STAT 410 **Practice Problems 15** Fall 2020
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1. Let X have a Poisson distribution with mean λ. The prior probability distribution of λ is

$$P(\lambda = 2) = 0.30, \qquad P(\lambda = 3) = 0.50, \qquad P(\lambda = 5) = 0.20.$$

Find the posterior distribution of λ, given that we observe $x = 4$.

2. **6.8-2** **9.2-2** 7.2-2 (STAT 400 textbook)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a gamma distribution with known α and $\theta = 1/\tau$. Say τ has a prior p.d.f. which is gamma with parameters α_0 and θ_0 so that the prior mean is $\alpha_0\theta_0$.

(a) Find the posterior p.d.f. of τ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

(b) Find the mean of this posterior distribution and write it as a function of the sample mean $\overline{X}$ and $\alpha_0\theta_0$.

(c) Explain how you would find a 95% interval estimate of τ if $n = 10, \alpha = 3, \alpha_0 = 10$, and $\theta_0 = 2$.

3.* Suppose that $S = \{ 1, 2 \}$, $\Omega = \{ 1, 2, 3 \}$, and the class of probability distribution for the response s is given by the following table.

	$s = 1$	$s = 2$
$f_1(s)$	1/2	1/2
$f_2(s)$	1/3	2/3
$f_3(s)$	3/4	1/4

If we use the prior $\pi(\theta)$ given by the table

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta)$	1/5	2/5	2/5

then determine the posterior distribution of θ for each possible sample of size …

a) … $n = 1$. b) … $n = 2$.

Answers:

1. Let X have a Poisson distribution with mean λ. The prior probability distribution of λ is

$$P(\lambda = 2) = 0.30, \qquad P(\lambda = 3) = 0.50, \qquad P(\lambda = 5) = 0.20.$$

Find the posterior distribution of λ, given that we observe $x = 4$.

$$P(X = 4 \mid \lambda = 2) = \frac{2^4 e^{-2}}{4!} \approx 0.090.$$

$$P(X = 4 \mid \lambda = 3) = \frac{3^4 e^{-3}}{4!} \approx 0.168.$$

$$P(X = 4 \mid \lambda = 5) = \frac{5^4 e^{-5}}{4!} \approx 0.175.$$

$$P(X = 4) = 0.30 \times 0.090 + 0.50 \times 0.168 + 0.20 \times 0.175 = 0.146.$$

$$P(\lambda = 2 \mid X = 4) = \frac{0.30 \times 0.090}{0.146} \approx \mathbf{0.185}.$$

$$P(\lambda = 3 \mid X = 4) = \frac{0.50 \times 0.168}{0.146} \approx \mathbf{0.575}.$$

$$P(\lambda = 5 \mid X = 4) = \frac{0.20 \times 0.175}{0.146} \approx \mathbf{0.240}.$$

2. **6.8-2** **9.2-2** 7.2-2 (STAT 400 textbook)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a gamma distribution with known α and $\theta = 1/\tau$. Say τ has a prior p.d.f. which is gamma with parameters α_0 and θ_0 so that the prior mean is $\alpha_0\theta_0$.

(a) Find the posterior p.d.f. of τ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

(b) Find the mean of this posterior distribution and write it as a function of the sample mean $\overline{X}$ and $\alpha_0\theta_0$.

(c) Explain how you would find a 95% interval estimate of τ if $n = 10, \alpha = 3, \alpha_0 = 10$, and $\theta_0 = 2$.

a) $$f(x_1, x_2, \ldots x_n \mid \tau) = f(x_1;\tau)\ f(x_2;\tau)\ \ldots\ f(x_n;\tau)$$

$$= \prod_{i=1}^{n} \left(\frac{\tau^{\alpha}}{\Gamma(\alpha)}\, x_i^{\alpha-1}\, e^{-\tau x_i} \right)$$

$$f(x_1, x_2, \ldots x_n, \lambda) = f(x_1, x_2, \ldots x_n \mid \lambda) \times \pi(\lambda)$$

$$= \prod_{i=1}^{n} \left(\frac{\tau^{\alpha}}{\Gamma(\alpha)}\, x_i^{\alpha-1}\, e^{-\tau x_i} \right) \times \frac{1}{\Gamma(\alpha_0)\,\theta_0^{\alpha_0}}\ \tau^{\alpha_0-1}\, e^{-\tau/\theta_0}$$

$$= \ \ldots\ \tau^{\alpha n+\alpha_0-1}\, e^{-\tau\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)}.$$

$\Rightarrow$ the posterior distribution of τ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$,

is **Gamma** with New $\alpha = \alpha n + \alpha_0$ and New $\theta = \dfrac{1}{\sum_{i=1}^{n} x_i + \dfrac{1}{\theta_0}}$.

b) $\left(\text{conditional mean of } \lambda, \text{ given } \mathrm{X}_1 = x_1, \mathrm{X}_2 = x_2, \dots, \mathrm{X}_n = x_n\right)$

$$= (\text{New } \alpha) \times (\text{New } \theta) = \frac{\alpha n + \alpha_0}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}}$$

$$= \frac{\alpha n}{\sum_{i=1}^{n} x_i} \cdot \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}} + \alpha_0 \theta_0 \cdot \frac{\frac{1}{\theta_0}}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}}$$

$$= \frac{\alpha}{\overline{x}} \cdot \frac{n \overline{x}}{n \overline{x} + \frac{1}{\theta_0}} + \alpha_0 \theta_0 \cdot \frac{\frac{1}{\theta_0}}{n \overline{x} + \frac{1}{\theta_0}}$$

