STAT 410 Fall 2020

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Exam 2 (After Party)

(due Wednesday, November 25, by 10:00 p.m. CST)

(10 points)

No credit will be given without supporting work.

You are welcome to use any result obtained on Exam 2.

2. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from a probability distribution with probability density function

$$f(x;\beta) = \frac{\beta}{2} e^{-\sqrt{\beta x}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

i) (i) Show that the method of moments estimator for β, $\tilde{\beta}$, is asymptotically normally distributed (as $n \to \infty$). Find the parameters.

Recall: Exam 2:

$$E(X^k) = \beta^{-k}\,\Gamma(2k+2) \quad \text{for } k > -1. \qquad \tilde{\beta} = \frac{6}{\overline{X}}.$$

$$\mu = E(X) = E(X^1) = \beta^{-1}\,\Gamma(4) = \frac{6}{\beta}.$$

$$E(X^2) = \beta^{-2}\,\Gamma(6) = \frac{120}{\beta^2}.$$

$$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{120}{\beta^2} - \left(\frac{6}{\beta}\right)^2 = \frac{84}{\beta^2}.$$

By CLT, $\sqrt{n}\,\left(\overline{X} - \mu\right) \xrightarrow{D} N\left(0, \sigma^2\right)$.

Consider $g(x) = \frac{6}{x}$. $\qquad g(\overline{\mathrm{X}}) = \tilde{\beta}, \qquad g(\mu) = \beta.$

$g'(x) = -\frac{6}{x^2}, \qquad g'(\mu) = -\frac{\beta^2}{6}.$

$$\sqrt{n}\left(g\left(\overline{\mathrm{X}}\right) - g\left(\mu\right)\right) \xrightarrow{D} \mathrm{N}\left(0, \left(g'(\mu)\right)^2 \cdot \sigma^2\right).$$

$$\Rightarrow \qquad \sqrt{n}\left(\tilde{\beta} - \beta\right) \xrightarrow{D} \mathrm{N}\left(0, \left(-\frac{\beta^2}{6}\right)^2 \cdot \frac{84}{\beta^2}\right) = \mathrm{N}\left(0, \frac{7\beta^2}{3}\right).$$

$\Rightarrow$ For large n, $\tilde{\beta}$ is approximately $\mathrm{N}\left(\beta, \frac{7\beta^2}{3n}\right)$.

(ii) Show that the maximum likelihood estimator for β, $\hat{\beta}$, is asymptotically normally distributed $(\text{as } n \to \infty)$. Find the parameters.

Recall: Exam 2:

$$\mathrm{I}(\beta) = \frac{1}{2\beta^2}.$$

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \xrightarrow{D} \mathrm{N}\left(0, \frac{1}{\mathrm{I}(\beta)}\right) = \mathrm{N}\left(0, 2\beta^2\right).$$

$\Rightarrow$ For large n, $\hat{\beta}$ is approximately $\mathrm{N}\left(\beta, \frac{2\beta^2}{n}\right)$.

OR

Recall: Exam 2:

$$W = \sqrt{X} \text{ has Gamma}\left(\alpha = 2,\ \theta = \frac{1}{\sqrt{\beta}} \right) \text{ distribution.}$$

$$\hat{\beta} = \frac{4n^2}{\left(\sum_{i=1}^{n} \sqrt{X_i} \right)^2}.$$

$$\hat{\beta} = \frac{4}{\left(\overline{W} \right)^2}.$$

$$\mu_W = E(W) = \alpha\,\theta = \frac{2}{\sqrt{\beta}}.$$

$$\sigma_W^2 = \operatorname{Var}(W) = \alpha\,\theta^2 = \frac{2}{\beta}.$$

By CLT, $\sqrt{n}\left(\overline{X} - \mu_W \right) \xrightarrow{D} N\left(0, \sigma_W^2 \right).$

Consider $g(x) = \dfrac{4}{x^2}.$ $\quad g(\overline{W}) = \hat{\beta},\quad g(\mu_W) = \beta.$

