Examples for 08/31/2020 (1)

Functions of One Random Variable

6.* Let Z be a $N(0,1)$ standard normal random variable.
Show that $X = Z^2$ has a chi-square distribution with 1 degree of freedom.

$$M_X(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(z^2/2\right)\cdot(1-2t)}\, dz = \frac{1}{(1-2t)^{1/2}}, \qquad t < 1/2,$$

since $\dfrac{(1-2t)^{1/2}}{\sqrt{2\pi}} e^{-\left(z^2/2\right)\cdot(1-2t)}$ is the p.d.f. of a $N\left(0, \dfrac{1}{1-2t}\right)$ random variable.

$\Rightarrow$ X has a $\chi^2(1)$ distribution.

OR

$$F_X(x) = P(X \le x) = P(Z^2 \le x) = P(-\sqrt{x} \le Z \le \sqrt{x})$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = F_Z(\sqrt{x}) - F_Z(-\sqrt{x}).$$

$$f_X(x) = F'_X(x) = \left(\frac{1}{2\sqrt{x}}\right) f_Z(\sqrt{x}) - \left(-\frac{1}{2\sqrt{x}}\right) f_Z(-\sqrt{x})$$

$$= \left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-x/2}\right) - \left(-\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-x/2}\right)$$

$$= \frac{1}{\sqrt{\pi}\ 2^{1/2}} x^{-1/2} e^{-x/2}$$

$$= \frac{1}{\Gamma(1/2)\, 2^{1/2}}\, x^{(1/2)-1}\, e^{-x/2}, \qquad x > 0.$$

$\Rightarrow$ X has a $\chi^2(1)$ distribution.

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du, \qquad x > 0 \qquad\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(x) = (x-1)\,\Gamma(x-1) \qquad\qquad \Gamma(n) = (n-1)! \quad \text{if } n \text{ is an integer}$$

Consider a continuous random variable X, with p.d.f. f and c.d.f. F, where F is strictly increasing on some interval I, $F = 0$ to the left of I, and $F = 1$ to the right of I. I may be a bounded interval or an unbounded interval such as the whole real line. $F^{-1}(u)$ is then well defined for $0 < u < 1$.

Fact 1:

Let U ~ Uniform$(0, 1)$, and let $X = F^{-1}(U)$. Then the c.d.f. of X is F.

Proof: $\quad P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x).$

Fact 2:

Let $U = F(X)$; then U has a Uniform$(0, 1)$ distribution.

Proof: $\quad P(U \le u) = P(F(X) \le u) = P(X \le F^{-1}(u)) = F(F^{-1}(u)) = u.$

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7. Let X have a uniform distribution on the interval $(0, 1)$.

a) Find the c.d.f. and the p.d.f. of $\mathrm{Y} = \dfrac{\mathrm{X}}{1-\mathrm{X}}$.

$$y = \frac{x}{1-x} \qquad 0 < x < 1 \qquad \Rightarrow \qquad 0 < y < \infty.$$

$$y = \frac{x}{1-x} \qquad x = \frac{y}{1+y} = g^{-1}(y) \qquad \frac{dx}{dy} = \frac{(1+y)-y}{(1+y)^2} = \frac{1}{(1+y)^2}$$

$$f_{\mathrm{Y}}(y) = f_{\mathrm{X}}(g^{-1}(y)) \left|\frac{dx}{dy}\right| = 1 \times \left|\frac{1}{(1+y)^2}\right| = \frac{1}{(1+y)^2}, \qquad 0 < y < \infty.$$

$$\mathrm{F_Y}(y) = \int_0^y \frac{1}{(1+u)^2}\,du = -\frac{1}{1+u}\Bigg|_0^y = 1 - \frac{1}{1+y}, \qquad 0 < y < \infty.$$

