STAT 410 | **Examples for 11/13/2020 (3)** | Fall 2020 A. Stepanov

1 – 2. Bert and Ernie noticed that the following are satisfied when Cookie Monster eats cookies:

(a) the number of cookies eaten during non-overlapping time intervals are independent;

(b) the probability of exactly one cookie eaten in a sufficiently short interval of length h is approximately λh;

(c) the probability of two or more cookies eaten in a sufficiently short interval is essentially zero.

Therefore, X_t, the number of cookies eaten by Cookie Monster by time t, is a Poisson process, and for any $t > 0$, the distribution of X_t is Poisson (λt).

However, Bert and Ernie could not agree on the value of λ, the average number of cookies that Cookie Monster eats per minute. Bert claimed that it equals 1.5, but Ernie insisted that it has been less than 1.5 ever since Cookie Monster was forced to eat broccoli and carrots. Thus, the two friends decided to test

$$H_0 : \lambda = 1.5 \quad \text{vs.} \quad H_1 : \lambda < 1.5.$$

Bert decided to count the number of cookies Cookie Monster would eat in 7 minutes, X_7, and then Reject H_0 if X_7 is too small. Ernie, who was the less patient of the two, decided to note how much time Cookie Monster needs to eat the first 4 cookies, T_4, and then Reject H_0 if T_4 is too large.

If X_t, the number of occurrences, follows a Poisson (λt) distribution, then the time of the k^{th} occurrence, T_k, follows a Gamma$\left(\alpha = k,\ \theta = 1/\lambda\right)$ distribution.

1. a) Help Bert to find the best (uniformly most powerful) Rejection Region with the significance level α closest to 0.05.

Hint 1: Reject H_0 if $X_7 \le c$. Hint 2: $X_7 \sim \text{Poisson}(7\lambda)$.

Hint 3: Want c such that $0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X_7 \le c \mid \lambda = 1.5)$.

b) Find the power of the test from part (a) if $\lambda = 1$.

Hint: $\text{Power}(\lambda = 1) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(X_7 \le c \mid \lambda = 1)$.

c) Bert decided to Reject H_0 if Cookie Monster eats at most 6 cookies in 7 minutes. Find the significance level α for this Rejection Region.

Hint: $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X_7 \le 6 \mid \lambda = 1.5)$.

d) Suppose Cookie Monster ate 4 cookies in 7 minutes. Find the p-value of the test.

2. a) Help Ernie to find the best (uniformly most powerful) Rejection Region with the significance level $\alpha = 0.05$.

Hint 1: Reject H_0 if $T_4 \geq c$. Hint 2: $T_4 \sim \text{Gamma}(\alpha = 4, \theta = 1/\lambda)$.

Hint 3: Want c such that $0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(T_4 \geq c \mid \lambda = 1.5)$.

Hint 4: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

b) Ernie decided to Reject H_0 if it takes Cookie Monster longer than 5 minutes to eat the first 4 cookies. Find the significance level α for this Rejection Region.

Hint 1: $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(T_4 \geq 5 \mid \lambda = 1.5)$.

Hint 2: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$F_T(t) = P(T \leq t) = P(Y \geq \alpha) \quad \text{and} \quad P(T > t) = P(Y \leq \alpha - 1),$$

where Y has a $\text{Poisson}(\lambda t)$ distribution.

c) Find the power of the test from part (b) if $\lambda = 1$.

Hint: $\text{Power}(\lambda = 1) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(T_4 \geq 5 \mid \lambda = 1)$.

d) It took Cookie Monster 6 minutes to eat the first 4 cookies. Find the p-value of the test.

3. The same STAT 410 instructor believes that only 30% of the students start working on their homework before Thursday. We wish to test $\mathrm{H}_0: p \le 0.30$ vs. $\mathrm{H}_1: p > 0.30$. A random sample of size $n = 15$ students is obtained.

a) Find the “best” Rejection Region with the significance level closest to 0.05.

Hint 1: Reject H_0 if $\mathrm{X} \ge c$. Hint 2: $\mathrm{X} \sim \text{Binomial}(n = 15, p)$.

Hint 3: Want c such that $0.05 \approx \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}(\mathrm{X} \ge c \mid p = 0.30)$.

