Examples for 11/04/2020 (1)

p.m.f. or p.d.f. $f(x;\theta)$, $\theta \in \Omega$.

$X_1, X_2, \dots, X_n$ are i.i.d. $f(x;\theta)$

- $\theta \neq \theta' \quad \Rightarrow \quad f(x;\theta) \neq f(x;\theta')$
- $f(x;\theta)$ have common support for all θ
- θ_0 is an interior point in Ω

Let θ_0 be the true parameter.

Then $\quad P[\, L(\theta_0 \mid X_1, X_2, \dots, X_n) > L(\theta \mid X_1, X_2, \dots, X_n)\,] \to 1$ as $n \to \infty$ for all $\theta \neq \theta_0$.

- $f(x;\theta)$ is differentiable as a function of θ

Then equation $\dfrac{d}{d\theta} L(\theta) = 0$ has a solution $\hat{\theta}$, such that $\hat{\theta} \xrightarrow{P} \theta_0$.

- $f(x;\theta)$ is twice differentiable as a function of θ
- $\int f(x;\theta)\,dx$ can be twice differentiable under the integral sign as a function of θ

$\dfrac{\partial}{\partial\theta} \ln f(x;\theta)$ is called the **score function** $\qquad E\left[\dfrac{\partial}{\partial\theta} \ln f(X;\theta)\right] = 0$

Fisher Information:

$$I(\theta) = \operatorname{Var}\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right] = E\left[\left(\frac{\partial}{\partial\theta} \ln f(X;\theta)\right)^2\right]$$

$$= -E\left[\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right].$$

Rao-Cramer Lower Bound:

$X_1, X_2, \ldots, X_n$ i.i.d. $f(x;\theta)$

$Y = u(X_1, X_2, \ldots, X_n)$ $\qquad\qquad$ $E(Y) = k(\theta)$

$$\Rightarrow \qquad \mathrm{Var}(Y) \ge \frac{\left(k'(\theta)\right)^2}{n \cdot \mathrm{I}(\theta)}$$

If $E(\hat{\theta}) = \theta$, then $\mathrm{Var}(\hat{\theta}) \ge \dfrac{1}{n \cdot \mathrm{I}(\theta)}$.

Let $\hat{\theta}$ be an unbiased estimator of θ. $\hat{\theta}$ is called an **efficient** estimator of θ if and only if the variance of $\hat{\theta}$ attains the Rao-Cramer lower bound.

- $\left| \dfrac{\partial^3}{\partial \theta^3} \ln f(x;\theta) \right| < M(x), \qquad E(M(X)) < \infty$

$\hat{\theta}$ – maximum likelihood estimator.

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{D} N\left(0, \frac{1}{\mathrm{I}(\theta)}\right).$$

That is, for large n, $\hat{\theta}$ is approximately $N\left(\theta, \dfrac{1}{n \cdot \mathrm{I}(\theta)}\right)$.

$\hat{\theta} \pm z_{\alpha/2} \sqrt{\dfrac{1}{n \cdot \mathrm{I}(\hat{\theta})}}$ would have an approximate $100(1-\alpha)\%$ confidence level for large n.

Example 1:

Consider $\mathrm{N}(\mu, \sigma^2)$ distribution. Determine the Fisher information $\mathrm{I}(\mu)$.

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\ln f(x;\mu,\sigma^2) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\left(\sigma^2\right) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\mu} \ln f(x;\mu,\sigma^2) = \frac{x-\mu}{\sigma^2} \qquad \frac{\partial^2}{\partial\mu^2} \ln f(x;\mu,\sigma^2) = -\frac{1}{\sigma^2}$$

$$\mathrm{I}(\mu) = -\mathrm{E}\left[\frac{\partial^2}{\partial\mu^2} f(x;\mu,\sigma^2)\right] = \frac{1}{\sigma^2}$$

OR

$$\mathrm{I}(\mu) = \mathrm{Var}\left[\frac{\partial}{\partial\mu} \ln f(x;\mu,\sigma^2)\right] = \frac{1}{\sigma^4}\mathrm{Var}\left[\mathrm{X}-\mu\right] = \frac{1}{\sigma^4}\sigma^2 = \frac{1}{\sigma^2}$$

$\hat{\mu} = \overline{\mathrm{X}}$ – maximum likelihood estimator of μ.

$$\mathrm{Var}(\hat{\mu}) = \mathrm{Var}(\overline{\mathrm{X}}) = \frac{\sigma^2}{n} = \frac{1}{n\cdot\mathrm{I}(\mu)}.$$

$\Rightarrow$ $\hat{\mu} = \overline{\mathrm{X}}$ is an efficient estimator of μ.

