STAT 410

STAT 400 Review (1)

Fall 2020
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random variables

discrete	continuous
probability **mass** function	probability **density** function
p.m.f.	p.d.f.
$p(x) = \mathrm{P}(\mathrm{X} = x)$	$f(x)$
$\forall x \quad 0 \le p(x) \le 1$	$\forall x \quad f(x) \ge 0$
$\sum\limits_{\text{all } x} p(x) = 1$	$\int\limits_{-\infty}^{\infty} f(x)\,dx = 1$

cumulative distribution function

c.d.f.

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$$

$$\mathrm{F}(x) = \sum_{y \le x} p(y) \qquad\qquad \mathrm{F}(x) = \int_{-\infty}^{x} f(y)\,dy$$

expected value

$$\mathrm{E}(\mathrm{X}) = \mu_{\mathrm{X}}$$

discrete	continuous
If $\sum\limits_{\text{all } x} \lvert x \rvert \cdot p(x) < \infty$,	If $\int\limits_{-\infty}^{\infty} \lvert x \rvert \cdot f(x)\,dx < \infty$,
$\mathrm{E}(\mathrm{X}) = \sum\limits_{\text{all } x} x \cdot p(x)$	$\mathrm{E}(\mathrm{X}) = \int\limits_{-\infty}^{\infty} x \cdot f(x)\,dx$

discrete	continuous
If $\sum\limits_{\text{all } x} \lvert g(x) \rvert \cdot p(x) < \infty$,	If $\int\limits_{-\infty}^{\infty} \lvert g(x) \rvert \cdot f(x)\,dx < \infty$,
$\mathrm{E}(g(\mathrm{X})) = \sum\limits_{\text{all } x} g(x) \cdot p(x)$	$\mathrm{E}(g(\mathrm{X})) = \int\limits_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$

variance

$$\mathrm{Var}(X) = \sigma_X^2 = E([X-\mu_X]^2) = E(X^2) - [E(X)]^2$$

discrete

$$\mathrm{Var}(X) = \sum_{\text{all } x} (x-\mu_X)^2 \cdot p(x)$$

$$= \sum_{\text{all } x} x^2 \cdot p(x) - [E(X)]^2$$

continuous

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f(x)\,dx$$

$$= \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - [E(X)]^2$$

moment-generating function

$$M_X(t) = E(e^{tX})$$

discrete

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot p(x)$$

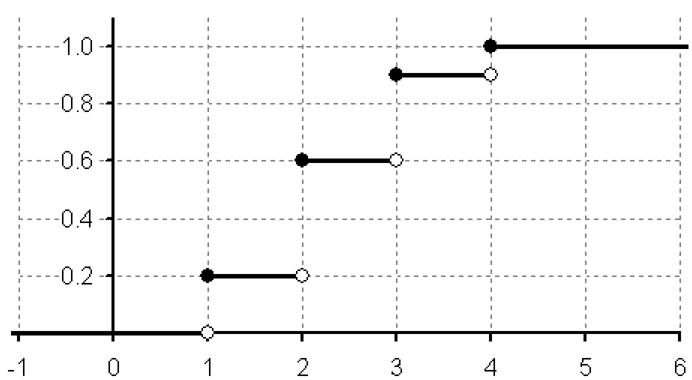
continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

Example 1:

x	$p(x)$	$F(x)$
1	0.2	0.2
2	0.4	0.6
3	0.3	0.9
4	0.1	1.0

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.2 & 1 \le x < 2 \\ 0.6 & 2 \le x < 3 \\ 0.9 & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$



x	$p(x)$	$x \cdot p(x)$
1	0.2	0.2
2	0.4	0.8
3	0.3	0.9
4	0.1	0.4
		2.3

$\mathrm{E}(\mathrm{X}) = \mu_{\mathrm{X}} = 2.3.$

x	$p(x)$	$x^2 \cdot p(x)$
1	0.2	0.2
2	0.4	1.6
3	0.3	2.7
4	0.1	1.6
		6.1

$\mathrm{E}(\mathrm{X}^2) = 6.1$

$\mathrm{Var}(\mathrm{X}) = 6.1 - 2.3^2 = 0.81$

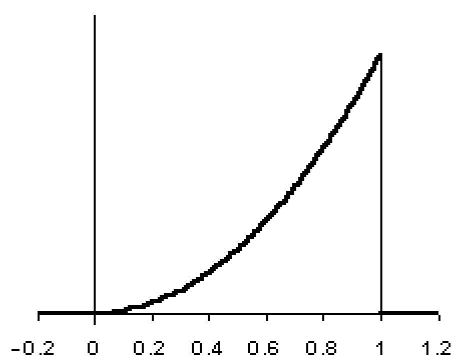
$$\mathrm{M}_{\mathrm{X}}(t) \;=\; \mathrm{E}(e^{t\mathrm{X}}) \;=\; 0.2\,e^{t} + 0.4\,e^{2t} + 0.3\,e^{3t} + 0.1\,e^{4t}$$

<u>Example 2</u>:

Let X be a continuous random variable with the probability density function

$f(x) = k \cdot x^2, \qquad 0 < x < 1,$

$f(x) = 0, \qquad$ otherwise.



a) What must the value of k be so that $f(x)$ is a probability density function?

1) $f(x) \ge 0,$ 2) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^1 k \cdot x^2\,\mathrm{d}x = k \cdot \int_0^1 x^2\,\mathrm{d}x$$

$$= k \cdot \left(\frac{x^3}{3} \right) \Bigg|_0^1 = k \cdot \left(\frac{1}{3} \right) = \frac{k}{3}. \qquad \Rightarrow \qquad k = \mathbf{3}.$$

b) Find the cumulative distribution function $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$.

