STAT 410 | **Practice Problems 8** | Fall 2020
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Practice Problems 8

1. Let X and Y be two independent Uniform$[0, 1]$ random variables. Find the probability distribution of $W = X + Y$. That is, find $f_{X+Y}(w)$.

2. Let X, Y, and Z be three independent Uniform$[0, 1]$ random variables. Find the probability distribution of $V = X + Y + Z$. That is, find $f_V(v) = f_{X+Y+Z}(v)$.

Hint: If $W = X + Y$, we know the p.d.f. of W, $f_W(w)$ (see Problem 1):

$$f_W(w) = w \qquad \text{if } 0 < w < 1,$$

$$f_W(w) = 2 - w \qquad \text{if } 1 < w < 2,$$

$$f_W(w) = 0 \qquad \text{otherwise.}$$

Now use convolution formula to find the p.d.f. of $V = W + Z$.

There will be 5 possible cases; two of them are "boring", two of them are "exciting", and one is "really exciting".

3. **2.1.7** (7th and 6th edition)

Let X and Y be two independent Uniform$[0, 1]$ random variables.

Find the c.d.f. and the p.d.f. of the product $Z = XY$.

4. Let X and Y be two independent Uniform$[0, 1]$ random variables.

Find the c.d.f. and the p.d.f. of $W = X - Y$.

"Hint": Consider two cases: $-1 < w < 0$ and $0 < w < 1$.

5. Let X and Y be two independent $\text{Uniform}[0, 1]$ random variables.

Find the c.d.f. and the p.d.f. of $V = X/Y$.

"Hint": Consider two cases: $0 < v < 1$ and $v > 1$.

6. Let X and Y be two independent $\text{Uniform}[0, 1]$ random variables. Let $V = \dfrac{X}{X+Y}$.

Find the p.d.f. of V, $f_V(v)$.

7. Let X be a $\text{Uniform}(0, 1)$ and Y be a $\text{Uniform}(0, 3)$ independent random variables. Let $W = X + Y$. Find and sketch the p.d.f. of W.

1. Let X and Y be two independent Uniform$[0, 1]$ random variables. Find the probability distribution of $W = X + Y$. That is, find $f_{X+Y}(w)$.

$$f_X(w) = f_Y(w) = \begin{cases} 1 & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(w-x) = \begin{cases} 1 & 0 \le w-x \le 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & w-1 \le x \le w \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $0 \le w \le 1$. Then $w - 1 \le 0$.

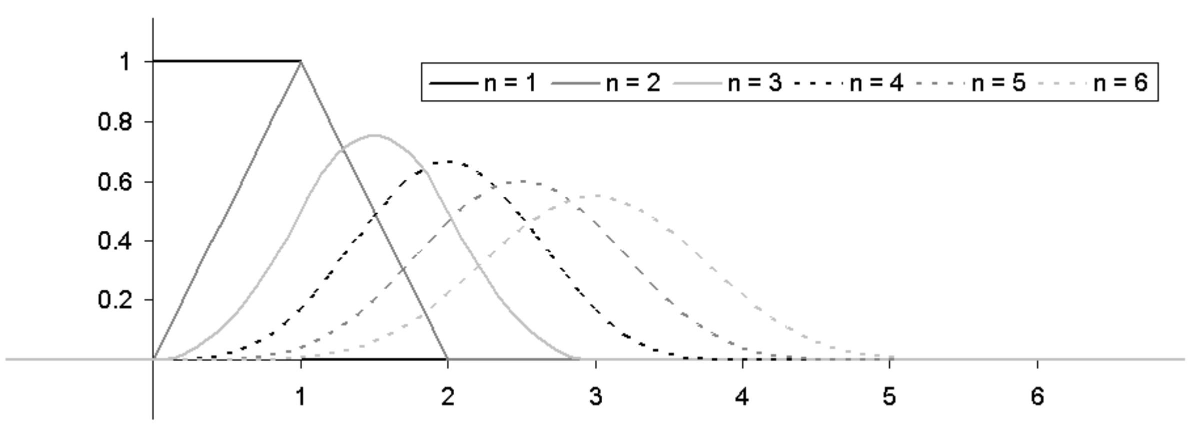
$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx = \int_0^w (1 \cdot 1)\,dx = w.$$

