Examples for 10/05/2020 (4)

1. In a grocery store in Pawnee, IN, the price of a pound of bacon (U) and the price of a dozen of eggs (V) vary from day to day and jointly follow a bivariate normal with

$$\mu_U = \$4.34, \quad \sigma_U = \$0.10, \quad \mu_V = \$2.22, \quad \sigma_V = \$0.03, \quad \rho_{UV} = 0.60.$$

a) Find the probability that on a given day, a pound of bacon costs more than \$4.30. That is, find $P(U > 4.30)$.

b) Suppose that on a given day, a dozen of eggs costs \$2.26. Find the probability that a pound of bacon costs more than \$4.30. That is, find $P(U > 4.30 \mid V = 2.26)$.

c) Suppose that on a given day, a dozen of eggs costs \$2.19. Find the probability that a pound of bacon costs more than \$4.30. That is, find $P(U > 4.30 \mid V = 2.19)$.

d) Find the probability that on a given day, a dozen of eggs costs more than \$2.25. That is, find $P(V > 2.25)$.

e) Suppose that on a given day, a pound of bacon costs \$4.30. Find the probability that a dozen of eggs costs more than \$2.25. That is, find $P(V > 2.25 \mid U = 4.30)$.

f) Find the probability that on a given day, a pound of bacon costs more than two dozen of eggs. That is, find $P(U > 2V)$.

g) Ron Swanson buys 5 pounds of bacon and 4 dozen of eggs. Find the probability that he paid more than \$30. That is, find $P(5U + 4V > 30)$.

1. (continued)

Suppose that the price of a pound of bacon (U), the price of a dozen of eggs (V), and the price of a pound of ham (W) [in dollars] jointly follow $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution with

$$\boldsymbol{\mu} = \begin{pmatrix} 4.34 \\ 2.22 \\ 3.31 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \alpha & \beta & 0.0024 \\ \gamma & \delta & 0.0009 \\ 0.0024 & 0.0009 & 0.0036 \end{pmatrix}.$$

h) What are the values of α, β, γ, and δ?

i) What are the values of ρ_{UW} and ρ_{VW}?

j) Find the probability that on a given day, a pound of ham costs more than \$3.34. That is, find $P(W > 3.34)$.

k) Ron Swanson buys 5 pounds of bacon, 4 dozen of eggs, and 3 pounds of ham. Find the probability that he paid more than \$40. That is, find $P(5U + 4V + 3W > 40)$.

l) Alex buys 0.5 pounds of bacon, 1 dozen of eggs, and 1.5 pounds of ham. Find the probability that he paid more than \$9.60. That is, find $P(0.5U + V + 1.5W > 9.60)$.

m) Sam-I-am (*Green Eggs and Ham* by Dr. Seuss) buys 6 dozen of eggs, and 5 pounds of ham. Find the probability that he paid more than \$30. That is, find $P(6V + 5W > 30)$.

Answers:

1. In a grocery store in Pawnee, IN, the price of a pound of bacon (U) and the price of a dozen of eggs (V) vary from day to day and jointly follow a bivariate normal with

$$\mu_U = \$4.34, \quad \sigma_U = \$0.10, \quad \mu_V = \$2.22, \quad \sigma_V = \$0.03, \quad \rho_{UV} = 0.60.$$

a) Find the probability that on a given day, a pound of bacon costs more than \$4.30. That is, find $P(U > 4.30)$.

$$P(U > 4.30) = P(Z > \frac{4.30 - 4.34}{0.10}) = P(Z > -0.40) = \mathbf{0.6554}.$$

b) Suppose that on a given day, a dozen of eggs costs \$2.26. Find the probability that a pound of bacon costs more than \$4.30. That is, find $P(U > 4.30 \mid V = 2.26)$.

Given V = 2.26, U has Normal distribution

with mean $4.34 + 0.6 \cdot \frac{0.10}{0.03} \cdot (2.26 - 2.22) = 4.42$

and variance $\left(1 - 0.6^2\right) \cdot 0.10^2 = 0.0064$ (standard deviation 0.08).

$$P(U > 4.30 \mid V = 2.26) = P(Z > \frac{4.30 - 4.42}{0.08}) = P(Z > -1.50) = \mathbf{0.9332}.$$

c) Suppose that on a given day, a dozen of eggs costs \$2.19. Find the probability that a pound of bacon costs more than \$4.30. That is, find $P(U > 4.30 \mid V = 2.19)$.

