STAT 410

Practice Problems 5

Fall 2020
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1. Once a car accident is reported to an insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint p.d.f.

$$f(x,y) = \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)}, \qquad x > 1, \quad y > 1.$$

a) Given that the initial claim estimated by the company is 1.5, determine the probability that the final settlement amount exceeds 2. (from Actuarial Science Exam P)

b) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$.

2. **2.3.11** (7th and 6th edition)

Let us choose at random a point from interval $(0, 1)$ and let the random variable X_1 be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0, x_1)$, where x_1 is the experimental value of X_1; and let the random variable X_2 be equal to the number which corresponds to this point.

(a) Make assumptions about the marginal pdf $f_1(x_1)$ and the conditional pdf $f_{2|1}(x_2|x_1)$.

Suggestion: Use Uniform distributions on interval $(0, 1)$ and $(0, x_1)$, respectively.

(b) Compute $\mathrm{P}(\mathrm{X}_1 + \mathrm{X}_2 \ge 1)$.

(c) Find the conditional mean $\mathrm{E}(\mathrm{X}_1 | x_2)$.

3. **2.3.6** (7th and 6th edition)

Let the joint pdf of X and Y be given by

$$f(x,y) = \begin{cases} \dfrac{2}{(1+x+y)^3} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Compute the marginal pdf of X and the conditional pdf of Y, given $\mathrm{X} = x$.

(b) For a fixed $\mathrm{X} = x$, compute $\mathrm{E}(1 + x + \mathrm{Y} \mid x)$ and use the result to compute $\mathrm{E}(\mathrm{Y} \mid x)$.

4. Let S and T have the joint probability density function

$$f_{S,T}(s,t) = \frac{1}{t}, \quad 0 < s < 1, \;\; s^2 < t < s.$$

a) Find $f_S(s)$ and $f_T(t)$.

b) Find $E(S)$ and $E(T)$.

c) Find $f_{S|T}(s|t)$ and $f_{T|S}(t|s)$.

d) Find $E(S \mid T = t)$ and $E(T \mid S = s)$.

e) Find the correlation coefficient ρ_{ST}.

5. **2.3.10** (7th and 6th editions)

Let X_1 and X_2 have joint pmf $p(x_1, x_2)$ described as follows:

(x_1, x_2)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$
$p(x_1, x_2)$	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{1}{18}$

and $p(x_1, x_2)$ is equal to zero elsewhere. Find the two marginal probability mass functions and the two conditional means.

"Hint": Write the probabilities in a rectangular array first.

6. Let X and Y have the joint probability density function $f_{XY}(x, y) = x$, $x > 0, \;\; 0 < y < e^{-x}$, zero elsewhere.

a) Find $f_X(x)$ and $f_Y(y)$.

b) Find $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

c) Find $E(X \mid Y = y)$ and $E(Y \mid X = x)$.

d) Find $E(X)$ and $E(Y)$.

e) Are X and Y independent?

7. $\sim$ **2.3.2** (7th and 6th editions)

Let $f_{X|Y}(x|y) = c_1 x/y^2$, $0<x<y$, $0<y<1$, zero elsewhere, and $f_Y(y) = c_2 y^4$, $0<y<1$, zero elsewhere, denote, respectively, the conditional p.d.f. of X, given $Y = y$, and the marginal p.d.f. of Y.

a) Determine the constants c_1 and c_2.

b) Find the joint pdf of X and Y.

c) Let $a>1$. Find $P(Y<aX)$.

d) Find $P\left(X>\frac{1}{3}\middle|Y=\frac{1}{2}\right)$.

e) Find $E(X|Y=y)$.

f) Find $P\left(Y<\frac{1}{2}\middle|X=\frac{1}{4}\right)$.

g) Find $P\left(Y>\frac{1}{3}\middle|X=\frac{1}{2}\right)$.

h) Find $E(Y|X=x)$.

i) Find $P\left(\frac{1}{4}<X<\frac{1}{2}\right)$.

j) Find $P\left(\frac{1}{4}<X<\frac{1}{2}\middle|Y=\frac{5}{8}\right)$.

