STAT 410 **Examples for 10/05/2020 (2)** Fall 2020
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1. Every month, the government of Neverland spends G million dollars purchasing guns, B million dollars purchasing butter, and P million dollars purchasing pants. Assume that $(\mathrm{G}, \mathrm{B}, \mathrm{P})$ jointly follow a $N_3(\mu, \Sigma)$, a 3- dimensional multivariate normal distribution with

$$\vec{\boldsymbol{\mu}} = \begin{pmatrix} 315 \\ 175 \\ 151 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1600 & -200 & -136 \\ -200 & 625 & -136 \\ -136 & -136 & 400 \end{pmatrix}.$$

a) Find the probability that the government of Neverland spends more than \$160 million on butter during a given month. That is, find $\mathrm{P}(\mathrm{B} > 160)$.

b) Suppose that the government of Neverland spends \$337 million on guns during a given month. Find the probability that the government of Neverland spends more than \$160 million on butter during this month. That is, find $\mathrm{P}(\mathrm{B} > 160 \mid \mathrm{G} = 337)$.

c) Find the probability that the government of Neverland spends more than \$160 million on pants during a given month. That is, find $\mathrm{P}(\mathrm{P} > 160)$.

d) Find the probability that the government of Neverland spends more on guns than twice the amount it spends on butter during a given month. That is, find $\mathrm{P}(\mathrm{G} > 2\,\mathrm{B})$.

e) Find the probability that the government of Neverland spends more on guns than it spends on butter and pants together during a given month. That is, find $\mathrm{P}(\mathrm{G} > \mathrm{B} + \mathrm{P})$.

f) Find the probability that the government of Neverland exceeds the \$600 million spending limit during a given month. That is, find $\mathrm{P}(\mathrm{G} + \mathrm{B} + \mathrm{P} > 600)$.

2. Ex-Coin and Why-Coin are two (much) lesser known cyber currencies; their prices \$X and \$Y (respectively) vary from day to day according to a bivariate normal distribution with parameters

$$\mu_X = 134, \qquad \sigma_X = 20, \qquad \mu_Y = 76, \qquad \sigma_Y = 8, \qquad \rho = 0.8.$$

a) What is the probability that on a given day the price of Ex-Coin is above \$150? That is, find $P(X > 150)$.

b) Suppose that on a given day the price of Why-Coin is \$78. What is the probability that the price of Ex-Coin is above \$150? That is, find $P(X > 150 \mid Y = 78)$.

c) What is the probability that on a given day the price of Why-Coin is below \$78? That is, find $P(Y < 78)$.

d) Suppose that on a given day the price of Ex-Coin is \$150. What is the probability that the price of Why-Coin is below \$78? That is, find $P(Y < 78 \mid X = 150)$.

e) What is the probability that 1 Ex-Coin is worth more than 2 Why-Coins? That is, find $P(X > 2Y)$.

f) Alex buys 5 Ex-Coins and 8 Why-Coins. What is the probability that the value of this portfolio exceeds \$1,200? That is, find $P(5X + 8Y > 1{,}200)$.

1. Every month, the government of Neverland spends G million dollars purchasing guns, B million dollars purchasing butter, and P million dollars purchasing pants. Assume that (G, B, P) jointly follow a $N_3(\mu, \Sigma)$, a 3- dimensional multivariate normal distribution with

$$\vec{\boldsymbol{\mu}} = \begin{pmatrix} 315 \\ 175 \\ 151 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1600 & -200 & -136 \\ -200 & 625 & -136 \\ -136 & -136 & 400 \end{pmatrix}.$$

a) Find the probability that the government of Neverland spends more than \$160 million on butter during a given month. That is, find $P(B > 160)$.

B has Normal distribution, $\mu_B = 175$, $\sigma_B = \sqrt{625} = 25$.

$$P(B > 160) = P(Z > \frac{160-175}{25}) = P(Z > -0.60) = \mathbf{0.7257}.$$

b) Suppose that the government of Neverland spends \$337 million on guns during a given month. Find the probability that the government of Neverland spends more than \$160 million on butter during this month. That is, find $P(B > 160 \mid G = 337)$.

(G, B) jointly follow a bivariate normal distribution.

$$\rho_{GB} = \frac{-200}{\sqrt{1600} \times \sqrt{625}} = \frac{-200}{40 \times 25} = -0.20.$$

Given $G = 337$, B has Normal distribution

with mean $175 - 0.20 \cdot \frac{25}{40} \cdot (337 - 315) = 172.25$

and variance $\left(1 - (-0.20)^2\right) \cdot 625 = 600.$

$$P(B > 160 \mid G = 337) = P(Z > \frac{160 - 172.25}{\sqrt{600}}) \approx P(Z > -0.50) = \mathbf{0.6915}.$$

c) Find the probability that the government of Neverland spends more than \$160 million on pants during a given month. That is, find $P(P > 160)$.

P has Normal distribution, $\mu_P = 151$, $\sigma_P = \sqrt{400} = 20$.

$$P(P > 160) = P(Z > \frac{160 - 151}{20}) = P(Z > 0.45) = \mathbf{0.3264}.$$

d) Find the probability that the government of Neverland spends more on guns than twice the amount it spends on butter during a given month. That is, find $P(G > 2B)$.

Want $P(G > 2B) = P(G - 2B > 0) = ?$

$G - 2B$ has Normal distribution,

$$E(G - 2B) = \mu_G - 2\mu_B = 315 - 2 \cdot 175 = -35,$$

$$\mathrm{Var}(G - 2B) = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1600 & -200 & -136 \\ -200 & 625 & -136 \\ -136 & -136 & 400 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2000 & -1450 & 136 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 4900.$$

$$P(G - 2B > 0) = P(Z > \frac{0 - (-35)}{\sqrt{4900}}) = P(Z > 0.50) = \mathbf{0.3085}.$$

e) Find the probability that the government of Neverland spends more on guns than it spends on butter and pants together during a given month. That is, find $P(G > B + P)$.

