Examples for 11/02/2020 (1)

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Theorem 1 (Factorization Theorem):

Let $X_1, X_2, \ldots, X_n$ denote random variables with joint p.d.f. or p.m.f. $f(x_1, x_2, \ldots, x_n; \theta)$, which depends on the parameter θ. The statistic $Y = u(X_1, X_2, \ldots, X_n)$ is **sufficient** for θ if and only if

$$f(x_1, x_2, \ldots, x_n; \theta) = \phi[u(x_1, x_2, \ldots, x_n); \theta] \cdot h(x_1, x_2, \ldots, x_n),$$

where ϕ depends on $x_1, x_2, \ldots, x_n$ only through $u(x_1, x_2, \ldots, x_n)$ and $h(x_1, x_2, \ldots, x_n)$ does not depend on θ.

½. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x; \lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

Find the sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for λ.

$$f(x_1, x_2, \ldots x_n; \lambda) = f(x_1; \lambda)\, f(x_2; \lambda) \,\ldots\, f(x_n; \lambda)$$

$$= \left[2^n \lambda^{2n} e^{-\lambda \sum_{i=1}^n x_i^2} \right] \left(\prod_{i=1}^n x_i^3 \right).$$

By Factorization Theorem, $Y = \sum_{i=1}^n X_i^2$ is a sufficient statistic for λ.

1. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a Poisson distribution with mean λ. That is,

$$f(k;\lambda) = P(X_1 = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}, \qquad k = 0, 1, 2, 3, \dots .$$

a) Use Factorization Theorem to find $Y = u(X_1, X_2, \dots, X_n)$, a sufficient statistic for λ.

$$f(x_1;\lambda)\ f(x_2;\lambda)\ \dots\ f(x_n;\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!}$$

$$= \lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda} \cdot \prod_{i=1}^{n} \frac{1}{x_i!}.$$

By Factorization Theorem,

$$Y = \sum_{i=1}^{n} X_i \quad \text{is a sufficient statistic for } \lambda.$$

[$\Rightarrow$ $\overline{X}$ is also a sufficient statistic for λ.]

b) Show that $P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n \mid Y = y)$ does not depend on λ.

Since $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $n\lambda$, if $\sum_{i=1}^{n} x_i = y$,

$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n \mid Y = y) =$

$$= \frac{\dfrac{\lambda^{x_1} \cdot e^{-\lambda}}{x_1!} \cdot \dfrac{\lambda^{x_2} \cdot e^{-\lambda}}{x_2!} \cdot \dots \cdot \dfrac{\lambda^{x_n} \cdot e^{-\lambda}}{x_n!}}{\dfrac{(n\lambda)^y \cdot e^{-n\lambda}}{y!}} = \frac{y!}{x_1!\, x_2! \dots x_n!} \cdot \left(\frac{1}{n}\right)^y$$

does not depend on λ.

[$P(X_1 = x_1, X_1 = x_1, \dots, X_n = x_n \mid Y = y) = 0$ if $\sum_{i=1}^{n} x_i \neq y$.]

1¼. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from an Exponential distribution with probability with mean θ.

$$\prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{\theta}\right)^n \cdot \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right).$$

By Factorization Theorem, $\sum\limits_{i=1}^{n} X_i$ is a sufficient statistic for θ.

OR

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} = \exp\left[-\frac{1}{\theta}\cdot x - \ln\theta\right]. \qquad K(x) = x.$$

$\Rightarrow \quad Y = \sum\limits_{i=1}^{n} X_i$ is a sufficient statistic for θ.

Since $\sum\limits_{i=1}^{n} X_i$ has Gamma distribution with $\alpha = n$, one can show that the conditional distribution of $X_1, X_2, \dots, X_n$ given $\sum\limits_{i=1}^{n} X_i = y$ does not depend on θ: the conditional p.d.f. is

$$\frac{\frac{1}{\theta} e^{-x_1/\theta} \cdot \frac{1}{\theta} e^{-x_2/\theta} \cdot \ldots \cdot \frac{1}{\theta} e^{-x_n/\theta}}{\frac{1}{\Gamma(n)\theta^n} y^{n-1} e^{-y/\theta}} = \frac{(n-1)!}{y^{n-1}} \qquad \text{if } \sum_{i=1}^{n} x_i = y,$$

zero otherwise.