Case 2: $1 \le w \le 2$. Then $0 \le w - 1$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\,dx = \int_{w-1}^{1} (1 \cdot 1)\,dx = 2 - w.$$

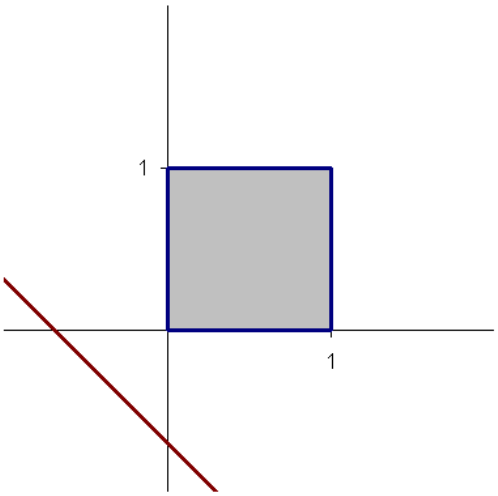
Case 3: $w < 0$ OR $w > 2$. $f_{X+Y}(w) = 0$.

p.d.f. of $X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are i.i.d. Uniform$[0, 1]$.



OR

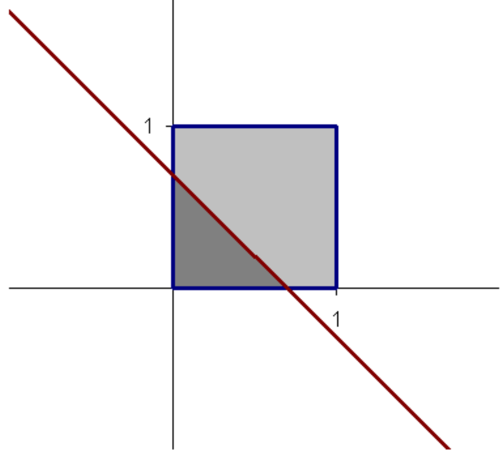
Case 1. $w < 0$.



$F_{X+Y}(w) = 0.$

$f_{X+Y}(w) = F'_{X+Y}(w) = 0.$

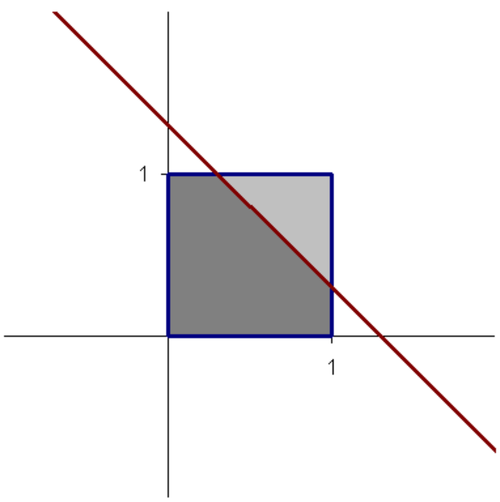
Case 2. $0 < w < 1$.



$F_{X+Y}(w) = \frac{1}{2}w^2.$

$f_{X+Y}(w) = F'_{X+Y}(w) = w.$

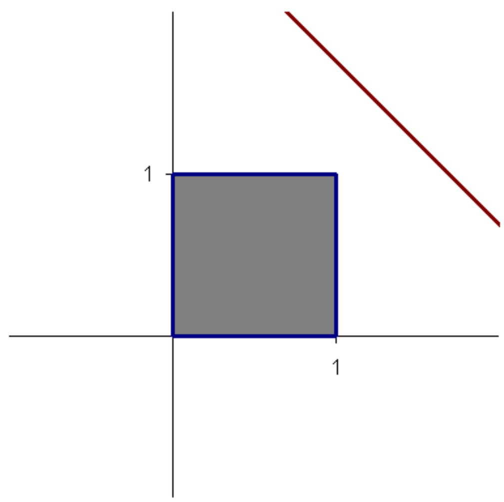
Case 3. $1 < w < 2$.



