STAT 410

# Examples for 09/25/2020 (2)

Fall 2020
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**1.** Let $X_1, X_2, X_3, X_4, X_5$ be a random sample (i.i.d.) of size $n = 5$ from a continuous random variable X with the probability density function

$$f_X(x) = \frac{x^3}{60}, \qquad 2 \le x \le 4, \qquad \text{zero elsewhere.}$$

Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the corresponding order statistics.

Recall (Examples for 08/26/2020 (2)): $\quad F_X(x) = P(X \le x) = \dfrac{x^4 - 16}{240}, \quad 2 \le x \le 4.$

"Hint": $\{Y_k \le x\} = \{$ at least $k$ of $X_1, X_2, \ldots, X_n$ are less than or equal to $x\}$
$= \{$ at most $n-k$ of $X_1, X_2, \ldots, X_n$ are greater than $x\}$.

$\{Y_k > x\} = \{$ at most $k-1$ of $X_1, X_2, \ldots, X_n$ are less than or equal to $x\}$.
$= \{$ at least $n-k+1$ of $X_1, X_2, \ldots, X_n$ are greater than $x\}$

a) Find $P(Y_5 > 3.8) = P(\max X_i > 3.8)$.

$$P(\max X_i > 3.8) = 1 - P(\max X_i \le 3.8) = 1 - \big(P(X \le 3.8)\big)^5 = 1 - \left(\frac{3.8^4 - 16}{240}\right)^5$$

$$= 1 - 0.80214^5 \approx \mathbf{0.667914}.$$

OR

$$P(\max X_i > 3.8) = \int_{3.8}^{4} 5\left(\frac{x^4 - 16}{240}\right)^{5-1} \frac{x^3}{60}\,dx = \ldots$$

Why would anyone do this?

b) Find $P(Y_1 \le 2.6) = P(\min X_i \le 2.6)$.

$$P(\min X_i \le 2.6) = 1 - P(\min X_i > 2.6) = 1 - \left(P(X > 2.6)\right)^5$$

$$= 1 - \left(1 - \frac{2.6^4 - 16}{240}\right)^5 = 1 - (1 - 0.12374)^5 = 1 - 0.87626^5 \approx \mathbf{0.483387}.$$

OR

$$P(\min X_i \le 2.6) = \int_2^{2.6} 5\left(1 - \frac{x^4 - 16}{240}\right)^{5-1} \frac{x^3}{60}\, dx = \ldots$$

Why would anyone do this?

c) Find $P(Y_2 \le 2.6)$.

| ≤ 2.6 | > 2.6 | |
|---:|:---|---|
| X **X** X X X | | ☺ |
| X **X** X X | X | ☺ |
| X **X** X | X X | ☺ |
| X **X** | X X X | ☺ |
| X | **X** X X X | ☹ |
| | X **X** X X X | ☹ |
| 2.6 | | |

$$P(X \le 2.6) = \frac{2.6^4 - 16}{240} = 0.12374.$$

$P(Y_2 \le 2.6) = P(\text{at least two of } X_1, X_2, \ldots, X_5 \text{ are less than or equal to } 2.6)$

$= 1 - P(\text{at most one of } X_1, X_2, \ldots, X_5 \text{ are less than or equal to } 2.6)$

$$= 1 - \binom{5}{0} 0.12374^0\, 0.87626^5 - \binom{5}{1} 0.12374^1\, 0.87626^4$$

$$\approx 1 - 0.516613 - 0.364764 = \mathbf{0.118623}.$$

```
> p=(2.6^4-16)/240
> 1-pbinom(1,5,p)
[1] 0.1186234
```

OR

$$P(Y_2 \le 2.6) = \int_2^{2.6} \frac{5!}{(2-1)!\,(5-2)!} \left(\frac{x^4-16}{240}\right)^{2-1} \left(1-\frac{x^4-16}{240}\right)^{5-2} \frac{x^3}{60}\,dx = \ldots$$

d) Find $P(Y_3 \le 3.2)$.

