2.3 Conditional Distributions and Expectations.

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_{\text{X}}(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_{\text{Y}}(y)$	0.40	0.40	0.20	

a) Find the conditional probability distributions $p_{\text{X}|\text{Y}}(x|y) = \dfrac{p(x,y)}{p_{\text{Y}}(y)}$ of X given $\text{Y} = y$, conditional expectation $\text{E}(\text{X}|\text{Y} = y)$ of X given $\text{Y} = y$, and $\text{E}(\text{E}(\text{X}|\text{Y}))$.

x	$p_{\text{X}\mid\text{Y}}(x\mid 0)$
1	$0.15/0.40 = 0.375$
2	$0.25/0.40 = 0.625$

x	$p_{\text{X}\mid\text{Y}}(x\mid 1)$
1	$0.10/0.40 = 0.25$
2	$0.30/0.40 = 0.75$

x	$p_{\text{X}\mid\text{Y}}(x\mid 2)$
1	$0.00/0.20 = 0.00$
2	$0.20/0.20 = 1.00$

$\text{E}(\text{X}|\text{Y} = 0) = 1.625$ $\text{E}(\text{X}|\text{Y} = 1) = 1.75$ $\text{E}(\text{X}|\text{Y} = 2) = 2.0$

Def

$$\text{E}(\text{X}|\text{Y}=y) = \sum_x x\,\text{P}(\text{X}=x|\text{Y}=y) = \sum_x x\,p_{\text{X}|\text{Y}}(x|y) \quad \text{– discrete}$$

$$\text{E}(\text{X}|\text{Y}=y) = \int_{-\infty}^{\infty} x \cdot f_{\text{X}|\text{Y}}(x\,|\,y)\,dx \quad \text{– continuous}$$

Denote by $\text{E}(\text{X}|\text{Y})$ that function of the random variable Y whose value at $\text{Y} = y$ is $\text{E}(\text{X}|\text{Y} = y)$. Note that $\text{E}(\text{X}|\text{Y})$ is itself a random variable, it depends on the (random) value of Y that occurs.

y	$E(X \mid Y=y)$	$p_Y(y)$	
0	1.625	0.40	0.65
1	1.75	0.40	0.70
2	2.0	0.20	0.40

$E(E(X \mid Y)) = 1.75.$

Recall: $E(X) = 1.75.$

- $E(a_1 X_1 + a_2 X_2 \mid Y) = a_1 E(X_1 \mid Y) + a_2 E(X_2 \mid Y)$
- $E[g(Y) \mid Y] = g(Y)$
- $E(E(X \mid Y)) = E(X)$
- $E[E(X \mid Y) \mid Y] = E(X \mid Y)$
- $E[g(Y) X \mid Y] = g(Y) E(X \mid Y)$

b) Find the conditional probability distributions $p_{Y|X}(y \mid x) = \dfrac{p(x,y)}{p_X(x)}$ of Y given $X = x$, conditional expectation $E(Y \mid X = x)$ of Y given $X = x$, and $E(E(Y \mid X))$.

y	$p_{Y \mid X}(y \mid 1)$
0	$0.15/0.25 = 0.60$
1	$0.10/0.25 = 0.40$
2	$0.00/0.25 = 0.00$

y	$p_{Y \mid X}(y \mid 2)$
0	$0.25/0.75 = 5/15$
1	$0.30/0.75 = 6/15$
2	$0.20/0.75 = 4/15$

$E(Y \mid X = 1) = 0.4 = 6/15$

$E(Y \mid X = 2) = 14/15$

x	$E(Y \mid X = x)$	$p_X(x)$
1	$6/15$	0.25
2	$14/15$	0.75

$$E(E(Y \mid X)) = \frac{6}{15} \cdot 0.25 + \frac{14}{15} \cdot 0.75$$
$$= 0.10 + 0.70 = 0.80.$$

Recall: $E(Y) = 0.80.$

2. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_{\mathrm{X}}(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1, \qquad \mathrm{E}(\mathrm{X}) = \dfrac{1}{2},$

$f_{\mathrm{Y}}(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1, \qquad \mathrm{E}(\mathrm{Y}) = \dfrac{1}{3}.$

a) Find the conditional probability density function $f_{\mathrm{Y}|\mathrm{X}}(y|x) = \dfrac{f(x,y)}{f_{\mathrm{X}}(x)}$ of Y given $\mathrm{X} = x, \quad 0 < x < 1.$

$$f_{\mathrm{Y}|\mathrm{X}}(y|x) = \frac{f(x,y)}{f_{\mathrm{X}}(x)} = \frac{60\,x^2\,y}{30\,x^2\,(1-x)^2} = \frac{2\,y}{(1-x)^2}, \qquad 0 < y < 1 - x.$$

b) Find $\mathrm{P}(\mathrm{Y} > 1/3 \mid \mathrm{X} = 1/2),\ \mathrm{P}(\mathrm{Y} > 1/4 \mid \mathrm{X} = 1/3),\ \mathrm{P}(\mathrm{Y} < 1/2 \mid \mathrm{X} = 1/3).$

