STAT 410 | Fall 2020 | A. Stepanov

Examples for 11/16/2020 (3)

1. $X_1, X_2, \dots, X_n$ are i.i.d. Exponential(mean θ).

That is, $X_1, X_2, \dots, X_n$ are i.i.d. $\quad f(x;\theta) = \frac{1}{\theta}\, e^{-x/\theta}, \qquad x > 0.$

We wish to test $H_0 : \theta = 3$ vs. $H_1 : \theta > 3$.

a) If $n = 5$, find a uniformly most powerful rejection region with the significance level $\alpha = 0.05$.

Let $\theta > 3$.

$$\frac{L(3)}{L(\theta)} = \frac{L(3;\, x_1, x_2, \dots, x_n)}{L(\theta;\, x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{3}\, e^{-x_i/3}}{\prod_{i=1}^{n} \frac{1}{\theta}\, e^{-x_i/\theta}}$$

$$= \left(\frac{\theta}{3}\right)^n \exp\left\{\left(\frac{1}{\theta} - \frac{1}{3}\right) \sum_{i=1}^{n} x_i\right\} \le k.$$

$$\Leftrightarrow \qquad \exp\left\{\left(\frac{1}{\theta} - \frac{1}{3}\right) \sum_{i=1}^{n} x_i\right\} \le k_1.$$

$$\Leftrightarrow \qquad \left(\frac{1}{\theta} - \frac{1}{3}\right) \sum_{i=1}^{n} x_i \le k_2.$$

$$\theta > 3 \qquad \Rightarrow \qquad \frac{1}{\theta} - \frac{1}{3} < 0$$

$$\Leftrightarrow \qquad \sum_{i=1}^{n} x_i \ge c.$$

$\sum_{i=1}^{n=5} X_i$ has a Gamma$(\alpha = 5, \theta)$ distribution. $\sum_{i=1}^{n=5} X_i = T_5$.

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{5} X_i \geq c \mid \theta = 3)$$

$$= P(T_5 \geq c \mid \theta = 3)$$

If T_α has a Gamma(α, θ) distribution, then $\frac{2}{\theta} T_\alpha$ has a $\chi^2(2\alpha)$ distribution.

$$= P(\frac{2}{\theta} T_5 \geq \frac{2}{\theta} c \mid \theta = 3) = P(\frac{2}{3} T_5 \geq \frac{2}{3} c \mid \theta = 3) = P(\chi^2(10) \geq \frac{2}{3} c).$$

$$\Rightarrow \quad \frac{2}{3} c = \chi^2_{0.05}(10) = 18.31. \qquad \Rightarrow \quad c = \mathbf{27.465}.$$

Reject H_0 if $\sum_{i=1}^{n=5} x_i \geq \mathbf{27.465}$.

```
> qgamma(0.95,5,1/3)
[1] 27.46056
>
> qchisq(0.95,10)
[1] 18.30704
> qchisq(0.95,10)*(3/2)
[1] 27.46056
```

$$\text{Power}(\theta = 5) = P(\text{Reject } H_0 \mid \theta = 5) = P(\sum_{i=1}^{5} X_i \geq 27.465 \mid \theta = 5)$$

$$= P(T_5 \geq 27.465 \mid \theta = 5)$$

$$= P(\frac{2}{5} T_5 \geq \frac{2}{5} \cdot 27.465 \mid \theta = 5) = P(\chi^2(10) \geq 10.986).$$

OR

$$= P(X_{27.465} \leq 5 - 1 \mid \theta = 5) = P(\text{Poisson}(\frac{27.465}{5}) \leq 4)$$

$$= P(\text{Poisson}(5.493) \leq 4).$$

```
> 1-pgamma(27.465,5,1/5)
[1] 0.3586098
> 1-pchisq(10.986,10)
[1] 0.3586098
> ppois(4,5.493)
[1] 0.3586098

> 1-pgamma(qgamma(0.95,5,1/3),5,1/5)
[1] 0.3587485
```

$$\int_{27.465}^{\infty} \frac{1}{24 \cdot 5^5} x^4 e^{-\frac{x}{5}} dx = 0.3586$$

b) Consider rejection region

Reject H_0 if $\sum_{i=1}^{n=5} x_i \geq 27$.

Find

(i) the significance level α;

(ii) Power$(\theta = 5)$.

