow \qquad \frac{1}{\theta} < y < 1$$

$$x = \frac{1}{y} \qquad {}^{dx}\!/_{dy} = -\frac{1}{y^2}$$

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{y}{\ln\theta} \cdot \frac{1}{y^2} = \frac{1}{y \ln\theta}, \qquad \frac{1}{\theta} < y < 1.$$

7. Suppose a random variable X has the following probability density function:

$$f_X(x) = \begin{cases} x e^{x} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = \int_0^1 e^{tx} \cdot x e^{x}\, dx = \int_0^1 x e^{(t+1)x}\, dx$$

$$= \left[\frac{1}{t+1} x e^{(t+1)x} - \frac{1}{(t+1)^2} e^{(t+1)x} \right]\Bigg|_0^1$$

$$= \frac{1}{t+1} e^{t+1} - \frac{1}{(t+1)^2} e^{t+1} + \frac{1}{(t+1)^2}$$

$$= \frac{t e^{t+1} + 1}{(t+1)^2}, \qquad t \neq -1.$$

$$M_X(-1) = \int_0^1 x\, dx = \frac{1}{2}.$$

b) Let $Y = e^X$. Find the probability distribution of Y.

$$y = g(x) = e^{x} \qquad x = g^{-1}(y) = \ln y \qquad \frac{dx}{dy} = \frac{1}{y}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (\ln y \cdot y) \cdot \frac{1}{y} = \ln y, \qquad 1 < y < e.$$

OR

$$F_X(x) = 1 + x\,e^{x} - e^{x}, \qquad 0 < x < 1.$$

$$F_Y(y) = P(Y \le y) = P(e^{X} \le y) = P(X \le \ln y) = F_X(\ln y) = 1 + \ln y \cdot y - y,$$

$$1 < y < e.$$

$$f_Y(y) = F_Y'(y) = \ln y, \qquad 1 < y < e.$$

c) Find $\operatorname{Var}(Y)$.

$$E(Y) = E(e^{X}) = M_X(1) = \frac{e^{2}+1}{4}.$$

$$E(Y^{2}) = E[(e^{X})^{2}] = E(e^{2X}) = M_X(2) = \frac{2e^{3}+1}{9}.$$

$$\operatorname{Var}(Y) = E(Y^{2}) - [E(Y)]^{2} = \frac{2e^{3}+1}{9} - \left(\frac{e^{2}+1}{4}\right)^{2} \approx 0.176.$$

OR

$$\operatorname{Var}(Y) = \int_{1}^{e} y^{2} \cdot \ln y\, dy - \left(\int_{1}^{e} y \cdot \ln y\, dy\right)^{2} = \ldots$$

8. Let X be a random variable with probability density function

$$f_X(x) = \frac{x+1}{8}, \qquad -1 < x < 3, \qquad \text{zero otherwise.}$$

Find the probability distribution of $Y = X^2$.

$y < 0$ $\qquad P(X^2 \le y) = 0 \qquad F_Y(y) = 0.$

$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

Case 1: $\quad 0 \le y < 1 \quad \Rightarrow \quad 0 \le \sqrt{y} < 1 \quad \Rightarrow \quad -1 < -\sqrt{y} \le \sqrt{y} < 3$

$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{x+1}{8}\,dx = \left(\frac{x^2}{16} + \frac{x}{8}\right)\Bigg|_{-\sqrt{y}}^{\sqrt{y}} = \frac{\sqrt{y}}{4}.$$

Case 2: $\quad 1 \le y < 9 \quad \Rightarrow \quad 1 \le \sqrt{y} < 3 \quad \Rightarrow \quad -\sqrt{y} \le -1 < \sqrt{y} < 3$

$$F_Y(y) = \int_{-1}^{\sqrt{y}} \frac{x+1}{8}\,dx = \left(\frac{x^2}{16} + \frac{x}{8}\right)\Bigg|_{-1}^{\sqrt{y}} = \frac{y}{16} + \frac{\sqrt{y}}{8} - \frac{1}{16} + \frac{1}{8} = \frac{\left(\sqrt{y}+1\right)^2}{16}.$$

