STAT 400 Review (6)

Fall 2020
A. Stepanov

Hypothesis Testing for the Population Mean μ

Null		Alternative	
$H_0 : \mu \geq \mu_0$	vs.	$H_1 : \mu < \mu_0$	Left - tailed.
$H_0 : \mu \leq \mu_0$	vs.	$H_1 : \mu > \mu_0$	Right - tailed.
$H_0 : \mu = \mu_0$	vs.	$H_1 : \mu \neq \mu_0$	Two - tailed.

	H_0 true	**H_0 false**
Accept H_0 (Do NOT Reject H_0)	☺	Type II Error
Reject H_0	Type I Error	☺

α = significance level = P (Type I Error) = P (Reject $H_0 \mid H_0$ is true)

β = P (Type II Error) = P (Do Not Reject $H_0 \mid H_0$ is NOT true)

Power = 1 – P (Type II Error) = P (Reject $H_0 \mid H_0$ is NOT true)

Test Statistic: $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$ OR $T = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}}$ OR $\overline{X}$

The **P-value** (**observed level of significance**) is the probability, computed assuming that H_0 is true, of obtaining a value of the test statistic as extreme as, or more extreme than, the observed value.

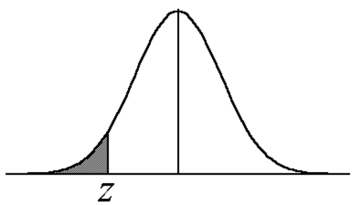
(The smaller the p-value is, the stronger is evidence against H_0 provided by the data.)

P-value > α Do Not Reject H_0 (Accept H_0).

P-value < α Reject H_0.

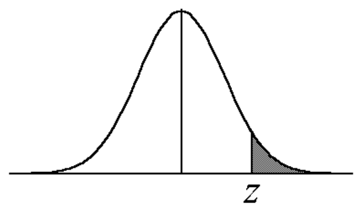
Computing P-value.

$H_0: \mu \geq \mu_0$
$H_1: \mu < \mu_0$
Left - tailed.



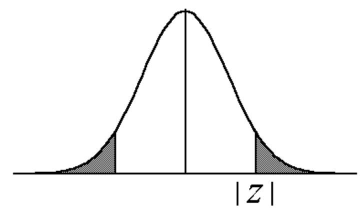
Area to the left of the observed test statistic

$H_0: \mu \leq \mu_0$
$H_1: \mu > \mu_0$
Right - tailed.



Area to the right of the observed test statistic

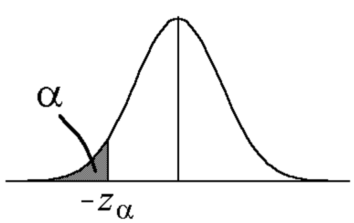
$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$
Two - tailed.



$2 \times$ area of the tail

Rejection Region:

$H_0: \mu \geq \mu_0$
$H_1: \mu < \mu_0$
Left - tailed.



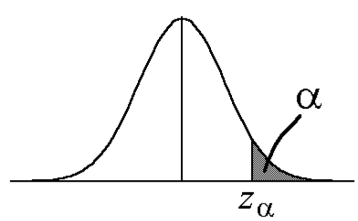
Reject H_0 if

$$Z < -z_\alpha$$

Reject H_0 if

$$\bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$$

$H_0: \mu \leq \mu_0$
$H_1: \mu > \mu_0$
Right - tailed.



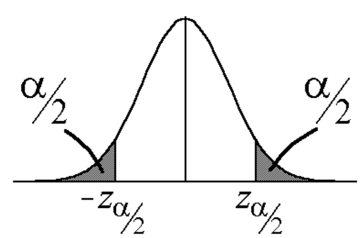
Reject H_0 if

$$Z > z_\alpha$$

Reject H_0 if

$$\bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$
Two - tailed.



Reject H_0 if

$$Z < -z_{\alpha/2}$$

or

$$Z > z_{\alpha/2}$$

Reject H_0 if

$$\bar{x} < \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

or

$$\bar{x} > \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Hypothesis Testing for the Population Standard Deviation σ

Null		Alternative	
$H_0: \sigma \geq \sigma_0$	vs.	$H_1: \sigma < \sigma_0$	
$H_0: \sigma^2 \geq \sigma_0^2$	vs.	$H_1: \sigma^2 < \sigma_0^2$	Left – tailed.
$H_0: \sigma \leq \sigma_0$	vs.	$H_1: \sigma > \sigma_0$	
$H_0: \sigma^2 \leq \sigma_0^2$	vs.	$H_1: \sigma^2 > \sigma_0^2$	Right – tailed.
$H_0: \sigma = \sigma_0$	vs.	$H_1: \sigma \neq \sigma_0$	
$H_0: \sigma^2 = \sigma_0^2$	vs.	$H_1: \sigma^2 \neq \sigma_0^2$	Two – tailed.

