STAT 410 | **Examples for 10/26/2020 (1)** | Fall 2020
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Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. Let F_{X_n} and F_X be, respectively, the c.d.f.s of X n and X. Let $C(F_X)$ denote the set of all points where F_X is continuous. We say that X_n **converges in distribution** to X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \qquad \text{for all } x \in C(F_X).$$

We denote this convergence by

$$X_n \xrightarrow{D} X.$$

Example 1:

Let X_n have p.d.f. $f_n(x) = n\,x^{n-1}$, for $0 < x < 1$, zero elsewhere.

Then $F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ x^n & 0 \le x < 1. \\ 1 & x \ge 1 \end{cases}$ $\qquad \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & x < 1 \\ 1 & x \ge 1 \end{cases}.$

Therefore, $X_n \xrightarrow{D} X$, where $P(X = 1) = 1$.

Recall that $X_n \xrightarrow{P} 1$, since

if $0 < \varepsilon \le 1$, $P(|X_n - 1| \ge \varepsilon) = (1-\varepsilon)^n \to 0$ as $n \to \infty$, and

if $\varepsilon > 1$, $P(|X_n - 1| \ge \varepsilon) = 0$.

Example 2:

Let X_n have p.d.f. $f_n(x) = n\,e^{-nx}$, for $x > 0$, zero otherwise.

Recall that $X_n \xrightarrow{P} 0$, since

if $\varepsilon > 0$, $P(|X_n - 0| \ge \varepsilon) = P(X_n \ge \varepsilon) = e^{-n\varepsilon} \to 0$ as $n \to \infty$.

Consider X with p.m.f. $P(X=0)=1$.

$$F_{X_n}(x)=\begin{cases}0 & x<0\\ 1-e^{-nx} & x\ge 0\end{cases} \qquad F_X(x)=\begin{cases}0 & x<0\\ 1 & x\ge 0\end{cases}$$

$\lim\limits_{n\to\infty} F_{X_n}(x)=F_X(x)$, for all $x\neq 0$.

$F_{X_n}(0)=0$ for all n, but $F_X(0)=1$. $\lim\limits_{n\to\infty} F_{X_n}(0)\neq F_X(0)$.

Since $0\notin C(F_X)$, $X_n \xrightarrow{D} X$.

Example 3:

Let X_n have p.m.f. $P(X_n=\frac{1}{n})=\frac{1}{2}-\frac{1}{n}$, $P(X_n=1)=\frac{1}{2}+\frac{1}{n}$.

Then $$F_{X_n}(x)=\begin{cases}0 & x<\frac{1}{n}\\ \frac{1}{2}-\frac{1}{n} & \frac{1}{n}\le x<1.\\ 1 & x\ge 1\end{cases} \qquad \lim_{n\to\infty} F_{X_n}(x)=\begin{cases}0 & x\le 0\\ \frac{1}{2} & 0<x<1.\\ 1 & x\ge 1\end{cases}$$

Consider X with p.m.f. $P(X=0)=\frac{1}{2}$, $P(X=1)=\frac{1}{2}$.

Then $$F_X(x)=\begin{cases}0 & x<0\\ \frac{1}{2} & 0\le x<1\\ 1 & x\ge 1\end{cases}$$

$\lim\limits_{n\to\infty} F_{X_n}(x)=F_X(x)$, for all $x\neq 0$.

$F_{X_n}(0)=0$ for all n, but $F_X(0)=\frac{1}{2}$. $\lim\limits_{n\to\infty} F_{X_n}(0)\neq F_X(0)$.

Since $0\notin C(F_X)$, $X_n \xrightarrow{D} X$.

<u>Example 4</u>:

Consider $\{X_n\}$ with p.m.f.s $P(X_n = 3) = 1 - \frac{1}{n}$, $P(X_n = 7) = \frac{1}{n}$.

Recall that $X_n \overset{P}{\to} 3$, since

if $0 < \varepsilon \le 4$, $P(|X_n - 3| \ge \varepsilon) = \frac{1}{n} \to 0$ as $n \to \infty$, and

if $\varepsilon > 4$, $P(|X_n - 3| \ge \varepsilon) = 0.$

Consider X with p.m.f. $P(X = 3) = 1.$

$$F_{X_n}(x) = \begin{cases} 0 & x < 3 \\ 1 - \frac{1}{n} & 3 \le x < 7 \\ 1 & x \ge 7 \end{cases} \qquad\qquad F_X(x) = \begin{cases} 0 & x < 3 \\ 1 & x \ge 3 \end{cases}$$

$\lim\limits_{n\to\infty} F_{X_n}(x) = F_X(x),$ for all $x \in \mathbf{R}$. $\Rightarrow$ $X_n \overset{D}{\to} X$.

<u>Example 5*</u>:

Let X have a Uniform distribution over $(0, 1)$.

Let $P(X_n = \frac{i}{n}) = \frac{1}{n}$, for $i = 1, 2, \ldots, n$.

Note that X is continuous, while X_n's are discrete.

