STAT 410 | **Examples for 10/07/2020 (Disc)** | Fall 2020
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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x^2 y, \qquad 0 < x < 2, \quad 0 < y < x + 1, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y). That is, sketch $\{\, 0 < x < 2, \quad 0 < y < x + 1 \,\}$.

b) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Are X and Y independent? *Justify your answer.*

f) Find $P(X \cdot Y \le 2)$.

g) Find $P(Y > 1.2 \mid X = 0.6)$.

h) Find $P(Y > 1.2 \mid X < 0.6)$.

i) Find $E(Y \mid X = x)$.

j) Find $P(X < 1.2 \mid Y = 0.2)$.

k) Find $P(X < 1.2 \mid Y = 2.0)$.

l) Find the probability distribution of $W = X + Y$.

m) Set up the integral(s) for the c.d.f. of $V = X \cdot Y$, $F_V(v)$.
You do NOT have to evaluate the integral(s).

n) Find the c.d.f. of $U = {}^{Y}/_{X}$, $F_U(u)$.

o) Let $V = X \cdot Y$.

Find the joint probability density function of (X, V), $f_{X,V}(x, v)$.

Sketch the support of (X, V).

OR

Find the joint probability density function of (Y, V), $f_{Y,V}(y, v)$.

Sketch the support of (Y, V).

p) Use (o) to find the p.d.f. of $V = X \cdot Y$, $f_V(v)$.

q) Let $U = {}^{Y}/_{X}$.

Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.

Sketch the support of (X, U).

OR

Find the joint probability density function of (Y, U), $f_{Y,U}(y, u)$.

Sketch the support of (Y, U).

r) Use (q) to find the p.d.f. of $U = {}^{Y}\!/_{X}$, $f_U(u)$.

s) Let $S = \min(X, Y)$. Find the probability distribution of S.

t) Let $T = \max(X, Y)$. Find the probability distribution of T.

2. Let X be a random variable with the probability density function

$$f_X(x) = \frac{324}{x^5}, \qquad x > 3, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = \ln\left(\frac{X}{3}\right)$.

b) Find the probability distribution of $Y = \frac{10}{X+2}$.

3. Let X_1, X_2, X_3, X_4 be a random sample (i.i.d.) of size $n = 4$ from a probability distribution with the p.d.f.

$$f_X(x) = \frac{324}{x^5}, \qquad x > 3.$$

a) Find $P(\max X_i < 6)$.

b) Find $P(\max X_i > 5)$.

c) Find $P(\min X_i < 4)$.

d) Find $E(\min X_i)$.

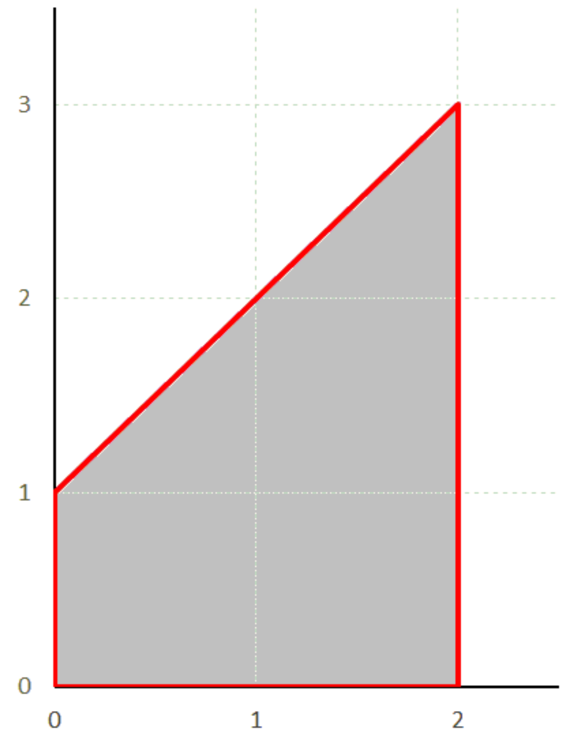
1. Let the joint probability density function for (X, Y) be

$$f(x,y) = C x^2 y, \qquad 0 < x < 2, \quad 0 < y < x+1, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y).

That is, sketch

$\{ 0 < x < 2, \quad 0 < y < x+1 \}$.



b) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^2 \left(\int_0^{x+1} C x^2 y \, dy \right) dx$$

$$= \int_0^2 \frac{C}{2} x^2 (x+1)^2 \, dx$$

$$= \frac{C}{2} \int_0^2 \left(x^4 + 2x^3 + x^2 \right) dx$$

$$= \frac{C}{2} \left(\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} \right) \Bigg|_0^2 = \frac{128\,C}{15}.$$

$$C = \mathbf{\frac{15}{128}}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{x+1} \frac{15}{128} x^2 y \, dy = \frac{15}{256} x^2 (x+1)^2, \qquad 0 < x < 2.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

If $0 < y < 1$,

$$f_Y(y) = \int_0^2 \frac{15}{128} x^2 y \, dx = \frac{5}{16} y.$$

If $1 < y < 3$,

$$f_Y(y) = \int_{y-1}^2 \frac{15}{128} x^2 y \, dx = \frac{5}{128} y \left(8 - (y-1)^3 \right)$$

$$= \frac{5}{128} \left(9y - 3y^2 + 3y^3 - y^4 \right).$$

$$f_Y(y) = \begin{cases} \frac{5}{16} y & 0 < y < 1 \\ \frac{5}{128} y \left(8 - (y-1)^3 \right) & 1 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

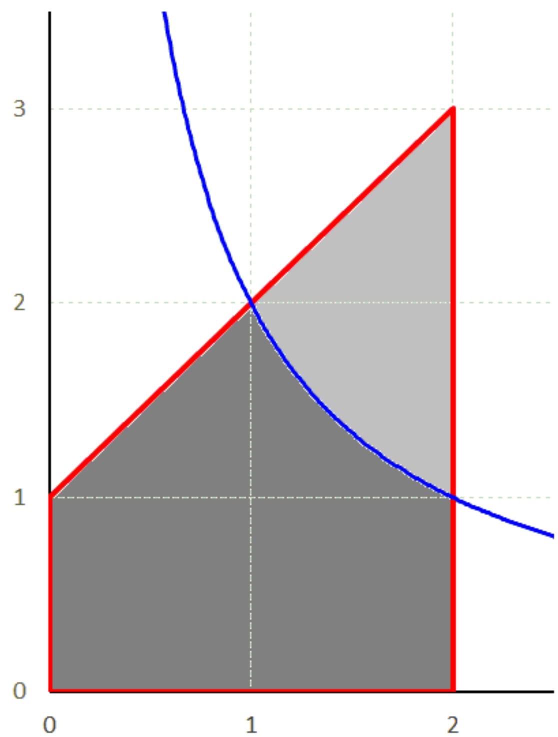
e) Are X and Y independent? *Justify your answer.*

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{X,Y}(x, y) \neq f_X(x) \times f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

f) Find $P(X \cdot Y \leq 2)$.



