STAT 410 **Examples for 10/14/2020 (1)** Fall 2020
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The Chi-Square Distribution $\chi^2(r)$

$$f(y) = \frac{1}{\Gamma(r/2)\,2^{r/2}}\, y^{\,r/2-1}\, e^{-y/2}, \qquad 0 \le y < \infty$$

$$\mathrm{E}(\mathrm{Y}) = r \qquad\qquad \mathrm{Var}(\mathrm{Y}) = 2r$$

2.5. a) Let X be a random variable with a chi-square distribution with r degrees of freedom. Show that X has a Gamma distribution. What are α and θ?

$$\mathrm{M}_{\mathrm{X}}(t) = \mathrm{M}_{\chi^2(r)}(t) = \frac{1}{(1-2t)^{r/2}}, \quad t < \frac{1}{2}.$$

$$\mathrm{M}_{\mathrm{Gamma}(\alpha,\,\theta)}(t) = \frac{1}{(1-\theta t)^{\alpha}}, \quad t < \frac{1}{\theta}.$$

If X has a chi-square distribution with r degrees of freedom, then X has a Gamma distribution with $\alpha = r/2$ and $\theta = 2$.

b) Let Y be a random variable with a Gamma distribution with parameters α and $\theta = 1/\lambda$. Assume α is an integer. Show that $2\mathrm{Y}/\theta$ has a chi-square distribution. What is the number of degrees of freedom?

$$\mathrm{M}_{\mathrm{Y}}(t) = \mathrm{M}_{\mathrm{Gamma}(\alpha,\,\theta)}(t) = \frac{1}{(1-\theta t)^{\alpha}}, \quad t < \frac{1}{\theta}.$$

If $\mathrm{W} = a\,\mathrm{Y} + b$, then $\mathrm{M}_{\mathrm{W}}(t) = e^{bt}\,\mathrm{M}_{\mathrm{Y}}(a\,t)$.

$$\mathrm{M}_{2\mathrm{Y}/\theta}(t) = \mathrm{M}_{\mathrm{Y}}\left(2t/\theta\right) = \frac{1}{(1-2t)^{\alpha}}, \quad t < \frac{1}{2}.$$

$2\mathrm{Y}/\theta$ has a chi-square distribution with $r = 2\alpha$ degrees of freedom.

3. Let Y be a random variable with a Gamma distribution with $\alpha = 5$ and $\theta = 3$.

a) Find the probability $P(Y > 18)$ …

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> 1-pgamma(18,5,1/3)
[1] 0.2850565
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=1-GAMMA.DIST(18,5,3,1)
0.285057

i) … by integrating the p.d.f. of the Gamma distribution;

$$P(Y > 18) = \int_{18}^{\infty} \frac{1}{\Gamma(5) \cdot 3^5} \cdot x^{5-1} \cdot e^{-x/3} dx = \int_{18}^{\infty} \frac{1}{5{,}832} \cdot x^4 \cdot e^{-x/3} dx = \ldots$$

ii) … by using the relationship between Gamma and Poisson distributions;

$$P(Y > 18) = P(X_{18} \le 4) = \mathbf{0.285}$$ where X_{18} is Poisson$(18/\theta = 6)$.

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> ppois(4,6)
[1] 0.2850565
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=POISSON.DIST(4,6,1)
0.285057

iii) … by using the relationship between Gamma and Chi-square distribution.

$$P(Y > 18) = P\left(\frac{2}{3} Y > \frac{2}{3} \cdot 18\right) = P(X > 12)$$ where X is $\chi^2(5 \cdot 2 = 10)$.

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> 1-pchisq(12,10)
[1] 0.2850565
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=1-CHISQ.DIST(12,10,1)
0.285057

=CHISQ.DIST.RT(12,10)
0.285057

b) Find a and b such that $P(a < Y < b) = 0.90$.

$\frac{2}{3} \cdot Y$ is $\chi^2(5 \cdot 2 = 10 \text{ degrees of freedom})$. Y is $\frac{3}{2} \cdot \chi^2(10)$.

$$P(3.940 < \chi^2(10) < 18.31) = 0.95 - 0.05 = 0.90.$$

$$\Rightarrow \quad P(3.940 \cdot \frac{3}{2} < Y < 18.31 \cdot \frac{3}{2}) = P(\mathbf{5.91} < Y < \mathbf{27.465}) = 0.90.$$

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> qgamma(0.05,5,1/3)
[1] 5.910449
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> qgamma(0.95,5,1/3)
[1] 27.46056
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=GAMMA.INV(0.05,5,3)

5.910449

=GAMMA.INV(0.95,5,3)