$F_{X+Y}(w) = 1 - \frac{1}{2}(2-w)^2.$

$f_{X+Y}(w) = F'_{X+Y}(w) = 2 - w.$

Case 4. $w > 2$.



$F_{X+Y}(w) = 1.$

$f_{X+Y}(w) = F'_{X+Y}(w) = 0.$

2. Let X, Y, and Z be three independent Uniform$[0, 1]$ random variables. Find the probability distribution of $V = X + Y + Z$. That is, find $f_V(v) = f_{X+Y+Z}(v)$.

Hint: If $W = X + Y$, we know the p.d.f. of W, $f_W(w)$ (see Problem 1):

$$f_W(w) = w \qquad \text{if } 0 < w < 1,$$

$$f_W(w) = 2 - w \qquad \text{if } 1 < w < 2,$$

$$f_W(w) = 0 \qquad \text{otherwise.}$$

Now use convolution formula to find the p.d.f. of $V = W + Z$.

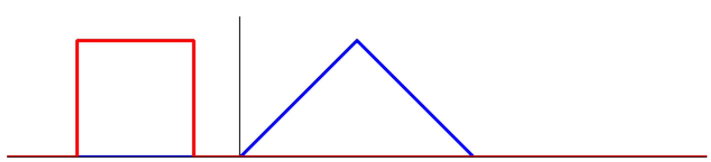
There will be 5 possible cases; two of them are "boring", two of them are "exciting", and one is "really exciting".

$$f_Z(z) = \begin{cases} 1 & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

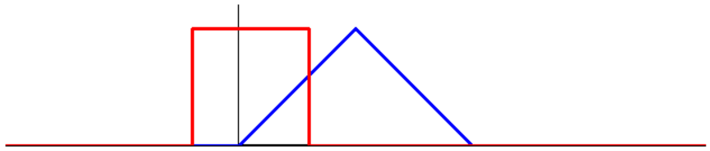
$$f_Z(v - w) = \begin{cases} 1 & 0 < v - w < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & v - 1 < w < v \\ 0 & \text{otherwise} \end{cases}$$

$$f_V(v) = f_{W+Z}(v) = \int_{-\infty}^{\infty} f_W(w) \cdot f_Z(v - w)\, dw \qquad \text{(convolution)}$$

Case 1: $v < 0$. $f_V(v) = 0$.

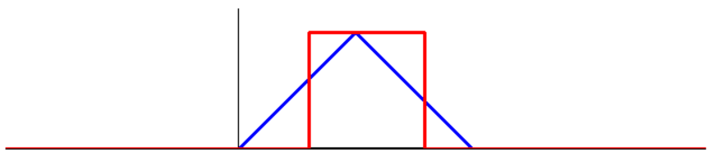


Case 2: $\quad 0 < v < 1. \quad$ Then $\quad v - 1 < 0.$



$$f_V(v) = \int_0^v (w \cdot 1)\,dw = \frac{v^2}{2}.$$

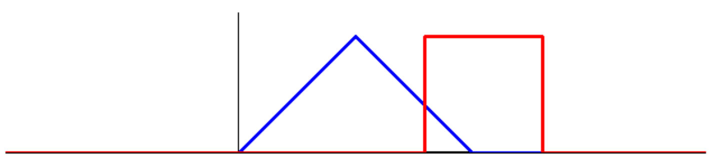
Case 3: $\quad 1 < v < 2. \quad$ Then $\quad 0 < v - 1 < 1.$



$$f_V(v) = \int_{v-1}^{1} (w \cdot 1)\,dw + \int_1^v ((2-w)\cdot 1)\,dw = -v^2 + 3v - \frac{3}{2}$$