8. Let $\lambda>0$. Consider the following joint probability distribution $p(x,y)$ of two random variables X and Y:

$$p(x,y)=\frac{\lambda^x e^{-\lambda}}{(x+1)!}, \qquad x,y \text{ – integers, } 0\le y\le x<\infty.$$

a) Verify that $p(x,y)$ is a legitimate probability mass function.

b) Find the marginal probability mass function for X.

c) Find $E(Y)$, $E(X\cdot Y)$.

d) Find the moment-generating function $M(t_1,t_2)$.

e) Find the conditional probability distribution $p_{Y|X}(y|x)$ of Y given $X=x$.

f) Find conditional expectation $E(Y|X)$ and use it to find $E(Y)$ and $E(X\cdot Y)$.

g) Find $\mathrm{Cov}(X,Y) = \sigma_{XY}$.

Answers:

1. Once a car accident is reported to an insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint p.d.f.

$$f(x,y) = \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)}, \qquad x > 1, \quad y > 1.$$

a) Given that the initial claim estimated by the company is 1.5, determine the probability that the final settlement amount exceeds 2. (from Actuarial Science Exam P)

$$f_X(x) = \int_1^{\infty} \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)}\, dy = \frac{2}{x^2(x-1)} \cdot \frac{1}{\frac{(2x-1)}{(x-1)} - 1} = \frac{2}{x^3}, \qquad x > 1.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x}{x-1} \cdot y^{-(2x-1)/(x-1)}, \qquad y > 1.$$

$$f_{Y|X}(y|x=1.5) = 3 \cdot y^{-4}, \qquad y > 1.$$

$$P(Y > 2 \mid X = 1.5) = \int_2^{\infty} 3 \cdot y^{-4}\, dy = \mathbf{\frac{1}{8}} = \mathbf{0.125}.$$

b) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \int_1^{\infty} y \cdot \frac{x}{x-1} \cdot y^{-(2x-1)/(x-1)}\, dy = \int_1^{\infty} \frac{x}{x-1} \cdot y^{-x/(x-1)}\, dy$$

$$= \frac{x}{x-1} \cdot \frac{1}{\frac{x}{(x-1)} - 1} = \boldsymbol{x}, \qquad x > 1.$$

2. **2.3.11** (7th and 6th edition)

Let us choose at random a point from interval $(0,1)$ and let the random variable X_1 be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0,x_1)$, where x_1 is the experimental value of X_1; and let the random variable X_2 be equal to the number which corresponds to this point.

(a) Make assumptions about the marginal pdf $f_1(x_1)$ and the conditional pdf $f_{2|1}(x_2|x_1)$.

Suggestion: Use Uniform distributions on interval $(0,1)$ and $(0,x_1)$, respectively.

(b) Compute $P(X_1+X_2\ge 1)$.

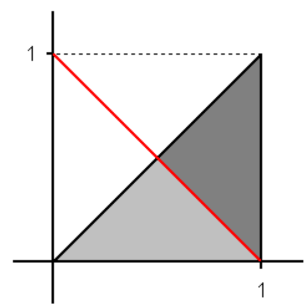
(c) Find the conditional mean $E(X_1|x_2)$.

(a) X_1 has a Uniform distribution on $(0,1)$: $\quad f_1(x_1)=1, \quad 0<x_1<1.$

$X_2|X_1=x_1$ has a Uniform distribution on $(0,x_1)$:

$$f_{2|1}(x_2|x_1)={}^1\!/_{x_1}, \qquad 0<x_2<x_1.$$

(b) $f(x_1,x_2)={}^1\!/_{x_1}, \qquad 0<x_2<x_1<1.$



$$P(X_1+X_2\ge 1) = \int_{0.5}^{1}\left(\int_{1-x_1}^{x_1}\frac{1}{x_1}\,dx_2\right)dx_1$$

$$= \int_{0.5}^{1}\frac{2x_1-1}{x_1}\,dx_1 = \int_{0.5}^{1}\left(2-\frac{1}{x_1}\right)dx_1$$

