STAT 410

Examples for 11/18/2020 (Disc)

Fall 2020
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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let $\beta > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0, \qquad \beta > 0.$$

Recall: $Y = \sum_{i=1}^{n} \sqrt{X_i}$ is a sufficient statistic for β;

$W = \sqrt{X}$ has $\text{Gamma}(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution;

$\Rightarrow \quad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has $\text{Gamma}(\alpha = 4n,\ \theta = \frac{1}{\beta})$ distribution;

We wish to test $H_0: \beta = 3$ vs. $H_1: \beta > 3$.

a) Suppose $n = 3$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

b) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.
Find the p-value for the test.

1. (continued)

Consider the rejection region Reject H_0 if $\sum_{i=1}^{3} \sqrt{x_i} \le 2.5$.

c) Find the significance level α of this rejection region.

d) Find the power of this rejection region if $\beta = 4$ and if $\beta = 6$.

1. (continued)

We wish to test $H_0 : \beta = 8$ vs. $H_1 : \beta < 8$.

e) Suppose $n = 3$. Find the uniformly most powerful rejection region with $\alpha = 0.05$.

f) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.
Find the p-value for the test.

1. (continued)

Consider the rejection region Reject H_0 if $\sum_{i=1}^{3} \sqrt{x_i} \ge 2.5$.

g) Find the significance level α of this rejection region.

h) Find the power of this rejection region if $\beta = 4$ and if $\beta = 6$.

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0, \qquad \beta > 0.$$

Recall: $Y = \sum_{i=1}^{n} \sqrt{X_i}$ is a sufficient statistic for β;

$W = \sqrt{X}$ has $\text{Gamma}(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution;

$\Rightarrow \quad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has $\text{Gamma}(\alpha = 4n,\ \theta = \frac{1}{\beta})$ distribution;

We wish to test $H_0 : \beta = 3$ vs. $H_1 : \beta > 3$.

a) Suppose $n = 3$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

Hint 1: Let $\beta > 3$. Start with

$$\frac{L(H_0;\, x_1, x_2, \ldots, x_n)}{L(H_1;\, x_1, x_2, \ldots, x_n)} = \frac{L(3;\, x_1, x_2, \ldots, x_n)}{L(\beta;\, x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} f(x_i; 3)}{\prod_{i=1}^{n} f(x_i; \beta)} \le k.$$

Simplify this. Since $Y = \sum_{i=1}^{n} \sqrt{X_i}$ is a sufficient statistic for β,

and the final form of the "best" rejection region should look like this:

$$\text{“ Reject } H_0 \text{ if } \sum_{i=1}^{n} \sqrt{x_i} = \sum_{i=1}^{3} \sqrt{x_i}\ [\le \text{or} \ge]\ c \text{ ”.}$$

The direction of the inequality sign is what you are trying to determine.

Hint 2: $Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has $\text{Gamma}(\alpha = 4n,\ \theta = \frac{1}{\beta})$ distribution.

Hint 3: Want c such that $0.10 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{n} \sqrt{X_i}\ ?\ c \mid \beta = 3)$.

Hint 4: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

Let $\beta > 3$.

$$\frac{L(H_0;\, x_1, x_2, \ldots, x_n)}{L(H_1;\, x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \frac{3^4}{12} x_i\, e^{-3\sqrt{x_i}}}{\prod_{i=1}^{n} \frac{\beta^4}{12} x_i\, e^{-\beta\sqrt{x_i}}} = \frac{3^{4n}}{\beta^{4n}} \exp\left((\beta - 3) \sum_{i=1}^{n} \sqrt{x_i} \right).$$

$$\frac{L(H_0)}{L(H_1)} \le k \quad \Leftrightarrow \quad \sum_{i=1}^{n} \sqrt{x_i} \le c \qquad (\text{since } \beta > 3).$$

Intuition: β is "λ". Large β $\Rightarrow$ small $\sqrt{X}$.
The sign is opposite from the sign in H_1.

$\sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has $\text{Gamma}(\alpha = 4n = 12,\ \theta = \frac{1}{\beta})$ distribution.