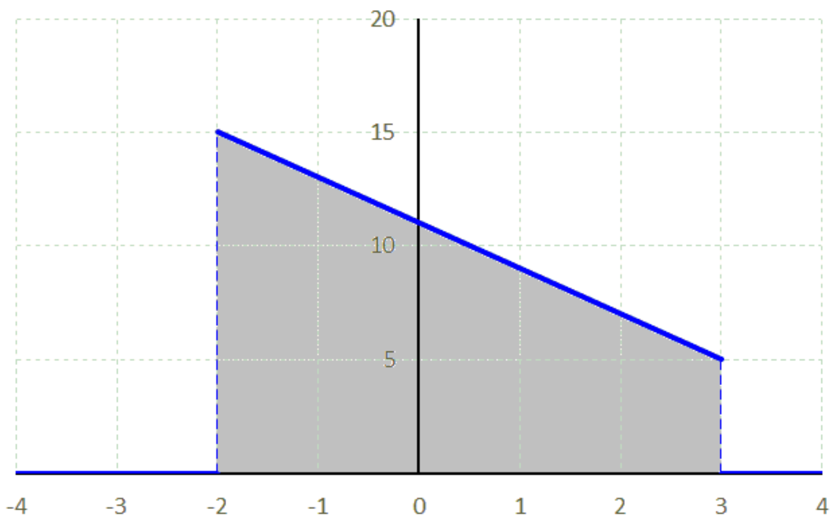
Case 3: $\quad y \ge 9 \qquad F_Y(y) = 1.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{\sqrt{y}}{4} & 0 \le y < 1 \\ \dfrac{\left(\sqrt{y}+1\right)^2}{16} & 1 \le y < 9 \\ 1 & y \ge 9 \end{cases} \qquad f_Y(y) = \begin{cases} \dfrac{1}{8\sqrt{y}} & 0 < y < 1 \\ \dfrac{\sqrt{y}+1}{16\sqrt{y}} & 1 < y < 9 \\ 0 & \text{o.w.} \end{cases}$$

9. Let X be a random variable with the probability density function

$$f_X(x) = \frac{11-2x}{50}, \qquad -2 < x < 3, \qquad \text{zero otherwise.}$$

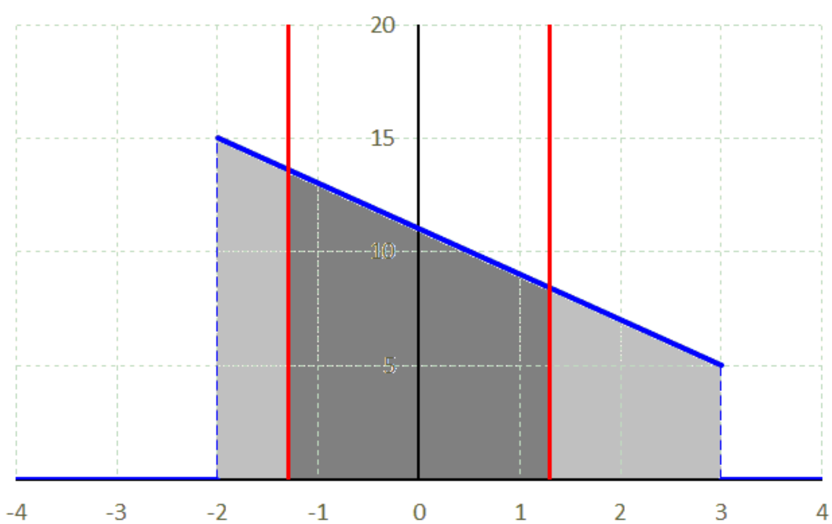
Find the probability distribution of $Y = X^2$.



$y < 0$ $\qquad P(X^2 \le y) = 0 \qquad F_Y(y) = 0.$

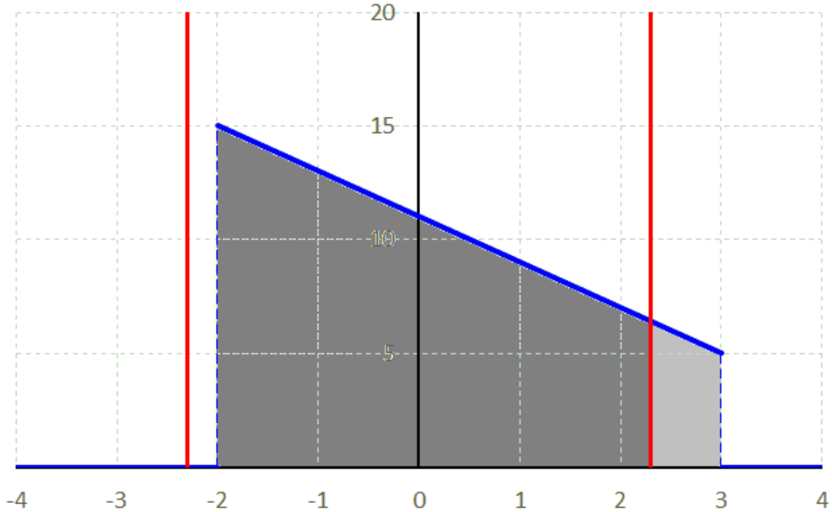
$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}\,)$

