STAT 410 | Fall 2020
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Examples for 11/18/2020 (6)

Examples for 11/06/2020 (5) (continued)

1 – 2. Let $\lambda > 0$. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

Note: Since $0 < x < 1$, $\ln x < 0$.

A better way to write this density function would be

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1} = \lambda^2 (-\ln x) \cdot x^{\lambda-1}, \qquad 0 < x < 1.$$

Recall: $W = -\ln X$ has a $\text{Gamma}\left(\alpha = 2, \theta = \frac{1}{\lambda}\right)$ distribution.

$-\sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for λ.

We wish to test $H_0 : \lambda = 2$ vs. $H_1 : \lambda > 2$.

1. m) Suppose $n = 4$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

n) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.
Find the p-value of this test.
State your decision (Reject H_0 or Do NOT Reject H_0) at $\alpha = 0.10$.

2. Consider the rejection region Reject H_0 if $-\sum_{i=1}^{n=4} \ln x_i \le 2$.

o) Find the significance level α of this rejection region.

p) Find the power of this rejection region if $\lambda = 3$ and if $\lambda = 4$.

Answers:

1 – 2. Let $\lambda > 0$. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = -\lambda^2 \ln x \cdot x^{\lambda-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

We wish to test $H_0 : \lambda = 2$ vs. $H_1 : \lambda > 2$.

Recall: $W = -\ln X$ has a Gamma$\left(\alpha = 2, \theta = \frac{1}{\lambda}\right)$ distribution.

$-\sum_{i=1}^{n} \ln X_i$ is a sufficient statistic for λ.

1. m) Suppose $n = 4$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

Let $\lambda > 2$.

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(H_0; x_1, x_2, \ldots, x_n)}{L(H_1; x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \left(-4 \ln x_i \cdot x_i\right)}{\prod_{i=1}^{n} \left(-\lambda^2 \ln x_i \cdot x_i^{\lambda-1}\right)}$$

$$= \left(\frac{2}{\lambda}\right)^{2n} \left(\prod_{i=1}^{n} x_i\right)^{2-\lambda}.$$

$$\lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad \prod_{i=1}^{n} x_i \ge k_1 \qquad (\text{since } \lambda > 2)$$

$$\Leftrightarrow \quad -\sum_{i=1}^{n} \ln x_i \le c.$$

Reject H_0 if $-\sum_{i=1}^{n} \ln x_i \le c$.

$-\sum_{i=1}^{n} \ln X_i$ has a Gamma distribution with $\alpha = 2n = 8$ and $\theta = \dfrac{1}{\lambda}$.

Then $-2\lambda \sum_{i=1}^{n} \ln X_i$ has a $\chi^2(2\alpha = 16$ degrees of freedom$)$ distribution.

$$0.10 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(-\sum_{i=1}^{n} \ln X_i \le c \mid \lambda = 2)$$

$$= P(-4\sum_{i=1}^{n} \ln X_i \le 4c \mid \lambda = 2) = P(\chi^2(16) \le 4c).$$

$\Rightarrow \quad 4c = \chi^2_{0.90}(16) = 9.312. \qquad \Rightarrow \quad c = 2.328.$

Reject H_0 if $-\sum_{i=1}^{4} \ln x_i \le \mathbf{2.328}$.

n) Suppose $n = 4$, and $x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.8$, $x_4 = 0.9$.
Find the p-value of this test.
State your decision (Reject H_0 or Do NOT Reject H_0) at $\alpha = 0.10$.

$$-\sum_{i=1}^{n} \ln x_i = -\ln 0.2016 \approx 1.60147.$$

$$\text{p-value} \approx P(-\sum_{i=1}^{4} \ln X_i \le 1.6 \mid \lambda = 2) = P(\text{Poisson}(1.6 \times 2) \ge 8)$$

$$= 1 - P(\text{Poisson}(3.2) \le 7) = 1 - 0.983 = \mathbf{0.017}.$$

$$\text{p-value} = P(\text{Poisson}(-2 \times \ln 0.2016) \ge 8) \approx 0.016912.$$

$$\text{p-value} = P(\chi^2(16) \le -4 \times \ln 0.2016) \approx 0.016912.$$

p-value = 0.017 $<$ 0.10 = α. **Reject H_0 at $\alpha = 0.10$**.

OR

1.60147 $<$ 2.328. **Reject H_0 at $\alpha = 0.10$**.

2. Consider the rejection region Reject H_0 if $-\sum_{i=1}^{n=4} \ln x_i \le 2$.

o) Find the significance level α of this rejection region.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(-\sum_{i=1}^{4} \ln X_i \le 2 \mid \lambda = 2)$$

$$= P(\text{Poisson}(2 \times 2) \ge 8) = 1 - P(\text{Poisson}(4) \le 7) = 1 - 0.949 = \mathbf{0.051}.$$

OR $$= P(\chi^2(16) \le 8) = 0.051134.$$

p) Find the power of this rejection region if $\lambda = 3$ and if $\lambda = 4$.

$$\text{Power}(\beta) = P(\text{Reject } H_0 \mid \beta) = P(-\sum_{i=1}^{4} \ln X_i \le 2 \mid \lambda)$$

$$= P(\text{Poisson}(2 \times \lambda) \ge 8) = 1 - P(\text{Poisson}(2\lambda) \le 7).$$

OR $$= P(\chi^2(16) \le 4\lambda).$$

$\text{Power}(3) = 1 - P(\text{Poisson}(6) \le 7) = 1 - 0.744 = \mathbf{0.256}$. 0.256020.

$\text{Power}(4) = 1 - P(\text{Poisson}(8) \le 7) = 1 - 0.453 = \mathbf{0.547}$. 0.547039.