$$= \left(2x_1-\ln x_1\right)\Big|_{0.5}^{1} = 1+\ln 0.5 = 1-\ln 2.$$

(c) $$f_2(x_2) = \int_{x_2}^{1}\frac{1}{x_1}\,dx_1 = -\ln x_2, \qquad 0<x_2<1.$$

$$f_{1|2}(x_1|x_2)=-{}^1\!/_{x_1\ln x_2}, \qquad x_2<x_1<1.$$

$$E(X_1|x_2) = -\int_{x_2}^{1} x_1\cdot\frac{1}{x_1\ln x_2}\,dx_1 = -\frac{1-x_2}{\ln x_2}, \qquad 0<x_2<1.$$

3. **2.3.6** (7th and 6th edition)

Let the joint pdf of X and Y be given by

$$f(x,y)=\begin{cases} \dfrac{2}{(1+x+y)^3} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Compute the marginal pdf of X and the conditional pdf of Y, given $X=x$.

$$f_X(x) = \int_0^\infty \frac{2}{(1+x+y)^3}\,dy = \int_{1+x}^\infty \frac{2}{u^3}\,du = \frac{1}{(1+x)^2}, \qquad 0<x<\infty.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2(1+x)^2}{(1+x+y)^3}, \qquad 0<y<\infty.$$

(b) For a fixed $X=x$, compute $E(1+x+Y\,|\,x)$ and use the result to compute $E(Y\,|\,x)$.

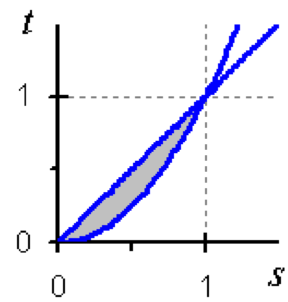
$$E(1+x+Y\,|\,x) = \int_0^\infty (1+x+y)\cdot\frac{2(1+x)^2}{(1+x+y)^3}\,dy = \int_0^\infty \frac{2(1+x)^2}{(1+x+y)^2}\,dy$$

$$= \int_{1+x}^\infty \frac{2(1+x)^2}{u^2}\,du = 2(1+x).$$

$$2(1+x) = E(1+x+Y\,|\,x) = 1+x+E(Y\,|\,x).$$

$$\Rightarrow \qquad E(Y\,|\,x) = 1+x.$$

4. $f_{S,T}(s,t) = \frac{1}{t}, \quad 0 < s < 1, \quad s^2 < t < s.$



a) $$f_S(s) = \int_{s^2}^{s} \frac{1}{t}\,dt = \left(\ln t\right)\Big|_{s^2}^{s}$$
$$= \ln s - \ln s^2 = -\ln s, \qquad 0 < s < 1.$$

$$f_T(t) = \int_{t}^{\sqrt{t}} \frac{1}{t}\,ds = \frac{1}{t}\left(\sqrt{t} - t\right) = \frac{1}{\sqrt{t}} - 1, \qquad 0 < t < 1.$$

b) $$E(S) = \int_0^1 s(-\ln s)\,ds = \left(-\frac{s^2}{2}\ln s + \frac{s^2}{4}\right)\Bigg|_0^1 = \mathbf{\frac{1}{4}}.$$

$$E(T) = \int_0^1 t\left(\frac{1}{\sqrt{t}} - 1\right)dt = \int_0^1 \left(\sqrt{t} - t\right)dt = \frac{2}{3} - \frac{1}{2} = \mathbf{\frac{1}{6}}.$$

c) $$f_{S|T}(s|t) = \frac{1/t}{\frac{1}{\sqrt{t}} - 1} = \frac{1}{\sqrt{t} - t}, \qquad t < s < \sqrt{t}, \qquad 0 < t < 1.$$

$S \mid T = t$ is Uniform$(t, \sqrt{t}\,)$

$$f_{T|S}(t|s) = \frac{1/t}{-\ln s} = \frac{1}{-t\ln s}, \qquad s^2 < t < s, \qquad 0 < s < 1.$$

d) $$E(S \mid T = t) = \frac{t + \sqrt{t}}{2}, \qquad 0 < t < 1,$$

since $S \mid T = t$ is Uniform$(t, \sqrt{t}\,)$.