Hint 4: Binomial distribution is discrete, integer-valued.

b) What is the power of the Rejection Region obtained in part (a) if $p = 0.40$? If $p = 0.50$?

Hint: $\text{Power}(p) = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is NOT true}) = \mathrm{P}(\mathrm{X} \ge c \mid p)$.

c) Suppose that 9 out of 15 students in the sample start working on their homework before Thursday. Find the p-value of the test.

Hint: $\text{p-value} = \mathrm{P}(\text{value of X as extreme or more extreme than } x_{\text{observed}} \mid \mathrm{H}_0 \text{ true})$.

_ _

1 – 2. Bert and Ernie noticed that the following are satisfied when Cookie Monster eats cookies:

(a) the number of cookies eaten during non-overlapping time intervals are independent;

(b) the probability of exactly one cookie eaten in a sufficiently short interval of length h is approximately λh;

(c) the probability of two or more cookies eaten in a sufficiently short interval is essentially zero.

Therefore, X_t, the number of cookies eaten by Cookie Monster by time t, is a Poisson process, and for any $t > 0$, the distribution of X_t is Poisson (λt).

However, Bert and Ernie could not agree on the value of λ, the average number of cookies that Cookie Monster eats per minute. Bert claimed that it equals 1.5, but Ernie insisted that it has been less than 1.5 ever since Cookie Monster was forced to eat broccoli and carrots. Thus, the two friends decided to test

$$H_0 : \lambda = 1.5 \quad \text{vs.} \quad H_1 : \lambda < 1.5.$$

Bert decided to count the number of cookies Cookie Monster would eat in 7 minutes, X_7, and then Reject H_0 if X_7 is too small. Ernie, who was the less patient of the two, decided to note how much time Cookie Monster needs to eat the first 4 cookies, T_4, and then Reject H_0 if T_4 is too large.

If X_t, the number of occurrences, follows a Poisson (λt) distribution, then the time of the k^{th} occurrence, T_k, follows a $\text{Gamma}(\alpha = k,\ \theta = 1/\lambda)$ distribution.

1. a) Help Bert to find the best (uniformly most powerful) Rejection Region with the significance level α closest to 0.05.

Hint 1: Reject H_0 if $X_7 \le c$. Hint 2: $X_7 \sim \text{Poisson}(7\lambda)$.

Hint 3: Want c such that $0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X_7 \le c \mid \lambda = 1.5)$.

$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X_7 \le c \mid \lambda = 1.5)$

$= P(\text{Poisson } (7 \cdot 1.5 = 10.5) \le c).$

$P(\text{Poisson } (10.5) \le 5) = 0.050.$

Reject H_0 if $\mathbf{X_7 \le 5}$.

b) Find the power of the test from part (a) if $\lambda = 1$.

Hint: $\text{Power}(\lambda = 1) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(X_7 \le c \mid \lambda = 1).$

$\text{Power}(\lambda = 1) = P(X_7 \le 5 \mid \lambda = 1) = P(\text{Poisson } (7) \le 5) = \mathbf{0.301}.$

c) Bert decided to Reject H_0 if Cookie Monster eats at most 6 cookies in 7 minutes. Find the significance level α for this Rejection Region.

Hint: $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X_7 \le 6 \mid \lambda = 1.5).$

$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X_7 \le 6 \mid \lambda = 1.5)$

$= P(\text{Poisson } (10.5) \le 6) = \mathbf{0.102}.$

d) Suppose Cookie Monster ate 4 cookies in 7 minutes. Find the p-value of the test.

$\text{P-value} = P(X_7 \le 4 \mid \lambda = 1.5) = P(\text{Poisson } (10.5) \le 4) = \mathbf{0.021}.$

2. a) Help Ernie to find the best (uniformly most powerful) Rejection Region with the significance level $\alpha = 0.05$.

Hint 1: Reject H_0 if $T_4 \ge c$. Hint 2: $T_4 \sim \text{Gamma}(\alpha = 4, \theta = 1/\lambda)$.

Hint 3: Want c such that $0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(T_4 \ge c \mid \lambda = 1.5)$.