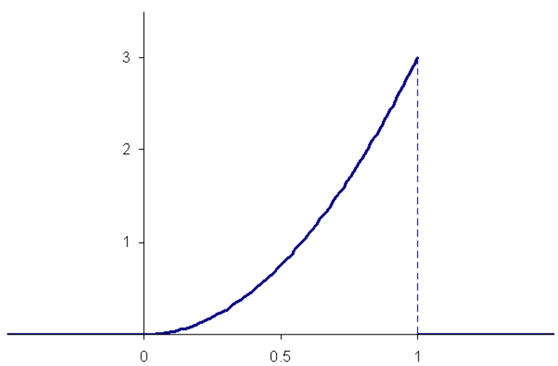
$$f_{\mathrm{X}}(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$x < 0$ $\quad \mathrm{F}_{\mathrm{X}}(x) = 0.$

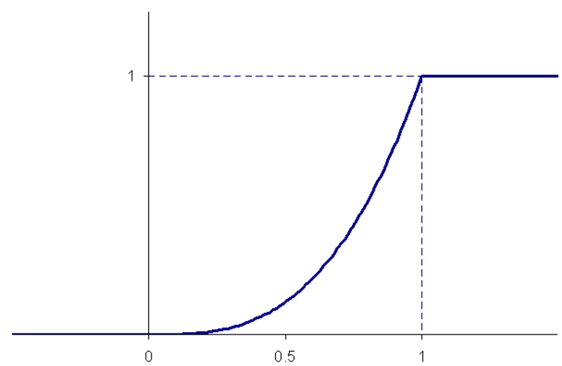
$0 \le x < 1$ $\quad \mathrm{F}_{\mathrm{X}}(x) = \int_0^x 3y^2\,dy = x^3.$

$x \ge 1$ $\quad \mathrm{F}_{\mathrm{X}}(x) = 1.$

$f_{\mathrm{X}}(x)$



$\mathrm{F}_{\mathrm{X}}(x)$

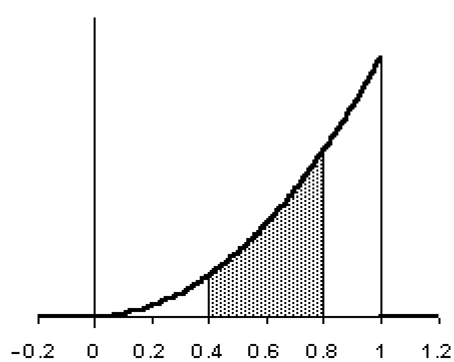


c) Find the probability $\mathrm{P}(\, 0.4 \le \mathrm{X} \le 0.8 \,)$.

$$\mathrm{P}(\, 0.4 \le \mathrm{X} \le 0.8 \,) = \int_{0.4}^{0.8} f(x)\,\mathrm{d}x$$

$$= \int_{0.4}^{0.8} 3 \cdot x^2\,\mathrm{d}x = x^3 \Bigg|_{0.4}^{0.8}$$

$$= 0.8^3 - 0.4^3 = \mathbf{0.448}.$$



OR

$$P(0.4 \le X \le 0.8) = F_X(0.8) - F_X(0.4-) = 0.8^3 - 0.4^3 = \mathbf{0.448}.$$

d) Find the median of the distribution of X.

Need $m = ?$ such that (Area to the left of m) $= \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$.

$$\frac{1}{2} = \int_{-\infty}^{m} f(x)\,dx = \int_{0}^{m} 3 \cdot x^2\,dx = x^3 \Big|_0^m = m^3.$$

$$m = \sqrt[3]{1/2} = \mathbf{0.7937}.$$

e) Find $\mu_X = E(X)$.

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{0}^{1} x \cdot \left(3 \cdot x^2\right) dx = 3 \cdot \int_{0}^{1} x^3\,dx.$$

$$= 3 \cdot \left(\frac{x^4}{4}\right)\Bigg|_0^1 = \frac{\mathbf{3}}{\mathbf{4}} = \mathbf{0.75}.$$

f) Find $\sigma_X = SD(X)$.

$$\operatorname{Var}(X) = \sigma_X^2 = \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - \left(\mu_X\right)^2 = \left[\int_{0}^{1} 3 \cdot x^4\,dx\right] - \left(\frac{3}{4}\right)^2$$

$$= 3 \cdot \left(\frac{x^5}{5}\right)\Bigg|_0^1 - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} = 0.0375.$$

$$\sigma_X = SD(X) = \sqrt{\operatorname{Var}(X)} = \sqrt{0.0375} = \mathbf{0.19365}.$$

g) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx = \int_0^1 e^{tx} \cdot 3x^2\,dx.$$

$$u = 3x^2, \qquad dv = e^{tx}dx,$$

$$du = 6x\,dx, \qquad v = \frac{1}{t}e^{tx}.$$

$$M_X(t) = \int_0^1 e^{tx} \cdot 3x^2\,dx = \left(3x^2 \cdot \frac{1}{t}e^{tx}\right)\Big|_0^1 - \int_0^1 \left(\frac{1}{t}e^{tx} \cdot 6x\right)dx$$

$$= \frac{3}{t}e^t - \int_0^1 \left(\frac{1}{t}e^{tx} \cdot 6x\right)dx$$

$$u = 6x, \qquad dv = = \frac{1}{t}e^{tx}dx,$$

$$du = 6\,dx, \qquad v = \frac{1}{t^2}e^{tx}.$$

$$M_X(t) = \frac{3}{t}e^t - \int_0^1 \left(\frac{1}{t}e^{tx} \cdot 6x\right)dx = \frac{3}{t}e^t - \left(6x \cdot \frac{1}{t^2}e^{tx}\right)\Big|_0^1 - \int_0^1 \left(\frac{1}{t^2}e^{tx} \cdot 6\right)dx$$

$$= \frac{3}{t}e^t - \frac{6}{t^2}e^t + \left(\frac{6}{t^3}e^{tx}\right)\Big|_0^1 = \frac{3}{t}e^t - \frac{6}{t^2}e^t + \frac{6}{t^3}e^t - \frac{6}{t^3}, \qquad t \neq 0.$$

$$M_X(0) = 1.$$

h) Find $E(\sqrt{X})$ and $E(\ln X)$.

$$E(\sqrt{X}) = \int_0^1 \sqrt{x} \cdot 3x^2\,dx = \frac{6}{7}. \qquad E(\ln X) = \int_0^1 \ln x \cdot 3x^2\,dx = -\frac{1}{3}.$$

Example 3:

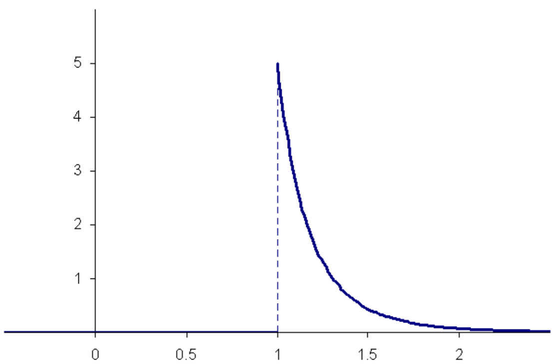
$$f_X(x) = \begin{cases} 5x^{-6} & x > 1 \\ 0 & \text{o.w.} \end{cases}$$

$x < 1$ $\qquad F_X(x) = 0.$

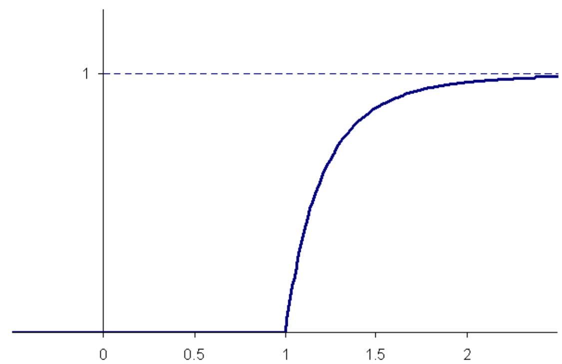
$x \ge 1$ $\qquad F_X(x) = \int_1^x 5y^{-6}\,dy$

$$= -y^{-5}\Big|_1^x = 1 - x^{-5}.$$

$f_X(x)$



$F_X(x)$



$$E(X) = \mu_X = \int_1^\infty x \cdot 5x^{-6}\,dx = \int_1^\infty 5x^{-5}\,dx = \frac{5}{4} = 1.25.$$

$$E(X^2) = \int_1^\infty x^2 \cdot 5x^{-6}\,dx = \int_1^\infty 5x^{-4}\,dx = \frac{5}{3}.$$

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \frac{5}{3} - \left(\frac{5}{4}\right)^2 = \frac{5}{48}.$$

$E(X^{10})$ does NOT exist since $\int_1^\infty x^{10} \cdot 5x^{-6}\,dx$ diverges.

Median: $\qquad F_X(m) = \frac{1}{2}. \qquad 1 - m^{-5} = \frac{1}{2}. \qquad m = \sqrt[5]{2} \approx 1.1487.$

30th percentile: $\qquad F_X(\pi_{0.30}) = 0.30. \qquad 1 - (\pi_{0.30})^{-5} = 0.30.$

$$\pi_{0.30} = \sqrt[5]{\frac{1}{0.70}} \approx 1.07394.$$

Example 4:

Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} k\,x & 2 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

a) Find the value of k that makes this is a valid probability distribution.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{3} k\,x\,dx = \frac{k}{2}\,x^2\Big|_2^3 = \frac{k}{2}\,(9-4) = \frac{5\,k}{2}.$$

$$\Rightarrow \qquad k = \mathbf{\frac{2}{5}}.$$

b) Find the cumulative distribution function $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$.

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = 0, \qquad x < 2.$$

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = \int_{-\infty}^{x} f(y)\,dy = \int_{2}^{x} \frac{2}{5}\,y\,dy = \frac{1}{5}\,y^2\Big|_2^x = \frac{1}{5}\left(x^2 - 4\right), \qquad 2 \le x < 3.$$

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = 1, \qquad x \ge 3.$$

c) Find $\mu_X = \mathrm{E}(\mathrm{X})$.

$$\mu_X = \mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{2}^{3} x \cdot \frac{2}{5}\,x\,dx = \frac{2}{15}\,x^3\Big|_2^3 = \mathbf{\frac{38}{15}} \approx 2.5333.$$

d) Find the median of the probability distribution of X.

$$\mathrm{F}(m) = \frac{1}{2}. \qquad \frac{1}{5}\left(m^2 - 4\right) = \frac{1}{2}. \qquad m = \sqrt{\mathbf{6.5}} \approx 2.5495.$$

Example 5:

(Standard) Cauchy distribution: $f_X(x) = \dfrac{1}{\pi\left(1+x^2\right)}, \quad -\infty < x < \infty.$

Even though $f_X(x)$ is symmetric about zero, $E(X)$ is undefined since

$$\int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi\left(1+x^2\right)} dx = \infty.$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi\left(1+y^2\right)} dy = \frac{1}{\pi} \arctan(x) + \frac{1}{2}, \qquad -\infty < x < \infty.$$

$$P(X < -1) = P(-1 < X < 0) = P(0 < X < 1) = P(X > 1) = 0.25.$$

$M_X(0) = 1.$ $\qquad$ $M_X(t)$ is undefined for all $t \neq 0$.

Theorem 1: $M_{X_1}(t) = M_{X_2}(t)$ for some interval containing 0

$$\Rightarrow \quad f_{X_1}(x) = f_{X_2}(x)$$

Theorem 2: $M_X'(0) = E(X)$ $\qquad$ $M_X''(0) = E(X^2)$

$$M_X^{(k)}(0) = E(X^k)$$

Theorem 3: Let $Y = aX + b$. Then $M_Y(t) = e^{bt} M_X(at)$

Example 6:

Suppose a discrete random variable X has the following probability distribution:

$$P(X=0)=p, \qquad P(X=k)=\frac{1}{2^k \cdot k!}, \quad k=1,2,3,\ldots$$

a) Find the value of p that would make this a valid probability distribution.