$$P(X \cdot Y \leq 2)$$

$$= 1 - \int_1^2 \left(\int_{2/x}^{x+1} \frac{15}{128} x^2 y \, dy \right) dx$$

$$= 1 - \int_1^2 \left(\frac{15}{256} x^2 (x+1)^2 - \frac{15}{64} \right) dx$$

$$= 1 - \frac{15}{256} \int_1^2 \left(x^4 + 2x^3 + x^2 - 4 \right) dx$$

$$= 1 - \frac{6x^5 + 15x^4 + 10x^3 - 120x}{512} \Bigg|_1^2$$

$$= 1 - \frac{272 - (-89)}{512}$$

$$= 1 - \frac{361}{512} = \mathbf{\frac{151}{512}} = \mathbf{0.294921875}.$$

OR

$$= \int_0^1 \left(\int_0^{x+1} \frac{15}{128} x^2 y \, dy \right) dx + \int_1^2 \left(\int_0^{2/x} \frac{15}{128} x^2 y \, dy \right) dx = \frac{31}{512} + \frac{15}{64} = \mathbf{\frac{151}{512}}.$$

OR

$$= \int_0^1 \left(\int_0^2 \frac{15}{128} x^2 y \, dx \right) dy + \int_1^2 \left(\int_{y-1}^{2/y} \frac{15}{128} x^2 y \, dx \right) dy = \frac{5}{32} + \frac{71}{512} = \mathbf{\frac{151}{512}}.$$

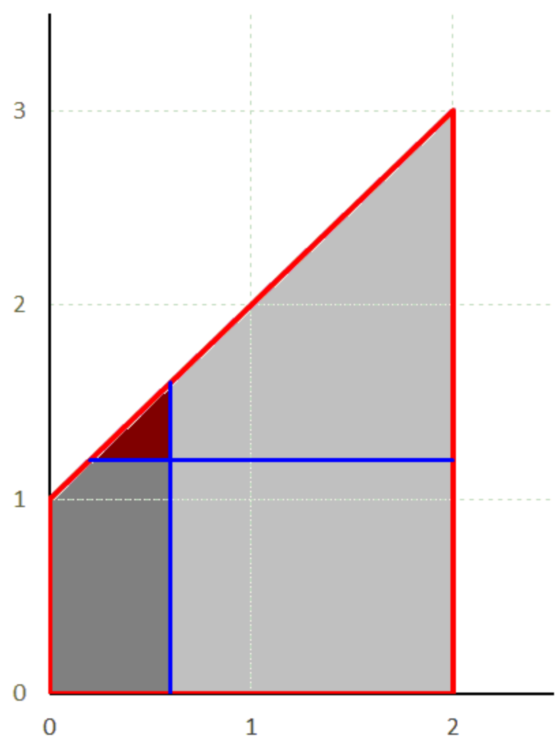
g) Find $P(Y > 1.2 \mid X = 0.6)$.

$$f_{Y|X}(y \mid x) = \frac{\frac{15}{128} x^2 y}{\frac{15}{256} x^2 (x+1)^2} = \frac{2y}{(x+1)^2}, \qquad 0 < y < x + 1.$$

$$f_{Y|X}(y \mid 0.6) = \frac{2y}{1.6^2}, \qquad 0 < y < 1.6.$$

$$P(Y > 1.2 \mid X = 0.6) = \int_{1.2}^{1.6} \frac{2y}{1.6^2}\, dx = \frac{1.6^2 - 1.2^2}{1.6^2} = \mathbf{\frac{7}{16}} = \mathbf{0.4375}.$$

h) Find $P(Y > 1.2 \mid X < 0.6)$.



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

provided $P(B) > 0$.

$$P(B) = P(X < 0.6)$$

$$= \int_0^{0.6} \frac{15}{256} x^2 (x+1)^2\, dx$$

$$= \frac{14{,}283}{1{,}600{,}000}$$

$$= 0.008926875.$$

$$P(A \cap B) = P(Y > 1.2 \cap X < 0.6) = \int_{0.2}^{0.6} \left(\int_{1.2}^{x+1} \frac{15}{128} x^2 y\, dy \right) dx$$

$$= 0.00287 = \frac{4{,}592}{1{,}600{,}000}.$$

$$\mathrm{P}(\mathrm{Y} > 1.2 \mid \mathrm{X} < 0.6) = \frac{\mathbf{4{,}592}}{\mathbf{14{,}283}} \approx 0.3215.$$

i) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$.

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x) = \int_0^{x+1} y \cdot \frac{2y}{(x+1)^2}\, dy = \frac{2}{3}(x+1).$$

j) Find $\mathrm{P}(\mathrm{X} < 1.2 \mid \mathrm{Y} = 0.2)$.

For $0 < y < 1$,

$$f_{\mathrm{X}\mid\mathrm{Y}}(x \mid y) = \frac{\frac{15}{128} x^2 y}{\frac{5}{16} y} = \frac{3x^2}{8}, \qquad 0 < x < 2.$$

$$\mathrm{P}(\mathrm{X} < 1.2 \mid \mathrm{Y} = 0.2) = \int_0^{1.2} \frac{3x^2}{8}\, dx = \frac{1.2^3}{8} = \mathbf{0.216}.$$

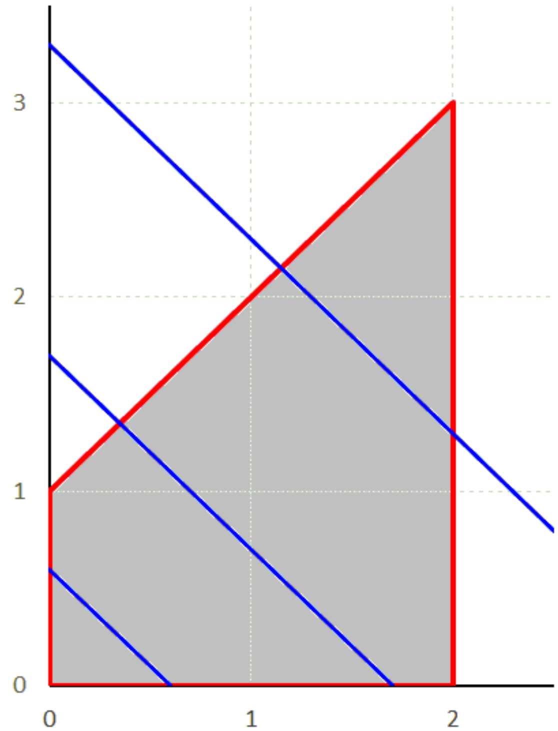
k) Find $\mathrm{P}(\mathrm{X} < 1.2 \mid \mathrm{Y} = 2.0)$.

