STAT 410

Homework #5

(due Friday, October 2, by 5:00 p.m. CDT)

Fall 2020
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Please include your name (with your last name underlined), your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

1. Grades on Fall 2020 STAT 410 Exam 1 were not very good*. Graphed, their distribution had a shape similar to the probability density function.

$$f_X(x) = \frac{\sqrt{x+6}}{468}, \qquad 3 \le x \le 75, \qquad \text{zero elsewhere.}$$

(Treat exam scores as a continuous (real numbers) random variable instead of discrete (integers).)

Recall (Homework #1): $\quad F_X(x) = \dfrac{(x+6)^{1.5} - 27}{702}, \qquad 3 \le x \le 75.$

Five exam papers were selected at random. That is, let X_1, X_2, X_3, X_4, X_5 be a be a random sample (i.i.d.) from the above probability distribution.

Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the corresponding order statistics.

h) Find the probability that the largest score of these 5 papers is above 63. That is, find $P(Y_5 > 63) = P(\max X_i > 63)$.

$$P(X \le 63) = F_X(63) = \frac{69^{1.5} - 27}{702} \approx 0.778.$$

$$P(\max X_i > 63) = 1 - P(\max X_i \le 63) = 1 - [P(X \le 63)]^5 \approx \mathbf{0.715}.$$

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* The probability distribution is fictional, the exam has not happened yet. Hopefully, the actual grades will be slightly better than these.

i) Find the probability that the lowest score of these 5 papers is below 23. That is, find $P(Y_1 < 23) = P(\min X_i < 23)$.

$$P(X \ge 23) = 1 - F_X(23) = 1 - \frac{29^{1.5} - 27}{702} \approx 0.816.$$

$$P(\min X_i < 23) = 1 - P(\min X_i \ge 23) = 1 - [P(X \ge 23)]^5 \approx \mathbf{0.6382}.$$

j) Find the probability that the second lowest score of these 5 papers is below 29. That is, $P(Y_2 < 29)$.

"Hint": At least two out of 5 scores are below 29.

$$P(X < 29) = F_X(29) = \frac{35^{1.5} - 27}{702} \approx 0.2565.$$

$$P(Y_2 < 29) = P(\text{at least two out of 5 scores are below 29})$$

$$= \binom{5}{2} 0.2565^2\ 0.7435^3 + \binom{5}{3} 0.2565^3\ 0.7435^2 + \binom{5}{4} 0.2565^4\ 0.7435^1 + \binom{5}{5} 0.2565^5\ 0.7435^0$$

$$\approx 0.2704 + 0.0933 + 0.0161 + 0.0011 = \mathbf{0.3809}.$$

OR

$$= 1 - \binom{5}{0} 0.2565^0\ 0.7435^5 - \binom{5}{1} 0.2565^1\ 0.7435^4$$

$$\approx 1 - 0.2272 - 0.3919 = \mathbf{0.3809}.$$

3. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Since the people become disobedient if they do not have food, the amount spent monthly on butter is always at least 2 million dollars. However, there is a law that prevents the government of Neverland spending more than 5 million dollars on guns in one month. In addition, the government of Neverland cannot afford to spend more than 8 million dollars per month on guns and butter. Suppose that the joint density function for (X, Y) is

$$f(x,y) = \frac{7x + 2y}{375}, \qquad x \ge 0, \quad y \ge 2, \quad x \le 5, \quad x + y \le 8, \qquad \text{zero otherwise.}$$

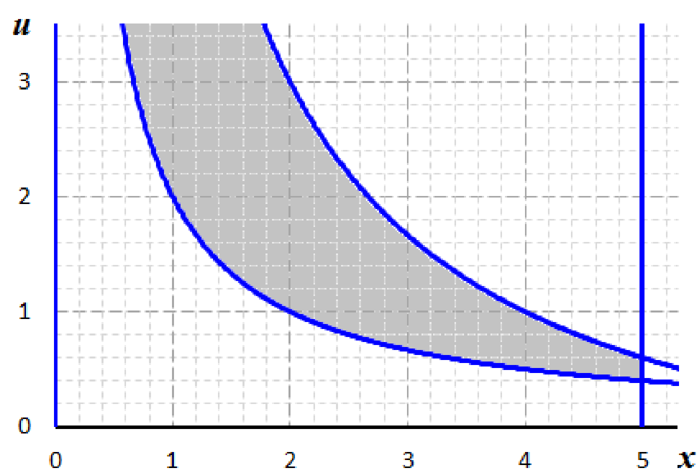
X – guns, Y – butter.

rx) Let $U = \dfrac{Y}{X} = \dfrac{\text{butter}}{\text{guns}}$.

Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.

Sketch the support of (X, U).

$X = X, \quad Y = U X.$



$x \ge 0$

$y \ge 2 \quad \Rightarrow \quad u x \ge 2$

$\Rightarrow \quad u \ge \dfrac{2}{x}, \quad x \ge \dfrac{2}{u}$

$x \le 5$

$x + y \le 8 \quad \Rightarrow \quad x + u x \le 8$

$\Rightarrow \quad u \le \dfrac{8}{x} - 1, \quad x \le \dfrac{8}{1+u}$

$$J = \begin{vmatrix} 1 & 0 \\ u & x \end{vmatrix} = x.$$

$$f_{X,U}(x, u) = f_{X,Y}(x, u x) \times |J| = \frac{7x + 2ux}{375} \times x = \frac{7 + 2u}{375} x^2.$$

ry) Let $U = \frac{Y}{X} = \frac{\text{butter}}{\text{guns}}$.

Find the joint probability density function of (Y, U), $f_{Y,U}(y, u)$.

Sketch the support of (Y, U).

$X = \frac{Y}{U}$, $Y = Y$.

$x \ge 0 \quad \Rightarrow \quad \frac{y}{u} \ge 0$

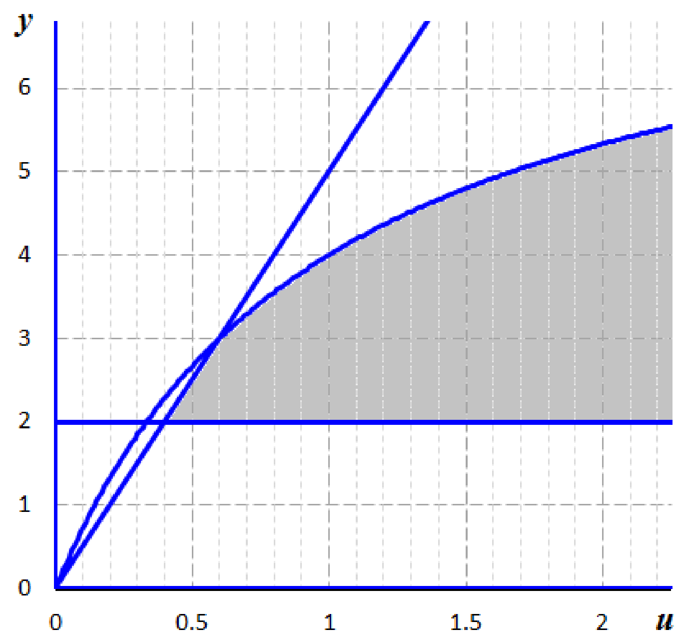
$y \ge 2$

$x \le 5 \quad \Rightarrow \quad \frac{y}{u} \le 5$

$\Rightarrow \quad y \le 5u$

$x + y \le 8 \quad \Rightarrow \quad \frac{y}{u} + y \le 8$

$\Rightarrow \quad y \le \frac{8u}{1+u}$



$$J = \begin{vmatrix} -\frac{y}{u^2} & \frac{1}{u} \\ 0 & 1 \end{vmatrix} = -\frac{y}{u^2}.$$

$$f_{Y,U}(y,u) = f_{X,Y}\left(\frac{y}{u}, y\right) \times |J| = \frac{7\frac{y}{u} + 2y}{375} \times \frac{y}{u^2} = \frac{7+2u}{375u^3}\, y^2.$$

sx) **Use part (r)** to find the p.d.f. of $U = \frac{Y}{X}$, $f_U(u)$.