$g'(x) = -\dfrac{8}{x^3},\quad g'(\mu) = -\sqrt{\beta^3}.$

$$\sqrt{n}\left(g\left(\overline{W} \right) - g\left(\mu_W \right) \right) \xrightarrow{D} N\left(0, \left(g'\left(\mu_W \right) \right)^2 \cdot \sigma_W^2 \right).$$

$$\Rightarrow \quad \sqrt{n}\left(\hat{\beta} - \beta \right) \xrightarrow{D} N\left(0, \left(-\sqrt{\beta^3} \right)^2 \cdot \frac{2}{\beta} \right) = N\left(0, 2\beta^2 \right).$$

$\Rightarrow$ For large n, $\hat{\beta}$ is approximately $N\left(\beta, \dfrac{2\beta^2}{n} \right).$

j) Recall that $\hat{\hat{\beta}} = \dfrac{(2n-1)(2n-2)}{\left(\sum_{i=1}^{n} \sqrt{X_i}\right)^2}$ is an unbiased estimator of β. $(n > 2)$

Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not efficient, find its efficiency.

Recall: Exam 2:

$$W = \sqrt{X} \text{ has Gamma}\left(\alpha = 2,\ \theta = \frac{1}{\sqrt{\beta}}\right) \text{ distribution.}$$

$$\Rightarrow \quad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i \text{ has a Gamma}\left(\alpha = 2n,\ \theta = \frac{1}{\sqrt{\beta}}\right) \text{ distribution.}$$

$$\hat{\hat{\beta}} = \frac{(2n-1)(2n-2)}{Y^2}.$$

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\, \Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\, \Gamma(\alpha)}, \qquad m > -\alpha.$$

$$E\left(\frac{1}{Y^2}\right) = E\left(Y^{-2}\right) = \frac{\Gamma(2n-2)}{\left(\sqrt{\beta}\right)^{-2} \Gamma(2n)} = \frac{\beta}{(2n-1)(2n-2)}.$$

$$E\left(\frac{1}{Y^4}\right) = E\left(Y^{-4}\right) = \frac{\Gamma(2n-4)}{\left(\sqrt{\beta}\right)^{-4} \Gamma(2n)}$$

$$= \frac{\beta^2}{(2n-1)(2n-2)(2n-3)(2n-4)}.$$

$$\operatorname{Var}\left(\frac{1}{Y^2}\right) = E\left(\frac{1}{Y^4}\right) - \left[E\left(\frac{1}{Y^2}\right)\right]^2$$

$$= \frac{\beta^2}{(2n-1)(2n-2)(2n-3)(2n-4)} - \frac{\beta^2}{(2n-1)^2(2n-2)^2}$$

$$= \frac{\beta^2 (8n-10)}{(2n-1)^2 (2n-2)^2 (2n-3)(2n-4)}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = (2n-1)^2 (2n-2)^2 \operatorname{Var}\left(\frac{1}{Y^2}\right) = \frac{\beta^2 (8n-10)}{(2n-3)(2n-4)}.$$

Recall: Exam 2:

$$\mathrm{I}(\beta) = \frac{1}{2\beta^2}.$$

$$\text{Rao-Cramér lower bound} = \frac{1}{n \cdot \mathrm{I}(\beta)} = \frac{2\beta^2}{n}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \frac{\beta^2 (8n-10)}{(2n-3)(2n-4)} = \frac{2\beta^2 (4n-5)}{\left(4n^2 - 14n + 12\right)} > \frac{2\beta^2}{n}.$$

$\operatorname{Var}(\hat{\hat{\beta}})$ does NOT attain its Rao-Cramér lower bound.

$\Rightarrow$ $\hat{\hat{\beta}}$ is NOT an efficient estimator of of β.

$$(\text{Efficiency of } \hat{\hat{\beta}}) = \frac{\dfrac{2\beta^2}{n}}{\dfrac{2\beta^2 (4n-5)}{\left(4n^2 - 14n + 12\right)}}$$

$$= \frac{4n^2 - 14n + 12}{4n^2 - 5n} = \frac{(2n-3)(2n-4)}{n(4n-5)}.$$

$(\text{Efficiency of } \hat{\hat{\beta}}) \to 1$ as $n \to \infty$.

$\Rightarrow$ $\hat{\hat{\beta}}$ is an asymptotically efficient estimator of β.