STAT 410 | **Examples for 09/09/2020 (Disc)** | Fall 2020
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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let the joint probability density function of X and Y be defined by

$$f(x,y) = \frac{x+4y}{C}, \qquad 0<y<1, \quad y<x<3, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y). That is, sketch $\{\, 0<y<1, \;\; y<x<3 \,\}$.

b) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

"Hint": $$f_X(x) = \begin{cases} \clubsuit(x) & 0<x<1 \\ \spadesuit(x) & 1<x<3 \end{cases}$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Are X and Y independent? *Justify your answer.*
If X and Y are not independent, find $\mathrm{Cov}(X, Y)$.

2. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C x^2 y, \qquad 0 < x < 4, \quad 0 < y < \sqrt{x}, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0 < x < 4, \quad 0 < y < \sqrt{x} \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Are X and Y independent? *Justify your answer.*
If X and Y are not independent, find $\text{Cov}(X, Y)$.

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} C x y^3 & 0 < x < 1, \ 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

b) Find $P(5Y > 4X)$.

c) Find $P(X + Y > 1)$.

d) Find the probability $P(X \cdot Y < 0.729)$.

e) Find the marginal probability density function of X, $f_X(x)$.

f) Find the marginal probability density function of Y, $f_Y(y)$.

g) Find $E(X)$, $E(Y)$, $E(X \cdot Y)$.

h) Find $\text{Cov}(X, Y)$.

4. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{C}{(2x+y)^3}, \qquad y > 1, \quad 0 < x < y, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ y > 1, \ 0 < x < y \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Let $1 < w < 2$. Find $P(X + Y \le w)$.

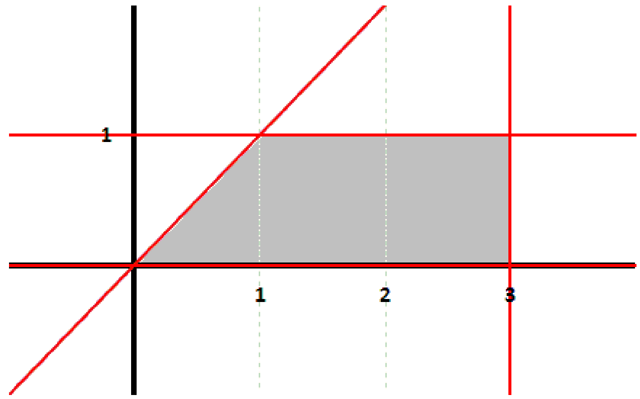
f) Let $w > 2$. Find $P(X + Y \le w)$.

g) Let $0 < u < 1$. Find $P(\frac{X}{Y} \le u)$.

1. Let the joint probability density function of X and Y be defined by

$$f(x,y) = \frac{x+4y}{C}, \qquad 0<y<1, \quad y<x<3, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y).
That is, sketch
$\{\, 0<y<1, \; y<x<3 \,\}$.



b) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^1 \left(\int_y^3 \frac{x+4y}{C}\, dx \right) dy = \frac{1}{C} \int_0^1 \left(\frac{x^2}{2} + 4xy \right) \Bigg|_{x=y}^{x=3} dy$$

$$= \frac{1}{C} \int_0^1 \frac{9+24y-9y^2}{2}\, dy = \frac{1}{C} \cdot \frac{9y+12y^2-3y^3}{2} \Bigg|_0^1 = \frac{9}{C} = 1.$$

OR

$$1 = \int_0^1 \left(\int_0^x \frac{x+4y}{C}\, dy \right) dx + \int_1^3 \left(\int_0^1 \frac{x+4y}{C}\, dy \right) dx$$

$$= \int_0^1 \left(\frac{xy+2y^2}{C} \right) \Bigg|_{y=0}^{y=x} dx + \int_1^3 \left(\frac{xy+2y^2}{C} \right) \Bigg|_{y=0}^{y=1} dx$$

$$= \int_0^1 \frac{3x^2}{C}\, dx + \int_1^3 \frac{x+2}{C}\, dx = \frac{1}{C} \cdot x^3 \Bigg|_0^1 + \frac{1}{C} \cdot \frac{x^2+4x}{2} \Bigg|_1^3$$

$$= \frac{1}{C} + \frac{8}{C} = \frac{9}{C} = 1.$$

$\Rightarrow \qquad C = \mathbf{9}.$

c) Find the marginal probability density function of X, $f_X(x)$.

