STAT 410

Homework #8

(due Friday, October 30, by 5:00 p.m. CDT)

Fall 2020
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No credit will be given without supporting work.

7. Let $\psi > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\psi) = \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall: $W = X^2$ has $\text{Gamma}(\alpha = \frac{1}{2}, \theta = \psi)$ distribution.

d) Recall that the maximum likelihood estimator for ψ is $\hat{\psi} = \frac{2}{n}\sum_{i=1}^{n} X_i^2 = 2\,\overline{X^2}$.

Help a pudding-brain lazy CourseHero worshiper determine if $\hat{\psi}$ an unbiased estimator of ψ? If $\hat{\psi}$ is not an unbiased estimator of ψ, help a pudding-brain lazy CourseHero worshiper construct an unbiased estimator of ψ based on $\hat{\psi}$.

"Hint": You need $E(X^2)$. Do NOT try to obtain it directly: $\int_0^{\infty} x^2 \cdot \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}\, dx$.

Use the fact that we know the probability distribution of $W = X^2$.
IF you insist on using this integral, instead of "fighting" it, compare it with the integral representing the variance of a random variable that is $N(0, \sigma^2)$, where $\psi = 2\sigma^2$.

$$\hat{\psi} = 2\,\overline{X^2} = 2\,\overline{W}.$$

$$E(\hat{\psi}) = E(2\overline{W}) = 2E(\overline{W}) = 2\mu_W = 2E(W) = 2(\alpha\theta) = 2(\tfrac{1}{2}\psi) = \psi.$$

$\hat{\psi}$ is an unbiased estimator of ψ.

OR

$$Y = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} W_i \quad \text{has a Gamma}\left(\alpha = \frac{n}{2}, \theta = \psi \right) \text{ distribution.}$$

$$E(Y) = \alpha\,\theta = \frac{n}{2}\,\psi.$$

or

$$E(Y) = E(Y^1) = \ldots \qquad\qquad \text{(unnecessarily complicated)}$$

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

$$\ldots = \frac{\psi^1\,\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\psi^1\,\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{n}{2}\,\psi.$$

$$\hat{\psi} = \frac{2}{n}\sum_{i=1}^{n} X_i^2 = \frac{2}{n}\,Y.$$

$$E(\hat{\psi}) = \frac{2}{n}\,E(Y) = \frac{2}{n}\cdot\frac{n}{2}\,\psi = \psi.$$

$\hat{\psi}$ is an unbiased estimator of ψ.

OR

$$\mathrm{E}(\mathrm{X}^2) = \int_0^\infty x^2 \cdot \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}\, dx.$$

Consider $\mathrm{V} \sim \mathrm{N}(0, \sigma^2)$, where $\psi = 2\sigma^2$ and $\sigma^2 = \frac{\psi}{2}$.

Then (since $\mathrm{E}(\mathrm{V}) = 0$ and the p.d.f. of V is symmetric about 0)

$$\frac{\psi}{2} = \sigma^2 = \mathrm{Var}(\mathrm{V}) = \mathrm{E}(\mathrm{V}^2) - [\mathrm{E}(\mathrm{V})]^2 = \mathrm{E}(\mathrm{V}^2)$$

$$= \int_{-\infty}^\infty x^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/(2\sigma^2)}\, dx$$

$$= \int_0^\infty x^2 \cdot \frac{2}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/(2\sigma^2)}\, dx$$

$$= \int_0^\infty x^2 \cdot \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}\, dx.$$

$$\Rightarrow \qquad \mathrm{E}(\mathrm{X}^2) = \frac{\psi}{2}.$$

$$\mathrm{E}(\hat{\psi}) = \mathrm{E}(2\,\overline{\mathrm{X}^2}) = 2\,\mathrm{E}(\overline{\mathrm{X}^2}) = 2\,\mathrm{E}(\mathrm{X}^2) = 2\,\frac{\psi}{2} = \psi.$$

$\hat{\psi}$ is an unbiased estimator of ψ.