For $1 < y < 3$,

$$f_{\mathrm{X}\mid\mathrm{Y}}(x \mid y) = \frac{\frac{15}{128} x^2 y}{\frac{5}{128} y \left(8 - (y-1)^3\right)} = \frac{3x^2}{8 - (y-1)^3}, \qquad y - 1 < x < 2.$$

$$\mathrm{P}(\mathrm{X} < 1.2 \mid \mathrm{Y} = 2.0) = \int_1^{1.2} \frac{3x^2}{8 - 1^3}\, dx = \frac{1.2^3 - 1^3}{7} = \mathbf{0.104}.$$

l) Find the probability distribution of $W = X + Y$.



$$F_W(w) = P(X + Y \le w) = \ldots$$

There are 3 cases:

$0 < w < 1, \quad 1 < w < 2, \quad 2 < w < 5.$

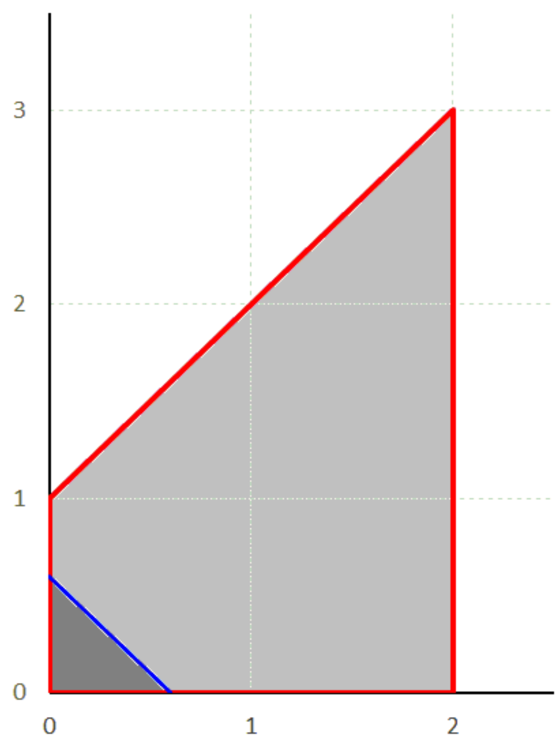
Technically, there are 5 cases:

$w < 0,$

$0 < w < 1, \quad 1 < w < 2, \quad 2 < w < 5,$

$w > 5,$

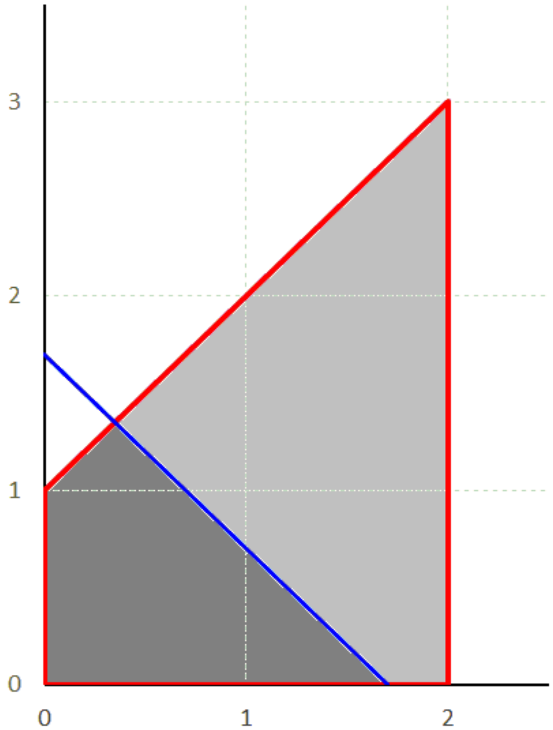
but $w < 0$ and $w > 5$ are boring.



Case 1: $0 \le w < 1.$

$$F_W(w) = \int_0^w \left(\int_0^{w-x} \frac{15}{128} x^2 y \, dy \right) dx$$

$$= \frac{w^5}{512}, \qquad 0 \le w < 1.$$



Case 2: $\quad 1 \le w < 2.$

$y = x + 1 \quad \& \quad x + y = w$

$$\Rightarrow \qquad x = \frac{w-1}{2}, \qquad y = \frac{w+1}{2}.$$

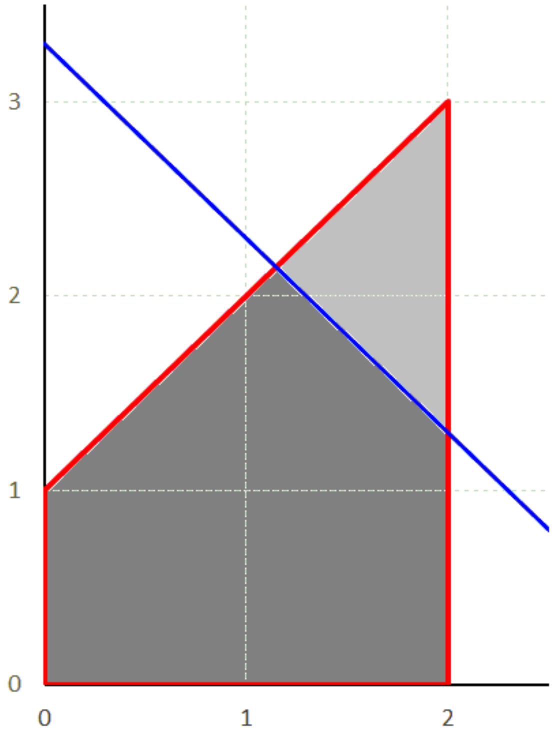
$$F_W(w) = \int_0^{\frac{w-1}{2}} \left(\int_0^{x+1} \frac{15}{128}\, x^2 y\, dy \right) dx$$

$$+ \int_{\frac{w-1}{2}}^{w} \left(\int_0^{w-x} \frac{15}{128}\, x^2 y\, dy \right) dx$$

OR

$$F_W(w) = \int_0^{1} \left(\int_0^{w-y} \frac{15}{128}\, x^2 y\, dx \right) dy + \int_1^{\frac{w+1}{2}} \left(\int_{y-1}^{w-y} \frac{15}{128}\, x^2 y\, dx \right) dy$$

$$= \frac{11w^5 + 15w^4 - 10w^3 - 10w^2 + 15w - 5}{8192}, \qquad 1 \le w < 2.$$



Case 3: $\quad 2 \le w < 5.$

$$F_W(w) = \int_0^{\frac{w-1}{2}} \left(\int_0^{x+1} \frac{15}{128}\, x^2 y \, dy \right) dx + \int_{\frac{w-1}{2}}^{2} \left(\int_0^{w-x} \frac{15}{128}\, x^2 y \, dy \right) dx$$

OR

$$F_W(w) = 1 - \int_{\frac{w-1}{2}}^{2} \left(\int_{w-x}^{x+1} \frac{15}{128}\, x^2 y \, dy \right) dx$$

$$= 1 - \frac{5w^5 - 15w^4 + 10w^3 - 1270w^2 + 3825w + 5125}{8192}$$

$$= \frac{-5w^5 + 15w^4 - 10w^3 + 1270w^2 - 3825w + 3067}{8192}$$

$$= 1 - \frac{5(5-w)^2 (w+1)\left(w^2 + 6w + 41\right)}{8192}, \qquad 2 \le w < 5.$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\, dx$$

$$f(x, w-x) = \frac{15}{128}\, x^2\, (w-x).$$

$0 < x < 2$

$0 < y \quad \Rightarrow \quad 0 < w - x \quad \Rightarrow \quad x < w$

$y < x + 1 \quad \Rightarrow \quad w - x < x + 1 \quad \Rightarrow \quad x > \dfrac{w-1}{2}$

$x > 0 \quad \& \quad x > \dfrac{w-1}{2} \qquad\qquad x < 2 \quad \& \quad x < w$