27.465

27

$$\alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}\Big(\sum_{i=1}^{5} \mathrm{X}_i \geq 27 \mid \theta = 3\Big) = \mathrm{P}(\mathrm{T}_5 \geq 27 \mid \theta = 3)$$

$$= \mathrm{P}(\mathrm{X}_{27} \leq 5 - 1 \mid \theta = 3) = \mathrm{P}\Big(\text{Poisson}\big(\tfrac{27}{3}\big) \leq 4\Big)$$

$$= \mathrm{P}(\text{Poisson}(9) \leq 4) = \mathbf{0.055}.$$

OR

$$= \mathrm{P}\Big(\tfrac{2}{3}\mathrm{T}_5 \geq \tfrac{2}{3} \cdot 27 \mid \theta = 3\Big) = \mathrm{P}(\chi^2(10) \geq 18).$$

OR

$$\int_{27}^{\infty} \frac{1}{24 \cdot 3^5} x^4 e^{-\frac{x}{3}} dx = \frac{3563}{8e^9} \quad (\text{Decimal: } 0.05496\ldots)$$

```
> 1-pgamma(27,5,1/3)
[1] 0.05496364
> ppois(5-1,27/3)
[1] 0.05496364
> 1-pchisq((2/3)*27,2*5)
[1] 0.05496364
```

$$\text{Power}(\theta = 5) = P(\text{Reject } H_0 \mid \theta = 5) = P\left(\sum_{i=1}^{5} X_i \ge 27 \mid \theta = 5\right)$$

$$= P(T_5 \ge 27 \mid \theta = 5)$$

$$= P(X_{27} \le 5 - 1 \mid \theta = 5) = P\left(\text{Poisson}\left(\frac{27}{5}\right) \le 4\right)$$

$$= P(\text{Poisson}(5.4) \le 4) = \mathbf{0.373}.$$

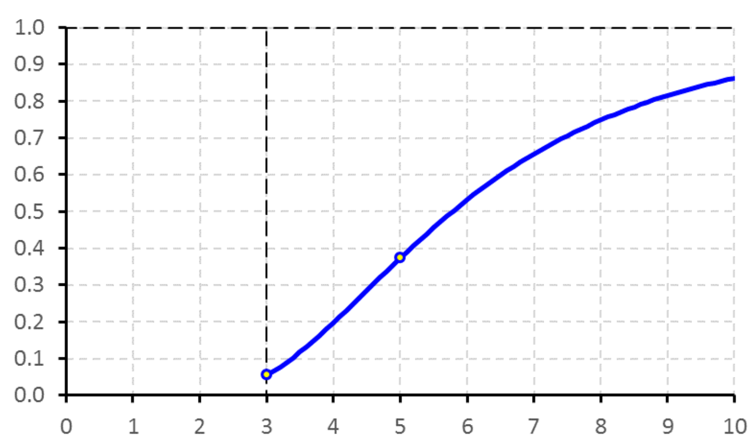
OR

$$= P\left(\frac{2}{5} T_5 \ge \frac{2}{5} \cdot 27 \mid \theta = 5\right) = P(\chi^2(10) \ge 10.8).$$

OR

$$\int_{27}^{\infty} \frac{1}{24 \cdot 5^5} x^4 e^{-\frac{x}{5}} dx = \frac{413267}{5000 e^{\frac{27}{5}}} \quad (\text{Decimal: } 0.37331\ldots)$$

```
> 1-pgamma(27,5,1/5)
[1] 0.3733108
> ppois(5-1,27/5)
[1] 0.3733108
> 1-pchisq((2/5)*27,2*5)
[1] 0.3733108
```



c) Suppose $\sum_{i=1}^{n=5} x_i = 24$. Find the p-value of this test.

$$\text{p-value} = \text{P}\left(\sum_{i=1}^{5} \text{X}_i \text{ as extreme or more extreme than } \left(\sum_{i=1}^{n=5} x_i \right)_{\text{observed}} \,\middle|\, \text{H}_0 \text{ true} \right)$$

$$= \text{P}\left(\sum_{i=1}^{5} \text{X}_i \ge 24 \,\middle|\, \theta = 3 \right)$$

$$= \text{P}(\text{X}_{24} \le 5 - 1 \mid \theta = 3) = \text{P}\left(\text{Poisson}\left(\frac{24}{3} \right) \le 4 \right)$$

$$= \text{P}(\text{Poisson}(8) \le 4) = \mathbf{0.100}.$$

OR

$$= \text{P}\left(\frac{2}{3} \text{T}_5 \ge \frac{2}{3} \cdot 24 \,\middle|\, \theta = 3 \right) = \text{P}(\chi^2(10) \ge 16).$$

OR

$$\int_{24}^{\infty} \frac{1}{24 \cdot 3^5} x^4 e^{-\frac{x}{3}} dx = \frac{297}{e^8} \quad (\text{Decimal: } 0.09963\ldots)$$

```
> 1-pgamma(24,5,1/3)
[1] 0.0996324
> ppois(5-1,24/3)
[1] 0.0996324
> 1-pchisq((2/3)*24,2*5)
[1] 0.0996324
```

2. $X_1, X_2, \dots, X_n$ are i.i.d. Exponential(mean θ).

