Examples for 09/30/2020 (1)

Bivariate Normal Distribution:

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 -2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\},$$

$-\infty < x < \infty,\quad -\infty < y < \infty.$

(a) the marginal distributions of X and Y are $\mathbf{N}\left(\mu_1,\sigma_1^2\right)$ and $\mathbf{N}\left(\mu_2,\sigma_2^2\right)$, respectively;

(b) the correlation coefficient of X and Y is $\rho_{XY}=\rho$, and X and Y are independent if and only if $\rho=0$;

(c) the conditional distribution of Y, given $X=x$, is

$$\mathbf{N}\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \left(1-\rho^2\right)\sigma_2^2\right);$$

(d) the conditional distribution of X, given $Y=y$, is

$$\mathbf{N}\left(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \left(1-\rho^2\right)\sigma_1^2\right).$$

(e) $a\mathrm{X}+b\mathrm{Y}$ is normally distributed with

mean $\mathrm{E}(a\mathrm{X}+b\mathrm{Y})=a\,\mu_1+b\,\mu_2$ and

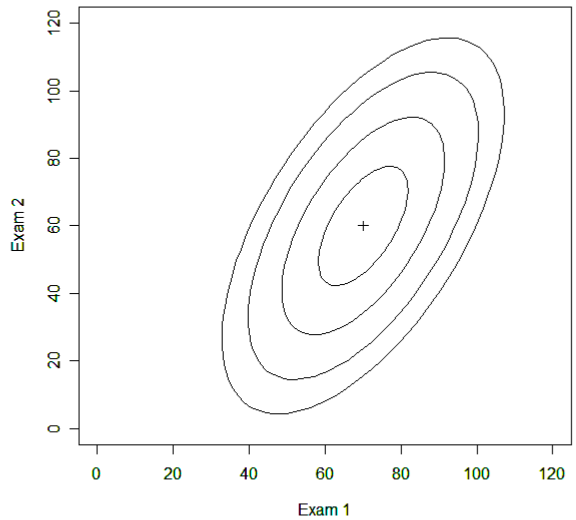
variance $\mathrm{Var}(a\mathrm{X}+b\mathrm{Y})=a^2\sigma_1^2+2\,a\,b\,\rho\,\sigma_1\sigma_2+b^2\sigma_2^2.$

1. A large class took two exams. Suppose the exam scores X (Exam 1) and Y (Exam 2) follow a bivariate normal distribution with

$$\mu_1 = 70, \qquad \sigma_1 = 10,$$

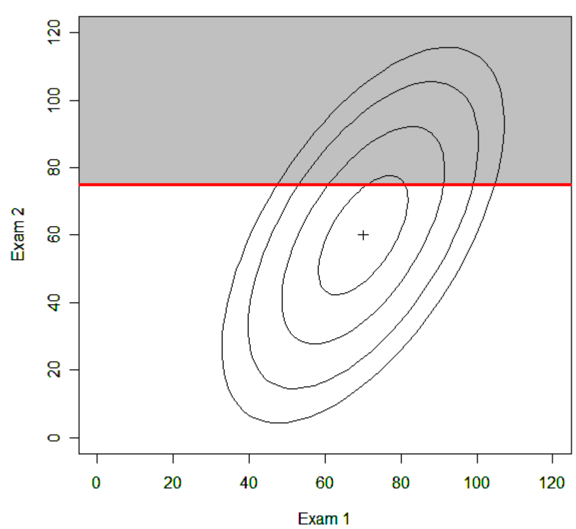
$$\mu_2 = 60, \qquad \sigma_2 = 15,$$

$$\rho = 0.6.$$



a) A students is selected at random. What is the probability that his/her score on Exam 2 is over 75?

$$P(Y > 75) = P\left(Z > \frac{75-60}{15}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$



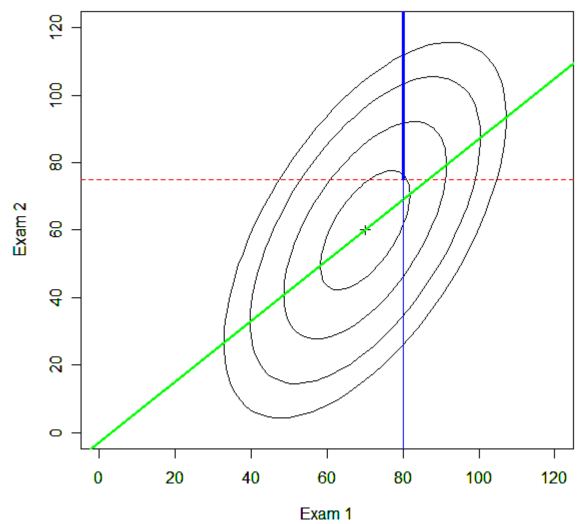
b) Suppose you're told that a student got a 80 on Exam 1. What is the probability that his/her score on Exam 2 is over 75?