Must have $p+\sum_{k=1}^{\infty}\frac{1}{2^k \cdot k!}=1.$

Since $\sum_{k=0}^{\infty}\frac{a^k}{k!}=e^a$, $\sum_{k=1}^{\infty}\frac{1}{2^k \cdot k!}=e^{1/2}-1.$

Therefore, $p+(e^{1/2}-1)=1$ and $p=2-e^{1/2}.$

b) Find $E(X)$.

$$E(X)=\sum_{\text{all } x} x\cdot p(x)=0\cdot\left(2-e^{1/2}\right)+\sum_{k=1}^{\infty}k\cdot\frac{1}{2^k\cdot k!}=\sum_{k=1}^{\infty}\frac{1}{2^k\cdot(k-1)!}$$

$$=\frac{1}{2}\cdot\sum_{k=1}^{\infty}\frac{1}{2^{k-1}\cdot(k-1)!}=\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{1}{2^n\cdot n!}=\frac{e^{1/2}}{2}.$$

c) Find the variance of X, $\mathrm{Var}(X)$.

$$E(X(X-1))=\sum_{k=2}^{\infty}k\cdot(k-1)\cdot\frac{1}{2^k\cdot k!}=\sum_{k=2}^{\infty}\frac{1}{2^k\cdot(k-2)!}$$

$$=\frac{1}{4}\cdot\sum_{k=2}^{\infty}\frac{1}{2^{k-2}\cdot(k-2)!}=\frac{1}{4}\cdot\sum_{n=0}^{\infty}\frac{1}{2^n\cdot n!}=\frac{e^{1/2}}{4}.$$

$$\mathrm{E}(\mathrm{X}^2) = \mathrm{E}(\mathrm{X}(\mathrm{X}-1)) + \mathrm{E}(\mathrm{X}) = \frac{3}{4} \cdot e^{1/2}.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = \frac{3}{4} \cdot e^{1/2} - \frac{1}{4} \cdot e.$$

d) Find the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$.

$$\mathrm{M}_{\mathrm{X}}(t) = \sum_{\text{all } x} e^{tx} \cdot p(x) = 1 \cdot \left(2 - e^{1/2}\right) + \sum_{k=1}^{\infty} e^{tk} \cdot \frac{1}{2^k \cdot k!}$$

$$= \left(2 - e^{1/2}\right) + \sum_{k=1}^{\infty} \frac{\left(e^t/2\right)^k}{k!} = \left(2 - e^{1/2}\right) + \left(e^{e^t/2} - 1\right)$$

$$= 1 - e^{1/2} + e^{e^t/2}.$$

e) Use the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$, to find $\mathrm{E}(\mathrm{X})$.

$$\mathrm{M}_{\mathrm{X}}'(t) = e^{e^t/2} \cdot e^t/2, \qquad \mathrm{E}(\mathrm{X}) = \mathrm{M}_{\mathrm{X}}'(0) = e^{1/2}/2.$$

f) Use the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$, to find the variance of X, $\mathrm{Var}(\mathrm{X})$.

$$\mathrm{M}_{\mathrm{X}}''(t) = e^{e^t/2} \cdot \left(e^t/2\right)^2 + e^{e^t/2} \cdot e^t/2,$$

$$\mathrm{E}(\mathrm{X}^2) = \mathrm{M}_{\mathrm{X}}''(0) = \frac{3}{4} \cdot e^{1/2}.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = \frac{3}{4} \cdot e^{1/2} - \frac{1}{4} \cdot e.$$

Example 7:

Let $a > 2$. Suppose a discrete random variable X has the following probability distribution:

$$p(0) = \mathrm{P}(\mathrm{X} = 0) = c,$$

$$p(k) = \mathrm{P}(\mathrm{X} = k) = \frac{1}{a^k}, \qquad k = 1, 2, 3, \dots .$$

a) Find the value of c (c will depend on a) that makes this is a valid probability distribution.

Must have $\sum\limits_{\text{all } x} p(x) = 1.$ $\qquad c + \sum\limits_{k=1}^{\infty} \frac{1}{a^k} = 1.$

$$\sum_{k=0}^{\infty} b^k = \frac{1}{1-b}, \qquad |b| < 1.$$

$$\sum_{k=1}^{\infty} \frac{1}{a^k} = \left[\sum_{k=0}^{\infty} \frac{1}{a^k} \right] - 1 = \frac{1}{1 - 1/a} - 1 = \frac{1}{a-1}.$$

OR

$$\sum_{k=1}^{\infty} \frac{1}{a^k} = \frac{1}{a} \cdot \sum_{k=0}^{\infty} \frac{1}{a^k} = \frac{1}{a} \cdot \frac{1}{1 - 1/a} = \frac{1}{a-1}.$$

$c + \frac{1}{a-1} = 1.$ $\qquad c = 1 - \frac{1}{a-1} = \frac{a-2}{a-1} = 2 - \frac{a}{a-1}.$

b) Find $\mathrm{P}(\text{odd})$.

$$\mathrm{P}(\text{odd}) = p(1) + p(3) + p(5) + \dots = \frac{1}{a^1} + \frac{1}{a^3} + \frac{1}{a^5} + \dots$$

$$= \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{1}{a}}{1 - \frac{1}{a^2}} = \frac{a}{a^2 - 1}.$$

c) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = E(e^{tX}) = 1 \cdot c + \sum_{k=1}^{\infty} e^{tk} \cdot \frac{1}{a^k}.$$

$$\sum_{k=0}^{\infty} b^k = \frac{1}{1-b}, \qquad |b| < 1.$$

$$\sum_{k=1}^{\infty} e^{tk} \cdot \frac{1}{a^k} = \left[\sum_{k=0}^{\infty} \left(\frac{e^t}{a} \right)^k \right] - 1 = \frac{1}{1 - e^t/a} - 1 = \frac{e^t}{a - e^t} = \frac{a}{a - e^t} - 1.$$

OR

$$\sum_{k=1}^{\infty} e^{tk} \cdot \frac{1}{a^k} = \frac{e^t}{a} \cdot \sum_{k=0}^{\infty} \left(\frac{e^t}{a} \right)^k = \frac{e^t}{a} \cdot \frac{1}{1 - e^t/a} = \frac{e^t}{a - e^t} = \frac{a}{a - e^t} - 1.$$

Need $\left| \frac{e^t}{a} \right| < 1. \qquad \Rightarrow \qquad t < \ln a.$

$$M_X(t) = \frac{a-2}{a-1} + \frac{e^t}{a - e^t} = \frac{a}{a - e^t} - \frac{1}{a-1}, \qquad t < \ln a.$$

d) Find $\mathrm{E}(\mathrm{X})$.

