STAT 410

Practice Problems 7

Fall 2020
A. Stepanov

1. When you leave you car at *Honest Harry's Car Repair Shop*, first it takes X weeks for needed parts to arrive, and then Y more weeks for the repairs to be finished. Thus the total wait is W = X + Y weeks. Suppose that X and Y are independent, the p.d.f. of X is $f_X(x) = 2x$, $0 < x < 1$, zero otherwise, and Y has a Uniform distribution on interval (0, 1). Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

2. If you take your car to *Dean McCoppin's Scrap Yard* instead of *Honest Harry's Car Repair Shop*, it will first take Dean X weeks to identify the problem with your car, and then Y weeks to fix it. Thus the total wait is W = X + Y weeks. Suppose that X and Y be two independent random variables, with probability density functions $f_X(x)$ and $f_Y(y)$, respectively.

$$f_X(x) = 2(1-x), \quad 0 < x < 1, \qquad\qquad f_Y(y) = \frac{3}{(1+y)^4}, \quad y > 0.$$

Find the p.d.f. $f_W(w)$ of W = X + Y.

3. Suppose that X and Y are independent,
X has a Uniform distribution on interval (0, 3),
and the p.d.f. of Y is

$$f_Y(y) = {}^{y}\!/_{2}, \quad 0 < y < 2, \qquad \text{zero otherwise.}$$

a) Find the probability density function of W = X + Y, $f_W(w) = f_{X+Y}(w)$.

b) Find the probability density function of V = X × Y, $f_V(v) = f_{X \times Y}(v)$.

c) Find the probability density function of U = ${}^{Y}\!/_{X}$, $f_U(u) = f_{Y/X}(u)$.

4. Let the joint pdf of X and Y be given by

$$f(x,y)=\begin{cases} \dfrac{2}{(1+x+y)^3} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{elsewhere.} \end{cases}$$

a) Let $W = X + Y$. Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

b) Let $V = Y/X$. Find the p.d.f. of V, $f_V(v)$.

5. **2.2.2** (7th and 6th edition)

Let X_1 and X_2 have the joint pmf $p_{X_1,X_2}(x_1,x_2) = x_1 x_2/36$, $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$, zero elsewhere. Find first the joint pmf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$, and then find the marginal pmf of Y_1.

Hint: X_1 and X_2 are discrete random variables. There are nine possible pairs (x_1, x_2).

6. Let X_1 and X_2 have the joint pmf $p_{X_1,X_2}(x_1,x_2) = x_1 x_2/36$, $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$, zero elsewhere. Find the probability distribution of $W = X_1 + X_2$.

1. When you leave you car at *Honest Harry's Car Repair Shop*, first it takes X weeks for needed parts to arrive, and then Y more weeks for the repairs to be finished. Thus the total wait is W = X + Y weeks. Suppose that X and Y are independent, the p.d.f. of X is $f_X(x) = 2x$, $0 < x < 1$, zero otherwise, and Y has a Uniform distribution on interval $(0, 1)$. Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

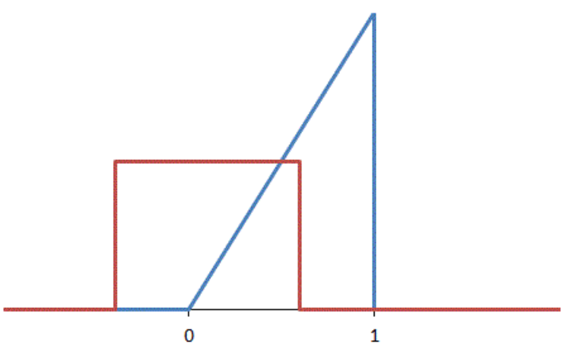
$0 < x < 1, \quad 0 < y < 1 \qquad \Rightarrow \qquad 0 < x + y < 2$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx.$$

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(w-x) = \begin{cases} 1 & 0 < w-x < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & w-1 < x < w \\ 0 & \text{otherwise} \end{cases}$$

