STAT 410

Homework #6

(due Friday, October 16, by 4:00 p.m.)

Fall 2020
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Please include your name (with your last name underlined), your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

6. The amount of time (in minutes) Bob and Carl spend on a STAT 410 homework problem jointly follow a bivariate normal distribution with

$$\mu_B = 84, \quad \sigma_B = 18, \quad \mu_C = 60, \quad \sigma_C = 15, \quad \rho_{BC} = 0.60.$$

a) What is the probability that Bob would spend over 102 minutes on a STAT 410 homework problem? That is, find $P(B > 102)$.

B has Normal distribution with mean $\mu_B = 84$ and standard deviation $\sigma_B = 18$.

$$P(B > 102) = P\left(Z > \frac{102-84}{18}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$

b) Suppose we know that Carl spent 57 minutes on a particular STAT 410 homework problem. What is the probability that Bob would spend over 102 minutes on that problem? That is, find $P(B > 102 \mid C = 57)$.

Given $C = 57$, B has Normal distribution

with mean $84 + 0.6 \cdot \frac{18}{15} \cdot (57 - 60) = 81.84$

and variance $\left(1 - 0.6^2\right) \cdot 18^2 = 207.36$ (standard deviation 14.4).

$$P(B > 102 \mid C = 57) = P\left(Z > \frac{102-81.84}{14.4}\right) = P(Z > 1.40) = \mathbf{0.0808}.$$

c) Suppose we know that Bob spent 100 minutes on a particular STAT 410 homework problem. What is the probability that Carl would spend over 65 minutes on that problem? That is, find $P(C > 65 \mid B = 100)$.

Given $B = 100$, C has Normal distribution

with mean $60 + 0.6 \cdot \frac{15}{18} \cdot (100 - 84) = 68$

and variance $\left(1 - 0.6^2\right) \cdot 15^2 = 144$ (standard deviation 12).

$$P(C > 65 \mid B = 100) = P\left(Z > \frac{65 - 68}{12}\right) = P(Z > -0.25) = \mathbf{0.5987}.$$

d) What is the probability that Bob would spend more time than Carl on a STAT 410 homework problem? That is, find $P(B > C)$.

Want $P(B > C) = P(B - C > 0) = ?$

$B - C$ has Normal distribution,

$E(B - C) = \mu_B - \mu_C = 84 - 60 = 24,$

$$\mathrm{Var}(B - C) = \sigma_B^2 - 2\,\sigma_{BC} + \sigma_C^2 = \sigma_B^2 - 2\,\rho\,\sigma_B\,\sigma_C + \sigma_C^2$$
$$= 18^2 - 2 \cdot 0.6 \cdot 18 \cdot 15 + 15^2 = 225 \quad (\text{standard deviation } 15).$$

$$P(B - C > 0) = P\left(Z > \frac{0 - 24}{15}\right) = P(Z > -1.60) = \mathbf{0.9452}.$$

e) What is the probability that Bob would spend more than twice the amount of time that Carl would spend a STAT 410 homework problem? That is, find $P(B > 2C)$.

Want $P(B > 2C) = P(B - 2C > 0) = ?$

$B - 2C$ has Normal distribution,

$E(B - 2C) = \mu_B - 2\mu_C = 84 - 2 \cdot 60 = -36,$

$Var(B - 2C) = \sigma_B^2 - 4\sigma_{BC} + 4\sigma_C^2 = \sigma_B^2 - 4\rho\sigma_B\sigma_C + 4\sigma_C^2$

$= 18^2 - 4 \cdot 0.6 \cdot 18 \cdot 15 + 4 \cdot 15^2 = 576$ (standard deviation 24).

$P(B - 2C > 0) = P\left(Z > \frac{0+36}{24}\right) = P(Z > 1.50) = \mathbf{0.0668}.$

f) What is the probability that the combined time Bob and Carl would spend on a STAT 410 homework problem is more than 170 minutes? That is, find $P(B + C > 170)$.

$B + C$ has Normal distribution,

$E(B + C) = \mu_B + \mu_C = 84 + 60 = 144,$

$Var(B + C) = \sigma_B^2 + 2\sigma_{BC} + \sigma_C^2 = \sigma_B^2 + 2\rho\sigma_B\sigma_C + \sigma_C^2$

$= 18^2 + 2 \cdot 0.6 \cdot 18 \cdot 15 + 15^2 = 873.$

$P(B + C > 170) = P\left(Z > \frac{170-144}{\sqrt{873}}\right) \approx P(Z > 0.88) = \mathbf{0.1894}.$

6. (continued)

The amount of time (in minutes) Alex, Bob, and Carl spend on a STAT 410 homework problem (A, B, C) jointly follow $N_3(\mu, \Sigma)$ distribution with

$$\mu = \begin{pmatrix} 134 \\ 84 \\ 60 \end{pmatrix} \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 400 & 195 & 129 \\ 195 & 324 & 162 \\ 129 & 162 & 225 \end{pmatrix}.$$

g) What is the probability that Alex would spend over 120 minutes on a STAT 410 homework problem? That is, find $P(A > 120)$.

