STAT 410

Examples for 10/12/2020 (2)

Fall 2020
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1. Suppose X follows a Gamma distribution with mean $\mu = 20$ and standard deviation $\sigma = 10$.

a) What are the parameters of this Gamma distribution, α and θ?

b) Find $P(X \le 25)$.

Suggestion: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer,

then $F_T(t) = P(T \le t) = P(X_t \ge \alpha)$ and $P(T > t) = P(X_t \le \alpha - 1)$,

where X_t has a $\text{Poisson}(\lambda t = t/\theta)$ distribution.

c) Find $P(10 \le X \le 30)$.

2. During a radio trivia contest, the radio station receives phone calls according to Poisson process with the average rate of five calls per minute.

a) Find the probability that we would have to wait less than two minutes for the ninth phone call.

b) Find the probability that the ninth phone call would arrive during the third minute.

3. Dementor attacks in Champaign County are relatively rare. Suppose they occur according to a Poisson process with the average rate of 0.4 attacks per month.

a) Find the probability that at most 2 dementor attacks occur in Champaign County in three months.

b) Find the probability that exactly 3 dementor attacks occur in Champaign County in one year. (1 year = 12 months)

c) Find the probability that there will be exactly 9 attack-free months (months without any dementor attacks) in one year.

d) Find the probability that the first dementor attack of a calendar year occurs in March. That is, find $P(2 < T_1 < 3)$.

e) Find the probability that the first dementor attack of a calendar year occurs before summer. That is, find $P(T_1 < 5)$.

"Hint": Spring: March, April, May. Summer: June, July August.

f) Find the probability that the third dementor attack of a calendar year occurs during summer.

g) Find the probability that the fifth dementor attack of a calendar year occurs during summer.

h) Find the probability that the fifth dementor attack of a calendar year occurs after Halloween (October 31). That is, find the probability that the fifth dementor attack of a calendar year occurs during the last two month of the calendar year.

1. Suppose X follows a Gamma distribution with mean $\mu = 20$ and standard deviation $\sigma = 10$.

a) What are the parameters of this Gamma distribution, α and θ?

$\mu = 20 = \alpha\,\theta.$ $\sigma^2 = 100 = \alpha\,\theta^2.$

$\Rightarrow$ $\alpha = \mathbf{4},$ $\theta = \mathbf{5}.$

b) Find $P(X \le 25)$.

Suggestion: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_T(t) = P(T \le t) = P(X_t \ge \alpha)$ and $P(T > t) = P(X_t \le \alpha - 1)$, where X_t has a $\text{Poisson}(\lambda t = t/\theta)$ distribution.

$$P(X \le 25) = P(T_4 \le 25) = \int_0^{25} \frac{1}{\Gamma(4) \cdot 5^4} \cdot t^{4-1} \cdot e^{-t/5}\, dt$$

