STAT 410 | **Examples for 10/14/2020 (2)** | Fall 2020 A. Stepanov

1. Suppose X follows a Gamma distribution with mean $\mu = 20$ and standard deviation $\sigma = 10$.

Recall: $\alpha = 4, \qquad \theta = 5.$

d) Find a such that $P(X > a) = 0.10$.

Hint: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

e) Find b and c such that $P(b < X < c) = 0.95$.

Suggestion: Find b and c such that $P(X < b) = 0.025$ and $P(X > c) = 0.025$.

TABLE IV
The Chi-Square Distribution

				$P(X \le x)$				
	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
r	$\chi^2_{0.99}(r)$	$\chi^2_{0.975}(r)$	$\chi^2_{0.95}(r)$	$\chi^2_{0.90}(r)$	$\chi^2_{0.10}(r)$	$\chi^2_{0.05}(r)$	$\chi^2_{0.025}(r)$	$\chi^2_{0.01}(r)$
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.34
4	0.297	0.484	0.711	1.064	7.779	9.488	11.14	13.28
5	0.554	0.831	1.145	1.610	9.236	11.07	12.83	15.09
6	0.872	1.237	1.635	2.204	10.64	12.59	14.45	16.81
7	1.239	1.690	2.167	2.833	12.02	14.07	16.01	18.48
8	1.646	2.180	2.733	3.490	13.36	15.51	17.54	20.09
9	2.088	2.700	3.325	4.168	14.68	16.92	19.02	21.67
10	2.558	3.247	3.940	4.865	15.99	18.31	20.48	23.21

4. Cars arrive to CampusTown Urgent Care, a walk-in no-appointment clinic on the corner of Wright Street and Green Street, for COVID-19 testing according to Poisson process with the average rate of 1 car per 15 minutes.

a) Find the probability that at most 2 cars arrive in 1 hour.

b) Find the probability that at least 2 cars arrive in 30 minutes.

c) Find the probability that exactly 7 cars arrive in 2 hours.

d) Find the probability that the 7th car arrives during the second hour.

Hint: If T has a Gamma$\left(\alpha, \theta = 1/\lambda\right)$ distribution, where α is an integer, then $F_{\mathrm{T}}(t) = \mathrm{P}(\mathrm{T} \le t) = \mathrm{P}(\mathrm{Y} \ge \alpha)$ and $\mathrm{P}(\mathrm{T} > t) = \mathrm{P}(\mathrm{Y} \le \alpha - 1)$, where Y has a Poisson(λt) distribution.

e) Find the probability that the 10th car arrives after 3 hours.

f) Find the probability that the 15th car arrives during the fourth hour.

g) Find the 95% upper prediction bound for the arrival time of the 7th car.
That is, find c such that $\mathrm{P}(\mathrm{T}_7 < c) = 0.95$.

Hint: If T has a Gamma$\left(\alpha, \theta = 1/\lambda\right)$ distribution, where α is an integer, then $2\mathrm{T}/\theta = 2\lambda\mathrm{T}$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

h) Construct an 80% prediction interval for the time when the 15th car arrives.
That is, find a and b such that $\mathrm{P}(a < \mathrm{T}_{15} < b) = 0.80$.

1. Suppose X follows a Gamma distribution with mean $\mu = 20$ and standard deviation $\sigma = 10$.

Recall: $\alpha = 4$, $\theta = 5$.

d) Find a such that $P(X > a) = 0.10$.

Hint: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$\frac{2\,T_4}{\theta} = \frac{2\,T_4}{5} \text{ has a } \chi^2(2\alpha = 8) \text{ distribution.}$$

$\chi^2_{0.10}(8) = 13.36.$ $\qquad P(\chi^2(8) > 13.36) = 0.10.$

$$0.10 = P(X > a) = P(T_4 > a) = P\left(\frac{2\,T_4}{5} > \frac{2\,a}{5}\right) = P\left(\chi^2(8) > \frac{2\,a}{5}\right).$$

$$\frac{2\,a}{5} = 13.36. \qquad \Rightarrow \qquad a = \mathbf{33.4}.$$

```
> qgamma(0.90,4,1/5)
[1] 33.40392
> qgamma(0.10,4,1/5, lower.tail=FALSE)
[1] 33.40392
>
> qchisq(0.90,2*4)
[1] 13.36157
> qchisq(0.90,2*4)*(5/2)
[1] 33.40392
```

=GAMMA.INV(0.90,4,5)	33.40392
=CHISQ.INV(0.90,2*4)	13.36157
=CHISQ.INV.RT(0.10,2*4)	13.36157
=CHISQ.INV(0.90,2*4)*(5/2)	33.40392

e) Find b and c such that $P(b < X < c) = 0.95$.

