STAT 410 **Homework #2** Fall 2020
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(due Friday, September 11, by 5:00 p.m. CDT)

Please include your name (with your last name underlined), your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

1. Several students have correctly pointed out that the exam scores in problem **1** of Homework #1 should have discrete (instead of continuous) nature. A continuous probability distribution was used as an approximation, since the alternative would have been dealing with a discrete random variable with 73 possible values (3, 4, 5, … , 74, 75), which is not nearly as much fun as I am describing it here.

Let's make this problem even more convoluted:

No matter how good or bad the exam grades are, there is usually a group of students who receive perfect scores. Consider a mixed random variable X for the exam scores with the p.m.f. of the discrete portion of the probability distribution

$$p(75) = c, \qquad \text{zero otherwise,}$$

and the p.d.f. of the continuous portion of the probability distribution

$$f(x) = \frac{\sqrt{x+6}}{500}, \qquad 3 \le x < 75, \qquad \text{zero elsewhere.}$$

(recall: equal signs are NOT important when dealing with a continuous random variable or a continuous portion of a probability distribution)

f) Find the value of c that would make this a valid probability distribution.

$$1 = \sum_{\text{all } x} p(x) + \int_{-\infty}^{\infty} f(x)\,dx$$

$$= c + \int_{3}^{75} \frac{\sqrt{x+6}}{500}\,dx = c + \left.\frac{2\,(x+6)^{1.5}}{1{,}500}\right|_{3}^{75}$$

$$= c + \frac{2\,(729-27)}{1{,}500} = c + \frac{117}{125} = c + 0.936.$$

$$\Rightarrow \qquad c = \mathbf{\frac{8}{125}} = \mathbf{0.064}.$$

g) Find $\mu = \mathrm{E}(\mathrm{X})$, the average exam score.

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= 75 \cdot 0.064 + \int_{3}^{75} x \cdot \frac{\sqrt{x+6}}{500}\, dx = 4.8 + \frac{1}{500} \cdot \int_{3}^{75} x \cdot \sqrt{x+6}\, dx.$$

$$\int_{3}^{75} x \cdot \sqrt{x+6}\, dx = \qquad\qquad u = x+6 \qquad du = dx \qquad x = u-6$$

$$= \int_{9}^{81} (u-6) \cdot \sqrt{u}\, du = \int_{9}^{81} \left(u^{1.5} - 6\, u^{0.5} \right) du$$

$$= \left(0.4\, u^{2.5} - 4\, u^{1.5} \right) \Big|_{9}^{81} = 20{,}703.6 + 10.8 = 20{,}714.4.$$

$$\mathrm{E}(\mathrm{X}) = 4.8 + \frac{20{,}714.4}{500} = \mathbf{46.2288}.$$

3. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Since the people become disobedient if they do not have food, the amount spent monthly on butter is always at least 2 million dollars. However, there is a law that prevents the government of Neverland spending more than 5 million dollars on guns in one month. In addition, the government of Neverland cannot afford to spend more than 8 million dollars per month on guns and butter. Suppose that the joint density function for (X, Y) is

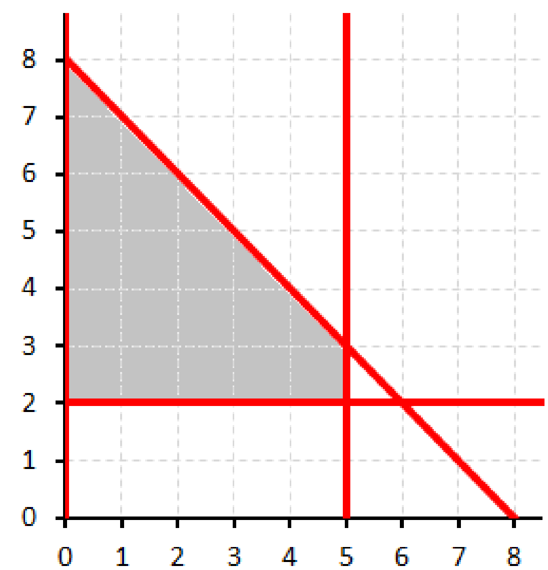
$$f(x,y) = \frac{7x + 2y}{C}, \qquad x \ge 0, \ \ y \ge 2, \ \ x \le 5, \ \ x + y \le 8, \qquad \text{zero otherwise.}$$

X – guns, Y – butter.

a) Sketch the support of (X, Y).

