STAT 410

Examples for 10/19/2020 (2)

Fall 2020
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1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

a) Obtain the maximum likelihood estimator for β, $\hat{\beta}$.

That is, find $\hat{\beta} = \arg\max L(\beta) = \arg\max \ln L(\beta)$, where $L(\beta) = \prod_{i=1}^{n} f(x_i;\beta)$.

① Multiply: $L(\beta) = f(x_1;\beta) \cdot f(x_2;\beta) \cdot \ldots \cdot f(x_n;\beta)$.

② Simplify. "Hint": $a^b \cdot a^c = a^{b+c}$, $a^c \cdot b^c = (a \cdot b)^c$, $(a^b)^c = a^{b \cdot c}$.

③ Take ln. "Hint": $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^b) = b \cdot \ln a$.

④ Take the derivative **with respect to β**.

⑤ Set equal to zero. Solve for β. Add a hat.

b) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.
Obtain the maximum likelihood estimate for β, $\hat{\beta}$.

c) Show that $W = \ln X$ has a Gamma distribution.
What are its parameters α and θ?

d) Is the maximum likelihood estimator of β, $\hat{\beta}$, an unbiased estimator of β?
If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\hat{\beta}$.

"Hint" 0: If U has a Gamma(α_1, θ) distribution, V has a Gamma(α_2, θ) distribution, U and V are independent, then U + V has a Gamma$(\alpha_1 + \alpha_2, \theta)$ distribution.

"Hint" 1: $E(a ☺) = a\,E(☺)$. "Hint" 2: $\frac{1}{♥} = ♥^{-1}$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m \, \Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m \, \Gamma(\alpha)}, \qquad m > -\alpha.$$

e) Find $\text{MSE}(\hat{\beta}) = (\text{bias}(\hat{\beta}))^2 + \text{Var}(\hat{\beta})$.

"Hint" 1: $\text{bias}(\hat{\beta}) = E(\hat{\beta}) - \beta$. You have $E(\hat{\beta})$ from part (d).

"Hint" 2: $\text{Var}(a☺) = a^2\, \text{Var}(☺)$. $\text{Var}(☺) = E(☺^2) - [E(☺)]^2$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m \, \Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m \, \Gamma(\alpha)}, \qquad m > -\alpha.$$

f) Assume $\beta > 1$. (We need this for the expected value $E(X)$ to exist.)

Obtain a method of moments estimator for β, $\tilde{\beta}$.

That is, if $E(X) = h(\beta)$, solve $\overline{X} = h(\tilde{\beta})$ for $\tilde{\beta}$.

① Find $E(X)$. It will depend on β, so it will be a function of β, say, $E(X) = h(\beta)$.

② Replace $E(X)$ with $\overline{X}$, so $\overline{X} = h(\beta)$.

③ Solve $\overline{X} = h(\beta)$ for β. Add a tilde.

g) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.

Obtain a method of moments estimate for β, $\tilde{\beta}$.

h) Suppose $\beta > 1$.

Is the method of moments estimator of β, $\tilde{\beta}$, an unbiased estimator of β?

If $\tilde{\beta}$ is not an unbiased estimator of β, does $\tilde{\beta}$ underestimate or overestimate β (on average)?

Hint: $\tilde{\beta} = g(\overline{X})$. Is $g(x)$ a linear function?

If it is not a linear function, does it curve up or down?

Answers:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

a) Obtain the maximum likelihood estimator for β, $\hat{\beta}$.

That is, find $\hat{\beta} = \arg\max \mathrm{L}(\beta) = \arg\max \ln \mathrm{L}(\beta)$, where $\mathrm{L}(\beta) = \prod_{i=1}^{n} f(x_i;\beta)$.

① Multiply: $\mathrm{L}(\beta) = f(x_1;\beta) \cdot f(x_2;\beta) \cdot \ldots \cdot f(x_n;\beta)$.

② Simplify. "Hint": $a^b \cdot a^c = a^{b+c}$, $a^c \cdot b^c = (a \cdot b)^c$, $(a^b)^c = a^{b \cdot c}$.

③ Take ln. "Hint": $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^b) = b \cdot \ln a$.

④ Take the derivative **with respect to β**.

⑤ Set equal to zero. Solve for β. Add a hat.

$$\mathrm{L}(\beta) = \prod_{i=1}^{n} \frac{\beta^2 (\ln x_i)}{x_i^{\beta+1}} = \beta^{2n} \cdot \left(\prod_{i=1}^{n} x_i\right)^{-\beta-1} \cdot \left(\prod_{i=1}^{n} \ln x_i\right).$$

$$\ln L(\beta) = 2n \ln \beta - (\beta+1) \cdot \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln \ln x_i .$$

$$\frac{d}{d\beta} \ln L(\beta) = \frac{2n}{\beta} - \sum_{i=1}^{n} \ln x_i = 0. \qquad \Rightarrow \qquad \hat{\beta} = \frac{2n}{\sum_{i=1}^{n} \ln X_i}.$$

b) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.