$$= \int_0^{25} \frac{1}{3! \cdot 5^4} \cdot t^3 \cdot e^{-t/5}\, dt = \ldots$$

OR

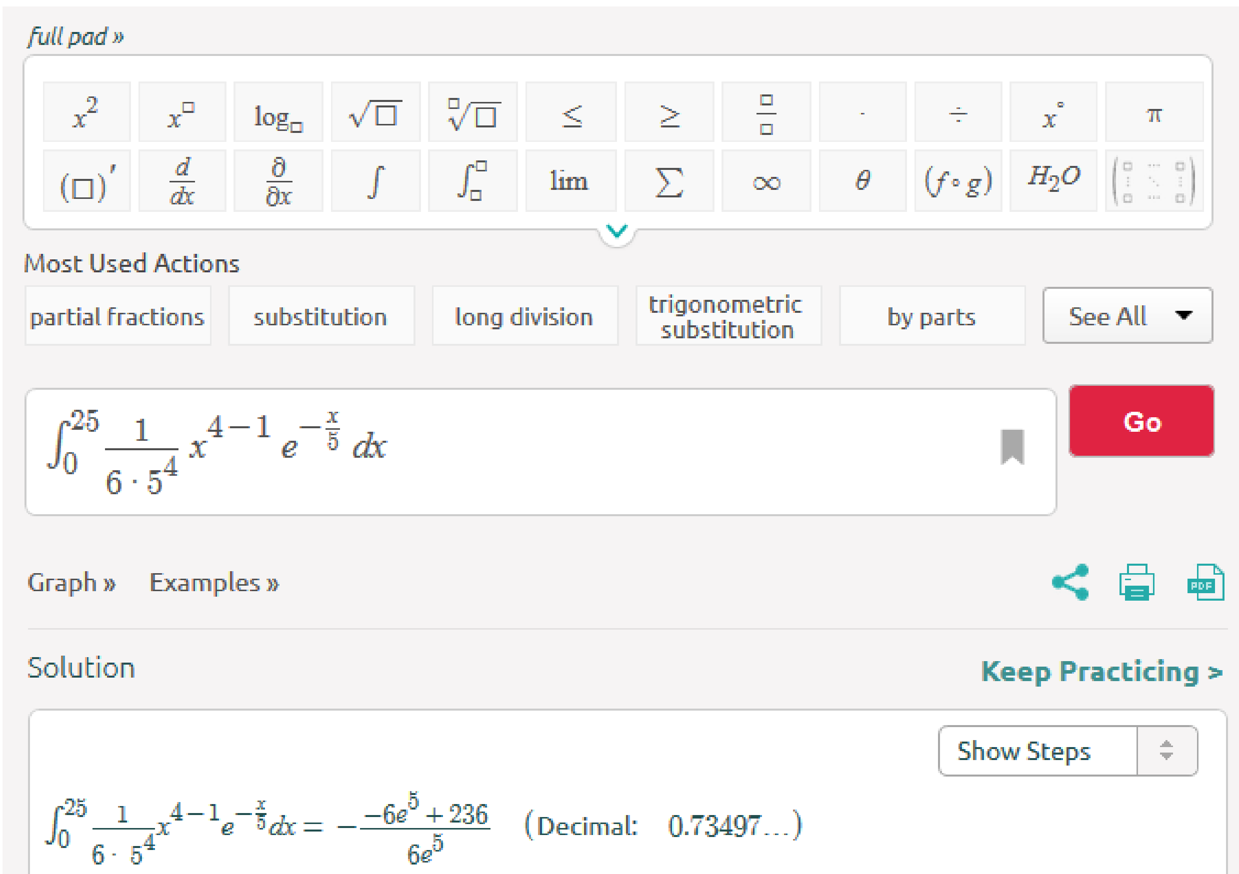
$$P(X \le 25) = P(T_4 \le 25) = P(X_{25} \ge 4) = 1 - P(X_{25} \le 3)$$

$$= 1 - P(\text{Poisson}(5) \le 3) = 1 - 0.265 = \mathbf{0.735}.$$

OR (Spoiler)

$$P(X \le 25) = P(T_4 \le 25) = P\left(\frac{2\,T_4}{5} \le 10\right) = P(\chi^2(8) \le 10) = \ldots$$

```
> pgamma(25,4,1/5)
[1] 0.7349741
> 1-ppois(3,5)
[1] 0.7349741
>
> ### spoiler
> pchisq(2*25/5,2*4)
[1] 0.7349741
```



=GAMMA.DIST(25,4,5,1) 0.734974

=1-POISSON.DIST(3,5,1) 0.734974

Spoiler:

=CHISQ.DIST(2*25/5,2*4,1) 0.734974

c) Find $P(10 \le X \le 30)$.

$$P(10 \le X \le 30) = P(10 \le T_4 \le 30) = \int_{10}^{30} \frac{1}{\Gamma(4) \cdot 5^4} \cdot t^{4-1} \cdot e^{-t/5}\, dt$$

$$= \int_{10}^{30} \frac{1}{3! \cdot 5^4} \cdot t^3 \cdot e^{-t/5}\, dt = \ldots$$

OR

$$P(10 \le X \le 30) = P(10 \le T_4 \le 30) = P(T_4 \ge 10) - P(T_4 > 30)$$

$$= P(X_{10} \le 3) - P(X_{30} \le 3) = P(\text{Poisson}(2) \le 3) - P(\text{Poisson}(6) \le 3)$$

