STAT 410

Examples for 12/02/2020 (2)

Fall 2020
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p.m.f. or p.d.f. $f(x;\theta)$, $\theta \in \Omega$.

$X_1, X_2, \dots, X_n$ are i.i.d. $f(x;\theta)$

- $\theta \neq \theta' \quad \Rightarrow \quad f(x;\theta) \neq f(x;\theta')$
- $f(x;\theta)$ have common support for all θ
- θ_0 is an interior point in Ω

Let θ_0 be the true parameter.

Then $\quad P[L(\theta_0 \mid X_1, X_2, \dots, X_n) > L(\theta \mid X_1, X_2, \dots, X_n)] \to 1$ as $n \to \infty$ for all $\theta \neq \theta_0$.

- $f(x;\theta)$ is differentiable as a function of θ

Then $\quad$ equation $\dfrac{d}{d\theta} L(\theta) = 0$ has a solution $\hat{\theta}$, such that $\hat{\theta} \overset{P}{\to} \theta_0$.

- $f(x;\theta)$ is twice differentiable as a function of θ
- $\int f(x;\theta)\, dx$ can be twice differentiable under the integral sign as a function of θ
- $\left| \dfrac{\partial^3}{\partial \theta^3} \ln f(x;\theta) \right| < M(x)$ with $E[M(X)] < \infty$

Then $\quad \sqrt{n}\left(\hat{\theta} - \theta\right)$ is approx. $N\left(0, \dfrac{1}{I(\theta)}\right)$ for large n.

That is, for large n, $\hat{\theta}$ is approximately $N\left(\theta, \dfrac{1}{n \cdot I(\theta)}\right)$.

Likelihood Ratio Test:

$\mathrm{H}_0 : \theta = \theta_0$ vs. $\mathrm{H}_1 : \theta \neq \theta_0$ Reject H_0 if $\Lambda = \dfrac{\mathrm{L}(\theta_0)}{\mathrm{L}(\hat{\theta})} \le k.$

$\Leftrightarrow \quad -2 \ln \Lambda \ge -2 \ln k = c.$

$-2 \ln \Lambda$ is approx. $\chi^2(1)$ for large n.

Reject H_0 if $-2 \ln \Lambda \ge \chi^2_\alpha(1)$ (for large n).

<u>Example 2</u>:

Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample of size n from $\mathrm{N}(\mu, \sigma^2)$ distribution (σ^2 known).

$\mathrm{H}_0 : \mu = \mu_0$ vs. $\mathrm{H}_1 : \mu \neq \mu_0$ $\hat{\mu} = \overline{\mathrm{X}}$.

$$\Lambda = \frac{\mathrm{L}(\mu_0)}{\mathrm{L}(\hat{\mu})} = \exp\left\{ -\frac{n(\overline{\mathrm{X}} - \mu_0)^2}{2\sigma^2} \right\}.$$

$$-2 \ln \Lambda = \frac{n(\overline{\mathrm{X}} - \mu_0)^2}{\sigma^2} = \left(\frac{\overline{\mathrm{X}} - \mu_0}{\sigma / \sqrt{n}} \right)^2 \text{ is } \chi^2(1).$$

Wald-type test: Reject H_0 if $\left\{ \sqrt{n \mathrm{I}(\hat{\theta})} \left(\hat{\theta} - \theta_0 \right) \right\}^2 \ge \chi^2_\alpha(1)$ (for large n).

Scores-type test: Reject H_0 if $\left(\dfrac{\frac{d}{d\theta} \ln \mathrm{L}(\theta_0)}{\sqrt{n \mathrm{I}(\theta_0)}} \right)^2 \ge \chi^2_\alpha(1)$ (for large n).