Obtain the maximum likelihood estimate for β, $\hat{\beta}$.

$n = 5$, $\qquad \ln 1.3 + \ln 1.4 + \ln 2.0 + \ln 3.0 + \ln 5.0 \approx 4.$

$$\hat{\beta} \approx \frac{2 \cdot 5}{4} = \mathbf{2.5}.$$

c) Show that $W = \ln X$ has a Gamma distribution.

What are its parameters α and θ?

$w = \ln x \qquad x = e^{w} \qquad \dfrac{dx}{dw} = e^{w}$

$$f_W(w) = \frac{\beta^2 w}{\left(e^{w}\right)^{\beta+1}} \cdot e^{w} = \beta^2 w\, e^{-\beta w}$$

$$= \frac{\beta^2}{\Gamma(2)} w^{2-1} e^{-\beta w}, \qquad w > 0.$$

$\Rightarrow$ W has a Gamma$\left(\alpha = \mathbf{2},\ \theta = \dfrac{\mathbf{1}}{\boldsymbol{\beta}}\right)$ distribution. $\quad (\lambda = \beta)$

d) Is the maximum likelihood estimator of β, $\hat{\beta}$, an unbiased estimator of β?
If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\hat{\beta}$.

"Hint" 0: If U has a Gamma(α_1, θ) distribution, V has a Gamma(α_2, θ) distribution, U and V are independent, then U + V has a Gamma$(\alpha_1 + \alpha_2, \theta)$ distribution.

"Hint" 1: $\mathrm{E}(a ☺) = a\,\mathrm{E}(☺)$. "Hint" 2: $\dfrac{1}{♥} = ♥^{-1}$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

$\mathrm{Y} = \sum_{i=1}^{n} \mathrm{W}_i = \sum_{i=1}^{n} \ln \mathrm{X}_i$ has a Gamma$(\alpha = 2n, \theta = \frac{1}{\beta})$ distribution. $(\lambda = \beta)$

$$\hat{\beta} = \frac{2n}{\sum_{i=1}^{n} \ln \mathrm{X}_i} = \frac{2n}{\mathrm{Y}}. \qquad a = 2n, \quad ☺ = \frac{1}{\mathrm{Y}}, \quad ♥ = \mathrm{Y}.$$

$$\mathrm{T}_\alpha \sim \text{Gamma}(\alpha,\ \theta = \frac{1}{\lambda}),$$

$$\Rightarrow \quad \mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

$$\mathrm{E}\left(\frac{1}{\mathrm{Y}}\right) = \mathrm{E}(\mathrm{Y}^{-1}) = \frac{\Gamma(\alpha-1)}{\lambda^{-1}\,\Gamma(\alpha)} = \frac{\lambda}{\alpha-1} = \frac{\beta}{2n-1}. \qquad \alpha = 2n, \quad m = -1.$$

$$\mathrm{E}(\hat{\beta}) = \mathrm{E}\left(\frac{2n}{\mathrm{Y}}\right) = 2n\,\mathrm{E}\left(\frac{1}{\mathrm{Y}}\right) = 2n \cdot \frac{\beta}{2n-1} = \frac{2n}{2n-1}\cdot\beta = \beta + \frac{\beta}{2n-1} \neq \beta.$$

$\hat{\beta}$ is NOT an unbiased estimator of β. $\quad \text{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta = \dfrac{\beta}{2n-1}$.

Consider $\quad \hat{\hat{\beta}} = \dfrac{2n-1}{2n} \cdot \hat{\beta} = \dfrac{2n-1}{\mathrm{Y}} = \dfrac{2n-1}{\sum\limits_{i=1}^{n} \ln \mathrm{X}_i}.$

Then $\quad \mathrm{E}(\hat{\hat{\beta}}) = \dfrac{2n-1}{2n} \cdot \mathrm{E}(\hat{\beta}) = \beta.$

e) Find $\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta})$.

"Hint" 1: $\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta$. You have $\mathrm{E}(\hat{\beta})$ from part (d).

"Hint" 2: $\mathrm{Var}(a \smiley) = a^2 \,\mathrm{Var}(\smiley)$. $\mathrm{Var}(\smiley) = \mathrm{E}(\smiley^2) - [\mathrm{E}(\smiley)]^2$.