$$\mathrm{P}(\mathrm{Y} > 1/3 \mid \mathrm{X} = 1/2) = \int_{1/3}^{1/2} \frac{2\,y}{(1/2)^2}\,dy = \int_{1/3}^{1/2} 8\,y\,dy = \mathbf{\frac{5}{9}}.$$

$$\mathrm{P}(\mathrm{Y} > 1/4 \mid \mathrm{X} = 1/3) = \int_{1/4}^{2/3} \frac{2\,y}{(2/3)^2}\,dy = \mathbf{\frac{55}{64}}.$$

$$\mathrm{P}(\mathrm{Y} < 1/2 \mid \mathrm{X} = 1/3) = \int_{0}^{1/2} \frac{2\,y}{(2/3)^2}\,dy = \mathbf{\frac{9}{16}} = \mathbf{0.5625}.$$

$$\mathrm{P}(\mathrm{Y} < 1/2 \mid \mathrm{X} = 2/3) = \int_{0}^{1/3} \frac{2\,y}{(1/3)^2}\,dy = \mathbf{1}.$$

c) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x),\ \mathrm{E}(\mathrm{Y} \mid \mathrm{X}),$ and $\mathrm{E}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X})).$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x) = \int_0^{1-x} y \cdot \frac{2\,y}{(1-x)^2}\,dy = \frac{2}{(1-x)^2} \cdot \int_0^{1-x} y^2\,dy = \frac{2}{3} \cdot (1-x), \quad 0 < x < 1.$$

$$E(Y|X) = \frac{2}{3}(1-X).$$

$$E(E(Y|X)) = \frac{2}{3}(1-E(X)) = \frac{2}{3}\left(1-\frac{1}{2}\right) = \frac{1}{3} = E(Y).$$

d) Find the conditional probability density function $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ of X given $Y = y$, $0 < y < 1$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{60x^2 y}{20y(1-y)^3} = \frac{3x^2}{(1-y)^3}, \qquad 0 < x < 1-y.$$

e) Find $P\left(X > \frac{1}{2} \,\middle|\, Y = \frac{1}{3}\right)$.

$$f_{X|Y}\left(x \,\middle|\, \frac{1}{3}\right) = \frac{81x^2}{8}, \qquad 0 < x < \frac{2}{3}.$$

$$P\left(X > \frac{1}{2} \,\middle|\, Y = \frac{1}{3}\right) = \int_{1/2}^{2/3} \frac{81x^2}{8}\,dx = \left(\frac{27x^3}{8}\right)\Bigg|_{1/2}^{2/3} = \mathbf{\frac{37}{64}}.$$

f) Find $E(X|Y=y)$, $E(X|Y)$, and $E(E(X|Y))$.

$$E(X|Y=y) = \int_0^{1-y} x \cdot \frac{3x^2}{(1-y)^3}\,dx = \frac{3}{(1-y)^3} \cdot \int_0^{1-y} x^3\,dx = \frac{3}{4}\cdot(1-y), \quad 0 < y < 1.$$

$$E(X|Y) = \frac{3}{4}(1-Y).$$

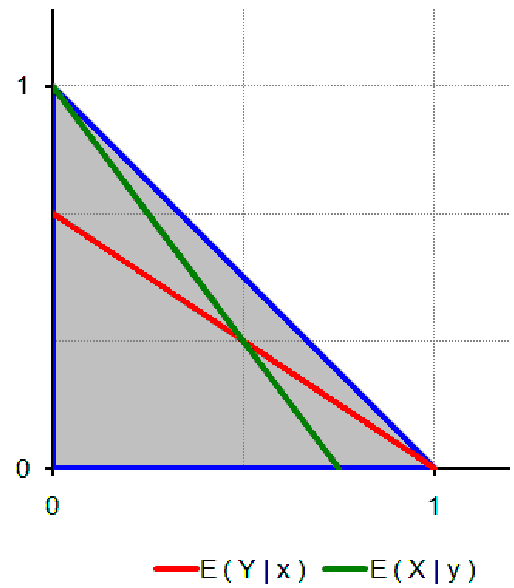
$$E(E(X|Y)) = \frac{3}{4}(1-E(Y)) = \frac{3}{4}\left(1-\frac{1}{3}\right) = \frac{1}{2} = E(X).$$

Recall:

$$\operatorname{Var}(X) = \frac{9}{252},$$

$$\operatorname{Var}(Y) = \frac{8}{252},$$

$$\rho_{XY} = -\frac{1}{\sqrt{2}}.$$



If $E(Y|X=x)$ is linear in x, then

$$E(Y|X=x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X).$$

$$E(Y|X=x) = \frac{1}{3} - \frac{1}{\sqrt{2}}\frac{\sqrt{8/252}}{\sqrt{9/252}}\left(x-\frac{1}{2}\right)$$

$$= \frac{1}{3}-\frac{2}{3}\left(x-\frac{1}{2}\right) = \frac{2}{3}-\frac{2}{3}x.$$

$$E(X|Y=y) = \frac{1}{2} - \frac{1}{\sqrt{2}}\frac{\sqrt{9/252}}{\sqrt{8/252}}\left(y-\frac{1}{3}\right) = \frac{1}{2}-\frac{3}{4}\left(y-\frac{1}{3}\right) = \frac{3}{4}-\frac{3}{4}y.$$