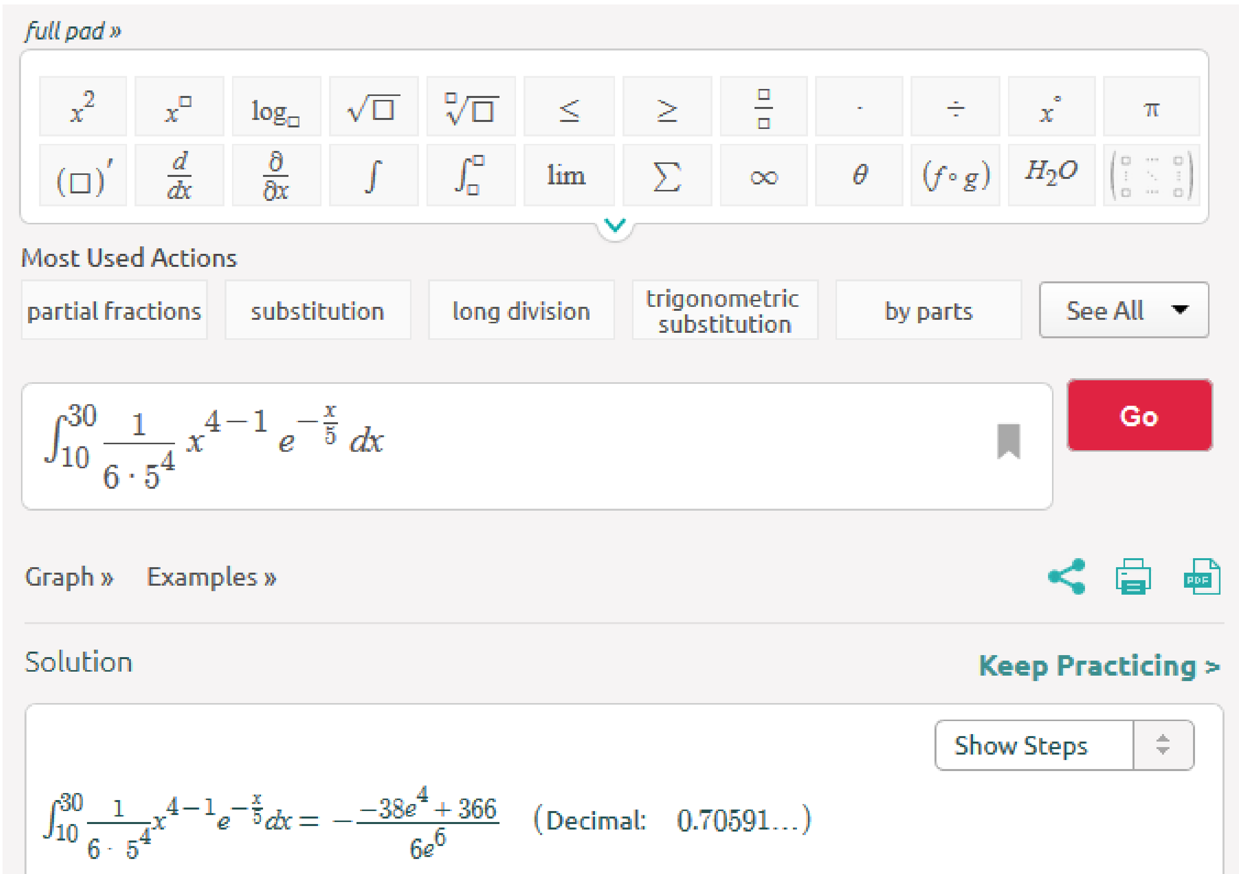
$$= 0.857 - 0.151 = \mathbf{0.706}.$$

OR (Spoiler)

$$P(10 \le X \le 30) = P(10 \le T_4 \le 30) = P\left(4 \le \frac{2\,T_4}{5} \le 12\right)$$

$$= P(4 \le \chi^2(8) \le 12) = \ldots$$

```
> pgamma(30,4,1/5)-pgamma(10,4,1/5)
[1] 0.7059196
> ppois(3,2)-ppois(3,6)
[1] 0.7059196
>
> ### spoiler
> pchisq(2*30/5,2*4)-pchisq(2*10/5,2*4)
[1] 0.7059196
```



=GAMMA.DIST(30,4,5,1)-GAMMA.DIST(10,4,5,1)	0.70592
=POISSON.DIST(3,2,1)-POISSON.DIST(3,6,1)	0.70592

2. During a radio trivia contest, the radio station receives phone calls according to Poisson process with the average rate of five calls per minute.