Test Statistic:

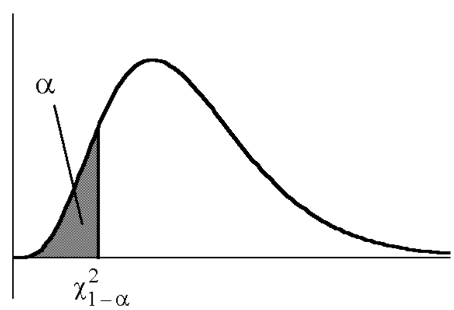
$$\chi^2 = \frac{(n-1)\cdot s^2}{\sigma_0^2}$$

Rejection Region:

$H_0: \sigma \geq \sigma_0$ vs. $H_a: \sigma < \sigma_0$

$H_0: \sigma^2 \geq \sigma_0^2$ vs. $H_a: \sigma^2 < \sigma_0^2$

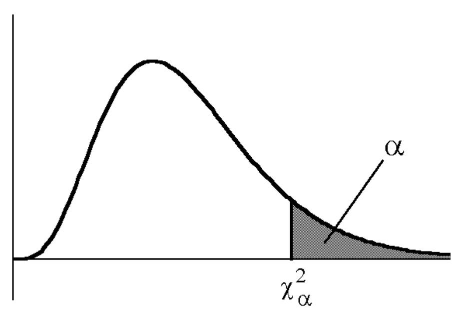
Left – tailed.



$H_0: \sigma \leq \sigma_0$ vs. $H_a: \sigma > \sigma_0$

$H_0: \sigma^2 \leq \sigma_0^2$ vs. $H_a: \sigma^2 > \sigma_0^2$

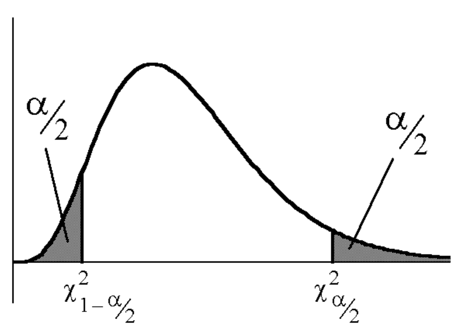
Right – tailed.



$H_0: \sigma = \sigma_0$ vs. $H_a: \sigma \neq \sigma_0$

$H_0: \sigma^2 = \sigma_0^2$ vs. $H_a: \sigma^2 \neq \sigma_0^2$

Two – tailed.



$n-1$ degrees of freedom

1. A coffee machine is regulated so that the amount of coffee dispensed is normally distributed. A random sample of 17 cups had an average of 7.91 ounces per cup and a sample standard deviation of 0.175 ounces.

a) The coffee machine manufacturer claims that the overall average amount of coffee dispensed is at least 8 ounces. Find the P-value of the appropriate test.

Claim : $\mu \geq 8$.

$H_0 : \mu \geq 8$ vs. $H_a : \mu < 8$. Left - tail.

$\overline{X} = 7.91$. $s = 0.175$. $n = 17$.

σ is unknown.

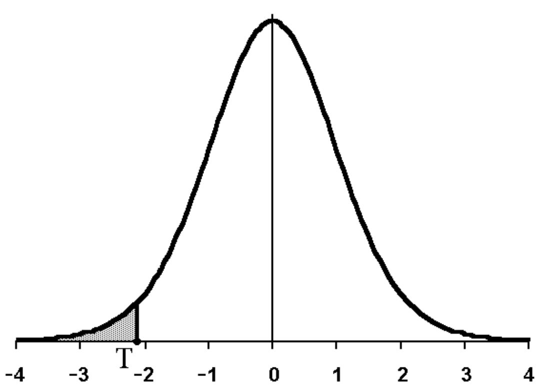
Test Statistic: $T = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}} = \dfrac{7.91 - 8}{0.175 / \sqrt{17}} = \mathbf{-2.120}$.

