STAT 410

Examples for 11/13/2020 (1)

Fall 2020
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1. Let $X_1, X_2, \ldots, X_{16}$ be a random sample of size $n = 16$ from a $N(\mu, \sigma^2)$ distribution. We are interested in testing $H_0: \sigma = 39$ vs. $H_1: \sigma > 39$.

Recall: If $X_1, X_2, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, then $\dfrac{(n-1)\cdot S^2}{\sigma^2}$ is $\chi^2(n-1)$.

a) Find the "best" critical (rejection) region with the significance level $\alpha = 0.05$.

Test Statistic: $\chi^2 = \dfrac{(n-1)\cdot s^2}{\sigma_0^2} = \dfrac{15 \cdot s^2}{39^2}$.

Reject H_0 if $\chi^2 > \chi^2_\alpha(n-1) = \chi^2_{0.05}(15) = 25.00$.

$\dfrac{15 \cdot s^2}{39^2} > 25.00 \qquad \Leftrightarrow \qquad s^2 > \mathbf{2535}$.

b) Find the power of the test from part (a) at $\sigma = 66.7$.

Power $= P(\text{Reject } H_0 \mid H_0 \text{ is not true}) = P(S^2 > 2535 \mid \sigma = 66.7)$

$= P\left(\dfrac{(n-1)\cdot S^2}{\sigma^2} > \dfrac{15 \cdot 2535}{66.7^2} \mid \sigma = 66.7\right) = P(\chi^2(15) > 8.547) = \mathbf{0.90}$.

c) What is the probability of Type II Error if $\sigma = 66.7$?

$P(\text{Type II Error}) = 1 - \text{Power} = \mathbf{0.10}$.

The Chi-Square Distribution

	$P(X \le x)$							
	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
r	$\chi^2_{0.99}(r)$	$\chi^2_{0.975}(r)$	$\chi^2_{0.95}(r)$	$\chi^2_{0.90}(r)$	$\chi^2_{0.10}(r)$	$\chi^2_{0.05}(r)$	$\chi^2_{0.025}(r)$	$\chi^2_{0.01}(r)$
15	5.229	6.262	7.261	8.547	22.31	25.00	27.49	30.58

d) Find the power of the test from part (a) at $\sigma = 50$.

$$\text{Power} = P(\text{Reject } H_0 \mid H_0 \text{ is not true}) = P(S^2 > 2535 \mid \sigma = 50)$$

$$= P\left(\frac{(n-1)\cdot S^2}{\sigma^2} > \frac{15 \cdot 2535}{50^2} \mid \sigma = 50\right) = P(\chi^2(15) > 15.21) \approx 0.4364.$$

=CHISQ.DIST.RT(15.21,15) 0.436399

e) Suppose we observe $s^2 = 2000$. Find the p-value for the test.

p-value $= P(\text{value of } S^2 \text{ as extreme or more extreme than } s^2 = 2000 \mid H_0 \text{ true})$

$$= P(S^2 > 2000 \mid \sigma = 39) = P\left(\chi^2(15) > \frac{15 \cdot 2000}{39^2}\right) \approx 0.1828.$$

=CHISQ.DIST.RT(15*2000/39^2,15) 0.182783

1.5. Let $X_1, X_2, \ldots, X_{25}$ be a random sample from a $N(\mu, \sigma^2)$ population, and suppose the null hypothesis $H_0: \sigma^2 = 10$ is to be tested against $H_1: \sigma^2 < 10$.

a) Find the "best" rejection region with the significance level $\alpha = 0.10$. That is, for which values of s^2 should H_0 be rejected?

Reject H_0 if $\chi^2 = \dfrac{(n-1)\cdot s^2}{\sigma_0^2} = \dfrac{24 \cdot s^2}{10} < \chi^2_{1-\alpha}(n-1) = \chi^2_{0.90}(24) = 15.66.$

$$\frac{24 \cdot s^2}{10} < 15.66 \quad \Leftrightarrow \quad s^2 < \mathbf{6.525}.$$

b) Find the power of the test from part (a) at $\sigma^2 = 4.3$.

$$\text{Power} = P(\text{Reject } H_0 \mid H_0 \text{ is not true}) = P(S^2 < 6.525 \mid \sigma^2 = 4.3)$$

$$= P\left(\frac{(n-1)\cdot S^2}{\sigma^2} < \frac{24 \cdot 6.525}{4.3} \mid \sigma^2 = 4.3\right) = P(\chi^2(24) < 36.42) = \mathbf{0.95}.$$

2. A scientist wishes to test if a new treatment has a better cure rate than the traditional treatment which cures only 60% of the patients. In order to test whether the new treatment is more effective or not, a test group of 20 patients were given the new treatment. Assume that each personal result is independent of the others. Let X denote the number of patients in the test group who were cured.