"Hint": $$f_X(x) = \begin{cases} \clubsuit(x) & 0 < x < 1 \\ \spadesuit(x) & 1 < x < 3 \end{cases}$$

For $0 < x < 1$,

$$f_X(x) = \int_0^x \frac{x+4y}{9}\,dy = \left(\frac{xy+2y^2}{9} \right) \Bigg|_{y=0}^{y=x} = \frac{x^2}{3}, \qquad 0 < x < 1.$$

For $1 < x < 3$,

$$f_X(x) = \int_0^1 \frac{x+4y}{9}\,dy = \left(\frac{xy+2y^2}{9} \right) \Bigg|_{y=0}^{y=1} = \frac{x+2}{9}, \qquad 1 < x < 3.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_y^3 \frac{x+4y}{9}\,dx = \frac{1}{9}\left(\frac{x^2}{2} + 4xy \right) \Bigg|_{x=y}^{x=3}$$

$$= \frac{3+8y-3y^2}{6} = \frac{1}{2} + \frac{4}{3}y - \frac{1}{2}y^2, \qquad 0 < y < 1.$$

e) Are X and Y independent? *Justify your answer.*
If X and Y are not independent, find $\operatorname{Cov}(X, Y)$.

$f(x,y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$\mathrm{E}(\mathrm{X}) = \int_0^1 x \cdot \frac{x^2}{3}\, dx + \int_1^3 x \cdot \frac{x+2}{9}\, dx = \frac{1}{12} + \frac{50}{27} = \frac{209}{108}.$$

$$\mathrm{E}(\mathrm{Y}) = \int_0^1 y \cdot \left(\frac{1}{2} + \frac{4}{3} y - \frac{1}{2} y^2 \right) dy = \frac{41}{72}.$$

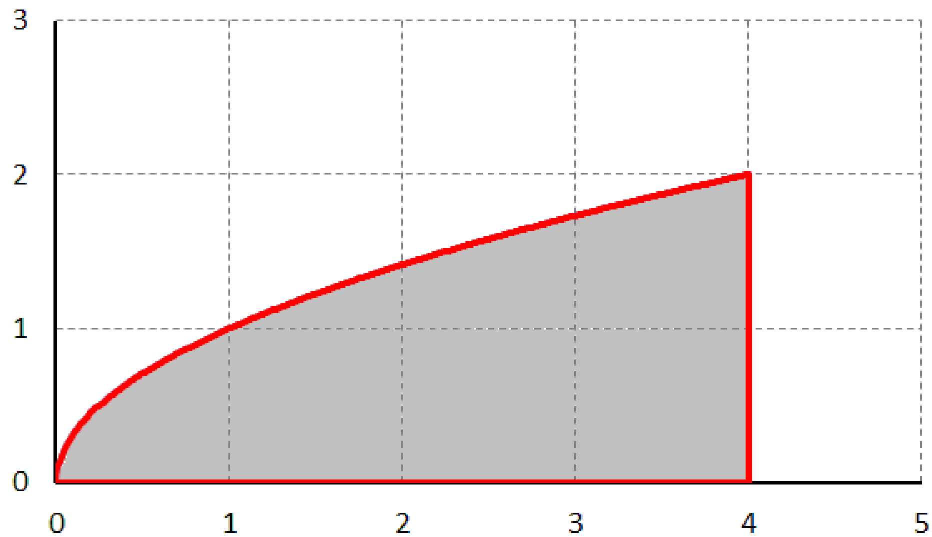
$$\mathrm{E}(\mathrm{XY}) = \int_0^1 \left(\int_y^3 xy \cdot \frac{x+4y}{9}\, dx \right) dy = \frac{301}{270}.$$

$$\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X}) \cdot \mathrm{E}(\mathrm{Y}) = \frac{301}{270} - \frac{209}{108} \cdot \frac{41}{72} = \mathbf{\frac{499}{38880}} \approx 0.012834.$$

2. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^2 y, \qquad 0 < x < 4, \quad 0 < y < \sqrt{x}, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{\, 0 < x < 4,\ \ 0 < y < \sqrt{x} \,\}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^4 \left(\int_0^{\sqrt{x}} C\,x^2\,y\,dy \right) dx = \int_0^4 \frac{C}{2}\,x^2\,y^2 \Big|_0^{\sqrt{x}}\,dx = \int_0^4 \frac{C}{2}\,x^3\,dx = \frac{C}{8}\,x^4 \Big|_0^4 = 32\,C.$$

$$\Rightarrow \qquad C = \mathbf{\frac{1}{32}}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{\sqrt{x}} \frac{1}{32}\,x^2\,y\,dy = \frac{1}{64}\,x^2\,y^2 \Big|_0^{\sqrt{x}} = \frac{1}{64}\,x^3, \qquad 0 < x < 4.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{y^2}^{4} \frac{1}{32} x^2 y \, dx = \frac{y}{96} \cdot x^3 \Big|_{y^2}^{4} = \frac{y}{96} \cdot \left(64 - y^6\right) = \frac{2}{3} y - \frac{1}{96} y^7, \qquad 0 < y < 2.$$

e) Are X and Y independent? *Justify your answer.*
If X and Y are not independent, find $\mathrm{Cov}(X, Y)$.

$f(x, y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_0^4 x \cdot \frac{1}{64} x^3 \, dx = \int_0^4 \frac{1}{64} x^4 \, dx = \frac{1}{320} x^5 \Big|_0^4 = \frac{16}{5} = 3.2.$$

$$E(Y) = \int_0^2 y \cdot \left(\frac{2}{3} y - \frac{1}{96} y^7 \right) dy = \int_0^2 \left(\frac{2}{3} y^2 - \frac{1}{96} y^8 \right) dy = \left(\frac{2}{9} y^3 - \frac{1}{864} y^9 \right) \Big|_0^2$$

$$= \frac{16}{9} - \frac{16}{27} = \frac{32}{27} \approx 1.1852.$$

$$E(XY) = \int_0^4 \left(\int_0^{\sqrt{x}} xy \cdot \frac{1}{32} x^2 y \, dy \right) dx = \int_0^4 \frac{1}{96} x^3 y^3 \Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{1}{96} x^{9/2} \, dx$$

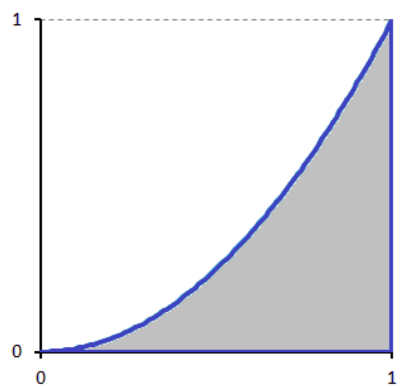
$$= \frac{1}{528} x^{11/2} \Big|_0^4 = \frac{128}{33} \approx 3.8788.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{128}{33} - \frac{16}{5} \cdot \frac{32}{27} = \mathbf{\frac{128}{1485}} \approx 0.0862.$$

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} C\,x\,y^3 & 0 < x < 1,\ 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.



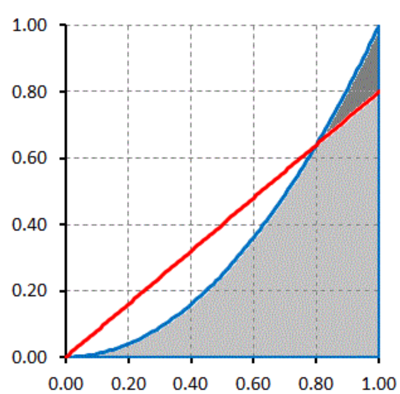
$$1 = \int_0^1 \left(\int_0^{x^2} C\,x\,y^3\,dy \right) dx$$

$$= \int_0^1 \frac{C}{4}\,x\,y^4 \Bigg|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 \frac{C}{4}\,x^9\,dx = \frac{C}{40}\,x^{10} \Bigg|_0^1 = \frac{C}{40}.$$

$$\Rightarrow \quad C = \mathbf{40}.$$

b) Find $P(5Y > 4X)$.



