STAT 410 Fall 2020

Examples for 11/04/2020 (Disc)

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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0, \qquad \beta > 0.$$

Recall: $W = \sqrt{X}$ has $\text{Gamma}(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution;

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i \text{ has } \text{Gamma}(\alpha = 4n,\ \theta = \tfrac{1}{\beta}) \text{ distribution;}$$

n) Find the sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for β.

o) Suggest a confidence interval for β with $(1-\alpha)\,100\,\%$ confidence level.

"Hint": Use $Y = \sum_{i=1}^{n} \sqrt{X_i}$.

p) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Use part (o) to construct a 95% confidence interval for β.

Recall: $\quad \mathrm{E}(\mathrm{X}^k) = \dfrac{\Gamma(2k+4)}{6\beta^{2k}}, \quad k > -2;$

$$\hat{\hat{\beta}} = \frac{4n-1}{\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}} \quad \text{and} \quad \breve{\beta} = \frac{3}{n} \cdot \sum_{i=1}^{n} \frac{1}{\sqrt{\mathrm{X}_i}} \quad \text{are unbiased estimators of } \beta.$$

q) Find the Fisher information $\mathrm{I}(\beta)$.

r) Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not an efficient estimator of β, find its efficiency. Is $\hat{\hat{\beta}}$ an asymptotically efficient estimator of β?

"Hint": Recall Examples for 10/21/2020 (Disc) part (e).

s) Is $\breve{\beta}$ an efficient estimator of β? If $\breve{\beta}$ is not an efficient estimator of β, find its efficiency. Is $\breve{\beta}$ an asymptotically efficient estimator of β?

"Hint": $\quad \mathrm{E}(\overline{\mathrm{V}}) = \mu_{\mathrm{V}} = \mathrm{E}(\mathrm{V}). \qquad \mathrm{Var}(\overline{\mathrm{V}}) = \dfrac{\sigma_{\mathrm{V}}^2}{n} = \dfrac{\mathrm{Var}(\mathrm{V})}{n}.$

$\mathrm{Var}(\mathrm{V}) = \mathrm{E}(\mathrm{V}^2) - [\mathrm{E}(\mathrm{V})]^2.$

$\mathrm{E}(a ☺) = a\,\mathrm{E}(☺). \qquad \mathrm{Var}(a ☺) = a^2\,\mathrm{Var}(☺).$

Answers:

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^4}{12}\, x\, e^{-\beta\sqrt{x}}, \qquad x > 0, \qquad \beta > 0.$$

Recall: $W = \sqrt{X}$ has Gamma($\alpha = 4$, $\theta = \frac{1}{\beta}$) distribution;

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} \sqrt{X_i} = \sum_{i=1}^{n} W_i \text{ has Gamma}(\ \alpha = 4n,\ \theta = \frac{1}{\beta}\ ) \text{ distribution;}$$

n) Find the sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for β.

$$f(x_1,x_2,\ldots x_n;\beta) = f_X(x_1;\beta)\cdot f_X(x_2;\beta)\cdot \ldots \cdot f_X(x_n;\beta)$$

$$= \left[\frac{\beta^{4n}}{12^n} e^{-\beta\sum_{i=1}^{n}\sqrt{x_i}}\right]\left(\prod_{i=1}^{n} x_i\right).$$

By Factorization Theorem, $Y = \sum_{i=1}^{n}\sqrt{X_i}$ is a sufficient statistic for β.

OR

$$f_X(x;\beta) = \exp\left\{-\beta\cdot\sqrt{x} + 4\ln\beta - \ln 12 + \ln x\right\}. \qquad \Rightarrow \qquad K(x) = \sqrt{x}\,.$$

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n}\sqrt{X_i} \text{ is a sufficient statistic for } \beta.$$

o) Suggest a confidence interval for β with $(1-\alpha)\,100\,\%$ confidence level.

"Hint": Use $\mathrm{Y} = \sum_{i=1}^{n} \sqrt{\mathrm{X}_i}$.

$\mathrm{W} = \sqrt{\mathrm{X}}$ has Gamma$(\alpha = 4,\ \theta = \frac{1}{\beta})$ distribution.

$\Rightarrow \quad \sum_{i=1}^{n} \sqrt{\mathrm{X}_i} = \sum_{i=1}^{n} \mathrm{W}_i$ has Gamma$(\alpha = 4n,\ \theta = \frac{1}{\beta})$ distribution.

$\Rightarrow \quad 2\beta \sum_{i=1}^{n} \sqrt{\mathrm{X}_i}$ has a $\chi^2(2\alpha = 8n)$ distribution.

$$\Rightarrow \quad \mathrm{P}\Big(\chi^2_{1-\alpha/2}(8n) < 2\beta \sum_{i=1}^{n} \sqrt{\mathrm{X}_i} < \chi^2_{\alpha/2}(8n) \Big) = 1-\alpha.$$

$$\Rightarrow \quad \mathrm{P}\Bigg(\frac{\chi^2_{1-\alpha/2}(8n)}{2\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}} < \beta < \frac{\chi^2_{\alpha/2}(8n)}{2\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}} \Bigg) = 1-\alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for β

$$\left(\frac{\chi^2_{1-\alpha/2}(8n)}{2\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}},\ \frac{\chi^2_{\alpha/2}(8n)}{2\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}} \right).$$

p) Suppose $n = 3$, and $x_1 = 0.25$, $x_2 = 0.36$, $x_3 = 0.81$.