Case 1: $\quad 0 < w < 1. \qquad$ Then $\dfrac{w-1}{2} < 0 < w < 2.$

$$f_W(w) = \int_0^w \frac{15}{128}\, x^2\, (w-x)\, dx = \frac{15}{128} \left(\frac{w x^3}{3} - \frac{x^4}{4} \right) \Bigg|_0^w = \frac{5 w^4}{512},$$

$0 < w < 1.$

Case 2: $\quad 1 < w < 2. \qquad$ Then $0 < \dfrac{w-1}{2} < w < 2.$

$$f_W(w) = \int_{\frac{w-1}{2}}^{w} \frac{15}{128}\, x^2\, (w-x)\, dx = \frac{15}{128} \left(\frac{w x^3}{3} - \frac{x^4}{4} \right) \Bigg|_{\frac{w-1}{2}}^{w}$$

$$= \frac{5 w^4}{512} - \frac{5 \left(8 w (w-1)^3 - 3 (w-1)^4 \right)}{8192}$$

$$= \frac{55 w^4 + 60 w^3 - 30 w^2 - 20 w + 15}{8192}, \qquad 1 < w < 2.$$

Case 3: $2 < w < 5.$ Then $0 < \frac{w-1}{2} < 2 < w.$

$$f_W(w) = \int_{\frac{w-1}{2}}^{2} \frac{15}{128} x^2 (w-x)\, dx = \frac{15}{128} \left(\frac{w x^3}{3} - \frac{x^4}{4} \right) \Bigg|_{\frac{w-1}{2}}^{2}$$

$$= \frac{5(2w-3)}{32} - \frac{5\left(8w(w-1)^3 - 3(w-1)^4\right)}{8192}$$

$$= \frac{-25w^4 + 60w^3 - 30w^2 + 2540w - 3825}{8192}, \qquad 2 < w < 5.$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy$$

$$f(w-y, y) = \frac{15}{128} (w-y)^2\, y.$$

$0 < x \quad \Rightarrow \quad 0 < w-y \quad \Rightarrow \quad y < w$

$x < 2 \quad \Rightarrow \quad w-y < 2 \quad \Rightarrow \quad y > w-2$

$0 < y$

$y < x+1 \quad \Rightarrow \quad y < w-y+1 \quad \Rightarrow \quad y < \frac{w+1}{2}$

$y > 0 \quad \& \quad y > w-2 \qquad\qquad y < w \quad \& \quad y < \frac{w+1}{2}$

Case 1: $0 < w < 1$. Then $w - 2 < 0 < w < \frac{w+1}{2}$.

$$f_W(w) = \int_0^w \frac{15}{128} (w-y)^2 \, y \, dx = \ldots, \qquad 0 < w < 1.$$

Case 2: $1 < w < 2$. Then $w - 2 < 0 < \frac{w+1}{2} < w$.

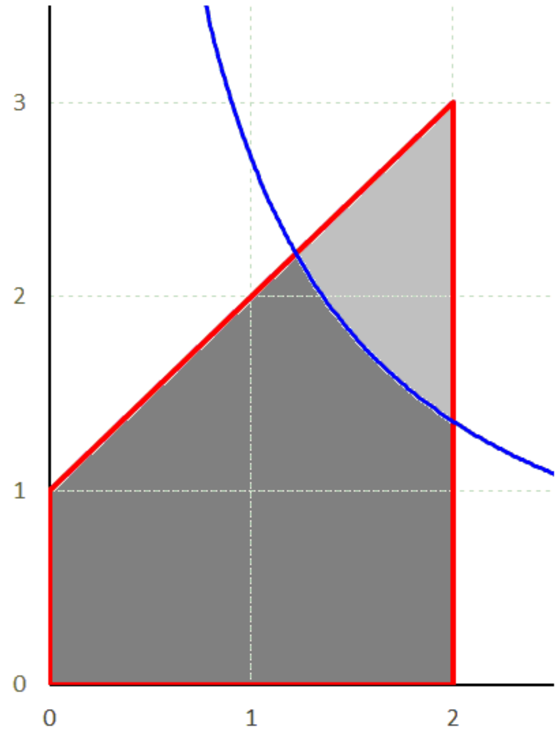
$$f_W(w) = \int_0^{\frac{w+1}{2}} \frac{15}{128} (w-y)^2 \, y \, dx = \ldots, \qquad 1 < w < 2.$$

Case 3: $2 < w < 5$. Then $0 < w - 2 < \frac{w+1}{2} < w$.

$$f_W(w) = \int_{w-2}^{\frac{w+1}{2}} \frac{15}{128} (w-y)^2 \, y \, dx = \ldots, \qquad 2 < w < 5.$$

m) Set up the integral(s) for the c.d.f. of $V = X \cdot Y$, $F_V(v)$.

You do NOT have to evaluate the integral(s).



$F_V(v) = P(X \cdot Y \le v) = \dots$

There is only 1 case:

$0 < v < 3.$

Technically, there are 3 cases:

$v < 0,$

$0 < v < 6,$

$v > 6,$

but $v < 0$ and $v > 6$ are boring.

$y = x + 1 \quad \& \quad x \cdot y = v \qquad \Rightarrow \qquad \frac{v}{x} = x + 1.$

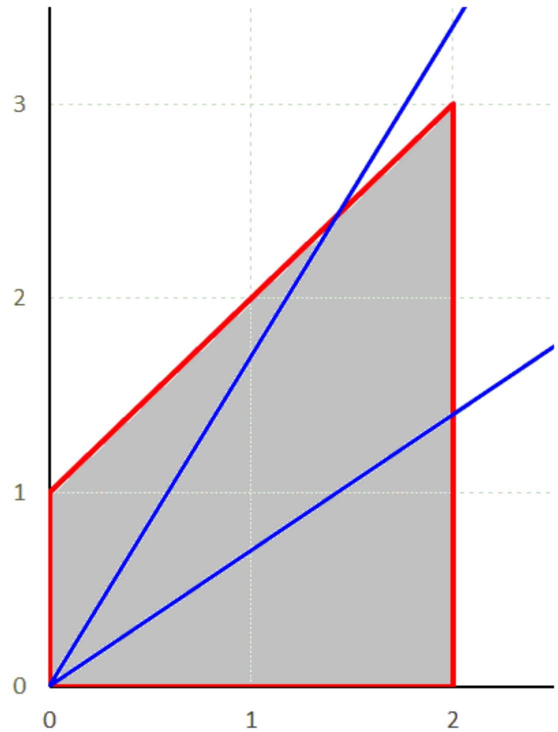
$x^2 + x - v = 0. \qquad \Rightarrow \qquad x = \frac{-1 \pm \sqrt{1+4v}}{2}.$

$0 < x < 2 \qquad \Rightarrow \qquad x = \frac{\sqrt{1+4v} - 1}{2}.$

$$F_V(v) = \int_0^{\frac{\sqrt{1+4v}-1}{2}} \left(\int_0^{x+1} \frac{15}{128} x^2 y \, dy \right) dx + \int_{\frac{\sqrt{1+4v}-1}{2}}^{2} \left(\int_0^{\frac{v}{x}} \frac{15}{128} x^2 y \, dy \right) dx$$

$$F_V(v) = 1 - \int_{\frac{\sqrt{1+4v}-1}{2}}^{2} \left(\int_{\frac{v}{x}}^{x+1} \frac{15}{128} x^2 y \, dy \right) dx$$

n) Find the c.d.f. of $U = {}^{Y}\!/_{X}$, $F_U(u)$.



