Examples for 08/26/2020 (2)

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x^3}{C}, \qquad 2 \le x \le 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(2) = 0$, $F_X(4) = 1$.

1. (continued)

Consider $Y = g(X) = X^2$.

c) Find the support (the range of possible values) of the probability distribution of Y.

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \ldots$.

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

f) Does μ_Y equal to $g(\mu_X)$? $\mu_X = E(X)$, $\mu_Y = E(Y)$.

1. (continued)

Consider $W = \dfrac{1}{X+6}$.

g) Find the support (the range of possible values) of the probability distribution of W.

h) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

i) Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

j) Find the moment-generating function of X, $M_X(t)$.

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3}{C}, \qquad x = 1, 2, 3, 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

2. (continued)

Consider $Y = g(X) = X^2$.

b) Find the probability distribution of Y.

c) Does μ_Y equal to $g(\mu_X)$? $\mu_X = E(X)$, $\mu_Y = E(Y)$.

d) Find the moment-generating function of X, $M_X(t)$.

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x^3}{C}, \qquad 2 \le x \le 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_2^4 \frac{x^3}{C}\,dx = \frac{x^4}{4C}\Bigg|_2^4 = \frac{256-16}{4C} = \frac{60}{C}.$$

$\Rightarrow \qquad C = \mathbf{60}.$

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(2) = 0$, $F_X(4) = 1$.

$F_X(x) = 0, \qquad x < 2,$

$$F_X(x) = P(X \le x) = \int_2^x \frac{u^3}{60}\,du = \frac{u^4}{240}\Bigg|_2^x = \frac{x^4 - 16}{240}, \qquad 2 \le x < 4,$$

$F_X(x) = 1, \qquad x \ge 4.$

1. (continued)

Consider $Y = g(X) = X^2$.

c) Find the support (the range of possible values) of the probability distribution of Y.

$2 \le x \le 4. \qquad 4 \le x^2 \le 16. \qquad \mathbf{4 \le y \le 16}.$

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}\,) = F_X(\sqrt{y}\,)$$

$$= \frac{\left(\sqrt{y}\right)^4 - 16}{240} = \frac{y^2 - 16}{240}, \qquad 4 \le y < 16.$$

$F_Y(y) = 0, \quad y < 4, \qquad\qquad F_Y(y) = 1, \quad y \ge 16.$

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F'_Y(y)$.

$y = x^2 \qquad x = \sqrt{y} = g^{-1}(y) \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{\left(\sqrt{y}\right)^3}{60} \cdot \left| \frac{1}{2\sqrt{y}} \right| = \frac{y}{120}, \qquad 4 \le y \le 16.$$

Indeed, $\frac{d}{dy}\left(\frac{y^2 - 16}{240}\right) = \frac{y}{120}$. ☺

f) Does μ_Y equal to $g(\mu_X)$? $\mu_X = E(X)$, $\mu_Y = E(Y)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{2}^{4} \frac{x^4}{60}\, dx = \left.\frac{x^5}{300}\right|_{2}^{4} = \frac{1{,}024 - 32}{300}$$

$$= \frac{992}{300} = \mathbf{\frac{248}{75}} \approx 3.306667.$$

$$E(Y) = \int_{4}^{16} \frac{y^2}{120}\, dy = \left.\frac{y^3}{360}\right|_{4}^{16} = \frac{4{,}096 - 64}{360} = \frac{4{,}032}{360} = \mathbf{\frac{56}{5}} = \mathbf{11.2}.$$

$$\left(\frac{248}{75}\right)^2 \neq 11.2. \qquad \mu_Y \neq g(\mu_X).$$

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X))$.

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X))$.

For fun:

Moment-generating function approach:

$$M_Y(t) = E(e^{tY}) = E(e^{tX^2}) = \int_{-\infty}^{\infty} e^{tx^2} \cdot f_X(x)\,dx = \int_{2}^{4} e^{tx^2} \cdot \frac{x^3}{60}\,dx$$

$$u = x^2 \qquad du = 2x\,dx$$

$$= \int_{4}^{16} e^{tu} \cdot \frac{u}{120}\,du = \frac{u}{120\,t}\,e^{tu} - \frac{1}{120\,t^2}\,e^{tu}\,\Bigg|_{4}^{16}$$

$$= \frac{16}{120\,t}\,e^{16t} - \frac{1}{120\,t^2}\,e^{16t} - \frac{4}{120\,t}\,e^{4t} + \frac{1}{120\,t^2}\,e^{4t}, \qquad t \neq 0.$$

$$M_Y(0) = 1.$$

Fortunately, in this case, we know that this particular moment-generating function belongs to a probability distribution with the probability density function

$$f(u) = \frac{u}{120}, \qquad 4 \le u \le 16, \qquad \text{zero otherwise.}$$

Indeed,

$$\int_{-\infty}^{\infty} f(u)\,du = \int_{4}^{16} \frac{u}{120}\,du = \frac{y^2}{240}\,\Bigg|_{4}^{16} = \frac{256-16}{240} = 1,$$