Hint 4: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$2\lambda T_4$ has a chi-square distribution with $2\alpha = 8$ degrees of freedom

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(T_4 \ge c \mid \lambda = 1.5)$$

$$= P(3T_4 \ge 3c \mid \lambda = 1.5) = P(\chi^2(8) \ge 3c).$$

$$\Rightarrow \quad 3c = \chi^2_{0.05}(8) = 15.51. \qquad \Rightarrow \quad c = 5.17.$$

Reject H_0 if $\mathbf{T_4 \ge 5.17}$.

```
> qgamma(0.95,4,1.5)
[1] 5.169104
>
> qchisq(0.95,8)
[1] 15.50731
> qchisq(0.95,8)/3
[1] 5.169104
```

b) Ernie decided to Reject H_0 if it takes Cookie Monster longer than 5 minutes to eat the first 4 cookies. Find the significance level α for this Rejection Region.

Hint 1: $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(T_4 \ge 5 \mid \lambda = 1.5)$.

Hint 2: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$F_T(t) = P(T \le t) = P(Y \ge \alpha)$ and $P(T > t) = P(Y \le \alpha - 1)$,

where Y has a $\text{Poisson}(\lambda t)$ distribution.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(T_4 \ge 5 \mid \lambda = 1.5) = P(\text{Poisson}(1.5 \cdot 5) \le 4 - 1)$$

$$= P(\text{Poisson}(7.5) \le 3) = \mathbf{0.059}.$$

```
> 1-pgamma(5,4,1.5)
[1] 0.05914546
> ppois(3,7.5)
[1] 0.05914546
> 1-pchisq(2*1.5*5,2*4)
[1] 0.05914546
```

c) Find the power of the test from part (b) if $\lambda = 1$.

Hint: $\text{Power}(\lambda = 1) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(T_4 \ge 5 \mid \lambda = 1)$.

$$\text{Power}(\lambda = 1) = P(T_4 \ge 5 \mid \lambda = 1) = P(\text{Poisson}(5) \le 3) = \mathbf{0.265}.$$

```
> 1-pgamma(5,4,1)
[1] 0.2650259
> ppois(3,5)
[1] 0.2650259
> 1-pchisq(2*1*5,2*4)
[1] 0.2650259
```

d) It took Cookie Monster 6 minutes to eat the first 4 cookies. Find the p-value of the test.

$$\text{P-value} = P(T_4 \ge 6 \mid \lambda = 1.5) = P(\text{Poisson}(1.5 \cdot 6) \le 3)$$

$$= P(\text{Poisson}(9) \le 3) = \mathbf{0.021}.$$

```
> 1-pgamma(6,4,1.5)
[1] 0.02122649
> ppois(3,9)
[1] 0.02122649
> 1-pchisq(2*1.5*6,2*4)
[1] 0.02122649
```

3. The same STAT 410 instructor believes that only 30% of the students start working on their homework before Thursday. We wish to test $H_0 : p \le 0.30$ vs. $H_1 : p > 0.30$. A random sample of size $n = 15$ students is obtained.

a) Find the "best" Rejection Region with the significance level closest to 0.05.

Hint 1: Reject H_0 if $X \ge c$. Hint 2: $X \sim \text{Binomial}(n = 15, p)$.

Hint 3: Want c such that $0.05 \approx \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X \ge c \mid p = 0.30)$.

Hint 4: Binomial distribution is discrete, integer-valued.

Decision rule: Reject H_0 if $X \ge c$. Want $P(\text{Type I error}) \approx 0.05$.

$$P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(X \ge c \mid p = 0.30)$$
$$= 1 - \text{CDF}(c - 1 \mid p = 0.30).$$

Want $(1 - \text{CDF}(c - 1 \mid p = 0.30)) \approx 0.05$, $\quad \text{CDF}(c - 1 \mid p = 0.30) \approx 0.95$.

$\text{CDF}(7 \mid p = 0.30) = 0.950$, $\quad c - 1 = 7$, $\quad c = 8$.

Decision rule: Reject H_0 if $\mathbf{X \ge 8}$.