$$F_U(u) = P\left({}^{Y}\!/_{X} \le u\right)$$

$$= P(Y \le u X) = \ldots$$

There are 2 cases:

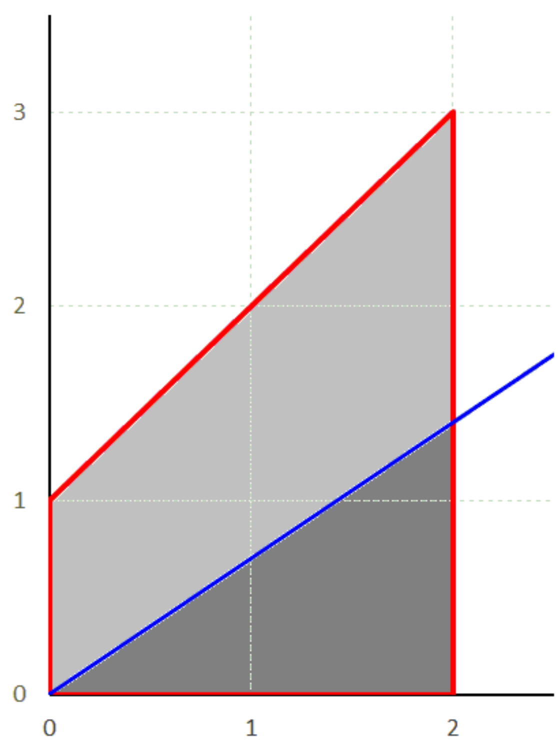
$0 < u < 1.5$, $\quad u > 1.5$.

Technically, there are 3 cases:

$u < 0$,

$0 < u < 1.5$, $\quad u > 1.5$.

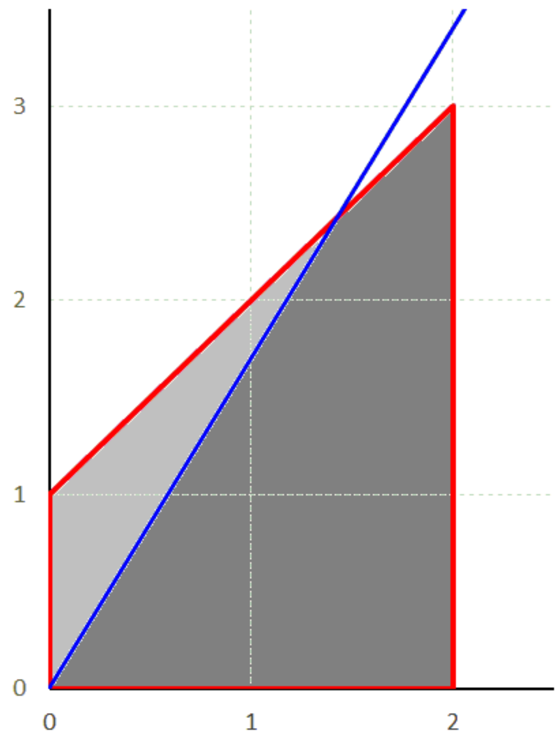
but $u < 0$ is boring.



Case 1: $\quad 0 \le u < 1.5$.

$$F_U(u) = \int_0^2 \left(\int_0^{ux} \frac{15}{128} x^2 y \, dy \right) dx$$

$$= \frac{3u^2}{8}, \qquad 0 \le u < 1.5.$$



Case 2: $u \geq 1.5.$

$$y = x + 1 \quad \& \quad y = u\,x$$

$$\Rightarrow \qquad x = \frac{1}{u-1}, \qquad y = \frac{u}{u-1}.$$

$$\mathrm{F}_{\mathrm{U}}(u) = \int_0^{\frac{1}{u-1}} \left(\int_0^{u\,x} \frac{15}{128}\, x^2 y\, dy \right) dx$$

$$+ \int_{\frac{1}{u-1}}^{2} \left(\int_0^{x+1} \frac{15}{128}\, x^2 y\, dy \right) dx$$

OR

$$\mathrm{F}_{\mathrm{U}}(u) = \int_0^{\frac{u}{u-1}} \left(\int_{\frac{y}{u}}^{2} \frac{15}{128}\, x^2 y\, dy \right) dx + \int_{\frac{u}{u-1}}^{3} \left(\int_{y-1}^{2} \frac{15}{128}\, x^2 y\, dy \right) dx$$

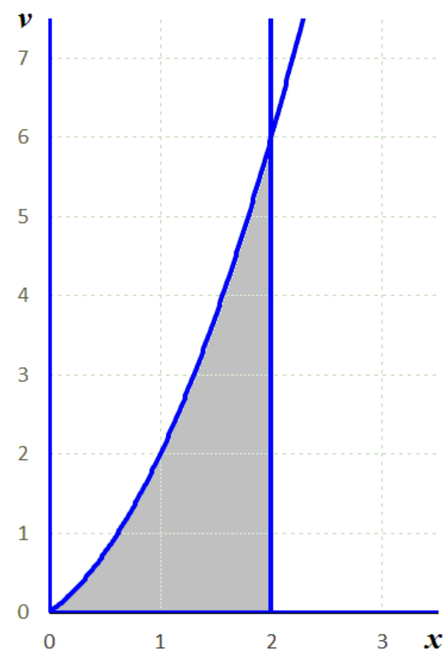
OR

$$\mathrm{F}_{\mathrm{U}}(u) = 1 - \int_0^{\frac{1}{u-1}} \left(\int_{u\,x}^{x+1} \frac{15}{128}\, x^2 y\, dy \right) dx = 1 - \frac{4\,u-1}{512\,(u-1)^4}, \qquad u \geq 1.5.$$

ox) Let $V = X \cdot Y$.

Find the joint probability density function of (X, V), $f_{X,V}(x, v)$.

Sketch the support of (X, V).



$$X = X, \qquad Y = V / X.$$

$$0 < x < 2,$$

$$0 < y \quad \Rightarrow \quad 0 < \frac{v}{x} \quad \Rightarrow \quad 0 < v,$$

$$y < x + 1 \quad \Rightarrow \quad \frac{v}{x} < x + 1 \quad \Rightarrow \quad v < x^2 + x.$$

$$J = \begin{vmatrix} 1 & 0 \\ -\dfrac{v}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{1}{x}.$$

$$f_{X,V}(x, v) = f_{X,Y}(x, \frac{v}{x}) \times |J| = \frac{15}{128} x^2 \frac{v}{x} \times \frac{1}{x} = \frac{15}{128} v,$$

$$0 < x < 2, \quad 0 < v < x^2 + x.$$

oy) Let $V = X \cdot Y$.

Find the joint probability density function of (Y, V), $f_{Y,V}(y, v)$.