$$M(t) = \int_{4}^{16} e^{tu} \cdot \frac{u}{120}\,du = \frac{u}{120\,t}\,e^{tu} - \frac{1}{120\,t^2}\,e^{tu}\,\Bigg|_{4}^{16}$$

$$= \frac{16}{120\,t}\,e^{16t} - \frac{1}{120\,t^2}\,e^{16t} - \frac{4}{120\,t}\,e^{4t} + \frac{1}{120\,t^2}\,e^{4t}, \qquad t \neq 0.$$

$$M(0) = 1.$$

Since there is one-to-one correspondence between probability distributions and their moment-generating functions, we now know that

$$f_Y(y) = \frac{y}{120}, \qquad 4 \le y \le 16, \qquad \text{zero otherwise.}$$

In this particular case, we were observant, and (fortunately) noticed the

$$\ldots = \int_4^{16} e^{tu} \cdot \frac{u}{120}\, du = \ldots \text{ step.}$$

IF all we had was the answer:

$$M_Y(t) = \frac{16}{120\,t} e^{16t} - \frac{1}{120\,t^2} e^{16t} - \frac{4}{120\,t} e^{4t} + \frac{1}{120\,t^2} e^{4t}, \qquad t \neq 0,$$

$$M_Y(0) = 1,$$

it could have been difficult to find the probability density function that matches this moment-generating function.

For continuous random variables,

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx.$$

Fun fact: Probability is also an expected value.

Define $$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then $$P(X \in A) = \int_A f(x)\, dx = \int_{-\infty}^{\infty} I_A(x) \cdot f(x)\, dx = E(I_A(X)).$$

Another fun fact: $I_A(x) \cdot I_B(x) = I_{A \cap B}(x).$

Consider $\mathrm{Y} = g(\mathrm{X})$ for a "nice" (one-to-one, differentiable) function $g(x)$.

Then $$\mathrm{E}(h(\mathrm{Y})) = \mathrm{E}(h(g(\mathrm{X}))) = \int_{-\infty}^{\infty} h\big(g(x)\big) \cdot f_{\mathrm{X}}(x)\, dx.$$

u-substitution: $u = g(x)$ $\quad x = g^{-1}(u)$ $\quad dx = \frac{d}{du} g^{-1}(u)\, du$

Or better: $y = g(x)$ $\quad x = g^{-1}(y)$ $\quad dx = \frac{d}{dy} g^{-1}(y)\, dy$

that is, $dx = \frac{dx}{dy}\, dy$

IF $g(x)$ is strictly increasing,

$$\mathrm{E}(h(\mathrm{Y})) = \int_{-\infty}^{\infty} h\big(g(x)\big) \cdot f_{\mathrm{X}}(x)\, dx = \int_{-\infty}^{\infty} h(y) \cdot f_{\mathrm{X}}\Big(g^{-1}(y)\Big)\, \frac{dx}{dy}\, dy.$$

IF $g(x)$ is strictly decreasing,

$$\mathrm{E}(h(\mathrm{Y})) = \int_{-\infty}^{\infty} h\big(g(x)\big) \cdot f_{\mathrm{X}}(x)\, dx = \int_{\infty}^{-\infty} h(y) \cdot f_{\mathrm{X}}\Big(g^{-1}(y)\Big)\, \frac{dx}{dy}\, dy$$

$$= \int_{-\infty}^{\infty} h(y) \cdot f_{\mathrm{X}}\Big(g^{-1}(y)\Big) \left(-\frac{dx}{dy}\right) dy.$$

$$\Rightarrow \quad \mathrm{E}(h(\mathrm{Y})) = \int_{-\infty}^{\infty} h(y) \cdot f_{\mathrm{X}}\Big(g^{-1}(y)\Big) \left|\frac{dx}{dy}\right| dy.$$

However, also $$\mathrm{E}(h(\mathrm{Y})) = \int_{-\infty}^{\infty} h(y) \cdot f_{\mathrm{Y}}(y)\, dy.$$

Change-of-variable technique is u-substitution from Calculus **!** ☺

1. (continued)

Consider $W = \dfrac{1}{X+6}$.

g) Find the support (the range of possible values) of the probability distribution of W.

$2 \le x \le 4.$ $\qquad \dfrac{1}{8} \ge \dfrac{1}{x+6} \ge \dfrac{1}{10}.$ $\qquad \mathbf{0.10 \le w \le 0.125}.$

h) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

$$F_W(w) = P(W \le w) = P\left(\frac{1}{X+6} \le w\right) = P\left(X \ge \frac{1}{w} - 6\right) = 1 - F_X\left(\frac{1}{w} - 6\right)$$

$$= 1 - \frac{\left(\frac{1}{w} - 6\right)^4 - 16}{240} = \frac{256 - \left(\frac{1}{w} - 6\right)^4}{240} = \frac{256\,w^4 - (1 - 6\,w)^4}{240\,w^4}$$

$$= 1 - \frac{1280\,w^4 - 864\,w^3 + 216\,w^2 - 24\,w + 1}{240\,w^4}$$

$$= \frac{-1040\,w^4 + 864\,w^3 - 216\,w^2 + 24\,w - 1}{240\,w^4}, \qquad 0.10 \le w < 0.125.$$