$$\mathrm{E}(\mathrm{T} \mid \mathrm{S}=s) = \int_{s^2}^{s} t\left(\frac{1}{-t\ln s}\right)dt = \int_{s^2}^{s} \frac{1}{-\ln s}\,dt = \frac{s-s^2}{-\ln s} = \frac{s^2-s}{\ln s},$$

$$0 < s < 1.$$

e) $$\mathrm{E}(\mathrm{S\,T}) = \int_0^1 \left(\int_{s^2}^{s} s\,t\,\frac{1}{t}\,dt\right)ds = \int_0^1 s\left(\int_{s^2}^{s} dt\right)ds$$

$$= \int_0^1 \left(s^2 - s^3\right)ds = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$\mathrm{Cov}(\mathrm{S},\mathrm{T}) = \frac{1}{12} - \frac{1}{4}\cdot\frac{1}{6} = \frac{1}{24}.$$

$$\mathrm{E}(\mathrm{S}^2) = \int_0^1 s^2(-\ln s)\,ds = \left(-\frac{s^3}{3}\ln s + \frac{s^3}{9}\right)\Bigg|_0^1 = \frac{1}{9}.$$

$$\mathrm{Var}(\mathrm{S}) = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}.$$

$$\mathrm{E}(\mathrm{T}^2) = \int_0^1 t^2\left(\frac{1}{\sqrt{t}} - 1\right)dt = \int_0^1 \left(t^{3/2} - t^2\right)dt = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}.$$

$$\mathrm{Var}(\mathrm{T}) = \frac{1}{15} - \frac{1}{36} = \frac{7}{180}.$$

$$\rho_{\mathrm{ST}} = \frac{1/24}{\sqrt{7/144}\times\sqrt{7/180}} = \frac{3\sqrt{5}}{7} \approx \mathbf{0.9583}.$$

5.

	x_1			
x_2	0	1	2	$p_2(x_2)$
0	$\frac{1}{18}$	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{11}{18}$
1	$\frac{3}{18}$	$\frac{3}{18}$	$\frac{1}{18}$	$\frac{7}{18}$
$p_1(x_1)$	$\frac{4}{18}$	$\frac{7}{18}$	$\frac{7}{18}$	

x_2	$p_{2\|1}(x_2\|0)$
0	$1/4$
1	$3/4$

x_2	$p_{2\|1}(x_2\|1)$
0	$4/7$
1	$3/7$

x_2	$p_{2\|1}(x_2\|2)$
0	$6/7$
1	$1/7$

$E(X_2|X_1=0)=3/4$ $\quad$ $E(X_2|X_1=1)=3/7$ $\quad$ $E(X_2|X_1=2)=1/7$

$E(X_2|X_1)$:

x_1	$E(X_2\|X_1=x_1)$	$p_1(x_1)$
0	$3/4$	$4/18$
1	$3/7$	$7/18$
2	$1/7$	$7/18$

x_1	$p_{1\|2}(x_1\|0)$
0	$1/11$
1	$4/11$
2	$6/11$

x_1	$p_{1\|2}(x_1\|1)$
0	$3/7$
1	$3/7$
2	$1/7$

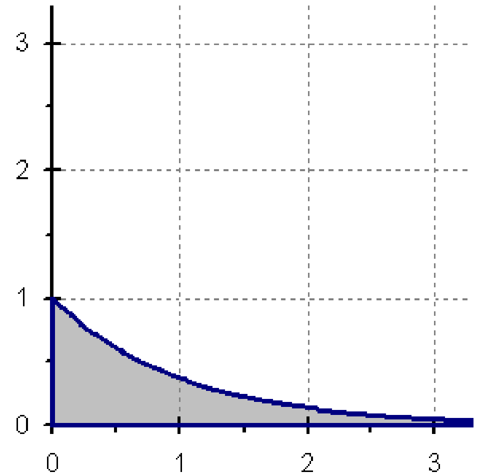
$E(X_1|X_2=0)=16/11$ $\quad$ $E(X_1|X_2=1)=5/7$