Also $\overline{\mathrm{X}}$ is $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.

$\overline{\mathrm{X}} \pm z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$ has $100(1-\alpha)\%$ confidence level. (recall STAT 400)

Example 1: (continued)

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a $N(\mu, \sigma^2)$ distribution.

a) Determine the Fisher information $I(\sigma^2)$.

That is, consider $N(\mu, \theta)$ distribution. Determine the Fisher information $I(\theta)$.

$$\ln f(x;\mu,\theta) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\theta) - \frac{(x-\mu)^2}{2\theta}$$

$$\frac{\partial}{\partial\theta}\ln f(x;\mu,\theta) = -\frac{1}{2\theta} + \frac{(x-\mu)^2}{2\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x;\mu,\theta) = \frac{1}{2\theta^2} - \frac{(x-\mu)^2}{\theta^3}$$

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\mu,\theta)\right] = -\frac{1}{2\theta^2} + \frac{E[(X-\mu)^2]}{\theta^3} = -\frac{1}{2\theta^2} + \frac{\theta}{\theta^3} = \frac{1}{2\theta^2}$$

$$I(\sigma^2) = \frac{1}{2\sigma^4}$$

b) Suppose that μ is unknown. We know that the sample variance S^2 is an unbiased estimator of σ^2. Is S^2 an efficient estimator of σ^2? If not, find its efficiency.

Hint 1: Recall that $\dfrac{(n-1)\cdot S^2}{\sigma^2} = \dfrac{\sum\left(X_i - \overline{X}\right)^2}{\sigma^2}$ has a $\chi^2(n-1)$ distribution.

Hint 2: Recall that $E(\chi^2(r)) = r$, $Var(\chi^2(r)) = 2r$.

$$Var(S^2) = \left(\frac{\sigma^2}{n-1}\right)^2 \cdot Var\left(\frac{(n-1)\cdot S^2}{\sigma^2}\right) = \left(\frac{\sigma^2}{n-1}\right)^2 \cdot 2(n-1) = \frac{2\sigma^4}{n-1}.$$

$\mathrm{E}(\mathrm{S}^2) = \sigma^2 = \theta = k(\theta).$ $\qquad\qquad k'(\theta) = 1.$

Rao-Cramer lower bound for S^2: $\quad \dfrac{(k'(\theta))^2}{n\,\mathrm{I}(\theta)} = \dfrac{1}{n\,\mathrm{I}(\theta)} = \dfrac{2\sigma^4}{n}.$

S^2 is NOT an efficient estimator of $\theta = \sigma^2$.

$$(\text{efficiency of } \mathrm{S}^2) = \frac{\dfrac{2\sigma^4}{n}}{\dfrac{2\sigma^4}{n-1}} = \frac{n-1}{n}.$$

note that $(\text{efficiency of } \mathrm{S}^2) \to 1$ as $n \to \infty$.

c) Suppose that μ is known.

i) Find the maximum likelihood estimator for σ^2.

$$\mathrm{L}\left(\sigma^2; x_1, x_2, \ldots, x_n\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}(x_i - \mu)^2 \right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta}} \right)^n \exp\left\{ -\frac{1}{2\theta} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}.$$

$$\ln \mathrm{L}(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i - \mu)^2.$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\ln \mathrm{L}(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i - \mu)^2 = 0.$$

$$\Rightarrow \qquad \hat{\theta} = \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(\mathrm{X}_i - \mu)^2.$$

ii) Is the maximum likelihood estimator for σ^2 unbiased?

