Examples for 08/28/2020 (2)

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0. Let $\beta > 0$ and consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{\beta}{(1+x)^{\beta+1}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Consider $W = \ln(1 + X)$. Find the probability distribution of W.

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x}{C}, \qquad 3 \le x \le 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(7) = 1$.

1. (continued)

Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \ldots$.

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x}{C}, \qquad x = 3, 4, 5, 6, 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

b) Consider $Y = g(X) = \frac{100}{X^2 + 1}$. Find the probability distribution of Y.

3. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{|x|}{C}, \qquad -2 \le x \le 1, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(-2) = 0$, $F_X(1) = 1$.

c) Consider $Y = g(X) = \frac{100}{X^2 + 1}$. Find the probability distribution of Y.

Answers:

0. Let $\beta > 0$ and consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{\beta}{(1+x)^{\beta+1}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Consider $W = \ln(1 + X)$. Find the probability distribution of W.

$$W = \ln(1+X) \qquad X = e^W - 1 \qquad \frac{dx}{dw} = e^w$$

$$f_W(w) = \beta\, e^{-(\beta+1)w} \cdot \left| e^w \right| = \beta\, e^{-\beta w}, \qquad w > 0.$$

$\Rightarrow$ W has Exponential distribution with mean $\frac{1}{\beta}$.

OR

$$F_X(x) = \int_0^x \frac{\beta}{(1+u)^{\beta+1}}\,du = -\frac{1}{(1+u)^{\beta}}\Bigg|_0^x = 1 - \frac{1}{(1+x)^{\beta}}, \qquad x > 0.$$

$$F_W(w) = P(\ln(1+X) \le w) = P(X \le e^w - 1) = F_X(e^w - 1) = 1 - e^{-\beta w},$$

$$w > 0.$$

$\Rightarrow$ W has Exponential distribution with mean $\frac{1}{\beta}$.

OR

$$M_W(t) = E(e^{tW}) = E(e^{t\ln(1+X)}) = E((1+X)^t)$$

$$= \int_0^\infty (1+x)^t \cdot \frac{\beta}{(1+x)^{\beta+1}}\,dx = \int_0^\infty \frac{\beta}{(1+x)^{\beta-t+1}}\,dx$$

$$= -\frac{\beta}{(\beta-t)(1+x)^{\beta-t}}\Bigg|_0^\infty = \frac{\beta}{\beta-t} = \frac{1}{1-\frac{1}{\beta}t}, \qquad t<\beta.$$

[$M_W(t)$ does not exist if $t \ge \beta$ (the integral diverges).]

$\Rightarrow$ W has Exponential distribution with mean $\frac{1}{\beta}$.

Exponential
$0 < \theta$

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}$$

$$\mu = \theta, \qquad \sigma^2 = \theta^2$$

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x}{C}, \qquad 3 \le x \le 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_3^7 \frac{x}{C}\,dx = \frac{x^2}{2C}\Bigg|_3^7 = \frac{49-9}{2C} = \frac{20}{C}.$$

$\Rightarrow \qquad C = \mathbf{20}.$

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(7) = 1$.

$$F_X(x) = 0, \qquad x < 3,$$

$$F_X(x) = P(X \le x) = \int_3^x \frac{u}{20}\,du = \frac{x^2}{40}\Bigg|_3^x = \frac{x^2-9}{40}, \qquad 3 \le x < 7,$$

$$F_X(x) = 1, \qquad x \ge 7.$$

1. (continued)

Consider $Y = g(X) = \dfrac{100}{X^2+1}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

$g(x) = \dfrac{100}{x^2+1}$ – strictly decreasing on $(3, 7)$.

$g(3) = 10,\qquad g(7) = 2.\qquad\qquad \mathbf{2 \le y \le 10}.$

$3 \le x \le 7.\qquad 9 \le x^2 \le 49.\qquad 10 \le x^2 + 1 \le 50.$

$10 \ge \dfrac{100}{x^2+1} \ge 2.\qquad\qquad \mathbf{2 \le y \le 10}.$

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

$$F_Y(y) = P(Y \le y) = P\left(\frac{100}{X^2+1} \le y\right) = P\left(X \ge \sqrt{\frac{100}{y}-1}\right) = 1 - F_X\left(\sqrt{\frac{100}{y}-1}\right)$$

$$= 1 - \frac{\left(\frac{100}{y}-1\right)-9}{40} = \frac{50-\frac{100}{y}}{40} = \frac{5y-10}{4y} = 1.25 - \frac{2.50}{y},\qquad 2 \le y < 10.$$