X_t = number of phone calls in t minutes. Poisson(λt)

T_k = time of the kth phone call. Gamma, $\alpha = k$.

five calls per minute $\Rightarrow$ $\lambda = 5$.

a) Find the probability that we would have to wait less than two minutes for the ninth phone call.

$$P(T_9 < 2) = P(X_2 \ge 9) = 1 - P(X_2 \le 8) = 1 - P(\text{Poisson}(10) \le 8) = 1 - 0.333 = \mathbf{0.667}.$$

OR

$$P(T_9 < 2) = \int_0^2 \frac{5^9}{\Gamma(9)} t^{9-1} e^{-5t}\, dt = \int_0^2 \frac{5^9}{8!} t^8 e^{-5t}\, dt = \ldots$$

b) Find the probability that the ninth phone call would arrive during the third minute.

$$P(2 < T_9 < 3) = P(T_9 > 2) - P(T_9 > 3) = P(X_2 \le 8) - P(X_3 \le 8)$$
$$= P(\text{Poisson}(10) \le 8) - P(\text{Poisson}(15) \le 8) = 0.333 - 0.037 = \mathbf{0.296}.$$

OR

$$P(2 < T_9 < 3) = \int_2^3 \frac{5^9}{\Gamma(9)} t^{9-1} e^{-5t}\, dt = \int_2^3 \frac{5^9}{8!} t^8 e^{-5t}\, dt = \ldots$$

3. Dementor attacks in Champaign County are relatively rare. Suppose they occur according to a Poisson process with the average rate of 0.4 attacks per month.

a) Find the probability that at most 2 dementor attacks occur in Champaign County in three months.

three months $\Rightarrow$ $\lambda = 3 \cdot 0.4 = 1.2.$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=1)$$

$$= \frac{1.2^{\,0} \cdot e^{-1.2}}{0\,!} + \frac{1.2^{\,1} \cdot e^{-1.2}}{1\,!} + \frac{1.2^{\,2} \cdot e^{-1.2}}{2\,!}$$

$$\approx 0.3012 + 0.3614 + 0.2169 = \mathbf{0.8795}.$$

```
> ppois(2,3*0.4)
[1] 0.8794871
```

=POISSON.DIST(2,3*0.4,1) 0.879487

b) Find the probability that exactly 3 dementor attacks occur in Champaign County in one year. (1 year = 12 months)

1 year = 12 months $\Rightarrow$ $\lambda = 12 \cdot 0.4 = 4.8.$

$$P(X=3) = \frac{4.8^{\,3} \cdot e^{-4.8}}{3\,!} \approx \mathbf{0.1517}.$$

```
> dpois(3,12*0.4)
[1] 0.1516907
```

=POISSON.DIST(3,12*0.4,0) 0.151691

c) Find the probability that there will be exactly 9 attack-free months (months without any dementor attacks) in one year.

Let W = the number of attack-free months (out of 12).

Then W has Binomial distribution, $n = 12$,

$$p = \mathrm{P}(\text{attack-free month}) = \frac{0.4^{\,0} \cdot e^{-0.4}}{0\,!} = 0.6703 \qquad (\text{Poisson}, \lambda = 0.40).$$

$$\mathrm{P}(\mathrm{W} = 9) = {}_{12}C_9 \cdot (0.6703)^9 \cdot (1 - 0.6703)^3 \approx \mathbf{0.2154}.$$

```
> p = dpois(0,0.4)
> p
[1] 0.67032
> dbinom(9,12,p)
[1] 0.2153973
```

=POISSON.DIST(0,0.4,0)	0.67032
=BINOM.DIST(9,12,0.67032,0)	0.215397
OR	
=BINOM.DIST(9,12,POISSON.DIST(0,0.4,0),0)	0.215397

X_t = number of dementor attack in t months. Poisson(λt)

T_k = time of the kth dementor attack. Gamma, $\alpha = k$.

the average rate of 0.4 attacks per month $\Rightarrow$ $\lambda = 0.4$.

d) Find the probability that the first dementor attack of a calendar year occurs in March. That is, find $P(2 < T_1 < 3)$.

