STAT 410 **Examples for 10/19/2020 (3)** Fall 2020
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1 – 4. The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0$, $\delta > 0$. Consider the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall (Examples for 08/28/2020 (3)):

$W = X^{\delta}$ has an Exponential$(\theta = \frac{1}{\beta})$ = Gamma$(\alpha = 1, \theta = \frac{1}{\beta})$ distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the above probability distribution.

$\Rightarrow \quad Y = \sum_{i=1}^{n} X_i^{\delta} = \sum_{i=1}^{n} W_i$ has a Gamma$(\alpha = n, \theta = \frac{1}{\beta})$ distribution. **! ! !**

1. Suppose δ is known.
Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\beta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find a closed-form expression for $E(X^k)$, $k > -\delta$.

"Hint" 1: $u = \beta x^{\delta}$ "Hint" 2: $\Gamma(a) = \int_0^{\infty} u^{a-1} e^{-u}\,du, \quad a > 0.$

b) Obtain a method of moments estimator for β, $\tilde{\beta}$.

c) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$.
Obtain a method of moments estimate for β, $\tilde{\beta}$.

"Hint": $\Gamma(a)$ R: `> gamma(a)`
Excel: `=GAMMA(a)`

2. Suppose δ is known.

Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

d) Obtain the maximum likelihood estimator for β, $\hat{\beta}$.

e) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$.

Obtain the maximum likelihood estimate for β, $\hat{\beta}$.

f) Is the maximum likelihood estimator $\hat{\beta}$ an unbiased estimator of β?

If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\hat{\beta}$.

"Hint" 1: $\mathrm{E}(a\,☺) = a\,\mathrm{E}(☺)$. "Hint" 2: $\frac{1}{♥} = ♥^{-1}$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(T_\alpha^k) = \frac{\theta^k\,\Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{\lambda^k\,\Gamma(\alpha)}, \qquad k > -\alpha.$$

g) Find $\mathrm{MSE}(\hat{\beta})$, where $\hat{\beta}$ the maximum likelihood estimator of β.

"Hint" 1: $\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta$. You have $\mathrm{E}(\hat{\beta})$ from part (f).

"Hint" 2: $\mathrm{Var}(a\,☺) = a^2\,\mathrm{Var}(☺)$. $\mathrm{Var}(☺) = \mathrm{E}(☺^2) - [\mathrm{E}(☺)]^2$.

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"Hint" 4: $\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta})$.

3. Suppose β is known.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

h) Obtain an equation for the maximum likelihood estimator for δ, $\hat{\delta}$.

"Hint": $\frac{d}{d\delta} \ln L(\delta) = 0.$

4. Suppose $\delta = 3$, $\beta = 4$, $n = 5$.

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size $n = 5$ from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

i) Find the probability $P\left(\sum_{i=1}^{5} X_i^3 > 2.5\right) = P\left(\sum_{i=1}^{n} X_i^{\delta} > 2.5\right)$.

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$P(T_\alpha \le t) = P(X_t \ge \alpha) \quad \text{and} \quad P(T_\alpha > t) = P(X_t \le \alpha - 1),$$

where X_t has a Poisson(λt) distribution.

j) Find a such that $P\left(\sum_{i=1}^{5} X_i^3 > a\right) = P\left(\sum_{i=1}^{n} X_i^{\delta} > a\right) = 0.10$.

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T_\alpha/\theta = 2\lambda T_\alpha$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

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$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall (Examples for 08/28/2020 (3)):

$\mathrm{W} = \mathrm{X}^{\delta}$ has an Exponential$\left(\theta = \frac{1}{\beta}\right)$ = Gamma$\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample from the above probability distribution.

$$\Rightarrow \qquad \mathrm{Y} = \sum_{i=1}^{n} \mathrm{X}_i^{\delta} = \sum_{i=1}^{n} \mathrm{W}_i \quad \text{has a Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right) \text{ distribution.} \qquad !!!$$

1. Suppose δ is known.
Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample from the distribution with the probability density function

$$f(x;\beta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find a closed-form expression for $\mathrm{E}(\mathrm{X}^k)$, $k > -\delta$.