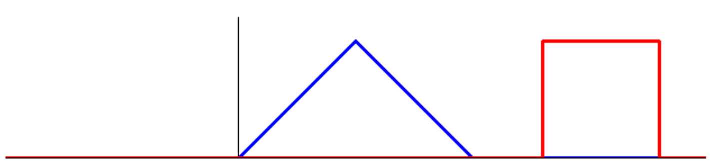
$$= (v-1)\cdot(2-v) + \frac{1}{2}.$$

Case 4: $\quad 2 < v < 3. \quad$ Then $\quad 1 < v - 1 < 2.$



$$f_V(v) = \int_{v-1}^{2} ((2-w)\cdot 1)\,dw = \frac{v^2}{2} - 3v + \frac{9}{2} = \frac{(3-v)^2}{2}.$$

Case 5: $\quad v > 3. \quad$ Then $\quad v - 1 > 2. \quad$ $f_V(v) = 0.$



3. **2.1.7** (7th and 6th edition)

Let X and Y be two independent Uniform$[0,1]$ random variables.

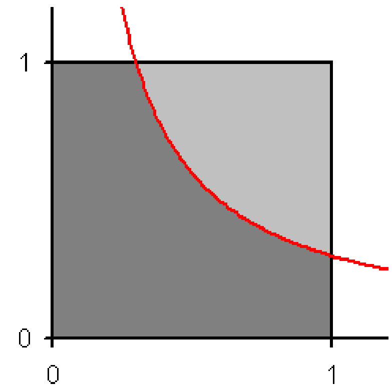
Find the c.d.f. and the p.d.f. of the product $Z = XY$.

$$F_Z(z) = P(Z \le z) = P(Y \le z/X)$$

$$= \int_0^z \left(\int_0^1 1\,dy \right) dx + \int_z^1 \left(\int_0^{z/x} 1\,dy \right) dx$$

$$= \int_0^z 1\,dx + \int_z^1 \frac{z}{x}\,dx$$

$$= z - z\ln z, \qquad 0 < z < 1.$$



OR

$$F_Z(z) = 1 - \int_z^1 \left(\int_{z/x}^1 1\,dy \right) dx = 1 - \int_z^1 \left(1 - \frac{z}{x} \right) dx = z - z\ln z, \qquad 0 < z < 1.$$

$$f_Z(z) = F_Z'(z) = -\ln z, \qquad 0 < z < 1.$$

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ z - z\ln z & 0 < z < 1 \\ 1 & z \ge 1 \end{cases} \qquad f_Z(z) = \begin{cases} -\ln z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

OR

Fact: Let X and Y be continuous random variables with joint p.d.f. $f(x, y)$.

Let $Z = XY$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \frac{1}{|x|}\, dx$$

Proof: Let $W = X$, $Z = XY$.

Then $X = W$, $Y = Z/W$.

$$J = \begin{vmatrix} 1 & 0 \\ -\frac{z}{w^2} & \frac{1}{w} \end{vmatrix} = \frac{1}{w}.$$

Then $f_{W,Z}(w, z) = f_{X,Y}\left(w, \frac{z}{w}\right) \frac{1}{|w|}$.

Therefore, $f_Z(z) = \int_{-\infty}^{\infty} f_{WZ}(w, z)\, dw = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{1}{|w|}\, dw$.

Let $W = X$, $Z = XY$. Then $X = W$, $Y = Z/W$.