$E(X_1|X_2)$:

x_2	$E(X_1\|X_2=x_2)$	$p_2(x_2)$
0	$16/11$	$11/18$
1	$5/7$	$7/18$

6. $f_{XY}(x,y) = x,$

$$x > 0, \quad 0 < y < e^{-x}.$$



a) $f_X(x) = x\,e^{-x}, \qquad x > 0.$

$$f_Y(y) = \frac{(\ln y)^2}{2}, \qquad 0 < y < 1.$$

b) $f_{X|Y}(x|y) = \dfrac{2x}{(\ln y)^2},$

$$0 < x < -\ln y, \quad 0 < y < 1.$$

$$f_{Y|X}(y|x) = e^{x}, \qquad 0 < y < e^{-x}, \quad x > 0.$$

c) $E(X\,|\,Y = y) = -\dfrac{2}{3}\ln y, \qquad 0 < y < 1.$

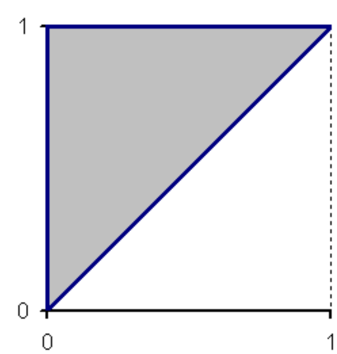
$$E(Y\,|\,X = x) = \frac{e^{-x}}{2}, \qquad x > 0.$$

d) $E(X) = \mathbf{2}. \qquad E(Y) = \dfrac{\mathbf{1}}{\mathbf{8}}.$

e) X and Y are **NOT independent**.

7. $f_Y(y) = c_2\,y^4, \quad 0 < y < 1.$

$$f_{X|Y}(x|y) = c_1\,x/y^2, \quad 0 < x < y, \quad 0 < y < 1.$$



a) $1 = \displaystyle\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_0^1 c_2\,y^4\,dy = \frac{c_2}{5}.$

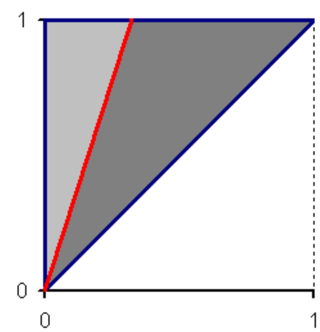
$\Rightarrow \qquad c_2 = \mathbf{5}.$

$$1 = \int_{-\infty}^{\infty} f_{X|Y}(x\,|\,y)\,dx = \int_0^y c_1\, x/y^2\, dx = \frac{c_1}{2}. \qquad \Rightarrow \qquad c_1 = \mathbf{2}.$$

b) $$f(x,y) = f_{X|Y}(x|y) \cdot f_Y(y) = 10\,x\,y^2, \qquad 0 < x < y < 1.$$

c) $$1 - \int_0^1 \left(\int_0^{y/a} 10\,x\,y^2\, dx \right) dy = 1 - \int_0^1 5\,\frac{y^4}{a^2}\, dy$$

$$= 1 - \frac{1}{a^2}.$$



OR $$1 - \int_0^{1/a} \left(\int_{a\,x}^{1} 10\,x\,y^2\, dy \right) dx = \ldots = 1 - \frac{1}{a^2}.$$

OR $$\int_0^1 \left(\int_{y/a}^{y} 10\,x\,y^2\, dx \right) dy = \ldots = 1 - \frac{1}{a^2}.$$

