STAT 410 | Fall 2020 | A. Stepanov

Homework #7

(due Friday, October 23, by 4:00 p.m.)

No credit will be given without supporting work.

7. Let $\psi > 0$ be a population parameter, and let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\psi) = \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}, \qquad x > 0, \qquad \text{zero otherwise.}$$

This is a Half-Normal distribution. Consider $|N(0,\sigma^2)|$, where $\psi = 2\sigma^2$.

a) (i) Obtain a method of moments estimator of ψ, $\tilde{\psi}$.

(ii) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of ψ.

① Find $E(X)$. It will depend on ψ, so it will be a function of ψ, say, $E(X) = h(\psi)$.

② Replace $E(X)$ with $\overline{X}$, so $\overline{X} = h(\psi)$.

③ Solve $\overline{X} = h(\psi)$ for ψ. Add a tilde.

(i) $$E(X) = \int_0^\infty x \cdot \frac{2}{\sqrt{\pi\psi}}\, e^{-x^2/\psi}\, dx = \dots \qquad u = x^2 \qquad du = 2x\,dx$$

$$= \frac{1}{\sqrt{\pi\psi}} \int_0^\infty e^{-u/\psi}\, du = \frac{\sqrt{\psi}}{\sqrt{\pi}}.$$

$$\overline{X} = \frac{\sqrt{\psi}}{\sqrt{\pi}}. \qquad \Rightarrow \qquad \tilde{\psi} = \pi\left(\overline{X}\right)^2.$$

(ii) $n = 4$ $\sum_{i=1}^{n} x_i = 0.2 + 0.6 + 1.1 + 1.7 = 3.6.$ $\overline{x} = 0.9.$

$\tilde{\psi} = \mathbf{0.81\,\pi \approx 2.54469}.$

b) (i) Obtain the maximum likelihood estimator of ψ, $\hat{\psi}$.

(ii) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

Find the maximum likelihood estimate of ψ.

That is, find $\hat{\psi} = \arg\max \mathrm{L}(\psi) = \arg\max \ln \mathrm{L}(\psi)$, where $\mathrm{L}(\psi) = \prod_{i=1}^{n} f(x_i\,;\psi)$.

① Multiply: $\mathrm{L}(\psi) = f(x_1;\psi) \cdot f(x_2;\psi) \cdot \ldots \cdot f(x_n;\psi)$.

② Simplify. "Hint": $a^b \cdot a^c = a^{b+c}$, $a^c \cdot b^c = (a \cdot b)^c$, $(a^b)^c = a^{b \cdot c}$.

③ Take ln. "Hint": $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^b) = b \cdot \ln a$.

④ Take the derivative **with respect to ψ**.

⑤ Set equal to zero. Solve for ψ. Add a hat.

(i) $$\mathrm{L}(\psi) = \prod_{i=1}^{n} f(x_i\,;\psi) = \prod_{i=1}^{n} \frac{2}{\sqrt{\pi\psi}}\, e^{-x_i^2/\psi}$$

$$= \left(\frac{2}{\sqrt{\pi\psi}}\right)^n \cdot \exp\left\{-\frac{1}{\psi}\sum_{i=1}^{n} x_i^2\right\}.$$

$$\ln \mathrm{L}(\psi) = n \cdot \ln 2 - \frac{n}{2} \cdot \ln \pi - \frac{n}{2} \cdot \ln \psi - \frac{1}{\psi}\sum_{i=1}^{n} x_i^2 .$$

$$\frac{d}{d\psi} \ln \mathrm{L}(\psi) = -\frac{n}{2\psi} + \frac{1}{\psi^2}\sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\psi} = \frac{2}{n}\sum_{i=1}^{n} \mathrm{X}_i^2 .$$

(ii) $n = 4$ $\qquad \sum_{i=1}^{n} x_i^2 = 0.2^2 + 0.6^2 + 1.1^2 + 1.7^2 = 4.5.$

$$\hat{\psi} = \frac{2}{4} \cdot 4.5 = \mathbf{2.25}.$$

c) Show that $W = X^2$ follows a Gamma distribution.

What are the parameters α and θ for this Gamma distribution?

No credit will be given without proper justification.

Let $W = X^2$ $\qquad X = \sqrt{W} = g^{-1}(W)$ $\qquad {}^{dx}/_{dw} = \frac{1}{2\sqrt{w}}$

$$f_W(w) = f_X(g^{-1}(y)) \left| \frac{dx}{dw} \right| = \frac{2}{\sqrt{\pi\psi}} e^{-w/\psi} \times \frac{1}{2\sqrt{w}}$$

$$= \frac{1}{\sqrt{\pi}\sqrt{\psi}} \frac{1}{\sqrt{w}} e^{-w/\psi} = \frac{1}{\Gamma\left(\frac{1}{2}\right) \psi^{\frac{1}{2}-1}} w^{\frac{1}{2}-1} e^{-w/\psi}, \qquad w > 0.$$

$\Rightarrow$ $\qquad W = X^2$ has Gamma$\left(\alpha = \mathbf{\frac{1}{2}}, \theta = \boldsymbol{\psi}\right)$ distribution.