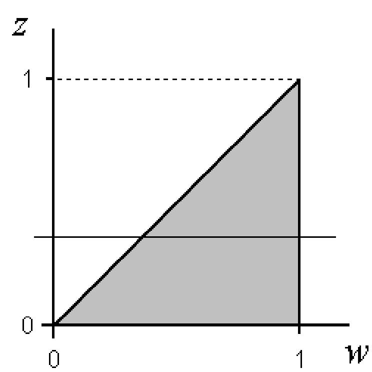
$0 < x < 1 \quad \Rightarrow \quad 0 < w < 1.$

$0 < y < 1 \quad \Rightarrow \quad 0 < z/w < 1 \quad \Rightarrow \quad 0 < z < w.$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{1}{|w|}\, dw$$

$$= \int_z^1 1 \cdot \frac{1}{w}\, dw = \left(\ln w\right)\Big|_z^1$$

$$= \ln 1 - \ln z = -\ln z, \qquad 0 < z < 1.$$

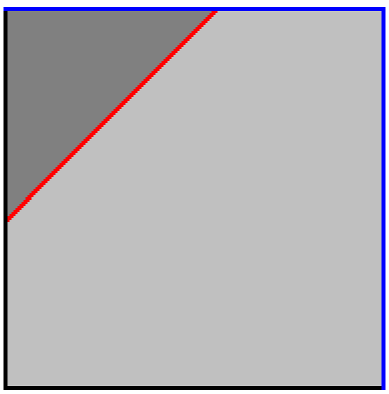


4. Let X and Y be two independent Uniform$[0, 1]$ random variables.
Find the c.d.f. and the p.d.f. of $W = X - Y$.

"Hint": Consider two cases: $-1 < w < 0$ and $0 < w < 1$.

$$F_W(w) = P(X - Y \le w) = P(Y \ge X - w) = \ldots$$

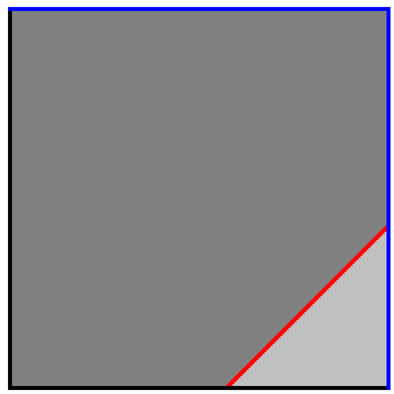
Case 1: $-1 < w < 0$.



$$\ldots = \frac{(1+w)^2}{2}, \quad -1 < w < 0.$$

$$f_W(w) = 1 + w, \quad -1 < w < 0.$$

Case 2: $0 < w < 1$.



$$\ldots = 1 - \frac{(1-w)^2}{2}, \quad 0 < w < 1.$$

$$f_W(w) = 1 - w, \quad 0 < w < 1.$$

$$f_W(w) = 0, \quad w < -1 \text{ or } w > 1.$$

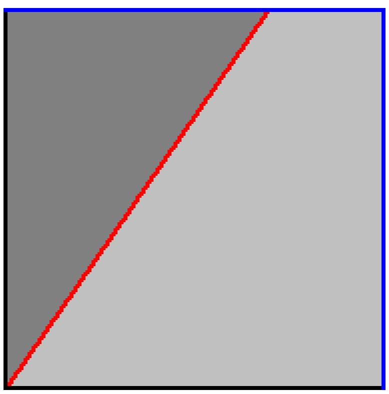
5. Let X and Y be two independent Uniform$[0, 1]$ random variables.

Find the c.d.f. and the p.d.f. of $V = {}^{X}/_{Y}$.