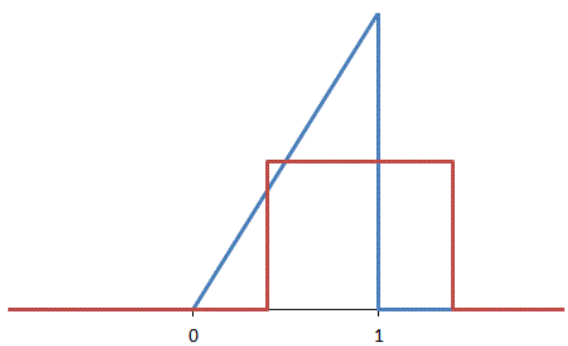
Case 1. $0 < w < 1 \quad \Rightarrow \quad w - 1 < 0$



$$f_{X+Y}(w) = \int_0^w (2x \cdot 1)\, dx = w^2,$$

$0 < w < 1.$

Case 2. $1 < w < 2 \quad \Rightarrow \quad 0 < w - 1 < 1$



$$f_{X+Y}(w) = \int_{w-1}^{1} (2x \cdot 1)\, dx$$

$$= 1 - (w-1)^2 = 2w - w^2,$$

$1 < w < 2.$

OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y) \cdot f_Y(y)\,dy .$$

Case 1. $0 < w < 1 \quad \Rightarrow \quad w - 1 < 0$

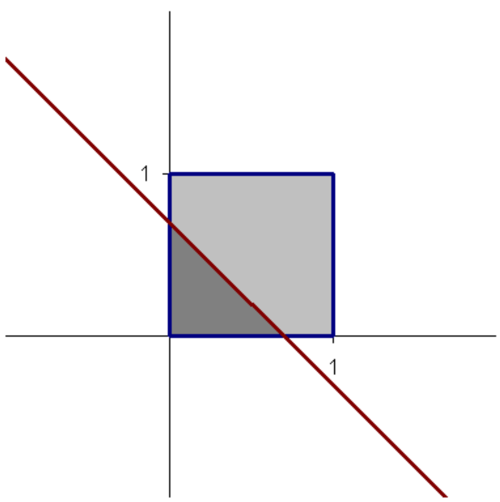
$$f_{X+Y}(w) = \int_0^w 1 \cdot 2(w-y)\,dy = w^2, \qquad 0 < w < 1.$$

Case 2. $1 < w < 2 \quad \Rightarrow \quad 0 < w - 1 < 1$

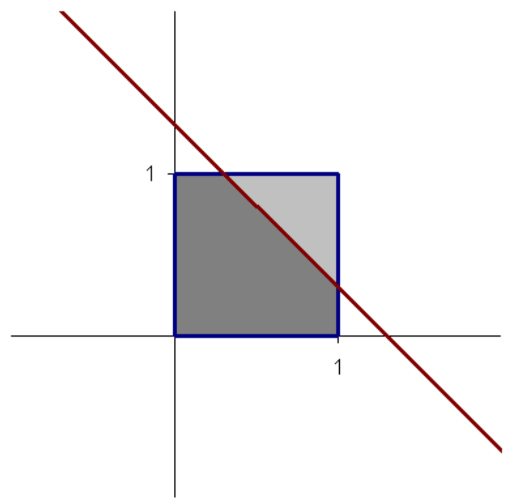
$$f_{X+Y}(w) = \int_{w-1}^{1} 1 \cdot 2(w-y)\,dy = 2w - w^2, \qquad 1 < w < 2.$$

OR

Case 1. $0 < w < 1.$



Case 2. $1 < w < 2.$



$$F_{X+Y}(w) = \int_0^w \left(\int_0^{w-x} 2x \cdot 1\,dy \right) dx = \frac{1}{3} w^3, \qquad 0 < w < 1.$$

$$F_{X+Y}(w) = 1 - \int_{w-1}^{1} \left(\int_{w-x}^{1} 2x \cdot 1\,dy \right) dx = \frac{1}{3} + (w-1) - \frac{1}{3}(w-1)^3, \qquad 1 < w < 2.$$

$$f_{X+Y}(w) = F'_{X+Y}(w) = w^2, \qquad 0 < w < 1.$$

$$f_{X+Y}(w) = F'_{X+Y}(w) = 1 - (w-1)^2 = 2w - w^2, \qquad 1 < w < 2.$$

2. If you take your car to *Dean McCoppin's Scrap Yard* instead of *Honest Harry's Car Repair Shop*, it will first take Dean X weeks to identify the problem with your car, and then Y weeks to fix it. Thus the total wait is W = X + Y weeks. Suppose that X and Y be two independent random variables, with probability density functions $f_X(x)$ and $f_Y(y)$, respectively.