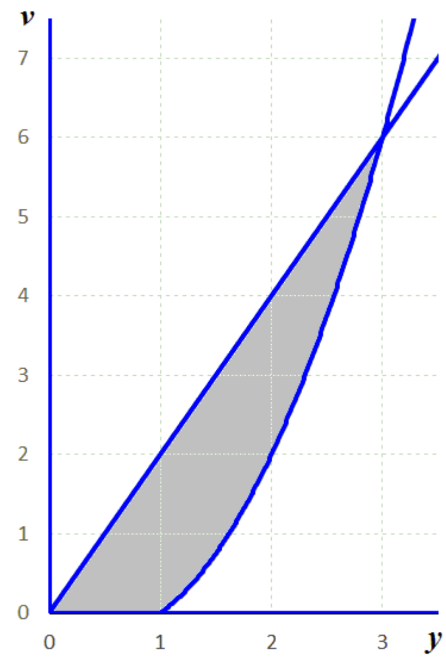
Sketch the support of (Y, V).

$X = V/Y, \qquad Y = Y.$

$0 < x < 2 \quad \Rightarrow \quad 0 < \dfrac{v}{y} < 2 \quad \Rightarrow \quad 0 < v < 2y,$

$0 < y,$

$y < x + 1 \quad \Rightarrow \quad y < \dfrac{v}{y} + 1 \quad \Rightarrow \quad v > y^2 - y.$



$$J = \begin{vmatrix} \dfrac{1}{y} & -\dfrac{v}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}.$$

$$f_{Y,V}(y, v) = f_{X,Y}\left(\frac{v}{y}, y\right) \times |J| = \frac{15}{128}\left(\frac{v}{y}\right)^2 y \times \frac{1}{y} = \frac{15}{128}\left(\frac{v}{y}\right)^2,$$

$$y > 0, \quad 0 < v < 2y, \quad v > y^2 - y.$$

px) Use (o) to find the p.d.f. of $V = X \cdot Y$, $f_V(v)$.

$$v < x^2 + x \qquad \Rightarrow \qquad x^2 + x - v > 0 \qquad \Rightarrow \qquad x > \frac{-1+\sqrt{1+4v}}{2}, \quad \text{since } x > 0.$$

$$f_V(v) = \int\limits_{\frac{-1+\sqrt{1+4v}}{2}}^{2} \frac{15}{128}\, v\, dx = \frac{15}{128}\, v \left[2 - \frac{-1+\sqrt{1+4v}}{2} \right] = \frac{15v\left(5-\sqrt{1+4v}\right)}{256},$$

$$0 < v < 6.$$

py) Use (o) to find the p.d.f. of $V = X \cdot Y$, $f_V(v)$.

$$v > y^2 - y \qquad \Rightarrow \qquad y^2 - y - v < 0 \qquad \Rightarrow \qquad y < \frac{1+\sqrt{1+4v}}{2}, \quad \text{since } y > 0.$$

$$f_V(v) = \int\limits_{\frac{v}{2}}^{\frac{1+\sqrt{1+4v}}{2}} \frac{15}{128} \left(\frac{v}{y}\right)^2 dy = \frac{15}{128}\, v^2 \left[\frac{2}{v} - \frac{2}{1+\sqrt{1+4v}} \right]$$

$$= \frac{15}{64}\, v \left[1 - \frac{v}{1+\sqrt{1+4v}} \right] = \frac{15}{64}\, v \left[1 - \frac{v\left(\sqrt{1+4v}-1\right)}{(1+4v)-1} \right]$$

$$= \frac{15}{64}\, v \left[1 - \frac{\sqrt{1+4v}-1}{4} \right] = \frac{15v\left(5-\sqrt{1+4v}\right)}{256},$$

$$0 < v < 6.$$

qx) Let $U = {}^{Y}\!/_{X}$.

Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.

Sketch the support of (X, U).

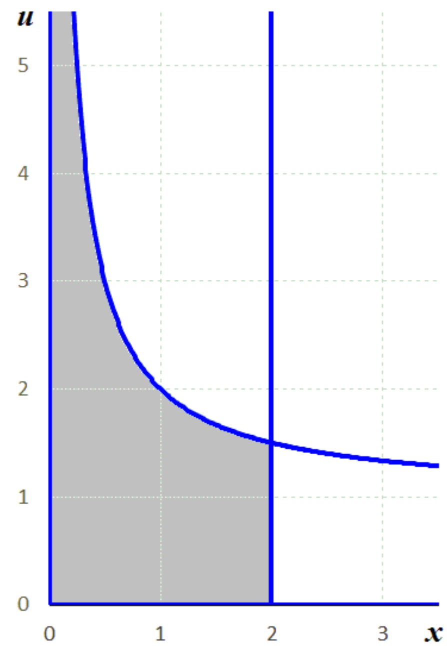
$X = X, \qquad Y = U\,X.$

$0 < x < 2,$

$0 < y \quad \Rightarrow \quad 0 < u\,x \quad \Rightarrow \quad 0 < u,$

$y < x + 1 \quad \Rightarrow \quad u\,x < x + 1$

$\Rightarrow \quad u < 1 + \dfrac{1}{x}, \quad x < \dfrac{1}{u-1}.$



$$J = \begin{vmatrix} 1 & 0 \\ u & x \end{vmatrix} = x.$$

$$f_{X,U}(x, u) = f_{X,Y}(x, u\,x) \times |J| = \frac{15}{128}\,x^2\,(u\,x) \times x = \frac{15}{128}\,x^4\,u,$$

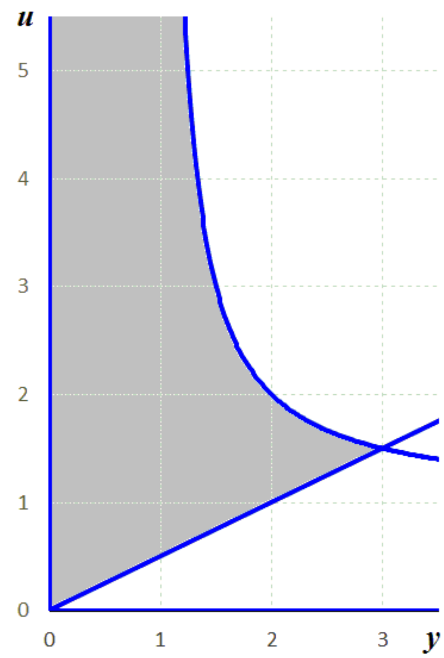
$$0 < x < 2, \quad 0 < u < 1 + \frac{1}{x}.$$

qy) Let $U = {}^{Y}\!/_{X}$.

Find the joint probability density function of (Y, U), $f_{Y,U}(y, u)$.

Sketch the support of (Y, U).

$X = Y/U, \qquad Y = Y.$



$0 < x < 2 \quad \Rightarrow \quad 0 < \dfrac{y}{u} < 2 \quad \Rightarrow \quad 0 < y < 2u,$

$0 < y$

$y < x + 1 \quad \Rightarrow \quad y < \dfrac{y}{u} + 1 \quad \Rightarrow \quad yu < y + u$

$\Rightarrow \quad u < \dfrac{y}{y-1}, \quad y < \dfrac{u}{u-1}.$

$$J = \begin{vmatrix} -\dfrac{y}{u^2} & \dfrac{1}{u} \\ 0 & 1 \end{vmatrix} = -\frac{y}{u^2}.$$

$$f_{Y,U}(y,u) = f_{X,Y}(x, ux) \times |J| = \frac{15}{128}\left(\frac{y}{u}\right)^2 y \times \frac{y}{u^2} = \frac{15}{128}\left(\frac{y}{u}\right)^4,$$

$$u > 0, \quad 0 < y < 2u, \quad 0 < y < \frac{u}{u-1}.$$

rx) Use (q) to find the p.d.f. of $U = {}^{Y}/_{X}$, $f_U(u)$.