Then $2\beta \sum_{i=1}^{n} \sqrt{X_i}$ has a $\chi^2(2\alpha = 8n = 24 \text{ degrees of freedom})$ distribution.

$$0.10 = P\left(\sum_{i=1}^{n} \sqrt{X_i} \le c \,\middle|\, \beta = 3\right) = P\left(2\beta \sum_{i=1}^{n} \sqrt{X_i} \le 2\beta c \,\middle|\, \beta = 3\right)$$

$$= P(\chi^2(24) \le 6c).$$

$\Rightarrow \quad 6c = \chi^2_{0.90}(24) = 15.66. \qquad \Rightarrow \quad c = \mathbf{2.61}.$

The uniformly most powerful rejection region is "Reject H_0 if $\sum_{i=1}^{n} \sqrt{x_i} \le 2.61$."

b) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Find the p-value for the test.

Hint 1: … $Y = \sum_{i=1}^{n} \sqrt{X_i}$ as extreme or more extreme than the observed $\sum_{i=1}^{n} \sqrt{x_i}$ …

Hint 2: For the p-value, go in the same direction as the "best" rejection region.

Hint 3: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$F_T(t) = P(T \le t) = P(X_t \ge \alpha)$ and $P(T > t) = P(X_t \le \alpha - 1)$,

where X_t has a $\text{Poisson}(\lambda t)$ distribution.

$$\sum_{i=1}^{n} \sqrt{x_i} = \sqrt{0.25} + \sqrt{0.36} + \sqrt{0.81} = 2.0.$$

$$\text{P-value} = P\left(\sum_{i=1}^{n} \sqrt{X_i} \le 2.0 \,\middle|\, \beta = 3\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{3}\right) \le 2.0\right)$$

$$= P(\text{Poisson}(2.0 \cdot 3) \ge 12) = 1 - P(\text{Poisson}(6) \le 11) = 1 - 0.980 = \mathbf{0.020}.$$

$$\text{P-value} = P\left(\sum_{i=1}^{n} \sqrt{X_i} \le 2.0 \,\middle|\, \beta = 3\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{3}\right) \le 2.0\right)$$

$$= P(\chi^2(24) \le 6 \cdot 2.0) = P(\chi^2(24) \le 12).$$

```
> pchisq(12,24)
[1] 0.02009196
```

```
> pgamma(2,12,3)
[1] 0.02009196
```

$$\text{P-value} = P\left(\sum_{i=1}^{n} \sqrt{X_i} \le 2.0 \,\middle|\, \beta = 3\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{3}\right) \le 2.0\right)$$

$$= \int_0^2 \frac{3^{12}}{\Gamma(12)} x^{12-1} e^{-3x}\, dx = \int_0^2 \frac{3^{12}}{11!} x^{11} e^{-3x}\, dx = \ldots$$

1. (continued)

Consider the rejection region Reject H_0 if $\sum_{i=1}^{3} \sqrt{x_i} \le 2.5$.

c) Find the significance level α of this rejection region.

Hint 1: $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{3} \sqrt{X_i} \le 2.5 \mid \beta = 3)$.

Hint 2: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$F_T(t) = P(T \le t) = P(X_t \ge \alpha)$ and $P(T > t) = P(X_t \le \alpha - 1)$,

where X_t has a Poisson(λt) distribution.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{n} \sqrt{X_i} \le 2.5 \mid \beta = 3)$$

$$= P(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{3}) \le 2.5) = P(\text{Poisson}(2.5 \cdot 3) \ge 12)$$

$$= 1 - P(\text{Poisson}(7.5) \le 11) = 1 - 0.921 = \mathbf{0.079}.$$

$$\alpha = P(\sum_{i=1}^{n} \sqrt{X_i} \le 2.5 \mid \beta = 3) = P(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{3}) \le 2.5)$$

$$= P(\chi^2(24) \le 6 \cdot 2.5) = P(\chi^2(24) \le 15).$$

```
> pchisq(15,24)
[1] 0.07924131
```

```
> pgamma(2.5,12,3)
[1] 0.07924131
```

$$\alpha = P(\sum_{i=1}^{n} \sqrt{X_i} \le 2.5 \mid \beta = 3) = P(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{3}) \le 2.5)$$

$$= \int_0^{2.5} \frac{3^{12}}{\Gamma(12)} x^{12-1} e^{-3x}\, dx = \int_0^{2.5} \frac{3^{12}}{11!} x^{11} e^{-3x}\, dx = \ldots$$

d) Find the power of this rejection region if $\beta = 4$ and if $\beta = 6$.