$$f_X(x) = 2\,(1-x), \qquad 0 < x < 1, \qquad\qquad f_Y(y) = \frac{3}{(1+y)^4}, \qquad y > 0.$$

Find the p.d.f. $f_W(w)$ of W = X + Y.

$$f_X(w-y) = 2\,(1-w+y) = 2\,(1+y) - 2\,w, \qquad\qquad 0 < w-y < 1$$

$$\Leftrightarrow \qquad w-1 < y < w.$$

<u>Case 1</u>: $0 < w < 1.$ Then $w-1 < 0 < w.$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y)\cdot f_Y(y)\,dy = \int_0^w [2(1+y)-2w]\cdot\frac{3}{(1+y)^4}\,dy$$

$$= \int_0^w \frac{6}{(1+y)^3}\,dy - \int_0^w \frac{6w}{(1+y)^4}\,dy = \left(-\frac{3}{(1+y)^2} + \frac{2w}{(1+y)^3}\right)\Bigg|_0^w$$

$$= 3 - 2w - \frac{3}{(1+w)^2} + \frac{2w}{(1+w)^3} = \frac{6w+3w^2-3w^3-2w^4}{(1+w)^3},$$

$$0 < w < 1.$$

<u>Case 2</u>: $w > 1.$ Then $w-1 > 0.$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y)\cdot f_Y(y)\,dy = \int_{w-1}^{w} [2(1+y)-2w]\cdot\frac{3}{(1+y)^4}\,dy$$

$$= \int_{w-1}^{w} \frac{6}{(1+y)^3}\,dy - \int_{w-1}^{w} \frac{6w}{(1+y)^4}\,dy = \left(-\frac{3}{(1+y)^2} + \frac{2w}{(1+y)^3}\right)\Bigg|_{w-1}^{w}$$

$$= \frac{1}{w^2} - \frac{3}{(1+w)^2} + \frac{2w}{(1+w)^3} = \frac{1+3w}{w^2\,(1+w)^3}, \qquad w > 1.$$

3. Suppose that X and Y are independent,

X has a Uniform distribution on interval $(0, 3)$,

and the p.d.f. of Y is

$$f_Y(y) = y/2, \quad 0 < y < 2, \quad \text{zero otherwise.}$$

a) Find the probability density function of $W = X + Y$, $f_W(w) = f_{X+Y}(w)$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(w-y) \cdot f_Y(y)\, dy$$

$$f_X(x) = \begin{cases} 1/3 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases} \qquad f_X(w-y) = \begin{cases} 1/3 & 0 < w-y < 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/3 & w-3 < y < w \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} y/2 & 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Case 1: $0 < w < 2$. $\qquad w - 3 < 0 < w < 2$.

$$f_W(w) = \int_0^w \frac{1}{3} \cdot \frac{y}{2}\, dy = \frac{w^2}{12}.$$

Case 2: $2 < w < 3$. $\qquad w - 3 < 0 < 2 < w$.

$$f_W(w) = \int_0^2 \frac{1}{3} \cdot \frac{y}{2}\, dy = \frac{1}{3}.$$

Case 3: $3 < w < 5$. $\qquad 0 < w - 3 < 2 < w$.

$$f_W(w) = \int_{w-3}^{2} \frac{1}{3} \cdot \frac{y}{2}\, dy = \frac{1}{3} - \frac{(w-3)^2}{12}.$$

$$f_{X+Y}(w) = \begin{cases} \dfrac{w^2}{12} & 0 < w < 2 \\[2ex] \dfrac{1}{3} & 2 < w < 3 \\[2ex] \dfrac{1}{3} - \dfrac{(w-3)^2}{12} & 3 < w < 5 \\[2ex] 0 & \text{otherwise} \end{cases}$$

OR

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx$$

Case 1: $0 < w < 2$. $w - 2 < 0 < w < 3$.