OR

$$\mathrm{E}(\mathrm{X}^2) = \int_0^\infty x^2 \cdot \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}\, dx$$

$$u = \frac{x^2}{\psi} \qquad x^2 = \psi u$$

$$x = \sqrt{\psi u} \qquad dx = \frac{\sqrt{\psi}}{2\sqrt{u}}\, du$$

$$= \int_0^\infty \psi u \cdot \frac{2}{\sqrt{\pi\psi}}\, e^{-u}\, \frac{\sqrt{\psi}}{2\sqrt{u}}\, du$$

$$= \frac{\psi}{\sqrt{\pi}} \int_0^{\infty} \sqrt{u}\, e^{-u}\, du \;=\; \frac{\psi}{\sqrt{\pi}}\, \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\psi}{\sqrt{\pi}} \cdot \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) \;=\; \frac{\psi}{\sqrt{\pi}} \cdot \frac{1}{2} \sqrt{\pi} \;=\; \frac{\psi}{2}.$$

$$\mathrm{E}(\hat{\psi}) \;=\; \mathrm{E}(2\,\overline{\mathrm{X}^2}) \;=\; 2\,\mathrm{E}(\overline{\mathrm{X}^2}) \;=\; 2\,\mathrm{E}(\mathrm{X}^2) \;=\; 2\,\frac{\psi}{2} \;=\; \psi.$$

$\hat{\psi}$ is an unbiased estimator of ψ.

e) Help a pudding-brain lazy CourseHero worshiper find $\mathrm{MSE}(\hat{\psi}) = (\mathrm{bias}(\hat{\psi}))^2 + \mathrm{Var}(\hat{\psi})$.

$\mathrm{bias}(\hat{\psi}) \;=\; 0.$

$$\mathrm{Var}(\hat{\psi}) \;=\; \mathrm{Var}(2\,\overline{\mathrm{W}}) \;=\; 4\,\mathrm{Var}(\overline{\mathrm{W}}) \;=\; 4\,\frac{\sigma_{\mathrm{W}}^2}{n}$$

$$= \frac{4}{n}(\alpha\,\theta^2) \;=\; \frac{4}{n}\left(\frac{1}{2}\,\psi^2\right) \;=\; \frac{2\,\psi^2}{n}.$$

$$\mathrm{MSE}(\hat{\psi}) \;=\; (\mathrm{bias}(\hat{\psi}))^2 + \mathrm{Var}(\hat{\psi}) \;=\; \mathrm{Var}(\hat{\psi}) \;=\; \frac{2\,\psi^2}{n}.$$

OR

$$\hat{\psi} \;=\; \frac{2}{n} \sum_{i=1}^{n} \mathrm{X}_i^2 \;=\; \frac{2}{n}\,\mathrm{Y}.$$

$$\mathrm{Var}(\mathrm{Y}) \;=\; \alpha\,\theta^2 \;=\; \frac{n}{2}\,\psi^2.$$

or

$$\mathrm{E}(\mathrm{Y}^2) = \frac{\psi^2\,\Gamma\left(\frac{n}{2}+2\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\psi^2\left(\frac{n}{2}+1\right)\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \psi^2\left(\frac{n}{2}+1\right)\frac{n}{2}.$$

$$\mathrm{Var}(\mathrm{Y}) = \mathrm{E}(\mathrm{Y}^2) - [\mathrm{E}(\mathrm{Y})]^2 = \psi^2\left(\frac{n}{2}+1\right)\frac{n}{2} - \left(\frac{n}{2}\,\psi\right)^2 = \frac{n}{2}\,\psi^2.$$

(unnecessarily complicated)

$$\mathrm{Var}(\hat{\psi}) = \left(\frac{2}{n}\right)^2 \mathrm{Var}(\mathrm{Y}) = \left(\frac{2}{n}\right)^2 \cdot \frac{n}{2}\,\psi^2 = \frac{2\,\psi^2}{n}.$$

$$\mathrm{MSE}(\hat{\psi}) = (\mathrm{bias}(\hat{\psi}))^2 + \mathrm{Var}(\hat{\psi}) = \mathrm{Var}(\hat{\psi}) = \frac{2\,\psi^2}{n}.$$

For fun:

$$\mathrm{E}(\mathrm{X}^k) = \mathrm{E}(\mathrm{W}^{k/2}) = \frac{\psi^{k/2}\,\Gamma\left(\frac{1}{2}+\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}, \qquad k > -\frac{1}{2}.$$

"Hint": $\mathrm{E}(\overline{\mathrm{V}}) = \mu_{\mathrm{V}} = \mathrm{E}(\mathrm{V}).$ $\qquad \mathrm{Var}(\overline{\mathrm{V}}) = \frac{\sigma_{\mathrm{V}}^2}{n} = \frac{\mathrm{Var}(\mathrm{V})}{n}.$

$\mathrm{Var}(\mathrm{V}) = \mathrm{E}(\mathrm{V}^2) - [\mathrm{E}(\mathrm{V})]^2.$

$\mathrm{E}(a\,☺) = a\,\mathrm{E}(☺).$ $\qquad \mathrm{Var}(a\,☺) = a^2\,\mathrm{Var}(☺).$

f) Recall that a method of moments estimator for ψ is $\tilde{\psi} = \pi\left(\overline{X}\right)^2$.

