Examples for 08/26/2020 (3)

1. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{3}{2{,}000}\sqrt{x}\,, \qquad 0 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) What was the class average, $E(X)$?

b) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

2. 1. (continued)

As a way of "curving" the results, the professor announces that he will replace each person's grade, X, with a new grade, $Y = g(X)$, where $g(X) = 10\sqrt{X}$.

c) Find the p.d.f. that describes the new grades, Y.

d) What is the new class average, $E(Y)$?

e) What is the new class median?

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* ☺ The probability distribution is fictional. The actual grades were slightly better than these.

3. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{1}{5{,}000}(100 - x), \qquad 0 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) What was the class average, $E(X)$?

b) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

c) What was the proportion of the students who received scores above 60? That is, find $P(X > 60)$?

4. 3. (continued)

As a way of "curving" the results, the professor announces that he will replace each person's grade, X, with a new grade, $Y = g(X)$, where $g(X) = 10\sqrt{X}$.

d) Find the p.d.f. that describes the new grades, Y.

e) Has the professor's strategy been successful in raising the class average above 60? What is the new class average, $E(Y)$?

f) What is the new class median?

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5. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{1}{(x+25)\ln 5}, \qquad 0 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": Should be $F_X(0) = 0$, $F_X(100) = 1$.

b) What was the class average, $E(X)$?

c) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

6. 5. (continued)

As a way of "curving" the results, the instructor announced that he would replace each person's grade, X, with a new grade, $Y = g(X)$, where $g(X) = 20\,(X+25)^{1/3}$.

d) Use part (a) and the c.d.f. approach to find the c.d.f. that describes the new grades, Y.

e) Use the change-of-variable technique to find the p.d.f. that describes the new grades, Y.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

f) What is the new class average, $E(Y)$?

g) What is the new class median?

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7. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{2x+5}{10{,}000}, \qquad 20 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": Should be $F_X(20) = 0$, $F_X(100) = 1$.

b) What was the class average, $E(X)$?

c) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

d) What was the proportion of the students who received grades above 80?
That is, find $P(X > 80)$?

8. 7. (continued)

As a way of "curving" the results, the instructor announced that he would replace each person's grade, X, with a new grade, $Y = g(X)$, where $g(x) = 4\sqrt{5x+125}$.

e) Use part (a) and the c.d.f. approach to find the c.d.f. that describes the new grades, Y.

f) Use the change-of-variable technique to find the p.d.f. that describes the new grades, Y.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

g) What is the new class average, $E(Y)$?

h) What is the new class median?

i) What is the proportion of the students with the new grades above 80?
That is, find $P(Y > 80)$?

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* ☹ Unfortunately, this is pretty close to the actual Fall 2017 STAT 410 Exam 1 grades.

Answers:

1. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{3}{2{,}000}\sqrt{x}, \qquad 0 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) What was the class average, $\mathrm{E}(X)$?

$$\mathrm{E}(X) = \int_0^{100} x \cdot \frac{3}{2{,}000}\sqrt{x}\,dx = \frac{3}{2{,}000} \cdot \int_0^{100} x^{3/2}\,dx = \frac{3}{2{,}000} \cdot \frac{2}{5} \cdot x^{5/2}\Big|_0^{100} = \mathbf{60}.$$

b) What was the class median? That is, find m such that $\mathrm{P}(X \le m) = \mathrm{P}(X \ge m) = \frac{1}{2}$.

$$\mathrm{F}_X(m) = \mathrm{P}(X \le m) = \int_0^{m} \frac{3}{2{,}000}\sqrt{x}\,dx = \frac{m^{3/2}}{1{,}000} = \frac{1}{2}.$$

$$\Rightarrow \qquad m = \mathbf{500^{2/3}} \approx 62.996.$$

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2. 1. (continued)

As a way of "curving" the results, the professor announces that he will replace each person's grade, X, with a new grade, $Y = g(X)$, where $g(X) = 10\sqrt{X}$.

c) Find the p.d.f. that describes the new grades, Y.