$0.40 < u < 0.60$, $$f_U(u) = \int_{\frac{2}{u}}^{5} \frac{7+2u}{375} x^2 \, dx = \frac{7+2u}{1{,}125} \left(125 - \frac{8}{u^3} \right).$$

$u > 0.60$, $$f_U(u) = \int_{\frac{2}{u}}^{\frac{8}{1+u}} \frac{7+2u}{375} x^2 \, dx = \frac{7+2u}{1{,}125} \left(\frac{512}{(1+u)^3} - \frac{8}{u^3} \right).$$

sy) **Use part (r)** to find the p.d.f. of $U = \frac{Y}{X}$, $f_U(u)$.

$0.40 < u < 0.60$, $$f_U(u) = \int_{2}^{5u} \frac{7+2u}{375u^3} y^2 \, dy = \frac{7+2u}{1{,}125u^3} \left(125u^3 - 8 \right).$$

$u > 0.60$, $$f_U(u) = \int_{2}^{\frac{8u}{1+u}} \frac{7+2u}{375u^3} y^2 \, dy = \frac{7+2u}{1{,}125u^3} \left(\frac{512u^3}{(1+u)^3} - 8 \right).$$

tx) Let $\mathrm{V} = \mathrm{X} \cdot \mathrm{Y}$.

Find the joint probability density function of (X, V), $f_{\mathrm{X,V}}(x, v)$.

Sketch the support of (X, V).

$\mathrm{X} = \mathrm{X}, \quad \mathrm{Y} = \dfrac{\mathrm{V}}{\mathrm{X}}$.

$x \geq 0$

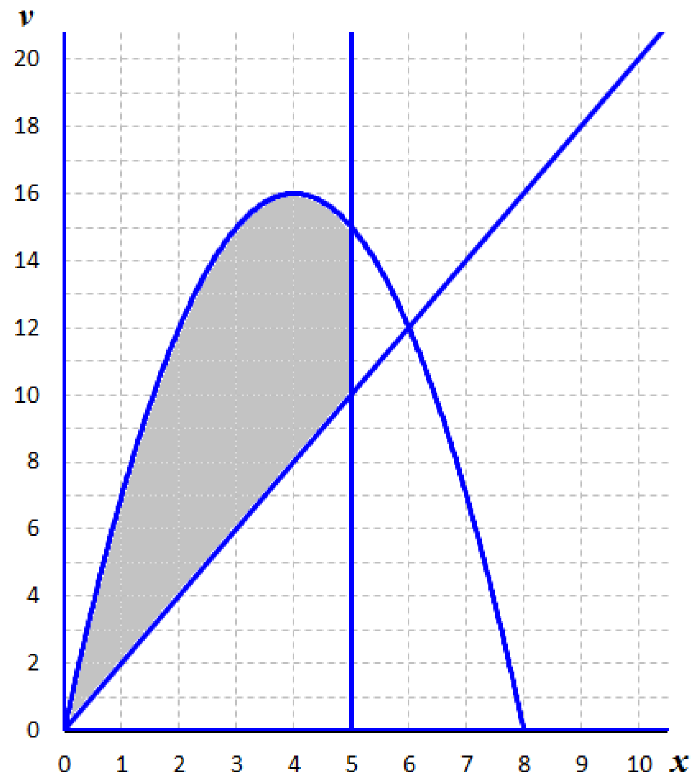
$y \geq 2 \quad \Rightarrow \quad \dfrac{v}{x} \geq 2$

$\Rightarrow \quad v \geq 2x, \quad x \leq \dfrac{v}{2}$

$x \leq 5$

$x + y \leq 8 \quad \Rightarrow \quad x + \dfrac{v}{x} \leq 8$

$\Rightarrow \quad v \leq 8x - x^2 = (8 - x)x$



$$\mathrm{J} = \begin{vmatrix} 1 & 0 \\ -\dfrac{v}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{1}{x}.$$

$$f_{\mathrm{X,V}}(x, v) = f_{\mathrm{X,Y}}\left(x, \frac{v}{x}\right) \times |\mathrm{J}| = \frac{7x + 2\dfrac{v}{x}}{375} \times \frac{1}{x} = \frac{7x^2 + 2v}{375x^2} = \frac{7}{375} + \frac{2v}{375x^2}.$$

$$x^2 - 8x + v = 0 \qquad \Rightarrow \qquad x = 4 \pm \sqrt{16 - v}.$$

ty) Let $V = X \cdot Y$.