That is, sketch

$$\{ (x,y) : \ x \ge 0, \ \ y \ge 2, \ \ x \le 5, \ \ x + y \le 8 \}.$$



b) Find the value of C so that $f(x,y)$ is a valid joint probability density function.

Must have $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

$$\int_0^5 \left(\int_2^{8-x} \frac{7x+2y}{C}\,dy \right) dx = \int_0^5 \left(\frac{7xy + y^2}{C} \right) \Bigg|_{y=2}^{y=8-x} dx$$

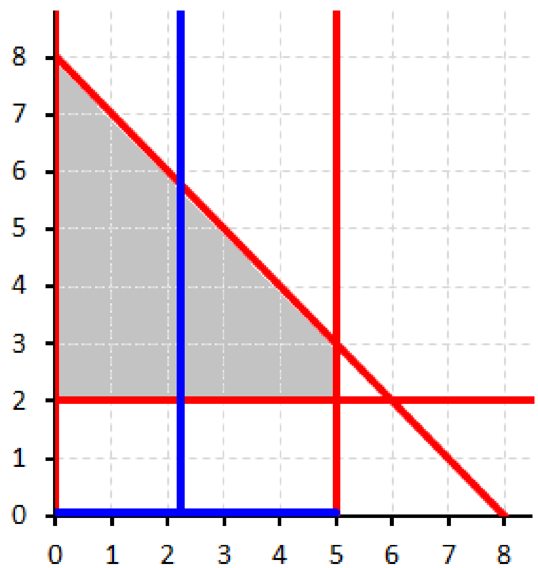
$$= \int_0^5 \frac{7x(8-x) + (8-x)^2 - 14x - 4}{C}\,dx$$

$$= \int_0^5 \frac{56x - 7x^2 + 64 - 16x + x^2 - 14x - 4}{C}\,dx = \int_0^5 \frac{60 + 26x - 6x^2}{C}\,dx$$

$$= \left(\frac{60x + 13x^2 - 2x^3}{C} \right) \Bigg|_0^5 = \frac{300 + 325 - 250}{C} = \frac{375}{C} = 1.$$

$\Rightarrow \quad C = \mathbf{375}.$

c) Find the marginal probability density function of X, $f_X(x)$.
Be sure to include its support.



$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

The range of possible values for X is $0 \le x \le 5$.

$$f_X(x) = \int_2^{8-x} \frac{7x + 2y}{375}\, dy = \left(\frac{7xy + y^2}{375} \right) \Bigg|_{y=2}^{y=8-x}$$

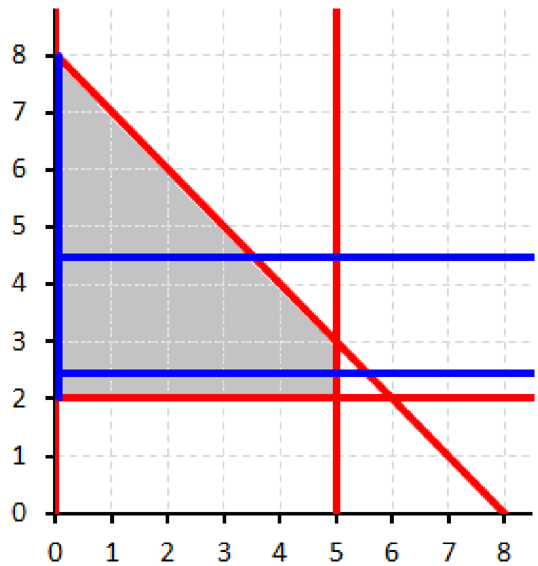
$$= \frac{7x(8-x) + (8-x)^2 - 14x - 4}{375}$$

$$= \frac{56x - 7x^2 + 64 - 16x + x^2 - 14x - 4}{375}$$

$$= \frac{60 + 26x - 6x^2}{375} = \frac{2(5+3x)(6-x)}{375}, \qquad 0 \le x \le 5.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.
Be sure to include its support.

