STAT 410 **Examples for 09/21/2020 (2)** Fall 2020
A. Stepanov

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{12}{5} x y^3, \quad 0 < y < 1, \quad y < x < 2, \quad \text{zero otherwise.}$$

r) Let $V = X \cdot Y$.

Find the joint probability density function of (X, V), $f_{X,V}(x, v)$.
Sketch the support of (X, V).

OR

Find the joint probability density function of (Y, V), $f_{Y,V}(y, v)$.
Sketch the support of (Y, V).

s) Use (r) to find the p.d.f. of $V = X \cdot Y$, $f_V(v)$.

t) Let $U = Y/X$.

Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.
Sketch the support of (X, U).

OR

Find the joint probability density function of (Y, U), $f_{Y,U}(y, u)$.
Sketch the support of (Y, U).

u) Use (t) to find the p.d.f. of $U = Y/X$, $f_U(u)$.

"Hint": To double-check your answer: Recall (Examples for 09/18/2020 (2)):

$$F_V(v) = \begin{cases} \frac{2}{5} v^3 - \frac{3}{40} v^4 & 0 \le v < 1 \\ -\frac{1}{5} + \frac{3}{5} v^2 - \frac{3}{40} v^4 & 1 \le v < 2 \end{cases} \qquad F_U(u) = \begin{cases} \frac{32}{5} u^4 & 0 \le u < \frac{1}{2} \\ \frac{6}{5} - \frac{1}{5u^2} & \frac{1}{2} \le u < 1 \end{cases}$$

v) Let $D = X - Y$.

Find the joint probability density function of (X, D), $f_{X,D}(x, d)$.

Sketch the support of (X, D).

OR

Find the joint probability density function of (Y, D), $f_{Y,D}(y, d)$.

Sketch the support of (Y, D).

w) Use (v) to find the p.d.f. of $D = X - Y$, $f_D(d)$.

Answers:

3. Let the joint probability density function for (X, Y) be

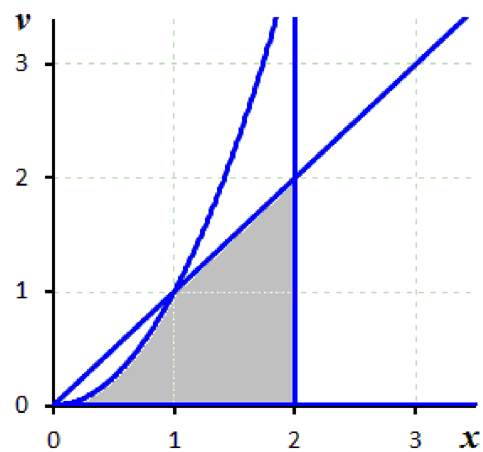
$$f(x,y) = \frac{12}{5} x y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

rx) Let $V = X \cdot Y$.

Find the joint probability density function of (X, V), $f_{X,V}(x, v)$.

Sketch the support of (X, V).

$X = X, \qquad Y = V / X.$



$0 < y < 1 \quad \Rightarrow \quad 0 < \frac{v}{x} < 1 \quad \Rightarrow \quad 0 < v < x,$

$y < x \quad \Rightarrow \quad \frac{v}{x} < x \quad \Rightarrow \quad v < x^2,$

$x < 2.$

$$J = \begin{vmatrix} 1 & 0 \\ -\frac{v}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x}.$$

$$f_{X,V}(x,v) = f_{X,Y}\left(x, \frac{v}{x}\right) \times |J| = \frac{12}{5} x \left(\frac{v}{x}\right)^3 \times \frac{1}{x} = \frac{12}{5} \left(\frac{v}{x}\right)^3,$$

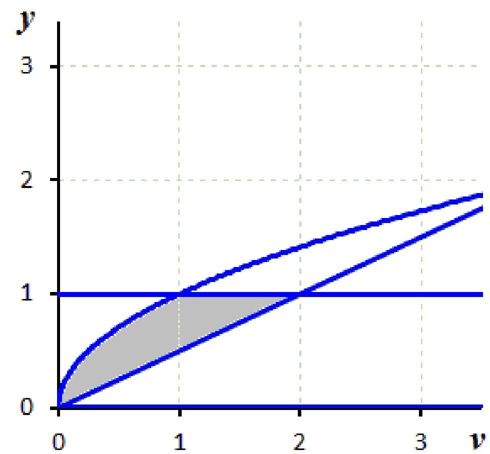
$$0 < v < 2, \quad \max(\sqrt{v}, v) < x < 2.$$

ry) Let $V = X \cdot Y$.