8. Let $\beta > 0$ be a population parameter, and let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\beta) = \beta\,(1-x)^{\beta-1}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) (i) Obtain a method of moments estimator for β, $\tilde{\beta}$.

(ii) Suppose $n = 3$, and $x_1 = 0.31$, $x_2 = 0.77$, $x_3 = 0.93$.
Find a method of moments estimate of β.

① Find $E(X)$. It will depend on β, so it will be a function of β, say, $E(X) = h(\beta)$.

② Replace $E(X)$ with $\overline{X}$, so $\overline{X} = h(\beta)$.

③ Solve $\overline{X} = h(\beta)$ for β. Add a tilde.

(i) $$E(X) = \int_0^1 x \cdot \beta\,(1-x)^{\beta-1}\,dx \qquad u = 1-x \qquad du = -\,dx$$

$$= -\int_1^0 (1-u)\cdot \beta\, u^{\beta-1}\,du = \int_0^1 \beta\, u^{\beta-1}\,du - \int_0^1 \beta\, u^{\beta}\,du$$

$$= 1 - \frac{\beta}{\beta+1} = \frac{1}{\beta+1}.$$

OR

$$E(X) = \int_0^1 x \cdot \beta\,(1-x)^{\beta-1}\,dx \qquad u = x \qquad dv = \beta\,(1-x)^{\beta-1}\,dx$$

$$du = dx \qquad v = -\,(1-x)^{\beta}$$

$$= -x\,(1-x)^{\beta}\Big|_0^1 + \int_0^1 (1-x)^{\beta}\,dx = \int_0^1 (1-x)^{\beta}\,dx = \frac{1}{\beta+1}.$$

OR

$$E(1-X) = \int_0^1 (1-x)\cdot\beta\,(1-x)^{\beta-1}\,dx = \beta\int_0^1 (1-x)^{\beta}\,dx = \frac{\beta}{\beta+1}.$$

$$1-E(X) = \frac{\beta}{\beta+1}. \qquad \Rightarrow \qquad E(X) = \frac{1}{\beta+1}.$$

OR

$$F_X(x) = \int_0^x \beta\,(1-u)^{\beta-1}\,du = -(1-u)^{\beta}\Big|_0^x = 1-(1-x)^{\beta}, \qquad 0 < x < 1.$$

Since X is a nonnegative random variable,

$$E(X) = \int_0^{\infty} \left(1 - F_X(x)\right) dx = \int_0^1 (1-x)^{\beta}\,dx = \frac{1}{\beta+1}.$$

OR

Beta distribution, $\alpha = 1$, $\beta = \beta$. $\Rightarrow$ $E(X) = \dfrac{\alpha}{\alpha+\beta} = \dfrac{1}{\beta+1}.$

$$\overline{X} = \frac{1}{\beta+1}. \qquad \Rightarrow \qquad \tilde{\beta} = \frac{1}{\overline{X}}-1 = \frac{1-\overline{X}}{\overline{X}}.$$

(ii) $n = 3$ $\quad \sum_{i=1}^{n} x_i = 0.31 + 0.77 + 0.93 = 2.01.$ $\quad \overline{x} = 0.67.$

$$\tilde{\beta} = = \frac{1}{0.67}-1 \approx \mathbf{0.49254}.$$

b) (i) Find the maximum likelihood estimator for β, $\hat{\beta}$.

(ii) Suppose $n = 3$, and $x_1 = 0.31$, $x_2 = 0.77$, $x_3 = 0.93$.

Find the maximum likelihood estimate for β, $\hat{\beta}$.

That is, find $\hat{\beta} = \arg\max \mathrm{L}(\beta) = \arg\max \ln \mathrm{L}(\beta)$, where $\mathrm{L}(\beta) = \prod_{i=1}^{n} f\left(x_i ; \beta\right)$.

① Multiply: $\mathrm{L}(\beta) = f(x_1 ; \beta) \cdot f(x_2 ; \beta) \cdot \ldots \cdot f(x_n ; \beta)$.

② Simplify. "Hint": $a^b \cdot a^c = a^{b+c}$, $a^c \cdot b^c = (a \cdot b)^c$, $(a^b)^c = a^{b \cdot c}$.

③ Take ln. "Hint": $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^b) = b \cdot \ln a$.

④ Take the derivative **with respect to** $\boldsymbol{\beta}$.

⑤ Set equal to zero. Solve for β. Add a hat.