Hint 3: Recall that $\dfrac{\sum\left(X_i-\mu\right)^2}{\sigma^2}$ has a $\chi^2(n)$ distribution.

$$E(\chi^2(n)) = n. \qquad \Rightarrow \qquad E\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = n.$$

$$\Rightarrow \qquad E(\hat{\sigma}^2) = \sigma^2. \qquad\qquad \hat{\sigma}^2 \text{ is unbiased for } \sigma^2.$$

OR

$$E(\hat{\sigma}^2) = E(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2) = \frac{1}{n}\sum_{i=1}^{n}E\left[(X_i-\mu)^2\right] = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \sigma^2.$$

iii) Is the maximum likelihood estimator for σ^2 efficient? If not, find its efficiency.

$$\operatorname{Var}(\chi^2(n)) = 2n. \qquad \Rightarrow \qquad \operatorname{Var}\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = 2n.$$

$$\operatorname{Var}(\hat{\sigma}^2) = \left(\frac{\sigma^2}{n}\right)^2 \cdot \operatorname{Var}\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = \left(\frac{\sigma^2}{n}\right)^2 \cdot 2n = \frac{2\sigma^4}{n}.$$

$$E(\hat{\sigma}^2) = \sigma^2 = \theta = k(\theta). \qquad\qquad k'(\theta) = 1.$$

$$I(\theta) = I(\sigma^2) = \frac{1}{2\sigma^4}.$$

Rao-Cramer lower bound for $\hat{\sigma}^2$: $\quad \dfrac{\left(k'(\theta)\right)^2}{n\,I(\theta)} = \dfrac{1}{n\,I(\theta)} = \dfrac{2\sigma^4}{n}.$

$\Rightarrow \qquad \hat{\sigma}^2$ is an efficient estimator of $\theta = \sigma^2$.

Example 1: (continued)

Consider $\mathrm{N}(\mu, \sigma^2)$ distribution. Determine the Fisher information $\mathrm{I}(\sigma)$.

$$\ln f(x; \mu, \sigma) = -\frac{1}{2}\ln(2\pi) - \ln(\sigma) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\sigma} \ln f(x; \mu, \sigma^2) = -\frac{1}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}$$

$$\frac{\partial^2}{\partial\sigma^2} \ln f(x; \mu, \sigma^2) = \frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4}$$

$$\mathrm{I}(\sigma) = -\mathrm{E}\left[\frac{\partial^2}{\partial\sigma^2} f(x; \mu, \sigma^2)\right] = -\frac{1}{\sigma^2} + \frac{3\,\mathrm{E}[(\mathrm{X}-\mu)^2]}{\sigma^4} = -\frac{1}{\sigma^2} + \frac{3\sigma^2}{\sigma^4} = \frac{2}{\sigma^2}$$

Example 2: Let X be an Exponential(θ) random variable. That is,

$$f(x;\theta) = \frac{1}{\theta}\cdot e^{-x/\theta}, \qquad x > 0. \qquad (\theta > 0)$$

$$\ln f(x; \theta) = -\ln\theta - \frac{x}{\theta}$$

$$\frac{\partial}{\partial\theta} \ln f(x; \theta) = -\frac{1}{\theta} + \frac{x}{\theta^2} \qquad \frac{\partial^2}{\partial\theta^2} \ln f(x; \theta) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$\mathrm{I}(\theta) = -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2} \ln f(\mathrm{X};\theta)\right] = -\frac{1}{\theta^2} + \frac{2\,\mathrm{E}(\mathrm{X})}{\theta^3} = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$$

OR

$$\mathrm{I}(\theta) = \mathrm{Var}\left[\frac{\partial}{\partial\theta} \ln f(\mathrm{X};\theta)\right] = \mathrm{Var}\left[-\frac{1}{\theta} + \frac{\mathrm{X}}{\theta^2}\right] = \frac{1}{\theta^4}\mathrm{Var}(\mathrm{X}) = \frac{1}{\theta^4}\theta^2 = \frac{1}{\theta^2}$$

Example 3:

Let X be a Poisson(λ) random variable. That is,

$$f(x;\lambda) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, 3, \ldots . \qquad (\lambda > 0)$$

$$\ln f(x;\lambda) = x \cdot \ln\lambda - \lambda - \ln x!$$

$$\frac{\partial}{\partial\lambda} \ln f(x;\lambda) = \frac{x}{\lambda} - 1 \qquad\qquad \frac{\partial^2}{\partial\lambda^2} \ln f(x;\lambda) = -\frac{x}{\lambda^2}$$

$$\mathrm{I}(\lambda) = -\mathrm{E}\left[\frac{\partial^2}{\partial\lambda^2} \ln f(\mathrm{X};\lambda)\right] = -\mathrm{E}\left[-\frac{\mathrm{X}}{\lambda^2}\right] = \frac{1}{\lambda^2}\mathrm{E}(\mathrm{X}) = \frac{1}{\lambda^2}\lambda = \frac{1}{\lambda}$$

OR

$$\mathrm{I}(\lambda) = \mathrm{Var}\left[\frac{\partial}{\partial\lambda} \ln f(\mathrm{X};\lambda)\right] = \mathrm{Var}\left[\frac{\mathrm{X}}{\lambda} - 1\right] = \frac{1}{\lambda^2}\mathrm{Var}(\mathrm{X}) = \frac{1}{\lambda^2}\lambda = \frac{1}{\lambda}$$

$\hat{\lambda} = \overline{\mathrm{X}}$ – maximum likelihood estimator of λ.

$$\mathrm{Var}(\hat{\lambda}) = \mathrm{Var}(\overline{\mathrm{X}}) = \frac{\sigma^2}{n} = \frac{\lambda}{n} = \frac{1}{n \cdot \mathrm{I}(\lambda)}.$$

$\Rightarrow \quad \hat{\lambda} = \overline{\mathrm{X}}$ is an efficient estimator of λ.

Consider S^2. $\qquad \mathrm{E}(\mathrm{S}^2) = \sigma^2 = \lambda$ – S^2 is an unbiased estimator of λ.

$$\mathrm{Var}(\mathrm{S}^2) = \frac{\lambda(2\lambda n + n - 1)}{n(n-1)} \qquad \Rightarrow$$

S^2 is NOT an efficient estimator of λ,

its efficiency $= \dfrac{n-1}{2\lambda n + n - 1}$.

Example 4:

Let X be a Bernoulli(p) random variable. That is,

$$f(x;p) = (1-p)^{1-x} p^{x}, \qquad x = 0, 1.$$

$$\ln f(x;p) = (1-x)\cdot\ln(1-p) + x\cdot\ln p$$

$$\frac{\partial}{\partial p}\ln f(x;p) = -\frac{1-x}{1-p} + \frac{x}{p} = \frac{x}{p(1-p)} - \frac{1}{1-p}$$

$$\frac{\partial^2}{\partial p^2}\ln f(x;p) = -\frac{1-x}{(1-p)^2} - \frac{x}{p^2}$$

$$\mathrm{I}(p) = -\mathrm{E}\left[\frac{\partial^2}{\partial p^2}\ln f(\mathrm{X};p)\right] = \frac{1-\mathrm{E}(\mathrm{X})}{(1-p)^2} + \frac{\mathrm{E}(\mathrm{X})}{p^2} = \frac{1}{1-p} + \frac{1}{p} = \frac{1}{p(1-p)}$$

OR

$$\mathrm{I}(p) = \mathrm{Var}\left[\frac{\partial}{\partial p}\ln f(\mathrm{X};p)\right] = \mathrm{Var}\left[\frac{\mathrm{X}}{p(1-p)} - \frac{1}{1-p}\right] = \frac{\mathrm{Var}(\mathrm{X})}{p^2(1-p)^2} = \frac{1}{p(1-p)}$$

$\hat{p} = \dfrac{\text{number of "Successes"}}{\text{number of attempts}} = \overline{\mathrm{X}}$ – maximum likelihood estimator of p.

$$\mathrm{Var}(\hat{p}) = \mathrm{Var}(\overline{\mathrm{X}}) = \frac{\sigma^2}{n} = \frac{p(1-p)}{n} = \frac{1}{n\cdot\mathrm{I}(p)}.$$

$\Rightarrow$ $\hat{p}$ is an efficient estimator of p.