Find the joint probability density function of (Y, V), $f_{Y,V}(y, v)$.

Sketch the support of (Y, V).

$X = V / Y, \qquad Y = Y.$



$0 < y < 1,$

$$y < x \quad \Rightarrow \quad y < \frac{v}{y} \quad \Rightarrow \quad v > y^2,$$

$$x < 2 \quad \Rightarrow \quad \frac{v}{y} < 2 \quad \Rightarrow \quad v < 2y.$$

$$J = \begin{vmatrix} \frac{1}{y} & -\frac{v}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}.$$

$$f_{V,Y}(v, y) = f_{X,Y}\left(\frac{v}{y}, y\right) \times |J| = \frac{12}{5} \frac{v}{y} y^3 \times \frac{1}{y} = \frac{12}{5} v y,$$

$$0 < y < 1, \quad y^2 < v < 2y.$$

$$\text{or} \qquad 0 < v < 2, \quad \frac{v}{2} < y < \min(1, \sqrt{v}).$$

sx) Use (r) to find the p.d.f. of $V = X \cdot Y$, $f_V(v)$.

$0 < v < 1$,
$$\int_{\sqrt{v}}^{2} \frac{12}{5}\left(\frac{v}{x}\right)^3 dx = \left(-\frac{6v^3}{5x^2}\right)\Bigg|_{x=\sqrt{v}}^{x=2} = \frac{6v^2}{5} - \frac{3v^3}{10}.$$

$1 < v < 2$,
$$\int_{v}^{2} \frac{12}{5}\left(\frac{v}{x}\right)^3 dx = \left(-\frac{6v^3}{5x^2}\right)\Bigg|_{x=v}^{x=2} = \frac{6v}{5} - \frac{3v^3}{10}.$$

sy) Use (r) to find the p.d.f. of $V = X \cdot Y$, $f_V(v)$.

$0 < v < 1$,
$$\int_{\frac{v}{2}}^{\sqrt{v}} \frac{12}{5} v\, y\, dy = \left(\frac{6}{5} v\, y^2\right)\Bigg|_{y=\frac{v}{2}}^{y=\sqrt{v}} = \frac{6v^2}{5} - \frac{3v^3}{10}.$$

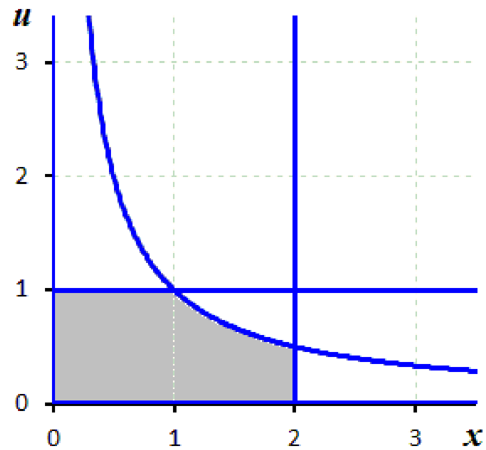
$1 < v < 2$,
$$\int_{\frac{v}{2}}^{1} \frac{12}{5} v\, y\, dy = \left(\frac{6}{5} v\, y^2\right)\Bigg|_{y=\frac{v}{2}}^{y=1} = \frac{6v}{5} - \frac{3v^3}{10}.$$

tx) Let $U = {}^{Y}/_{X}$.

Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.

Sketch the support of (X, U).

$X = X, \qquad Y = U\,X.$



$0 < y < 1 \quad \Rightarrow \quad 0 < u\,x < 1,$

$y < x \quad \Rightarrow \quad u\,x < x \quad \Rightarrow \quad u < 1,$

$x < 2.$

$$J = \begin{vmatrix} 1 & 0 \\ u & x \end{vmatrix} = x.$$

$$f_{X,U}(x, u) = f_{X,Y}(x, u\,x) \times |J| = \frac{12}{5}\,x\,(u\,x)^3 \times x = \frac{12}{5}\,x^5\,u^3,$$

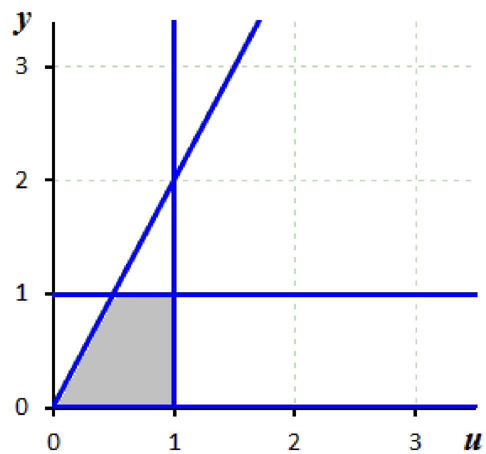
$$0 < u < 1, \quad 0 < x < \min\left(2, \frac{1}{u}\right).$$

ty) Let $\mathrm{U} = {}^{\mathrm{Y}}\!/\!_{\mathrm{X}}$.