$y = 0.80\,x$ and $y = x^2$

$\Rightarrow \quad x = 0.80$

$$P(5Y > 4X) = P(Y > 0.80\,X)$$

$$= \int_{0.80}^1 \left(\int_{0.80\,x}^{x^2} 40\,x\,y^3\,dy \right) dx$$

$$= \int_{0.80}^1 10\,x\left(y^4\right) \Bigg|_{0.80\,x}^{x^2} dx$$

$$= \int_{0.80}^1 \left(10\,x^9 - 4.096\,x^5\right) dx$$

$$= \left(x^{10} - \frac{256}{375}\,x^6 \right) \Bigg|_{0.80}^1 \approx 0.3889.$$

OR

$$\int_{0.64}^{0.80}\left(\int_{\sqrt{y}}^{1.25\,y} 40\,x\,y^3\,dx\right)dy \;+\; \int_{0.80}^{1}\left(\int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx\right)dy \;=\; \ldots$$

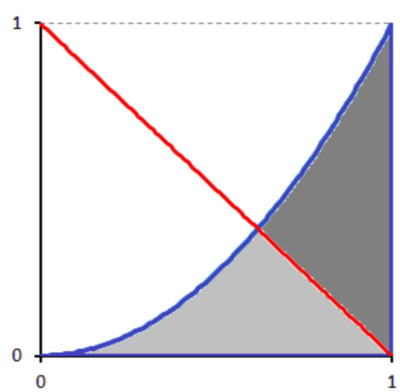
OR

$$1 \;-\; \int_{0}^{0.80}\left(\int_{0}^{x^2} 40\,x\,y^3\,dy\right)dx \;-\; \int_{0.80}^{1}\left(\int_{0}^{0.80\,x} 40\,x\,y^3\,dy\right)dx \;=\; \ldots$$

OR

$$1 \;-\; \int_{0}^{0.64}\left(\int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx\right)dy \;-\; \int_{0.64}^{0.80}\left(\int_{1.25\,y}^{1} 40\,x\,y^3\,dx\right)dy \;=\; \ldots$$

c) Find $P(X+Y>1)$.

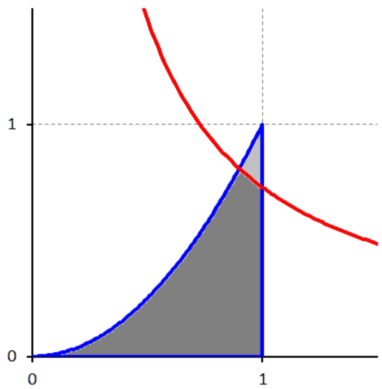


$$y = 1-x \quad\text{and}\quad y = x^2 \qquad\Rightarrow\qquad x = \frac{\sqrt{5}-1}{2}$$

$$P(X+Y>1) \;=\; \int_{\frac{\sqrt{5}-1}{2}}^{1}\left(\int_{1-x}^{x^2} 40\,x\,y^3\,dy\right)dx \;=\; \ldots$$

$$= \frac{509}{6} - \frac{75\sqrt{5}}{2} \;\approx\; 0.9808.$$

d) Find the probability $P(X \cdot Y < 0.729)$.