That is, $X_1, X_2, \dots, X_n$ are i.i.d. $\qquad f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \qquad x > 0.$

We wish to test $H_0: \theta = 3$ vs. $H_1: \theta < 3$.

a) If $n = 5$, find a uniformly most powerful rejection region with the significance level $\alpha = 0.05$.

Let $\theta > 3$.

$$\frac{L(3)}{L(\theta)} = \frac{L(3;\, x_1, x_2, \dots, x_n)}{L(\theta;\, x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{3} e^{-x_i/3}}{\prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}}$$

$$= \left(\frac{\theta}{3}\right)^n \exp\left\{\left(\frac{1}{\theta} - \frac{1}{3}\right) \sum_{i=1}^{n} x_i\right\} \le k.$$

$$\Leftrightarrow \qquad \exp\left\{\left(\frac{1}{\theta} - \frac{1}{3}\right) \sum_{i=1}^{n} x_i\right\} \le k_1.$$

$$\Leftrightarrow \qquad \left(\frac{1}{\theta} - \frac{1}{3}\right) \sum_{i=1}^{n} x_i \le k_2.$$

$$\theta < 3 \qquad \Rightarrow \qquad \frac{1}{\theta} - \frac{1}{3} > 0$$

$$\Leftrightarrow \qquad \sum_{i=1}^{n} x_i \le c.$$

$\sum_{i=1}^{n=5} X_i$ has a Gamma($\alpha = 5, \theta$) distribution. $\qquad \sum_{i=1}^{n=5} X_i = T_5.$

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{5} X_i \le c \mid \theta = 3\right)$$

$$= P(T_5 \le c \mid \theta = 3)$$

If T_α has a $\text{Gamma}(\alpha, \theta)$ distribution, then $\frac{2}{\theta} T_\alpha$ has a $\chi^2(2\alpha)$ distribution.

$$= P\left(\frac{2}{\theta} T_5 \le \frac{2}{\theta} c \mid \theta = 3\right) = P\left(\frac{2}{3} T_5 \le \frac{2}{3} c \mid \theta = 3\right) = P\left(\chi^2(10) \le \frac{2}{3} c\right).$$

$$\Rightarrow \quad \frac{2}{3} c = \chi^2_{0.95}(10) = 3.94. \qquad \Rightarrow \quad c = \mathbf{5.91}.$$

Reject H_0 if $\sum_{i=1}^{n=5} x_i \le \mathbf{5.91}$.

```
> qgamma(0.05,5,1/3)
[1] 5.910449
>
> qchisq(0.05,10)
[1] 3.940299
> qchisq(0.05,10)*(3/2)
[1] 5.910449
```

$$\text{Power}(\theta = 1) = P(\text{Reject } H_0 \mid \theta = 1) = P\left(\sum_{i=1}^{5} X_i \le 5.91 \mid \theta = 1\right)$$

$$= P(T_5 \le 5.91 \mid \theta = 1)$$

$$= P\left(\frac{2}{1} T_5 \le \frac{2}{1} \cdot 5.91 \mid \theta = 1\right) = P(\chi^2(10) \le 11.82).$$

OR

$$= P(X_{5.91} \ge 5 \mid \theta = 1) = P\left(\text{Poisson}\left(\frac{5.91}{1}\right) \ge 5\right)$$

$$= 1 - P(\text{Poisson}(5.91) \le 4).$$

OR

$$\int_0^{5.91} \frac{1}{24 \cdot 1^5} x^4 e^{-\frac{x}{1}} dx = 0.70271\ldots$$

```
> pgamma(5.91,5,1)
[1] 0.7027161
> pchisq(11.82,10)
[1] 0.7027161
> 1-ppois(4,5.91)
[1] 0.7027161

> pgamma(qgamma(0.05,5,1/3),5,1)
[1] 0.702778
```

b) Consider rejection region

$$\text{Reject } H_0 \text{ if } \sum_{i=1}^{n=5} x_i \le 6.$$

Find

(i) the significance level α;