$$f_W(w) = \int_0^w \frac{1}{3} \cdot \frac{w-x}{2}\, dx = \frac{w^2}{12}.$$

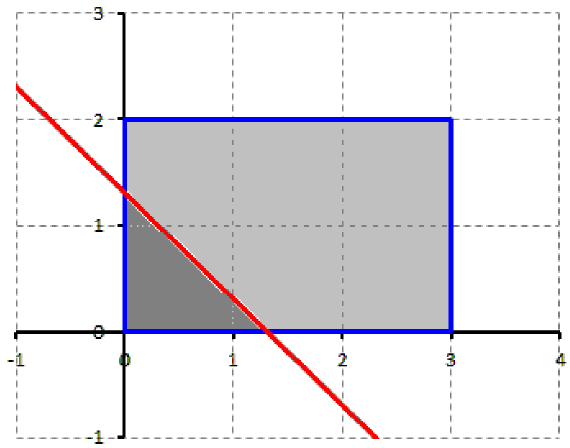
Case 2: $2 < w < 3$. $0 < w - 2 < w < 3$.

$$f_W(w) = \int_{w-2}^{w} \frac{1}{3} \cdot \frac{w-x}{2}\, dx = \frac{1}{3}.$$

Case 3: $3 < w < 5$. $0 < w - 2 < 3 < w$.

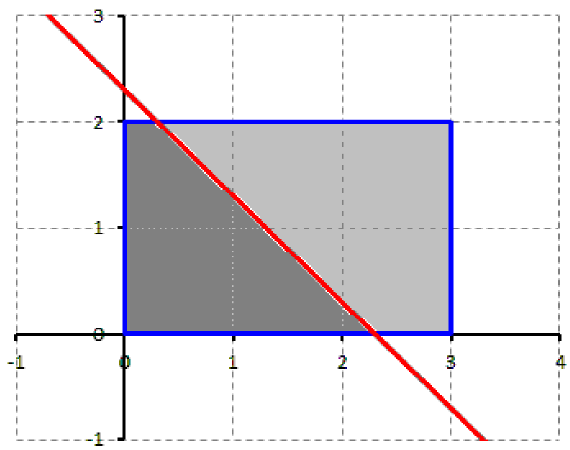
$$f_W(w) = \int_{w-2}^{3} \frac{1}{3} \cdot \frac{w-x}{2}\, dx = \frac{1}{3} - \frac{(w-3)^2}{12}.$$

OR



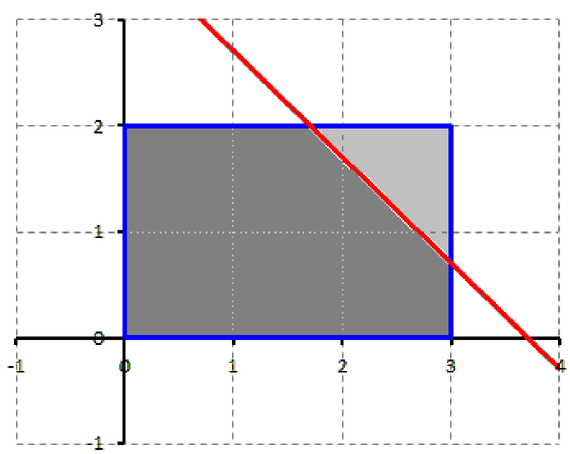
Case 1: $0 < w < 2$.

$$F_W(w) = \int_0^w \int_0^{w-y} \frac{1}{3} \cdot \frac{y}{2}\, dx\, dy = \ldots$$



Case 2: $2 < w < 3$.

$$F_W(w) = \int_0^2 \int_0^{w-y} \frac{1}{3} \cdot \frac{y}{2}\, dx\, dy = \ldots$$