Help a pudding-brain lazy CourseHero worshiper determine if $\tilde{\psi}$ an unbiased estimator of ψ? If $\tilde{\psi}$ is not an unbiased estimator of ψ, help a pudding-brain lazy CourseHero worshiper construct an unbiased estimator of ψ based on $\tilde{\psi}$.

"Hint": $E[(\overline{X})^2] = \operatorname{Var}(\overline{X}) + [E(\overline{X})]^2 = \dfrac{\sigma^2}{n} + \mu^2$.

$$\mu = E(X) = \frac{\sqrt{\psi}}{\sqrt{\pi}}. \qquad\qquad E(X^2) = \frac{\psi}{2}.$$

$$\sigma^2 = \operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \frac{\psi}{2} - \frac{\psi}{\pi} = \psi\left(\frac{\pi-2}{2\pi}\right).$$

$$E[(\overline{X})^2] = \frac{\sigma^2}{n} + \mu^2 = \frac{\psi}{n}\left(\frac{\pi-2}{2\pi}\right) + \frac{\psi}{\pi} = \psi\left(\frac{2n+\pi-2}{2\pi n}\right).$$

$$E(\tilde{\psi}) = E[\pi\left(\overline{X}\right)^2] = \pi E[(\overline{X})^2]$$

$$= \psi\left(\frac{2n+\pi-2}{2n}\right) = \psi + \psi\left(\frac{\pi-2}{2n}\right) \neq \psi.$$

$\tilde{\psi}$ is NOT an unbiased estimator of ψ.

$$\operatorname{bias}(\tilde{\psi}) = E(\tilde{\psi}) - \psi = \psi\left(\frac{\pi-2}{2n}\right).$$

Consider $\tilde{\tilde{\psi}} = \dfrac{2n}{2n+\pi-2}\cdot\tilde{\psi} = \dfrac{2\pi n}{2n+\pi-2}\left(\overline{X}\right)^2$.

Then $E(\tilde{\tilde{\psi}}) = \dfrac{2n}{2n+\pi-2}\cdot E(\tilde{\psi}) = \psi$.

$\tilde{\tilde{\psi}}$ is an unbiased estimator of ψ.

"OR"

$\tilde{\psi} = \pi \left(\overline{\mathrm{X}} \right)^2.$

Consider $g(x) = \pi x^2.$

Then $g(\overline{\mathrm{X}}) = \tilde{\psi}.$

$g'(x) = 2 \pi x.$

$g''(x) = 2 \pi > 0$ for $x > 0.$

Since $g(x) = \pi x^2,$ $x > 0,$ is strictly convex (that is, it curves up), and $\overline{\mathrm{X}}$ is not a constant random variable, by Jensen's Inequality (Theorem 1.10.5),

$$\mathrm{E}(\tilde{\psi}) = \mathrm{E}(g(\overline{\mathrm{X}})) > g(\mathrm{E}(\overline{\mathrm{X}})) = g(\mu).$$

$$g(\mu) = g\left(\frac{\sqrt{\psi}}{\sqrt{\pi}} \right) = \psi.$$

$\Rightarrow \quad \mathrm{E}(\tilde{\psi}) > \psi.$

$\tilde{\psi}$ is NOT an unbiased estimator for ψ.

On average, $\tilde{\psi}$ overestimates ψ.

However, this does not help us help us find an unbiased estimator for ψ. ☹

For fun: (fair game for the exam)

g) Is $\hat{\psi}$ a consistent estimator of ψ? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

$$\hat{\psi} = 2\,\overline{X^2} = 2\,\overline{W}.$$

By WLLN, $\overline{W} \xrightarrow{P} E(W) = \alpha\theta = \frac{\psi}{2}$.

$$\Rightarrow \quad \hat{\psi} = 2\,\overline{W} \xrightarrow{P} 2\,\frac{\psi}{2} = \psi.$$

$\hat{\psi}$ is a consistent estimator of ψ.

h) Is $\tilde{\psi}$ a consistent estimator of ψ? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

By WLLN, $\overline{X} \xrightarrow{P} E(X) = \frac{\sqrt{\psi}}{\sqrt{\pi}}$.

$$\spadesuit \xrightarrow{P} a, \; g \text{ is continuous at } a \;\Rightarrow\; g(\spadesuit) \xrightarrow{P} g(a)$$

Consider $g(x) = \pi x^2$. Then $g(\overline{X}) = \tilde{\psi}$.

