Examples for 11/06/2020 (5)

Fall 2020
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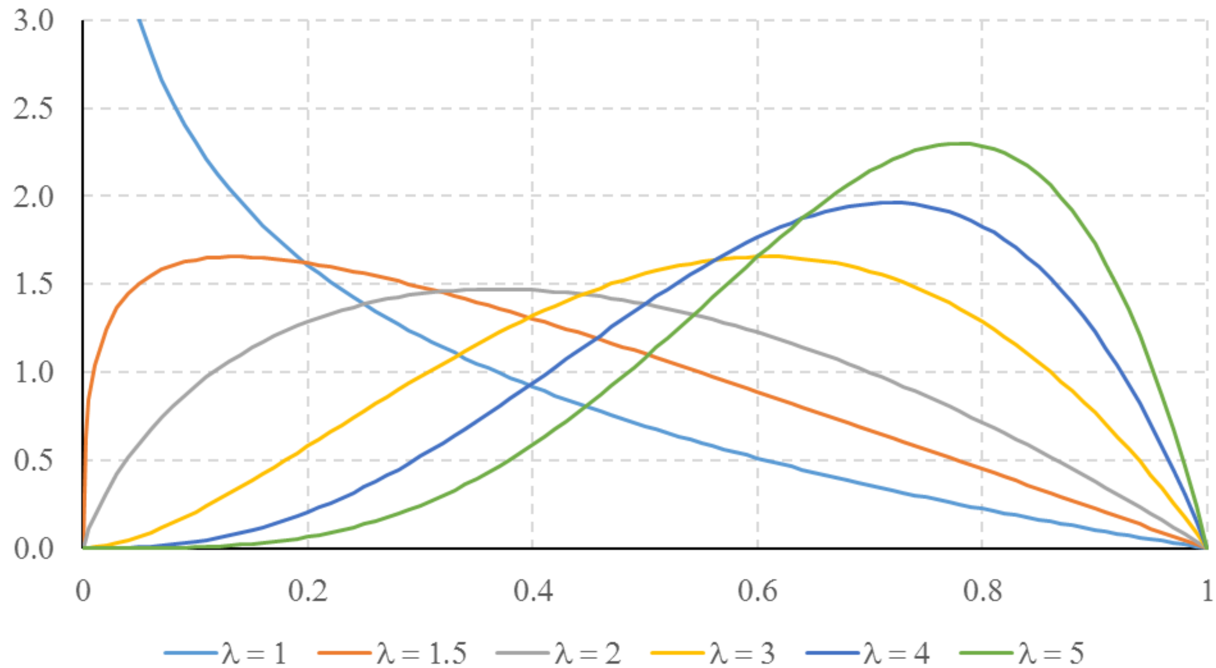
1. Let $\lambda > 0$. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

Note: Since $0 < x < 1$, $\ln x < 0$.

A better way to write this density function would be

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1} = \lambda^2 (-\ln x) \cdot x^{\lambda-1}, \qquad 0 < x < 1.$$



a) Obtain a method of moments estimator for λ, $\tilde{\lambda}$.

b) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Obtain a method of moments estimate for λ, $\tilde{\lambda}$.

c) Show that $\tilde{\lambda}$ is a consistent estimator of λ.

(NOT enough to say "because it is a method of moments estimator")

d) Obtain the maximum likelihood estimator for λ, $\hat{\lambda}$.

e) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Obtain the maximum likelihood estimate for λ, $\hat{\lambda}$.

f) Show that $W = -\ln X$ has a Gamma distribution. What are its parameters?

$$\Rightarrow \qquad Y = -\sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} W_i \quad \text{has a Gamma distribution.} \qquad !!!$$

g) Show that $\hat{\lambda}$ is a consistent estimator of λ.

(NOT enough to say "because it is the maximum likelihood estimator")

h) Is the maximum likelihood estimator $\hat{\lambda}$ an unbiased estimator of λ?
If $\hat{\lambda}$ is not an unbiased estimator of λ, construct an unbiased estimator of λ based on $\hat{\lambda}$.

i) Suggest a confidence interval for λ with $(1 - \alpha)100\%$ confidence level.

j) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Use part (i) to construct a 95% confidence interval for λ.

k) Find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for λ.

l) Find the Fisher information $I(\lambda)$.

Answers:

1. Let $\lambda > 0$. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

Note: Since $0 < x < 1$, $\ln x < 0$.