$H_0: p = 0.60$ vs. $H_1: p > 0.60$. Right-tailed. $n = 20$.

0a) If the new treatment has the same success rate as the traditional, what is the probability that at least 14 out of 20 patients (14 or more) will be cured?

$P(X \geq 14 \mid p = 0.60) = 1 - CDF(13 \mid p = 0.60) = 1 - 0.750 = \mathbf{0.250}$.

0b) Suppose that 14 out of 20 patients in the test group were cured. Based on the answer for part (a), is there a reason to believe that the new treatment has a better cure rate than the traditional treatment?

If $p = 0.60$, then 25% of all possible samples would have 14 or more patients cured (out of 20). Thus, it is not unusual to see 14 out of 20 patients cured for a treatment that cures 60% of the patients. We have no reason to believe that the new treatment has a better cure rate than the traditional treatment if $X = 14$.

0c) If the new treatment has the same success rate as the traditional, what is the probability that at least 17 out of 20 patients (17 or more) will be cured?

$P(X \geq 17 \mid p = 0.60) = 1 - CDF(16 \mid p = 0.60) = 1 - 0.984 = \mathbf{0.016}$.

0d) Suppose that 17 out of 20 patients in the test group were cured. Based on the answer for part (c), is there a reason to believe that the new treatment has a better cure rate than the traditional treatment?

If $p = 0.60$, then only 1.6% of all possible samples would have 17 or more patients cured (out of 20). Thus, it is fairly unusual to see 17 out of 20 patients cured for a treatment that cures 60% of the patients. We have a good reason to believe that the new treatment has a better cure rate than the traditional treatment if $X = 17$.

a) Suppose we decided to use the rejection region "Reject H_0 if $X \ge 15$."
Find the significance level α associated with this Rejection Region.

$$\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(X \ge 15 \mid p = 0.60)$$

$$= 1 - \text{CDF}(14 \mid p = 0.60) = 1 - 0.874 = \mathbf{0.126}.$$

b) Find the "best" rejection region with the significance level α closest to 0.05.

Rejection Region for a Right – tailed test:

Find b such that $P(X \le b - 1) \approx 1 - \alpha$.
Then the Rejection Region is "Reject H_0 if $X \ge b$."

Decision rule: Reject H_0 if $X \ge b$.

Want $P(\text{Type I error}) = 0.05$.

$$P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(X \ge b \mid p = 0.60)$$

$$= 1 - \text{CDF}(b - 1 \mid p = 0.60).$$

Want $\left(1 - \text{CDF}(b - 1 \mid p = 0.60)\right) \approx 0.05$, $\quad \text{CDF}(b - 1 \mid p = 0.60) \approx 0.95$.

$\text{CDF}(15 \mid p = 0.60) = 0.949$, $\qquad b - 1 = 15$, $\qquad b = 16$.

Decision rule: Reject H_0 if $\mathbf{X \ge 16}$.

c) What is the power of this test (using the rejection region from part (b))
if $p = 0.70$? If $p = 0.80$? If $p = 0.90$? If $p = 1.00$?