Since $g(x) = \pi x^2$ is continuous at $\frac{\sqrt{\psi}}{\sqrt{\pi}}$,

$$\tilde{\psi} = \pi\left(\overline{X}\right)^2 = g(\overline{X}) \xrightarrow{P} g\left(\frac{\sqrt{\psi}}{\sqrt{\pi}}\right) = \pi\left(\frac{\sqrt{\psi}}{\sqrt{\pi}}\right)^2 = \psi.$$

$\tilde{\psi}$ is a consistent estimator of ψ.

8. Let $\beta > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\beta) = \beta\,(1-x)^{\beta-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

Recall: $W = -\ln(1-X)$ has an $\text{Exponential}\left(\theta = \frac{1}{\beta}\right)$

$= \text{Gamma}\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

e) Recall that the maximum likelihood estimator for β is $\hat{\beta} = \dfrac{n}{\sum_{i=1}^{n}\left(-\ln(1-X_i)\right)}$.

Help a pudding-brain lazy CourseHero worshiper determine if $\hat{\beta}$ an unbiased estimator of β? If $\hat{\beta}$ is not an unbiased estimator of β, help a pudding-brain lazy CourseHero worshiper construct an unbiased estimator of β based on $\hat{\beta}$.

"Hint" 0: If U has a $\text{Gamma}(\alpha_1, \theta)$ distribution, V has a $\text{Gamma}(\alpha_2, \theta)$ distribution, U and V are independent, then U + V has a $\text{Gamma}(\alpha_1 + \alpha_2, \theta)$ distribution.

"Hint" 1: $E(a ☺) = a\,E(☺)$.

"Hint" 2: $\dfrac{1}{♥} = ♥^{-1}$.

"Hint" 3: If T_α has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

$Y = \sum_{i=1}^{n}\left(-\ln(1-X_i)\right) = \sum_{i=1}^{n} W_i$ has a $\text{Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right)$ distribution.

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n}\left(-\ln(1-X_i)\right)} = \frac{n}{Y}. \qquad a = n, \quad ☺ = \frac{1}{Y}, \quad ♥ = Y.$$

$$\mathrm{E}\left(\frac{1}{\mathrm{Y}}\right) = \mathrm{E}\left(\mathrm{Y}^{-1}\right) = \frac{\Gamma(n-1)}{\beta^{-1}\,\Gamma(n)} = \frac{\beta}{n-1}.$$

$$\mathrm{E}(\hat{\beta}) = \mathrm{E}\left(\frac{n}{\mathrm{Y}}\right) = n\,\mathrm{E}\left(\frac{1}{\mathrm{Y}}\right) = n\cdot\frac{\beta}{n-1} = \frac{n}{n-1}\cdot\beta = \beta+\frac{\beta}{n-1} \neq \beta.$$

$\hat{\beta}$ is NOT an unbiased estimator of β. $\quad\quad$ $\operatorname{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta})-\beta = \dfrac{\beta}{n-1}.$

Consider $\quad \hat{\hat{\beta}} = \dfrac{n-1}{n}\cdot\hat{\beta} = \dfrac{n-1}{\sum\limits_{i=1}^{n}\left(-\ln(1-\mathrm{X}_i)\right)}.$

Then $\quad \mathrm{E}(\hat{\hat{\beta}}) = \dfrac{n-1}{n}\cdot\mathrm{E}(\hat{\beta}) = \beta.$ $\quad\quad$ $\hat{\hat{\beta}}$ is an unbiased estimator of β.

f) Help a pudding-brain lazy CourseHero worshiper find $\operatorname{MSE}(\hat{\beta}) = (\operatorname{bias}(\hat{\beta}))^2+\operatorname{Var}(\hat{\beta})$.

"Hint" 1: $\operatorname{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta})-\beta.$ $\quad\quad$ You have $\mathrm{E}(\hat{\beta})$ from part (e).

"Hint" 2: $\operatorname{Var}(a\,☺) = a^2\operatorname{Var}(☺).$ $\quad\quad$ $\operatorname{Var}(☺) = \mathrm{E}(☺^2)-[\mathrm{E}(☺)]^2.$