A better way to write this density function would be

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1} = \lambda^2 \left(-\ln x\right) \cdot x^{\lambda-1}, \qquad 0 < x < 1.$$

a) Obtain a method of moments estimator for λ, $\tilde{\lambda}$.

$$1 = \int_0^1 \left(-\lambda^2 \ln x \cdot x^{\lambda-1}\right) dx. \qquad \Rightarrow \qquad \int_0^1 \left(-\ln x \cdot x^{\lambda-1}\right) dx = \frac{1}{\lambda^2}, \qquad \lambda > 0.$$

$$E(X^k) = \int_0^1 x^k \cdot \left(-\lambda^2 \ln x \cdot x^{\lambda-1}\right) dx = \lambda^2 \int_0^1 \left(-\ln x \cdot x^{\lambda+k-1}\right) dx = \frac{\lambda^2}{(\lambda+k)^2},$$

$$k > -\lambda.$$

$$\mu = E(X) = E(X^1) = \frac{\lambda^2}{(\lambda+1)^2}.$$

OR $$\mu = E(X) = \int_0^1 x \cdot \left(-\lambda^2 \ln x \cdot x^{\lambda-1}\right) dx = \ldots \text{ by parts } \ldots$$

$$\overline{X} = \frac{\tilde{\lambda}^2}{\left(\tilde{\lambda}+1\right)^2}. \qquad \Rightarrow \qquad \sqrt{\overline{X}} = \frac{\tilde{\lambda}}{\tilde{\lambda}+1}. \qquad \Rightarrow \qquad \tilde{\lambda} = \frac{\sqrt{\overline{X}}}{1-\sqrt{\overline{X}}}.$$

b) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Obtain a method of moments estimate for λ, $\tilde{\lambda}$.

$\overline{x} = 0.70.$

$$\tilde{\lambda} = \frac{\sqrt{0.70}}{1-\sqrt{0.70}} \approx \mathbf{5.1222}.$$

c) Show that $\tilde{\lambda}$ is a consistent estimator of λ.

(NOT enough to say "because it is a method of moments estimator")

By WLLN, $$\overline{X} \xrightarrow{P} \mu = \frac{\lambda^2}{(\lambda+1)^2}.$$

Since $g(x) = \dfrac{\sqrt{x}}{1-\sqrt{x}}$ is continuous at μ,

$$\tilde{\lambda} = \frac{\sqrt{\overline{X}}}{1-\sqrt{\overline{X}}} = g(\overline{X}) \xrightarrow{P} g(\mu) = \frac{\sqrt{\mu}}{1-\sqrt{\mu}} = \frac{\dfrac{\lambda}{\lambda+1}}{1-\dfrac{\lambda}{\lambda+1}} = \lambda.$$

$\tilde{\lambda}$ is a consistent estimator of λ.

d) Obtain the maximum likelihood estimator for λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda) = \prod_{i=1}^{n} \lambda^2 (-\ln x_i)\cdot x_i^{\lambda-1} = \lambda^{2n} \prod_{i=1}^{n} (-\ln x_i)\cdot\left(\prod_{i=1}^{n} x_i\right)^{\lambda-1}.$$

$$\ln L(\lambda) = 2n\ln\lambda + \sum_{i=1}^{n} \ln(-\ln x_i) + (\lambda-1)\sum_{i=1}^{n} \ln x_i.$$

$$\frac{d}{d\lambda}\ln L(\lambda) = \frac{2n}{\lambda} + \sum_{i=1}^{n} \ln x_i = 0. \qquad \Rightarrow \qquad \hat{\lambda} = -\frac{2n}{\sum_{i=1}^{n} \ln X_i}.$$

e) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Obtain the maximum likelihood estimate for λ, $\hat{\lambda}$.

$$\sum_{i=1}^{n} \ln x_i \approx -1.60147. \qquad \hat{\lambda} \approx -\frac{8}{-1.60147} \approx \mathbf{4.9954}.$$

f) Show that $W = -\ln X$ has a Gamma distribution. What are its parameters?

Let $W = -\ln X$.

$$x = e^{-w} \qquad \frac{dx}{dw} = -e^{-w}$$

$$f_W(w) = f_X(e^{-w})\cdot\left|\frac{dx}{dw}\right| = \lambda^2 w e^{-(\lambda-1)w}\cdot e^{-w} = \frac{\lambda^2}{\Gamma(2)} w^{2-1} e^{-\lambda w}, \qquad w > 0.$$

$\Rightarrow$ $W = -\ln X$ has a Gamma$\left(\alpha = 2, \theta = \frac{1}{\lambda}\right)$ distribution.

g) Show that $\hat{\lambda}$ is a consistent estimator of λ.