"Hint": Consider two cases: $0 < v < 1$ and $v > 1$.

$$F_V(v) = P\left({}^{X}/_{Y} \le v\right) = P\left(Y \ge {}^{X}/_{v}\right) = \dots$$

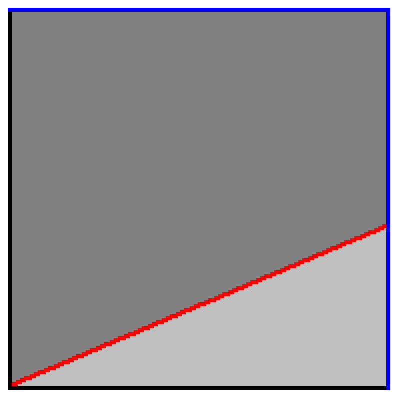
Case 1: $0 < v < 1$.



$$\dots = \frac{v}{2}, \quad 0 < v < 1.$$

$$f_V(v) = \frac{1}{2}, \quad 0 < v < 1.$$

Case 2: $v > 1$.



$$\dots = 1 - \frac{1}{2v}, \quad v > 1.$$

$$f_V(v) = \frac{1}{2v^2}, \quad v > 1.$$

$$f_V(v) = 0, \quad v < 0.$$

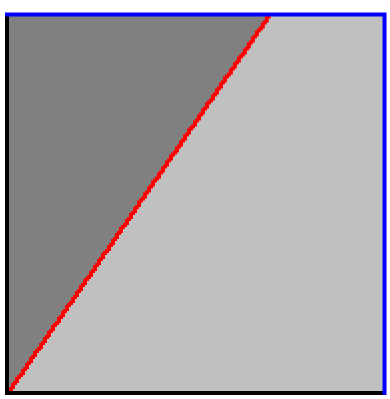
6. Let X and Y be two independent Uniform$[0,1]$ random variables. Let $V = \dfrac{X}{X+Y}$. Find the p.d.f. of V, $f_V(v)$.

$0 < x < 1, \quad 0 < y < 1 \qquad \Rightarrow \qquad 0 < v < 1.$

$$F_V(v) = P(V \le v) = P\left(\frac{X}{X+Y} \le v\right) = P(X \le v(X+Y)) = P\left(Y \ge \frac{1-v}{v}X\right) = \ldots$$

Case 1: $0 < v < \dfrac{1}{2}$.

$\Rightarrow \quad \dfrac{1-v}{v} > 1.$

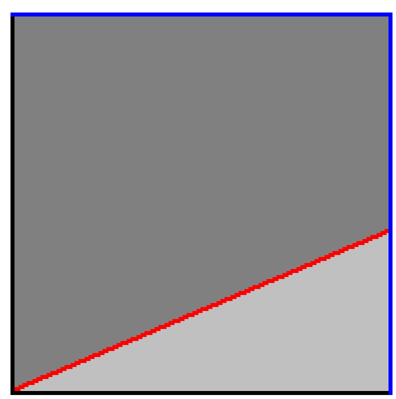


$$\ldots = \frac{v}{2(1-v)}, \qquad 0 < v < \frac{1}{2}.$$

$$f_V(v) = \frac{1}{2(1-v)^2}, \qquad 0 < v < \frac{1}{2}.$$

Case 2: $\dfrac{1}{2} < v < 1$.

$\Rightarrow \quad 0 < \dfrac{1-v}{v} < 1.$



$$\ldots = 1 - \frac{1-v}{2v}, \qquad \frac{1}{2} < v < 1.$$

$$f_V(v) = \frac{1}{2v^2}, \qquad \frac{1}{2} < v < 1.$$

OR

$$V = \frac{X}{X+Y} = \frac{\frac{X}{Y}}{\frac{X}{Y}+1} = \frac{V_5}{V_5+1},$$ where V_5 is V from problem 5.

$$V_5 = \frac{V}{1-V}. \qquad \frac{dv_5}{dv} = \frac{1}{(1-v)^2}$$

$$f_V(v) = f_{V_5}\left(\frac{v}{1-v}\right) \times \frac{1}{(1-v)^2}.$$

Case 1: $0 < v_5 < 1$.

$$\Rightarrow \quad 0 < v < \frac{1}{2}.$$

$$f_{V_5}(v_5) = \frac{1}{2}, \quad 0 < v_5 < 1.$$

$$f_V(v) = \frac{1}{2(1-v)^2}, \qquad 0 < v < \frac{1}{2}.$$

Case 2: $v_5 > 1$.