$F_Y(y) = 0,\qquad y < 2,\qquad\qquad F_Y(y) = 1,\qquad y \ge 10.$

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = \frac{100}{x^2+1} \qquad x = \sqrt{\frac{100}{y}-1} \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{\frac{100}{y}-1}} \cdot \left(-\frac{100}{y^2}\right)$$

$$f_Y(y) = \frac{\sqrt{\frac{100}{y}-1}}{20} \times \left| \frac{1}{2\sqrt{\frac{100}{y}-1}} \cdot \left(-\frac{100}{y^2}\right) \right| = \frac{5}{2y^2} = \frac{2.50}{y^2},$$

$$2 \le y \le 10.$$

Indeed, $\dfrac{d}{dy}\left(1.25 - \dfrac{2.50}{y}\right) = \dfrac{2.50}{y^2}$. ☺

To check: $\displaystyle\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_2^{10} \frac{2.50}{y^2}\,dy = -\frac{2.50}{y}\Big|_2^{10}$

$= 1.25 - 0.25 = 1.$ ☺

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x}{C}, \qquad x = 3, 4, 5, 6, 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

$$1 = \sum_{\text{all } x} p_X(x) = \frac{3}{C} + \frac{4}{C} + \frac{5}{C} + \frac{6}{C} + \frac{7}{C} = \frac{25}{C}.$$

$\Rightarrow \quad C = \mathbf{25}.$

b) Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$. Find the probability distribution of Y.

x	$p_X(x)$
3	$\frac{3}{25} = 0.12$
4	$\frac{4}{25} = 0.16$
5	$\frac{5}{25} = 0.20$
6	$\frac{6}{25} = 0.24$
7	$\frac{7}{25} = 0.28$

$\Rightarrow$

y	$p_Y(y)$
$\frac{100}{10} = 10$	0.12
$\frac{100}{17} \approx 5.882$	0.16
$\frac{100}{26} \approx 3.846$	0.20
$\frac{100}{37} \approx 2.703$	0.24
$\frac{100}{50} = 2$	0.28

OR

$$p_Y(y) = \frac{\sqrt{\frac{100}{y} - 1}}{25}, \qquad y = 2, \frac{100}{37}, \frac{100}{26}, \frac{100}{17}, 10.$$

3. Consider a continuous random variable X with the probability density function

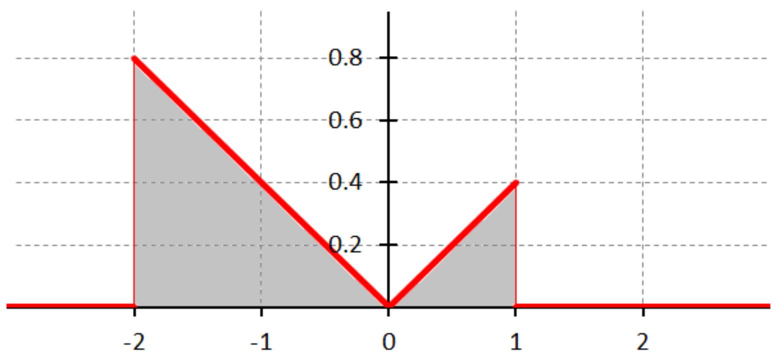
$$f_X(x) = \frac{|x|}{C}, \qquad -2 \le x \le 1, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-2}^{1} \frac{|x|}{C}\,dx = \int_{-2}^{0} \frac{-x}{C}\,dx + \int_{0}^{1} \frac{x}{C}\,dx$$

$$= \left.\frac{-x^2}{2C}\right|_{-2}^{0} + \left.\frac{x^2}{2C}\right|_{0}^{1} = \frac{4}{2C} + \frac{1}{2C} = \frac{5}{2C} = \frac{2.5}{C}.$$

$\Rightarrow \qquad C = \mathbf{2.5}.$



b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(-2) = 0$, $F_X(1) = 1$.

$$F_X(x) = 0, \qquad x < -2,$$

$$F_X(x) = \int_{-2}^{x} \left(-\frac{u}{2.5}\right) du = \frac{4 - x^2}{5} = \frac{4}{5} - \frac{x^2}{5}, \qquad -2 \le x < 0,$$

$$F_X(x) = \int_{-2}^{0} \left(-\frac{u}{2.5}\right) du + \int_{0}^{x} \left(\frac{u}{2.5}\right) du = \frac{4}{5} + \frac{x^2}{5}, \qquad 0 \le x < 1,$$

$$F_X(x) = 1, \qquad x \ge 1.$$

c) Consider $Y = g(X) = \dfrac{100}{X^2+1}$. Find the probability distribution of Y.