For $0 < x < 1$, $F_X(x) = x$, $F_{X_n}(x) = \frac{\lfloor nx \rfloor}{n}$, where

$\lfloor x \rfloor$ = the greatest integer less than or equal to x.

Therefore, $X_n \overset{D}{\to} X$, since $|F_{X_n}(x) - F_X(x)| \le \frac{1}{n}$ for all x.

Example 6:

Suppose $P(X_n = i) = \dfrac{n+i}{3n+6}$, for $i = 1, 2, 3$.

Find the limiting distribution of X_n.

$$F_{X_n}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{n+1}{3n+6} & 1 \le x < 2 \\ \dfrac{2n+3}{3n+6} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases} \qquad \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{1}{3} & 1 \le x < 2 \\ \dfrac{2}{3} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Then $X_n \xrightarrow{D} X$, where $P(X = i) = \dfrac{1}{3}$, for $i = 1, 2, 3$.

Example 7:

Let X_n have p.d.f. $f_n(x) = \dfrac{1 + x/n}{1 + 1/2n}$, for $0 < x < 1$, zero elsewhere,

$n = 1, 2, 3, \ldots$. Find the limiting distribution of X_n.

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x + x^2/2n}{1 + 1/2n} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Then $X_n \xrightarrow{D} X$, where X has a Uniform distribution over $(0, 1)$.

Example 8:

5.2.5 (7th edition) **4.3.5** (6th edition)

Let the pmf of Y_n be $p_n(y) = 1,\ y = n,$ zero elsewhere. Show that Y_n does not have a limiting distribution. (In this case, the probability has "escaped" to infinity.)

$$F_{Y_n}(y) = \begin{cases} 0 & y < n \\ 1 & y \ge n \end{cases}$$

Let $y \in \mathbf{R}$. Let $N = \lfloor y \rfloor + 1$.

($\lfloor y \rfloor$ = the greatest integer less than or equal to y.)

Then $F_{Y_n}(y) = 0$ for all $n \ge N$.

Therefore, $\lim_{n\to\infty} F_{Y_n}(y) = 0$ for all $y \in \mathbf{R}$.

However, $F(y) = 0$ for all $y \in \mathbf{R}$ is not a c.d.f.

Therefore, Y_n does not have a limiting distribution.

Example 9:

Let $X_1, X_2, \ldots$ be i.i.d. Uniform$(0, \theta)$. Let $Y_n = \max(X_1, X_2, \ldots, X_n)$.

We already know that $Y_n \xrightarrow{P} \theta$.

Find the limiting distribution of $Z_n = n(\theta - Y_n)$.

$$F_{Y_n}(x) = F_{\max X_i}(x) = \left(\frac{x}{\theta}\right)^n, \quad 0 < x < \theta.$$

$$F_{Z_n}(z) = P(n(\theta - Y_n) \le z) = P(Y_n > \theta - \frac{z}{n}) = 1 - (1 - \frac{z}{n\theta})^n,$$

$$0 < z < n\theta.$$

$$F_{Z_n}(z) \to 1 - e^{-z/\theta}, \quad z > 0, \qquad \text{as } n \to \infty.$$

$Z_n \xrightarrow{D} X$, where X has Exponential distribution with mean θ.

Example 10:

Let $X_1, X_2, \dots$ be i.i.d. with mean μ and standard deviation σ.

Let $\overline{X}_n = \dfrac{X_1 + \dots + X_n}{n}, \quad n = 1, 2, \dots$.

We already know that $\overline{X}_n \xrightarrow{P} \mu$.

Then for all $\varepsilon > 0$, $P\left(\left|\overline{X}_n - \mu\right| \ge \varepsilon\right) \to 0$, $P\left(\left|\overline{X}_n - \mu\right| < \varepsilon\right) \to 1$ as $n \to \infty$.

We wish to show that $\overline{X}_n \xrightarrow{D} \mu$.

$$F_\mu(x) = \begin{cases} 0 & x < \mu \\ 1 & x \ge \mu \end{cases}$$

Since $F_\mu(x)$ is not continuous at μ, we need to show

① if $x < \mu$, $\lim\limits_{n\to\infty} F_{\overline{X}_n}(x) = 0$,

② if $x > \mu$, $\lim\limits_{n\to\infty} F_{\overline{X}_n}(x) = 1$.

① If $x < \mu$, then $\exists\, \varepsilon > 0$ such that $x \le \mu - \varepsilon$.

Then $F_{\overline{X}_n}(x) \le P\left(\overline{X}_n \le \mu - \varepsilon\right) \le P\left(\left|\overline{X}_n - \mu\right| \ge \varepsilon\right) \to 0$ as $n \to \infty$.

② If $x > \mu$, then $\exists\, \varepsilon > 0$ such that $x \ge \mu + \varepsilon$.

Then $F_{\overline{X}_n}(x) \ge P\left(\overline{X}_n \le \mu + \varepsilon\right) \ge P\left(\left|\overline{X}_n - \mu\right| < \varepsilon\right) \to 1$ as $n \to \infty$.

Therefore, $\overline{X}_n \xrightarrow{D} \mu$.