| ≤ 3.2 | > 3.2 | |
|---:|:---|:---:|
| X X **X** X X | | ☺ |
| X X **X** X | X | ☺ |
| X X **X** | X X | ☺ |
| X X | **X** X X | ☹ |
| X | X **X** X X | ☹ |
| | X X **X** X X | ☹ |

3.2

$$P(X \le 3.2) = \frac{3.2^4-16}{240} = 0.37024.$$

$P(Y_3 \le 3.2) = P(\text{at least three of } X_1, X_2, \ldots, X_5 \text{ are less than or equal to } 3.2)$

$= 1 - P(\text{at most two of } X_1, X_2, \ldots, X_5 \text{ are less than or equal to } 3.2)$

$$= \binom{5}{3} 0.37024^3\, 0.62976^2 + \binom{5}{4} 0.37024^4\, 0.62976^1 + \binom{5}{5} 0.37024^5\, 0.62976^0$$

$$= 1 - \binom{5}{0} 0.37024^0\, 0.62976^5 - \binom{5}{1} 0.37024^1\, 0.62976^4 - \binom{5}{2} 0.37024^2\, 0.62976^3$$

$\approx 0.201280 + 0.059167 + 0.006957 = \mathbf{0.267404}$.

$\approx 1 - 0.099055 - 0.291175 - 0.342367 = \mathbf{0.267403}$.

```
> p=(3.2^4-16)/240
> 1-pbinom(2,5,p)
[1] 0.2674035
```

OR

$$P(Y_3 \le 3.2) = \int_2^{3.2} \frac{5!}{(3-1)!\,(5-3)!} \left(\frac{x^4-16}{240}\right)^{3-1} \left(1-\frac{x^4-16}{240}\right)^{5-3} \frac{x^3}{60}\,dx = \dots$$

e) Find $P(Y_4 > 3.4)$.

| $\le 3.4$ | $> 3.4$ | |
|---|---|---|
| X X X **X** X | | ☹ |
| X X X **X** | X | ☹ |
| X X X | **X** X | ☺ |
| X X | X **X** X | ☺ |
| X | X X **X** X | ☺ |
| | X X X **X** X | ☺ |

3.4

$$P(X \le 3.4) = \frac{3.4^4 - 16}{240} = 0.49014.$$

$P(Y_4 > 3.4) = P(\text{at least two of } X_1, X_2, \dots, X_5 \text{ are greater than } 3.4)$

$= 1 - P(\text{at most one of } X_1, X_2, \dots, X_5 \text{ are greater than } 3.4)$

$= P(\text{at most three of } X_1, X_2, \dots, X_5 \text{ are less than or equal to } 3.4)$

$= 1 - P(\text{at least four of } X_1, X_2, \dots, X_5 \text{ are less than or equal to } 3.4)$

$$= 1 - \binom{5}{4} 0.49014^4\, 0.50986^1 - \binom{5}{5} 0.49014^5\, 0.50986^0$$

$\approx 1 - 0.147130 - 0.028288 = \mathbf{0.824582}.$

```
> p=(3.4^4-16)/240
> pbinom(3,5,p)
[1] 0.824582
```

OR

$$P(Y_4 > 3.4) = \int_{3.4}^{4} \frac{5!}{(4-1)!\,(5-4)!} \left(\frac{x^4-16}{240}\right)^{4-1} \left(1-\frac{x^4-16}{240}\right)^{5-4} \frac{x^3}{60}\,dx = \dots$$

IF Alex is a terrible person ( he IS ), …

f) Find $E(Y_5) = E(\max X_i)$.

$$E(\max X_i) = \int_2^4 x \cdot 5 \left( \frac{x^4 - 16}{240} \right)^{5-1} \frac{x^3}{60}\, dx = \ldots$$ ☹

g) Find $E(Y_1) = E(\min X_i)$.

$$E(\min X_i) = \int_2^4 x \cdot 5 \left( 1 - \frac{x^4 - 16}{240} \right)^{5-1} \frac{x^3}{60}\, dx = \ldots$$ ☹

☹ Obviously, you would have to evaluate all integrals on an exam.