Suggestion: Find b and c such that $P(X < b) = 0.025$ and $P(X > c) = 0.025$.

$\chi^2_{0.975}(8) = 2.180$, $\chi^2_{0.025}(8) = 17.54$.

$P(2.180 < \chi^2(8) < 17.54) = 0.95$.

$\dfrac{2b}{5} = 2.180$, $\dfrac{2c}{5} = 17.54$.

$\Rightarrow$ $b = \mathbf{5.45}$, $c = \mathbf{43.85}$.

OR

We do NOT have to do what Alex suggests! This is America!

$\chi^2_{0.95}(8) = 2.733$.

$P(2.733 < \chi^2(8) < \infty) = 0.95$.

$\dfrac{2b}{5} = 2.733$. $\dfrac{2c}{5} = \infty$.

$\Rightarrow$ $b = \mathbf{6.8325}$, $c = \infty$.

$\chi^2_{0.05}(8) = 15.51$.

$P(0 < \chi^2(8) < 15.51) = 0.95$.

$\dfrac{2b}{5} = 0$. $\dfrac{2c}{5} = 15.51$.

$\Rightarrow$ $b = \mathbf{0}$, $c = \mathbf{38.775}$.

```
> qgamma(0.025,4,1/5)
[1] 5.449327
> qgamma(0.975,4,1/5)
[1] 43.83637
```

```
> qchisq(0.025,2*4)
[1] 2.179731
> qchisq(0.025,2*4)*(5/2)
[1] 5.449327
> qchisq(0.975,2*4)
[1] 17.53455
> qchisq(0.975,2*4)*(5/2)
[1] 43.83637
```

4. Cars arrive to CampusTown Urgent Care, a walk-in no-appointment clinic on the corner of Wright Street and Green Street, for COVID-19 testing according to Poisson process with the average rate of 1 car per 15 minutes.

a) Find the probability that at most 2 cars arrive in 1 hour.

1 hour $\Rightarrow$ $\lambda = 4$.

$$\mathrm{P}(\text{at most 2 cars}) = \mathrm{P}(0 \text{ cars}) + \mathrm{P}(1 \text{ cars}) + \mathrm{P}(2 \text{ cars})$$

$$= \frac{4^{0} \cdot e^{-4}}{0\,!} + \frac{4^{1} \cdot e^{-4}}{1\,!} + \frac{4^{2} \cdot e^{-4}}{2\,!}$$

$$\approx 0.018316 + 0.073263 + 0.146525 = \mathbf{0.238104}.$$

Using Cumulative Poisson Probabilities table: $\mathrm{P}(\text{Poisson}(4) \le 2) = \mathbf{0.238}$.

R:

```
> ppois(2,4)
[1] 0.2381033
```

Excel:

=POISSON.DIST(2,4,1) **0.238103**

b) Find the probability that at least 2 cars arrive in 30 minutes.

30 minutes = 0.5 hour $\Rightarrow$ $\lambda = 0.5 \times 4 = 2$.

$$\text{P}(\text{ at least 2 cars }) = 1 - \text{P}(\text{ at most 1 car })$$

$$= 1 - \frac{2^{\,0} \cdot e^{-2}}{0\,!} - \frac{2^{\,1} \cdot e^{-2}}{1\,!}$$

$$\approx 1 - 0.135335 - 0.270671 = \mathbf{0.593994}.$$

Using Cumulative Poisson Probabilities table:

$$\text{P}(\text{Poisson}(2) \geq 2) = 1 - \text{P}(\text{Poisson}(2) \leq 1) = 1 - 0.406 = \mathbf{0.594}.$$

R:

```
> 1-ppois(1,2)
[1] 0.5939942
```

Excel:

=1-POISSON.DIST(1,2,1) **0.593994**

c) Find the probability that exactly 7 cars arrive in 2 hours.

2 hours $\Rightarrow$ $\lambda = 2 \times 4 = 8$.

$$\text{P}(\text{exactly 7 cars}) = \frac{8^{7} \cdot e^{-8}}{7\,!} \approx \mathbf{0.139587}.$$

Using Cumulative Poisson Probabilities table:

$$\text{P}(\text{Poisson}(8) = 7) = \text{P}(\text{Poisson}(8) \le 7) - \text{P}(\text{Poisson}(8) \le 6)$$

$$= 0.453 - 0.313 = \mathbf{0.140}.$$

R:

```
> dpois(7,8)
[1] 0.1395865
> ppois(7,8)-ppois(6,8)
[1] 0.1395865
```

Excel:

=POISSON.DIST(7,8,0)	**0.139587**
=POISSON.DIST(7,8,1)-POISSON.DIST(6,8,1)	**0.139587**

d) Find the probability that the 7th car arrives during the second hour.