"Hint": It would be **wise** to break this problem into **piece**s.



$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

The range of possible values for Y is $2 \le y \le 8$.

$f_Y(y)$ will be a **piecewise-defined** function, since the limits of this integral will not be the same for $2 \le y \le 3$ and for $3 \le y \le 8$.

If $2 \le y \le 3$,

$$f_Y(y) = \int_0^5 \frac{7x + 2y}{375}\, dx = \left(\frac{7x^2 + 4xy}{750}\right)\Bigg|_{x=0}^{x=5} = \frac{175 + 20y}{750} = \frac{35 + 4y}{150},$$

$2 \le y \le 3$.

If $3 \le y \le 8$,

$$f_Y(y) = \int_0^{8-y} \frac{7x + 2y}{375}\, dx = \left(\frac{7x^2 + 4xy}{750}\right)\Bigg|_{x=0}^{x=8-y}$$

$$= \frac{7(8-y)^2 + 4y(8-y)}{750} = \frac{448 - 112y + 7y^2 + 32y - 4y^2}{750}$$

$$= \frac{448 - 80y + 3y^2}{750} = \frac{(56 - 3y)(8 - y)}{750},$$

$3 \le y \le 8$.

$$f_Y(y) = \begin{cases} \dfrac{35+4y}{150} & 2 \le y \le 3 \\[2ex] \dfrac{448-80y+3y^2}{750} & 3 \le y \le 8 \\[2ex] 0 & \text{otherwise} \end{cases}$$

e) Are X and Y independent? *Justify your answer.*

$f(x, y) \ne f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

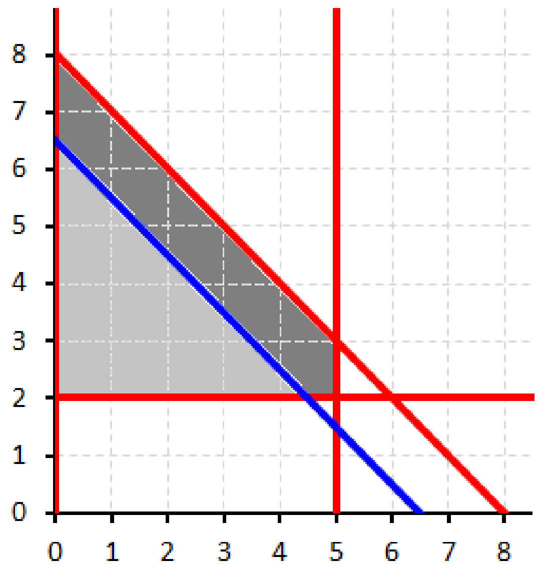
OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f(x, y)$ cannot be written as a product of two functions, one of x only,
the other of y only. $\Rightarrow$ X and Y are **NOT independent**.

f) Find the probability that the total amount spent monthly on guns and butter exceeds 6.5 million dollars. That is, find $P(X+Y>6.5)$.



$$P(X+Y>6.5) = \ldots$$

$$\ldots = 1 - \int_0^{4.5}\left(\int_2^{6.5-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$\ldots = 1 - \int_2^{6.5}\left(\int_0^{6.5-y}\frac{7x+2y}{375}\,dx\right)dy$$

$$\ldots = \int_0^{4.5}\left(\int_{6.5-x}^{8-x}\frac{7x+2y}{375}\,dy\right)dx + \int_{4.5}^{5}\left(\int_2^{8-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$\ldots = \int_2^{3}\left(\int_{6.5-y}^{5}\frac{7x+2y}{375}\,dx\right)dy + \int_3^{6.5}\left(\int_{6.5-y}^{8-y}\frac{7x+2y}{375}\,dx\right)dy + \int_{6.5}^{8}\left(\int_0^{8-y}\frac{7x+2y}{375}\,dx\right)dy$$

$$1 - \int_0^{4.5}\left(\int_2^{6.5-x}\frac{7x+2y}{375}\,dy\right)dx = 1 - \int_0^{4.5}\frac{38.25+18.5x-6x^2}{375}\,dx$$

$$= 1 - 0.4725 = \mathbf{0.5275}.$$

$$1 - \int_2^{6.5}\left(\int_0^{6.5-y}\frac{7x+2y}{375}\,dx\right)dy = 1 - \int_2^{6.5}\frac{147.875-32.5y+1.5y^2}{375}\,dy$$

$$= 1 - 0.4725 = \mathbf{0.5275}.$$

$$\int_0^{4.5}\left(\int_{6.5-x}^{8-x}\frac{7x+2y}{375}\,dy\right)dx + \int_{4.5}^{5}\left(\int_{2}^{8-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$= \int_0^{4.5}\frac{21.75+7.5x}{375}\,dx + \int_{4.5}^{5}\frac{60+26x-6x^2}{375}\,dx$$