Hint: $\text{Power}(\beta) = \text{P}(\text{Reject } \text{H}_0 \mid \text{H}_0 \text{ is NOT true}) = \text{P}(\sum_{i=1}^{3} \sqrt{\text{X}_i} \le 2.5 \mid \beta)$.

$$\text{Power}(\beta = 4) = \text{P}(\sum_{i=1}^{n} \sqrt{\text{X}_i} \le 2.5 \mid \beta = 4) = \text{P}(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{4}) \le 2.5)$$

$$= \text{P}(\text{Poisson}(2.5 \cdot 4) \ge 12) = 1 - \text{P}(\text{Poisson}(10) \le 11) = 1 - 0.697 = \mathbf{0.303}.$$

$$\text{Power}(\beta = 4) = \text{P}(\sum_{i=1}^{n} \sqrt{\text{X}_i} \le 2.5 \mid \beta = 4) = \text{P}(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{4}) \le 2.5)$$

$$= \text{P}(\chi^2(24) \le 8 \cdot 2.5) = \text{P}(\chi^2(24) \le 20).$$

```
> pchisq(15,24)
[1] 0.3032239
```

```
> pgamma(2.5,12,4)
[1] 0.3032239
```

$$\text{Power}(\beta = 6) = \text{P}(\sum_{i=1}^{n} \sqrt{\text{X}_i} \le 2.5 \mid \beta = 6) = \text{P}(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{6}) \le 2.5)$$

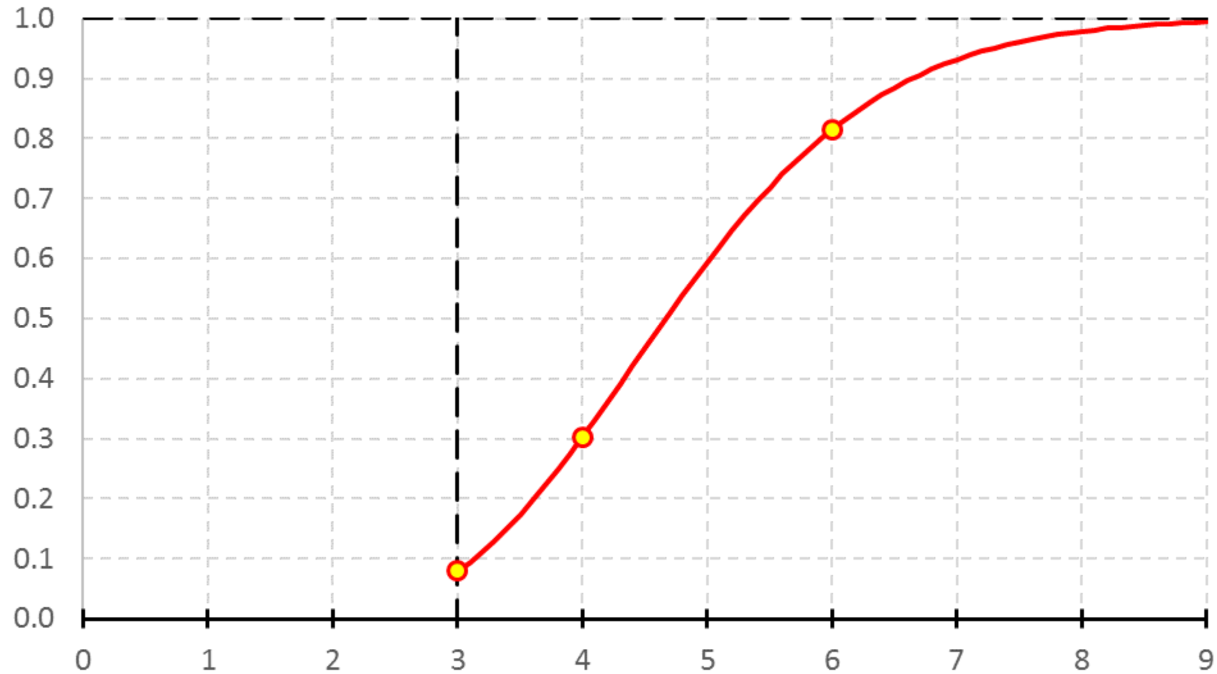
$$= \text{P}(\text{Poisson}(2.5 \cdot 6) \ge 12) = 1 - \text{P}(\text{Poisson}(15) \le 11) = 1 - 0.185 = \mathbf{0.815}.$$