P-value = left tail.

$n - 1 = \mathbf{16}$ degrees of freedom.

Area to the right of **2.120** is **0.025**.

P-value = Area to the left of $T = \mathbf{-2.120}$ = **0.025**.



b) State your decision (Accept H_0 or Reject H_0) for the significance level $\alpha = 0.07$.

P-value > α $\Rightarrow$ Accept H_0 P-value < α $\Rightarrow$ Reject H_0

Since $0.025 < 0.07$, **Reject H_0 at $\alpha = 0.07$**.

c) (Type I Error, Type II Error, a correct decision) was made in part (b).

Do NOT know μ $\Rightarrow$ Do NOT know if H_0 is true or not.

$\Rightarrow$ **Cannot determine** if our decision was correct or not.

d) Suppose that the true overall average amount of coffee dispensed is 7.9 ounces. Then (Type I Error, Type II Error, a correct decision) was made in part (b).

$\mu = 7.9$ makes $H_0 : \mu \geq 8$ false. In part (b), H_0 was rejected.

	H_0 true	H_0 false
Accept H_0	☺	Type II Error
Reject H_0	Type I Error	☺

Therefore, a **correct decision** was made.

e) Suppose that the true overall average amount of coffee dispensed is 8 ounces. Then (Type I Error, Type II Error, a correct decision) was made in part (b).

$\mu = 8$ makes $H_0 : \mu \geq 8$ true. In part (b), H_0 was rejected.

Therefore, a **Type I Error** was made.

f) Find the rejection region with the significance level $\alpha = 0.05$.

Rejection Region:

$$T = \frac{\overline{X} - \mu_0}{s / \sqrt{n}} < -t_{\alpha}(n-1). \qquad T = \frac{\overline{X} - 8}{s / \sqrt{17}} < -t_{0.05}(16).$$

$$T = \frac{\overline{X} - 8}{s / \sqrt{17}} < -1.746.$$

g) Construct a 90% confidence interval for the overall average amount of coffee dispensed by the machine.

$\overline{X} = 7.91.$ $s = 0.175.$ $n = 17.$ $\alpha = 0.10.$

σ is unknown. The confidence interval : $\overline{X} \pm t_{\alpha/2} \dfrac{s}{\sqrt{n}}$.

$\alpha/2 = \mathbf{0.05}$. number of degrees of freedom $= n - 1 = 17 - 1 = \mathbf{16}$.

$t_{\alpha/2} = \mathbf{1.746}$. $7.91 \pm 1.746 \cdot \dfrac{0.175}{\sqrt{17}}$ $\mathbf{7.91 \pm 0.074}$ $\mathbf{(7.836, 7.984)}$

h) Construct a 90% confidence upper bound for μ.

The confidence upper bound : $\overline{X} + t_{\alpha} \dfrac{s}{\sqrt{n}}$.

$\alpha = \mathbf{0.10}$. number of degrees of freedom $= n - 1 = 17 - 1 = \mathbf{16}$.

$t_{\alpha} = \mathbf{1.337}$. $7.91 + 1.337 \cdot \dfrac{0.175}{\sqrt{17}}$ $\mathbf{7.91 + 0.057}$ $\mathbf{(0, 7.967)}$

i) If the overall standard deviation of the amounts dispensed exceeds 0.15, the machine needs some adjusting. Perform the appropriate test at a 10% significance level.

Claim : $\sigma > 0.15$.

$H_0 : \sigma \le 0.15$ vs. $H_a : \sigma > 0.15$. Right - tail.

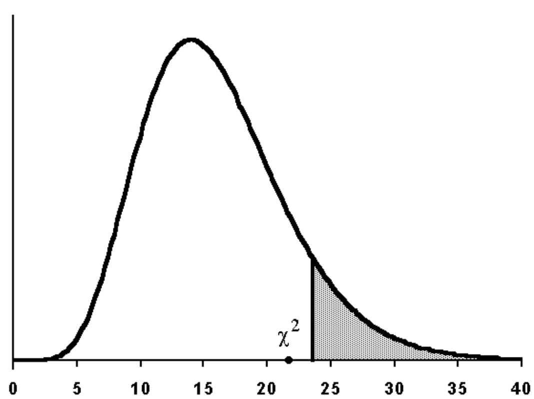
Test Statistic: $\chi^2 = \dfrac{(n-1) \cdot s^2}{\sigma_0^2} = \dfrac{(17-1) \cdot 0.175^2}{0.15^2} = \mathbf{21.7778}$.