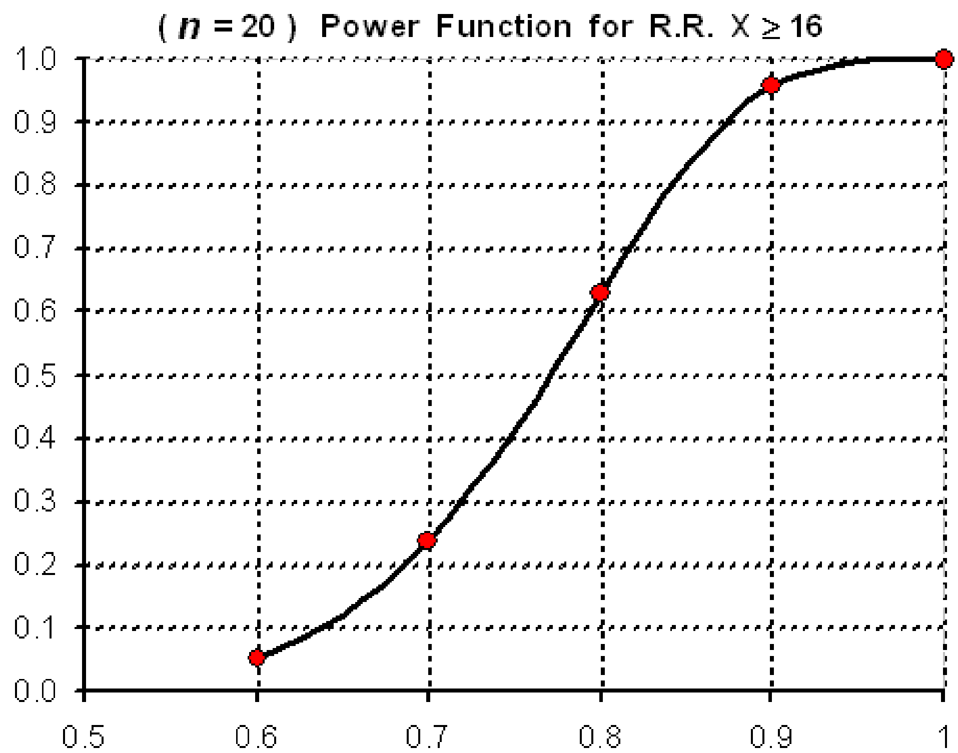
$$\text{Power} = P(\text{Reject } H_0) = P(X \ge 16) = 1 - \text{CDF}(15).$$

$$\text{Power}(0.70) = P(\text{Reject } H_0 \mid p = 0.70) = P(X \ge 16 \mid p = 0.70) = 1 - 0.762 = \mathbf{0.238},$$

$$\text{Power}(0.80) = P(\text{Reject } H_0 \mid p = 0.80) = P(X \ge 16 \mid p = 0.80) = 1 - 0.370 = \mathbf{0.630},$$

$$\text{Power}(0.90) = P(\text{Reject } H_0 \mid p = 0.90) = P(X \ge 16 \mid p = 0.90) = 1 - 0.043 = \mathbf{0.957},$$

$$\text{Power}(1.00) = P(\text{Reject } H_0 \mid p = 1.00) = P(X \ge 16 \mid p = 1.00) = \mathbf{1.00}.$$



d) Suppose that 17 out of 20 patients in the test group were cured. Find the p-value of the test. (That is, if the new treatment has the same success rate as the traditional, what is the probability that 17 or more individuals out of 20 would be cured?) Would you Reject H_0 at $\alpha = 0.05$ if $X = 17$?

p-value = P(value of X as extreme or more extreme than $X = 17 \mid H_0$ true)

$$= P(X \geq 17 \mid p = 0.60) = \mathbf{0.016}.$$

p-value $> \alpha$ $\Leftrightarrow$ Do Not Reject H_0

p-value $< \alpha$ $\Leftrightarrow$ Reject H_0

Since $0.016 < 0.05$, we **Reject $\mathbf{H_0}$** at $\alpha = 0.05$ if $X = 17$.

e)* Find the "best" (randomized) rejection region with the significance level $\alpha = 0.10$.

$$0.051 = \mathrm{P}(\mathrm{X} \geq 16 \mid p = 0.60) < 0.10 < \mathrm{P}(\mathrm{X} \geq 15 \mid p = 0.60) = 0.126.$$

$$\mathrm{P}(\mathrm{X} = 15 \mid p = 0.60) = 0.949 - 0.874 = 0.126 - 0.051 = 0.075.$$

$$0.10 = \mathrm{P}(\mathrm{X} \geq 16 \mid p = 0.60) + \mathrm{P}(\mathrm{X} = 15 \mid p = 0.60) \cdot p^{\star}.$$

$$0.10 = 0.051 + 0.075 \cdot p^{\star}. \qquad p^{\star} = \frac{0.049}{0.075} \approx 0.6533.$$

Reject H_0 if $\mathrm{X} \geq 16$,

Reject H_0 with probability $p^{\star} \approx 0.6533$ if $\mathrm{X} = 15$,

Do NOT Reject H_0 if $\mathrm{X} \leq 14$.