Use part (o) to construct a 95% confidence interval for β.

$$\sum_{i=1}^{n} \sqrt{x_i} = 2. \qquad \chi^2_{0.975}(24) = 12.40, \qquad \chi^2_{0.025}(24) = 39.36.$$

$$\left(\frac{12.40}{2 \cdot 2}, \frac{39.36}{2 \cdot 2} \right) \qquad \mathbf{(3.10, 9.84)}$$

Recall (Examples for 10/21/2020 (Bonus)):

$$\hat{\beta} = \frac{4n}{\sum_{i=1}^{n} \sqrt{x_i}} = 6.$$

Recall: $$\mathrm{E}(\mathrm{X}^k) = \frac{\Gamma(2k+4)}{6\beta^{2k}}, \quad k > -2;$$

$$\hat{\hat{\beta}} = \frac{4n-1}{\sum_{i=1}^{n} \sqrt{\mathrm{X}_i}} \quad \text{and} \quad \breve{\beta} = \frac{3}{n} \cdot \sum_{i=1}^{n} \frac{1}{\sqrt{\mathrm{X}_i}} \quad \text{are unbiased estimators of } \beta.$$

q) Find the Fisher information $\mathrm{I}(\beta)$.

$$\ln f(x;\beta) = -\beta \cdot \sqrt{x} + 4\ln\beta - \ln 12 + \ln x$$

$$\frac{\partial}{\partial\beta} \ln f(x;\beta) = -\sqrt{x} + \frac{4}{\beta} \qquad\qquad \frac{\partial^2}{\partial\beta^2} \ln f(x;\beta) = -\frac{4}{\beta^2}$$

$$\mathrm{I}(\beta) = -\mathrm{E}\left[\frac{\partial^2}{\partial\beta^2} \ln f(\mathrm{X};\beta)\right] = -\mathrm{E}\left[-\frac{4}{\beta^2}\right] = \frac{4}{\beta^2}.$$

OR

$$\mathrm{I}(\beta) = \mathrm{Var}\left[\frac{\partial}{\partial\beta} \ln f(\mathrm{X};\beta)\right] = \mathrm{Var}\left[-\sqrt{\mathrm{X}} + \frac{4}{\beta}\right] = \mathrm{Var}(\mathrm{W}) = \alpha\,\theta^2 = \frac{4}{\beta^2}.$$

r) Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not an efficient estimator of β, find its efficiency. Is $\hat{\hat{\beta}}$ an asymptotically efficient estimator of β?

"Hint": Recall Examples for 10/21/2020 (Disc) part (e).

Examples for 10/21/2020 (Disc) part (e):

$$\operatorname{Var}\left(\frac{1}{Y}\right) = E\left(\frac{1}{Y^2}\right) - \left[E\left(\frac{1}{Y}\right)\right]^2 = \frac{\beta^2}{(4n-2)(4n-1)} - \left(\frac{\beta}{4n-1}\right)^2 = \frac{\beta^2}{(4n-2)(4n-1)^2}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = (4n-1)^2 \operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{4n-2}.$$

$$\text{Rao-Cramer Lower Bound} = \frac{1}{n \cdot I(\beta)} = \frac{\beta^2}{4n}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \frac{\beta^2}{4n-2} > \frac{\beta^2}{4n}.$$

$\operatorname{Var}(\hat{\hat{\beta}})$ does NOT attain its Rao-Cramer lower bound.

$\Rightarrow$ $\hat{\hat{\beta}}$ is NOT an efficient estimator of β,

its efficiency $= \dfrac{4n-2}{4n} \to 1$ as $n \to \infty$,

$\Rightarrow$ $\hat{\hat{\beta}}$ IS an asymptotically efficient estimator of β.

s) Is $\breve{\beta}$ an efficient estimator of β? If $\breve{\beta}$ is not an efficient estimator of β, find its efficiency. Is $\breve{\beta}$ an asymptotically efficient estimator of β?

Let $V = \dfrac{1}{\sqrt{X}}$. Then $\breve{\beta} = 3\,\overline{V}$.

$$E(V) = E\left(\frac{1}{\sqrt{X}}\right) = E(X^{-1/2}) = \frac{\Gamma(3)}{6\,\beta^{-1}} = \frac{\beta}{3}.$$

$$E(V^2) = E\left(\frac{1}{X}\right) = E(X^{-1}) = \frac{\Gamma(2)}{6\,\beta^{-2}} = \frac{\beta^2}{6}.$$

$$\operatorname{Var}(V) = E(V^2) - [E(V)]^2 = \frac{\beta^2}{6} - \left(\frac{\beta}{3}\right)^2 = \frac{\beta^2}{18}.$$

$$\operatorname{Var}(\breve{\beta}) = \operatorname{Var}(3\,\overline{V}) = 9\operatorname{Var}(\overline{V}) = 9\,\frac{\operatorname{Var}(V)}{n} = \frac{\beta^2}{2n}.$$

$$\text{Rao-Cramer Lower Bound} = \frac{1}{n \cdot I(\beta)} = \frac{\beta^2}{4n}.$$

$$\operatorname{Var}(\breve{\beta}) = \frac{\beta^2}{2n} > \frac{\beta^2}{4n}.$$

$\operatorname{Var}(\breve{\beta})$ does NOT attain its Rao-Cramer lower bound.

$\Rightarrow$ $\breve{\beta}$ is NOT an efficient estimator of β,

its efficiency $= \dfrac{2n}{4n} = \dfrac{1}{2}$,

$\Rightarrow$ $\breve{\beta}$ is NOT an asymptotically efficient estimator of β.