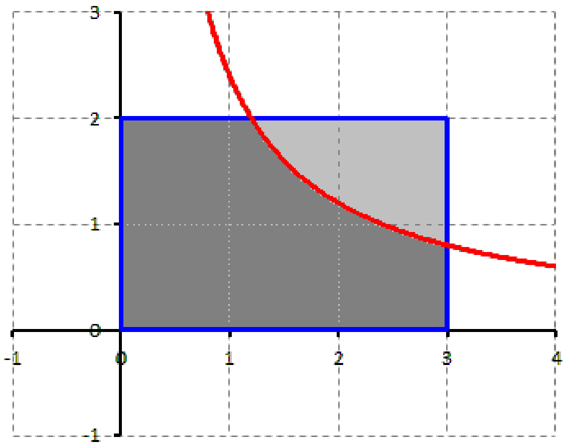
Case 3: $3 < w < 5$.

$$F_W(w) = 1 - \int_{w-3}^2 \int_{w-y}^3 \frac{1}{3} \cdot \frac{y}{2}\, dx\, dy = \ldots$$

$$f_W(w) = F_W'(w) = \ldots$$

b) Find the probability density function of $V = X \times Y$, $f_V(v) = f_{X \times Y}(v)$.

$$F_V(v) = P(V \le v) = P(XY \le v)$$



$$= 1 - \int_{v/3}^{2} \int_{v/y}^{3} \frac{1}{3} \cdot \frac{y}{2}\, dx\, dy = 1 - \int_{v/3}^{2} \frac{y}{6} \cdot \left(3 - \frac{v}{y}\right) dy = 1 - \int_{v/3}^{2} \left(\frac{y}{2} - \frac{v}{6}\right) dy$$

$$= 1 - \left(\frac{y^2}{4} - \frac{v\,y}{6}\right)\Bigg|_{v/3}^{2} = 1 - \left(1 - \frac{v}{3} - \frac{v^2}{36} + \frac{v^2}{18}\right) = \frac{v}{3} - \frac{v^2}{36},$$

$0 < v < 6.$

$$f_V(v) = F_V'(v) = \frac{1}{3} - \frac{v}{18}, \qquad 0 < v < 6.$$

OR

$$f_V(v) = \int_{-\infty}^{\infty} f\left(x, \frac{v}{x}\right) \frac{1}{|x|} dx$$

$0 < x < 3$

$0 < y < 2 \qquad \Rightarrow \qquad 0 < \frac{v}{x} < 2 \qquad \Rightarrow \qquad x > \frac{v}{2}$

$$f_V(v) = \int_{-\infty}^{\infty} f\left(x, \frac{v}{x}\right) \frac{1}{|x|} dx = \int_{v/2}^{3} \frac{v}{6x^2} dx = \left(-\frac{v}{6x}\right)\Bigg|_{v/2}^{3} = \frac{1}{3} - \frac{v}{18}, \qquad 0 < v < 6.$$

OR

$$f_V(v) = \int_{-\infty}^{\infty} f\left(\frac{v}{y}, y\right) \frac{1}{|y|} dy$$

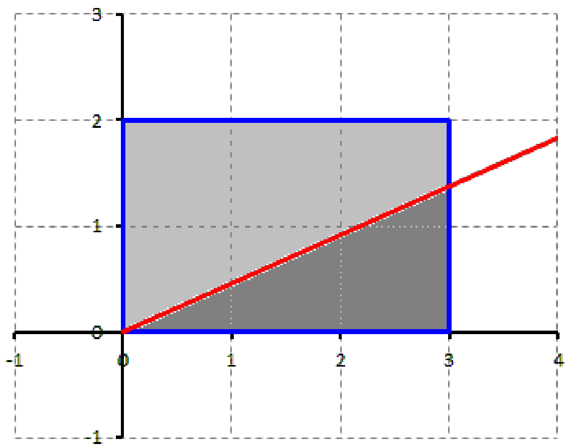
$0 < x < 3 \qquad \Rightarrow \qquad 0 < \frac{v}{y} < 3 \qquad \Rightarrow \qquad y > \frac{v}{3}$

$0 < y < 2$

$$f_V(v) = \int_{-\infty}^{\infty} f\left(\frac{v}{y}, y\right) \frac{1}{|y|} dy = \int_{v/3}^{2} \frac{1}{6} dx = \frac{1}{3} - \frac{v}{18}, \qquad 0 < v < 6.$$

c) Find the probability density function of $U = {}^{Y}\!/_{X}$, $f_U(u) = f_{Y/X}(u)$.