Cumulative Binomial Probabilities

		p												
n	**x**	**0.01**	**0.05**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**0.95**	**0.99**
20	0	0.818	0.358	0.122	0.012	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	1	0.983	0.736	0.392	0.069	0.008	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2	0.999	0.925	0.677	0.206	0.035	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3	1.000	0.984	0.867	0.411	0.107	0.016	0.001	0.000	0.000	0.000	0.000	0.000	0.000
	4	1.000	0.997	0.957	0.630	0.238	0.051	0.006	0.000	0.000	0.000	0.000	0.000	0.000
	5	1.000	1.000	0.989	0.804	0.416	0.126	0.021	0.002	0.000	0.000	0.000	0.000	0.000
	6	1.000	1.000	0.998	0.913	0.608	0.250	0.058	0.006	0.000	0.000	0.000	0.000	0.000
	7	1.000	1.000	1.000	0.968	0.772	0.416	0.132	0.021	0.001	0.000	0.000	0.000	0.000
	8	1.000	1.000	1.000	0.990	0.887	0.596	0.252	0.057	0.005	0.000	0.000	0.000	0.000
	9	1.000	1.000	1.000	0.997	0.952	0.755	0.412	0.128	0.017	0.001	0.000	0.000	0.000
	10	1.000	1.000	1.000	0.999	0.983	0.872	0.588	0.245	0.048	0.003	0.000	0.000	0.000
	11	1.000	1.000	1.000	1.000	0.995	0.943	0.748	0.404	0.113	0.010	0.000	0.000	0.000
	12	1.000	1.000	1.000	1.000	0.999	0.979	0.868	0.584	0.228	0.032	0.000	0.000	0.000
	13	1.000	1.000	1.000	1.000	1.000	0.994	0.942	0.750	0.392	0.087	0.002	0.000	0.000
	14	1.000	1.000	1.000	1.000	1.000	0.998	0.979	0.874	0.584	0.196	0.011	0.000	0.000
	15	1.000	1.000	1.000	1.000	1.000	1.000	0.994	0.949	0.762	0.370	0.043	0.003	0.000
	16	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.984	0.893	0.589	0.133	0.016	0.000
	17	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.965	0.794	0.323	0.075	0.001
	18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.992	0.931	0.608	0.264	0.017
	19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.988	0.878	0.642	0.182

3. A state legislature says that it is going to decrease its funding of a state university because, according to its sources, 40% of the university's graduates move out of the state within three years of graduation. In an attempt to save the university's funding, you want to show that the proportion of graduates who move out of state is less than 0.40, and decide to test $H_0 : p = 0.40$ vs. $H_1 : p < 0.40$. You plan to obtain a random sample of 20 graduates and choose the probability of Type I error equal to 0.05. Let X denote the number of graduates in the sample who move out of state within three years of graduation.

$H_0 : p = 0.40$ vs. $H_1 : p < 0.40$. Left-tailed. $n = 20$.

a) Find the decision rule which has a probability of Type I error closest to the desired value.

Decision rule: Reject H_0 if $X \le a$.

Want $P(\text{Type I error}) = 0.05$.

$$P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ true}) = P(X \le a \mid p = 0.40)$$

$$= \text{CDF}(a \mid p = 0.40).$$

Want $\text{CDF}(a \mid p = 0.40) \approx 0.05$.

$\text{CDF}(4 \mid p = 0.40) = 0.051$, $a = 4$.

Decision rule: Reject H_0 if $\mathbf{X \le 4}$.

b) What is the power of this test if $p = 0.30$? If $p = 0.20$?

$$\text{Power} = P(\text{Reject } H_0) = P(X \le 4) = \text{CDF}(4).$$

$$\text{Power}(0.30) = P(\text{Reject } H_0 \mid p = 0.30) = P(X \le 4 \mid p = 0.30) = \mathbf{0.238},$$

$$\text{Power}(0.20) = P(\text{Reject } H_0 \mid p = 0.20) = P(X \le 4 \mid p = 0.20) = \mathbf{0.630}.$$

c) Suppose that 6 out of 20 graduates in your sample moved out of state within three years of graduation. Compute the p-value.

$$\text{p-value} = P(\text{value of } X \text{ as extreme or more extreme than } X = 6 \mid H_0 \text{ true})$$

$$= P(X \le 6 \mid p = 0.40) = \text{CDF}(6 \mid p = 0.40) = \mathbf{0.250}.$$

3.5. Alex wants to test whether a coin is fair or not. Suppose he decides to use 25 tosses. Let p denote the probability of obtaining heads.

$$\mathrm{H}_0: p = 0.50 \qquad \text{vs.} \qquad \mathrm{H}_1: p \neq 0.50.$$

a) Find the Rejection Rule with the significance level α closest to 0.05.