$-2 \le x \le 1$ $\qquad$ $-2 \le x \le 0$ $\qquad$ $0 \le x \le 1$

$y = \dfrac{100}{x^2+1}$ $\qquad$ $20 \le y \le 100$ $\qquad$ $100 \ge y \ge 50$

$F_Y(y) = 0, \quad y < 20, \qquad F_Y(y) = 1, \quad y \ge 100.$

$20 \le y \le 100$

$$F_Y(y) = P(Y \le y) = P\left(\frac{100}{X^2+1} \le y\right) = P\left(X^2 \ge \frac{100}{y} - 1\right)$$

$$= P\left(X \le -\sqrt{\frac{100}{y}-1}\right) + P\left(X \ge \sqrt{\frac{100}{y}-1}\right)$$

$$= F_X\left(-\sqrt{\frac{100}{y}-1}\right) + 1 - F_X\left(\sqrt{\frac{100}{y}-1}\right).$$

Case 1. $\quad 20 \le y < 50. \qquad 4 \ge \dfrac{100}{y} - 1 > 1 \qquad 1 < \sqrt{\dfrac{100}{y}-1} \le 2.$

$$F_X\left(-\sqrt{\frac{100}{y}-1}\right) = \frac{4}{5} - \frac{\frac{100}{y}-1}{5} = 1 - \frac{20}{y}, \qquad F_X\left(\sqrt{\frac{100}{y}-1}\right) = 1.$$

$$F_Y(y) = F_X\left(-\sqrt{\frac{100}{y}-1}\right) + 1 - F_X\left(\sqrt{\frac{100}{y}-1}\right) = 1 - \frac{20}{y}, \qquad 20 \le y < 50.$$

Case 2. $50 \le y < 100.$ $1 \ge \frac{100}{y} - 1 > 0$ $0 < \sqrt{\frac{100}{y} - 1} \le 1.$

$$\mathrm{F}_{\mathrm{X}}\left(-\sqrt{\frac{100}{y} - 1} \right) = \frac{4}{5} - \frac{\frac{100}{y} - 1}{5} = 1 - \frac{20}{y},$$

$$\mathrm{F}_{\mathrm{X}}\left(\sqrt{\frac{100}{y} - 1} \right) = \frac{4}{5} + \frac{\frac{100}{y} - 1}{5} = \frac{3}{5} + \frac{20}{y}.$$

$$\mathrm{F}_{\mathrm{Y}}(y) = \mathrm{F}_{\mathrm{X}}\left(-\sqrt{\frac{100}{y} - 1} \right) + 1 - \mathrm{F}_{\mathrm{X}}\left(\sqrt{\frac{100}{y} - 1} \right) = \frac{7}{5} - \frac{40}{y}, \qquad 50 \le y < 100.$$

c.d.f.

$$\mathrm{F}_{\mathrm{Y}}(y) = \begin{cases} 0 & y < 20 \\ 1 - \frac{20}{y} & 20 \le y < 50 \\ \frac{7}{5} - \frac{40}{y} & 50 \le y < 100 \\ 1 & y \ge 100 \end{cases}$$

p.d.f.

$$f_{\mathrm{Y}}(y) = \mathrm{F}'_{\mathrm{Y}}(y) = \begin{cases} \frac{20}{y^2} & 20 < y < 50 \\ \frac{40}{y^2} & 50 < y < 100 \\ 0 & \text{otherwise} \end{cases}$$

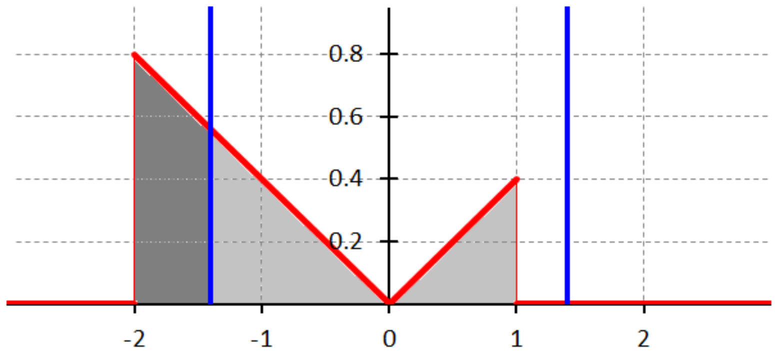
OR

$20 \le y \le 100 \qquad F_Y(y) = P\left(X \le -\sqrt{\frac{100}{y} - 1}\right) + P\left(X \ge \sqrt{\frac{100}{y} - 1}\right).$