$0 < u < 1.5$, $$\int_0^2 \frac{15}{128} x^4 u \, dx = \frac{3}{128} x^5 u \Big|_{x=0}^{x=2} = \frac{3u}{4}.$$

$u > 1.5$, $$\int_0^{\frac{1}{u-1}} \frac{15}{128} x^4 u \, dx = \frac{3}{128} x^5 u \Big|_{x=0}^{x=\frac{1}{u-1}} = \frac{3u}{128\,(u-1)^5}.$$

ry) Use (q) to find the p.d.f. of $U = {}^{Y}/_{X}$, $f_U(u)$.

$0 < u < 1.5$, $$\int_0^{2u} \frac{15}{128} \left(\frac{y}{u}\right)^4 dy = \frac{3y^5}{128u^4} \Big|_{y=0}^{y=2u} = \frac{3u}{4}.$$

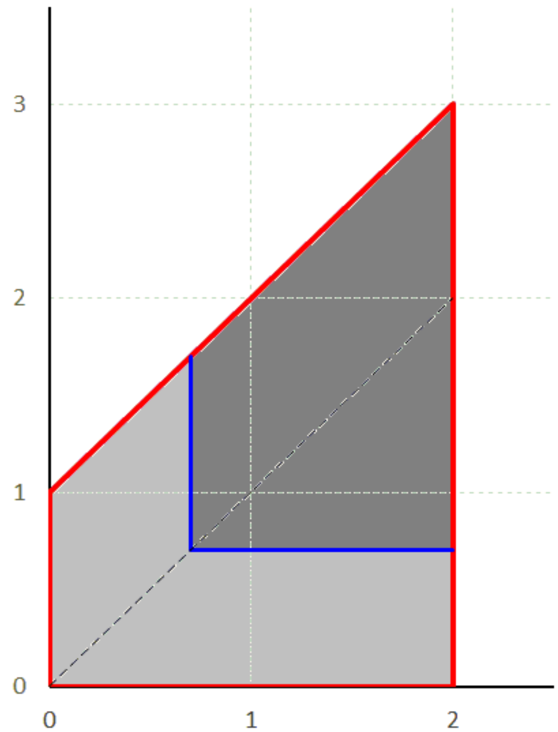
$u > 1.5$, $$\int_0^{\frac{u}{u-1}} \frac{15}{128} \left(\frac{y}{u}\right)^4 dy = \frac{3y^5}{128u^4} \Big|_{y=0}^{y=\frac{u}{u-1}} = \frac{3u}{128\,(u-1)^5}.$$

To double-check your answers, recall part (n):

$$F_U(u) = \frac{3u^2}{8}, \qquad 0 \le u < 1.5.$$

$$F_U(u) = 1 - \frac{4u-1}{512\,(u-1)^4}, \qquad u \ge 1.5.$$

s) Let $S = \min(X, Y)$. Find the probability distribution of S.



"Hint": $F_S(s) = 1 - P(X > s, Y > s)$.

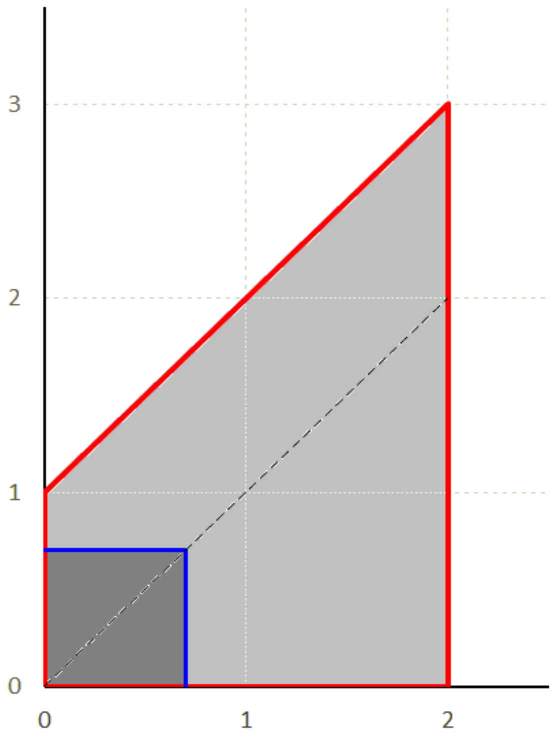
$$F_S(s) = 1 - P(X > s, Y > s)$$

$$= 1 - \int_s^2 \left(\int_s^{x+1} \frac{15}{128} x^2 y \, dy \right) dx$$

$$= \frac{80 s^2 + 10 s^3 + 15 s^4 - 4 s^5}{512},$$

$$0 \le s < 2.$$

t) Let $\mathrm{T} = \max(\mathrm{X}, \mathrm{Y})$. Find the probability distribution of T.



"Hint 1": $\mathrm{F_T}(t) = \mathrm{P}(\mathrm{X} \le t, \mathrm{Y} \le t)$.

"Hint 2": Consider three cases:

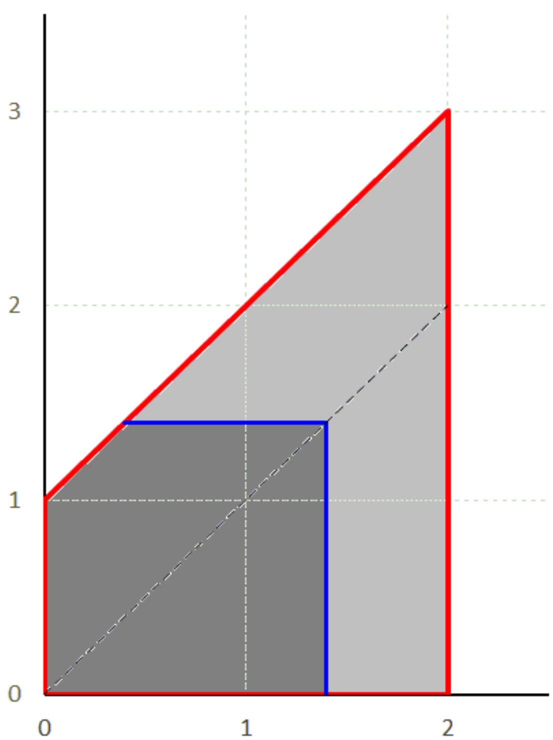
$0 < t < 1, \quad 1 < t < 2, \quad 2 < t < 3.$

Case 1: $0 < t < 1$.

$$\mathrm{F_T}(t) = \mathrm{P}(\mathrm{X} \le t, \mathrm{Y} \le t)$$

$$= \int_0^t \left(\int_0^t \frac{15}{128} x^2 y \, dy \right) dx$$

$$= \frac{10 t^5}{512} = \frac{5 t^5}{256}, \qquad 0 \le t < 1.$$



Case 2: $1 < t < 2$.