Hint: If T has a $\mathrm{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then
$F_{\mathrm{T}}(t) = P(\mathrm{T} \le t) = P(\mathrm{Y} \ge \alpha)$ and $P(\mathrm{T} > t) = P(\mathrm{Y} \le \alpha - 1)$,
where Y has a $\mathrm{Poisson}(\lambda t)$ distribution.

$$P(1 < \mathrm{T}_7 < 2) = P(\mathrm{T}_7 > 1) - P(\mathrm{T}_7 > 2) = P(\mathrm{X}_1 \le 6) - P(\mathrm{X}_2 \le 6)$$
$$= P(\mathrm{Poisson}(4) \le 6) - P(\mathrm{Poisson}(8) \le 6) = 0.889 - 0.313 = \mathbf{0.576}.$$

OR

$$P(1 < \mathrm{T}_7 < 2) = \int_1^2 \frac{4^7}{\Gamma(7)} t^{7-1} e^{-4t} \, dt = \int_1^2 \frac{4^7}{6!} t^6 e^{-4t} \, dt = \ldots$$

OR

$$P(1 < \mathrm{T}_7 < 2) = P(2\lambda \cdot 1 < 2\lambda \mathrm{T}_7 < 2\lambda \cdot 2) = P(8 < \chi^2(14) < 16) = \ldots$$

R:

```
> pgamma(2,7,4)-pgamma(1,7,4)
[1] 0.5759517
>
> ppois(6,4)-ppois(6,8)
[1] 0.5759517
>
> pchisq(16,14)-pchisq(8,14)
[1] 0.5759517
```

Excel:

=GAMMA.DIST(2,7,1/4,1)-GAMMA.DIST(1,7,1/4,1)	**0.575952**
=CHISQ.DIST(16,14,1)-CHISQ.DIST(8,14,1)	**0.575952**
=CHISQ.DIST.RT(8,14)-CHISQ.DIST.RT(16,14)	**0.575952**

e) Find the probability that the 10th car arrives after 3 hours.

$$P(T_{10} > 3) = P(X_3 \le 9) = P(\text{Poisson}(12) \le 9) = \mathbf{0.242}.$$

OR

$$P(T_{10} > 3) = \int_3^{\infty} \frac{4^{10}}{\Gamma(10)}\, t^{10-1} e^{-4t}\, dt = \int_3^{\infty} \frac{4^{10}}{9!}\, t^{9} e^{-4t}\, dt = \ldots$$

OR

$$P(T_{10} > 3) = P(2\lambda T_{10} > 2\lambda \cdot 3) = P(\chi^2(20) > 24) = \ldots$$

R:

```
> 1-pgamma(3,10,4)
[1] 0.2423922
>
> ppois(9,12)
[1] 0.2423922
>
> 1-pchisq(24,20)
[1] 0.2423922
```

Excel:

=1-GAMMA.DIST(3,10,1/4,1)	**0.242392**
=1-CHISQ.DIST(24,20,1)	**0.242392**
=CHISQ.DIST.RT(24,20)	**0.242392**

f) Find the probability that the 15th car arrives during the fourth hour.

$$P(3 < T_{15} < 4) = P(T_{15} > 3) - P(T_{15} > 4) = P(X_3 \le 14) - P(X_4 \le 14)$$

$$= P(\text{Poisson}(12) \le 14) - P(\text{Poisson}(16) \le 14) = 0.772 - 0.368 = \mathbf{0.404}.$$

OR

$$P(3 < T_{15} < 4) = \int_3^4 \frac{4^{15}}{\Gamma(15)} t^{15-1} e^{-4t}\,dt = \int_3^4 \frac{4^{15}}{14!} t^{14} e^{-4t}\,dt = \ldots$$

OR

$$P(3 < T_{15} < 4) = P(2\lambda \cdot 3 < 2\lambda T_{15} < 2\lambda \cdot 4) = P(24 < \chi^2(30) < 32) = \ldots$$

R:

```
> pgamma(4,15,4)-pgamma(3,15,4)
[1] 0.4044972
>
> ppois(14,12)-ppois(14,16)
[1] 0.4044972
>
> pchisq(32,30)-pchisq(24,30)
[1] 0.4044972
```

Excel:

=GAMMA.DIST(4,15,1/4,1)-GAMMA.DIST(3,15,1/4,1) **0.404497**

=CHISQ.DIST(32,30,1)-CHISQ.DIST(24,30,1) **0.404497**

=CHISQ.DIST.RT(24,30)-CHISQ.DIST.RT(32,30) **0.404497**

g) Find the 95% upper prediction bound for the arrival time of the 7th car.
That is, find c such that $\mathrm{P}(\mathrm{T}_7 < c) = 0.95$.

