STAT 410 **Examples for 11/18/2020 (4)** Fall 2020
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Examples for 10/19/2020 (4) & 10/21/2020 (2) & 10/23/2020 (3)

& Examples for 11/06/2020 (2) (continued)

1. The Pareto probability distribution has many applications in economics, biology, and physics. Let $\beta > 0$ and $\delta > 0$ be the population parameters, and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta,\delta) = \frac{\delta \cdot \beta^{\delta}}{x^{\delta+1}}, \qquad x > \beta, \qquad \text{zero otherwise.}$$

Suppose β is known.

Recall: $W = \ln\left(\frac{X}{\beta}\right) = \ln X - \ln \beta$ has an Exponential$\left(\theta = \frac{1}{\delta}\right)$ distribution;

$Y = \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) = \sum_{i=1}^{n} W_i$ has a Gamma$\left(\alpha = n, \theta = \frac{1}{\delta}\right)$ distribution;

IF β is known, $Y = \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right)$ is a sufficient statistic for δ.

We wish to test $H_0: \delta \ge 2$ vs. $H_1: \delta < 2$.

Suppose $n = 5$, $\beta = 3$.

af) Find the uniformly most powerful rejection region with the significance level $\alpha = 0.05$.

ag) Suppose $x_1 = 3.9$, $x_2 = 4.2$, $x_3 = 6$, $x_4 = 9$, $x_5 = 15$. Find the p-value of the test.

ah) Consider the rejection "Reject H_0 if $\sum_{i=1}^{5} \ln\left(\frac{X_i}{3}\right) \geq 5$". Find …

i) … the significance level α;

ii) … power if $\delta = 1.5$;

iii) … power if $\delta = 1.0$.

"Hint": Since β is known and $Y = \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right)$ is a sufficient statistic for δ,

and the final form of the "best" rejection region should look like this:

$$\text{" Reject } H_0 \text{ if } \sum_{i=1}^{n} \ln\left(\frac{x_i}{\beta}\right) = \sum_{i=1}^{5} \ln\left(\frac{x_i}{3}\right) \left[\leq \text{or} \geq\right] c \text{ ".}$$

1. The Pareto probability distribution has many applications in economics, biology, and physics. Let $\beta > 0$ and $\delta > 0$ be the population parameters, and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

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IF β is known, $Y = \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right)$ is a sufficient statistic for δ.

We wish to test $H_0 : \delta \ge 2$ vs. $H_1 : \delta < 2$.

Suppose $n = 5$, $\beta = 3$.

af) Find the uniformly most powerful rejection region with the significance level $\alpha = 0.05$.

Let $\delta < 2$.

$$\frac{L(H_0;\, x_1, x_2, \ldots, x_n)}{L(H_1;\, x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \frac{2 \cdot \beta^2}{x_i^3}}{\prod_{i=1}^{n} \frac{\delta \cdot \beta^{\delta}}{x_i^{\delta+1}}} = \frac{2^n}{\delta^n} \left(\prod_{i=1}^{n} \frac{x_i}{\beta} \right)^{\delta-2}.$$

$$\frac{L(H_0)}{L(H_1)} \le k \quad \Leftrightarrow \quad \prod_{i=1}^{n} \frac{x_i}{\beta} \ge c \qquad (\text{since } \delta < 2)$$

$$\Leftrightarrow \qquad \sum_{i=1}^{n} \ln\left(\frac{x_i}{\beta}\right) \geq \tilde{c}.$$

Intuition: δ is "λ". Small δ $\Rightarrow$ large $W = \ln\left(\frac{X}{\beta}\right)$.

The sign is opposite from the sign in H_1.