Case 1: $\quad 0 \le y < 4 \quad \Rightarrow \quad 0 \le \sqrt{y} < 2 \quad \Rightarrow \quad -2 < -\sqrt{y} \le \sqrt{y} < 3$



$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{11-2x}{50}\,dx = \left.\frac{11x-x^2}{50}\right|_{-\sqrt{y}}^{\sqrt{y}} = \frac{11\sqrt{y}}{25}, \qquad 0 \le y < 4.$$

Case 2: $4 \le y < 9 \quad \Rightarrow \quad 2 \le \sqrt{y} < 3 \quad \Rightarrow \quad -\sqrt{y} \le -2 < \sqrt{y} < 3$



$$F_Y(y) = \int_{-2}^{\sqrt{y}} \frac{11-2x}{50}\,dx = \left.\frac{11x-x^2}{50}\right|_{-2}^{\sqrt{y}} = \frac{26+11\sqrt{y}-y}{50}, \qquad 4 \le y < 9.$$

$y \ge 9 \qquad F_Y(y) = 1.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{11\sqrt{y}}{25} & 0 \le y < 4 \\ \dfrac{26+11\sqrt{y}-y}{50} & 4 \le y < 9 \\ 1 & y \ge 9 \end{cases}$$

$$f_Y(y) = F_Y'(y)$$

$$f_Y(y) = \begin{cases} \dfrac{11}{50\sqrt{y}} & 0 < y < 4 \\ \dfrac{11-2\sqrt{y}}{100\sqrt{y}} & 4 < y < 9 \\ 0 & \text{otherwise} \end{cases}$$

OR

$$-2 < x < 0$$

$$f_X(x) = \frac{11 - 2x}{50}$$

$$Y = g(X) = X^2$$

$$x = -\sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$$

$$0 < y < 4$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\frac{11 + 2\sqrt{y}}{50} \cdot \left| -\frac{1}{2\sqrt{y}} \right| = \frac{11 + 2\sqrt{y}}{100\sqrt{y}}$$

$$0 < x < 3$$

$$f_X(x) = \frac{11 - 2x}{50}$$

$$Y = g(X) = X^2$$

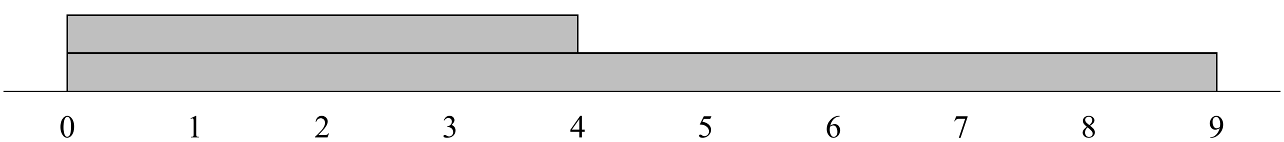
$$x = \sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$0 < y < 9$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\frac{11 - 2\sqrt{y}}{50} \cdot \left| \frac{1}{2\sqrt{y}} \right| = \frac{11 - 2\sqrt{y}}{100\sqrt{y}}$$



$$f_Y(y) = \begin{cases} \dfrac{11 + 2\sqrt{y}}{100\sqrt{y}} + \dfrac{11 - 2\sqrt{y}}{100\sqrt{y}} & 0 < y < 4 \\[2ex] 0 + \dfrac{11 - 2\sqrt{y}}{100\sqrt{y}} & 4 < y < 9 \\[2ex] 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{11}{50\sqrt{y}} & 0 < y < 4 \\[2ex] \dfrac{11 - 2\sqrt{y}}{100\sqrt{y}} & 4 < y < 9 \\[2ex] 0 & \text{otherwise} \end{cases}$$

OR

$$F_X(x) = 0, \qquad x < -2.$$

$$F_X(x) = \int_{-2}^{x} \frac{11-2u}{50}\,du = \left.\frac{11u-u^2}{50}\right|_{-2}^{x} = \frac{11x-x^2+26}{50}, \qquad -2 \le x < 3.$$

$$F_X(x) = 1, \qquad x \ge 3.$$

$y < 0$ $\qquad P(X^2 \le y) = 0 \qquad F_Y(y) = 0.$

$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \begin{cases} \dfrac{11\sqrt{y}-y+26}{50} - \dfrac{-11\sqrt{y}-y+26}{50} & 0 \le y < 4 \\[2ex] \dfrac{11\sqrt{y}-y+26}{50} - 0 & 4 \le y < 9 \\[2ex] 1-0 & y \ge 9 \end{cases}$$

$$= \begin{cases} \dfrac{11\sqrt{y}}{25} & 0 \le y < 4 \\[2ex] \dfrac{11\sqrt{y}-y+26}{50} & 4 \le y < 9 \\[2ex] 1 & y \ge 9 \end{cases}$$