# Examples for 12/07/2020 (2)

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$f(x;\theta) = f(x\,|\,\theta)$ – p.d.f. (or p.m.f.) of $x$ for given $\theta$.

$\pi(\theta)$ – prior distribution of $\theta$.

$f(x,\theta) = f(x\,|\,\theta) \times \pi(\theta)$ – joint p.d.f. of $x$ and $\theta$.

$f(x)$ – marginal p.d.f. of $x$.

$$\pi(\theta\,|\,x) = \frac{f(x,\theta)}{f(x)} = \frac{f(x|\theta) \times \pi(\theta)}{f(x)}$$ – posterior distribution of $\theta$, given $x$.

$H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_1$ $\qquad \Theta_0 \cap \Theta_1 = \varnothing$.

Reject $H_0$ if $P(\theta \in \Theta_0\,|\,x) < P(\theta \in \Theta_1\,|\,x)$.

**1.** Let X have a Poisson distribution with mean $\lambda$. The prior probability distribution of $\lambda$ is

$$P(\lambda = 4) = 0.80, \qquad P(\lambda = 7) = 0.20.$$

Given that we observe $x = 8$, test $H_0 : \lambda = 4$ versus $H_1 : \lambda = 7$.

$$P(X = 8\,|\,\lambda = 4) = \frac{4^8 e^{-4}}{8!} \approx 0.03.$$

$$P(X = 8\,|\,\lambda = 7) = \frac{7^8 e^{-7}}{8!} \approx 0.13.$$

$$P(X = 8) = 0.80 \times 0.03 + 0.20 \times 0.13 = 0.024 + 0.026 = 0.050.$$

$$P(\lambda = 4\,|\,X = 8) = \frac{0.80 \times 0.03}{0.050} = \frac{0.024}{0.050} = 0.48.$$

$$P(\lambda = 7\,|\,X = 8) = \frac{0.20 \times 0.13}{0.050} = \frac{0.026}{0.050} = 0.52.$$

$P(\lambda = 4\,|\,X = 8) < P(\lambda = 7\,|\,X = 8)$. **Reject $\mathbf{H_0}$.**

**2.** Let X have a Poisson distribution with mean $\lambda$. The prior probability distribution of $\lambda$ is

$$P(\lambda = 4) = 0.90, \qquad P(\lambda = 7) = 0.10.$$

Given that we observe $x = 8$, test $H_0 : \lambda = 4$ versus $H_1 : \lambda = 7$.

$$P(X = 8) = 0.90 \times 0.03 + 0.10 \times 0.13 = 0.027 + 0.013 = 0.040.$$

$$P(\lambda = 4 \mid X = 8) = \frac{0.90 \times 0.03}{0.040} = \frac{0.027}{0.040} = 0.675.$$

$$P(\lambda = 7 \mid X = 8) = \frac{0.10 \times 0.13}{0.040} = \frac{0.013}{0.040} = 0.325.$$

$P(\lambda = 4 \mid X = 8) > P(\lambda = 7 \mid X = 8)$. **Do NOT Reject $H_0$.**

$$\frac{P(H_0 \mid data)}{P(H_1 \mid data)} = \frac{\dfrac{P(H_0) \times P(data \mid H_0)}{P(data)}}{\dfrac{P(H_1) \times P(data \mid H_1)}{P(data)}} = \frac{P(H_0)}{P(H_1)} \times \frac{P(data \mid H_0)}{P(data \mid H_1)}.$$

$$\begin{pmatrix} posterior \\ odds \end{pmatrix} = \begin{pmatrix} prior \\ odds \end{pmatrix} \times \begin{pmatrix} Bayes \\ factor \end{pmatrix},$$ where *Bayes factor* is $B = \dfrac{P(data \mid H_0)}{P(data \mid H_1)}$.

| | |
|---|---|
| $B < \frac{1}{10}$ | Strong against $H_0$ |
| $\frac{1}{10} < B < \frac{1}{3}$ | Substantial against $H_0$ |
| $\frac{1}{3} < B < 1$ | Barely against $H_0$ |
| $1 < B < 3$ | Barely for $H_0$ |
| $3 < B < 10$ | Substantial for $H_0$ |
| $B > 10$ | Strong for $H_0$ |

In **1** and **2**,

$$B = \frac{0.03}{0.13} \approx 0.23.$$