(i) Likelihood function:

$$\mathrm{L}(\beta) = \prod_{i=1}^{n} \beta\left(1-x_i\right)^{\beta-1} = \beta^n \cdot \left(\prod_{i=1}^{n}\left(1-x_i\right)\right)^{\beta-1}.$$

$$\ln \mathrm{L}(\beta) = n \cdot \ln \beta + (\beta-1) \sum_{i=1}^{n} \ln\left(1-x_i\right).$$

$$\frac{d}{d\beta}\left(\ln \mathrm{L}(\beta)\right) = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1-x_i\right) = 0.$$

$$\Rightarrow \quad \hat{\beta} = -\frac{n}{\sum_{i=1}^{n} \ln\left(1-\mathrm{X}_i\right)}.$$

(ii) $n = 3$ $\quad \sum_{i=1}^{n} \ln\left(1-x_i\right) = \ln 0.69 + \ln 0.23 + \ln 0.07 \approx -4.5.$

$$\hat{\beta} = -\frac{3}{-4.5} = \mathbf{\frac{2}{3}} \approx \mathbf{0.66667}.$$

c) Show that $W = -\ln(1-X)$ follows a Gamma distribution.

What are the parameters α and θ for this Gamma distribution?

No credit will be given without proper justification.

Let $W = -\ln(1-X)$. $\quad 0 < x < 1 \qquad \infty > w > 0.$

$$F_X(x) = \int_0^x \beta(1-u)^{\beta-1}\,du = -(1-u)^{\beta}\Big|_0^x = 1-(1-x)^{\beta}, \qquad 0 < x < 1.$$

$$F_W(w) = P(W \le w) = P(-\ln(1-X) \le w) = P(X \le 1-e^{-w})$$

$$= 1-e^{-\beta w}, \qquad 0 < w < \infty.$$

OR

$$X = 1-e^{-W} = g^{-1}(W) \qquad {}^{dx}\!/_{dw} = e^{-w}$$

$$f_W(w) = f_X(g^{-1}(y))\left|\frac{dx}{dw}\right| = \beta\left(e^{-w}\right)^{\beta-1} \times \left|e^{-w}\right| = \beta e^{-\beta w},$$

$$0 < w < \infty.$$

$\Rightarrow \qquad W = -\ln(1-X)$ has an Exponential$\left(\theta = \frac{1}{\beta}\right)$

$= \text{Gamma}\left(\alpha = \mathbf{1}, \theta = \frac{\mathbf{1}}{\boldsymbol{\beta}}\right)$ distribution.

d) Suppose $n = 3$ and $\beta = 0.8$. Find $P\left(-\sum_{i=1}^{3} \ln\left(1 - X_i\right) > 4.5 \right)$.

$$\sum_{i=1}^{n} W_i = -\sum_{i=1}^{n} \ln\left(1 - X_i\right) \text{ has a Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right) \text{ distribution.}$$

$$\Rightarrow \quad -\sum_{i=1}^{3} \ln\left(1 - X_i\right) \text{ has a Gamma}\left(\alpha = 3, \theta = \frac{1}{0.8} = 1.25\right) \text{ distribution.}$$

$$P\left(-\sum_{i=1}^{3} \ln\left(1 - X_i\right) > 4.5 \right) = P\left(T_3 > 4.5\right) = \int_{4.5}^{\infty} \frac{0.8^3}{\Gamma(3)} x^{3-1} e^{-0.8x} \, dx = \ldots$$

OR

If T has a Gamma$\left(\alpha, \theta = 1/\lambda\right)$ distribution, where α is an integer, then
$F_T(t) = P(T \le t) = P(Y \ge \alpha)$ and $P(T > t) = P(Y \le \alpha - 1)$,
where Y has a Poisson(λt) distribution.

$$P\left(-\sum_{i=1}^{3} \ln\left(1 - X_i\right) > 4.5 \right) = P\left(T_3 > 4.5\right) = P\left(\text{Poisson}\left(\beta \cdot 4.5\right) \le 3 - 1\right)$$

$$= P\left(\text{Poisson}\left(0.8 \cdot 4.5\right) \le 2\right) = P\left(\text{Poisson}(3.6) \le 2\right) = \mathbf{0.303}.$$

OR

If T has a Gamma$\left(\alpha, \theta = 1/\lambda\right)$ distribution, where α is an integer, then
$2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$P\left(-\sum_{i=1}^{3} \ln\left(1 - X_i\right) > 4.5 \right) = P\left(T_3 > 4.5\right) = P\left(\chi^2(6) > 7.2\right) = \ldots$$

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> 1-pgamma(4.5,3,0.8)
[1] 0.3027468
>
> ppois(3-1,0.8*4.5)
[1] 0.3027468
>
> 1-pchisq(2*0.8*4.5,2*3)
[1] 0.3027468
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