Rejection Region: $\chi^2 > \chi^2_{\alpha}$

number of degrees of freedom

$n - 1 = 17 - 1 = \mathbf{16}$.

$\chi^2_{0.10} = \mathbf{23.5418}$.



The value of the test statistic is **not** in the Rejection Region.

Do NOT reject H_0 at $\alpha = 0.10$

(Accept H_0 at $\alpha = 0.10$).

(P-value ≈ 0.1505)

j) Suppose that the overall variance of the amounts dispensed is 0.03 ounces^2. Then in part (i), a (Type I Error, Type II Error, a correct decision) was made.

Given : $\sigma^2 = \mathbf{0.03}$. $\sigma = \sqrt{0.03} \approx \mathbf{0.1732}$.

$\sigma \approx \mathbf{0.1732}$ makes $H_0 : \sigma \le 0.15$ **false**. In part (i), H_0 was **not** rejected.

	H_0 true	H_0 false
Accept H_0	☺	Type II Error
Reject H_0	Type I Error	☺

Therefore, a **Type II Error** was made.

k) Construct a 95% confidence interval for the overall standard deviation of the amounts of coffee dispensed by the machine.

$s = 0.175$. $n = 17$.

The confidence interval : $\left(\sqrt{\dfrac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}}, \sqrt{\dfrac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}}} \right)$.

95% confidence level $\alpha = 0.05$ $\alpha/2 = \mathbf{0.025}$.

$n - 1 = \mathbf{16}$ degrees of freedom.

$\chi^2_{\alpha/2} = \chi^2_{0.025} = \mathbf{28.84}$. $\chi^2_{1-\alpha/2} = \chi^2_{0.975} = \mathbf{6.908}$.

$$\left(\sqrt{\frac{(17-1)\cdot 0.175^2}{28.84}}, \sqrt{\frac{(17-1)\cdot 0.175^2}{6.908}} \right) \qquad \mathbf{(\ 0.1303\ ,\ 0.2663\ )}$$

l) Construct a 95% confidence lower bound for σ.

$s = 0.175$. $n = 17$.

The confidence lower bound : $\left(\sqrt{\dfrac{(n-1)\cdot s^2}{\chi^2_{\alpha}}}, \infty \right)$.

95% confidence level $\alpha = \mathbf{0.05}$.

$n - 1 = \mathbf{16}$ degrees of freedom.

$$\chi^2_{\alpha} = \chi^2_{0.05} = \mathbf{26.30}.$$

$$\left(\sqrt{\frac{(17-1)\cdot 0.175^2}{26.30}}, \infty \right) \qquad \mathbf{(\ 0.1365\ ,\ \infty\ )}$$

2. The overall standard deviation of the diameters of the ball bearings is $\sigma = 0.005$ mm. The overall mean diameter of the ball bearings must be 4.300 mm. A sample of 81 ball bearings had a sample mean diameter of 4.299 mm. Is there a reason to believe that the actual overall mean diameter of the ball bearings is not 4.300 mm?

a) Perform the appropriate test using a 10% level of significance.

Claim: $\mu \neq 4.300$

$H_0: \mu = 4.300$ vs. $H_1: \mu \neq 4.300$

Test Statistic: σ is known

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{4.299 - 4.300}{0.005 / \sqrt{81}} = \mathbf{-\ 1.80}.$$

Rejection Region: 2 – tailed.

Reject H_0 if $Z < -z_{\alpha/2}$ or $Z > z_{\alpha/2}$

$\alpha = 0.10$ $\alpha/2 = 0.05$. $z_{0.05} = \mathbf{1.645}$.

Reject H_0 if $Z < -1.645$ or $Z > 1.645$.

Decision:

The value of the test statistic **does** fall into the Rejection Region.

Reject H_0 at $\alpha = 0.10$.

OR

P-value:

p-value = $P(Z \le -1.80) + P(Z \ge 1.80) = 0.0359 + 0.0359 = \mathbf{0.0718}$.

Decision:

$0.0718 < 0.10$. P-value $< \alpha$.

Reject H_0 at $\alpha = 0.10$.

OR

Confidence Interval:

σ is known. The confidence interval : $\overline{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$.

90% conf. level. $\alpha = 0.10$ $\alpha/2 = \mathbf{0.05}$. $z_{0.05} = \mathbf{1.645}$.