"Hint" 3: If T_α has a $\mathrm{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m \,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m \,\Gamma(\alpha)}, \qquad m > -\alpha.$$

$$\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta = \frac{\beta}{2n-1}.$$

$$\mathrm{E}\left(\frac{1}{\mathrm{Y}^2}\right) = \mathrm{E}\left(\mathrm{Y}^{-2}\right) = \frac{\Gamma(\alpha-2)}{\lambda^{-2}\,\Gamma(\alpha)} = \frac{\lambda^2}{(\alpha-1)(\alpha-2)} = \frac{\beta^2}{(2n-1)(2n-2)}.$$

$$\alpha = 2n, \quad m = -2.$$

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}\left(\frac{2n}{\mathrm{Y}}\right) = 4n^2 \,\mathrm{Var}\left(\frac{1}{\mathrm{Y}}\right) = 4n^2 \left[\mathrm{E}\left(\frac{1}{\mathrm{Y}^2}\right) - \left(\mathrm{E}\left(\frac{1}{\mathrm{Y}}\right)\right)^2 \right]$$

$$= 4n^2 \left[\frac{\beta^2}{(2n-1)(2n-2)} - \left(\frac{\beta}{2n-1}\right)^2 \right] = \frac{4n^2 \beta^2}{(2n-1)^2 (2n-2)}.$$

$$\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta}) = \left(\frac{\beta}{2n-1}\right)^2 + \frac{4n^2\beta^2}{(2n-1)^2(2n-2)}$$

$$= \frac{\left(4n^2+2n-2\right)\beta^2}{(2n-1)^2(2n-2)} = \frac{(2n+2)\beta^2}{(2n-1)(2n-2)}.$$

f) Assume $\beta > 1$. (We need this for the expected value $\mathrm{E}(\mathrm{X})$ to exist.)

Obtain a method of moments estimator for β, $\tilde{\beta}$.

That is, if $\mathrm{E}(\mathrm{X}) = h(\beta)$, solve $\overline{\mathrm{X}} = h(\tilde{\beta})$ for $\tilde{\beta}$.

① Find $\mathrm{E}(\mathrm{X})$. It will depend on β, so it will be a function of β, say, $\mathrm{E}(\mathrm{X}) = h(\beta)$.

② Replace $\mathrm{E}(\mathrm{X})$ with $\overline{\mathrm{X}}$, so $\overline{\mathrm{X}} = h(\beta)$.

③ Solve $\overline{\mathrm{X}} = h(\beta)$ for β. Add a tilde.

$$\mathrm{E}(\mathrm{X}) = \int_1^\infty x \cdot \frac{\beta^2 \ln x}{x^{\beta+1}}\,dx = \beta^2 \cdot \int_1^\infty \ln x \cdot \frac{1}{x^\beta}\,dx \qquad \text{by parts}$$

$$= \beta^2 \cdot \left[\ln x \cdot \left(-\frac{1}{(\beta-1)x^{\beta-1}}\right)\Bigg|_1^\infty - \int_1^\infty \frac{1}{x} \cdot \left(-\frac{1}{(\beta-1)x^{\beta-1}}\right)dx\right]$$

$$= \beta^2 \cdot \left[\frac{1}{(\beta-1)} \cdot \int_1^\infty \frac{1}{x^\beta}\,dx\right] = \beta^2 \cdot \left[\frac{1}{(\beta-1)} \cdot \left(-\frac{1}{(\beta-1)x^{\beta-1}}\right)\Bigg|_1^\infty\right]$$

$$= \frac{\beta^2}{(\beta-1)^2}.$$

OR

Since $\quad f(x;\beta) = \dfrac{\beta^2 \ln x}{x^{\beta+1}}, \quad x > 1, \quad \beta > 0,$ is a probability density function,

$$\int_1^\infty \frac{\beta^2 \ln x}{x^{\beta+1}}\, dx = 1, \qquad \text{and} \qquad \int_1^\infty \frac{\ln x}{x^{\beta+1}}\, dx = \frac{1}{\beta^2}, \quad \beta > 0.$$

$$\Rightarrow \qquad \mathrm{E}(\mathrm{X}) = \int_1^\infty x \cdot \frac{\beta^2 \ln x}{x^{\beta+1}}\, dx = \beta^2 \cdot \int_1^\infty \frac{\ln x}{x^{(\beta-1)+1}}\, dx = \frac{\beta^2}{(\beta-1)^2}.$$

$$\overline{\mathrm{X}} = \frac{\beta^2}{(\beta-1)^2}. \qquad \Rightarrow \qquad \sqrt{\overline{\mathrm{X}}} = \frac{\beta}{\beta-1}.$$

$$\Rightarrow \qquad \tilde{\beta} = \frac{\sqrt{\overline{\mathrm{X}}}}{\sqrt{\overline{\mathrm{X}}} - 1} = 1 + \frac{1}{\sqrt{\overline{\mathrm{X}}} - 1}.$$

