STAT 410

Examples for 10/30/2020 (1)

Fall 2020
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Central Limit Theorem

$X_1, X_2, \ldots, X_n$ are i.i.d. with mean μ and variance σ^2.

$$\sqrt{n}\left(\overline{X}-\mu\right)/\sigma = \left(\sum_{i=1}^{n} X_i - n\mu\right)\Big/\sqrt{n}\,\sigma \xrightarrow{D} Z, \qquad Z \sim N(0,1).$$

Δ-Method

$$\sqrt{n}\,(X_n - \theta) \xrightarrow{D} N\left(0, \sigma^2\right)$$

$g(x)$ is differentiable at θ and $g'(\theta) \neq 0$

$$\Rightarrow \qquad \sqrt{n}\,(g(X_n) - g(\theta)) \xrightarrow{D} N\left(0, \left(g'(\theta)\right)^2 \sigma^2\right)$$

Intuition:

By CLT, $\overline{X} - \mu$ is approximately $N\left(0, \frac{\sigma^2}{n}\right)$ for large n.

If $g(x)$ is differentiable at μ and x is "close" to μ,

$$g(x) \approx g(\mu) + g'(\mu)(x-\mu).$$

Therefore, if $g'(\mu) \neq 0$,

$g(\overline{X})$ is approximately $N\left(g(\mu), [g'(\mu)]^2 \frac{\sigma^2}{n}\right)$ for large n.

1. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{1}{\theta} \cdot x^{\frac{1}{\theta}-1}, \qquad 0 < x < 1, \qquad 0 < \theta < \infty.$$

a) Recall that the method of moments estimator of θ, $\tilde{\theta} = \frac{1-\overline{X}}{\overline{X}} = \frac{1}{\overline{X}} - 1$, is a consistent estimator of θ. Show that $\tilde{\theta}$ is asymptotically normally distributed (as $n \to \infty$). Find the parameters.

$$E(X) = \frac{1}{1+\theta}. \qquad E(X^2) = \int_0^1 \left(x^2 \cdot \frac{1}{\theta} \cdot x^{\frac{1}{\theta}-1} \right) dx = \frac{1}{1+2\theta}.$$

$$\sigma^2 = \operatorname{Var}(X) = \frac{1}{1+2\theta} - \left(\frac{1}{1+\theta}\right)^2 = \frac{\theta^2}{(1+2\theta)(1+\theta)^2}.$$

By CLT, $\sqrt{n}\left(\overline{X} - \mu\right) \xrightarrow{D} N\left(0, \sigma^2\right)$.

Since $g(x) = \frac{1}{x} - 1$ is differentiable at $\mu = \frac{1}{1+\theta}$, $g'(\mu) = -(1+\theta)^2 \neq 0$,

$$\Rightarrow \quad \sqrt{n}\left(g(\overline{X}) - g(\mu)\right) \xrightarrow{D} N\left(0, \left(-(1+\theta)^2\right)^2 \cdot \frac{\theta^2}{(1+2\theta)(1+\theta)^2}\right).$$

$$\Rightarrow \quad \sqrt{n}\left(\tilde{\theta} - \theta\right) \xrightarrow{D} N\left(0, \frac{\theta^2 (1+\theta)^2}{(1+2\theta)}\right).$$

$$\Rightarrow \quad \text{For large } n, \qquad \tilde{\theta} \sim N\left(\theta, \frac{\theta^2 (1+\theta)^2}{(1+2\theta) n}\right).$$

b) Suggest a $100(1-\alpha)\%$ confidence interval for θ.

For large n, $\tilde{\theta} \sim N\left(\theta, \dfrac{\theta^2(1+\theta)^2}{(1+2\theta)n}\right)$. $\tilde{\theta} = \dfrac{1-\overline{X}}{\overline{X}}$

$\tilde{\theta} \pm z_{\alpha/2} \dfrac{\tilde{\theta}\left(1+\tilde{\theta}\right)}{\sqrt{\left(1+2\tilde{\theta}\right)n}}$ would have an approximate $100(1-\alpha)\%$ confidence level for large n.

c) Recall that the maximum likelihood estimator of θ, $\hat{\theta} = -\dfrac{1}{n}\cdot\sum\limits_{i=1}^{n} \ln X_i$, is a consistent estimator of θ. Show that $\hat{\theta}$ is asymptotically normally distributed (as $n \to \infty$). Find the parameters.

Let $W_i = -\ln X_i$, $i = 1, 2, \dots, n$. Then $E(W) = \theta$, $Var(W) = \theta^2$.

By CLT, $\sqrt{n}\left(\overline{W} - \mu_W\right) \xrightarrow{D} N\left(0, \sigma_W^2\right)$.

$\Rightarrow$ $\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{D} N\left(0, \theta^2\right)$.