$$y = 10\sqrt{x} \qquad x = \frac{y^2}{100} \qquad \frac{dx}{dy} = \frac{y}{50}$$

$$f_Y(y) = \frac{3}{2{,}000}\sqrt{\frac{y^2}{100}} \cdot \left|\frac{y}{50}\right| = \frac{3y^2}{1{,}000{,}000}, \qquad 0 \le y \le 100.$$

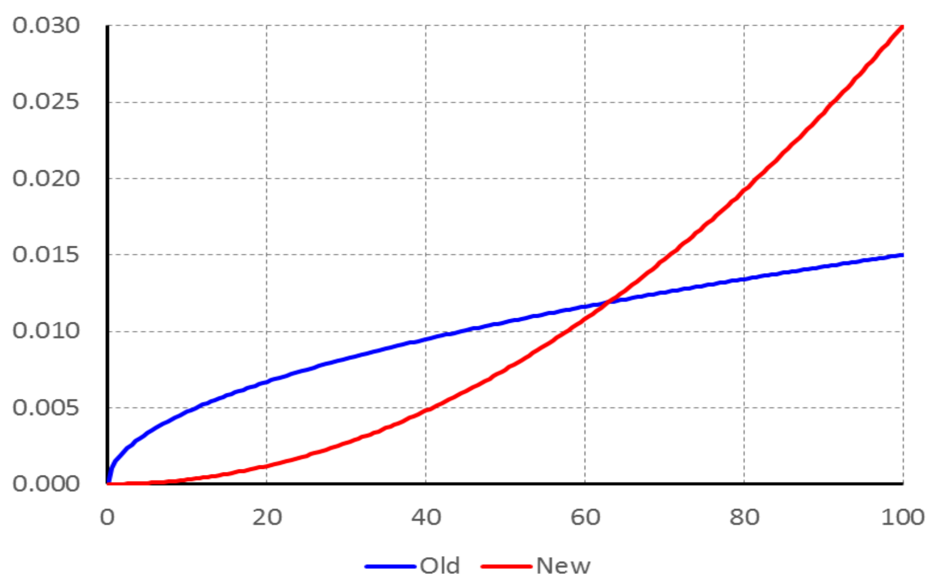
OR

$$F_X(x) = P(X \le x) = \frac{x^{3/2}}{1{,}000}, \qquad 0 \le x \le 100.$$

$$F_Y(y) = P(Y \le y) = P(10\sqrt{X} \le y) = P\left(X \le \frac{y^2}{100}\right)$$

$$= F_X\left(\frac{y^2}{100}\right) = \frac{y^3}{1{,}000{,}000}, \qquad 0 \le y \le 100.$$

$$\Rightarrow \qquad f_Y(y) = F_Y'(y) = \frac{3y^2}{1{,}000{,}000}, \qquad 0 \le y \le 100.$$



d) What is the new class average, $\mathrm{E}(\mathrm{Y})$?

$$\mathrm{E}(\mathrm{Y}) = \int_0^{100} y \cdot \frac{3y^2}{1{,}000{,}000}\, dy = \frac{3}{1{,}000{,}000} \cdot \frac{1}{4} \cdot y^4 \Big|_0^{100} = \mathbf{75}.$$

OR

$$\mathrm{E}(\mathrm{Y}) = \int_0^{100} 10\sqrt{x} \cdot \frac{3}{2{,}000}\sqrt{x}\, dx = \int_0^{100} \frac{3}{200}\, x\, dx = \frac{3}{200} \cdot \frac{1}{2} \cdot x^2 \Big|_0^{100} = \mathbf{75}.$$

e) What is the new class median?