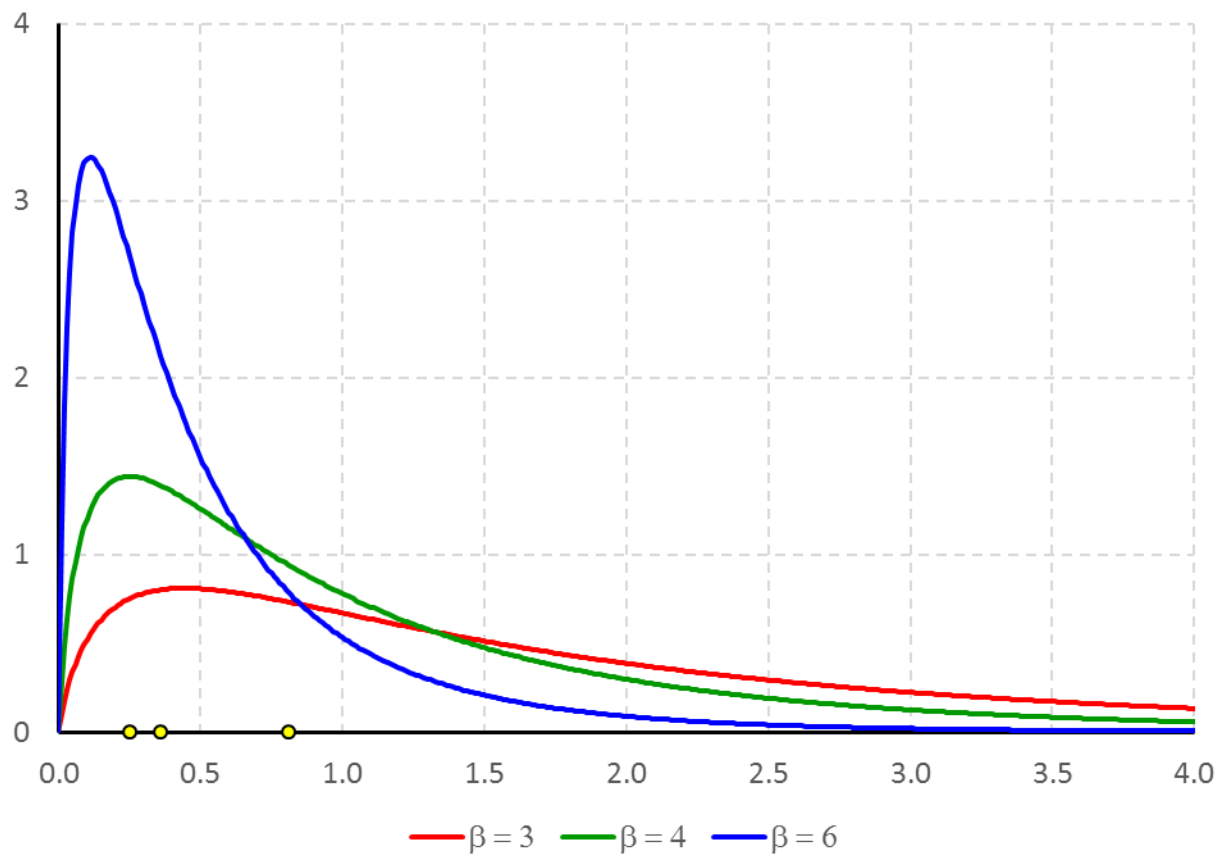
$$\text{Power}(\beta = 6) = \text{P}(\sum_{i=1}^{n} \sqrt{\text{X}_i} \le 2.5 \mid \beta = 6) = \text{P}(\text{Gamma}(\alpha = 12,\ \theta = \frac{1}{6}) \le 2.5)$$

$$= \text{P}(\chi^2(24) \le 12 \cdot 2.5) = \text{P}(\chi^2(24) \le 30).$$

```
> pchisq(15,24)
[1] 0.8152482
```

```
> pgamma(2.5,12,6)
[1] 0.8152482
```