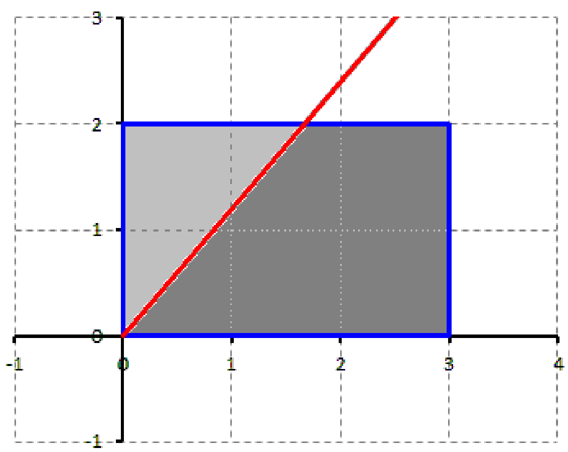
$$F_U(u) = P(U \le u) = P(Y \le uX) = \ldots$$



Case 1: $0 < u < \dfrac{2}{3}$.

$$\ldots = \int_0^3 \int_0^{ux} \frac{1}{3} \cdot \frac{y}{2}\, dy\, dx$$

$$= \int_0^3 \frac{u^2 x^2}{12}\, dx = \frac{3}{4} u^2 .$$



Case 2: $u > \dfrac{2}{3}$.

$$\ldots = 1 - \int_0^2 \int_0^{y/u} \frac{1}{3} \cdot \frac{y}{2}\, dx\, dy$$

$$= 1 - \int_0^2 \frac{y^2}{6u}\, dy = 1 - \frac{8}{18u} .$$

$$f_U(u) = F_U'(u) = \begin{cases} \dfrac{3}{2} u & 0 < w < \dfrac{2}{3} \\ \dfrac{8}{18u^2} & w > \dfrac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

OR

$$f_U(u) = \int_{-\infty}^{\infty} f(x, xu)\,|x|\,dx$$

$0 < x < 3$

$0 < y < 2 \quad \Rightarrow \quad 0 < x\,u < 2 \quad \Rightarrow \quad 0 < x < \dfrac{2}{u}$

Case 1: $\quad 0 < u < \dfrac{2}{3}$. $\quad$ Then $3 < \dfrac{2}{u}$.

$$f_U(u) = \int_{-\infty}^{\infty} f(x, xu)\,|x|\,dx = \int_0^3 \frac{u\,x^2}{6}\,dx = \frac{3}{2}u, \qquad 0 < u < \frac{2}{3}.$$

Case 2: $\quad u > \dfrac{2}{3}$. $\quad$ Then $3 > \dfrac{2}{u}$.

$$f_U(u) = \int_{-\infty}^{\infty} f(x, xu)\,|x|\,dx = \int_0^{2/u} \frac{u\,x^2}{6}\,dx = \frac{8}{18\,u^2}, \qquad u > \frac{2}{3}.$$

4. Let the joint pdf of X and Y be given by

$$f(x,y) = \begin{cases} \dfrac{2}{(1+x+y)^3} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

a) Let W = X + Y. Find the p.d.f. of W, $f_W(w) = f_{X+Y}(w)$.

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$x > 0$

$y > 0 \quad \Rightarrow \quad w - x > 0 \quad \Rightarrow \quad x < w$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx = \int_0^w \frac{2}{(1+w)^3}\, dx = \frac{2w}{(1+w)^3}, \qquad w > 0.$$

OR

$$F_W(w) = P(W \le w) = P(X + Y \le w) = \int_0^w \left(\int_0^{w-x} \frac{2}{(1+x+y)^3}\, dy \right) dx$$

$$= \int_0^w \left(\frac{1}{(1+x)^2} - \frac{1}{(1+w)^2} \right) dx = 1 - \frac{1}{(1+w)} - \frac{w}{(1+w)^2} = \frac{w^2}{(1+w)^2},$$

$w > 0.$

$$f_W(w) = F_W'(w) = \frac{2w}{(1+w)^3}, \qquad w > 0.$$

b) Let $V = Y/X$. Find the p.d.f. of V, $f_V(v)$.