Also $\hat{p}$ is approximately $\mathrm{N}\left(p, \dfrac{p(1-p)}{n}\right)$ for large n. (recall STAT 400)

$\hat{p} \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ has an approximate $100(1-\alpha)\%$ confidence level for large n.

Example 4.5:

Consider $\text{Geometric}(p)$ distribution. That is,

$$p_X(k) = p \cdot (1-p)^{k-1}, \qquad k = 1, 2, 3, \ldots .$$

Determine the Fisher information $I(p)$.

$$\ln p_X(k) = \ln p + (k-1) \cdot \ln(1-p)$$

$$\frac{\partial}{\partial p} \ln p_X(k) = \frac{1}{p} - \frac{k-1}{1-p} = \frac{1}{p(1-p)} - \frac{k}{1-p}$$

$$\frac{\partial^2}{\partial p^2} \ln p_X(k) = -\frac{1}{p^2} - \frac{k-1}{(1-p)^2}$$

$$I(p) = -E\left[\frac{\partial^2}{\partial p^2} \ln p_X(k) \right] = \frac{1}{p^2} + \frac{E(X)-1}{(1-p)^2} = \frac{1}{p^2} + \frac{\frac{1}{p}-1}{(1-p)^2}$$

$$= \frac{1}{p^2} + \frac{1-p}{p(1-p)^2} = \frac{1}{p^2(1-p)}$$

OR

$$I(p) = \operatorname{Var}\left[\frac{\partial}{\partial p} \ln f(X;p) \right] = \operatorname{Var}\left[\frac{1}{p(1-p)} - \frac{X}{1-p} \right] = \frac{\operatorname{Var}(X)}{(1-p)^2}$$

$$= \frac{\frac{1-p}{p^2}}{(1-p)^2} = \frac{1}{p^2(1-p)}$$

$\Rightarrow$ For large n, $\hat{p}$ is approximately $N\left(p, \frac{1}{n \cdot I(p)}\right) = N\left(p, \frac{p^2(1-p)}{n}\right)$.

(recall Examples for 10/30/2020 (1) Example **1½**)

Example 5: $f(x;\theta) = \frac{1}{\theta} \cdot x^{\frac{1}{\theta}-1}$, $0 < x < 1$.

$\hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^{n} \ln X_i$ is an unbiased estimator of θ. (Examples for 10/16/2020 (1))

Is $\hat{\theta}$ an efficient estimator of θ? If not, find its efficiency.

Recall: Let $W_i = -\ln X_i$, $i = 1, 2, \ldots, n$.

Then $W_1, W_2, \ldots, W_n$ are i.i.d. Exponential(θ).

$\hat{\theta} = \overline{W}$.

$$E(\hat{\theta}) = \mu_W = \theta. \qquad \mathrm{Var}(\hat{\theta}) = \frac{\sigma_W^2}{n} = \frac{\theta^2}{n}.$$

$$\ln f(x;\theta) = -\ln\theta + \left(\frac{1}{\theta} - 1\right) \cdot \ln x$$

$$\frac{\partial}{\partial\theta} \ln f(x;\theta) = -\frac{1}{\theta} - \frac{1}{\theta^2} \cdot \ln x \qquad \frac{\partial^2}{\partial\theta^2} \ln f(x;\theta) = \frac{1}{\theta^2} + \frac{2}{\theta^3} \cdot \ln x$$

$$I(\theta) = \mathrm{Var}\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right] = \frac{1}{\theta^4} \cdot \mathrm{Var}(-\ln X) = \frac{1}{\theta^2}$$

OR

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right] = -\frac{1}{\theta^2} + \frac{2}{\theta^3} \cdot E(-\ln X) = \frac{1}{\theta^2}$$

Rao-Cramer lower bound $= \dfrac{1}{n \cdot I(\theta)} = \dfrac{\theta^2}{n}$.

$\mathrm{Var}(\hat{\theta})$ DOES attain its Rao-Cramer lower bound. $\hat{\theta}$ is an efficient estimator of θ.

