Examples for 11/18/2020 (1)

$X_1, X_2, \ldots, X_n$ i.i.d. p.d.f. or p.m.f. $f(x;\theta)$. $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$.

Likelihood Ratio:

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(\theta_0; x_1, x_2, \ldots, x_n)}{L(\theta_1; x_1, x_2, \ldots, x_n)}.$$

Neyman-Pearson Lemma:

$$C = \{ (x_1, x_2, \ldots, x_n) : \lambda(x_1, x_2, \ldots, x_n) \le k \}.$$

("Reject H_0 if $\lambda(x_1, x_2, \ldots, x_n) \le k$ ")

is the best (most powerful) rejection region.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\,\lambda^2\, x^3\, e^{-\lambda x^2} \qquad x > 0 \qquad \lambda > 0.$$

We wish to test $H_0: \lambda = 5$ vs. $H_1: \lambda = 3$.

a) Find the form of the most powerful rejection region.

Reject H_0 if

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(H_0; x_1, x_2, \ldots, x_n)}{L(H_1; x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n}\left(2 \cdot 5^2\, x_i^3\, e^{-5 x_i^2} \right)}{\prod_{i=1}^{n}\left(2 \cdot 3^2\, x_i^3\, e^{-3 x_i^2} \right)} \le k.$$

Since $\lambda(x_1, x_2, \ldots, x_n) = \left(\frac{25}{9} \right)^n \cdot \exp\left\{ -2 \cdot \sum_{i=1}^{n} x_i^2 \right\}$,

$$\lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad \sum_{i=1}^{n} x_i^2 \ge c.$$

b) Suppose $n = 4$. Find the most powerful rejection region with a 5% level of significance. significance.

Recall: W has $\text{Gamma}(\alpha = 2, \theta = \frac{1}{\lambda})$ distribution. To show this:

Let $W = X^2 = u(X)$ $\qquad X = \sqrt{W} = v(W)$ $\qquad v'(w) = \frac{1}{2\sqrt{w}}$

$$f_W(w) = f_X(v(w)) \cdot |v'(w)| = 2\lambda^2 w^{3/2} e^{-\lambda w} \cdot \frac{1}{2\sqrt{w}} = \lambda^2 w e^{-\lambda w}$$

$$= \frac{\lambda^2}{\Gamma(2)} w^{2-1} e^{-\lambda w}, \qquad w > 0.$$

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} W_i \text{ has } \text{Gamma}(\alpha = 2n, \theta = \frac{1}{\lambda}) \text{ distribution.}$$

If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$\Rightarrow \qquad 2\lambda \sum_{i=1}^{n} X_i^2 \text{ has a } \chi^2(2\alpha = 4n) \text{ distribution.}$$

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{n} X_i^2 \ge c \mid \lambda = 5)$$

$$= P(10 \cdot \sum_{i=1}^{n} X_i^2 \ge 10c \mid \lambda = 5) = P(\chi^2(16) \ge 10c).$$

$\chi^2_{0.05}(16) = 26.30 = 10c. \qquad \Rightarrow \qquad c = \mathbf{2.63}.$

Reject H_0 if $\sum_{i=1}^{n} x_i^2 \ge 2.63$.

c) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{4} x_i^2 \geq 2.5$". Find the significance level of this test.

If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $P(T \geq t) = P(Y \leq \alpha - 1)$, where Y has a Poisson$(\lambda t)$ distribution.

$\sum_{i=1}^{n} X_i^2$ has Gamma$(\alpha = 2n = 8, \theta = \frac{1}{\lambda})$ distribution.

α = significance level = $P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{4} X_i^2 \geq 2.5 \mid \lambda = 5)$

$= P(Y \leq 7)$ where Y has a Poisson$(5 \times 2.5 = 12.5)$ distribution

$=$ **0.070**.

d) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{4} x_i^2 \geq 2.5$". Find the power of this test.

Power $= P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(\sum_{i=1}^{4} X_i^2 \geq 2.5 \mid \lambda = 3)$

$= P(Y \leq 7)$ where Y has a Poisson$(3 \times 2.5 = 7.5)$ distribution

$=$ **0.525**.

e) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{4} x_i^2 \leq 0.8$". Find the significance level of this test.

significance level $= P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(\sum_{i=1}^{4} X_i^2 \leq 0.8 \mid \lambda = 5)$

$= P(Y \geq 8)$ where Y has a Poisson$(5 \times 0.8 = 4.0)$ distribution

$$= 1 - P(Y \le 7) = 1 - 0.949 = \mathbf{0.051}.$$

f) Consider the rejection region "Reject H_0 if $\sum_{i=1}^{4} x_i^2 \le 0.8$". Find the power of this test.

