STAT 410 | **Examples for 11/04/2020 (2)** | Fall 2020
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Examples for 10/19/2020 (2) and 10/23/2020 (2) and 10/30/2020 (3) (continued)

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

Recall: $W = \ln X$ has a Gamma$(\alpha = 2, \theta = \frac{1}{\beta})$ distribution.

$\Rightarrow$ $Y = \sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} W_i$ has a Gamma$(\alpha = 2n, \theta = \frac{1}{\beta})$ distribution.

n) Find a sufficient statistic $u(X_1, X_2, \ldots, X_n)$ for β.

o) Find the Fisher information $I(\beta)$.

(After you are done with part (o), glance back at part (m).)

Recall: $\hat{\hat{\beta}} = \dfrac{2n-1}{\sum_{i=1}^{n} \ln X_i}$ is an unbiased estimator for β.

p) Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

① Find $\operatorname{Var}(\hat{\hat{\beta}})$. ("Hint": Recall **1** (e) of Examples for 10/19/2020 (2).)

② Find the Rao-Cramér lower bound.

③ Is $\hat{\hat{\beta}}$ an efficient estimator of β? Does $\operatorname{Var}(\hat{\hat{\beta}})$ attain the R.C.L.B.?

If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

Answers:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

Recall: $W = \ln X$ has a Gamma$(\alpha = 2, \theta = \frac{1}{\beta})$ distribution.

$\Rightarrow$ $Y = \sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} W_i$ has a Gamma$(\alpha = 2n, \theta = \frac{1}{\beta})$ distribution.

n) Find a sufficient statistic $u(X_1, X_2, \ldots, X_n)$ for β.

$$f(x_1, x_2, \ldots x_n; \beta) = f(x_1;\beta)\, f(x_2;\beta)\, \ldots\, f(x_n;\beta)$$

$$= \prod_{i=1}^{n} \frac{\beta^2 (\ln x_i)}{x_i^{\beta+1}} = \left[\beta^{2n} \cdot \left(\prod_{i=1}^{n} x_i \right)^{-\beta-1} \right] \cdot \left(\prod_{i=1}^{n} \ln x_i \right).$$

By Factorization Theorem, $Y = \prod_{i=1}^{n} X_i$ is a sufficient statistic for β.

$\prod_{i=1}^{n} \frac{1}{X_i}$ is also a sufficient statistic for β.

OR

$$f(x;\beta) = \exp\{ -\beta \cdot \ln x + 2 \ln \beta + \ln \ln x - \ln x \}. \qquad K(x) = \ln x.$$

$\Rightarrow$ $Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for β.

o) Find the Fisher information $\mathrm{I}(\beta)$.

(After you are done with part (o), glance back at part (m).)

$$\ln f(x;\beta) = -\beta \cdot \ln x + 2\ln\beta + \ln\ln x - \ln x.$$

$$\frac{\partial}{\partial\beta}\ln f(x;\beta) = -\ln x + \frac{2}{\beta}. \qquad\qquad \frac{\partial^2}{\partial\beta^2}\ln f(x;\beta) = -\frac{2}{\beta^2}.$$

$$\begin{aligned}\mathrm{I}(\beta) &= \mathrm{Var}\left[\frac{\partial}{\partial\beta}\ln f(\mathrm{X};\beta)\right]\\ &= \mathrm{Var}\left[-\ln \mathrm{X} + \frac{1}{\beta}\right]\\ &= \mathrm{Var}(\mathrm{W})\\ &= \alpha\,\theta^2 = \frac{2}{\beta^2}.\end{aligned}$$

$$\begin{aligned}\mathrm{I}(\beta) &= -\mathrm{E}\left[\frac{\partial^2}{\partial\beta^2}\ln f(\mathrm{X};\beta)\right]\\ &= -\mathrm{E}\left[-\frac{2}{\beta^2}\right]\\ &= \frac{2}{\beta^2}.\end{aligned}$$

Glancing back at part (m):

$\hat{\beta}$ is the maximum likelihood estimator for β.

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \xrightarrow{D} N\left(0, \frac{1}{\mathrm{I}(\beta)}\right) = N\left(0, \frac{\beta^2}{2}\right).$$

Recall: $\hat{\hat{\beta}} = \dfrac{2n-1}{\sum\limits_{i=1}^{n} \ln X_i}$ is an unbiased estimator for β.

p) Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

① Find $\operatorname{Var}(\hat{\hat{\beta}})$. ("Hint": Recall **1** (e) of Examples for 10/19/2020 (2).)

② Find the Rao-Cramér lower bound.

③ Is $\hat{\hat{\beta}}$ an efficient estimator of β? Does $\operatorname{Var}(\hat{\hat{\beta}})$ attain the R.C.L.B.?

If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

Recall: $$\operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{(2n-1)(2n-2)} - \left(\frac{\beta}{2n-1}\right)^2 = \frac{\beta^2}{(2n-1)^2 (2n-2)}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \operatorname{Var}\left(\frac{2n-1}{Y}\right) = (2n-1)^2 \operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{2n-2}.$$

$$\text{Rao-Cramer lower bound} = \frac{1}{n \cdot I(\beta)} = \frac{\beta^2}{2n}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \frac{\beta^2}{2n-2} > \frac{\beta^2}{2n}.$$

$\operatorname{Var}(\hat{\hat{\beta}})$ does NOT attain its Rao-Cramer lower bound.

$\Rightarrow$ $\hat{\hat{\beta}}$ is NOT an efficient estimator of β,

its efficiency $= \dfrac{2n-2}{2n} = \dfrac{n-1}{n} \to 1$ as $n \to \infty$.

$\Rightarrow$ $\hat{\hat{\beta}}$ is an asymptotically efficient estimator of β.