$\Rightarrow$ For large n, $\hat{\theta} \sim N\left(\theta, \dfrac{\theta^2}{n}\right)$.

d) Suggest a $100(1-\alpha)\%$ confidence interval for θ.

For large n, $\hat{\theta} \sim N\left(\theta, \dfrac{\theta^2}{n}\right)$. $\hat{\theta} = -\dfrac{1}{n}\cdot\sum\limits_{i=1}^{n} \ln X_i$

$$\Rightarrow \quad P\left(-z_{\alpha/2} < \frac{\hat{\theta}-\theta}{\theta/\sqrt{n}} < z_{\alpha/2}\right) \approx 1-\alpha.$$

$$\Rightarrow \quad P\left(\frac{\hat{\theta}}{1+\frac{z_{\alpha/2}}{\sqrt{n}}} < \theta < \frac{\hat{\theta}}{1-\frac{z_{\alpha/2}}{\sqrt{n}}} \right) \approx 1-\alpha.$$

$$\Rightarrow \quad \left(\frac{\hat{\theta}}{1+\frac{z_{\alpha/2}}{\sqrt{n}}}, \frac{\hat{\theta}}{1-\frac{z_{\alpha/2}}{\sqrt{n}}} \right)$$

would have an approximate $100(1-\alpha)\%$ confidence level for large n.

OR

For large n, $\quad \hat{\theta} \sim N\left(\theta, \frac{\theta^2}{n} \right)$. $\quad \hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^{n} \ln X_i$

$\hat{\theta} \pm z_{\alpha/2} \frac{\hat{\theta}}{\sqrt{n}}$ would have an approximate $100(1-\alpha)\%$ confidence level for large n.

OR

$F_X(x) = x^{1/\theta}, \qquad 0 < x < 1.$

Let $W_i = -\ln X_i, \quad i = 1, 2, \ldots, n.$

Then $F_W(w) = P(W \le w) = P(X \ge e^{-w})$

$$= 1 - F_X(e^{-w}) = 1 - e^{-w/\theta}, \qquad w > 0.$$

$\Rightarrow \quad W_1, W_2, \ldots, W_n$ are i.i.d. Exponential(θ).

$$M_{\hat{\theta}}(t) = M_{\overline{W}}(t) = \left[M_W\left(\frac{t}{n} \right) \right]^n = \frac{1}{\left(1 - \frac{\theta}{n} t \right)^n}, \qquad t < n/\theta.$$

$\Rightarrow \quad \hat{\theta}$ is $\text{Gamma}\left(n, \dfrac{\theta}{n} \right)$. $\quad \Rightarrow \quad \dfrac{2n\hat{\theta}}{\theta}$ is $\chi^2(2n)$.

$$\Rightarrow \quad P\left(\chi^2_{1-\alpha/2} < \frac{2n\hat{\theta}}{\theta} < \chi^2_{\alpha/2} \right) = 1 - \alpha.$$

$2n$ degrees of freedom

$$\Rightarrow \quad P\left(\frac{2n\hat{\theta}}{\chi^2_{1-\alpha/2}} > \theta > \frac{2n\hat{\theta}}{\chi^2_{\alpha/2}} \right) = 1 - \alpha.$$

$2n$ degrees of freedom

$$\Rightarrow \quad \left(\frac{2n\hat{\theta}}{\chi^2_{\alpha/2}}, \frac{2n\hat{\theta}}{\chi^2_{1-\alpha/2}} \right)$$

would have an exact $100(1-\alpha)\%$ confidence level for any n.

$2n$ degrees of freedom

1½. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Geometric(p) distribution (the number of independent trials until the first "success"). That is,

$$P(X_1 = k) = (1-p)^{k-1}\, p, \quad k = 1, 2, 3, \ldots .$$

Show that $\hat{p} = \tilde{p} = 1/\overline{X}$ is asymptotically normally distributed (as $n \to \infty$).

$$\operatorname{Var}(X) = \frac{1-p}{p^2}.$$

By CLT, $\sqrt{n}\left(\overline{X} - 1/p\right) \xrightarrow{D} N\left(0, \frac{1-p}{p^2}\right)$.

$g(x) = 1/x$ is differentiable at $1/p$, $g'(1/p) = -p^2 \neq 0$.

$$\Rightarrow \quad \sqrt{n}\left(g(\overline{X}) - g\left(1/p\right)\right) \xrightarrow{D} N\left(0, \left(-p^2\right)^2 \cdot \frac{1-p}{p^2}\right).$$

$$\Rightarrow \quad \sqrt{n}\,(\hat{p} - p) \xrightarrow{D} N\left(0, p^2(1-p)\right).$$

$\Rightarrow$ For large n, $\hat{p} \sim N\left(p, \frac{p^2(1-p)}{n}\right)$.