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7
8
9


4
3
2
1
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
β = 3
β = 4
β = 6

1. (continued)

We wish to test $H_0 : \beta = 8$ vs. $H_1 : \beta < 8$.

e) Suppose $n = 3$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

Let $\beta < 8$.

$$\frac{L(H_0\,;\,x_1, x_2, \ldots, x_n)}{L(H_1\,;\,x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \frac{8^4}{12}\, x_i\, e^{-8\sqrt{x_i}}}{\prod_{i=1}^{n} \frac{\beta^4}{12}\, x_i\, e^{-\beta\sqrt{x_i}}} = \frac{8^{4n}}{\beta^{4n}} \exp\left((\beta - 8) \sum_{i=1}^{n} \sqrt{x_i} \right).$$

$$\frac{L(H_0)}{L(H_1)} \le k \quad \Leftrightarrow \quad \sum_{i=1}^{n} \sqrt{x_i} \ge c \qquad (\text{since } \beta < 8).$$

Intuition: β is "λ". Small β $\Rightarrow$ large $\sqrt{X}$.
The sign is opposite from the sign in H_1.

$\sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has Gamma$\left(\alpha = 4n = 20,\ \theta = \frac{1}{\beta}\right)$ distribution.

Then $2\beta \sum_{i=1}^{n} \sqrt{X_i}$ has a $\chi^2(2\alpha = 8n = 24 \text{ degrees of freedom})$ distribution.

$$0.10 = P\left(\sum_{i=1}^{n} \sqrt{X_i} \ge c \,\middle|\, \beta = 8\right) = P\left(2\beta \sum_{i=1}^{n} \sqrt{X_i} \ge 2\beta c \,\middle|\, \beta = 8\right)$$
$$= P(\chi^2(24) \ge 16c).$$

$\Rightarrow$ $16c = \chi^2_{0.10}(24) = 33.20.$ $\Rightarrow$ $c = \mathbf{2.075}$.

The uniformly most powerful rejection region is "Reject H_0 if $\sum_{i=1}^{n} \sqrt{x_i} \ge 2.075$."

f) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Find the p-value for the test.

$$\sum_{i=1}^{n} \sqrt{x_i} = \sqrt{0.25} + \sqrt{0.36} + \sqrt{0.81} = 2.0.$$

$$\text{P-value} = \mathrm{P}\left(\sum_{i=1}^{n} \sqrt{\mathrm{X}_i} \ge 2.0 \,\middle|\, \beta = 8 \right) = \mathrm{P}\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{8} \right) \ge 2.0 \right)$$

$$= \mathrm{P}\left(\text{Poisson}\left(2.0 \cdot 8 \right) \le 11 \right) = \mathrm{P}\left(\text{Poisson}\left(16 \right) \le 11 \right) = \mathbf{0127}.$$

$$\text{P-value} = \mathrm{P}\left(\sum_{i=1}^{n} \sqrt{\mathrm{X}_i} \ge 2.0 \,\middle|\, \beta = 8 \right) = \mathrm{P}\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{8} \right) \ge 2.0 \right)$$

$$= \mathrm{P}\left(\chi^2(24) \ge 16 \cdot 2.0 \right) = \mathrm{P}\left(\chi^2(24) \ge 32 \right).$$

```
> 1-pchisq(32,24)
[1] 0.1269927
```

```
> 1-pgamma(2,12,8)
[1] 0.1269927
```

$$\text{P-value} = \mathrm{P}\left(\sum_{i=1}^{n} \sqrt{\mathrm{X}_i} \ge 2.0 \,\middle|\, \beta = 8 \right) = \mathrm{P}\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{8} \right) \ge 2.0 \right)$$

$$= \int_2^{\infty} \frac{8^{12}}{\Gamma(12)} x^{12-1} e^{-8x}\, dx = \int_2^{\infty} \frac{8^{12}}{11!} x^{11} e^{-8x}\, dx = \ldots$$