Find the joint probability density function of (Y, U), $f_{\mathrm{Y,U}}(y, u)$.

Sketch the support of (Y, U).

$\mathrm{X} = \mathrm{Y}/\mathrm{U}, \qquad \mathrm{Y} = \mathrm{Y}.$



$0 < y < 1,$

$y < x \quad \Rightarrow \quad y < \dfrac{y}{u} \quad \Rightarrow \quad u < 1,$

$x < 2 \quad \Rightarrow \quad \dfrac{y}{u} < 2 \quad \Rightarrow \quad y < 2u.$

$$\mathrm{J} = \begin{vmatrix} -\dfrac{y}{u^2} & \dfrac{1}{u} \\ 0 & 1 \end{vmatrix} = -\dfrac{y}{u^2}.$$

$$f_{\mathrm{Y,U}}(y, u) = f_{\mathrm{X,Y}}\left(\frac{y}{u}, y\right) \times |\mathrm{J}| = \frac{12}{5}\,\frac{y}{u}\,y^3 \times \frac{y}{u^2} = \frac{12}{5}\,\frac{y^5}{u^3},$$

$$0 < y < 1, \qquad \frac{y}{2} < u < 1.$$

or $\qquad 0 < u < 1, \qquad 0 < y < \min(1, 2u).$

ux) Use (t) to find the p.d.f. of $U = Y/X$, $f_U(u)$.

$0 < u < \frac{1}{2}$, $$\int_0^2 \frac{12}{5} x^5 u^3 \, dx = \left(\frac{2}{5} x^6 u^3 \right) \Big|_{x=0}^{x=2} = \frac{128}{5} u^3.$$

$\frac{1}{2} < u < 1$, $$\int_0^{\frac{1}{u}} \frac{12}{5} x^5 u^3 \, dx = \left(\frac{2}{5} x^6 u^3 \right) \Big|_{x=0}^{x=\frac{1}{u}} = \frac{2}{5u^3}.$$

uy) Use (t) to find the p.d.f. of $U = Y/X$, $f_U(u)$.

$0 < u < \frac{1}{2}$, $$\int_0^{2u} \frac{12}{5} \frac{y^5}{u^3} \, dy = \left(\frac{2}{5} \frac{y^6}{u^3} \right) \Big|_{y=0}^{y=2u} = \frac{128}{5} u^3.$$

$\frac{1}{2} < u < 1$, $$\int_0^1 \frac{12}{5} \frac{y^5}{u^3} \, dy = \left(\frac{2}{5} \frac{y^6}{u^3} \right) \Big|_{y=0}^{y=1} = \frac{2}{5u^3}.$$

“Hint”: To double-check your answer: Recall (Examples for 09/18/2020 (2)):

$$F_V(v) = \begin{cases} \frac{2}{5}v^3 - \frac{3}{40}v^4 & 0 \le v < 1 \\ -\frac{1}{5} + \frac{3}{5}v^2 - \frac{3}{40}v^4 & 1 \le v < 2 \end{cases} \qquad F_U(u) = \begin{cases} \frac{32}{5}u^4 & 0 \le u < \frac{1}{2} \\ \frac{6}{5} - \frac{1}{5u^2} & \frac{1}{2} \le u < 1 \end{cases}$$

Indeed, $$\frac{d}{dv}\left(\frac{2}{5}v^3 - \frac{3}{40}v^4\right) = \frac{6v^2}{5} - \frac{3v^3}{10},$$

$$\frac{d}{dv}\left(-\frac{1}{5} + \frac{3}{5}v^2 - \frac{3}{40}v^4\right) = \frac{6v}{5} - \frac{3v^3}{10}.$$ ☺