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m>-\alpha.$$

$$\operatorname{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta})-\beta = \frac{\beta}{n-1}.$$

$$\mathrm{E}\left(\frac{1}{\mathrm{Y}^2}\right) = \mathrm{E}\left(\mathrm{Y}^{-2}\right) = \frac{\Gamma(n-2)}{\beta^{-2}\,\Gamma(n)} = \frac{\beta^2}{(n-1)\,(n-2)}.$$

$$\operatorname{Var}(\hat\beta) = \operatorname{Var}\left(\frac{n}{\mathrm{Y}}\right) = n^2 \operatorname{Var}\left(\frac{1}{\mathrm{Y}}\right) = n^2\left[\frac{\beta^2}{(n-1)(n-2)} - \frac{\beta^2}{(n-1)^2}\right]$$

$$= \frac{n^2\beta^2}{(n-1)^2(n-2)}.$$

$$\operatorname{MSE}(\hat\beta) = (\operatorname{bias}(\hat\beta))^2 + \operatorname{Var}(\hat\beta) = \frac{\beta^2}{(n-1)^2} + \frac{n^2\beta^2}{(n-1)^2(n-2)}$$

$$= \frac{\left(n^2+n-2\right)\beta^2}{(n-1)^2\ (n-2)} = \frac{(n+2)\beta^2}{(n-1)(n-2)}.$$

g) Recall that a method of moments estimator for β is $\tilde\beta = \frac{1}{\overline{\mathrm{X}}} - 1$.

Help a pudding-brain lazy CourseHero worshiper determine if $\tilde\beta$ an unbiased estimator of β? If $\tilde\beta$ is not an unbiased estimator of β, help a pudding-brain lazy CourseHero worshiper determine if $\tilde\beta$ underestimates or overestimates β (on average).

Hint: $\tilde\beta = g(\overline{\mathrm{X}})$. Is $g(x)$ a linear function?

If it is not a linear function, does it curve up or down?

$$\tilde\beta = \frac{1}{\overline{\mathrm{X}}} - 1.$$

Consider $g(x) = \frac{1}{x} - 1$.

Then $g(\overline{\mathrm{X}}) = \tilde\beta$.

$$g'(x) = -\frac{1}{x^2}.$$

$$g''(x) = \frac{2}{x^3} > 0 \qquad \text{for } 0 < x < 1.$$

Since $g(x) = \frac{1}{x} - 1$, $0 < x < 1$, is strictly convex (that is, it curves up),

and $\overline{X}$ is not a constant random variable, by Jensen's Inequality (Theorem 1.10.5),

$$E(\tilde{\beta}) = E(g(\overline{X})) > g(E(\overline{X})) = g(\mu).$$

$$g(\mu) = g\left(\frac{1}{\beta+1}\right) = \frac{1}{\frac{1}{\beta+1}} - 1 = \beta.$$

$$\Rightarrow \qquad E(\tilde{\beta}) > \beta.$$

$\tilde{\beta}$ is NOT an unbiased estimator for β.

On average, $\tilde{\beta}$ **overestimates** β.

For fun: (fair game for the exam)

h) Is $\hat{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n}\left(-\ln\left(1-X_i\right)\right)} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left(-\ln\left(1-X_i\right)\right)} = \frac{1}{\overline{\overline{W}}}.$$

By WLLN, $\overline{W} \xrightarrow{P} E(W) = \alpha\theta = \frac{1}{\beta}$.

$$\spadesuit \xrightarrow{P} a,\ g \text{ is continuous at } a \Rightarrow g(\spadesuit) \xrightarrow{P} g(a)$$

Consider $g(x) = \frac{1}{x}$. Then $g(\overline{W}) = \hat{\beta}$.

Since $g(x) = \frac{1}{x}$ is continuous at $\frac{1}{\beta}$,

$$\hat{\beta} = \frac{1}{\overline{\overline{W}}} = g(\overline{W}) \xrightarrow{P} g\left(\frac{1}{\beta}\right) = \frac{1}{\frac{1}{\beta}} = \beta.$$

$\hat{\beta}$ is a consistent estimator of β.

i) Is $\tilde{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

By WLLN, $\quad \overline{X} \xrightarrow{P} E(X) = \dfrac{1}{\beta+1}.$

$\spadesuit \xrightarrow{P} a$, g is continuous at a $\Rightarrow$ $g(\spadesuit) \xrightarrow{P} g(a)$

Consider $g(x) = \dfrac{1}{x} - 1.$ $\qquad$ Then $g(\overline{X}) = \tilde{\beta}.$

Since $g(x) = \dfrac{1}{x} - 1$ is continuous at $\dfrac{1}{\beta+1}$,

$$\tilde{\beta} = \frac{1}{\overline{X}} - 1 = g(\overline{X}) \xrightarrow{P} g\left(\frac{1}{\beta+1}\right) = \frac{1}{\frac{1}{\beta+1}} - 1 = \beta.$$

$\tilde{\beta}$ is a consistent estimator of β.