Want $P(G > B + P) = P(G - B - P > 0) = ?$

$G - B - P$ has Normal distribution,

$E(G - B - P) = \mu_G - \mu_B - \mu_P = 315 - 175 - 151 = -11,$

$$\mathrm{Var}(G - B - P) = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1600 & -200 & -136 \\ -200 & 625 & -136 \\ -136 & -136 & 400 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1936 & -689 & -400 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 3025.$$

$$P(G - B - P > 0) = P\left(Z > \frac{0 - (-11)}{\sqrt{3025}}\right) = P(Z > 0.20) = \mathbf{0.4207}.$$

f) Find the probability that the government of Neverland exceeds the \$600 million spending limit during a given month. That is, find $P(G + B + P > 600)$.

$G + B + P$ has Normal distribution,

$E(G + B + P) = \mu_G + \mu_B + \mu_P = 315 + 175 + 151 = 641,$

$$\mathrm{Var}(G + B + P) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1600 & -200 & -136 \\ -200 & 625 & -136 \\ -136 & -136 & 400 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1264 & 289 & 128 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1681.$$

$$P(G + B + P > 600) = P\left(Z > \frac{600 - 641}{\sqrt{1681}}\right) = P(Z > -1.00) = \mathbf{0.8413}.$$

2. Ex-Coin and Why-Coin are two (much) lesser known cyber currencies; their prices \$X and \$Y (respectively) vary from day to day according to a bivariate normal distribution with parameters

$$\mu_X = 134, \qquad \sigma_X = 20, \qquad \mu_Y = 76, \qquad \sigma_Y = 8, \qquad \rho = 0.8.$$

a) What is the probability that on a given day the price of Ex-Coin is above \$150? That is, find $P(X > 150)$.

$$P(X > 150) = P\left(Z > \frac{150-134}{20}\right) = P(Z > 0.80) = \mathbf{0.2119}.$$

b) Suppose that on a given day the price of Why-Coin is \$78. What is the probability that the price of Ex-Coin is above \$150? That is, find $P(X > 150 \mid Y = 78)$.

Given $Y = 78$, X has Normal distribution

with mean $\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y) = 134 + 0.8 \cdot \frac{20}{8} \cdot (78-76) = 138$

and variance $\left(1-\rho^2\right) \cdot \sigma_X^2 = \left(1-0.8^2\right) \cdot 20^2 = 144$

(standard deviation = 12).

$$P(X > 150 \mid Y = 78) = P\left(Z > \frac{150-138}{12}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$

c) What is the probability that on a given day the price of Why-Coin is below \$78? That is, find $P(Y < 78)$.

$$P(Y < 78) = P\left(Z < \frac{78-76}{8}\right) = P(Z < 0.25) = \mathbf{0.5987}.$$

d) Suppose that on a given day the price of Ex-Coin is \$150. What is the probability that the price of Why-Coin is below \$78? That is, find $P(Y < 78 \mid X = 150)$.

Given $X = 150$, Y has Normal distribution

with mean $\quad \mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X) = 76 + 0.8 \cdot \dfrac{8}{20} \cdot (150 - 134) = 81.12$

and variance $\quad \left(1-\rho^2\right) \cdot \sigma_Y^2 = \left(1 - 0.8^2\right) \cdot 8^2 = 23.04$

(standard deviation = 4.8).

$$P(Y < 78 \mid X = 150) = P\left(Z < \frac{78-81.12}{4.8}\right) = P(Z < -0.65) = \mathbf{0.2578}.$$

e) What is the probability that 1 Ex-Coin is worth more than 2 Why-Coins? That is, find $P(X > 2Y)$.

$P(X > 2Y) = P(X - 2Y > 0)$.

$X - 2Y$ has Normal distribution,

$$E(X - 2Y) = \mu_X - 2\mu_Y = 134 - 2 \cdot 76 = -18,$$

$$\begin{aligned} Var(X - 2Y) &= \sigma_X^2 - 4\sigma_{XY} + 4\sigma_Y^2 \\ &= \sigma_X^2 - 4\rho\sigma_X\sigma_Y + 4\sigma_Y^2 \\ &= 20^2 - 4 \cdot 0.8 \cdot 20 \cdot 8 + 4 \cdot 8^2 = 144 \end{aligned}$$

(standard deviation = 12).

$$P(X - 2Y > 0) = P\left(Z > \frac{0 - (-18)}{12}\right) = P(Z > 1.50) = \mathbf{0.0668}.$$

f) Alex buys 5 Ex-Coins and 8 Why-Coins. What is the probability that the value of this portfolio exceeds \$1,200? That is, find $P(5X + 8Y > 1{,}200)$.

$5X + 8Y$ has Normal distribution,

$$E(5X + 8Y) = 5\mu_X + 8\mu_Y = 5 \cdot 134 + 8 \cdot 76 = 1278,$$

$$\begin{aligned} Var(5X + 8Y) &= 25\sigma_X^2 + 80\sigma_{XY} + 64\sigma_Y^2 \\ &= 25\sigma_X^2 + 80\rho\sigma_X\sigma_Y + 64\sigma_Y^2 \\ &= 25 \cdot 20^2 + 80 \cdot 0.8 \cdot 20 \cdot 8 + 64 \cdot 8^2 = 24336 \end{aligned}$$

(standard deviation = 156).

$$P(5X + 8Y > 1200) = P\left(Z > \frac{1200 - 1278}{156}\right) = P(Z > -0.50) = \mathbf{0.6915}.$$