1. (continued)

Consider the rejection region Reject H_0 if $\sum_{i=1}^{3} \sqrt{x_i} \ge 2.5.$

g) Find the significance level α of this rejection region.

$$\alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}(\sum_{i=1}^{n} \sqrt{\mathrm{X}_i} \ge 2.5 \mid \beta = 8)$$

$$= \mathrm{P}(\mathrm{Gamma}(\alpha = 12,\ \theta = \frac{1}{8}) \ge 2.5) = \mathrm{P}(\mathrm{Poisson}(2.5 \cdot 8) \le 11)$$

$$= \mathrm{P}(\mathrm{Poisson}(20) \le 11).$$

```
> ppois(11,20)
[1] 0.02138682
```

$$\alpha = \mathrm{P}(\sum_{i=1}^{n} \sqrt{\mathrm{X}_i} \ge 2.5 \mid \beta = 8) = \mathrm{P}(\mathrm{Gamma}(\alpha = 12,\ \theta = \frac{1}{8}) \ge 2.5)$$

$$= \mathrm{P}(\chi^2(24) \ge 16 \cdot 2.5) = \mathrm{P}(\chi^2(24) \ge 40).$$

```
> 1-pchisq(40,24)
[1] 0.02138682
```

$$\alpha = \mathrm{P}(\sum_{i=1}^{n} \sqrt{\mathrm{X}_i} \ge 2.5 \mid \beta = 8) = \mathrm{P}(\mathrm{Gamma}(\alpha = 12,\ \theta = \frac{1}{8}) \ge 2.5)$$

$$= \int_{2.5}^{\infty} \frac{8^{12}}{\Gamma(12)} x^{12-1} e^{-8x}\, dx = \int_{2.5}^{\infty} \frac{8^{12}}{11!} x^{11} e^{-8x}\, dx = \ldots$$

```
> 1-pgamma(2.5,12,8)
[1] 0.02138682
```

h) Find the power of this rejection region if $\beta = 4$ and if $\beta = 6$.

$$\text{Power}(\beta = 4) = P\left(\sum_{i=1}^{n} \sqrt{X_i} \ge 2.5 \,\middle|\, \beta = 4\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{4}\right) \ge 2.5\right)$$

$$= P(\text{Poisson}(2.5 \cdot 4) \le 11) = P(\text{Poisson}(10) \le 11) = \mathbf{0.697}.$$

$$\text{Power}(\beta = 4) = P\left(\sum_{i=1}^{n} \sqrt{X_i} \ge 2.5 \,\middle|\, \beta = 4\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{4}\right) \ge 2.5\right)$$

$$= P(\chi^2(24) \ge 8 \cdot 2.5) = P(\chi^2(24) \ge 20).$$

```
> 1-pchisq(20,24)
[1] 0.6967761
```

```
> 1-pgamma(2.5,12,4)
[1] 0.6967761
```

$$\text{Power}(\beta = 6) = P\left(\sum_{i=1}^{n} \sqrt{X_i} \ge 2.5 \,\middle|\, \beta = 6\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{6}\right) \ge 2.5\right)$$

$$= P(\text{Poisson}(2.5 \cdot 6) \le 11) = P(\text{Poisson}(15) \le 11) = \mathbf{0.185}.$$

$$\text{Power}(\beta = 6) = P\left(\sum_{i=1}^{n} \sqrt{X_i} \ge 2.5 \,\middle|\, \beta = 6\right) = P\left(\text{Gamma}\left(\alpha = 12,\ \theta = \frac{1}{6}\right) \ge 2.5\right)$$

$$= P(\chi^2(24) \ge 12 \cdot 2.5) = P(\chi^2(24) \ge 30).$$

```
> 1-pchisq(30,24)
[1] 0.1847518
```

```
> 1-pgamma(2.5,12,6)
[1] 0.1847518
```