STAT 410

Examples for 12/09/2020 (Disc)

Fall 2020
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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0, \qquad \beta > 0.$$

Recall: the maximum likelihood estimator of β is $\hat{\beta} = \dfrac{4n}{\sum\limits_{i=1}^{n} \sqrt{X_i}}$.

Let β have a prior p.d.f. which is gamma with parameters α and θ.

a) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

b) Show that $E(\beta \mid x_1, x_2, \ldots, x_n)$ is a weighted average of the maximum likelihood estimate $\hat{\beta}$ and the prior mean $\alpha\,\theta$.

c) Construct a $(1-\gamma)100\%$ credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots,$ $X_n = x_n$.

d) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Let $\alpha = 1$, $\theta = 2$. That is, the prior distribution of β is

$$\text{Gamma}(\alpha = 1,\ \theta = 2) = \text{Exponential}(\theta = 2) = \chi^2(2 \text{ d.f.}) \text{ distribution.}$$

(i) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

(ii) Construct a 95% credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0, \qquad \beta > 0.$$

Recall: the maximum likelihood estimator of β is $\hat{\beta} = \dfrac{4n}{\sum_{i=1}^{n}\sqrt{X_i}}$.

Let β have a prior p.d.f. which is gamma with parameters α and θ.

a) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$f(x_1, x_2, \ldots x_n \mid \beta) = f_X(x_1;\beta) \cdot f_X(x_2;\beta) \cdot \ldots \cdot f_X(x_n;\beta)$$

$$= \left[\frac{\beta^{4n}}{12^n} e^{-\beta\sum_{i=1}^{n}\sqrt{x_i}}\right]\left(\prod_{i=1}^{n} x_i\right).$$

$$f(x_1, x_2, \ldots x_n, \beta) = f(x_1, x_2, \ldots x_n \mid \beta) \times \pi(\beta)$$

$$= \left[\frac{\beta^{4n}}{12^n} e^{-\beta\sum_{i=1}^{n}\sqrt{x_i}}\right]\left(\prod_{i=1}^{n} x_i\right) \times \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,\beta^{\alpha-1} e^{-\beta/\theta}$$

$$= \ldots\, \beta^{4n+\alpha-1} e^{-\beta\left(\sum_{i=1}^{n}\sqrt{x_i} + \frac{1}{\theta}\right)}.$$

$\Rightarrow$ the posterior distribution of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$,

is **Gamma** with New $\alpha = 4n + \alpha$ and New $\theta = \dfrac{1}{\sum_{i=1}^{n}\sqrt{x_i} + \frac{1}{\theta}}$.

$\Rightarrow \qquad$ (conditional mean of β, given $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$)

$$= (\text{New } \alpha) \times (\text{New } \theta) = \frac{4n + \alpha}{\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}}.$$

b) Show that $E(\beta \mid x_1, x_2, \dots, x_n)$ is a weighted average of the maximum likelihood estimate $\hat{\beta}$ and the prior mean $\alpha\,\theta$.

$$\frac{4n + \alpha}{\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}} = \hat{\beta} \cdot \frac{\sum_{i=1}^{n} \sqrt{x_i}}{\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}} + \alpha\,\theta \cdot \frac{\frac{1}{\theta}}{\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}}.$$

c) Construct a $(1 - \gamma)100\%$ credible interval for β, given $X_1 = x_1, X_2 = x_2, \dots,$ $X_n = x_n$.

$2\left(\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}\right)(\beta \mid x_1, x_2, \dots, x_n)$ has a $\chi^2(8n + 2\alpha)$ distribution.

$$P\left(\chi^2_{1-\gamma/2}(8n + 2\alpha) < 2\left(\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}\right)(\beta \mid x_1, x_2, \dots, x_n) < \chi^2_{\gamma/2}(8n + 2\alpha)\right) = 1 - \gamma.$$

$$P\left(\frac{\chi^2_{1-\gamma/2}(8n + 2\alpha)}{2\left(\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}\right)} < (\beta \mid x_1, x_2, \dots, x_n) < \frac{\chi^2_{\gamma/2}(8n + 2\alpha)}{2\left(\sum_{i=1}^{n} \sqrt{x_i} + \frac{1}{\theta}\right)}\right) = 1 - \gamma.$$

$$\left(\frac{\chi^2_{1-\gamma/2}(8n+2\alpha)}{2\sum_{i=1}^{n}\sqrt{x_i}+\frac{2}{\theta}}, \frac{\chi^2_{\gamma/2}(8n+2\alpha)}{2\sum_{i=1}^{n}\sqrt{x_i}+\frac{2}{\theta}} \right)$$

is a $(1-\gamma)\,100\%$ credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

d) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Let $\alpha = 1$, $\theta = 2$. That is, the prior distribution of β is

Gamma$(\alpha = 1,\ \theta = 2)$ = Exponential$(\theta = 2)$ = $\chi^2(2\text{ d.f.})$ distribution.

Recall: Examples for 10/21/2020 (Disc) $\sum_{i=1}^{n}\sqrt{x_i} = 2.$ $\hat{\beta} = \frac{12}{2} = 6.$

Prior mean = $\alpha\,\theta = 2$.

(i) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$E(\beta \mid x_1, x_2, \ldots, x_n) = \frac{4n+\alpha}{\sum_{i=1}^{n}\sqrt{x_i}+\frac{1}{\theta}} = \frac{13}{2.5} = \mathbf{5.2}.$$

(ii) Construct a 95% credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$\chi^2_{0.975}(26) = 13.84$, $\chi^2_{0.025}(26) = 41.92$.

$\left(\frac{13.84}{2\cdot 2.5}, \frac{41.92}{2\cdot 2.5}\right)$ **(2.768, 8.384)**