OR $$\int_0^{1/a} \left(\int_x^{a\,x} 10\,x\,y^2\, dy \right) dx + \int_{1/a}^{1} \left(\int_x^{1} 10\,x\,y^2\, dy \right) dx = \ldots = 1 - \frac{1}{a^2}.$$

d) $$f_{X|Y}(x\,|\,y) = \frac{2\,x}{y^2}, \qquad 0 < x < y. \qquad f_{X|Y}(x\,|\,y = \tfrac{1}{2}) = 8\,x, \qquad 0 < x < \frac{1}{2}.$$

$$P\left(X > \frac{1}{3} \,\middle|\, Y = \frac{1}{2} \right) = \int_{1/3}^{1/2} 8\,x\,dx = \mathbf{\frac{5}{9}}.$$

e) $$E(X\,|\,Y = y) = \int_0^y x \cdot \frac{2\,x}{y^2}\, dx = \frac{2\,y}{3}, \qquad 0 < y < 1.$$

f) $$f_X(x) = \int_x^1 10\,x\,y^2\,dy = \frac{10}{3}\left(x - x^4\right), \qquad 0 < x < 1.$$

$$f_{Y|X}(y\,|\,x) = \frac{f(x,y)}{f_X(x)} = \frac{3\,y^2}{1 - x^3}, \qquad x < y < 1.$$

$$f_{Y|X}(y\,|\,x = \tfrac{1}{4}) = \frac{192\,y^2}{63}, \qquad \frac{1}{4} < y < 1.$$

$$P\left(Y < \frac{1}{2} \,\middle|\, X = \frac{1}{4}\right) = \int_{1/4}^{1/2} \frac{192\,y^2}{63}\,dy = \mathbf{\frac{1}{9}}.$$

g) Since the support of (X, Y) is $0 < \underline{x < y} < 1$, $\quad P\left(Y > \frac{1}{3} \,\middle|\, X = \frac{1}{2}\right) = 1.$

OR

$$f_{Y|X}(y\,|\,x = \tfrac{1}{2}) = \frac{24\,y^2}{7}, \qquad \frac{1}{2} < y < 1.$$

$$P\left(Y > \frac{1}{3} \,\middle|\, X = \frac{1}{2}\right) = \int_{1/2}^{1} \frac{24}{7}\,y^2\,dy = 1.$$

h) $$E(Y\,|\,X = x) = \int_x^1 y \cdot \frac{3\,y^2}{1 - x^3}\,dy = \frac{3}{4} \cdot \frac{1 - x^4}{1 - x^3} = \frac{3}{4} \cdot \frac{1 + x + x^2 + x^3}{1 + x + x^2}, \qquad 0 < x < 1.$$

i) $$P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} \left(\int_x^1 10\,x\,y^2\,dy\right) dx = \int_{1/4}^{1/2} \frac{10}{3}\,x\left(1 - x^3\right) dx$$

$$= \frac{10}{3} \cdot \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_{1/4}^{1/2} = \mathbf{\frac{449}{1536}} \approx 0.2923.$$

j) $$P\left(\frac{1}{4} < X < \frac{1}{2} \,\middle|\, Y = \frac{5}{8}\right) = \int_{1/4}^{1/2} \frac{2\,x}{(5/8)^2}\,dx = \mathbf{\frac{12}{25}} = \mathbf{0.48}.$$

8. Let $\lambda > 0$. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

$$p(x, y) = \frac{\lambda^x e^{-\lambda}}{(x+1)!}, \qquad x, y - \text{integers}, \ 0 \le y \le x.$$

a) Verify that $p(x, y)$ is a legitimate probability mass function.

1. $p(x, y) \ge 0$ for all (x, y). ✓

2. $$\sum_{x=0}^{\infty} \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda}}{(x+1)!} = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot (x+1) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = 1.$$ ✓

b) Find the marginal probability mass function for X.

$$p_X(x) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda}}{(x+1)!} = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x - \text{integer}, \ x \ge 0.$$

X has a Poisson(λ) distribution.

$\Rightarrow$ $\quad \mathrm{E}(\mathrm{X}) = \lambda. \quad \mathrm{Var}(\mathrm{X}) = \lambda. \quad \mathrm{E}(\mathrm{X}^2) = \lambda^2 + \lambda.$

c) Find $\mathrm{E}(\mathrm{Y})$, $\mathrm{E}(\mathrm{X} \cdot \mathrm{Y})$.