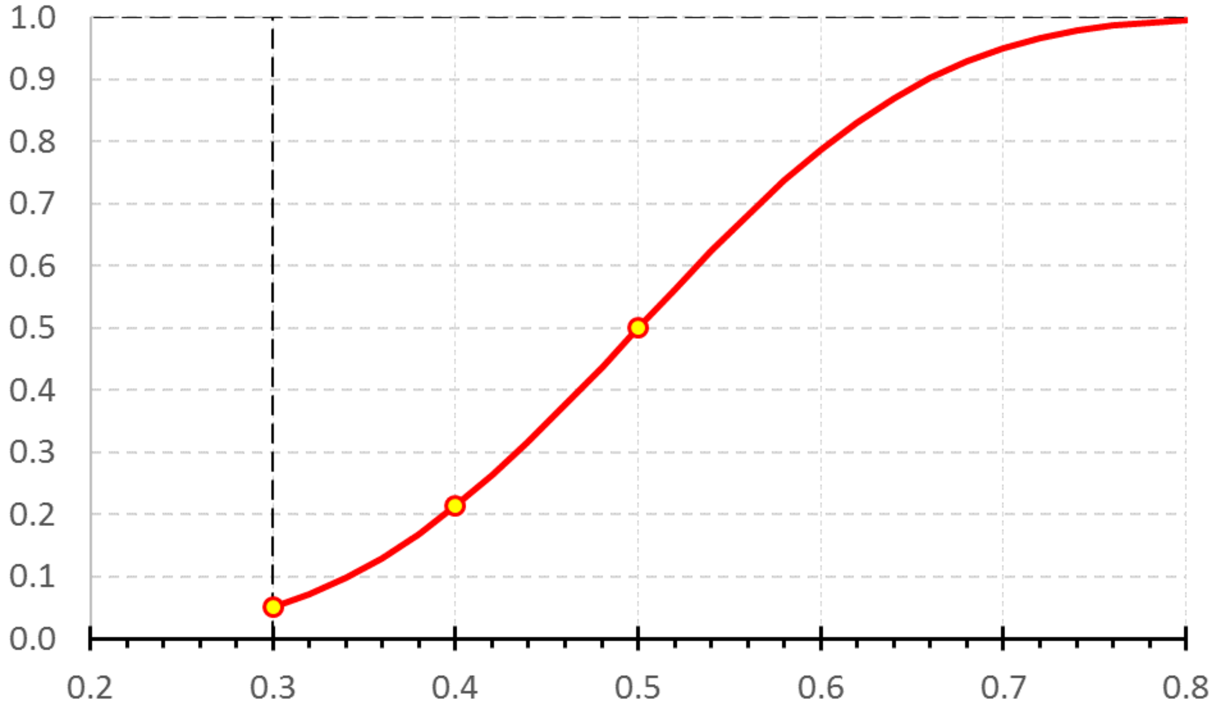
b) What is the power of the Rejection Region obtained in part (a) if $p = 0.40$? If $p = 0.50$?

Hint: $\text{Power}(p) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(X \ge c \mid p)$.

$$\text{Power}(p) = P(\text{Reject } H_0 \mid p) = P(X \ge 8 \mid p) = 1 - \text{CDF}(7 \mid p).$$

$$\text{Power}(0.40) = P(\text{Reject } H_0 \mid p = 0.40) = P(X \ge 8 \mid p = 0.40)$$
$$= 1 - P(X \le 7 \mid p = 0.40) = 1 - 0.787 = \mathbf{0.213}.$$

$$\text{Power}(0.50) = P(\text{Reject } H_0 \mid p = 0.50) = P(X \ge 8 \mid p = 0.50)$$
$$= 1 - P(X \le 7 \mid p = 0.50) = 1 - 0.500 = \mathbf{0.500}.$$



c) Suppose that 9 out of 15 students in the sample start working on their homework before Thursday. Find the p-value of the test.

Hint: p-value = P(value of X as extreme or more extreme than $x_{\text{observed}} \mid H_0$ true).

p-value = P(value of X as extreme or more extreme than X = 11 $\mid H_0$ true)

$$= P(X \geq 9 \mid p = 0.30) = 1 - \text{CDF}(8 \mid p = 0.30) = 1 - 0.985 = \mathbf{0.015}.$$