Example 6:

Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

Recall: $\quad Y = \sum_{i=1}^{n} X_i^2$ has Gamma$(\alpha = 2n,\ \theta = \frac{1}{\lambda})$ distribution.

$$\hat{\hat{\lambda}} = \frac{2n-1}{\sum_{i=1}^{n} X_i^2}$$ is an unbiased estimator of λ. (Examples for 10/19/2020 (2))

Is $\hat{\hat{\lambda}}$ an efficient estimator of λ? If not, find its efficiency.

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{x} \cdot \frac{\lambda^{2n}}{\Gamma(2n)} x^{2n-1} e^{-\lambda x}\,dx$$

$$= \frac{\lambda}{(2n-1)} \cdot \int_0^\infty \frac{\lambda^{2n-1}}{\Gamma(2n-1)} x^{2n-2} e^{-\lambda x}\,dx = \frac{\lambda}{2n-1}.$$

$$E\left[\left(\frac{1}{Y}\right)^2\right] = \int_0^\infty \frac{1}{x^2} \cdot \frac{\lambda^{2n}}{\Gamma(2n)} x^{2n-1} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^2}{(2n-1)(2n-2)} \cdot \int_0^\infty \frac{\lambda^{2n-2}}{\Gamma(2n-2)} x^{2n-3} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^2}{(2n-1)(2n-2)}.$$

$$\operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\lambda^2}{(2n-1)(2n-2)} - \frac{\lambda^2}{(2n-1)^2} = \frac{\lambda^2}{(2n-1)^2(2n-2)}.$$

$$\operatorname{Var}(\hat{\hat{\lambda}}) = (2n-1)^2 \times \operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\lambda^2}{2n-2}.$$

$$\ln f(x;\lambda) = -\lambda\cdot x^2 + \ln 2 + 2\ln\lambda + 3\ln x$$

$$\frac{\partial}{\partial\lambda}\ln f(x;\lambda) = -x^2 + \frac{2}{\lambda}$$

$$\frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda) = -\frac{2}{\lambda^2}$$

$$\mathrm{I}(\lambda) = -\mathrm{E}\left[\frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda)\right] = \frac{2}{\lambda^2}.$$

OR

$$\mathrm{I}(\lambda) = \mathrm{Var}\left[\frac{\partial}{\partial\lambda}\ln f(x;\lambda)\right] = \mathrm{Var}\left[-\mathrm{X}^2 + \frac{2}{\lambda}\right] = \mathrm{Var}(\mathrm{X}^2) = \alpha\,\theta^2 = \frac{2}{\lambda^2}.$$

OR $$= \mathrm{E}(\mathrm{X}^4) - [\mathrm{E}(\mathrm{X}^2)]^2 = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2}.$$

Rao-Cramer lower bound $= \dfrac{1}{n\cdot\mathrm{I}(\lambda)} = \dfrac{\lambda^2}{2n}.$

$$\mathrm{Var}(\hat{\hat{\lambda}}) = \frac{\lambda^2}{2n-2} > \frac{\lambda^2}{2n}.$$

$\mathrm{Var}(\hat{\hat{\lambda}})$ does NOT attain its Rao-Cramer lower bound.

$\Rightarrow$ $\hat{\hat{\lambda}}$ is NOT an efficient estimator of λ,

its efficiency $= \dfrac{2n-2}{2n} = \dfrac{n-1}{n} \to 1$ as $n\to\infty$.

$\Rightarrow$ For large n, $\hat{\lambda}$ is approximately $\mathrm{N}\left(\lambda,\ \dfrac{1}{n\cdot\mathrm{I}(\lambda)}\right) = \mathrm{N}\left(\lambda,\ \dfrac{\lambda^2}{2n}\right).$

(recall Examples for 10/30/2020 (1) Example **2**)

Example 7: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Gamma$(\alpha,\ \beta =$ "usual θ"$)$ distribution. That is,

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\, x^{\alpha-1} e^{-x/\beta}, \qquad 0 < x < \infty.$$

a) Find a sufficient statistics for α.

$$\prod_{i=1}^{n} f(x_i;\alpha,\beta) = \left(\frac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\right)^{n} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \exp\left\{-\frac{1}{\beta}\sum_{i=1}^{n} x_i\right\}.$$

$\Rightarrow$ By Factorization Theorem, $Y = \prod_{i=1}^{n} X_i$ is a sufficient statistic for α.