$$\mathrm{E}(\mathrm{Y}) = \sum_{x=0}^{\infty} \sum_{y=0}^{x} y \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot \sum_{y=0}^{x} y$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot \frac{x \cdot (x+1)}{2} = \frac{1}{2} \cdot \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = \frac{\mathrm{E}(\mathrm{X})}{2} = \frac{\lambda}{2}.$$

$$\mathrm{E}(\mathrm{X} \cdot \mathrm{Y}) = \sum_{x=0}^{\infty} \sum_{y=0}^{x} xy \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot \sum_{y=0}^{x} y$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot \frac{x \cdot (x+1)}{2} = \frac{1}{2} \cdot \sum_{x=0}^{\infty} x^2 \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \frac{\mathrm{E}(\mathrm{X}^2)}{2} = \frac{\lambda + \lambda^2}{2}.$$

d) Find the moment-generating function $\mathrm{M}(t_1, t_2)$.

$$\mathrm{M}(t_1, t_2) = \sum_{x=0}^{\infty} \sum_{y=0}^{x} e^{t_1 x + t_2 y} \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} = \sum_{x=0}^{\infty} e^{t_1 x} \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot \sum_{y=0}^{x} e^{t_2 y}$$

$$= \sum_{x=0}^{\infty} e^{t_1 x} \cdot \frac{\lambda^x e^{-\lambda}}{(x+1)!} \cdot \frac{e^{t_2 (x+1)} - 1}{e^{t_2} - 1}$$

$$= \frac{1}{e^{t_2} - 1} \cdot \frac{e^{-\lambda}}{\lambda \cdot e^{t_1}} \cdot \left[\sum_{x=0}^{\infty} \frac{\left(\lambda \cdot e^{t_1 + t_2}\right)^{(x+1)}}{(x+1)!} - \sum_{x=0}^{\infty} \frac{\left(\lambda \cdot e^{t_1}\right)^{(x+1)}}{(x+1)!} \right]$$

$$= \frac{1}{e^{t_2} - 1} \cdot \frac{e^{-\lambda}}{\lambda \cdot e^{t_1}} \cdot \left[e^{\lambda e^{t_1 + t_2}} - e^{\lambda e^{t_1}} \right].$$

e) Find the conditional probability distribution $p_{\mathrm{Y}|\mathrm{X}}(y|x)$ of Y given $\mathrm{X} = x$.

f) Find conditional expectation $\mathrm{E}(\mathrm{Y}|\mathrm{X})$ and use it to find $\mathrm{E}(\mathrm{Y})$ and $\mathrm{E}(\mathrm{X} \cdot \mathrm{Y})$.

$$p_{\mathrm{Y}|\mathrm{X}}(y|x) = \frac{p(x,y)}{p_{\mathrm{X}}(x)} = \frac{1}{x+1}, \qquad y \text{ – integers, } 0 \le y \le x.$$

$\Rightarrow$ Y | X has Uniform distribution in integers $0, 1, 2, \dots, x$.

$\Rightarrow$ $\mathrm{E}(\mathrm{Y}|\mathrm{X}=x) = \dfrac{x}{2}$. $\qquad \Rightarrow$ $\mathrm{E}(\mathrm{Y}|\mathrm{X}) = \dfrac{\mathrm{X}}{2}$.

$\Rightarrow$ $\mathrm{E}(\mathrm{Y}) = \mathrm{E}(\mathrm{E}(\mathrm{Y}|\mathrm{X})) = \dfrac{\mathrm{E}(\mathrm{X})}{2} = \dfrac{\lambda}{2}$.

$$\mathrm{E}(\mathrm{X} \cdot \mathrm{Y}) = \mathrm{E}(\mathrm{X} \cdot \mathrm{E}(\mathrm{Y}|\mathrm{X})) = \frac{\mathrm{E}\left(\mathrm{X}^2\right)}{2} = \frac{\lambda + \lambda^2}{2}.$$

g) Find $\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \sigma_{\mathrm{XY}}$.

$$\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \mathrm{E}(\mathrm{X} \cdot \mathrm{Y}) - \mathrm{E}(\mathrm{X}) \cdot \mathrm{E}(\mathrm{Y}) = \frac{\lambda + \lambda^2}{2} - \lambda \cdot \frac{\lambda}{2} = \frac{\lambda}{2}.$$