Given X = 80, Y has Normal distribution

with mean $60 + 0.6 \cdot \dfrac{15}{10} \cdot (\ 80 - 70\ ) = 69$

and variance $\left(1 - 0.6^{2}\right) \cdot 15^{2} = 144$ (standard deviation 12).

$P\,(\,Y > 75 \mid X = 80\,) = P\,(\,Z > \dfrac{75 - 69}{12}\,) = P\,(\,Z > 0.50\,) =$ **0.3085**.



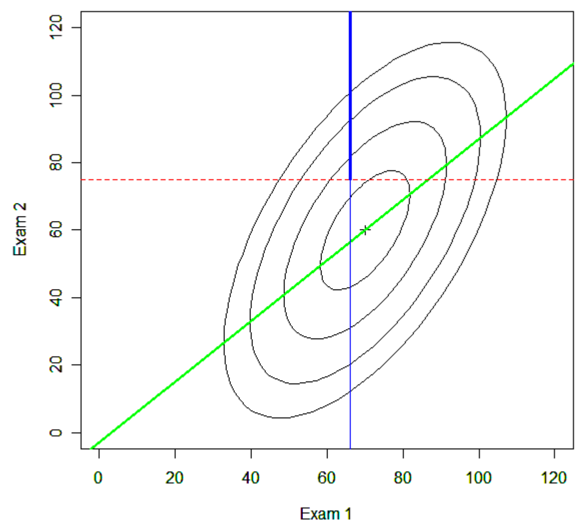
c) Suppose you're told that a student got a 66 on Exam 1. What is the probability that his/her score on Exam 2 is over 75?

Given X = 66, Y has Normal distribution

with mean $60 + 0.6 \cdot \frac{15}{10} \cdot (\ 66 - 70\ ) = 56.4$

and variance $\left(1 - 0.6^2\right) \cdot 15^2 = 144$ (standard deviation 12).

$P\,(\,Y > 75 \mid X = 66\,) = P\,(\,Z > \frac{75 - 56.4}{12}\,) = P\,(\,Z > 1.55\,) =$ **0.0606**.



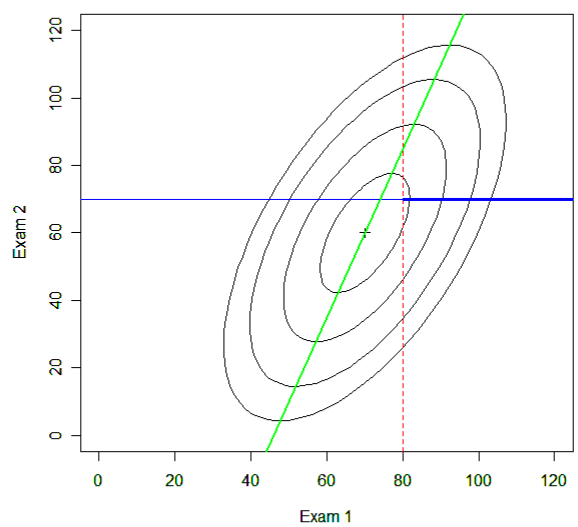
d) Suppose you're told that a student got a 70 on Exam 2. What is the probability that his/her score on Exam 1 is over 80?

Given Y = 70, X has Normal distribution

with mean $70 + 0.6 \cdot \frac{10}{15} \cdot (70 - 60) = 74$

and variance $\left(1 - 0.6^2\right) \cdot 10^2 = 64$ (standard deviation 8).

$P(X > 80 \mid Y = 70) = P(Z > \frac{80 - 74}{8}) = P(Z > 0.75) = \mathbf{0.2266}$.



e) A students is selected at random. What is the probability that the sum of his/her Exam 1 and Exam 2 scores is over 150?

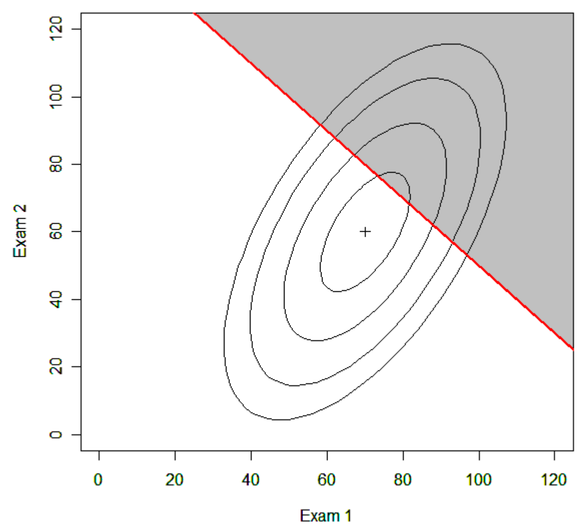
$X + Y$ has Normal distribution,

$E(X+Y) = \mu_X + \mu_Y = 70 + 60 = 130,$

$\mathrm{Var}(X+Y) = \sigma_X^2 + 2\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2$

$= 10^2 + 2 \cdot 0.6 \cdot 10 \cdot 15 + 15^2 = 505$ (standard deviation ≈ 22.4722).