A has Normal distribution with mean $\mu_A = 134$ and variance $\sigma_A^2 = 400$.

$$P(A > 120) = P\left(Z > \frac{120-134}{\sqrt{400}}\right) = P(Z > -0.70) = \mathbf{0.7580}.$$

h) Suppose we know that Carl spent 57 minutes on a particular STAT 410 homework problem. What is the probability that Alex would spend over 120 minutes on that problem? That is, find $P(A > 120 \mid C = 57)$.

"Hint": (A, C) jointly follow a bivariate normal distribution.

You have μ_A, σ_A, μ_C, σ_C.

You have $\sigma_{AC} = \text{Cov}(A, C)$, so you can find ρ_{AC}.

$$\rho_{AC} = \frac{\sigma_{AC}}{\sigma_A \sigma_C} = \frac{129}{\sqrt{400} \cdot \sqrt{225}} = 0.43.$$

Given $C = 57$, A has Normal distribution

with mean $134 + 0.43 \cdot \frac{20}{15} \cdot (57 - 60) = 132.28$

and variance $\left(1 - 0.43^2\right) \cdot 20^2 = 326.04.$

$$P(A > 120 \mid C = 57) = P\left(Z > \frac{120 - 132.28}{\sqrt{326.04}}\right) \approx P(Z > -0.68) = \mathbf{0.7517}.$$

i) What is the probability that Alex would spend more than twice the amount of time that Carl would spend a STAT 410 homework problem? That is, find $P(A > 2C)$.

Want $P(A > 2C) = P(A - 2C > 0) = ?$

$A - 2C$ has Normal distribution,

$E(A - 2C) = \mu_A - 2\mu_C = 134 - 2 \cdot 60 = 14,$

$\mathrm{Var}(A - 2C) = \sigma_A^2 - 4\sigma_{AC} + 4\sigma_C^2$

$= 400 - 4 \cdot 129 + 4 \cdot 225 = 784$ (standard deviation 28).

OR

$$\mathrm{Var}(A - 2C) = \begin{pmatrix} 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 400 & 195 & 129 \\ 195 & 324 & 162 \\ 129 & 162 & 225 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 142 & -129 & -321 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 784$$

(standard deviation 28).

$$P(A - 2C > 0) = P\left(Z > \frac{0 - 14}{28}\right) = P(Z > -0.50) = \mathbf{0.6915}.$$

j) What is the probability that Alex would spend more than the combined time of Bob and Carl on a STAT 410 homework problem? That is, find $P(A > B + C)$.

Want $P(A > B + C) = P(A - B - C > 0) = ?$

$A - B - C$ has Normal distribution,

$$E(A - B - C) = \mu_A - \mu_B - \mu_C = 134 - 84 - 60 = -10,$$

$$\begin{aligned} Var(A - B - C) &= \sigma_A^2 + \sigma_B^2 + \sigma_C^2 - 2\sigma_{AB} - 2\sigma_{AC} + 2\sigma_{BC} \\ &= 400 + 324 + 225 - 2 \cdot 195 - 2 \cdot 129 + 2 \cdot 162 = 625 \end{aligned}$$

(standard deviation 25).

OR

$$Var(A - B - C) = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 400 & 195 & 129 \\ 195 & 324 & 162 \\ 129 & 162 & 225 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 76 & -291 & -258 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 625$$

(standard deviation 25).

$$P(A - B - C > 0) = P\left(Z > \frac{0 + 10}{25}\right) = P(Z > 0.40) = \mathbf{0.3446}.$$

k) What is the probability that the combined time Alex, Bob, and Carl would spend on a STAT 410 homework problem is more than 310 minutes? That is, find $P(A+B+C>310)$.

$A+B+C$ has Normal distribution,

$$E(A+B+C) = \mu_A+\mu_B+\mu_C = 134+84+60 = 278,$$

$$\begin{aligned}\mathrm{Var}(A+B+C) &= \sigma_A^2+\sigma_B^2+\sigma_C^2+2\sigma_{AB}+2\sigma_{AC}+2\sigma_{BC}\\ &= 400+324+225+2\cdot 195+2\cdot 129+2\cdot 162 = 1921\end{aligned}$$

OR

$$\mathrm{Var}(A+B+C) = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}400 & 195 & 129\\ 195 & 324 & 162\\ 129 & 162 & 225\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}724 & 681 & 516\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 1921.$$

$$P(A+B+C>310) = P\left(Z>\frac{310-278}{\sqrt{1921}}\right) \approx P(Z>0.73) = \mathbf{0.2327}.$$