$$\Rightarrow \quad \frac{1}{2} < v < 1.$$

$$f_{V_5}(v_5) = \frac{1}{2v_5^2}, \quad v_5 > 1.$$

$$f_V(v) = \frac{1}{2\left(\frac{v}{1-v}\right)^2} \times \frac{1}{(1-v)^2}$$

$$= \frac{1}{2v^2}, \qquad \frac{1}{2} < v < 1.$$

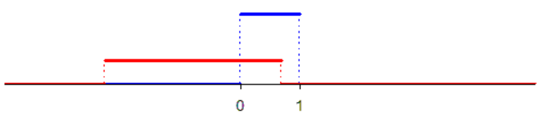
7. Let X be a Uniform$(0,1)$ and Y be a Uniform$(0,3)$ independent random variables. Let $W = X + Y$. Find and sketch the p.d.f. of W.

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad f_Y(y) = \begin{cases} 1/3 & 0 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx$$

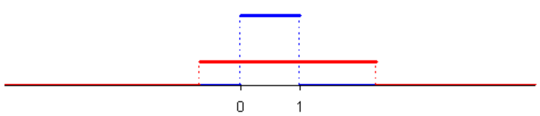
$$f_Y(y) = \begin{cases} 1/3 & 0 < y < 3 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(w-x) = \begin{cases} 1/3 & 0 < w-x < 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/3 & w-3 < x < w \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $0 < w < 1$.



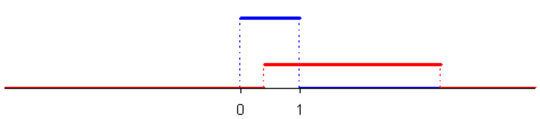
$$f_W(w) = \int_0^w 1 \cdot \frac{1}{3}\, dx = \frac{1}{3} w.$$

Case 2: $1 < w < 3$.

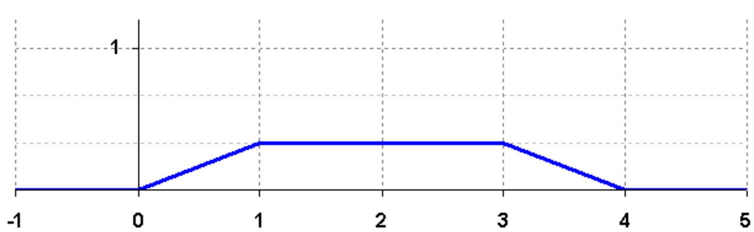


$$f_W(w) = \int_0^1 1 \cdot \frac{1}{3}\, dx = \frac{1}{3}.$$

Case 3: $3 < w < 4$.



$$f_W(w) = \int_{w-3}^{1} 1 \cdot \frac{1}{3}\, dx = \frac{1}{3}(4-w).$$

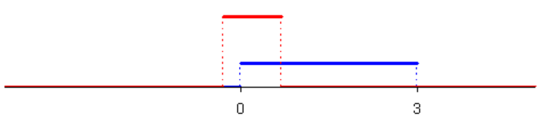


OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y) \cdot f_Y(y)\,dy$$

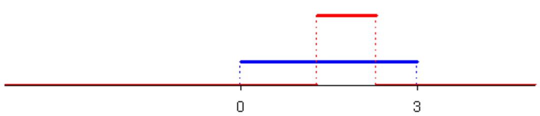
$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_X(w-y) = \begin{cases} 1 & 0 < w-y < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & w-1 < y < w \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $0 < w < 1$.



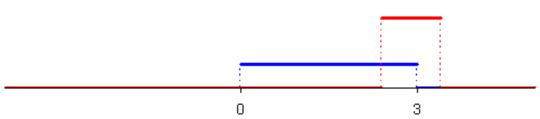
$$f_W(w) = \int_0^w 1 \cdot \frac{1}{3}\,dy = \frac{1}{3}w.$$

Case 2: $1 < w < 3$.