(ii) $\text{Power}(\theta = 1)$.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{5} X_i \le 6 \mid \theta = 3\right) = P(T_5 \le 6 \mid \theta = 3)$$

$$= P(X_6 \ge 5 \mid \theta = 3) = P\left(\text{Poisson}\left(\frac{6}{3}\right) \ge 5\right)$$

$$= 1 - P(\text{Poisson}(2) \le 4) = 1 - 0.947 = \mathbf{0.053}.$$

OR

$$= P\left(\frac{2}{3} T_5 \le \frac{2}{3} \cdot 6 \mid \theta = 3\right) = P(\chi^2(10) \le 4).$$

OR

$$\int_0^6 \frac{1}{24 \cdot 3^5} x^4 e^{-\frac{x}{3}} dx = \frac{-40824 + 5832e^2}{5832e^2} \quad (\text{Decimal: } 0.05265\ldots)$$

```
> pgamma(6,5,1/3)
[1] 0.05265302
> 1-ppois(5-1,6/3)
[1] 0.05265302
> pchisq((2/3)*6,2*5)
[1] 0.05265302
```

$$\text{Power}(\theta = 1) = P(\text{Reject } H_0 \mid \theta = 1) = P\left(\sum_{i=1}^{5} X_i \le 6 \mid \theta = 1\right)$$

$$= P(T_5 \le 6 \mid \theta = 1)$$

$$= P(X_6 \ge 5 \mid \theta = 1) = P\left(\text{Poisson}\left(\frac{6}{1}\right) \ge 5\right)$$

$$= 1 - P(\text{Poisson}(6) \le 4) = 1 - 0.285 = \mathbf{0.715}.$$

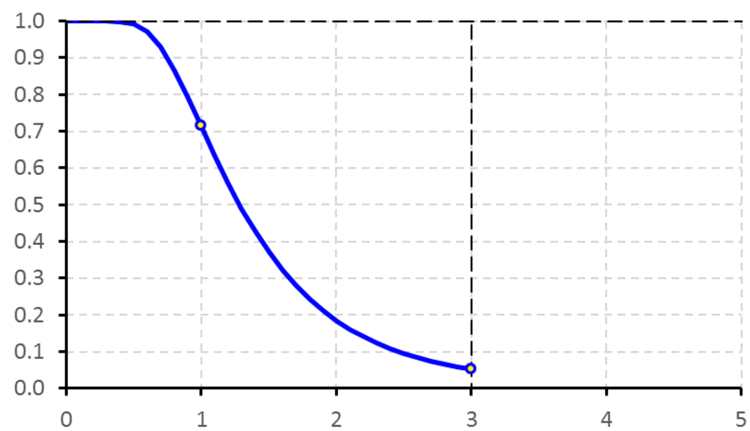
OR

$$= P\left(\frac{2}{1} T_5 \le \frac{2}{1} \cdot 6 \mid \theta = 1\right) = P(\chi^2(10) \le 12).$$

OR

$$\int_0^6 \frac{1}{24 \cdot 1^5} x^4 e^{-\frac{x}{1}} dx = \frac{-2760 + 24e^6}{24e^6} \quad (\text{Decimal:} \quad 0.71494\ldots)$$

```
> pgamma(6,5,1)
[1] 0.7149435
> 1-ppois(5-1,6/1)
[1] 0.7149435
> pchisq((2/1)*6,2*5)
[1] 0.7149435
```



c) Suppose $\sum_{i=1}^{n=5} x_i = 5.4$. Find the p-value of this test.

$$\text{p-value} = P\left(\sum_{i=1}^{5} X_i \text{ as extreme or more extreme than } \left(\sum_{i=1}^{n=5} x_i\right)_{\text{observed}} \,\middle|\, H_0 \text{ true}\right)$$

$$= P\left(\sum_{i=1}^{5} X_i \le 5.4 \mid \theta = 3\right)$$

$$= P(X_{5.4} \ge 5 \mid \theta = 3) = P\left(\text{Poisson}\left(\frac{5.4}{3}\right) \ge 5\right)$$

$$= 1 - P(\text{Poisson}(1.8) \le 4) = 1 - 0.964 = \mathbf{0.036}.$$

OR

$$= P\left(\frac{2}{3} T_5 \le \frac{2}{3} \cdot 5.4 \mid \theta = 3\right) = P(\chi^2(10) \le 3.6).$$

OR

$$\int_0^{5.4} \frac{1}{24 \cdot 3^5} x^4 e^{-\frac{x}{3}} dx = 0.03640\ldots$$

```
> pgamma(5.4,5,1/3)
[1] 0.03640666
> 1-ppois(5-1,5.4/3)
[1] 0.03640666
> pchisq((2/3)*5.4,2*5)
[1] 0.03640666
```