OR

$$\mathrm{F_X}(x) = x, \qquad 0 < x < 1.$$

$$\mathrm{F_Y}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}\left(\frac{\mathrm{X}}{1-\mathrm{X}} \le y\right) = \mathrm{P}\left(\mathrm{X} \le \frac{y}{1+y}\right) = \frac{y}{1+y}, \qquad y > 0.$$

$$f_{\mathrm{Y}}(y) = \mathrm{F}'_{\mathrm{Y}}(y) = \frac{(1+y)-y}{(1+y)^2} = \frac{1}{(1+y)^2}, \qquad y > 0.$$

b) Find the c.d.f. and the p.d.f. of $\mathrm{W} = \ln \mathrm{Y}$.

$$w = \ln y \qquad 0 < y < \infty \qquad \Rightarrow \qquad -\infty < w < \infty.$$

$$w = \ln y \qquad y = e^{w} = g^{-1}(w) \qquad \frac{dy}{dw} = e^{w}$$

$$f_{\mathrm{W}}(w) = f_{\mathrm{Y}}(g^{-1}(w)) \left| \frac{dy}{dw} \right| = \frac{1}{\left(1+e^{w}\right)^{2}} \times \left| e^{w} \right|$$

$$= \frac{e^{w}}{\left(1+e^{w}\right)^{2}} = \frac{e^{-w}}{\left(1+e^{-w}\right)^{2}}, \qquad -\infty < w < \infty.$$

$$\mathrm{F}_{\mathrm{W}}(w) = \int_{-\infty}^{w} \frac{e^{u}}{\left(1+e^{u}\right)^{2}}\,du = -\frac{1}{1+e^{u}} \Bigg|_{-\infty}^{w}$$

$$= 1 - \frac{1}{1+e^{w}} = \frac{e^{w}}{1+e^{w}} = \frac{1}{1+e^{-w}}, \qquad -\infty < w < \infty.$$

OR

$$\mathrm{F}_{\mathrm{Y}}(y) = 1 - \frac{1}{1+y} = \frac{y}{1+y}, \qquad 0 < y < \infty.$$

$$\mathrm{F}_{\mathrm{W}}(w) = \mathrm{P}(\mathrm{W} \le w) = \mathrm{P}(\ln \mathrm{Y} \le w) = \mathrm{P}(\mathrm{Y} \le e^{w}) = \mathrm{F}_{\mathrm{Y}}(e^{w})$$

$$= \frac{e^{w}}{1+e^{w}} = \frac{1}{1+e^{-w}}, \qquad -\infty < w < \infty.$$

$$f_{\mathrm{W}}(w) = \mathrm{F}_{\mathrm{W}}'(w) = \frac{e^{w}\left(1+e^{w}\right) - e^{w} \cdot e^{w}}{\left(1+e^{w}\right)^{2}} = \frac{e^{w}}{\left(1+e^{w}\right)^{2}} = \frac{e^{-w}}{\left(1+e^{-w}\right)^{2}},$$

$$-\infty < w < \infty.$$

These are the transformations in the logistic regression:

Frist, odds ratio $= \dfrac{\text{probability}}{1-\text{probability}}$.

Then $\ln(\text{odds ratio}) = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + \varepsilon$.

The distribution of W is called the (standard) logistic distribution.

8. Let X have a **logistic distribution** with p.d.f.

$$f(x) = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}, \qquad -\infty < x < \infty.$$

Show that

$$\mathrm{Y} = \frac{1}{1+e^{-\mathrm{X}}}$$

has a $\mathrm{U}(0, 1)$ distribution.

$$\mathrm{F}_{\mathrm{X}}(x) = \frac{1}{1+e^{-x}}, \qquad -\infty < x < \infty.$$

$\Rightarrow$ $\quad \mathrm{Y} = \mathrm{F}_{\mathrm{X}}(\mathrm{X})$ has a Uniform$(0, 1)$ distribution by Fact 2.

OR

It is easier to note that

$$\frac{dy}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{(1+e^{-x})^2}{e^{-x}}.$$

Say the solution of x in terms of y is given by x^*. Then the p.d.f. of Y is

$$g(y) = \frac{e^{-x^*}}{(1+e^{-x^*})^2}\left|\frac{(1+e^{-x^*})^2}{e^{-x^*}}\right| = 1, \qquad 0 < y < 1,$$

as $-\infty < x < \infty$ maps onto $0 < y < 1$. Thus Y is $U(0,1)$.