$$\mathrm{F}_{\mathrm{Y}}(m) = \int_0^{m} \frac{3y^2}{1{,}000{,}000}\, dy = \frac{m^3}{1{,}000{,}000} = \frac{1}{2}.$$

$$\Rightarrow \quad m = \mathbf{500{,}000^{1/3}} \approx 79.37.$$

OR

$$\mathrm{P}(\mathrm{X} \le m_{\mathrm{X}}) = \mathrm{P}(10\sqrt{\mathrm{X}} \le 10\sqrt{m_{\mathrm{X}}}) = \mathrm{P}(\mathrm{Y} \le 10\sqrt{m_{\mathrm{X}}}).$$

$$m_{\mathrm{Y}} = 10\sqrt{m_{\mathrm{X}}} = \mathbf{10 \cdot 500^{1/3}} \approx 79.37.$$

3. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{1}{5{,}000}(100 - x), \qquad 0 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) What was the class average, $E(X)$?

$$E(X) = \int_0^{100} x \cdot \frac{1}{5{,}000}(100 - x)\,dx = \mathbf{33.33\overline{3}}.$$

b) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

$$P(X \ge m) = \int_m^{100} \frac{1}{5{,}000}(100 - x)\,dx = \frac{(100 - m)^2}{10{,}000} = \frac{1}{2}.$$

$$\Rightarrow \qquad m = 100\left(1 - \frac{1}{\sqrt{2}}\right) = 50\left(2 - \sqrt{2}\right) \approx 29.2893.$$

c) What was the proportion of the students who received scores above 60? That is, find $P(X > 60)$?

$$P(X > 60) = \int_{60}^{100} \frac{1}{5{,}000}(100 - x)\,dx = \mathbf{0.16}.$$

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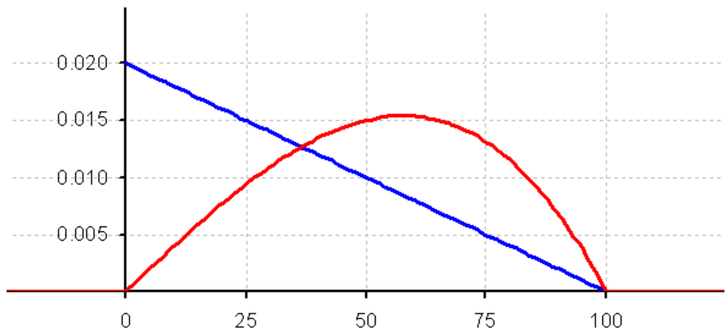
4. 3. (continued)

As a way of "curving" the results, the professor announces that he will replace each person's grade, X, with a new grade, $\mathrm{Y} = g(\mathrm{X})$, where $g(\mathrm{X}) = 10\sqrt{\mathrm{X}}$.

d) Find the p.d.f. that describes the new grades, Y.

$$y = 10\sqrt{x} \qquad x = \frac{y^2}{100} \qquad \frac{dx}{dy} = \frac{y}{50}$$

$$f_{\mathrm{Y}}(y) = \frac{1}{5{,}000}\left(100 - \frac{y^2}{100}\right) \cdot \frac{y}{50} = \frac{1}{25{,}000{,}000}\, y\left(10{,}000 - y^2\right), \qquad 0 \le y \le 100.$$



e) Has the professor's strategy been successful in raising the class average above 60? What is the new class average, $\mathrm{E}(\mathrm{Y})$?

$$\mathrm{E}(\mathrm{Y}) = \int_0^{100} 10\sqrt{x} \cdot \frac{1}{5{,}000}(100 - x)\, dx = \mathbf{53.33\bar{3}}.$$

OR

$$\mathrm{E}(\mathrm{Y}) = \int_0^{100} y \cdot \frac{1}{25{,}000{,}000}\, y\left(10{,}000 - y^2\right) dy = \mathbf{53.33\bar{3}}.$$

f) What is the new class median?

$$\mathrm{P}(\mathrm{X} \ge m) = \frac{1}{2}. \qquad \Rightarrow \qquad \mathrm{P}(\mathrm{Y} \ge 10\sqrt{m}) = \frac{1}{2}.$$

$$m = 50\left(2 - \sqrt{2}\right) \approx 29.2893. \qquad 10\sqrt{m} \approx 54.1196.$$

5. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{1}{(x+25)\ln 5}, \qquad 0 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": Should be $F_X(0) = 0$, $F_X(100) = 1$.