March $=$ 3[rd] month. Need $P(2 < T_1 < 3)$.

```
> pexp(3,0.4)-pexp(2,0.4)
[1] 0.1481348
> pgamma(3,1,0.4)-pgamma(2,1,0.4)
[1] 0.1481348
> ppois(0,2*0.4)-ppois(0,3*0.4)
[1] 0.1481348
```

$$P(2 < T_1 < 3) = P(T_1 > 2) - P(T_1 > 3) = P(X_2 \le 0) - P(X_3 \le 0)$$

$$= P(\text{Poisson}(0.8) \le 0) - P(\text{Poisson}(1.2) \le 0) = 0.449 - 0.301 = \mathbf{0.148}.$$

OR

$$P(2 < T_1 < 3) = \int_2^3 \frac{0.4^1}{\Gamma(1)} t^{1-1} e^{-0.4t}\, dt = \int_2^3 0.4\, e^{-0.4t}\, dt$$

$$= e^{-0.8} - e^{-1.2} \approx 0.148135.$$

OR

January		February		March		
no attacks	$\cap$	no attacks	$\cap$	attack(s)		
$\frac{0.4^0 e^{-0.4}}{0!}$	$\times$	$\frac{0.4^0 e^{-0.4}}{0!}$	$\times$	$1 - \frac{0.4^0 e^{-0.4}}{0!}$		
0.67032	$\times$	0.67032	$\times$	0.32968	$\approx$	**0.148**.

e) Find the probability that the first dementor attack of a calendar year occurs before summer. That is, find $P(T_1 < 5)$.

"Hint": Spring: March, April, May. Summer: June, July August.

```
> pexp(5,0.4)
[1] 0.8646647
> pgamma(5,1,0.4)
[1] 0.8646647
> 1-ppois(0,5*0.4)
[1] 0.8646647
```

$$P(T_1 < 5) = P(X_5 \ge 1) = 1 - P(\text{Poisson}(2.0) \le 0) = 1 - 0.135 = \mathbf{0.865}.$$

OR $$P(T_1 < 5) = 1 - P(T_1 > 5) = 1 - P(X_5 \le 0) = \ldots$$

OR

$$P(T_1 < 5) = \int_0^5 \frac{0.4^1}{\Gamma(1)} t^{1-1} e^{-0.4t}\, dt = \int_0^5 0.4\, e^{-0.4t}\, dt$$

$$= 1 - e^{-2.0} \approx 0.864665.$$

f) Find the probability that the third dementor attack of a calendar year occurs during summer.

June $=$ 6^{th} month, August $=$ 8^{th} month. Need $\mathrm{P}(5 < \mathrm{T}_3 < 8)$.

```
> pgamma(8,3,0.4)-pgamma(5,3,0.4)
[1] 0.2967727
> ppois(2,5*0.4)-ppois(2,8*0.4)
[1] 0.2967727
```

$$\mathrm{P}(5 < \mathrm{T}_3 < 8) = \mathrm{P}(\mathrm{T}_3 > 5) - \mathrm{P}(\mathrm{T}_3 > 8) = \mathrm{P}(\mathrm{X}_5 \le 2) - \mathrm{P}(\mathrm{X}_8 \le 2)$$

$$= \mathrm{P}(\text{Poisson}(2.0) \le 2) - \mathrm{P}(\text{Poisson}(3.2) \le 2) = 0.677 - 0.380 = \mathbf{0.297}.$$

OR

$$\mathrm{P}(5 < \mathrm{T}_3 < 8) = \int_5^8 \frac{0.4^3}{\Gamma(3)}\, t^{3-1} e^{-0.4t}\, dt = \int_5^8 \frac{0.4^3}{2}\, t^2 e^{-0.4t}\, dt \approx 0.296773.$$

OR

$\mathrm{P}(2 \text{ attacks in } 5 \text{ months}) \times \mathrm{P}(\text{at least } 1 \text{ attack in the next } 3 \text{ months})$

$+ \ \mathrm{P}(1 \text{ attack in } 5 \text{ months}) \times \mathrm{P}(\text{at least } 2 \text{ attacks in the next } 3 \text{ months})$