$$= (\text{MLE}) \cdot \frac{n \overline{x}}{n \overline{x} + \frac{1}{\theta_0}} + (\text{prior mean}) \cdot \frac{\frac{1}{\theta_0}}{n \overline{x} + \frac{1}{\theta_0}}.$$

c) If T has a $\text{Gamma}\left(\alpha, \theta = {}^{1}\!/_{\lambda}\right)$ distribution, where α is an integer, then ${}^{2\mathrm{T}}\!/_{\theta} = 2\lambda \mathrm{T}$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$2\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)\left(\tau \mid x_1, x_2, \dots, x_n\right) \quad \text{has a} \quad \chi^2(2\alpha n + 2\alpha_0) \text{ distribution.}$$

$$\mathrm{P}\left(\chi^2_{1-\gamma/2}(2\alpha n + 2\alpha_0) < 2\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)\left(\tau \mid x_1, x_2, \dots, x_n\right) < \chi^2_{\gamma/2}(2\alpha n + 2\alpha_0)\right)$$

$$= 1 - \gamma.$$

$$P\left(\frac{\chi^2_{1-\gamma/2}(2\alpha n + 2\alpha_0)}{2\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)} < (\tau \mid x_1, x_2, \ldots, x_n) < \frac{\chi^2_{\gamma/2}(2\alpha n + 2\alpha_0)}{2\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)} \right) = 1 - \gamma.$$

$$\Rightarrow \quad \left(\frac{\chi^2_{1-\gamma/2}(2\alpha n + 2\alpha_0)}{2\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)}, \ \frac{\chi^2_{\gamma/2}(2\alpha n + 2\alpha_0)}{2\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)} \right)$$

is a $(1-\gamma)\,100\%$ interval estimate for τ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$n = 10, \quad \alpha = 3, \quad \alpha_0 = 10, \quad \theta_0 = 2.$

$2\alpha n + 2\alpha_0 = 80$ degrees of freedom.

$\chi^2_{0.975}(80) = 57.15, \qquad \chi^2_{0.025}(80) = 106.6.$

$$\Rightarrow \quad \left(\frac{57.15}{2\sum_{i=1}^{n} x_i + 1}, \ \frac{106.6}{2\sum_{i=1}^{n} x_i + 1} \right)$$

is a 95% interval estimate for τ, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

3.* Suppose that $S = \{1, 2\}$, $\Omega = \{1, 2, 3\}$, and the class of probability distribution for the response s is given by the following table.

	$s = 1$	$s = 2$
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If we use the prior $\pi(\theta)$ given by the table

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta)$	1/5	2/5	2/5

then determine the posterior distribution of θ for each possible sample of size …

a) … $n = 1$.

$$P(s=1) = \frac{1}{5}\cdot\frac{1}{2}+\frac{2}{5}\cdot\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4} = \frac{3+4+9}{30} = \frac{16}{30}.$$

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta \mid s=1)$	$\mathbf{\frac{3}{16}}$	$\mathbf{\frac{4}{16}}$	$\mathbf{\frac{9}{16}}$

$$P(s=2) = \frac{1}{5}\cdot\frac{1}{2}+\frac{2}{5}\cdot\frac{2}{3}+\frac{2}{5}\cdot\frac{1}{4} = \frac{3+8+3}{30} = \frac{14}{30}.$$

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta \mid s=2)$	$\mathbf{\frac{3}{14}}$	$\mathbf{\frac{8}{14}}$	$\mathbf{\frac{3}{14}}$

b) … $n = 2$.

$(1, 1)$

$$P((1,1)) = \frac{1}{4}\times\frac{1}{5}+\frac{1}{9}\times\frac{2}{5}+\frac{9}{16}\times\frac{2}{5} = \frac{1}{20}+\frac{2}{45}+\frac{9}{40} = \frac{18}{360}+\frac{16}{360}+\frac{81}{360} = \frac{115}{360}$$

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta \mid (1,1))$	$\frac{18}{115}$	$\frac{16}{115}$	$\frac{81}{115}$

$(1, 2)$

$$P((1,2)) = \frac{1}{4}\times\frac{1}{5}+\frac{2}{9}\times\frac{2}{5}+\frac{3}{16}\times\frac{2}{5} = \frac{1}{20}+\frac{4}{45}+\frac{3}{40} = \frac{18}{360}+\frac{32}{360}+\frac{27}{360} = \frac{77}{360}$$

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta \mid (1,2))$	$\frac{18}{77}$	$\frac{32}{77}$	$\frac{27}{77}$

$(2, 1)$

$$P((2,1)) = \frac{1}{4}\times\frac{1}{5}+\frac{2}{9}\times\frac{2}{5}+\frac{3}{16}\times\frac{2}{5} = \frac{1}{20}+\frac{4}{45}+\frac{3}{40} = \frac{18}{360}+\frac{32}{360}+\frac{27}{360} = \frac{77}{360}$$

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta \mid (2,1))$	$\frac{18}{77}$	$\frac{32}{77}$	$\frac{27}{77}$

(same as the one for $(1, 2)$)

$(2, 2)$

$$P((2,2)) = \frac{1}{4}\times\frac{1}{5}+\frac{4}{9}\times\frac{2}{5}+\frac{1}{16}\times\frac{2}{5} = \frac{1}{20}+\frac{8}{45}+\frac{1}{40} = \frac{18}{360}+\frac{64}{360}+\frac{9}{360} = \frac{91}{360}$$

	$\theta = 1$	$\theta = 2$	$\theta = 3$
$\pi(\theta \mid (2,2))$	$\frac{18}{91}$	$\frac{64}{91}$	$\frac{9}{91}$