"Hint" 1: $u = \beta x^{\delta}$ "Hint" 2: $\Gamma(a) = \int_0^{\infty} u^{a-1} e^{-u}\,du, \quad a > 0.$

$$\mathrm{E}(\mathrm{X}^k) = \int_0^{\infty} x^k\,\beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}\,dx \qquad u = \beta x^{\delta} \qquad du = \beta\,\delta\,x^{\delta-1}\,dx$$

$$= \int_0^{\infty} \left(\frac{u}{\beta}\right)^{k/\delta} e^{-u}\,du = \frac{1}{\beta^{k/\delta}}\,\Gamma\left(\frac{k}{\delta} + 1\right), \qquad k > -\delta.$$

OR

$$\mathrm{E}(\mathrm{X}^k) = \int_0^{\infty} x^k \, \beta \, \delta \, x^{\delta-1} e^{-\beta x^{\delta}} dx \qquad u = x^{\delta} \qquad du = \delta \, x^{\delta-1} dx$$

$$= \int_0^{\infty} u^{k/\delta} \, \beta \, e^{-\beta u} du = \frac{\Gamma\left(\frac{k}{\delta}+1\right)}{\beta^{k/\delta}} \int_0^{\infty} \frac{\beta^{k/\delta\,+\,1}}{\Gamma\left(\frac{k}{\delta}+1\right)} u^{k/\delta\,+\,1\,-\,1} \, \beta \, e^{-\beta u} du$$

$$= \frac{1}{\beta^{k/\delta}} \Gamma\left(\frac{k}{\delta}+1\right),$$

since $\dfrac{\beta^{k/\delta\,+\,1}}{\Gamma\left(\frac{k}{\delta}+1\right)} u^{k/\delta\,+\,1\,-\,1} \, \beta \, e^{-\beta u}$ is the p.d.f. of

Gamma$\left(\alpha = \dfrac{k}{\delta}+1, \theta = \dfrac{1}{\beta}\right)$ distribution.

OR

$\mathrm{W} = \mathrm{X}^{\delta}$ has an Exponential$\left(\theta = \dfrac{1}{\beta}\right)$ = Gamma$\left(\alpha = 1, \theta = \dfrac{1}{\beta}\right)$ distribution.

If T_{α} has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_{\alpha}^{k}) = \frac{\theta^k \, \Gamma(\alpha + k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha + k)}{\lambda^k \, \Gamma(\alpha)}, \qquad k > -\alpha.$$

$$\mathrm{E}(\mathrm{X}^k) = \mathrm{E}(\mathrm{W}^{k/\delta}) = \frac{\Gamma\left(1+\frac{k}{\delta}\right)}{\beta^{k/\delta} \, \Gamma(1)} = \frac{1}{\beta^{k/\delta}} \Gamma\left(\frac{k}{\delta}+1\right), \qquad \frac{k}{\delta} > -1.$$

b) Obtain a method of moments estimator for β, $\tilde{\beta}$.

$$\mathrm{E}(\mathrm{X}) = \frac{1}{\beta^{1/\delta}}\,\Gamma\left(\frac{1}{\delta}+1\right). \qquad \overline{\mathrm{X}} = \frac{1}{\tilde{\beta}^{1/\delta}}\,\Gamma\left(\frac{1}{\delta}+1\right).$$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{1}{\delta}+1\right)}{\overline{\mathrm{X}}}\right)^{\delta}.$$

c) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$.
Obtain a method of moments estimate for β, $\tilde{\beta}$.

"Hint": $\Gamma(a)$ R: `> gamma(a)`
Excel: `=GAMMA(a)`

$$\bar{x} = \frac{2.8}{5} = 0.56. \qquad \Gamma\left(\frac{4}{3}\right) \approx 0.89298.$$

$$\tilde{\beta} = \left(\frac{0.89298}{0.56}\right)^{3} \approx \mathbf{4.0547}.$$

OR

b) $E(X^2) = \dfrac{1}{\beta^{2/\delta}} \Gamma\left(\dfrac{2}{\delta}+1\right).$ $\overline{X^2} = \dfrac{1}{\tilde{\beta}^{2/\delta}} \Gamma\left(\dfrac{2}{\delta}+1\right).$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{2}{\delta}+1\right)}{\overline{X^2}}\right)^{\delta/2}.$$

c) $\overline{x^2} = \dfrac{2.42}{5} = 0.484.$ $\Gamma\left(\dfrac{5}{3}\right) \approx 0.902745.$

$$\tilde{\beta}_2 = \left(\frac{0.902745}{0.484}\right)^{1.5} \approx \mathbf{2.5473}.$$

OR

b) $E(X^3) = \dfrac{1}{\beta^{3/\delta}} \Gamma\left(\dfrac{3}{\delta}+1\right).$ $\overline{X^3} = \dfrac{1}{\tilde{\beta}^{3/\delta}} \Gamma\left(\dfrac{3}{\delta}+1\right).$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{3}{\delta}+1\right)}{\overline{X^3}}\right)^{\delta/3}.$$

c) $\overline{x^3} = \dfrac{2.5}{5} = 0.50.$ $\Gamma(2) = 1.$

$$\tilde{\beta}_3 = \left(\frac{1}{0.50}\right)^{1} = \mathbf{2}.$$

OR

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2. Suppose δ is known.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

d) Obtain the maximum likelihood estimator for β, $\hat{\beta}$.