$\Rightarrow$ $\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}^2(1-\hat{p})}{n}}$ would have an approximate $100(1-\alpha)\%$ confidence level for large n.

2. Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

Recall: the maximum likelihood estimator of λ is $\hat{\lambda} = \dfrac{2n}{\sum\limits_{i=1}^{n} X_i^2}$;

$W = X^2$ has Gamma$(\alpha = 2, \theta = \frac{1}{\lambda})$ distribution.

Show that $\hat{\lambda}$ is asymptotically normally distributed (as $n \to \infty$).

Find the parameters.

$$E(W) = \alpha\theta = \frac{2}{\lambda}. \qquad \operatorname{Var}(W) = \alpha\theta^2 = \frac{2}{\lambda^2}.$$

Central Limit Theorem: $\sqrt{n}\left(\overline{W} - \dfrac{2}{\lambda}\right) \xrightarrow{D} N\left(0, \dfrac{2}{\lambda^2}\right)$.

$\hat{\lambda} = \dfrac{2}{\overline{W}}$. Consider $g(x) = \dfrac{2}{x}$. Then $g'(x) = -\dfrac{2}{x^2}$,

$$g(\overline{W}) = \hat{\lambda}, \qquad g\left(\frac{2}{\lambda}\right) = \lambda, \qquad g'\left(\frac{2}{\lambda}\right) = -\frac{\lambda^2}{2}.$$

By the Δ-method, $\sqrt{n}\left(g(\overline{W}) - g\left(\dfrac{2}{\lambda}\right)\right) \xrightarrow{D} N\left(0, \left(g'\left(\dfrac{2}{\lambda}\right)\right)^2 \cdot \dfrac{2}{\lambda^2}\right)$.

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow{D} N\left(0, \left(-\frac{\lambda^2}{2}\right)^2 \cdot \frac{2}{\lambda^2}\right) = N\left(0, \frac{\lambda^2}{2}\right).$$

$\Rightarrow$ For large n, $\hat{\lambda}$ is approximately $N\left(\lambda, \dfrac{\lambda^2}{2n}\right)$.

3. Let X_n be $\chi^2(n)$.

What is the limiting distribution of $W_n = \sqrt{X_n} - \sqrt{n}$?

Hint: We already know that (a) $Y_n = \dfrac{X_n}{n} \xrightarrow{P} 1$ and

(b) $Z_n = \dfrac{X_n - n}{\sqrt{2n}} = \dfrac{\sqrt{n}}{\sqrt{2}}\left(\dfrac{X_n}{n} - 1\right) \xrightarrow{D} N(0,1)$.

$$W_n = \sqrt{X_n} - \sqrt{n} = \sqrt{n}\left(\sqrt{\frac{X_n}{n}} - 1\right).$$

$$Z_n = \frac{X_n - n}{\sqrt{2n}} = \frac{\sqrt{n}}{\sqrt{2}}\left(\frac{X_n}{n} - 1\right) \xrightarrow{D} N(0,1).$$

$$\Rightarrow \quad \sqrt{n}\left(\frac{X_n}{n} - 1\right) \xrightarrow{D} N(0,2).$$

Let $g(x) = \sqrt{x}$. Then $g(x)$ is differentiable, $g'(x) = \dfrac{1}{2\sqrt{x}}$, and $g'(1) = \dfrac{1}{2}$.

$$\Rightarrow \quad \sqrt{n}\left(\sqrt{\frac{X_n}{n}} - \sqrt{1}\right) \xrightarrow{D} N\left(0, \left(g'(1)\right)^2 \cdot 2\right) = N\left(0, \left(\frac{1}{2}\right)^2 \cdot 2\right).$$

$$\Rightarrow \quad W_n = \sqrt{X_n} - \sqrt{n} = \sqrt{n}\left(\sqrt{\frac{X_n}{n}} - 1\right) \xrightarrow{D} N\left(0, \frac{1}{2}\right).$$

3¼. 5.2.16 (7th edition) **4.3.16** (6th edition)

a) $M_{X_1}(t) = (1-t)^{-1}, \qquad t < 1.$

$$M_{Y_n}(t) = E\left(e^{t\sqrt{n}\left(\overline{X}_n - 1\right)}\right) = e^{-t\sqrt{n}}E\left(e^{t\left(X_1 + X_2 + \ldots + X_n\right)/\sqrt{n}}\right)$$

$$= e^{-t\sqrt{n}}\left(M_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n = e^{-t\sqrt{n}}\left(1 - \frac{t}{\sqrt{n}}\right)^{-n}$$

$$= \left(e^{t/\sqrt{n}} - \frac{t}{\sqrt{n}}e^{t/\sqrt{n}}\right)^{-n}, \qquad \frac{t}{\sqrt{n}} < 1.$$

b) $e^{t/\sqrt{n}} = 1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2n} + o\left(\frac{1}{n}\right).$

$$\Rightarrow \qquad M_{Y_n}(t) = \left(1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2n} - \frac{t}{\sqrt{n}} - \frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)^{-n} = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^{-n}.$$