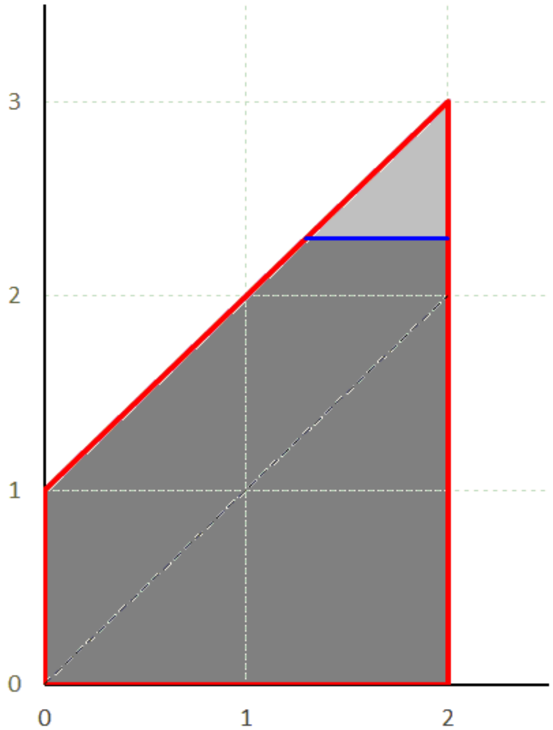
$$\mathrm{F_T}(t) = \mathrm{P}(\mathrm{X} \le t, \mathrm{Y} \le t)$$

$$= \int_0^{t-1} \left(\int_0^{x+1} \frac{15}{128} x^2 y \, dy \right) dx$$

$$+ \int_{t-1}^{t} \left(\int_0^t \frac{15}{128} x^2 y \, dy \right) dx$$

$$= \frac{6 t^5 + 15 t^4 - 20 t^3 + 10 t^2 - 1}{512},$$

$$1 \le t < 2.$$



Case 3: $2 < t < 3.$

$$F_T(t) = P(X \le t, Y \le t)$$

$$= 1 - \int_{t-1}^{2} \left(\int_{t}^{x+1} \frac{15}{128} x^2 y \, dy \right) dx$$

$$= \frac{-4t^5 + 15t^4 - 20t^3 + 90t^2 - 1}{512},$$

$2 \le t < 3.$

2. Let X be a random variable with the probability density function

$$f_X(x) = \frac{324}{x^5}, \qquad x > 3, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = \dfrac{10}{X+2}$.

$$y = \frac{10}{x+2} \qquad \infty > x > 3 \qquad \Rightarrow \qquad 0 < y < 2.$$

$$y = \frac{10}{x+2} \qquad x = \frac{10}{y} - 2 = g^{-1}(y) \qquad \frac{dx}{dy} = -\frac{10}{y^2}$$

p.d.f. $$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{324}{\left(\frac{10}{y} - 2 \right)^5} \times \left| -\frac{10}{y^2} \right| = \frac{405\,y^3}{4\,(5-y)^5}, \qquad 0 < y < 2.$$

OR

$$F_X(x) = P(X \le x) = \int_3^x \frac{324}{u^5}\,du = -\frac{81}{u^4}\Bigg|_3^x = 1 - \frac{81}{x^4} = 1 - \frac{3^4}{x^4}, \qquad x > 3.$$

Recall: There is no such thing as a negative cumulative distribution function!

c.d.f. $$F_Y(y) = P(Y \le y) = P\left(\frac{10}{X+2} \le y \right) = P\left(X \ge \frac{10}{y} - 2 \right)$$

$$= 1 - F_X\left(\frac{10}{y} - 2 \right) = \frac{81}{\left(\frac{10}{y} - 2 \right)^4} = \frac{81\,y^4}{16\,(5-y)^4}, \qquad 0 < y < 2.$$

b) Find the probability distribution of $Y = \ln\left(\frac{X}{3}\right)$.

$$y = \ln\left(\frac{x}{3}\right) \qquad x > 3 \qquad \Rightarrow \qquad y > \ln 1 = 0.$$

$$y = \ln\left(\frac{x}{3}\right) \qquad x = 3\,e^{y} = g^{-1}(y) \qquad \frac{dx}{dy} = 3\,e^{y}$$

p.d.f. $$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = \frac{324}{\left(3\,e^{y}\right)^5}\times\left|\,3\,e^{y}\,\right| = 4\,e^{-4y}, \qquad y > 0.$$

Y has an Exponential$\left(\theta = \frac{1}{4}\right)$ distribution.

OR

$$F_X(x) = P(X \le x) = \int_3^x \frac{324}{u^5}\,du = -\frac{81}{u^4}\Bigg|_3^x = 1-\frac{81}{x^4} = 1-\frac{3^4}{x^4}, \qquad x > 3.$$

Recall: There is no such thing as a negative cumulative distribution function!

c.d.f. $$F_Y(y) = P(Y \le y) = P\left(\ln\left(\frac{X}{3}\right) \le y\right) = P(X \le 3\,e^{y})$$

$$= F_X(3\,e^{y}) = 1-\frac{81}{\left(3\,e^{y}\right)^4} = 1 - e^{-4y}, \qquad y > 0.$$

p.d.f. $$f_Y(y) = F_Y'(y) = 4\,e^{-4y}, \qquad y > 0.$$

Y has an Exponential$\left(\theta = \frac{1}{4}\right)$ distribution.

3. Let X_1, X_2, X_3, X_4 be a random sample (i.i.d.) of size $n = 4$ from a probability distribution with the p.d.f.

$$f_X(x) = \frac{324}{x^5}, \qquad x > 3.$$

$$F_X(x) = P(X \le x) = \int_3^x \frac{324}{u^5}\,du = -\frac{81}{u^4}\Bigg|_3^x = 1 - \frac{81}{x^4} = 1 - \frac{3^4}{x^4}, \qquad x > 3.$$

Recall: There is no such thing as a negative cumulative distribution function!

a) Find $P(\max X_i < 6)$.

$$P(\max X_i < 6) = [P(X < 6)]^n$$

$$= [F_X(6)]^4 = [1 - 0.50^4]^4 = 0.9375^4 \approx 0.772476.$$

b) Find $P(\max X_i > 5)$.

$$P(\max X_i > 5) = 1 - P(\max X_i \le 5) = 1 - [P(X \le 5)]^n$$

$$= 1 - [F_X(5)]^4 = 1 - [1 - 0.60^4]^4 = 1 - 0.8704^4 \approx 0.426048.$$

c) Find $P(\min X_i < 4)$.

$$P(\min X_i < 4) = 1 - P(\min X_i \ge 4) = 1 - [P(X \ge 4)]^n$$

$$= 1 - [1 - F_X(4)]^4 = 1 - [0.75^4]^4 \approx 0.9899774.$$

d) Find $E(\min X_i)$.

$$f_{\min X_i}(x) = n \cdot \left(1 - F_X(x)\right)^{n-1} \cdot f_X(x)$$

$$= 4 \cdot \left(\frac{3^4}{x^4}\right)^{4-1} \cdot \frac{324}{x^5} = \frac{16 \cdot 3^{16}}{x^{17}}, \qquad x > 3.$$

$$E(\min X_i) = \int_3^\infty x \cdot \frac{16 \cdot 3^{16}}{x^{17}}\, dx = \frac{16 \cdot 3^{16}}{15 \cdot x^{15}} \Bigg|_3^\infty = \mathbf{\frac{16}{5}} = \mathbf{3.2}.$$