Let S denote the number of "successes" (heads) in 25 tosses.

Decision rule: Reject H_0 if $S \leq a$ or $S \geq b$.

Want P(Type I error) = 0.05.

$$\text{P(Type I error)} = \text{P(Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ true)} = \text{P(} S \leq a \text{ or } S \geq b \mid p = 0.50 \text{)}$$
$$= \text{CDF(} a \mid p = 0.50 \text{)} + \big(1 - \text{CDF(} b - 1 \mid p = 0.50 \text{)} \big).$$

Want $\text{CDF}(a \mid p = 0.50) \approx 0.025$,

$\big(1 - \text{CDF}(b - 1 \mid p = 0.50) \big) \approx 0.025$, $\quad \text{CDF}(b - 1 \mid p = 0.50) \approx 0.975$.

$\text{CDF}(7 \mid p = 0.50) = 0.022$, $\qquad a = 7$.

$\text{CDF}(17 \mid p = 0.50) = 0.978$, $\qquad b - 1 = 17$, $\qquad b = 18$.

Decision rule: Reject H_0 if $\boldsymbol{S \leq 7}$ **or** $\boldsymbol{S \geq 18}$.

b) What is the actual value of the significance level α associated with the Rejection Rule obtained in part (a)?

$$\text{P(Type I error)} = \text{P(Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ true)} = \text{P(} S \leq 7 \text{ or } S \geq 18 \mid p = 0.50 \text{)}$$
$$= \text{CDF(7} \mid p = 0.50 \text{)} + \big(1 - \text{CDF(17} \mid p = 0.50 \text{)} \big)$$
$$= 0.022 + (1 - 0.978) = \mathbf{0.044}.$$

c) What is the power of this test (using the Rejection Rule obtained in part (a)) if $p = 0.60$?

$$\text{Power} = \text{P(Reject } \mathrm{H}_0 \text{)} = \text{P(} S \leq 7 \text{ or } S \geq 18 \text{)} = \text{CDF(7)} + \big(1 - \text{CDF(17)} \big).$$

$$\text{Power(0.60)} = \text{CDF(7} \mid p = 0.60 \text{)} + \big(1 - \text{CDF(17} \mid p = 0.60 \text{)} \big)$$
$$= 0.001 + (1 - 0.846) = \mathbf{0.155}.$$

d) Suppose Alex got 9 heads and 16 tails. Find the p-value of the test.

$n \cdot p_0 = 25 \cdot 0.50 = 12.5.$ $S = 9$ is in the left tail.

p-value = P(value of S as extreme or more extreme than $S = 9 \mid \mathrm{H}_0$ true)

$= \mathrm{P}(S \leq 9 \mid p = 0.50) \cdot 2$ (since it is a two-tail test)

$= \left(\mathrm{CDF}(9 \mid p = 0.50) \right) \cdot 2 = 0.115 \cdot 2 = \mathbf{0.230}.$

e) Suppose Alex got 20 heads and 5 tails. Find the p-value of the test.

$n \cdot p_0 = 25 \cdot 0.50 = 12.5.$ $S = 20$ is in the right tail.

p-value = P(value of S as extreme or more extreme than $S = 20 \mid \mathrm{H}_0$ true)

$= \mathrm{P}(S \geq 20 \mid p = 0.50) \cdot 2$ (since it is a two-tail test)

$= \left(1 - \mathrm{CDF}(19 \mid p = 0.50) \right) \cdot 2 = \left(1 - 0.998 \right) \cdot 2 = 0.002 \cdot 2 = \mathbf{0.004}.$

4. At *Initech*, server failures follow a Poisson process with the average rate of occurrence λ failures per month. We plan to look at the 2-year history of the server's performance. (That is, let $X_1, X_2, \ldots, X_{24}$ be a random sample of size $n = 24$ from a Poisson(λ) distribution.) We wish to test

$$H_0: \lambda = {}^1\!/_6 \quad \text{vs.} \quad H_1: \lambda > {}^1\!/_6.$$

a) Find the "best" Rejection Region with the significance level closest to 0.05.