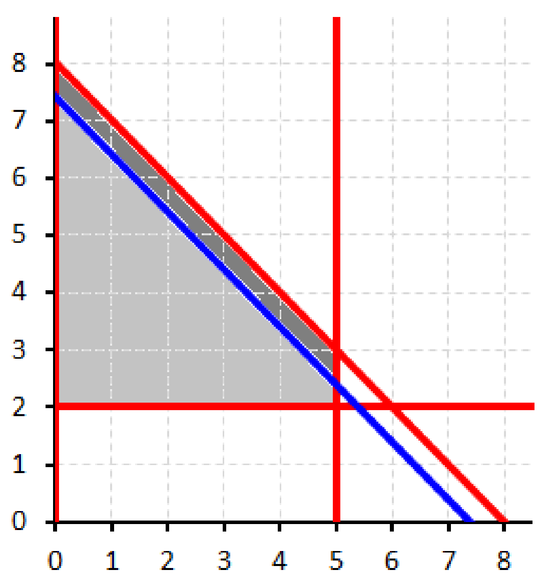
$$= 0.4635 + 0.0640 = \mathbf{0.5275}.$$

$$\int_2^{3}\left(\int_{6.5-y}^{5}\frac{7x+2y}{375}\,dx\right)dy + \int_{3}^{6.5}\left(\int_{6.5-y}^{8-y}\frac{7x+2y}{375}\,dx\right)dy + \int_{6.5}^{8}\left(\int_{0}^{8-y}\frac{7x+2y}{375}\,dx\right)dy$$

$$= \int_2^{3}\frac{-60.375+42.5y-1.5y^2}{375}\,dy + \int_{3}^{6.5}\frac{76.125-7.5y}{375}\,dy + \int_{6.5}^{8}\frac{224-40y+1.5y^2}{375}\,dy$$

$$= 0.0970 + 0.3780 + 0.0525 = \mathbf{0.5275}.$$

g) Find the probability that the total amount spent monthly on guns and butter exceeds 7.4 million dollars. That is, find $P(X+Y>7.4)$.



$$P(X+Y>7.4) = \ldots$$

$$\ldots = \int_0^{5}\left(\int_{7.4-x}^{8-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$\ldots = 1 - \int_0^{5}\left(\int_{2}^{7.4-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$\ldots = 1 - \int_2^{2.4}\left(\int_{0}^{5}\frac{7x+2y}{375}\,dx\right)dy$$

$$- \int_{2.4}^{7.4}\left(\int_{0}^{7.4-y}\frac{7x+2y}{375}\,dx\right)dy$$

$$\ldots = \int_{2.4}^{3} \left(\int_{7.4-y}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{3}^{7.4} \left(\int_{7.4-y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$+ \int_{7.4}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$\int_{0}^{5} \left(\int_{7.4-x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx = \int_{0}^{5} \frac{9.24+3x}{375}\, dx = \mathbf{0.2232}.$$

$$1 - \int_{0}^{5} \left(\int_{2}^{7.4-x} \frac{7x+2y}{375}\, dy \right) dx = 1 - \int_{0}^{5} \frac{50.76+23x-6x^2}{375}\, dx = 1 - 0.7768 = \mathbf{0.2232}.$$

$$1 - \int_{2}^{2.4} \left(\int_{0}^{5} \frac{7x+2y}{375}\, dx \right) dy - \int_{2.4}^{7.4} \left(\int_{0}^{7.4-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$= 1 - \int_{2}^{2.4} \frac{87.5+10y}{375}\, dy - \int_{2.4}^{7.4} \frac{191.66-37y+1.5y^2}{375}\, dy$$

