STAT 410 **Examples for 10/28/2020 (Disc)** Fall 2020
A. Stepanov

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0.$$

Recall: $$E(X^k) = \frac{\Gamma(2k+4)}{6\beta^{2k}}, \quad k > -2;$$

$W = \sqrt{X}$ has Gamma($\alpha = 4,\ \theta = \frac{1}{\beta}$) distribution;

$$\Rightarrow \quad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i \text{ has Gamma}\left(\alpha = 4n,\ \theta = \frac{1}{\beta}\right) \text{ distribution;}$$

the maximum likelihood estimator of β is $\hat{\beta} = \dfrac{4n}{\sum_{i=1}^{n}\sqrt{X_i}}$;

a method of moments estimator of β is $\tilde{\beta} = \sqrt{\dfrac{20}{\overline{X}}} = \sqrt{\dfrac{20\cdot n}{\sum_{i=1}^{n} X_i}}$.

i) Is $\hat{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

j) Is $\widetilde{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

k) Find a such that $\tilde{\tilde{\beta}} = \sqrt[4]{\dfrac{a \cdot n}{\sum_{i=1}^{n} X_i^2}}$ is a consistent estimator of β.

l) Construct a consistent estimator $\breve{\beta}$ of β based on $\sum_{i=1}^{n} \dfrac{1}{\sqrt{X_i}} = \sum_{i=1}^{n} X_i^{-1/2}$.

m) Is $\breve{\beta}$ an unbiased estimator of β? *Justify your answer.*

Answers:

1. Let $\beta > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0.$$

Recall: $\quad \mathrm{E}(X^k) = \dfrac{\Gamma(2k+4)}{6\beta^{2k}}, \quad k > -2;$

$W = \sqrt{X}$ has Gamma$(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution;

$\Rightarrow \quad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i$ has Gamma$(\alpha = 4n,\ \theta = \frac{1}{\beta})$ distribution;

the maximum likelihood estimator of β is $\hat{\beta} = \dfrac{4n}{\sum_{i=1}^{n} \sqrt{X_i}}$;

a method of moments estimator of β is $\tilde{\beta} = \sqrt{\dfrac{20}{\overline{X}}} = \sqrt{\dfrac{20 \cdot n}{\sum_{i=1}^{n} X_i}}$.

i) Is $\hat{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is the maximum likelihood estimator")

$$\mathrm{E}(\sqrt{X}) = \frac{\Gamma(5)}{6\beta} = \frac{4!}{6\beta} = \frac{4}{\beta}.$$

OR

$W = \sqrt{X}$ has Gamma$(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution.

$$\mathrm{E}(\sqrt{X}) = \mathrm{E}(W) = \alpha\,\theta = \frac{4}{\beta}.$$

By WLLN, $\overline{\sqrt{X}} = \frac{1}{n} \cdot \sum_{i=1}^{n} \sqrt{X_i} \xrightarrow{P} E(\sqrt{X}) = \frac{4}{\beta}$.

$X_n \xrightarrow{P} a$, g is continuous at a $\Rightarrow$ $g(X_n) \xrightarrow{P} g(a)$

Since $g(x) = \frac{4}{x}$ is continuous at $\frac{4}{\beta}$,

$\hat{\beta} = g(\overline{\sqrt{X}}) \xrightarrow{P} g(\frac{4}{\beta}) = \beta$. $\hat{\beta}$ is a consistent estimator of β.

j) Is $\tilde{\beta}$ a consistent estimator of β? *Justify your answer.*

(NOT enough to say "because it is a method of moments estimator")

By WLLN, $\overline{X} \xrightarrow{P} \mu = \frac{20}{\beta^2}$.

Since $g(x) = \sqrt{\frac{20}{x}}$ is continuous at $\frac{20}{\beta^2}$,

$\tilde{\beta} = g(\overline{X}) \xrightarrow{P} g(\frac{20}{\beta^2}) = \beta$. $\tilde{\beta}$ is a consistent estimator of β.

k) Find a such that $\tilde{\tilde{\beta}} = \sqrt[4]{\frac{a \cdot n}{\sum_{i=1}^{n} X_i^2}}$ is a consistent estimator of β.

$E(X^2) = \frac{\Gamma(8)}{6\beta^4} = \frac{7!}{6\beta^4} = \frac{840}{\beta^4}$.

By WLLN, $\overline{X^2} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i^2 \xrightarrow{P} E(X^2) = \frac{840}{\beta^4}$.

Consider $g(x) = \sqrt[4]{\frac{840}{x}}$. Since $g(x) = \sqrt[4]{\frac{840}{x}}$ is continuous at $\frac{840}{\beta^4}$,

$$\sqrt[4]{\frac{840 \cdot n}{\sum_{i=1}^{n} X_i^2}} = \sqrt[4]{\frac{840}{\overline{X^2}}} = g(\overline{X^2}) \xrightarrow{P} g\left(\frac{840}{\beta^4}\right) = \beta.$$

$a = \mathbf{840}$.

l) Construct a consistent estimator $\breve{\beta}$ of β based on $\sum_{i=1}^{n} \frac{1}{\sqrt{X_i}} = \sum_{i=1}^{n} X_i^{-1/2}$.

$$E\left(\frac{1}{\sqrt{X}}\right) = E(X^{-1/2}) = \frac{\Gamma(3)}{6\,\beta^{-1}} = \frac{\beta}{3}.$$

WLLN: $\frac{1}{n} \cdot \sum_{i=1}^{n} \frac{1}{\sqrt{X_i}} \xrightarrow{P} E\left(\frac{1}{\sqrt{X}}\right) = \frac{\beta}{3}$.

$$\Rightarrow \quad \breve{\beta} = \frac{3}{n} \cdot \sum_{i=1}^{n} \frac{1}{\sqrt{X_i}} = 3 \cdot \overline{\frac{1}{\sqrt{X}}} \xrightarrow{P} \beta.$$

m) Is $\breve{\beta}$ an unbiased estimator of β? *Justify your answer.*

Let $V = \frac{1}{\sqrt{X}}$. Then $\breve{\beta} = 3\,\overline{V}$.

$$E(\breve{\beta}) = E(3\,\overline{V}) = 3\,E(\overline{V}) = 3\,\mu_V = \beta.$$

$\breve{\beta}$ IS an unbiased estimator of β.