Want $\qquad 0.05 \approx P(\sum_{i=1}^{24} X_i \ge c \mid \lambda = {}^1\!/_6)$.

$\sum_{i=1}^{24} X_i \mid \lambda = {}^1\!/_6$ is distributed Poisson$(24\lambda = 4)$.

$P(\text{Poisson}(4) \le 7) = 0.949$. $\qquad$ $P(\text{Poisson}(4) \ge 8) = 0.051$.

Reject H_0 if $\sum_{i=1}^{24} x_i \ge \mathbf{8}$.

b) What is the power of the test from part (a) if $\lambda = {}^1\!/_4$? If $\lambda = {}^1\!/_3$? If $\lambda = {}^1\!/_2$?

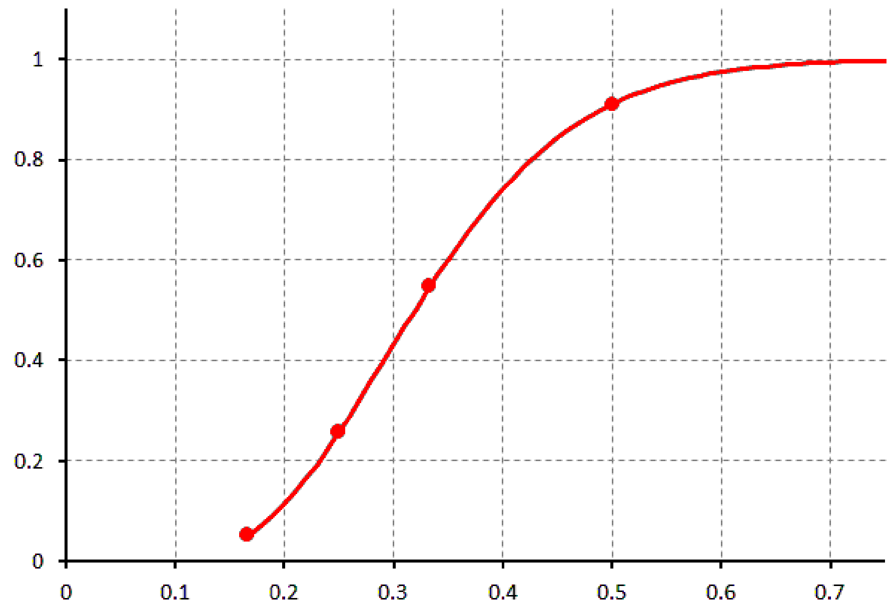
Power(λ) = $P(\text{Reject } H_0 \mid \lambda)$ = $P(\sum_{i=1}^{24} X_i \ge 8 \mid \lambda)$.

$\sum_{i=1}^{24} X_i \mid \lambda$ is distributed Poisson(24λ).

Power$(\lambda = {}^1\!/_4)$ = $P(\text{Poisson}(6) \ge 8)$ = $1 - 0.744$ = $\mathbf{0.256}$.

Power$(\lambda = {}^1\!/_3)$ = $P(\text{Poisson}(8) \ge 8)$ = $1 - 0.453$ = $\mathbf{0.547}$.

Power$(\lambda = {}^1\!/_2)$ = $P(\text{Poisson}(12) \ge 8)$ = $1 - 0.090$ = $\mathbf{0.910}$.



c) Suppose the server failed 9 times during the two-year period. Find the p-value.

$$\text{p-value} = P(\text{ as extreme or more extreme than } (\textstyle\sum_{i=1}^{24} x_i)_{\text{observed}} \mid H_0 \text{ true})$$

$$= P(\textstyle\sum_{i=1}^{24} X_i \ge 9 \mid \lambda = {}^1\!/_6) = P(\text{Poisson}(4) \ge 9) = 1 - 0.979 = \mathbf{0.021}.$$

d) We decide to use the rejection rule: Reject H_0 if $\sum_{i=1}^{24} x_i \ge 7$.