$\mathrm{E}(\mathrm{X}) = \mathrm{M}_{\mathrm{X}}'(0)$.

$$\mathrm{M}_{\mathrm{X}}'(t) = \frac{d}{dt}\left(\frac{a-2}{a-1} + \frac{e^t}{a-e^t}\right) = \frac{e^t\left(a-e^t\right) - e^t\left(-e^t\right)}{\left(a-e^t\right)^2} = \frac{a \cdot e^t}{\left(a-e^t\right)^2}.$$

OR

$$\mathrm{M}_{\mathrm{X}}'(t) = \frac{d}{dt}\left(\frac{a}{a-e^t} - \frac{1}{a-1}\right) = -\frac{a}{\left(a-e^t\right)^2} \cdot \left(-e^t\right) = \frac{a \cdot e^t}{\left(a-e^t\right)^2}.$$

$$\mathrm{E}(\mathrm{X}) = \mathrm{M}_{\mathrm{X}}'(0) = \frac{a}{(a-1)^2}.$$

OR

$$\mathrm{E}(\mathrm{X}) = \sum_{k=1}^{\infty} k \cdot \frac{1}{a^k} = \frac{1}{a} \cdot \frac{a}{a-1} \cdot \left[\sum_{k=1}^{\infty} k \cdot \left(\frac{1}{a}\right)^{k-1} \cdot \frac{a-1}{a}\right] = \frac{1}{a-1} \cdot \mathrm{E}(\mathrm{Y}),$$

where Y is a Geometric random variable with probability of "success" $\frac{a-1}{a}$.

$$\mathrm{E}(\mathrm{Y}) = \frac{1}{p} = \frac{a}{a-1}.$$

Therefore, $\mathrm{E}(\mathrm{X}) = \dfrac{a}{(a-1)^2}$.

OR

$$\mathrm{E}(\mathrm{X}) = 1\cdot\frac{1}{a^1}+2\cdot\frac{1}{a^2}+3\cdot\frac{1}{a^3}+4\cdot\frac{1}{a^4}+5\cdot\frac{1}{a^5}+6\cdot\frac{1}{a^6}+\ldots$$

$$\frac{1}{a}\cdot\mathrm{E}(\mathrm{X}) = \qquad 1\cdot\frac{1}{a^2}+2\cdot\frac{1}{a^3}+3\cdot\frac{1}{a^4}+4\cdot\frac{1}{a^5}+5\cdot\frac{1}{a^6}+\ldots$$

$$\Rightarrow \quad \left(1-\frac{1}{a}\right)\cdot\mathrm{E}(\mathrm{X}) = \frac{1}{a^1}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\frac{1}{a^5}+\frac{1}{a^6}+\ldots = \sum_{k=1}^{\infty}\frac{1}{a^k} = \frac{1}{a-1}.$$

Therefore, $\mathrm{E}(\mathrm{X}) = \dfrac{a}{(a-1)^2}$.

e) Find the cumulative distribution function $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$.

If $x < 0$, $\quad \mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = 0.$

If $k = 0, 1, 2, 3, \ldots$,

$$1-\mathrm{F}(k) = \mathrm{P}(\mathrm{X} > k) = \sum_{n=k+1}^{\infty}\frac{1}{a^n} = \frac{1}{a^{k+1}}\cdot\frac{1}{1-1/a} = \frac{1}{a^k(a-1)}$$

$$\Rightarrow \quad \mathrm{F}(k) = 1-\frac{1}{a^k(a-1)}.$$

Since X is a discrete integer-valued random variable, if $k \le x < k+1$,

$$\mathrm{F}(x) = \mathrm{F}(k) = 1-\frac{1}{a^k(a-1)}.$$

Therefore,

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ 1-\dfrac{1}{a^k(a-1)} & k \le x < k+1 \qquad k = 0, 1, 2, 3, \ldots \end{cases}$$

Example 8:

Let $\lambda > 0$. Suppose the probability density function of X is $f_X(x) = \lambda\, e^{-\lambda x}$, $x > 0$.
(Exponential distribution.)

a) Find the moment-generating function of X.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = \int_{0}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \cdot \int_{0}^{\infty} e^{x(t-\lambda)}\,dx = \lambda \cdot \left(\frac{e^{x(t-\lambda)}}{t-\lambda} \right) \Bigg|_{0}^{\infty} = \frac{\lambda}{\lambda - t}, \qquad t < \lambda.$$

b) Use the moment-generating function of X to find E(X).

$$M_X'(t) = \frac{\lambda}{(\lambda - t)^2}, \quad t < \lambda. \qquad\qquad E(X) = M_X'(0) = \frac{1}{\lambda}.$$

Example 9:

Let X be a discrete Binomial(n, p) random variable. That is, suppose the p.m.f. of X is $p_X(k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$, $k = 0, 1, 2, \ldots, n$.

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot \left(p \cdot e^t\right)^k \cdot (1-p)^{n-k} = [(1-p) + p\,e^t]^n.$$

Example 10:

Let X be a discrete Geometric(p) random variable. That is, suppose the probability mass function of X is $p_X(x) = (1-p)^{x-1}\, p, \quad x = 1, 2, 3, \ldots$.

a) Find the moment-generating function of X.

$$M_X(t) = \sum_{k=1}^{\infty} e^{tk} \cdot (1-p)^{k-1} \cdot p = p \cdot e^t \cdot \sum_{k=1}^{\infty} e^{t(k-1)} \cdot (1-p)^{k-1}$$

$$= p \cdot e^t \cdot \sum_{n=0}^{\infty} \left[(1-p) \cdot e^t\right]^n = \frac{p \cdot e^t}{1-(1-p) \cdot e^t}, \quad t < -\ln(1-p).$$

b) Use the moment-generating function of X to find $E(X)$.

$$M_X'(t) = \frac{p \cdot e^t \cdot \left(1-(1-p) \cdot e^t\right) - p \cdot e^t \cdot \left(-(1-p) \cdot e^t\right)}{\left(1-(1-p) \cdot e^t\right)^2}$$

$$= \frac{p \cdot e^t}{\left(1-(1-p) \cdot e^t\right)^2}, \qquad t < -\ln(1-p).$$

$$E(X) = M_X'(0) = \frac{p}{(p)^2} = \frac{1}{p}.$$

Example 11:

Let X be a random variable distributed uniformly over the interval $[a, b]$.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \cdot \left(\frac{e^{tx}}{t}\right)\Bigg|_a^b = \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \neq 0.$$

$$M_X(0) = 1.$$

Example 12:

Let X be a Poisson(λ) random variable. That is,

$$P(X=k)=\frac{\lambda^k\cdot e^{-\lambda}}{k!}, \qquad k=0,1,2,3,\ldots .$$

a) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk}\cdot\frac{\lambda^k\cdot e^{-\lambda}}{k!} = e^{-\lambda}\cdot\sum_{k=0}^{\infty}\frac{\left(\lambda\cdot e^t\right)^k}{k!}$$

$$= e^{-\lambda}\cdot e^{\lambda\cdot e^t} = e^{\lambda\cdot(e^t-1)}.$$

1.9.17 in the 7th edition **(1.9.16** in the 6th edition**)**

Let $\psi(t)=\ln M(t)$, where $M(t)$ is the m.g.f. of a distribution.

Prove that $\psi_X{}'(0)=\mu$ and $\psi_X{}''(0)=\sigma^2$. The function $\psi(t)$ is

called the **cumulant generating function**.

$$(\ln M_X(t))' = \frac{M_X'(t)}{M_X(t)} \qquad\qquad (\ln M_X(t))'' = \frac{M_X''(t)\cdot M_X(t)-\left[M_X'(t)\right]^2}{[M_X(t)]^2}$$

Since $M_X(0)=1$, $M_X{}'(0)=E(X)$, $M_X{}''(0)=E(X^2)$,

$$\psi_X{}'(0) = (\ln M_X(t))'\big|_{t=0} = E(X) = \mu_X$$

$$\psi_X{}''(0) = (\ln M_X(t))''\big|_{t=0} = E(X^2)-[E(X)]^2 = \mathrm{Var}(X) = \sigma_X^2$$

b) Find $E(X)$ and $\mathrm{Var}(X)$.

$\ln M_X(t)=\lambda\,(e^t-1)$.

$(\ln M_X(t))'=\lambda\,e^t.$ $\qquad (\ln M_X(t))'\big|_{t=0}=E(X)=\lambda.$

$(\ln M_X(t))''=\lambda\,e^t.$ $\qquad (\ln M_X(t))''\big|_{t=0}=\mathrm{Var}(X)=\lambda.$

Example 13:

Let Y denote a random variable with probability density function given by

$$f(y) = \frac{1}{2} e^{-|y|}, \qquad -\infty < y < \infty. \qquad (\text{double exponential p.d.f.})$$

a) Find the moment-generating function of Y. For which values of t does it exist?

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} \cdot \frac{1}{2} e^{-|y|} dy = \int_{-\infty}^{0} e^{ty} \cdot \frac{1}{2} e^{-|y|} dy + \int_{0}^{\infty} e^{ty} \cdot \frac{1}{2} e^{-|y|} dy$$

$$= \int_{-\infty}^{0} e^{ty} \cdot \frac{1}{2} e^{y} dy + \int_{0}^{\infty} e^{ty} \cdot \frac{1}{2} e^{-y} dy$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{y(t+1)} dy + \frac{1}{2} \int_{0}^{\infty} e^{y(t-1)} dy$$

Note that the first integral converges only if $t + 1 > 0$,

and the second integral converges only if $t - 1 < 0$.

Therefore, the moment-generating function is only defined for $-1 < t < 1$.

$$M_Y(t) = \frac{1}{2(t+1)} e^{y(t+1)} \Big|_{-\infty}^{0} + \frac{1}{2(t-1)} e^{y(t-1)} \Big|_{0}^{\infty} = \frac{1}{2(t+1)} - \frac{1}{2(t-1)}$$

$$= \frac{(t-1)-(t+1)}{2(t+1)(t-1)} = \frac{-2}{2\left(t^2-1\right)} = \frac{1}{1-t^2}, \qquad -1 < t < 1.$$

b) Find E(Y).

$$M_Y'(t) = -(1-t^2)^{-2}(-2t) = 2t(1-t^2)^{-2}$$

$$\Rightarrow \qquad E(Y) = M_Y'(0) = \mathbf{0}.$$

OR

$$E(Y) = \int_{-\infty}^{\infty} y \cdot \frac{1}{2} e^{-|y|} dy = \mathbf{0}, \qquad \text{since } y \cdot \frac{1}{2} e^{-|y|} \text{ is an odd function.}$$

c) Find $\operatorname{Var}(Y)$.

$$M_Y''(t) = 2(1-t^2)^{-2} + 2t(-2)(1-t^2)^{-3}(-2t) = \frac{2+6t^2}{\left(1-t^2\right)^3}$$

$$\Rightarrow \quad E(Y^2) = M_Y''(0) = 2. \qquad \Rightarrow \quad \operatorname{Var}(Y) = 2 - 0^2 = \mathbf{2}.$$

OR

$$\operatorname{Var}(Y) = E(Y^2) - [E(Y)]^2 = \int_{-\infty}^{\infty} y^2 \cdot \frac{1}{2} e^{-|y|}\, dy = \ldots = \mathbf{2}.$$

d) Find the cumulative distribution function $F(y) = P(Y \le y)$.