$$L(\beta) = \prod_{i=1}^{n}\left(\beta\,\delta\,x_i^{\delta-1}\,e^{-\beta x_i^{\delta}}\right).$$

$$\ln L(\beta) = n\cdot\ln\beta + n\cdot\ln\delta + (\delta-1)\cdot\sum_{i=1}^{n}\ln x_i - \beta\cdot\sum_{i=1}^{n}x_i^{\delta}.$$

$$\frac{d}{d\beta}\ln L(\beta) = \frac{n}{\beta} - \sum_{i=1}^{n}x_i^{\delta} = 0. \qquad \Rightarrow \qquad \hat{\beta} = \frac{n}{\sum_{i=1}^{n}X_i^{\delta}}.$$

e) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$.

Obtain the maximum likelihood estimate for β, $\hat{\beta}$.

$$\sum_{i=1}^{n}x_i^3 = 2.5. \qquad \hat{\beta} = \frac{5}{2.5} = \mathbf{2}.$$

f) Is the maximum likelihood estimator $\hat{\beta}$ an unbiased estimator of β?
If $\hat{\beta}$ is not an unbiased estimator of β, construct an unbiased estimator of β based on $\hat{\beta}$.

"Hint" 1: $E(a ☺) = a E(☺)$. "Hint" 2: $\frac{1}{♥} = ♥^{-1}$.

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^k) = \frac{\theta^k \, \Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{\lambda^k \, \Gamma(\alpha)}, \qquad k > -\alpha.$$

$Y = \sum_{i=1}^{n} X_i^\delta = \sum_{i=1}^{n} W_i$ has a Gamma$(\alpha = n, \theta = \frac{1}{\beta})$ distribution.

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} X_i^\delta} = \frac{n}{Y}. \qquad a = n, \quad ☺ = \frac{1}{Y}, \quad ♥ = Y.$$

$$E\left(\frac{1}{Y}\right) = E\left(Y^{-1}\right) = \frac{\Gamma(n-1)}{\beta^{-1}\,\Gamma(n)} = \frac{\beta}{n-1}.$$

$$E(\hat{\beta}) = E\left(\frac{n}{Y}\right) = n\,E\left(\frac{1}{Y}\right) = n \cdot \frac{\beta}{n-1} = \frac{n}{n-1} \cdot \beta = \beta + \frac{\beta}{n-1} \neq \beta.$$

$\hat{\beta}$ is NOT an unbiased estimator of β. $\text{bias}(\hat{\beta}) = E(\hat{\beta}) - \beta = \frac{\beta}{n-1}$.

Consider $\hat{\hat{\beta}} = \frac{n-1}{n} \cdot \hat{\beta} = \frac{n-1}{\sum_{i=1}^{n} X_i^\delta}$. Then $E(\hat{\hat{\beta}}) = \frac{n-1}{n} \cdot E(\hat{\beta}) = \beta$.

$\hat{\hat{\beta}}$ is an unbiased estimator of β.

g) Find $\mathrm{MSE}(\hat{\beta})$, where $\hat{\beta}$ the maximum likelihood estimator of β.

"Hint" 1: $\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta.$ You have $\mathrm{E}(\hat{\beta})$ from part (f).

"Hint" 2: $\mathrm{Var}(a ☺) = a^2\,\mathrm{Var}(☺).$ $\mathrm{Var}(☺) = \mathrm{E}(☺^2) - [\mathrm{E}(☺)]^2.$

"Hint" 3: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^k) = \frac{\theta^k\,\Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{\lambda^k\,\Gamma(\alpha)}, \qquad k > -\alpha.$$

"Hint" 4: $\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta}).$

$$\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta = \frac{\beta}{n-1}.$$

$$\mathrm{E}\Big(\frac{1}{\mathrm{Y}^2}\Big) = \mathrm{E}\big(\mathrm{Y}^{-2}\big) = \frac{\Gamma(n-2)}{\beta^{-2}\,\Gamma(n)} = \frac{\beta^2}{(n-1)\,(n-2)}.$$

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}\Big(\frac{n}{\mathrm{Y}}\Big) = n^2\,\mathrm{Var}\Big(\frac{1}{\mathrm{Y}}\Big) = n^2\left[\frac{\beta^2}{(n-1)(n-2)} - \frac{\beta^2}{(n-1)^2}\right]$$

$$= \frac{n^2\beta^2}{(n-1)^2\,(n-2)}.$$

$$\mathrm{MSE}(\hat{\beta}) = (\mathrm{bias}(\hat{\beta}))^2 + \mathrm{Var}(\hat{\beta}) = \frac{\beta^2}{(n-1)^2} + \frac{n^2\beta^2}{(n-1)^2\,(n-2)}$$

$$= \frac{\left(n^2+n-2\right)\beta^2}{(n-1)^2\,(n-2)} = \frac{(n+2)\beta^2}{(n-1)(n-2)}.$$

3. Suppose β is known.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

h) Obtain an equation for the maximum likelihood estimator for δ, $\hat{\delta}$.