$+ \ \mathrm{P}(0 \text{ attacks in } 5 \text{ months}) \times \mathrm{P}(\text{at least } 3 \text{ attacks in the next } 3 \text{ months})$

$$= \frac{2.0^2 e^{-2.0}}{2!} \times \left(1 - \frac{1.2^0 e^{-1.2}}{0!}\right)$$

$$+ \frac{2.0^1 e^{-2.0}}{1!} \times \left(1 - \frac{1.2^0 e^{-1.2}}{0!} - \frac{1.2^1 e^{-1.2}}{1!}\right)$$

$$+ \frac{2.0^0 e^{-2.0}}{0!} \times \left(1 - \frac{1.2^0 e^{-1.2}}{0!} - \frac{1.2^1 e^{-1.2}}{1!} - \frac{1.2^2 e^{-1.2}}{2!}\right)$$

$$= 0.27067 \times 0.69881 + 0.27067 \times 0.33737 + 0.13534 \times 0.12051 \approx \mathbf{0.2968}.$$

g) Find the probability that the fifth dementor attack of a calendar year occurs during summer.

Need $P(5 < T_5 < 8)$.

```
> pgamma(8,5,0.4)-pgamma(5,5,0.4)
[1] 0.1667345
> ppois(4,5*0.4)-ppois(4,8*0.4)
[1] 0.1667345
```

$$P(5 < T_5 < 8) = P(T_5 > 5) - P(T_5 > 8) = P(X_5 \le 4) - P(X_8 \le 4)$$

$$= P(\text{Poisson}(2.0) \le 4) - P(\text{Poisson}(3.2) \le 4) = 0.947 - 0.781 = \mathbf{0.166}.$$

OR

$$P(5 < T_5 < 8) = \int_5^8 \frac{0.4^5}{\Gamma(5)} t^{5-1} e^{-0.4t}\, dt = \int_5^8 \frac{0.4^4}{24} t^4 e^{-0.4t}\, dt \approx 0.166734.$$

OR

$P(\text{4 attacks in 5 months}) \times P(\text{at least 1 attack in the next 3 months})$

$+ \; P(\text{3 attacks in 5 months}) \times P(\text{at least 2 attacks in the next 3 months})$

$+ \; P(\text{2 attacks in 5 months}) \times P(\text{at least 3 attacks in the next 3 months})$

$+ \; P(\text{1 attack in 5 months}) \times P(\text{at least 4 attacks in the next 3 months})$

$+ \; P(\text{0 attacks in 5 months}) \times P(\text{at least 5 attacks in the next 3 months})$

$= \ldots$

h) Find the probability that the fifth dementor attack of a calendar year occurs after Halloween (October 31). That is, find the probability that the fifth dementor attack of a calendar year occurs during the last two month of the calendar year.

Need $P(10 < T_5 < 12)$.

```
> pgamma(12,5,0.4)-pgamma(10,5,0.4)
[1] 0.1525782
> ppois(4,10*0.4)-ppois(4,12*0.4)
[1] 0.1525782
```

$$P(10 < T_5 < 12) = P(T_5 > 10) - P(T_5 > 12) = P(X_{10} \le 4) - P(X_{12} \le 4)$$

$$= P(\text{Poisson}(4.0) \le 4) - P(\text{Poisson}(4.8) \le 4) = 0.629 - 0.476 = \mathbf{0.153}.$$

OR

$$P(10 < T_5 < 12) = \int_{10}^{12} \frac{0.4^5}{\Gamma(5)} t^{5-1} e^{-0.4t}\, dt = \int_{10}^{12} \frac{0.4^4}{24} t^4 e^{-0.4t}\, dt \approx 0.152578.$$

OR

$P(4 \text{ attacks in } 10 \text{ months}) \times P(\text{at least } 1 \text{ attack in the next } 2 \text{ months})$

$+ P(3 \text{ attacks in } 10 \text{ months}) \times P(\text{at least } 2 \text{ attacks in the next } 2 \text{ months})$

$+ P(2 \text{ attacks in } 10 \text{ months}) \times P(\text{at least } 3 \text{ attacks in the next } 2 \text{ months})$

$+ P(1 \text{ attack in } 10 \text{ months}) \times P(\text{at least } 4 \text{ attacks in the next } 2 \text{ months})$

$+ P(0 \text{ attacks in } 10 \text{ months}) \times P(\text{at least } 5 \text{ attacks in the next } 2 \text{ months})$

$= \ldots$