If $y < 0$, $\quad F(y) = \int_{-\infty}^{y} \frac{1}{2} e^{-|x|}\, dx = \int_{-\infty}^{y} \frac{1}{2} e^{x}\, dx = \frac{1}{2} e^{y}.$

If $y \ge 0$, $\quad F(y) = \int_{-\infty}^{y} \frac{1}{2} e^{-|x|}\, dx = \int_{-\infty}^{0} \frac{1}{2} e^{x}\, dx + \int_{0}^{y} \frac{1}{2} e^{-x}\, dx$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - e^{-y}\right) = 1 - \frac{1}{2} e^{-y}.$$

Therefore,

$$F(y) = \begin{cases} \frac{1}{2} e^{y} & y < 0 \\ 1 - \frac{1}{2} e^{-y} & y \ge 0 \end{cases}$$

e) Find $E(Y^k)$ for positive integer k.

$$E(Y^k) = \int_{-\infty}^{\infty} y^k \cdot \frac{1}{2} e^{-|y|}\, dy = \int_{-\infty}^{0} y^k \cdot \frac{1}{2} e^{y}\, dy + \int_{0}^{\infty} y^k \cdot \frac{1}{2} e^{-y}\, dy = \ldots$$

k odd $\qquad \ldots = 0.$

k even $\qquad \ldots = 2 \cdot \int_{0}^{\infty} y^k \cdot \frac{1}{2} e^{-y}\, dy = \int_{0}^{\infty} y^k \cdot e^{-y}\, dy = \Gamma(k+1) = k!.$

OR

Taylor Formula:

$$M_Y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} M_Y^{(k)}(0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(Y^k).$$

On the other hand,

$$M_Y(t) = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}.$$

$\Rightarrow$ If k odd, $\qquad E(Y^k) = 0.$

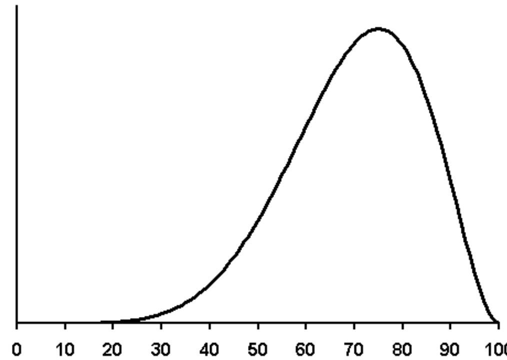
If k even, $\qquad k = 2n, \qquad E(Y^k) = k!.$

Example 14:

A simple model for describing mortality in the general population in a particular country is given by the probability density function

$$f(y) = \frac{252}{10^{18}}\ y^6\,(100-y)^2,$$

$$0 < y < 100.$$



a) Verify that $f(y)$ is a valid probability density function.

1. $f(y) \ge 0$ for each y; ✓

2. $\int\limits_{-\infty}^{\infty} f(y)\,dy = 1.$

$$\int\limits_{-\infty}^{\infty} f(y)\,dy = \int\limits_{0}^{100} \frac{252}{10^{18}}\ y^6\,(100-y)^2\,dy = \int\limits_{0}^{1} 252\ x^6\,(1-x)^2\,dx$$

$$= 252 \cdot \left[\frac{1}{7}x^7 - 2\cdot\frac{1}{8}x^8 + \frac{1}{9}x^9\right]\Bigg|_0^1 = 252 \cdot \frac{2}{504} = 1. \quad ✓$$

b) Based on this model, which event is more likely

A: a person dies between the ages of 70 and 80

or

B: a person lives past age 80?

A: $$\int\limits_{70}^{80} \frac{252}{10^{18}}\ y^6\,(100-y)^2\,dy = \int\limits_{0.7}^{0.8} 252\ x^6\,(1-x)^2\,dx$$

$$= 252 \cdot \left[\frac{1}{7}x^7 - 2\cdot\frac{1}{8}x^8 + \frac{1}{9}x^9\right]\Bigg|_{0.7}^{0.8}$$

$$\approx 0.7382 - 0.4628 = 0.2754.$$

B: $$\int_{80}^{100} \frac{252}{10^{18}}\, y^6 (100-y)^2\, dy = \int_{0.8}^{1.0} 252\, x^6 (1-x)^2\, dx$$

$$= 252 \cdot \left[\frac{1}{7} x^7 - 2 \cdot \frac{1}{8} x^8 + \frac{1}{9} x^9 \right] \Bigg|_{0.8}^{1.0}$$

$$\approx 1 - 0.7382 = 0.2618.$$

A is more likely.

c) Given that a randomly selected individual just celebrated his 60th birthday, find the probability that he will live past age 80.

$$\text{P}(\text{ over } 80 \mid \text{over } 60\,) = \frac{\text{P}(\text{ over } 80 \cap \text{over } 60\,)}{\text{P}(\text{ over } 60\,)} = \frac{\int_{80}^{100} \frac{252}{10^{18}}\, y^6 (100-y)^2\, dy}{\int_{60}^{100} \frac{252}{10^{18}}\, y^6 (100-y)^2\, dy}$$

$$\approx \frac{1-0.7382}{1-0.2318} = \frac{0.2618}{0.7682} \approx \mathbf{0.3408}.$$

d) Find the value of y that maximizes $f(y)$ (**mode**).

$$f'(y) = \frac{252}{10^{18}} \left[6\, y^5 (100-y)^2 - 2\, y^6 (100-y) \right]$$

$$= \frac{252}{10^{18}}\, y^5 (100-y) [\, 6(100-y) - 2\, y\,]$$

$$= \frac{252}{10^{18}}\, y^5 (100-y) [\, 600 - 8\, y\,] = 0.$$

$\Rightarrow$ $y = 0,\ y = 100$ (not max), $y = \mathbf{75}$ years (max).

e) Find the (average) life expectancy.

$$\mathrm{E}(\mathrm{Y}) = \int_{-\infty}^{\infty} y \cdot f(y)\, dy = \int_{0}^{100} \frac{252}{10^{18}}\ y^{7}\,(100-y)^{2}\ dy = \int_{0}^{1} 252 \cdot 100\ x^{7}\,(1-x)^{2}\ dx$$

$$= 252 \cdot 100 \cdot \left[\frac{1}{8} x^{8} - 2 \cdot \frac{1}{9} x^{9} + \frac{1}{10} x^{10} \right] \Bigg|_{0}^{1} = 252 \cdot 100 \cdot \frac{2}{720} = \mathbf{70} \text{ years.}$$

OR

Consider $\mathrm{X} = \dfrac{\mathrm{Y}}{100}$. Then $\mathrm{Y} = 100\,\mathrm{X}$, and X has the probability density function

$$f(x) = 252\ x^{6}\,(1-x)^{2}, \quad 0 < x < 1.$$

Then X has Beta distribution with $\alpha = 7$ and $\beta = 3$.