"Hint": $\dfrac{d}{d\delta}\ln L(\delta) = 0.$

$$L(\delta) = \prod_{i=1}^{n}\left(\beta\,\delta\,x_i^{\delta-1}\,e^{-\beta x_i^{\delta}}\right).$$

$$\ln L(\delta) = n\cdot\ln\beta + n\cdot\ln\delta + (\delta-1)\cdot\sum_{i=1}^{n}\ln x_i - \beta\cdot\sum_{i=1}^{n} x_i^{\delta}.$$

$$\frac{d}{d\delta}\ln L(\delta) = \frac{n}{\delta} + \sum_{i=1}^{n}\ln x_i - \beta\cdot\sum_{i=1}^{n} x_i^{\delta}\ln x_i = 0.$$

This equation cannot be solved algebraically for δ in closed form.

The solution could be approximated for given $x_1, x_2, \ldots, x_n$ by using iterative numerical procedures.

4. Suppose $\delta = 3$, $\beta = 4$, $n = 5$.

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size $n = 5$ from the distribution with the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

i) Find the probability $P\left(\sum_{i=1}^{5} X_i^3 > 2.5\right) = P\left(\sum_{i=1}^{n} X_i^{\delta} > 2.5\right)$.

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$P(T_\alpha \le t) = P(X_t \ge \alpha) \quad \text{and} \quad P(T_\alpha > t) = P(X_t \le \alpha - 1),$$

where X_t has a Poisson(λt) distribution.

$$Y = \sum_{i=1}^{5} X_i^3 = \sum_{i=1}^{n} X_i^{\delta} \text{ has a Gamma}\left(\alpha = n = 5, \theta = \frac{1}{\beta} = \frac{1}{4}\right) \text{ distribution.}$$

$$P\left(\sum_{i=1}^{5} X_i^3 > 2.5\right) = P(T_5 > 2.5) = P(\text{Poisson}(\beta \cdot 2.5) \le 5 - 1)$$

$$= P(\text{Poisson}(4 \cdot 2.5) \le 4) = P(\text{Poisson}(10) \le 4) = \mathbf{0.029}.$$

OR

$$P(T_5 > 2.5) = \int_{2.5}^{\infty} \frac{4^5}{\Gamma(5)}\, t^{5-1}\, e^{-4t}\, dt = \int_{2.5}^{\infty} \frac{4^5}{4!}\, t^4\, e^{-4t}\, dt = \ldots \approx 0.029253.$$

OR $$P(T_5 > 2.5) = P(2\beta T_5 > 2\beta\, 2.5) = P(\chi^2(10) > 20) \approx 0.029253.$$

j) Find a such that $P\left(\sum_{i=1}^{5} X_i^3 > a\right) = P\left(\sum_{i=1}^{n} X_i^{\delta} > a\right) = 0.10.$

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T_\alpha/\theta = 2\lambda T_\alpha$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$Y = \sum_{i=1}^{5} X_i^3 = \sum_{i=1}^{n} X_i^{\delta}$ has a Gamma$\left(\alpha = n = 5, \theta = \frac{1}{\beta} = \frac{1}{4}\right)$ distribution.

$2Y/\theta = 2\beta \sum_{i=1}^{n} X_i^3$ has a chi-square distribution with $r = 2\alpha = 2n = 10$ d.f.

$$\Rightarrow \quad P\left(2\beta \sum_{i=1}^{5} X_i^3 > \chi^2_{0.10}(10)\right) = 0.10.$$

$\chi^2_{0.10}(10) = 15.99.$ $\Rightarrow$ $2\beta a = 8a = 15.99.$

$\Rightarrow$ $a = \mathbf{1.99875}.$

OR

$$P\left(\sum_{i=1}^{5} X_i^3 > a\right) = P(T_5 > a) = P(\text{Poisson}(\beta \cdot a) \le 5-1) = P(\text{Poisson}(4 \cdot a) \le 4).$$

$P(\text{Poisson}(8) \le 4) = 0.10.$ $\Rightarrow$ $4a = 8.$

$\Rightarrow$ $a = \mathbf{2}.$

$$a \approx \frac{15.98718}{8} \approx 1.9984.$$