$4.299 \pm 1.645 \cdot \frac{0.005}{\sqrt{81}}$ $\mathbf{4.299 \pm 0.0009138889}$

Decision:

90% confidence interval for μ **does not** cover 4.300.

Reject H_0 at $\alpha = 0.10$.

Two-tailed test	same α	Confidence Interval
Accept H_0	$\Leftrightarrow$	Covers μ_0
Reject H_0	$\Leftrightarrow$	Does not cover μ_0

b) State your decision (Accept H_0 or Reject H_0) for the significance level $\alpha = 0.05$.

$0.0718 > 0.05$. P-value $> \alpha$.

Do NOT Reject H_0 (Accept H_0) at $\alpha = 0.05$.

3. A random sample of size $n = 9$ from a normal distribution is obtained:

4.4 3.7 5.1 4.3 4.7 3.7 3.5 4.6 4.7

a) Compute the sample mean $\overline{x}$ and the sample standard deviation s.

$$\overline{x} = \frac{\sum x}{n} = \frac{4.4+3.7+5.1+4.3+4.7+3.7+3.5+4.6+4.7}{9} = \frac{38.7}{9} = \mathbf{4.3}.$$

x	x^2
4.4	19.36
3.7	13.69
5.1	26.01
4.3	18.49
4.7	22.09
3.7	13.69
3.5	12.25
4.6	21.16
4.7	22.09
	168.83

OR

x	$x-\overline{x}$	$\left(x-\overline{x}\right)^2$
4.4	0.1	0.01
3.7	-0.6	0.36
5.1	0.8	0.64
4.3	0	0.00
4.7	0.4	0.16
3.7	-0.6	0.36
3.5	-0.8	0.64
4.6	0.3	0.09
4.7	0.4	0.16
	0	2.42

$$s^2 = \frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1} = \frac{168.83 - \frac{(38.7)^2}{9}}{8}$$

$$= \frac{2.42}{8} = 0.3025.$$

$$s^2 = \frac{\sum\left(x-\overline{x}\right)^2}{n-1} = \frac{2.42}{8} = 0.3025.$$

$$s = \sqrt{s^2} = \sqrt{0.3025} = \mathbf{0.55}.$$

b) Construct a 95% (two-sided) confidence interval for the overall (population) mean μ.

σ is unknown. $n = 9$ – small. The confidence interval : $\overline{X} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$.

$n - 1 = 9 - 1 = 8$ degrees of freedom.

95% confidence level, $\alpha = 0.05$, $\alpha/2 = 0.025$, $t_{\alpha/2}(8) = 2.306$.

$4.3 \pm 2.306 \frac{0.55}{\sqrt{9}}$ **4.30 ± 0.423** **(3.877 ; 4.723)**

c) Construct a 90% one-sided confidence interval for μ that provides an upper bound for μ.

$t_{0.10}(8) = 1.397.$ $\overline{X} + t_{\alpha} \cdot \frac{s}{\sqrt{n}} = 4.3 + 1.397 \cdot \frac{0.55}{\sqrt{9}} = \mathbf{4.556}.$

$$\mathbf{(-\infty, 4.556)}$$

d) Construct a 95% one-sided confidence interval for μ that provides a lower bound for μ.

$t_{0.05}(8) = 1.860.$ $\overline{X} - t_{\alpha} \cdot \frac{s}{\sqrt{n}} = 4.3 - 1.860 \cdot \frac{0.55}{\sqrt{9}} = \mathbf{3.959}.$

$$\mathbf{(3.959, \infty)}$$

e) Construct a 95% (two-sided) confidence interval for the overall standard deviation σ.

Confidence Interval for σ^2: $\left(\frac{(n-1) \cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha/2}} \right).$

$\alpha = 0.05.$ $\alpha/2 = 0.025.$ $1 - \alpha/2 = 0.975.$

number of degrees of freedom = $n - 1 = 9 - 1 = 8$.

$\chi^2_{\alpha/2} = 17.54.$ $\chi^2_{1-\alpha/2} = 2.180.$

$\left(\frac{(9-1) \cdot 0.3025}{17.54}, \frac{(9-1) \cdot 0.3025}{2.180} \right)$ (0.13797 ; 1.11009)

Confidence Interval for σ: $\left(\sqrt{0.13797}, \sqrt{1.11009} \right) = \mathbf{(0.3714 ; 1.0536)}$

f) Construct a 90% one-sided confidence interval for σ that provides an upper bound for σ.