(Each X_i is the time of the first occurrence of some random event or the time between two consecutive occurrences, $\sum\limits_{i=1}^{n} X_i$ is the time of the nth occurrence. If we wait until the nth occurrence, knowing when the first $n-1$ occurrences were does not help us estimate θ, the average time between occurrences, better than if we just knew the time of the nth occurrence.)

1½. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Geometric distribution with probability of "success" p.

$$f(x_1;p)\ f(x_2;p)\ \ldots\ f(x_n;p) = p^n\,(1-p)^{\Sigma x_i - n}.$$

By Factorization Theorem, $\sum\limits_{i=1}^{n} X_i$ is a sufficient statistic for p.

$[\ \Rightarrow\ \overline{X}$ is also a sufficient statistic for p. $]$

$[\ \Rightarrow\ 1/\overline{X}$ is also a sufficient statistic for p. $]$

OR

$$f(x;\theta) = p\cdot(1-p)^{x-1} = \exp[\,\ln(1-p)\cdot x - \ln(1-p) + \ln p\,].$$

$K(x) = x \qquad \Rightarrow \qquad Y = \sum\limits_{i=1}^{n} X_i$ is a sufficient statistic for p.

Since $\sum\limits_{i=1}^{n} X_i$ has Negative Binomial distribution with $r = n$, one can show that the conditional distribution of $X_1, X_2, \ldots, X_n$ given $\sum\limits_{i=1}^{n} X_i = y$ does not depend on p: the conditional p.m.f. is

$$\frac{p(1-p)^{x_1-1}\cdot p(1-p)^{x_2-1}\cdot\ldots\cdot p(1-p)^{x_n-1}}{\binom{y-1}{n-1}\,p^n\,(1-p)^{y-n}} = \frac{1}{\binom{y-1}{n-1}} \qquad \text{if } \sum_{i=1}^{n} x_i = y,$$

zero otherwise.

(Each X_i is the number of attempts to get the first "success", $\sum\limits_{i=1}^{n} X_i$ is the number of attempts to the first n "successes". If we wait until the nth "success", knowing when the first $n-1$ "successes" were does not help us estimate p better that if we just know how many attempts it took to get n "successes".)

2. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a Uniform$(0, \theta)$ distribution. That is,

$$f(x;\theta) = \frac{1}{\theta}, \quad 0 < x < \theta, \qquad \text{zero elsewhere.}$$

a) Use Factorization Theorem to find $Y = u(X_1, X_2, \dots, X_n)$, a sufficient statistic for θ.

Define
$$I\{A\} = \begin{cases} 1 & \text{if A is true} \\ 0 & \text{if A is false} \end{cases}.$$

Then
$$f(x;\theta) = \frac{1}{\theta} \cdot I\{x < \theta\} \cdot I\{x > 0\}.$$

$$f(x_1;\theta)\, f(x_2;\theta)\, \dots\, f(x_n;\theta) = \frac{1}{\theta^n} \cdot \prod_{i=1}^{n} I\{x_i < \theta\} \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

$$= \frac{1}{\theta^n} \cdot I\{\max x_i < \theta\} \cdot I\{\min x_i > 0\}.$$

By Factorization Theorem,

$\max X_i$ is a sufficient statistic for θ.

b) Show that the conditional distribution of $X_1, X_2, \dots, X_n$ given $Y = y$ does not depend on θ.

Since $f_{\max X_i}(y) = \dfrac{n \cdot y^{n-1}}{\theta^n}$, $0 < y < \theta$, zero elsewhere, if $\max x_i = y$,

$$\frac{f(x_1;\theta)\, f(x_2;\theta)\, \dots\, f(x_n;\theta)}{f_{\max X_i}(y)} = \frac{\frac{1}{\theta} \cdot \frac{1}{\theta} \cdot \dots \cdot \frac{1}{\theta}}{\frac{n \cdot y^{n-1}}{\theta^n}} = \frac{1}{n \cdot y^{n-1}}$$

does not depend on θ.