27.46056

$$P(0 < \chi^2(10) < 15.99) = 0.90 - 0.00 = 0.90.$$

$$\Rightarrow \quad P(0 \cdot \frac{3}{2} < Y < 15.99 \cdot \frac{3}{2}) = P(\mathbf{0} < Y < \mathbf{23.985}) = 0.90.$$

$$P(4.865 < \chi^2(10) < \infty) = 1.00 - 0.10 = 0.90.$$

$$\Rightarrow \quad P(4.865 \cdot \frac{3}{2} < Y < \infty \cdot \frac{3}{2}) = P(\mathbf{7.2975} < Y < \boldsymbol{\infty}) = 0.90.$$

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du, \qquad x > 0 \qquad\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(x) = (x-1)\,\Gamma(x-1) \qquad\qquad \Gamma(n) = (n-1)! \quad \text{if } n \text{ is an integer}$$

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4. Let T_7 be a random variable with a Gamma distribution with $\alpha = 7$ and $\theta = 5$. Find the probability $P(20 < T_7 < 30)$.

[Text messages arrive according to Poisson process, on average once every 5 minutes. Find the probability that we would have to wait more than 20 minutes but less than 30 minutes for the 7th text message.]

$$P(20 < T_7 < 30) = \int_{20}^{30} \frac{1}{\Gamma(7) \cdot 5^7} \cdot t^{7-1} \cdot e^{-t/5} dt = \int_{20}^{30} \frac{1}{6! \cdot 5^7} \cdot t^6 \cdot e^{-t/5} dt = \ldots$$

OR

$$P(20 < T_7 < 30) = P(T_7 > 20) - P(T_7 > 30) = P(X_{20} \le 6) - P(X_{30} \le 6)$$
$$= P(\text{Poisson}(4) \le 6) - P(\text{Poisson}(6) \le 6) = 0.889 - 0.606 = \mathbf{0.283}.$$

[If the 7th text message arrives after 20 minutes, then we could have received at most 6 text messages during the first 20 minutes. If the average time between the text messages is 5 minutes, then the expected number of text messages in 20 minutes is 4.
If the 7th text message arrives after 30 minutes, then we could have received at most 6 text messages during the first 30 minutes. If the average time between the text messages is 5 minutes, then the expected number of text messages in 30 minutes is 6.]

	A	B
1	=POISSON.DIST(6,4,1)	
2	=POISSON.DIST(6,6,1)	
3	=A1-A2	
4		

$\Rightarrow$

	A	B
1	0.889326	
2	0.606303	
3	0.283023	
4		

OR

$$P(20 < T_7 < 30) = \left(\frac{2}{5} \cdot 20 < \frac{2}{5} \cdot T_7 < \frac{2}{5} \cdot 30\right) = P(8 < \chi^2(2 \cdot 7 = 14) < 12)$$

	A	B
1	=CHISQ.DIST.RT(8,14)	
2	=CHISQ.DIST.RT(12,14)	
3	=A1-A2	
4		

$\Rightarrow$

	A	B
1	0.889326	
2	0.606303	
3	0.283023	
4		

5. Let X be a random variable with a Gamma distribution with $\alpha = 3$ and $\theta = 5$ (i.e., $\lambda = 0.2$). Find the probability $P(X > 31.48)$ …

a) … by integrating the p.d.f. of the Gamma distribution;

$$P(X > 31.48) = \int_{31.48}^{\infty} \frac{1}{2 \cdot 5^3} \cdot x^2 \cdot e^{-x/5} dx$$

$$= \frac{1}{2 \cdot 5^3} \cdot \left(-5 \cdot x^2 \cdot e^{-x/5} - 2 \cdot 5^2 \cdot x \cdot e^{-x/5} - 2 \cdot 5^3 \cdot e^{-x/5} \right) \Bigg|_{31.48}^{\infty}$$

$$= \mathbf{0.04999}.$$

b) … by using the relationship between Gamma and Poisson distributions;

Hint: If X has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_X(t) = P(X \le t) = P(Y \ge \alpha)$ and $P(X > t) = P(Y \le \alpha - 1)$, where Y has a Poisson$(\lambda t)$ distribution.

$$P(X > 31.48) = 1 - P(X \le 31.48) = 1 - P(Y \ge 3) = P(Y \le 2)$$

where Y has a Poisson$(31.48/5 = 6.296)$ distribution.

$$= \frac{6.296^0 \cdot e^{-6.296}}{0!} + \frac{6.296^1 \cdot e^{-6.296}}{1!} + \frac{6.296^2 \cdot e^{-6.296}}{2!}$$

$$= 0.00184 + 0.01161 + 0.03654 = \mathbf{0.04999}.$$

c) … by using the relationship between Gamma and Chi-square distribution.

Hint: If X has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2X/\theta$ has a chi-square distribution with 2α degrees of freedom.

$$P(X > 31.48) = P(2X/5 > 12.592) = P(\chi^2(6) > 12.592) \approx \mathbf{0.05}.$$