(NOT enough to say "because it is the maximum likelihood estimator")

By WLLN, $\overline{\mathrm{W}} \xrightarrow{P} \mu_{\mathrm{W}} = \alpha\theta = \dfrac{2}{\lambda}$.

OR $\mu_{\mathrm{W}} = \mathrm{E}(-\ln \mathrm{X}) = \int_0^1 \lambda^2 (\ln x)^2 \cdot x^{\lambda-1}\, dx = \ldots$ by parts … twice …

Since $g(x) = \dfrac{2}{x}$ is continuous at μ_{W},

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} \mathrm{W}_i} = \frac{2}{\overline{\mathrm{W}}} = g(\overline{\mathrm{W}}) \xrightarrow{P} g(\mu_{\mathrm{W}}) = g\left(\frac{2}{\lambda}\right) = \lambda.$$

$\hat{\lambda}$ is a consistent estimator of λ.

h) Is the maximum likelihood estimator $\hat{\lambda}$ an unbiased estimator of λ?
If $\hat{\lambda}$ is not an unbiased estimator of λ, construct an unbiased estimator of λ based on $\hat{\lambda}$.

$\mathrm{Y} = -\sum_{i=1}^{n} \ln \mathrm{X}_i = \sum_{i=1}^{n} \mathrm{W}_i$ has a Gamma$\left(\alpha = 2n, \theta = \dfrac{1}{\lambda}\right)$ distribution. ! ! !

$$\hat{\lambda} = -\frac{2n}{\sum_{i=1}^{n} \ln \mathrm{X}_i} = \frac{2n}{\mathrm{Y}}.$$

If T_α has a Gamma$(\alpha, \theta = {}^1\!/\!_\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^k) = \frac{\theta^k\, \Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{\lambda^k\, \Gamma(\alpha)}, \qquad k > -\alpha.$$

$$E\left(\frac{1}{Y}\right) = E\left(Y^{-1}\right) = \frac{\Gamma(2n-1)}{\lambda^{-1}\,\Gamma(2n)} = \frac{\lambda}{2n-1}.$$

$$E(\hat{\beta}) = E\left(\frac{n}{Y}\right) = 2n\,E\left(\frac{1}{Y}\right) = 2n\cdot\frac{\lambda}{2n-1} = \frac{2n}{2n-1}\cdot\lambda = \lambda + \frac{\lambda}{2n-1} \neq \lambda.$$

$\hat{\lambda}$ is NOT an unbiased estimator of λ. $\quad$ $\operatorname{bias}(\hat{\lambda}) = E(\hat{\lambda}) - \lambda = \dfrac{\lambda}{2n-1}.$

Consider $\quad \hat{\hat{\lambda}} = \dfrac{2n-1}{2n}\cdot\hat{\lambda} = -\dfrac{2n-1}{\sum_{i=1}^{n} \ln X_i}.$

Then $\quad E(\hat{\hat{\lambda}}) = \dfrac{2n-1}{2n}\cdot E(\hat{\lambda}) = \lambda.$ $\quad$ $\hat{\hat{\lambda}}$ is an unbiased estimator of λ.

i) Suggest a confidence interval for λ with $(1-\alpha)100\%$ confidence level.

$Y = -\sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} W_i$ has a Gamma distribution with $\alpha = 2n$ and $\theta = \dfrac{1}{\lambda}$.

If T_α has a Gamma$\left(\alpha, \theta = 1/\lambda\right)$ distribution, where α is an integer, then $2T_\alpha/\theta = 2\lambda T_\alpha$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$2Y/\theta = -2\lambda \sum_{i=1}^{n} \ln X_i$ has a chi-square distribution with $r = 2\alpha = 4n$ d.f.

$$\Rightarrow \quad P\left(\chi^2_{1-\alpha/2}(4n) < -2\lambda \sum_{i=1}^{n} \ln X_i < \chi^2_{\alpha/2}(4n)\right) = 1-\alpha.$$

$$\Rightarrow \qquad P\left(\frac{\chi^2_{1-\alpha/2}(4n)}{-2\sum\limits_{i=1}^{n} \ln X_i} < \beta < \frac{\chi^2_{\alpha/2}(4n)}{-2\sum\limits_{i=1}^{n} \ln X_i} \right) = 1-\alpha.$$

Note: Since $0 < x < 1$, $\ln x < 0$.