$F_W(w) = 0, \quad w < 0.10,$ $\qquad F_W(w) = 1, \quad w \ge 0.125.$

OR $\quad \ldots = P\left(X \ge \dfrac{1}{w} - 6\right) = \displaystyle\int_{\frac{1}{w}-6}^{4} \frac{x^3}{60}\,dx = \frac{256 - \left(\frac{1}{w} - 6\right)^4}{240} = \ldots$

i) Use the change-of-variable technique to find the p.d.f. of W, $f_{\mathrm{W}}(w)$.

$$w = \frac{1}{x+6} \qquad x = \frac{1}{w} - 6 \qquad \frac{dx}{dw} = -\frac{1}{w^2}$$

$$f_{\mathrm{W}}(w) = \frac{\left(\frac{1}{w}-6\right)^3}{60} \cdot \left| -\frac{1}{w^2} \right| = \frac{(1-6w)^3}{60\,w^5}$$

$$= \frac{-216\,w^3 + 108\,w^2 - 18\,w + 1}{60\,w^5}, \qquad 0.10 \le w \le 0.125.$$

Indeed, $$\frac{d}{dw}\left(\frac{256\,w^4 - (1-6w)^4}{240\,w^4}\right) = \frac{(1-6w)^3}{60\,w^5},$$ ☺

$$\frac{d}{dw}\left(\frac{-1040\,w^4 + 864\,w^3 - 216\,w^2 + 24\,w - 1}{240\,w^4}\right) = \frac{-216\,w^3 + 108\,w^2 - 18\,w + 1}{60\,w^5}.$$

j) Find the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$.

$$\mathrm{M}_{\mathrm{X}}(t) = \mathrm{E}(e^{t\mathrm{X}}) = \int_{-\infty}^{\infty} e^{tx} \cdot f_{\mathrm{X}}(x)\,dx = \int_{2}^{4} e^{tx} \cdot \frac{x^3}{60}\,dx \underset{\text{three times}}{\overset{\text{by parts}}{=}}$$

$$= \frac{\left(32\,t^3 - 24\,t^2 + 12\,t - 3\right)e^{4t} - \left(4\,t^3 - 6\,t^2 + 6\,t - 3\right)e^{2t}}{30\,t^4},$$

$t \neq 0.$

$\mathrm{M}_{\mathrm{X}}(0) = 1.$

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3}{C}, \qquad x = 1, 2, 3, 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

$$1 = \sum_{\text{all } x} p_X(x) = \frac{1^3}{C} + \frac{2^3}{C} + \frac{3^3}{C} + \frac{4^3}{C} = \frac{100}{C}.$$

$$\Rightarrow \qquad C = \mathbf{100}.$$

2. (continued)

Consider $Y = g(X) = X^2$.

b) Find the probability distribution of Y.

x	$p_X(x)$
1	$\frac{1}{100} = 0.01$
2	$\frac{8}{100} = 0.08$
3	$\frac{27}{100} = 0.27$
4	$\frac{64}{100} = 0.64$

$\Rightarrow$

y	$p_Y(y)$
$1^2 = 1$	0.01
$2^2 = 4$	0.08
$3^2 = 9$	0.27
$4^2 = 16$	0.64

OR $$p_Y(y) = \frac{y^{3/2}}{100}, \qquad y = 1, 4, 9, 16.$$

c) Does μ_Y equal to $g(\mu_X)$? $\mu_X = E(X)$, $\mu_Y = E(Y)$.

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x).$$

x	$p_X(x)$	$x \cdot p_X(x)$
1	0.01	0.01
2	0.08	0.16
3	0.27	0.81
4	0.64	2.56
		3.54
		$E(X)$

y	$p_Y(y)$	$y \cdot p_Y(y)$
1	0.01	0.01
4	0.08	0.32
9	0.27	2.43
16	0.64	10.24
		13
		$E(Y)$

$3.54^2 \neq 13.$ $\mu_Y \neq g(\mu_X).$

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X)).$

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X)).$

d) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } x} e^{tx} \cdot p_X(x)$$

$$= 0.01\, e^{t} + 0.08\, e^{2t} + 0.27\, e^{3t} + 0.64\, e^{4t}.$$

For fun:

Moment-generating function approach:

$$M_Y(t) = E(e^{tY}) = E(e^{tX^2}) = \sum_{\text{all } x} e^{tx^2} \cdot p_X(x)$$

$$= 0.01\, e^{t} + 0.08\, e^{4t} + 0.27\, e^{9t} + 0.64\, e^{16t}.$$

$\Rightarrow$

y	$p_Y(y)$
1	0.01
4	0.08
9	0.27
16	0.64

For discrete random variables, the possible values are isolated points on the number line.

$\Rightarrow$ no derivatives. $\Rightarrow$ no $\dfrac{dx}{dy}$.