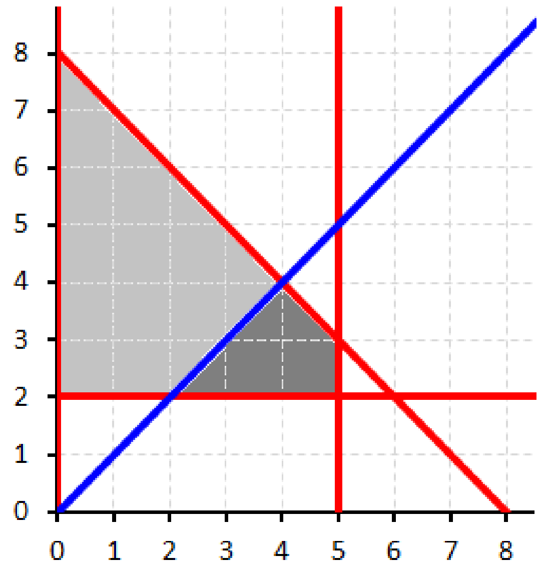
$$= 1 - 0.1168 - 0.6600 = \mathbf{0.2232}.$$

$$\int_{2.4}^{3} \left(\int_{7.4-y}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{3}^{7.4} \left(\int_{7.4-y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy + \int_{7.4}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$= \int_{2.4}^{3} \frac{-104.16+47y-1.5y^2}{375}\, dy + \int_{3}^{7.4} \frac{32.34-3y}{375}\, dy + \int_{7.4}^{8} \frac{224-40y+1.5y^2}{375}\, dy$$

$$= 0.018816 + 0.196416 + 0.007968 = \mathbf{0.2232}.$$

h) Find the probability that the government of Neverland spends more purchasing guns than purchasing butter in a given month. That is, find $P(X > Y)$.



$$P(X > Y) = \ldots$$

$$\ldots = \int_2^4 \left(\int_2^x \frac{7x + 2y}{375}\, dy \right) dx + \int_4^5 \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$\ldots = \int_2^3 \left(\int_y^5 \frac{7x + 2y}{375}\, dx \right) dy + \int_3^4 \left(\int_y^{8-y} \frac{7x + 2y}{375}\, dx \right) dy$$

$$\ldots = 1 - \int_0^2 \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx - \int_2^4 \left(\int_x^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$\ldots = 1 - \int_2^4 \left(\int_0^y \frac{7x + 2y}{375}\, dx \right) dy - \int_4^8 \left(\int_0^{8-y} \frac{7x + 2y}{375}\, dx \right) dy$$

$$\int_2^4 \left(\int_2^x \frac{7x + 2y}{375}\, dy \right) dx + \int_4^5 \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$= \int_2^4 \frac{-4 - 14x + 8x^2}{375}\, dx + \int_4^5 \frac{60 + 26x - 6x^2}{375}\, dx$$

$$= \frac{172}{1125} + \frac{11}{75} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

$$\int_2^3 \left(\int_y^5 \frac{7x+2y}{375}\, dx \right) dy \;+\; \int_3^4 \left(\int_y^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$= \int_2^3 \frac{87.5+10y-5.5y^2}{375}\, dy \;+\; \int_3^4 \frac{224-40y-4y^2}{375}\, dy$$

$$= \frac{233}{1125} + \frac{104}{1125} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

$$1 - \int_0^2 \left(\int_2^{8-x} \frac{7x+2y}{375}\, dy \right) dx \;-\; \int_2^4 \left(\int_x^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

$$= 1 - \int_0^2 \frac{60+26x-6x^2}{375}\, dx \;-\; \int_2^4 \frac{64+40x-14x^2}{375}\, dx$$

$$= 1 - \frac{52}{125} - \frac{64}{225} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

$$1 - \int_2^4 \left(\int_0^y \frac{7x+2y}{375}\, dx \right) dy \;-\; \int_4^8 \left(\int_0^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$= 1 - \int_2^4 \frac{5.5y^2}{375}\, dy \;-\; \int_4^8 \frac{224-40y+1.5y^2}{375}\, dy$$

$$= 1 - \frac{308}{1125} - \frac{32}{75} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$