Given V = 2.19, U has Normal distribution

with mean $4.34 + 0.6 \cdot \dfrac{0.10}{0.03} \cdot (2.19 - 2.22) = 4.28$

and variance $\left(1 - 0.6^2\right) \cdot 0.10^2 = 0.0064$ (standard deviation 0.08).

$$P(U > 4.30 \mid V = 2.19) = P\left(Z > \frac{4.30 - 4.28}{0.08}\right) = P(Z > 0.25) = \mathbf{0.4013}.$$

d) Find the probability that on a given day, a dozen of eggs costs more than \$2.25. That is, find $P(V > 2.25)$.

$$P(V > 2.25) = P\left(Z > \frac{2.25 - 2.22}{0.03}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$

e) Suppose that on a given day, a pound of bacon costs \$4.30. Find the probability that a dozen of eggs costs more than \$2.25. That is, find $P(V > 2.25 \mid U = 4.30)$.

Given U = 4.30, V has Normal distribution

with mean $2.22 + 0.6 \cdot \dfrac{0.03}{0.10} \cdot (4.30 - 4.34) = 2.2128$

and variance $\left(1 - 0.6^2\right) \cdot 0.03^2 = 0.000576$ (standard deviation 0.024).

$$P(V > 2.25 \mid U = 4.30) = P\left(Z > \frac{2.25 - 2.2128}{0.024}\right) = P(Z > 1.55) = \mathbf{0.0606}.$$

f) Find the probability that on a given day, a pound of bacon costs more than two dozen of eggs. That is, find $P(U > 2V)$.

Want $P(U > 2V) = P(U - 2V > 0) = ?$

$U - 2V$ has Normal distribution,

$$E(U - 2V) = \mu_U - 2\mu_V = 4.34 - 2 \cdot 2.22 = -0.10,$$

$$\mathrm{Var}(U - 2V) = \sigma_U^2 - 4\sigma_{UV} + 4\sigma_V^2 = \sigma_U^2 - 2\rho\sigma_U\sigma_V + \sigma_V^2$$

$$= 0.10^2 - 4 \cdot 0.6 \cdot 0.10 \cdot 0.03 + 4 \cdot 0.03^2 = 0.0064 \quad (\text{standard deviation } 0.08).$$

$$P(U - 2V > 0) = P(Z > \frac{0 + 0.10}{0.08}) = P(Z > 1.25) = \mathbf{0.1056}.$$

g) Ron Swanson buys 5 pounds of bacon and 4 dozen of eggs. Find the probability that he paid more than \$30. That is, find $P(5U + 4V > 30)$.

$5U + 4V$ has Normal distribution,

$$E(5U + 4V) = 5\mu_U + 4\mu_V = 5 \cdot 4.34 + 4 \cdot 2.22 = 30.58,$$

$$\mathrm{Var}(5U + 3V) = 25\sigma_U^2 + 40\sigma_{UV} + 16\sigma_V^2 = 25\sigma_U^2 + 40\rho\sigma_U\sigma_V + 16\sigma_V^2$$

$$= 25 \cdot 0.10^2 + 40 \cdot 0.6 \cdot 0.10 \cdot 0.03 + 16 \cdot 0.03^2 = 0.3364$$

(standard deviation 0.58).

$$P(5U + 4V > 30) = P(Z > \frac{30 - 30.58}{0.58}) = P(Z > -1.00) = \mathbf{0.8413}.$$

1. (continued)

Suppose that the price of a pound of bacon (U), the price of a dozen of eggs (V), and the price of a pound of ham (W) [in dollars] jointly follow $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution with

$$\boldsymbol{\mu} = \begin{pmatrix} 4.34 \\ 2.22 \\ 3.31 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \alpha & \beta & 0.0024 \\ \gamma & \delta & 0.0009 \\ 0.0024 & 0.0009 & 0.0036 \end{pmatrix}.$$

h) What are the values of α, β, γ, and δ?

$$\alpha = \sigma_{\text{U}}^2 = 0.10^2 = \mathbf{0.0100}.$$

$$\beta = \gamma = \sigma_{\text{UV}} = 0.6 \cdot 0.10 \cdot 0.03 = \mathbf{0.0018}.$$

$$\delta = \sigma_{\text{V}}^2 = 0.03^2 = \mathbf{0.0009}.$$

i) What are the values of ρ_{UW} and ρ_{VW}?