Find the joint probability density function of (Y, V), $f_{Y,V}(y, v)$.

Sketch the support of (Y, V).

$$X = \frac{V}{Y}, \quad Y = Y.$$

$$x \ge 0 \quad \Rightarrow \quad \frac{v}{y} \ge 0$$

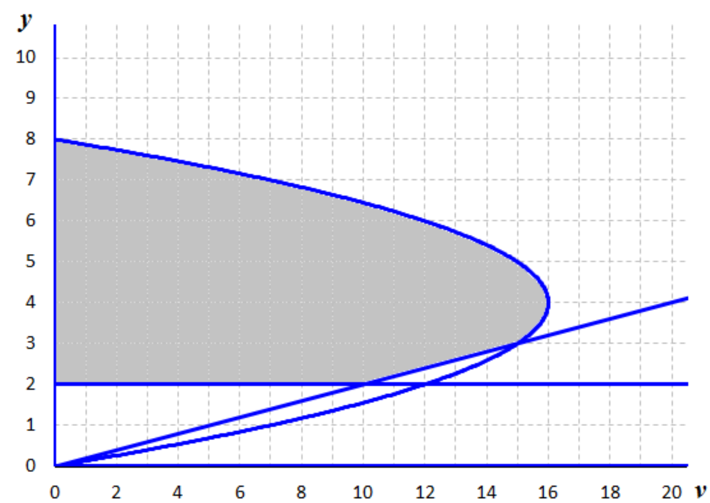
$$y \ge 2$$

$$x \le 5 \quad \Rightarrow \quad \frac{v}{y} \le 5$$

$$\Rightarrow \quad y \ge \frac{v}{5}$$

$$x + y \le 8 \quad \Rightarrow \quad \frac{v}{y} + y \le 8$$

$$\Rightarrow \quad v \le 8y - y^2 = (8 - y)y$$



$$J = \begin{vmatrix} \frac{1}{y} & -\frac{v}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}.$$

$$f_{Y,V}(y, v) = f_{X,Y}\left(\frac{v}{y}, y\right) \times |J| = \frac{7\frac{v}{y} + 2y}{375} \times \frac{1}{y} = \frac{7v + 2y^2}{375y^2} = \frac{7v}{375y^2} + \frac{2}{375}.$$

$$y^2 - 8y + v = 0 \quad \Rightarrow \quad y = 4 \pm \sqrt{16 - v}.$$

ux) **Use part (t)** to set up the integral(s) for the p.d.f. of of V, $f_V(v)$.

*You do NOT have to evaluate the integral(s). **

$0 < v < 10,$ $$f_V(v) = \int_{4-\sqrt{16-v}}^{v/2} \frac{7x^2 + 2v}{375x^2}\, dx.$$

$10 < v < 15,$ $$f_V(v) = \int_{4-\sqrt{16-v}}^{5} \frac{7x^2 + 2v}{375x^2}\, dx.$$

$15 < v < 16,$ $$f_V(v) = \int_{4-\sqrt{16-v}}^{4+\sqrt{16-v}} \frac{7x^2 + 2v}{375x^2}\, dx.$$

ux) **Use part (t)** to set up the integral(s) for the p.d.f. of of V, $f_V(v)$.

*You do NOT have to evaluate the integral(s). **

$0 < v < 10,$ $$f_V(v) = \int_{2}^{4+\sqrt{16-v}} \frac{7v + 2y^2}{375y^2}\, dy.$$

$10 < v < 15,$ $$f_V(v) = \int_{v/5}^{4+\sqrt{16-v}} \frac{7v + 2y^2}{375y^2}\, dy.$$

$15 < v < 16,$ $$f_V(v) = \int_{4-\sqrt{16-v}}^{4+\sqrt{16-v}} \frac{7x^2 + 2v}{375x^2}\, dx.$$

* Obviously, you would have to evaluate all integrals on an exam.