$$F_V(v) = P(V \le v) = P(Y/X \le v) = P(Y \le vX) = 1 - P(Y > vX)$$

$$= 1 - \int_0^{\infty}\left(\int_{vx}^{\infty} \frac{2}{(1+x+y)^3}\,dy\right)dx = 1 - \int_0^{\infty} \frac{1}{(1+x+vx)^2}\,dx$$

$$= 1 - \frac{1}{v+1}, \qquad v > 0.$$

OR

$$F_V(v) = P(V \le v) = P(Y/X \le v) = P(X \ge Y/v)$$

$$= \int_0^{\infty}\left(\int_{y/v}^{\infty} \frac{2}{(1+x+y)^3}\,dx\right)dy = \int_0^{\infty} \frac{1}{\left(1+\frac{y}{v}+y\right)^2}\,dy$$

$$= \int_0^{\infty} \frac{1}{\left(1+\frac{v+1}{v}\,y\right)^2}\,dy = \frac{v}{v+1}, \qquad v > 0.$$

$$f_V(v) = F_V'(v) = \frac{1}{(1+v)^2}, \qquad w > 0.$$

5. **2.2.2** (7th and 6th edition)

Let X_1 and X_2 have the joint pmf $p_{X_1,X_2}(x_1,x_2) = x_1 x_2/36$, $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$, zero elsewhere. Find first the joint pmf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$, and then find the marginal pmf of Y_1.

Hint: X_1 and X_2 are discrete random variables. There are nine possible pairs (x_1, x_2).

$p_{X_1,X_2}(x_1,x_2) = x_1 x_2/36$, $\qquad x_1 = 1, 2, 3, \quad x_2 = 1, 2, 3.$

$Y_1 = X_1 X_2 \qquad\qquad Y_2 = X_2.$

x_2	x_1		
	1	2	3
1	$p_{X_1,X_2}(x_1,x_2) = 1/36$ $y_1 = 1 \quad y_2 = 1$	$p_{X_1,X_2}(x_1,x_2) = 2/36$ $y_1 = 2 \quad y_2 = 1$	$p_{X_1,X_2}(x_1,x_2) = 3/36$ $y_1 = 3 \quad y_2 = 1$
2	$p_{X_1,X_2}(x_1,x_2) = 2/36$ $y_1 = 2 \quad y_2 = 2$	$p_{X_1,X_2}(x_1,x_2) = 4/36$ $y_1 = 4 \quad y_2 = 2$	$p_{X_1,X_2}(x_1,x_2) = 6/36$ $y_1 = 6 \quad y_2 = 2$
3	$p_{X_1,X_2}(x_1,x_2) = 3/36$ $y_1 = 3 \quad y_2 = 3$	$p_{X_1,X_2}(x_1,x_2) = 6/36$ $y_1 = 6 \quad y_2 = 3$	$p_{X_1,X_2}(x_1,x_2) = 9/36$ $y_1 = 9 \quad y_2 = 3$

$p_{Y_1,Y_2}(y_1,y_2)$:

y_1	y_2			
	1	2	3	$p_{Y_1}(y_1)$
1	$1/36$	0	0	$1/36$
2	$2/36$	$2/36$	0	$4/36$
3	$3/36$	0	$3/36$	$6/36$
4	0	$4/36$	0	$4/36$
6	0	$6/36$	$6/36$	$12/36$
9	0	0	$9/36$	$9/36$

6. Let X_1 and X_2 have the joint pmf $p_{X_1,X_2}(x_1,x_2) = {}^{x_1 x_2}\!/_{36}$, $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$, zero elsewhere. Find the probability distribution of $W = X_1 + X_2$.

x_2	x_1					
	1		2		3	
1	$^1/_{36}$	$w = 2$	$^2/_{36}$	$w = 3$	$^3/_{36}$	$w = 4$
2	$^2/_{36}$	$w = 3$	$^4/_{36}$	$w = 4$	$^6/_{36}$	$w = 5$
3	$^3/_{36}$	$w = 4$	$^6/_{36}$	$w = 5$	$^9/_{36}$	$w = 6$

w	$p_W(w)$
2	$^1/_{36}$
3	$^4/_{36}$
4	$^{10}/_{36}$
5	$^{12}/_{36}$
6	$^9/_{36}$