Hint: If T has a $\text{Gamma}(\alpha, \theta = {}^1\!/\!_\lambda)$ distribution, where α is an integer, then ${}^{2\mathrm{T}}\!/\!_\theta = 2\lambda\mathrm{T}$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

Using the Chi-Square Distribution table:

$2\lambda\mathrm{T}_7 = 8\,\mathrm{T}_7$ has a $\chi^2(2\alpha = 14)$ distribution.

$\chi^2_{0.05}(14) = 23.68.$

$\mathrm{P}(\chi^2(14) < 23.68) = 0.95.$

$c = \dfrac{23.68}{8} = \mathbf{2.96}.$

R:

```
> qgamma(0.95,7,4)
[1] 2.960599
>
> qchisq(0.95,14); qchisq(0.95,14)/8
[1] 23.68479
[1] 23.68479
```

Excel:

=GAMMA.INV(0.95,7,1/4)	**2.960599**
=CHISQ.INV(0.95,14)	23.68479
=CHISQ.INV(0.95,14)/8	**2.960599**
=CHISQ.INV.RT(0.05,14)	23.68479
=CHISQ.INV.RT(0.05,14)/8	**2.960599**

h) Construct an 80% prediction interval for the time when the 15th car arrives.

That is, find a and b such that $P(a < T_{15} < b) = 0.80$.

Using the Chi-Square Distribution table:

$2\lambda T_{15} = 8 T_{15}$ has a $\chi^2(2\alpha = 30)$ distribution.

$\chi^2_{0.90}(30) = 20.60,$ $\qquad\qquad$ $\chi^2_{0.10}(30) = 40.26.$

$P(20.60 < \chi^2(30) < 40.26) = 0.80.$

$a = \dfrac{20.60}{8} = \mathbf{2.575},$ $\qquad\qquad$ $b = \dfrac{40.26}{8} = \mathbf{5.0325}.$

R:

```
> qgamma(0.10,15,4); qgamma(0.90,15,4)
[1] 2.574904
[1] 5.032003
>
> qchisq(0.10,30); qchisq(0.10,30)/8
[1] 20.59923
[1] 2.574904
> qchisq(0.90,30); qchisq(0.90,30)/8
[1] 40.25602
[1] 5.032003
```

Excel:

=GAMMA.INV(0.1,15,1/4)	**2.574904**
=GAMMA.INV(0.9,15,1/4)	**5.032003**
=CHISQ.INV(0.1,30)	20.59923
=CHISQ.INV(0.1,30)/8	**2.574904**
=CHISQ.INV.RT(0.1,30)	40.25602
=CHISQ.INV.RT(0.1,30)/8	**5.03203**

There are infinitely many possible answers:

left	a	b	right
0	**0**	**4.531273**	0.20
0.01	**1.869182**	**4.571269**	0.19
0.02	**2.038272**	**4.612823**	0.18
0.03	**2.150953**	**4.656114**	0.17
0.04	**2.238535**	**4.701351**	0.16
0.05	**2.311583**	**4.748781**	0.15
0.06	**2.375048**	**4.798704**	0.14
0.07	**2.431679**	**4.851481**	0.13
0.08	**2.483171**	**4.907558**	0.12
0.09	**2.530650**	**4.967494**	0.11
0.10	**2.574904**	**5.032003**	0.10
0.11	**2.616509**	**5.102019**	0.09
0.12	**2.655897**	**5.178795**	0.08
0.13	**2.693405**	**5.264076**	0.07
0.14	**2.729299**	**5.360391**	0.06
0.15	**2.763793**	**5.471621**	0.05
0.16	**2.797063**	**5.604195**	0.04
0.17	**2.829256**	**5.769992**	0.03
0.18	**2.860495**	**5.995225**	0.02
0.19	**2.890884**	**6.361523**	0.01
0.20	**2.920514**	∞	0