$$\mathrm{E}(\mathrm{X}) = \frac{\alpha}{\alpha+\beta} = \frac{7}{7+3} = 0.70. \qquad \mathrm{E}(\mathrm{Y}) = 100\,\mathrm{E}(\mathrm{X}) = \mathbf{70} \text{ years.}$$

f) Find the standard deviation of the lifetimes.

$$\mathrm{Var}(\mathrm{X}) = \frac{7 \cdot 3}{11 \cdot 10^{2}} = \frac{21}{1100}. \qquad \mathrm{Var}(\mathrm{Y}) = 100^{2}\,\mathrm{Var}(\mathrm{X}) = \frac{2100}{11}.$$

$$\mathrm{SD}(\mathrm{Y}) \approx 13.817.$$

OR

$$\mathrm{E}(\mathrm{Y}^{2}) = \int_{-\infty}^{\infty} y^{2} \cdot f(y)\, dy = \int_{0}^{100} \frac{252}{10^{18}}\ y^{8}\,(100-y)^{2}\ dy = \ldots$$

$$\mathrm{Var}(\mathrm{Y}) = \mathrm{E}(\mathrm{Y}^{2}) - [\mathrm{E}(\mathrm{Y})]^{2} = \ldots$$

$$\mathrm{SD}(\mathrm{Y}) = \sqrt{\mathrm{Var}(\mathrm{Y})} = \ldots$$

Example 15:

Suppose a random variable X has the following probability density function:

$$f(x) = \begin{cases} C \cdot e^{-x} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) What must the value of C be so that $f(x)$ is a probability density function?

For $f(x)$ to be a probability density function, we must have:

1) $f(x) \ge 0$, 2) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^1 C \cdot e^{-x}\,\mathrm{d}x = C \cdot \int_0^1 e^{-x}\,\mathrm{d}x$$

$$= C \cdot \left(-e^{-x}\right)\Big|_0^1 = C \cdot \left(1 - e^{-1}\right) = C \cdot \left(\frac{e-1}{e}\right).$$

Therefore, $C = \left(\frac{e}{e-1}\right) \approx \mathbf{1.5819767}$.

$$f(x) = \begin{cases} \left(\frac{e}{e-1}\right) \cdot e^{-x} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

b) Find $\mu_X = \mathrm{E}(X)$.

$$\mu_X = \mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x = \int_0^1 x \cdot \left(\frac{e}{e-1}\right) \cdot e^{-x}\,\mathrm{d}x = \left(\frac{e}{e-1}\right) \cdot \int_0^1 x \cdot e^{-x}\,\mathrm{d}x.$$

Integrating by parts,

$$\int_0^1 x \cdot e^{-x}\,\mathrm{d}x = \left[(x) \cdot \left(-e^{-x}\right)\right]\Big|_0^1 - \int_0^1 \left(-e^{-x}\right)\mathrm{d}x$$

$$= -e^{-1} + \int_0^1 e^{-x}\,\mathrm{d}x = -e^{-1} + \left(-e^{-x}\right)\Big|_0^1 = 1 - 2 \cdot e^{-1} = \frac{e-2}{e}.$$

Therefore,

$$\mu_X = E(X) = \left(\frac{e}{e-1}\right)\cdot\int_0^1 x\cdot e^{-x}\,dx = \left(\frac{e}{e-1}\right)\cdot\left(\frac{e-2}{e}\right) = \frac{e-2}{e-1} \approx \mathbf{0.418}.$$

c) Find the cumulative distribution function $F(x) = P(X \le x)$.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy.$$

$F(x) = 0$ for $x < 0$.

$F(x) = 1$ for $x > 1$.

For $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_0^x \left(\frac{e}{e-1}\right)\cdot e^{-y}\,dy = \left(\frac{e}{e-1}\right)\cdot\left(-e^{-y}\right)\Big|_0^x$$

$$= \left(\frac{e}{e-1}\right)\cdot\left(1-e^{-x}\right)$$

$$= \left(\frac{e}{e-1}\right)\cdot\left(\frac{e^x-1}{e^x}\right).$$

$$F(x) = \begin{cases} 0 & x<0 \\ \left(\frac{e}{e-1}\right)\cdot\left(\frac{e^x-1}{e^x}\right) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

d) Find the median of the probability distribution of X.

Need $m = ?$ such that $P(X \le m) = P(X \ge m) = 1/2$.

Thus, $1/2 = F(m) = \left(\frac{e}{e-1}\right)\cdot\left(\frac{e^m-1}{e^m}\right)$.

$\Rightarrow \quad e^m - 1 = \left(\frac{e-1}{2\cdot e}\right)\cdot e^m. \qquad \Rightarrow \quad e^m - \left(\frac{e-1}{2\cdot e}\right)\cdot e^m = 1.$

$\Rightarrow \quad \left(\frac{e+1}{2\cdot e}\right)\cdot e^m = 1. \qquad \Rightarrow \quad e^m = \frac{2\cdot e}{e+1}.$

$\Rightarrow \quad m = \ln\left(\frac{2\cdot e}{e+1}\right) \approx \mathbf{0.3799}.$

e) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx} \cdot \left(\frac{e}{e-1}\right) \cdot e^{-x}\, dx = \left(\frac{e}{e-1}\right) \cdot \int_0^1 e^{(t-1)x}\, dx$$

$$= \left(\frac{e}{e-1}\right) \cdot \left(\frac{1}{t-1} \cdot e^{(t-1)x}\right)\Bigg|_0^1 = \left(\frac{e}{e-1}\right) \cdot \frac{1}{t-1} \cdot \left(e^{t-1} - 1\right)$$

$$= \frac{e^t - e}{(e-1)\cdot(t-1)}, \qquad t \neq 1.$$

$$M_X(1) = \frac{e}{e-1}.$$

f) Find $E(2^X)$.

$$E(2^X) = E(e^{\ln 2 \cdot X}) = M_X(\ln 2) = \frac{e^{\ln 2} - e}{(e-1)\cdot(\ln 2 - 1)} = \frac{e-2}{(e-1)\cdot(1 - \ln 2)}.$$