A $(1-\alpha)\,100\,\%$ confidence interval for λ: $\left(\dfrac{\chi^2_{1-\alpha/2}(4n)}{-2\sum\limits_{i=1}^{n} \ln X_i}, \dfrac{\chi^2_{\alpha/2}(4n)}{-2\sum\limits_{i=1}^{n} \ln X_i} \right)$.

j) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Use part (i) to construct a 95% confidence interval for λ.

$$\sum_{i=1}^{n} \ln x_i \approx -1.60147.$$

$$\chi^2_{0.975}(16) = 6.908, \qquad \chi^2_{0.025}(16) = 28.84.$$

$$\left(\frac{6.908}{2 \cdot 1.60147}, \frac{28.84}{2 \cdot 1.60147} \right) \qquad \mathbf{(2.157, 9.004)}$$

k) Find a sufficient statistic $Y = u(X_1, X_2, \dots, X_n)$ for λ.

$$\prod_{i=1}^{n} f(x_i;\lambda) = \prod_{i=1}^{n} \lambda^2 (-\ln x_i) \cdot x_i^{\lambda-1} = \lambda^{2n} \prod_{i=1}^{n} (-\ln x_i) \cdot \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1}.$$

By Factorization Theorem, $\prod_{i=1}^{n} X_i$ is a sufficient statistic for λ.

OR

$$f(x;\lambda) = \exp\{(\lambda-1)\ln x + 2\ln\lambda + \ln(-\ln x)\}. \qquad K(x) = \ln x.$$

$\Rightarrow \quad \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for λ.

$\Rightarrow \quad -\sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for λ.

l) Find the Fisher information $\mathrm{I}(\lambda)$.

$$\ln f(x;\lambda) = 2\ln\lambda + \ln(-\ln x) + (\lambda-1)\ln x.$$

$$\frac{\partial}{\partial\lambda}\ln f(x;\lambda) = \frac{2}{\lambda} + \ln x. \qquad\qquad \frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda) = -\frac{2}{\lambda^2}.$$

$$\begin{aligned}\mathrm{I}(\lambda) &= \mathrm{Var}\Big[\frac{\partial}{\partial\lambda}\ln f(\mathrm{X};\lambda)\Big]\\ &= \mathrm{Var}\Big[\frac{2}{\lambda}+\ln \mathrm{X}\Big]\\ &= \mathrm{Var}(\mathrm{W})\\ &= \alpha\,\theta^2 = \frac{2}{\lambda^2}.\end{aligned}$$

$$\begin{aligned}\mathrm{I}(\lambda) &= -\mathrm{E}\Big[\frac{\partial^2}{\partial\lambda^2}\ln f(\mathrm{X};\lambda)\Big]\\ &= -\mathrm{E}\Big[-\frac{2}{\lambda^2}\Big]\\ &= \frac{2}{\lambda^2}.\end{aligned}$$

For fun:

$$\mathrm{E}\left(\frac{1}{-\ln X}\right) = \int_0^1 \frac{1}{-\ln x} \cdot \left(-\lambda^2 \ln x \cdot x^{\lambda-1}\right) dx = \int_0^1 \lambda^2 x^{\lambda-1}\, dx = \lambda.$$

Consider $\hat{\tilde{\lambda}} = \overline{\dfrac{1}{-\ln X}} = \dfrac{1}{n} \cdot \displaystyle\sum_{i=1}^{n} \dfrac{1}{-\ln X_i} = \dfrac{1}{n} \cdot \displaystyle\sum_{i=1}^{n} \dfrac{1}{\ln \dfrac{1}{X_i}}.$

By WLLN, $\hat{\tilde{\lambda}} = \overline{\dfrac{1}{-\ln X}} \xrightarrow{P} \mathrm{E}\left(\dfrac{1}{-\ln X}\right) = \lambda.$

$\hat{\tilde{\lambda}}$ is a consistent estimator of λ.

$$\mathrm{E}(\hat{\tilde{\lambda}}) = \mathrm{E}\left(\overline{\frac{1}{-\ln X}}\right) = \mathrm{E}\left(\frac{1}{-\ln X}\right) = \lambda.$$

$\hat{\tilde{\lambda}}$ is an unbiased estimator of λ.

Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.

Then $\hat{\tilde{\lambda}} \approx 4.4669.$