$$\rho_{\text{UW}} = \frac{\sigma_{\text{UW}}}{\sigma_{\text{U}}\,\sigma_{\text{W}}} = \frac{0.0024}{\sqrt{0.0100} \times \sqrt{0.0036}} = \mathbf{0.40}.$$

$$\rho_{\text{VW}} = \frac{\sigma_{\text{VW}}}{\sigma_{\text{V}}\,\sigma_{\text{W}}} = \frac{0.0009}{\sqrt{0.0009} \times \sqrt{0.0036}} = \mathbf{0.50}.$$

j) Find the probability that on a given day, a pound of ham costs more than \$3.34. That is, find $\text{P}(\text{W} > 3.34)$.

$$\text{P}(\text{W} > 3.34) = \text{P}\left(\text{Z} > \frac{3.34 - 3.31}{\sqrt{0.0036}}\right) = \text{P}(\text{Z} > 0.50) = \mathbf{0.3085}.$$

k) Ron Swanson buys 5 pounds of bacon, 4 dozen of eggs, and 3 pounds of ham. Find the probability that he paid more than \$40. That is, find $P(5U + 4V + 3W > 40)$.

$5U + 4V + 3W$ has Normal distribution,

$$E(5U + 4V + 3W) = 5\mu_U + 4\mu_V + 3\mu_W = 5 \cdot 4.34 + 4 \cdot 2.22 + 3 \cdot 3.31 = 40.51,$$

$$\mathrm{Var}(5U + 4V + 3W) = \begin{pmatrix} 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 0.0100 & 0.0018 & 0.0024 \\ 0.0018 & 0.0009 & 0.0009 \\ 0.0024 & 0.0009 & 0.0036 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0644 & 0.0153 & 0.0264 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = 0.4624$$

(standard deviation 0.68).

$$P(5U + 4V + 3W > 40) = P(Z > \frac{40 - 40.51}{0.68}) = P(Z > -0.75) = \mathbf{0.7734}.$$

l) Alex buys 0.5 pounds of bacon, 1 dozen of eggs, and 1.5 pounds of ham. Find the probability that he paid more than \$9.60. That is, find $P(0.5U + V + 1.5W > 9.60)$.

$0.5U + V + 1.5W$ has Normal distribution,

$$E(0.5U + V + 1.5W) = 0.5\mu_U + \mu_V + 1.5\mu_W = 0.5 \cdot 4.34 + 2.22 + 1.5 \cdot 3.31 = 9.355,$$

$$\mathrm{Var}(0.5U + V + 1.5W) = \begin{pmatrix} 0.5 & 1 & 1.5 \end{pmatrix} \begin{pmatrix} 0.0100 & 0.0018 & 0.0024 \\ 0.0018 & 0.0009 & 0.0009 \\ 0.0024 & 0.0009 & 0.0036 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \\ 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0104 & 0.00315 & 0.0075 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \\ 1.5 \end{pmatrix} = 0.0196$$

(standard deviation 0.14).

$$P(0.5U + V + 1.5W > 9.60) = P(Z > \frac{9.60 - 9.355}{0.14}) = P(Z > 1.75) = \mathbf{0.0401}.$$

m) Sam-I-am (*Green Eggs and Ham* by Dr. Seuss) buys 6 dozen of eggs, and 5 pounds of ham. Find the probability that he paid more than \$30. That is, find $P(6V + 5W > 30)$.

$6V + 5W$ has Normal distribution,

$$E(6V+5W) = 6\mu_V + 5\mu_W = 6 \cdot 2.22 + 5 \cdot 3.31 = 29.87,$$

$$\mathrm{Var}(6V+5W) = \begin{pmatrix} 0 & 6 & 5 \end{pmatrix} \begin{pmatrix} 0.0100 & 0.0018 & 0.0024 \\ 0.0018 & 0.0009 & 0.0009 \\ 0.0024 & 0.0009 & 0.0036 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0228 & 0.0099 & 0.0234 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 5 \end{pmatrix} = 0.1764$$

(standard deviation 0.42).

OR

$$\mathrm{Var}(6V+5W) = 36\,\sigma_V^2 + 60\,\sigma_{VW} + 25\,\sigma_W^2$$

$$= 36 \cdot 0.0009 + 60 \cdot 0.0009 + 25 \cdot 0.0036 = 0.1764$$

(standard deviation 0.42).

$$P(6V+5W>30) = P\left(Z > \frac{30-29.87}{0.42}\right) \approx P(Z > 0.31) = \mathbf{0.3783}.$$