$y = x^2$ and $x \cdot y = 0.729$

$\Rightarrow \quad x = 0.9, \; y = 0.81$

$$1 - \int_{0.9}^{1} \left(\int_{\frac{0.729}{x}}^{x^2} 40\, x\, y^3 \, dy \right) dx$$

$$= 1 - \int_{0.9}^{1} 10\, x \left(x^8 - \frac{0.9^{12}}{x^4} \right) dx$$

$$= 1 - \int_{0.9}^{1} \left(10\, x^9 - 10 \cdot \frac{0.9^{12}}{x^3} \right) dx$$

$$= 6 \cdot 0.9^{10} - 5 \cdot 0.9^{12} \approx \mathbf{0.6799}.$$

OR $$\int_{0}^{0.9} \left(\int_{0}^{x^2} 40\, x\, y^3 \, dy \right) dx + \int_{0.9}^{1} \left(\int_{0}^{\frac{0.729}{x}} 40\, x\, y^3 \, dy \right) dx$$

OR $$\int_{0}^{0.729} \left(\int_{\sqrt{y}}^{1} 40\, x\, y^3 \, dx \right) dy + \int_{0.729}^{0.81} \left(\int_{\sqrt{y}}^{\frac{0.729}{y}} 40\, x\, y^3 \, dx \right) dy$$

e) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_{0}^{x^2} 40\, x\, y^3 \, dy = 10\, x^9, \qquad 0 < x < 1.$$

f) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx = 20\,y^3 - 20\,y^4 = 20\,y^3(1-y), \qquad 0 < y < 1.$$

g) Find $E(X)$, $E(Y)$, $E(X \cdot Y)$.

$$E(X) = \int_0^1 x \cdot 10\,x^9\,dx = \mathbf{\frac{10}{11}}.$$

$$E(Y) = \int_0^1 y \cdot \left(20\,y^3 - 20\,y^4\right) dy = \frac{20}{5} - \frac{20}{6} = \mathbf{\frac{2}{3}}.$$

OR $$E(Y) = \int_0^1 \left(\int_0^{x^2} y \cdot 40\,x\,y^3\,dy \right) dx = \int_0^1 8\,x^{11}\,dx = \frac{8}{12} = \mathbf{\frac{2}{3}}.$$

OR Y has Beta distribution with $\alpha = 4$, $\beta = 2$. $\qquad E(Y) = \frac{4}{4+2} = \mathbf{\frac{2}{3}}.$

$$E(XY) = \int_0^1 \left(\int_0^{x^2} x\,y \cdot 40\,x\,y^3\,dy \right) dx = \int_0^1 8\,x^{12}\,dx = \mathbf{\frac{8}{13}}.$$

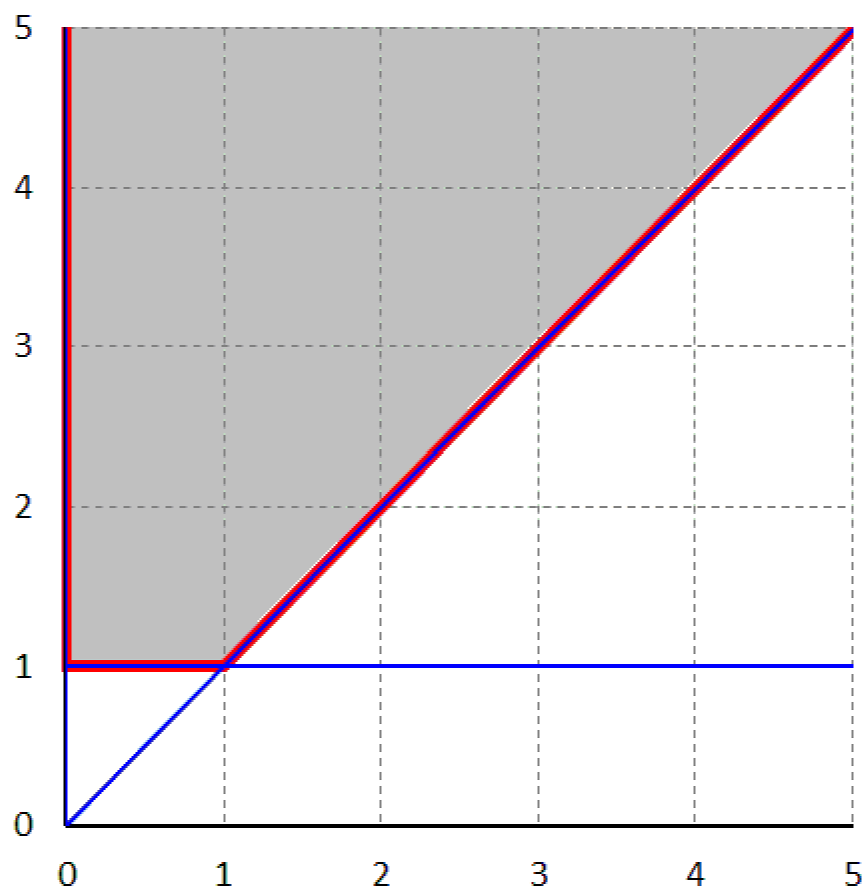
h) Find $\mathrm{Cov}(X, Y)$.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\,E(Y) = \frac{8}{13} - \frac{10}{11} \cdot \frac{2}{3} = \mathbf{\frac{4}{429}} \approx 0.009324.$$

4. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{C}{(2x+y)^3}, \qquad y > 1, \quad 0 < x < y, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ y > 1, \ 0 < x < y \}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_1^\infty \left(\int_0^y \frac{C}{(2x+y)^3} \, dx \right) dy = \int_1^\infty \left(-\frac{C}{4(2x+y)^2} \right) \Bigg|_0^y \, dy$$

$$= \int_1^\infty \left(-\frac{C}{36y^2} + \frac{C}{4y^2} \right) dy = \frac{2C}{9} \int_1^\infty \frac{1}{y^2} \, dy = \frac{2C}{9} \left(-\frac{1}{y} \right) \Bigg|_1^\infty = \frac{2C}{9}.$$

$\Rightarrow \qquad C = \dfrac{\mathbf{9}}{\mathbf{2}} = \mathbf{4.5}.$

c) Find the marginal probability density function of X, $f_X(x)$.

For $0 < x < 1$,

$$f_X(x) = \int_1^{\infty} \frac{9}{2(2x+y)^3}\,dy = -\frac{9}{4(2x+y)^2}\Bigg|_1^{\infty} = \frac{9}{4(2x+1)^2}, \qquad 0 < x < 1.$$

For $1 < x < \infty$,

$$f_X(x) = \int_x^{\infty} \frac{9}{2(2x+y)^3}\,dy = -\frac{9}{4(2x+y)^2}\Bigg|_x^{\infty} = \frac{1}{4x^2}, \qquad 1 < x < \infty.$$

Check:

$$\int_0^1 \frac{9}{4(2x+1)^2}\,dx + \int_1^{\infty} \frac{1}{4x^2}\,dx = \left(-\frac{9}{8(2x+1)}\right)\Bigg|_0^1 + \left(-\frac{1}{4x}\right)\Bigg|_1^{\infty}$$

$$= \left(-\frac{3}{8}+\frac{9}{8}\right)+\left(-0+\frac{1}{4}\right) = 1.$$

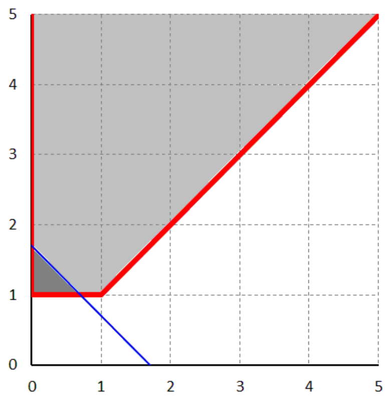
☺

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^{y} \frac{9}{2(2x+y)^3}\,dx = -\frac{9}{8(2x+y)^2}\Bigg|_0^{y}$$

$$= -\frac{1}{8y^2}+\frac{9}{8y^2} = \frac{1}{y^2}, \qquad 1 < y < \infty.$$

e) Let $1 < w < 2$. Find $P(X+Y \le w)$.



$$P(X+Y \le w) = \int_0^{w-1} \left(\int_1^{w-x} \frac{9}{2(2x+y)^3}\, dy \right) dx$$