As $n \to \infty$, $\qquad M_{Y_n}(t) \to \exp\left\{\frac{t^2}{2}\right\} = M_Z(t),$

where Z has Standard Normal $N(0, 1)$ distribution.

$$\Rightarrow \qquad Y_n \xrightarrow{D} Z, \qquad Z \sim N(0, 1).$$

3½. 5.2.17 (7th edition) **4.3.17** (6th edition)

From 5.2.16 (4.3.16), $\qquad Y_n = \sqrt{n}\left(\overline{X}_n - 1\right) \xrightarrow{D} N(0, 1).$

Let $g(x) = \sqrt{x}$. Then $g(x)$ is differentiable, $g'(x) = \frac{1}{2\sqrt{x}}$, and $g'(1) = \frac{1}{2}$.

$$\Rightarrow \qquad \sqrt{n}\left(\sqrt{\overline{X}_n} - 1\right) \xrightarrow{D} N\left(0, \left(g'(1)\right)^2 \cdot 1\right) = N\left(0, \frac{1}{4}\right).$$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a uniform distribution on the interval $(0, \theta)$.

$$f(x;\theta) = \begin{cases} \dfrac{1}{\theta} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{F}(x;\theta) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{\theta} & 0 < x < \theta \\ 1 & x > \theta \end{cases}$$

$$\mathrm{E}(\mathrm{X}) = \frac{\theta}{2} \qquad \mathrm{Var}(\mathrm{X}) = \frac{\theta^2}{12}$$

Recall that the method of moments estimator of θ is $\tilde{\theta} = 2\overline{\mathrm{X}}$.

By CLT, $\sqrt{n}\left(\overline{\mathrm{X}} - \mu\right) \xrightarrow{D} N\left(0, \sigma^2\right)$.

$\Rightarrow$ For large n, $\tilde{\theta} \sim \mathrm{N}\left(\theta, \dfrac{\theta^2}{3n}\right)$.

$\Rightarrow$ $\left(\dfrac{\tilde{\theta}}{1 + \dfrac{z_{\alpha/2}}{\sqrt{3n}}}, \dfrac{\tilde{\theta}}{1 - \dfrac{z_{\alpha/2}}{\sqrt{3n}}}\right)$ would have an approximate $100(1-\alpha)\%$ confidence level for large n.

OR

$\tilde{\theta} \pm z_{\alpha/2} \dfrac{\tilde{\theta}}{\sqrt{3n}}$ would have an approximate $100(1-\alpha)\%$ confidence level for large n.

OR

Recall that the maximum likelihood estimator of θ is $\hat{\theta} = \max X_i$.

$$F_{\max X_i}(x) = (F(x))^n = (x/\theta)^n, \quad 0 < x < \theta.$$

$$f_{\max X_i}(x) = \frac{n \cdot x^{n-1}}{\theta^n}, \quad 0 < x < \theta.$$

$$P(c\theta < \max X_i < \theta) = F_{\max X_i}(\theta) - F_{\max X_i}(c\theta) = 1 - c^n.$$

$$\Rightarrow \quad P\left(\max X_i < \theta < \frac{\max X_i}{c}\right) = 1 - c^n.$$

$$\Rightarrow \quad \left(\max X_i, \frac{\max X_i}{\alpha^{1/n}}\right) \quad \text{has a } 100(1-\alpha)\% \text{ confidence level for any } n.$$

OR

Recall $\quad n(\theta - \max X_i) \xrightarrow{D}$ Exponential distribution with mean θ.

(Example 9 from Examples for 10/26/2020 (1))

$$\Rightarrow \quad P\left(-\theta \ln\left(1 - \frac{\alpha}{2}\right) < n(\theta - \max X_i) < -\theta \ln\left(\frac{\alpha}{2}\right)\right) \approx 1 - \alpha \qquad \text{for large } n.$$

$$\Rightarrow \quad P\left(\frac{\max X_i}{1 + \frac{1}{n}\ln\left(1 - \frac{\alpha}{2}\right)} < \theta < \frac{\max X_i}{1 + \frac{1}{n}\ln\left(\frac{\alpha}{2}\right)}\right) \approx 1 - \alpha \qquad \text{for large } n.$$

$$\left(\frac{\max X_i}{1 + \frac{1}{n}\ln\left(1 - \frac{\alpha}{2}\right)}, \frac{\max X_i}{1 + \frac{1}{n}\ln\left(\frac{\alpha}{2}\right)}\right)$$

would have an approximate $100(1-\alpha)\%$ confidence level for large n.