OR

$$f(x;\alpha,\beta) = \exp\left\{(\alpha-1)\cdot\ln x - \frac{1}{\beta}\cdot x - \ln\Gamma(\alpha) - \alpha\cdot\ln\beta\right\}. \qquad \Rightarrow \qquad K(x) = \ln x.$$

$\Rightarrow$ $Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for α.

b) Find a sufficient statistics for β.

$$\prod_{i=1}^{n} f(x_i;\alpha,\beta) = \left(\frac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\right)^{n} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \exp\left\{-\frac{1}{\beta}\sum_{i=1}^{n} x_i\right\}.$$

$\Rightarrow$ By Factorization Theorem, $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for β.

OR

$$f(x;\alpha,\beta) = \exp\left\{(\alpha-1)\cdot\ln x - \frac{1}{\beta}\cdot x - \ln\Gamma(\alpha) - \alpha\cdot\ln\beta\right\}. \qquad \Rightarrow \qquad K(x) = x.$$

$\Rightarrow$ $Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for β.

c) Determine the Fisher information $\mathrm{I}(\beta)$.

$$\ln f(x;\theta) = -\ln\Gamma(\alpha) - \alpha\cdot\ln\theta + (\alpha-1)\cdot\ln x - \frac{x}{\theta}.$$

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = -\frac{\alpha}{\theta} + \frac{x}{\theta^2}. \qquad \frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = \frac{\alpha}{\theta^2} - \frac{2x}{\theta^3}.$$

$$\mathrm{I}(\theta) = -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\ln f(\mathrm{X};\theta)\right] = -\mathrm{E}\left[\frac{\alpha}{\theta^2} - \frac{2\mathrm{X}}{\theta^3}\right] = -\frac{\alpha}{\theta^2} + \frac{2\,\mathrm{E}(\mathrm{X})}{\theta^3} = -\frac{\alpha}{\theta^2} + \frac{2\alpha\theta}{\theta^3} = \frac{\alpha}{\theta^2}.$$

OR

$$\mathrm{I}(\theta) = \mathrm{Var}\left[\frac{\partial}{\partial\theta}\ln f(\mathrm{X};\theta)\right] = \mathrm{Var}\left[-\frac{\alpha}{\theta} + \frac{\mathrm{X}}{\theta^2}\right] = \frac{\mathrm{Var}(\mathrm{X})}{\theta^4} = \frac{\alpha\theta^2}{\theta^4} = \frac{\alpha}{\theta^2}.$$

Suppose α is known.

d) Find the maximum likelihood estimator $\hat{\beta}$ of β.

$$\mathrm{L}(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\right)^{n}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}\exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right\}.$$

$$\ln\mathrm{L}(\theta) = n\ln\Gamma(\alpha) - n\alpha\ln\theta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \frac{1}{\theta}\sum_{i=1}^{n} x_i.$$

$$\frac{d}{d\theta}\ln\mathrm{L}(\theta) = -\frac{n\alpha}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0. \qquad \hat{\theta} = \frac{1}{n\alpha}\sum_{i=1}^{n}\mathrm{X}_i = \frac{\overline{\mathrm{X}}}{\alpha}.$$

e) Is $\hat{\beta}$ an unbiased estimator for β? *Justify your answer.*

$$E(\hat{\theta}) = \frac{\mu}{\alpha} = \frac{\alpha\theta}{\alpha} = \theta.$$

$\hat{\theta}$ is an unbiased estimator for θ.

f) Is $\hat{\beta}$ an efficient estimator for β? *Justify your answer.*

$$\operatorname{Var}(\hat{\theta}) = \frac{1}{\alpha^2} \cdot \frac{\sigma^2}{n} = \frac{1}{\alpha^2} \cdot \frac{\alpha\theta^2}{n} = \frac{\theta^2}{n\alpha}.$$

Rao-Cramer Lower Bound: $$\frac{1}{n \cdot I(\theta)} = \frac{\theta^2}{n\alpha}.$$

$\hat{\theta}$ is an efficient estimator for θ. (Variance of $\hat{\theta}$ attains Rao-Cramer Lower Bound)