Find the significance level α of the test.

$$\text{significance level } \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

$$= P(\textstyle\sum_{i=1}^{24} X_i \ge 7 \mid \lambda = {}^1\!/_6).$$

$\sum_{i=1}^{24} X_i \mid \lambda = {}^1\!/_6$ is distributed Poisson(4).

$$P(\text{Poisson}(4) \ge 7) = 1 - P(\text{Poisson}(4) \le 6) = 1 - 0.889 = \mathbf{0.111}.$$

Poisson(λ) distribution CDF, $P(X \le x)$:

	λ										
x	2	3	4	5	6	7	8	9	10	11	12
0	0.135	0.050	0.018	0.007	0.002	0.001	0.000	0.000	0.000	0.000	0.000
1	0.406	0.199	0.092	0.040	0.017	0.007	0.003	0.001	0.000	0.000	0.000
2	0.677	0.423	0.238	0.125	0.062	0.030	0.014	0.006	0.003	0.001	0.001
3	0.857	0.647	0.433	0.265	0.151	0.082	0.042	0.021	0.010	0.005	0.002
4	0.947	0.815	0.629	0.440	0.285	0.173	0.100	0.055	0.029	0.015	0.008
5	0.983	0.916	0.785	0.616	0.446	0.301	0.191	0.116	0.067	0.038	0.020
6	0.995	0.966	0.889	0.762	0.606	0.450	0.313	0.207	0.130	0.079	0.046
7	0.999	0.988	0.949	0.867	0.744	0.599	0.453	0.324	0.220	0.143	0.090
8	1.000	0.996	0.979	0.932	0.847	0.729	0.593	0.456	0.333	0.232	0.155
9	1.000	0.999	0.992	0.968	0.916	0.830	0.717	0.587	0.458	0.341	0.242
10	1.000	1.000	0.997	0.986	0.957	0.901	0.816	0.706	0.583	0.460	0.347
11	1.000	1.000	0.999	0.995	0.980	0.947	0.888	0.803	0.697	0.579	0.462
12	1.000	1.000	1.000	0.998	0.991	0.973	0.936	0.876	0.792	0.689	0.576
13	1.000	1.000	1.000	0.999	0.996	0.987	0.966	0.926	0.864	0.781	0.682
14	1.000	1.000	1.000	1.000	0.999	0.994	0.983	0.959	0.917	0.854	0.772
15	1.000	1.000	1.000	1.000	0.999	0.998	0.992	0.978	0.951	0.907	0.844

5. **4.5.11** (7th edition) **5.5.11** (6th edition)

Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with a p.d.f. $f(x;\theta) = 1/\theta$, for $0 \le x \le \theta$, zero elsewhere, where $0 < \theta$. The hypothesis $H_0 : \theta = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \ge c$.

a) Find the constant c so that the significance level is $\alpha = 0.05$.

Recall: Let $X_1, X_2, \ldots, X_n$ be i.i.d. Uniform$(0, \theta)$. $Y_n = \max X_i$.

$$F_{\max X_i}(x) = \left(\frac{x}{\theta}\right)^n, \qquad f_{\max X_i}(x) = \frac{n \cdot x^{n-1}}{\theta^n}, \qquad 0 < x < \theta.$$

$$F_{Y_4}(x) = P(Y_4 \le x) = P(X_1 \le x, X_2 \le x, X_3 \le x, X_4 \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot P(X_3 \le x) \cdot P(X_4 \le x) = \left(\frac{x}{\theta}\right)^4, \qquad 0 < x < \theta.$$

Want $0.05 = \alpha = P(Y_4 \geq c \mid \theta = 1) = 1 - c^4.$

$\Rightarrow \quad c^4 = 0.95. \qquad \Rightarrow \quad c = 0.95^{1/4} \approx 0.98726.$

b) Determine the power function of the test.

$$\text{Power}(\theta) = P(Y_4 \geq c \mid \theta) = 1 - \left(\frac{c}{\theta}\right)^4 = 1 - \frac{0.95}{\theta^4}, \qquad \theta > 1.$$

c)* The hypothesis $H_0 : \theta = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \geq 1$. Find the significance level α of this test.

$$\alpha = P(Y_4 \geq 1 \mid \theta = 1) = \mathbf{0}.$$

d)* Determine the power function of the test in part (c).

$$\text{Power}(\theta) = P(Y_4 \geq 1 \mid \theta) = 1 - \frac{1}{\theta^4}, \qquad \theta > 1.$$

In general, $c = (1-\alpha)^{1/n}, \qquad \alpha \geq 0.$

$\alpha = 1 - c^n, \qquad c \leq 1.$

$$\text{Power}(\theta) = 1 - \frac{1-\alpha}{\theta^n}, \qquad \theta > 1.$$