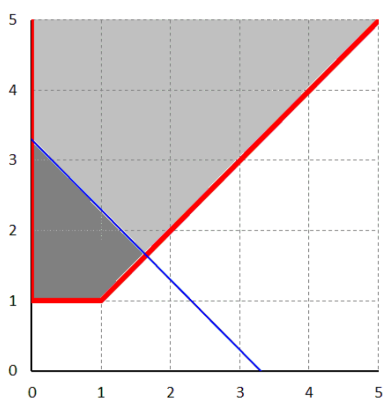
$$= \int_0^{w-1} \left(-\frac{9}{4(2x+y)^2} \right) \Bigg|_1^{w-x} dx$$

$$= \int_0^{w-1} \left(\frac{9}{4(2x+1)^2} - \frac{9}{4(x+w)^2} \right) dx$$

$$= \left(-\frac{9}{8(2x+1)} + \frac{9}{4(x+w)} \right) \Bigg|_0^{w-1}$$

$$= \frac{9}{8(2w-1)} + \frac{9}{8} - \frac{9}{4w}.$$

f) Let $w > 2$. Find $P(X+Y \le w)$.



$$P(X+Y \le w) = 1 - \int_0^{w/2} \left(\int_{w-x}^{\infty} \frac{9}{2(2x+y)^3}\, dy \right) dx$$

$$- \int_{w/2}^{\infty} \left(\int_x^{\infty} \frac{9}{2(2x+y)^3}\, dy \right) dx$$

$$= 1 - \int_0^{w/2} \frac{9}{4(x+w)^2}\, dx - \int_{w/2}^{\infty} \frac{1}{4x^2}\, dx$$

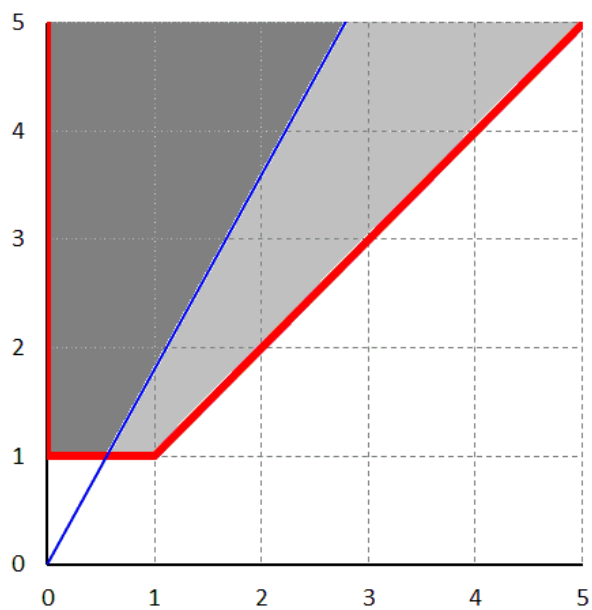
$$= 1 + \frac{9}{4(x+w)} \Bigg|_0^{w/2} + \frac{1}{4x} \Bigg|_{w/2}^{\infty}$$

$$= 1 + \frac{3}{2w} - \frac{9}{4w} - \frac{1}{2w} = 1 - \frac{5}{4w}.$$

Technically, there are 3 cases: $w < 1$, $1 < w < 2$, $w > 2$,

but $w < 1$ is boring.

g) Let $0 < u < 1$. Find $P\left(\frac{X}{Y} \le u\right)$.



$$P\left(\frac{X}{Y} \le u\right) = P(X \le uY)$$

$$= \int_1^\infty \left(\int_0^{uy} \frac{9}{2(2x+y)^3}\, dx \right) dy$$

$$= \int_1^\infty -\frac{9}{8(2x+y)^2} \Bigg|_0^{uy} dy$$

$$= \int_1^\infty \left(\frac{9}{8y^2} - \frac{9}{8(2u+1)^2 y^2} \right) dy$$

$$= \frac{9}{8} - \frac{9}{8(2u+1)^2}.$$

Technically, there are 3 cases: $u < 0$, $0 < u < 1$, $u > 1$,

but $u < 0$ and $u > 1$ are boring.