$$F_X(x) = P(X \le x) = \int_0^x \frac{1}{(u+25)\ln 5}\,du = \frac{1}{\ln 5}\ln(u+25)\Big|_0^x$$

$$= \frac{\ln(x+25) - \ln 25}{\ln 5} = \frac{\ln(x+25)}{\ln 5} - 2, \qquad 0 \le x \le 100.$$

b) What was the class average, $E(X)$?

$$E(X) = \int_0^{100} x \frac{1}{(x+25)\ln 5}\,dx = \int_0^{100} \frac{(x+25)}{(x+25)\ln 5}\,dx - \int_0^{100} \frac{25}{(x+25)\ln 5}\,dx$$

$$= \frac{100}{\ln 5} - 25 \approx 37.1335.$$

c) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

$$F_X(m) = \frac{\ln(m+25)}{\ln 5} - 2 = \frac{1}{2}. \qquad m = 5^{2.5} - 25 \approx 30.9017.$$

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6. 5. (continued)

As a way of "curving" the results, the instructor announced that he would replace each person's grade, X, with a new grade, $\mathrm{Y} = g(\mathrm{X})$, where $g(\mathrm{X}) = 20\,(\mathrm{X}+25)^{1/3}$.

d) Use part (a) and the c.d.f. approach to find the c.d.f. that describes the new grades, Y.

$0 \le x \le 100 \qquad y = 20\,(x+25)^{1/3} \qquad \Rightarrow \qquad 20 \cdot 5^{2/3} \le y \le 100$

$20 \cdot 5^{2/3} \approx 58.48035$

$$\mathrm{F_Y}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}(20\,(\mathrm{X}+25)^{1/3} \le y) = \mathrm{P}\left(\mathrm{X} \le \left(\frac{y}{20}\right)^3 - 25\right)$$

$$= \mathrm{F_X}\left(\left(\frac{y}{20}\right)^3 - 25\right) = \frac{3\ln y - 3\ln 20 - \ln 25}{\ln 5} = \frac{3\ln y - \ln 200{,}000}{\ln 5},$$

$$20 \cdot 5^{2/3} \le y \le 100.$$

e) Use the change-of-variable technique to find the p.d.f. that describes the new grades, Y.

"Hint": To double-check your answer: should be $f_{\mathrm{Y}}(y) = \mathrm{F}'_{\mathrm{Y}}(y)$.

$y = 20\,(x+25)^{1/3} \qquad x = \left(\frac{y}{20}\right)^3 - 25 \qquad \frac{dx}{dy} = \frac{3\,y^2}{20^3}$

$$f_{\mathrm{Y}}(y) = \frac{1}{\left(\frac{y}{20}\right)^3 \ln 5} \cdot \left|\frac{3\,y^2}{20^3}\right| = \frac{3}{y\,\ln 5}, \qquad 20 \cdot 5^{2/3} \le y \le 100.$$

f) What is the new class average, $\mathrm{E}(\mathrm{Y})$?

$$\mathrm{E}(\mathrm{Y}) = \int_{20 \cdot 5^{2/3}}^{100} y\,\frac{3}{y\,\ln 5}\,dy = \frac{3}{\ln 5}\,(100 - 20 \cdot 5^{2/3}) \approx 77.3928.$$