$\left(0, \sqrt{\frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha}}} \right)$ $\chi^2_{1-\alpha} = \chi^2_{0.90} = 3.490.$

$\left(0, \sqrt{\frac{(9-1) \cdot 0.3025}{3.490}} \right)$ $\mathbf{(0, 0.8327)}$

g) Test $H_0 : \mu = 4$ vs. $H_1 : \mu > 4$ at a 5% level of significance.
What is the p-value of the test? (You may give a range.)

Test Statistic: σ is unknown

$$T = \frac{\overline{X} - \mu_0}{s/\sqrt{n}} = \frac{4.3 - 4}{0.55/\sqrt{9}} = \mathbf{1.636}.$$

P-value:

Since $1.397 < 1.636 < 1.860$ $\qquad n - 1 = 8$ degrees of freedom

$t_{0.10} < T < t_{0.05}$

0.10 > P-value > 0.05. (p-value $\approx$ 0.0702.)

Decision:

P-value > α = 0.05. **Do NOT Reject H_0 at $\alpha = 0.05$**.

h) Test $H_0 : \sigma = 0.4$ vs. $H_1 : \sigma > 0.4$ at a 5% level of significance.
What is the p-value of the test? (You may give a range.)

Test Statistic: $$\chi^2 = \frac{(n-1)\cdot s^2}{\sigma_0^2} = \frac{(9-1)\cdot 0.55^2}{0.4^2} = \mathbf{15.125}.$$

P-value:

Since $13.36 < 15.125 < 15.51$ $\qquad n - 1 = 8$ degrees of freedom

$\chi^2_{0.10} < \chi^2 < \chi^2_{0.05}$

0.10 > P-value > 0.05. (p-value $\approx$ 0.05676.)

Decision:

P-value > α = 0.05. **Do NOT Reject H_0 at $\alpha = 0.05$**.

4. A contractor assumes that construction workers are idle for at most 75 minutes per day (on average). A sample of 16 construction workers had a mean of 83 minutes per day. The sample standard deviation was 20 minutes. Assume that the time the workers are idle is approximately normally distributed.

a) Use $\alpha = 0.05$ to perform the appropriate test. State the null and alternative hypothesis for this test in terms of the relevant parameter. Report the value of the test statistic, the critical value, and the decision.

$\overline{X} = 83, \quad s = 20, \quad n = 16.$

$\mathbf{H_0 : \mu \le 75}$ vs. $\mathbf{H_1 : \mu > 75}$. Right – tailed.

Test Statistic: $T = \dfrac{\overline{X} - \mu_0}{s / \sqrt{n}} = \dfrac{83 - 75}{20 / \sqrt{16}} = \mathbf{1.60}.$

Rejection Region: $T > t_{0.05}(15) = 1.753.$

The value of the test statistic is **not** in the Rejection Region.

Do NOT Reject $\mathbf{H_0}$ at $\alpha = 0.05$.

b) Using the *t* distribution table only, what is the p-value of the test in part (a)? (You may give a range.)

$1.753 > 1.60 > 1.341.$

$\mathbf{0.05} <$ p-value $< \mathbf{0.10}$.

c) Use a computer to find the p-value of the test in part (a).

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=T.DIST.RT(1.6,15,1)            > 1-pt(1.6,15)
```

0.06522251.

5. In the past, the average guest check at a local restaurant was \$17.85. After the menu has been redesigned, a random sample of 20 guest checks was taken, the sample mean was \$19.35 with the sample standard deviation of \$3.88. Assume that the guest check amounts are approximately normally distributed. Is there enough evidence that the average guest check has changed? State the null and alternative hypothesis for this test in terms of the relevant parameter. Report the value of the test statistic and find the p-value of the appropriate test.

$\mathbf{H_0 : \mu = 17.85}$ vs. $\mathbf{H_1 : \mu \neq 17.85}$. 2-tailed.

Test Statistic: $T = \dfrac{\overline{X} - \mu_0}{s/\sqrt{n}} = \dfrac{19.35 - 17.85}{3.88/\sqrt{20}} = \mathbf{1.729}$.

$n - 1 = 19$ degrees of freedom

$t_{0.05}(19) = 1.729$ p-value $= 2 \times 0.05 = \mathbf{0.10}$.