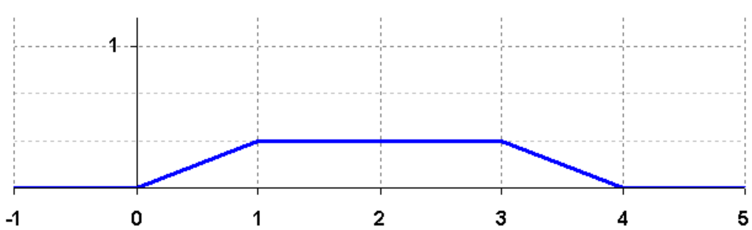


$$f_W(w) = \int_{w-1}^{w} 1 \cdot \frac{1}{3}\,dy = \frac{1}{3}.$$

Case 3: $3 < w < 4$.



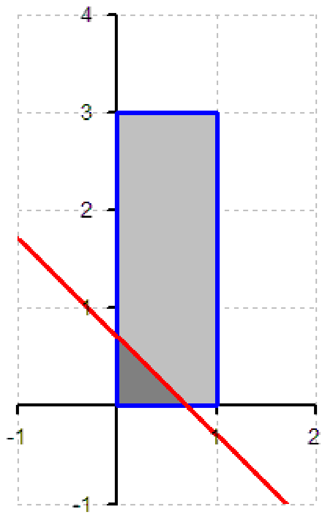
$$f_W(w) = \int_{w-1}^{3} 1 \cdot \frac{1}{3}\,dy = \frac{1}{3}(4-w).$$



OR

$$F_W(w) = P(W \le w) = P(X + Y \le w) = \ldots$$

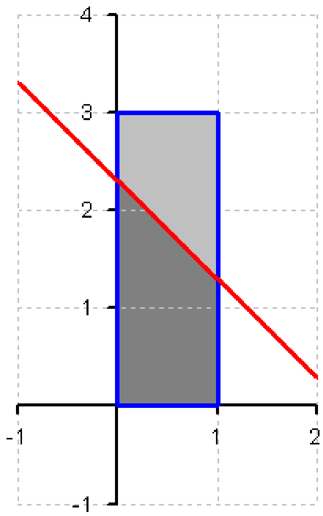
Case 1: $0 < w < 1$.



$$\ldots = \int_0^w \left(\int_0^{w-x} 1 \cdot \frac{1}{3}\, dy \right) dx$$

$$= \frac{1}{6} w^2.$$

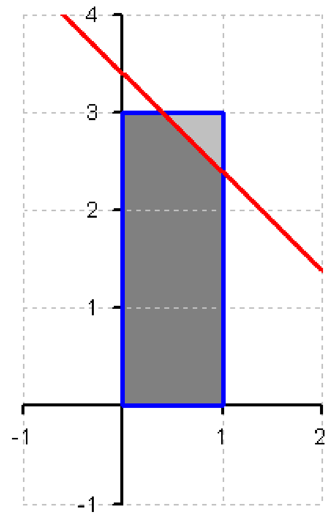
Case 2: $1 < w < 3$.



$$\ldots = \int_0^1 \left(\int_0^{w-x} 1 \cdot \frac{1}{3}\, dy \right) dx$$

$$= \frac{1}{6}(2w - 1).$$

Case 3: $3 < w < 4$.



$$\ldots = 1 - \int_{w-3}^1 \left(\int_{w-x}^3 1 \cdot \frac{1}{3}\, dy \right) dx$$

$$= 1 - \frac{1}{6}(4 - w)^2.$$

$$f_W(w) = F_W'(w) = \ldots$$

$$\ldots = \frac{1}{3} w, \quad 0 < w < 1.$$

$$\ldots = \frac{1}{3}, \quad 1 < w < 3.$$

$$\ldots = \frac{1}{3}(4 - w), \quad 3 < w < 4.$$