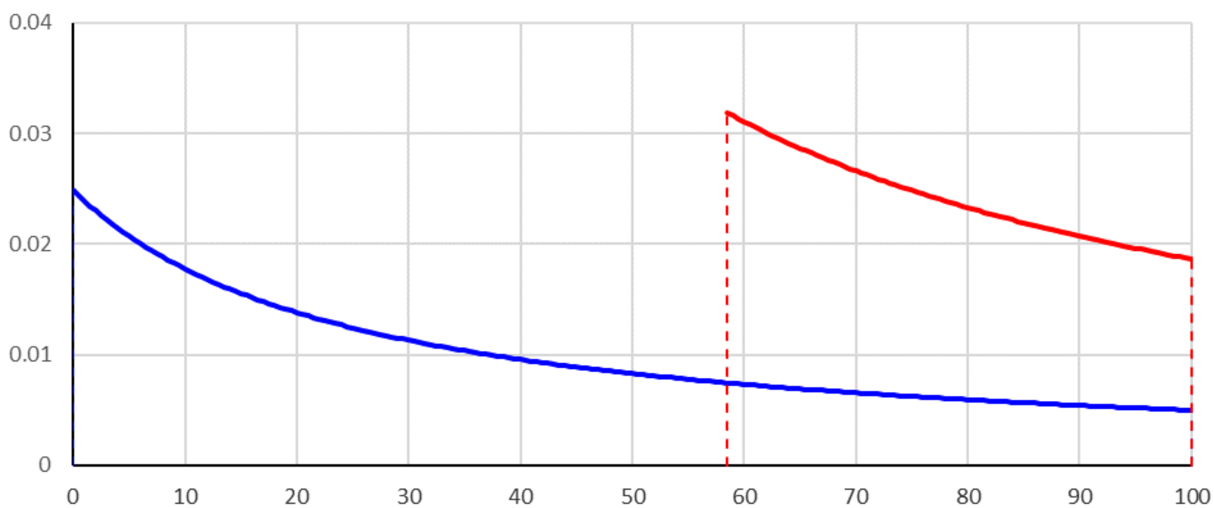
g) What is the new class median?

$$m_Y = 20\,(m_X + 25)^{1/3} = 20 \cdot 5^{5/6} \approx 76.47245.$$

OR

$$F_Y(m_Y) = \frac{3 \ln m_Y - \ln 200{,}000}{\ln 5} = \frac{1}{2}.$$

$$m_Y = 200{,}000^{1/3} \cdot 5^{1/6} \approx 76.47245.$$



Old **New**

7. Grades on the last STAT 410 exam were not very good*. Graphed, their distribution had a shape similar to the p.d.f.

$$f_X(x) = \frac{2x+5}{10{,}000}, \qquad 20 \le x \le 100, \qquad \text{zero elsewhere.}$$

a) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": Should be $F_X(20) = 0$, $F_X(100) = 1$.

$$F_X(x) = P(X \le x) = \int_{20}^{x} \frac{2u+5}{10{,}000}\,du = \left.\frac{u^2+5u}{10{,}000}\right|_{20}^{x}$$

$$= \frac{x^2+5x-500}{10{,}000} = \frac{(x-20)(x+25)}{10{,}000}, \qquad 20 \le x \le 100.$$

b) What was the class average, $E(X)$?

$$E(X) = \int_{20}^{100} x \cdot \frac{2x+5}{10{,}000}\,dx = \int_{20}^{100} \frac{2x^2+5x}{10{,}000}\,dx = \left.\left(\frac{2x^3}{30{,}000} + \frac{5x^2}{20{,}000}\right)\right|_{20}^{100}$$

$$= \left(\frac{200}{3} + \frac{5}{2}\right) - \left(\frac{8}{15} + \frac{1}{10}\right) = \frac{415}{6} - \frac{19}{30} = \mathbf{\frac{1{,}028}{15}} \approx 68.5333.$$

c) What was the class median? That is, find m such that $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

$$F_X(m) = \frac{m^2+5m-500}{10{,}000} = \frac{1}{2}. \qquad \Rightarrow \qquad m^2 + 5m - 5{,}500 = 0.$$

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* ☹ Unfortunately, this is pretty close to the actual Fall 2017 STAT 410 Exam 1 grades.

$$\Rightarrow \qquad m = \frac{-5 \pm \sqrt{25 + 22{,}000}}{2}$$

Since $20 \le m \le 100$, $\qquad m = \dfrac{\sqrt{22{,}025} - 5}{2} \approx 71.7041.$

d) What was the proportion of the students who received grades above 80?
That is, find $P(X > 80)$?

