STAT 410 **Examples for 09/23/2020 (3)** Fall 2020
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1. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{4}{3}x^3 y, \qquad 0<x<1, \quad 0<y<3x, \qquad \text{zero otherwise.}$$

m) Find the probability density function of $V = X \times Y$, $f_V(v)$.

$$\begin{aligned} X &= X \\ V &= XY \end{aligned} \qquad \begin{aligned} X &= X \\ Y &= \frac{V}{X} \end{aligned} \qquad J = \begin{array}{c|cc} & x & v \\ \hline x & 1 & 0 \\ y & -\frac{v}{x^2} & \frac{1}{x} \end{array} = \frac{1}{x}$$

$$\begin{aligned} f_{X,V}(x,v) &= f_{X,Y}\left(x, \frac{v}{x}\right) \cdot \left|\frac{1}{x}\right| \\ &= \frac{4}{3}x^3\left(\frac{v}{x}\right) \cdot \frac{1}{x} = \frac{4}{3}xv. \end{aligned}$$

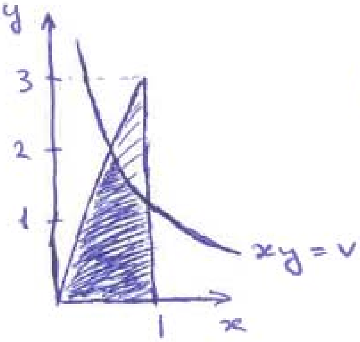
$0<x<1$

$0<y<3x \;\Rightarrow\; 0<\frac{v}{x}<3x \;\Rightarrow\; 0<v<3x^2$

$\Rightarrow\; x>\sqrt{\frac{v}{3}}$

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{X,V}(x,v)\,dx \\ &= \int_{\sqrt{v/3}}^{1} \frac{4}{3}xv\,dx = \left(\frac{2}{3}x^2 v\right)\Big|_{\sqrt{v/3}}^{1} \\ &= \frac{2}{3}v - \frac{2}{9}v^2, \qquad 0<v<3. \end{aligned}$$

OR



$$F_V(v) = P(V \le v) = P(XY \le v) = \dots$$

$$y = 3x \ \& \ xy = v$$

$$\Rightarrow x = \sqrt{\tfrac{v}{3}}, \quad y = \sqrt{3v}.$$

$$\dots = 1 - \int_{\sqrt{\frac{v}{3}}}^{1} \left(\int_{\frac{v}{x}}^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$= 1 - \int_{\sqrt{\frac{v}{3}}}^{1} \frac{2}{3} x^3 \left(9x^2 - \frac{v^2}{x^2} \right) dx$$

$$= 1 - \int_{\sqrt{\frac{v}{3}}}^{1} \left(6x^5 - \frac{2}{3} x v^2 \right) dx$$

$$= 1 - \left(x^6 - \frac{1}{3} x^2 v^2 \right) \Big|_{\sqrt{\frac{v}{3}}}^{1}$$

$$= \frac{1}{3} v^2 - \frac{2}{27} v^3, \qquad 0 \le v < 3.$$

$$f_V(v) = F_V'(v) = \frac{2}{3} v - \frac{2}{9} v^2, \qquad 0 < v < 3.$$

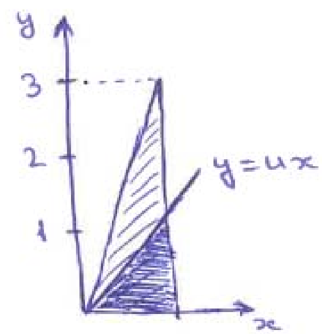
$$\dots = \int_0^{\sqrt{\frac{v}{3}}} \left(\int_0^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{\sqrt{\frac{v}{3}}}^{1} \left(\int_0^{\frac{v}{x}} \frac{4}{3} x^3 y \, dy \right) dx$$

$$= \int_0^{\sqrt{\frac{v}{3}}} 6x^5 \, dx + \int_{\sqrt{\frac{v}{3}}}^{1} \frac{2}{3} x v^2 \, dx$$

$$= \frac{v^3}{27} + \frac{1}{3} v^2 - \frac{1}{9} v^3 = \frac{1}{3} v^2 - \frac{2}{27} v^3, \quad 0 \le v < 3.$$

$$f_V(v) = F_V'(v) = \frac{2}{3} v - \frac{2}{9} v^2, \qquad 0 < v < 3.$$

n) Find the probability density function of $U = {}^{Y}\!/_{X}$, $f_U(u)$.



$$F_U(u) = P(U \le u) = P\left(\frac{Y}{X} \le u\right)$$

$$= P(Y \le uX)$$

$$= \int_0^1 \left(\int_0^{ux} \frac{4}{3} x^3 y \, dy \right) dx$$

$$= \int_0^1 \frac{2}{3} x^3 (ux)^2 \, dx$$

$$= \int_0^1 \frac{2}{3} x^5 u^2 \, dx = \frac{u^2}{9},$$

$$0 \le u < 3.$$

$$f_U(u) = F_U'(u) = \frac{2u}{9}, \quad 0 < u < 3.$$

OR

$$X = X \qquad X = X$$
$$U = \frac{Y}{X} \qquad Y = UX$$

$$J = \begin{array}{c|cc|} & x & u \\ x & 1 & 0 \\ y & u & x \end{array} = x$$

$$f_{X,U}(x,u) = f_{X,Y}(x, ux) \cdot |x|$$

$$= \frac{4}{3} x^3 (ux) \cdot x = \frac{4}{3} x^5 u.$$

$0 < x < 1$

$0 < y < 3x \quad \Rightarrow \quad 0 < ux < 3x \quad \Rightarrow \quad u < 3$

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,U}(x,u)\,dx$$

$$= \int_0^1 \frac{4}{3} x^5 u \, dx = \frac{2}{9} u, \qquad 0 < u < 3.$$