OR

$$\overline{\mathrm{X}} = \frac{\beta^2}{(\beta-1)^2}. \qquad \Rightarrow \qquad (\overline{\mathrm{X}} - 1)\,\beta^2 - 2\,\overline{\mathrm{X}}\,\beta + \overline{\mathrm{X}} = 0.$$

$$\tilde{\beta} = \frac{2\,\overline{\mathrm{X}} \pm \sqrt{4\,\overline{\mathrm{X}}^2 - 4\,\overline{\mathrm{X}}\left(\overline{\mathrm{X}}-1\right)}}{2\left(\overline{\mathrm{X}}-1\right)} = \frac{\overline{\mathrm{X}} \pm \sqrt{\overline{\mathrm{X}}}}{\overline{\mathrm{X}}-1} = \frac{\overline{\mathrm{X}}\left(\sqrt{\overline{\mathrm{X}}} \pm 1\right)}{\left(\sqrt{\overline{\mathrm{X}}} - 1\right)\left(\sqrt{\overline{\mathrm{X}}} + 1\right)}$$

$$= \frac{\sqrt{\overline{\mathrm{X}}}}{\sqrt{\overline{\mathrm{X}}} - 1} \quad \text{or} \quad \frac{\sqrt{\overline{\mathrm{X}}}}{\sqrt{\overline{\mathrm{X}}} + 1}.$$

However, $\beta > 1$, and $\dfrac{\sqrt{\overline{\mathrm{X}}}}{\sqrt{\overline{\mathrm{X}}} + 1} < 1.$

$$\Rightarrow \qquad \tilde{\beta} = \frac{\sqrt{\overline{\mathrm{X}}}}{\sqrt{\overline{\mathrm{X}}} - 1} = 1 + \frac{1}{\sqrt{\overline{\mathrm{X}}} - 1}.$$

g) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.

Obtain a method of moments estimate for β, $\tilde{\beta}$.

$n = 5$, $1.3 + 1.4 + 2.0 + 3.0 + 5.0 = 12.7$.

$$\bar{x} = \frac{12.7}{5} = 2.54. \qquad \tilde{\beta} = \frac{\sqrt{2.54}}{\sqrt{2.54} - 1} = 1 + \frac{1}{\sqrt{2.54} - 1} \approx \mathbf{2.684}.$$

h) Suppose $\beta > 1$.

Is the method of moments estimator of β, $\tilde{\beta}$, an unbiased estimator of β?

If $\tilde{\beta}$ is not an unbiased estimator of β, does $\tilde{\beta}$ underestimate or overestimate β (on average)?

Hint: $\tilde{\beta} = g(\bar{X})$. Is $g(x)$ a linear function?

If it is not a linear function, does it curve up or down?

$$\tilde{\beta} = \frac{\sqrt{\bar{X}}}{\sqrt{\bar{X}} - 1} = 1 + \frac{1}{\sqrt{\bar{X}} - 1}.$$

Consider $g(x) = 1 + \dfrac{1}{\sqrt{x} - 1}$.

Then $g(\bar{X}) = \tilde{\beta}$.

$$g'(x) = -\frac{1}{\left(\sqrt{x} - 1\right)^2} \cdot \frac{1}{2\sqrt{x}}.$$

$$g''(x) = \frac{1}{\left(\sqrt{x} - 1\right)^3} \cdot \frac{1}{2x} + \frac{1}{\left(\sqrt{x} - 1\right)^2} \cdot \frac{1}{4x^{3/2}} > 0 \qquad \text{for } x > 1.$$

Since $g(x) = 1 + \dfrac{1}{\sqrt{x} - 1}$, $x > 1$, is strictly convex (that is, it curves up),

and $\bar{X}$ is not a constant random variable, by Jensen's Inequality (Theorem 1.10.5),

$$\mathrm{E}(\tilde{\beta}) = \mathrm{E}(g(\overline{\mathrm{X}})) > g(\mathrm{E}(\overline{\mathrm{X}})) = g(\mu).$$

$$g(\mu) = g\left(\frac{\beta^2}{(\beta-1)^2}\right) = 1 + \frac{1}{\frac{\beta}{\beta-1} - 1} = \beta.$$

$$\Rightarrow \qquad \mathrm{E}(\tilde{\beta}) > \beta.$$

$\tilde{\beta}$ is NOT an unbiased estimator for β.

On average, $\tilde{\beta}$ **overestimates** β.