$$P(X > 80) = 1 - F_X(80) = 1 - 0.63 = \mathbf{0.37}.$$

OR

$$P(X > 80) = \int_{80}^{100} \frac{2x + 5}{10{,}000}\,dx = \left.\frac{x^2 + 5x}{10{,}000}\right|_{80}^{100} = 1.05 - 0.68 = \mathbf{0.37}.$$

8. 7. (continued)

As a way of "curving" the results, the instructor announced that he would replace each person's grade, X, with a new grade, $Y = g(X)$, where $g(x) = 4\sqrt{5x + 125}$.

e) Use part (a) and the c.d.f. approach to find the c.d.f. that describes the new grades, Y.

$$F_Y(y) = P(Y \le y) = P(4\sqrt{5X + 125} \le y) = P(\sqrt{5X + 125} \le \frac{y}{4})$$

$$= P(5X + 125 \le \frac{y^2}{16}) = P(X \le \frac{y^2}{80} - 25) = F_X(\frac{y^2}{80} - 25)$$

$$= \frac{\left(\frac{y^2}{80} - 45\right)\left(\frac{y^2}{80}\right)}{10{,}000} = \frac{y^2\left(y^2 - 3{,}600\right)}{64{,}000{,}000} = \frac{y^4 - 3{,}600\,y^2}{64{,}000{,}000},$$

$$20 \le x \le 100 \qquad \Rightarrow \qquad 4\sqrt{5 \cdot 20 + 125} \le y \le 4\sqrt{5 \cdot 100 + 125}$$

$$60 \le y \le 100.$$

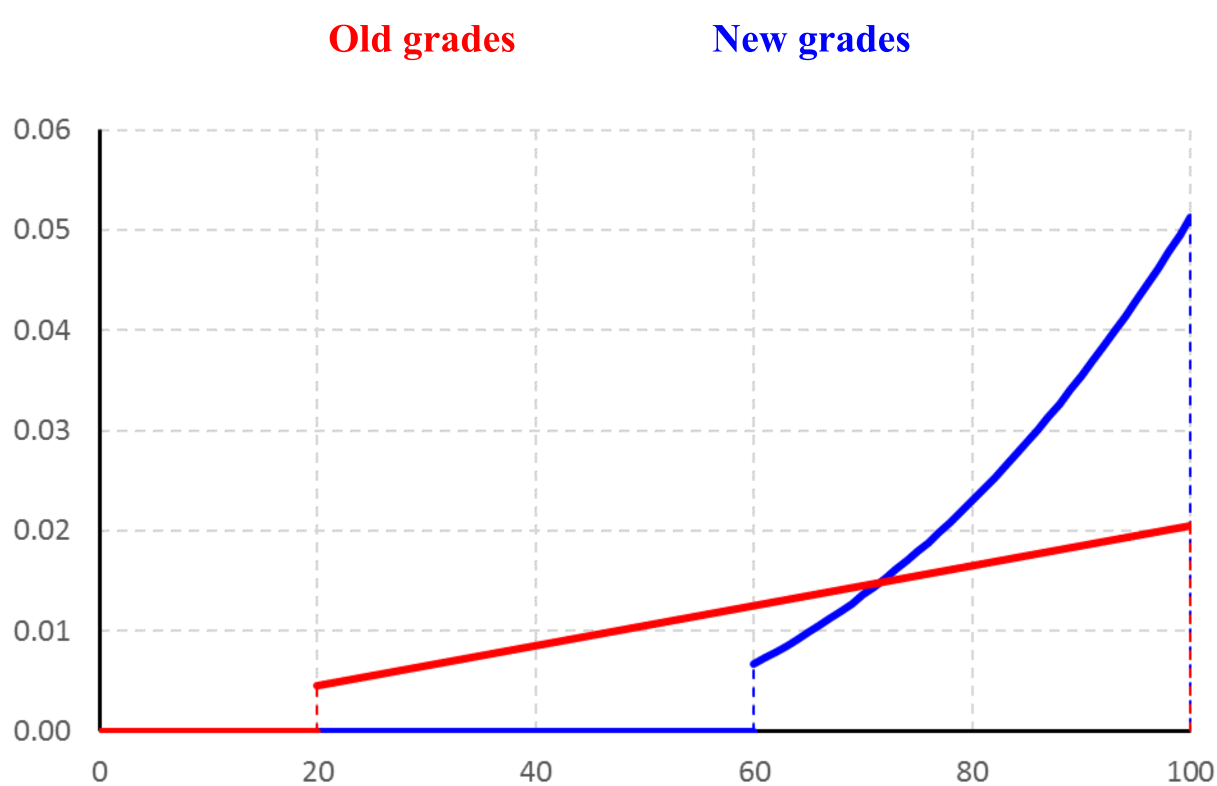
f) Use the change-of-variable technique to find the p.d.f. that describes the new grades, Y.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = 4\sqrt{5x+125} \qquad\qquad x = \frac{y^2}{80} - 25 \qquad\qquad \frac{dx}{dy} = \frac{y}{40}$$