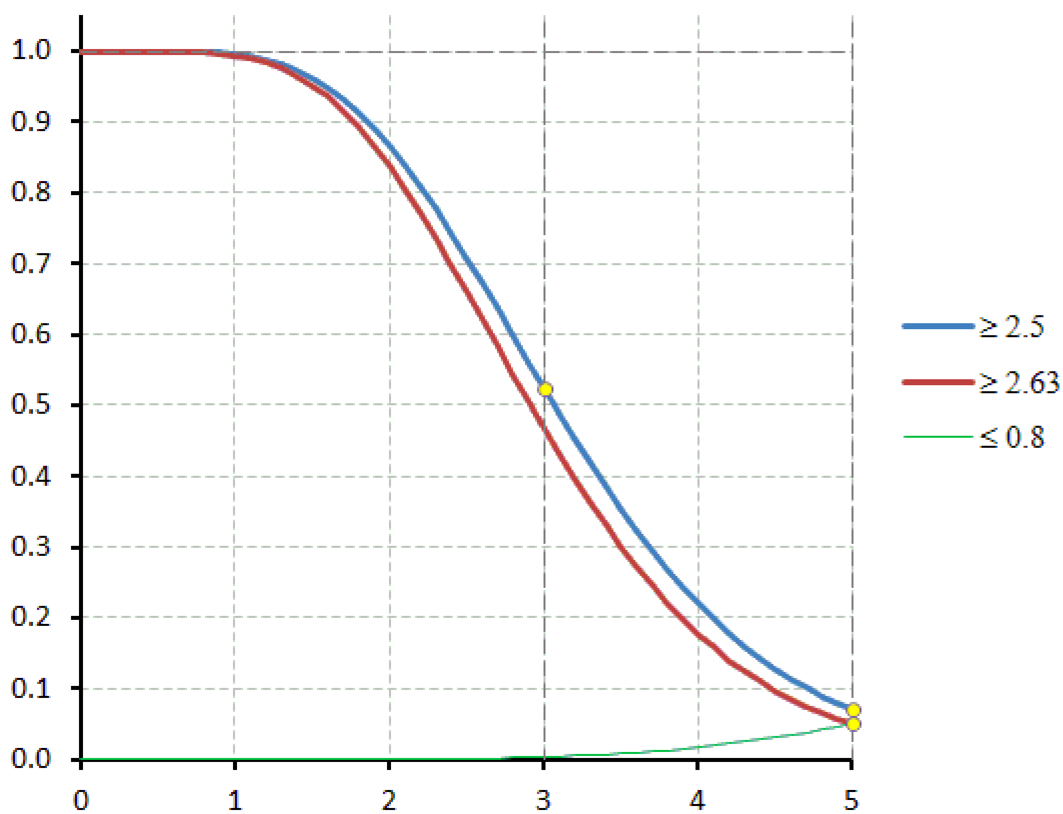
$$\text{Power} = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P\left(\sum_{i=1}^{4} X_i^2 \le 0.8 \mid \lambda = 3\right)$$

$$= P(Y \ge 8) \qquad \text{where } Y \text{ has a Poisson}(3 \times 0.8 = 2.4) \text{ distribution}$$

$$= 1 - P(Y \le 7) = 1 - 0.997 = \mathbf{0.003}.$$

Should NOT have the power of a test smaller than the significance level!

Recall: Best (most powerful) rejection region: Reject H_0 if $\sum_{i=1}^{n} x_i^2 \ge c$.



g) Suppose $\sum_{i=1}^{4} x_i^2 = 3.2$. Find the p-value of this test.

$$\text{p-value} = \mathrm{P}\Big(\sum_{i=1}^{4} \mathrm{X}_i^2 \text{ as extreme or more extreme than } \Big(\sum_{i=1}^{4} x_i^2\Big)_{\text{observed}} \,\Big|\, \mathrm{H}_0 \text{ true}\Big)$$

$$= \mathrm{P}\Big(\sum_{i=1}^{4} \mathrm{X}_i^2 \ge 3.2 \,\Big|\, \lambda = 5\Big) = \mathrm{P}(\mathrm{Y} \le 7)$$

where Y has a Poisson$(5 \times 3.2 = 16.0)$ distribution

$= \mathbf{0.010}.$

OR

$$\text{p-value} = \mathrm{P}\Big(\sum_{i=1}^{4} \mathrm{X}_i^2 \text{ as extreme or more extreme than } \Big(\sum_{i=1}^{4} x_i^2\Big)_{\text{observed}} \,\Big|\, \mathrm{H}_0 \text{ true}\Big)$$

$$= \mathrm{P}\Big(\sum_{i=1}^{4} \mathrm{X}_i^2 \ge 3.2 \,\Big|\, \lambda = 5\Big) = \mathrm{P}(\chi^2(16) \ge 32) = \mathbf{0.01}.$$

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2} \qquad x > 0 \qquad \lambda > 0.$$

$\mathrm{H}_0: \lambda = 5$ vs. $\mathrm{H}_1: \lambda = 3$.