Examples for 10/02/2020 (1)

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$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{pmatrix} \quad – \quad n\text{-dimensional random vector} \qquad\qquad E(\mathbf{X}) = \boldsymbol{\mu}_{\mathbf{X}} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_n \end{pmatrix}$$

Covariance matrix:

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}$$

$\sigma_{ij} = \operatorname{Cov}(X_i, X_j)$

$\boldsymbol{\Sigma}_{\mathbf{X}} = E[\,(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T\,]$

$\boldsymbol{\Sigma}_{\mathbf{X}}$ – symmetric, nonnegative-definite

$n = 1$

$\mathbf{X} = (X)$

$\boldsymbol{\mu}_{\mathbf{X}} = (\mu)$

$\boldsymbol{\Sigma}_{\mathbf{X}} = (\sigma^2)$

$n = 2$

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad \boldsymbol{\mu}_{\mathbf{X}} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{pmatrix} \operatorname{Var}(X_1) & \operatorname{Cov}(X_1, X_2) \\ \operatorname{Cov}(X_1, X_2) & \operatorname{Var}(X_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\,\sigma_1\sigma_2 \\ \rho\,\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

Let $\mathbf{Y} = \mathbf{A}\,\mathbf{X} + \mathbf{b}$, where $\mathbf{A}$ – $m \times n$ (non-random) matrix,

$\mathbf{b} \in \mathbf{R}^m$ – (non-random) vector

Then $\quad E(\mathbf{Y}) = \boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\,\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b} \qquad\qquad \boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{A}\,\boldsymbol{\Sigma}_{\mathbf{X}}\,\mathbf{A}^T$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$$

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n = \mathbf{a}^T \mathbf{X}$$

$$E(\mathbf{a}^T \mathbf{X}) = \mathbf{a}^T \boldsymbol{\mu}_X \qquad\qquad \operatorname{Var}(\mathbf{a}^T \mathbf{X}) = \mathbf{a}^T \boldsymbol{\Sigma}_{\mathbf{X}}\,\mathbf{a} \geq 0$$

Example: Consider a random vector $\vec{\mathbf{X}} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ with mean $E(\vec{\mathbf{X}}) = \vec{\boldsymbol{\mu}} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$

and variance-covariance matrix $\mathrm{Cov}(\vec{\mathbf{X}}) = \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & 16 \end{pmatrix}$.

Then $\mathrm{Var}(X_1) = 9, \quad \mathrm{Var}(X_2) = 4, \quad \mathrm{Var}(X_3) = 16,$

$$\rho_{12} = \frac{2}{\sqrt{9}\cdot\sqrt{4}} = \frac{1}{3}, \qquad \rho_{13} = \frac{-3}{\sqrt{9}\cdot\sqrt{16}} = -\frac{1}{4}, \qquad \rho_{23} = \frac{-2}{\sqrt{4}\cdot\sqrt{16}} = -\frac{1}{4}.$$

Consider $2X_1 - 3X_2 - X_3 = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

$$E(2X_1 - 3X_2 - X_3) = 2\mu_1 - 3\mu_2 - \mu_3 = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = 2\cdot 5 - 3\cdot 1 - 1\cdot 3 = 4.$$

$$\mathrm{Var}(2X_1 - 3X_2 - X_3) = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 9 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & -6 & -16 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 64.$$

OR

$$\mathrm{Var}(2X_1 - 3X_2 - X_3) = 4\,\mathrm{Var}(X_1) + 9\,\mathrm{Var}(X_2) + \mathrm{Var}(X_3)$$

$$- 12\,\mathrm{Cov}(X_1, X_2) - 4\,\mathrm{Cov}(X_1, X_3) + 6\,\mathrm{Cov}(X_2, X_3)$$

$$= 36 + 36 + 16 - 24 + 12 - 12 = 64.$$