$Y = \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) = \sum_{i=1}^{n} W_i$ has a $\text{Gamma}(\alpha = n, \theta = \frac{1}{\delta})$ distribution;

Then $2\delta \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right)$ has a $\chi^2(2\alpha = 2n = 10 \text{ degrees of freedom})$ distribution.

$$0.05 = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \geq \tilde{c} \;\middle|\; \delta = 2\right) = P\left(2\delta \sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \geq 2\delta\,\tilde{c} \;\middle|\; \delta = 2\right)$$

$$= P(\chi^2(10) \geq 4\,\tilde{c}).$$

$\Rightarrow \qquad 4\,\tilde{c} = \chi^2_{0.05}(10) = 18.31. \qquad \Rightarrow \qquad \tilde{c} = \mathbf{4.5775}.$

The uniformly most powerful rejection region is

"Reject H_0 if $\sum_{i=1}^{n} \ln\left(\frac{x_i}{\beta}\right) = \sum_{i=1}^{5} \ln\left(\frac{x_i}{3}\right) \geq 4.5775$"

"Reject H_0 if $\sum_{i=1}^{5} \ln x_i \geq 10.07056$"

"Reject H_0 if $\prod_{i=1}^{5} x_i \geq 23{,}636.8318$".

ag) Suppose $x_1 = 3.9,\ \ x_2 = 4.2,\ \ x_3 = 6,\ \ x_4 = 9,\ \ x_5 = 15$. Find the p-value of the test.

$$\sum_{i=1}^{n} \ln\left(\frac{x_i}{\beta}\right) = \sum_{i=1}^{n} \ln\left(\frac{x_i}{3}\right) \approx 4.$$

$$\text{P-value} = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 4 \,\middle|\, \delta = 2\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{2}\right) \ge 4\right)$$

$$= P(\text{Poisson}(4 \cdot 2) \le 5 - 1) = P(\text{Poisson}(8) \le 4) = \mathbf{0.100}.$$

$$\text{P-value} = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 4 \,\middle|\, \delta = 2\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{2}\right) \ge 4\right)$$

$$= P(\chi^2(10) \ge 4 \cdot 4) = P(\chi^2(10) \ge 16) \approx 0.099632.$$

ah) Consider the rejection "Reject H_0 if $\sum_{i=1}^{5} \ln\left(\frac{X_i}{3}\right) \ge 5$". Find …

i) … the significance level α;

$$\alpha = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 5 \,\middle|\, \delta = 2\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{2}\right) \ge 5\right)$$

$$= P(\text{Poisson}(5 \cdot 2) \le 5 - 1) = P(\text{Poisson}(10) \le 4) = \mathbf{0.029}.$$

$$\alpha = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 5 \,\middle|\, \delta = 2\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{2}\right) \ge 5\right)$$

$$= P(\chi^2(10) \ge 4 \cdot 5) = P(\chi^2(10) \ge 20) \approx 0.029253.$$

ii) … power if $\delta = 1.5$;

$$\text{Power}(\delta = 1.5) = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 5 \,\middle|\, \delta = 1.5\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{1.5}\right) \ge 5\right)$$

$$= P(\text{Poisson}(5 \cdot 1.5) \le 5 - 1) = P(\text{Poisson}(7.5) \le 4) = \mathbf{0.132}.$$

$$\text{Power}(\delta = 1.5) = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 5 \,\middle|\, \delta = 1.5\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{1.5}\right) \ge 5\right)$$

$$= P(\chi^2(10) \ge 3 \cdot 5) = P(\chi^2(10) \ge 15) \approx 0.132062.$$

iii) … power if $\delta = 1.0$.

$$\text{Power}(\delta = 1.0) = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 5 \,\middle|\, \delta = 1.0\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{1.0}\right) \ge 5\right)$$

$$= P(\text{Poisson}(5 \cdot 1.0) \le 5 - 1) = P(\text{Poisson}(5) \le 4) = \mathbf{0.440}.$$

$$\text{Power}(\delta = 1.0) = P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{\beta}\right) \ge 5 \,\middle|\, \delta = 1.0\right) = P\left(\text{Gamma}\left(\alpha = 5,\ \theta = \frac{1}{1.0}\right) \ge 5\right)$$

$$= P(\chi^2(10) \ge 2 \cdot 5) = P(\chi^2(10) \ge 10) \approx 0.440493.$$