Theorem 2:

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with a p.d.f. or p.m.f. of the exponential form

$$f(x;\theta) = \exp[\,p(\theta)\,K(x) + S(x) + q(\theta)\,],$$

on a support free of θ. The statistic $Y = \sum_{i=1}^{n} K(X_i)$ is sufficient for θ.

½. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0, \qquad \text{zero elsewhere.}$$

b) Use Theorem 2 to find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for λ.

$$f(x;\lambda) = \exp\left\{-\lambda \cdot x^2 + \ln 2 + 2\ln\lambda + 3\ln x\right\}. \qquad \Rightarrow \qquad K(x) = x^2.$$

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} X_i^2 \text{ is a sufficient statistic for } \lambda.$$

1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Poisson distribution with mean λ.

c) Use Theorem 2 to find $Y = u(X_1, X_2, \ldots, X_n)$, a sufficient statistic for λ.

$$f(x;\lambda) = \exp[\,\ln\lambda \cdot x - \ln(x!) - \lambda\,]. \qquad K(x) = x.$$

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} X_i \text{ is a sufficient statistic for } \lambda.$$

[$\Rightarrow$ $\overline{X}$ is also a sufficient statistic for λ.]

Cannot use Theorem 2 for Example 2, Uniform$(0, \theta)$, since the support depends on θ.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\theta) = \frac{1}{\theta} \cdot x^{\frac{1}{\theta}-1}, \qquad 0 < x < 1, \qquad 0 < \theta < \infty.$$

Find a sufficient statistic for θ.

$$f(x) = \exp\left\{ \frac{1-\theta}{\theta} \ln x - \ln \theta \right\}. \qquad K(x) = \ln x.$$

$$\Rightarrow \qquad Y_1 = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} \ln X_i \quad \text{is a sufficient statistic for } \theta.$$

$$\Rightarrow \qquad Y_2 = e^{Y_1} = \prod_{i=1}^{n} X_i \quad \text{is also a sufficient statistic for } \theta.$$

OR

$$\prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{\theta^n} \cdot \left(\prod_{i=1}^{n} x_i \right)^{1-\theta/\theta}.$$

By Factorization Theorem, $\qquad Y_2 = \prod_{i=1}^{n} X_i$ is sufficient statistic for θ.

$$\Rightarrow \qquad Y_1 = \ln Y_2 = \ln \prod_{i=1}^{n} X_i = \sum_{i=1}^{n} \ln X_i \quad \text{is also a sufficient statistic for } \theta.$$

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a $N(\mu, \sigma^2)$ distribution. Find joint sufficient statistics for μ and σ.

$$\prod_{i=1}^{n} f(x_i;\mu,\sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{n\mu^2}{2\sigma^2}\right\}.$$

$\Rightarrow$ $\sum_{i=1}^{n} X_i^2$ and $\sum_{i=1}^{n} X_i$ are joint sufficient statistics for μ and σ.

$\Rightarrow$ $\sum_{i=1}^{n} X_i^2$ and $\overline{X}$ are joint sufficient statistics for μ and σ.

$\Rightarrow$ $\sum_{i=1}^{n} X_i^2 - n(\overline{X})^2$ and $\overline{X}$ are joint sufficient statistics for μ and σ.

$\Rightarrow$ S^2 and $\overline{X}$ are joint sufficient statistics for μ and σ.

$\Rightarrow$ S and $\overline{X}$ are joint sufficient statistics for μ and σ.

Recall: $$S^2 = \frac{\sum(X_i - \overline{X})^2}{n-1} = \frac{\sum X_i - n(\overline{X})^2}{n-1}.$$

5. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Uniform(a, b) distribution. Find joint sufficient statistics for a and b.

$$f(x;a,b) = \frac{1}{b-a} \cdot I\{x<b\} \cdot I\{x>a\}.$$

$$\prod_{i=1}^{n} f(x_i;a,b) = \frac{1}{(b-a)^n} \cdot \prod_{i=1}^{n} I\{x_i<b\} \cdot \prod_{i=1}^{n} I\{x_i>a\}$$

$$= \frac{1}{(b-a)^n} \cdot I\{\max x_i < b\} \cdot I\{\min x_i > a\}.$$

$\Rightarrow$ $\min X_i$ and $\max X_i$ are joint sufficient statistics for a and b.