Indeed, $$\frac{d}{du}\left(\frac{32}{5}u^4\right) = \frac{128}{5}u^3,$$

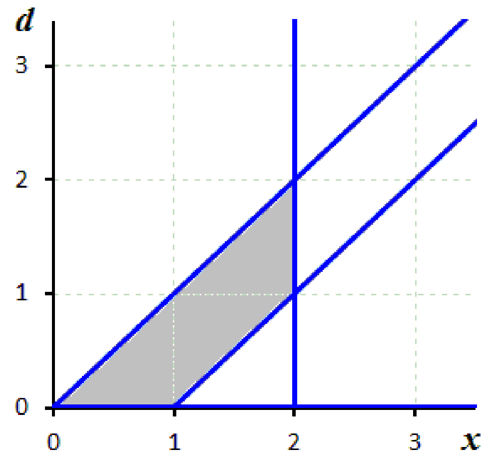
$$\frac{d}{du}\left(\frac{6}{5} - \frac{1}{5u^2}\right) = \frac{2}{5u^3}.$$ ☺

vx) Let $\mathrm{D} = \mathrm{X} - \mathrm{Y}$.

Find the joint probability density function of (X, D), $f_{\mathrm{X,D}}(x, d)$.

Sketch the support of (X, D).

$\mathrm{X} = \mathrm{X}, \qquad \mathrm{Y} = \mathrm{X} - \mathrm{D}.$



$0 < y < 1 \quad \Rightarrow \quad 0 < x - d < 1$

$\quad\quad\quad\quad \Rightarrow \quad d < x < d + 1, \quad x - 1 < d < x,$

$y < x \quad \Rightarrow \quad x - d < x \quad \Rightarrow \quad d > 0,$

$x < 2.$

$$\mathrm{J} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1.$$

$$f_{\mathrm{X,D}}(x, d) = f_{\mathrm{X,Y}}(x, x-d) \times |\,\mathrm{J}\,| = \frac{12}{5}\, x\,(x-d)^3 \times 1 = \frac{12}{5}\, x\,(x-d)^3,$$

$0 < d < 2, \quad d < x < \min(2, d+1)$

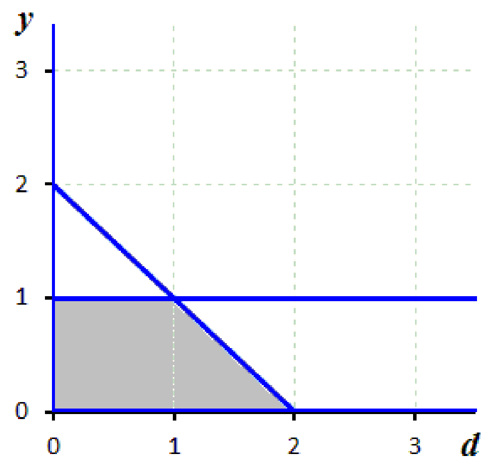
or $\quad 0 < x < 2, \quad \max(0, x-1) < d < x.$

vy) Let $\mathrm{D} = \mathrm{X} - \mathrm{Y}$.

Find the joint probability density function of (Y, D), $f_{\mathrm{Y,D}}(y, d)$.

Sketch the support of (Y, D).

$\mathrm{X} = \mathrm{D} + \mathrm{Y}, \qquad \mathrm{Y} = \mathrm{Y}.$



$0 < y < 1,$

$y < x \quad \Rightarrow \quad y < d + y \quad \Rightarrow \quad d > 0,$

$x < 2 \quad \Rightarrow \quad d + y < 2 \quad \Rightarrow \quad y < 2 - d.$

$$\mathrm{J} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

$$f_{\mathrm{Y,D}}(y, d) = f_{\mathrm{X,Y}}(d + y, y) \times |\mathrm{J}| = \frac{12}{5}\,(d + y)\,y^3 \times 1 = \frac{12}{5}\,(d + y)\,y^3,$$

$0 < y < 1, \quad 0 < d < 2 - y.$

or $\quad 0 < d < 2, \quad 0 < y < \min(1, 2 - d).$

w) Use (v) to find the p.d.f. of $D = X - Y$, $f_D(d)$.

$0 < d < 1$,

$$\int_0^1 \frac{12}{5}\,(d+y)\,y^3\,dy = \frac{12}{5}\left(\frac{d\,y^4}{4}+\frac{y^5}{5}\right)\Bigg|_{y=0}^{y=1} = \frac{15\,d + 12}{25}.$$

$1 < d < 2$,

$$\int_0^{2-d} \frac{12}{5}\,(d+y)\,y^3\,dy = \frac{12}{5}\left(\frac{d\,y^4}{4}+\frac{y^5}{5}\right)\Bigg|_{y=0}^{y=2-d}$$

$$= \frac{15\,d\,(2-d)^4 + 12\,(2-d)^5}{25}$$

$$= \frac{3\,d^5 - 120\,d^3 + 480\,d^2 - 720\,d + 384}{25}.$$