$P(X+Y > 150) = P(Z > \frac{150-130}{22.4722}) = P(Z > 0.89) = \mathbf{0.1867}.$



f) What proportion of students did better on Exam 1 than on Exam 2?

Want $P(X > Y) = P(X - Y > 0) = ?$

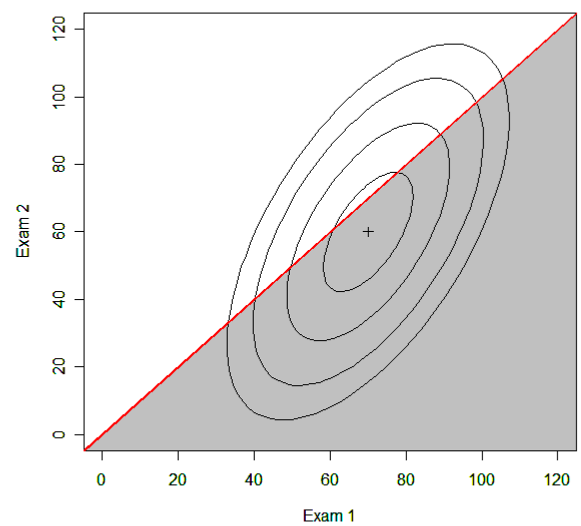
$X - Y$ has Normal distribution,

$E(X - Y) = \mu_X - \mu_Y = 70 - 60 = 10$,

$$\mathrm{Var}(X - Y) = \sigma_X^2 - 2\,\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 - 2\,\rho\,\sigma_X\,\sigma_Y + \sigma_Y^2$$

$$= 10^2 - 2 \cdot 0.6 \cdot 10 \cdot 15 + 15^2 = 145 \quad (\text{standard deviation} \approx 12.0416).$$

$$P(X - Y > 0) = P\left(Z > \frac{0 - 10}{12.0416}\right) = P(Z > -0.83) = \mathbf{0.7967}.$$



g) Find $P(2X + 3Y > 350)$.

$2X + 3Y$ has Normal distribution,

$E(2X + 3Y) = 2\mu_X + 3\mu_Y = 2 \times 70 + 3 \times 60 = 320$,

$\mathrm{Var}(2X + 3Y) = 4\sigma_X^2 + 12\sigma_{XY} + 9\sigma_Y^2 = 4\sigma_X^2 + 12\rho\sigma_X\sigma_Y + 9\sigma_Y^2$

$= 4 \times 10^2 + 12 \cdot 0.6 \cdot 10 \cdot 15 + 9 \times 15^2 = 3505$

(standard deviation ≈ 59.203).

$$P(2X + 3Y > 350) = P\left(Z > \frac{350 - 320}{59.203}\right) = P(Z > 0.5067) \approx \mathbf{0.3050}.$$

h) Find $P(5X + 3Y < 570)$.

$5X + 3Y$ has Normal distribution,

$E(5X + 3Y) = 5\mu_X + 3\mu_Y = 5 \times 70 + 3 \times 60 = 530$,

$\mathrm{Var}(5X + 3Y) = 25\sigma_X^2 + 30\sigma_{XY} + 9\sigma_Y^2 = 25\sigma_X^2 + 30\rho\sigma_X\sigma_Y + 9\sigma_Y^2$

$= 25 \times 10^2 + 30 \cdot 0.6 \cdot 10 \cdot 15 + 9 \times 15^2 = 7225$

(standard deviation $= 85$).

$$P(5X + 3Y < 570) = P\left(Z < \frac{570 - 530}{85}\right) = P(Z < 0.47) = \mathbf{0.6808}.$$

i) Find $P(5X - 4Y > 150)$.

$5X - 4Y$ has Normal distribution,

$E(5X - 4Y) = 5\mu_X - 4\mu_Y = 5 \times 70 - 4 \times 60 = 110$,

$\mathrm{Var}(5X - 4Y) = 25\sigma_X^2 - 40\sigma_{XY} + 16\sigma_Y^2 = 25\sigma_X^2 - 40\rho\sigma_X\sigma_Y + 16\sigma_Y^2$

$= 25 \times 10^2 - 40 \cdot 0.6 \cdot 10 \cdot 15 + 16 \times 15^2 = 2500$

(standard deviation $= 50$).

$$P(5X - 4Y > 150) = P\left(Z > \frac{150 - 110}{50}\right) = P(Z > 0.80) = \mathbf{0.2119}.$$