Case 1. $\quad 20 \le y < 50.$

$4 \ge \frac{100}{y} - 1 > 1.$

$1 < \sqrt{\frac{100}{y} - 1} \le 2.$

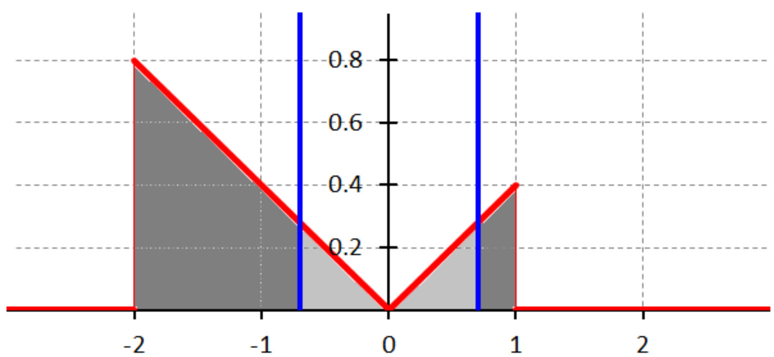


$$F_Y(y) = \int_{-2}^{-\sqrt{\frac{100}{y}-1}} \left(-\frac{x}{2.5} \right) dx = \frac{4 - \left(\frac{100}{y} - 1 \right)}{5} = 1 - \frac{20}{y}, \qquad 20 \le y < 50.$$

Case 2. $\quad 50 \le y < 100.$

$1 \ge \frac{100}{y} - 1 > 0.$

$0 < \sqrt{\frac{100}{y} - 1} \le 1.$



$$F_Y(y) = \int_{-2}^{-\sqrt{\frac{100}{y}-1}} \left(-\frac{x}{2.5} \right) dx + \int_{\sqrt{\frac{100}{y}-1}}^{1} \left(\frac{x}{2.5} \right) dx$$

$$= \frac{4 - \left(\frac{100}{y} - 1 \right)}{5} + \frac{1 - \left(\frac{100}{y} - 1 \right)}{5} = \frac{7}{5} - \frac{40}{y}, \qquad 20 \le y < 50.$$

OR

$-2 \le x < 0$

$$f_X(x) = -\frac{x}{2.5}$$

$$Y = g(X) = \frac{100}{X^2 + 1}$$

$20 \le y < 100$

$$x = -\sqrt{\frac{100}{y} - 1} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right)$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$-\frac{-\sqrt{\frac{100}{y} - 1}}{2.5} \times \left| -\frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right) \right|$$

$$\frac{20}{y^2}$$

$0 < x \le 1$

$$f_X(x) = \frac{x}{2.5}$$

$$Y = g(X) = \frac{100}{X^2 + 1}$$

$100 > y \ge 50$

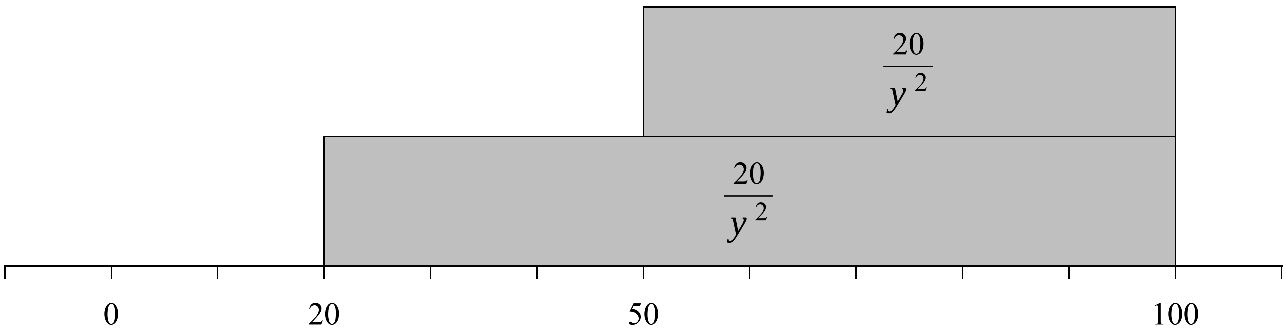
$$x = \sqrt{\frac{100}{y} - 1} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right)$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\frac{\sqrt{\frac{100}{y} - 1}}{2.5} \times \left| \frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right) \right|$$

$$\frac{20}{y^2}$$



$$f_Y(y) = \frac{20}{y^2}, \quad 20 < y < 50. \qquad f_Y(y) = \frac{20}{y^2} + \frac{20}{y^2} = \frac{40}{y^2}, \quad 50 < y < 100.$$

p.d.f.

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{20}{y^2} & 20 < y < 50 \\ \frac{40}{y^2} & 50 < y < 100 \\ 0 & \text{otherwise} \end{cases}$$

To check:

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_{20}^{50} \frac{20}{y^2}\,dy + \int_{50}^{100} \frac{40}{y^2}\,dy$$

$$= \left(-\frac{20}{y}\right)\Bigg|_{20}^{50} + \left(-\frac{40}{y}\right)\Bigg|_{50}^{100}$$

$$= -\frac{20}{50} + \frac{20}{20} - \frac{40}{100} + \frac{40}{50} = 1.$$

☺