$$f_Y(y) = \frac{2\left(\dfrac{y^2}{80} - 25\right) + 5}{10{,}000} \cdot \left|\frac{y}{40}\right| = \frac{\dfrac{y^2}{40} - 45}{10{,}000} \cdot \frac{y}{40} = \frac{y^3 - 1{,}800\,y}{16{,}000{,}000},$$

$$60 \le y \le 100.$$



g) What is the new class average, $\mathrm{E}(\mathrm{Y})$?

$$\mathrm{E}(\mathrm{Y}) = \int_{60}^{100} y \cdot \frac{y^3 - 1{,}800\,y}{16{,}000{,}000}\,dy = \int_{60}^{100} \frac{y^4 - 1{,}800\,y^2}{16{,}000{,}000}\,dy$$

$$= \left(\frac{y^5}{80{,}000{,}000} - \frac{600\,y^3}{16{,}000{,}000} \right) \Bigg|_{60}^{100} = (125 - 37.5) - (9.72 - 8.1) = \mathbf{85.88}.$$

h) What is the new class median?

$$\frac{1}{2} = \mathrm{P}(\mathrm{X} \le m_{\mathrm{X}}) = \mathrm{P}\left(4\sqrt{5\mathrm{X}+125} \le 4\sqrt{5m_{\mathrm{X}}+125}\right) = \mathrm{P}\left(\mathrm{Y} \le 4\sqrt{5m_{\mathrm{X}}+125}\right).$$

$$\Rightarrow \qquad m_{\mathrm{Y}} = 4\sqrt{5m_{\mathrm{X}}+125} \approx 87.9564.$$

OR

$$\mathrm{F}_{\mathrm{Y}}(m_{\mathrm{Y}}) = \frac{m_{\mathrm{Y}}^4 - 3{,}600\,m_{\mathrm{Y}}^2}{64{,}000{,}000} = \frac{1}{2}.$$

$$\Rightarrow \qquad m_{\mathrm{Y}}^4 - 3{,}600\,m_{\mathrm{Y}}^2 - 32{,}000{,}000 = 0.$$

$$\Rightarrow \qquad m_{\mathrm{Y}}^2 = 1{,}800 \pm \sqrt{1{,}800^2 + 32{,}000{,}000}$$

Since $60 \le m_{\mathrm{Y}} \le 100$, $\qquad m_{\mathrm{Y}} = \sqrt{1{,}800 + \sqrt{35{,}240{,}000}} \approx 87.9564.$

i) What is the proportion of the students with the new grades above 80?
That is, find $\mathrm{P}(\mathrm{Y} > 80)$?

$$\mathrm{P}(\mathrm{Y} > 80) = 1 - \mathrm{F}_{\mathrm{Y}}(80) = 1 - 0.28 = \mathbf{0.72}.$$

OR

$$\mathrm{P}(\mathrm{X} > 80) = \int_{80}^{100} \frac{y^3 - 1{,}800\,y}{16{,}000{,}000}\,dy = \frac{y^4 - 3{,}600\,y^2}{64{,}000{,}000} \Bigg|_{80}^{100} = 1 - 0.28 = \mathbf{0.72}.$$