STAT 410

Practice Problems 14

Fall 2020
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1. Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with the p.d.f.

$$f(x;\lambda) = \lambda\, e^{-\lambda x}, \qquad x > 0.$$

We wish to test $H_0 : \lambda = 25$ vs. $H_1 : \lambda > 25$.

a) If $n = 3$, find a uniformly most powerful rejection region with the significance level $\alpha = 0.05$ that is based on the statistic $\sum_{i=1}^{3} X_i$.

Hint: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

b) Find the power or the test in part (a) if $\lambda = 33.7$.

c) Suppose $\sum_{i=1}^{3} x_i = 0.06$. Find the p-value.

Hint: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_T(t) = P(T \le t) = P(Y \ge \alpha)$, where Y has a Poisson$(\lambda t)$ distribution.

d) Find the significance level of the rejection rule "Reject H_0 if $\sum_{i=1}^{3} X_i \le 0.04$".

Hint: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_T(t) = P(T \le t) = P(Y \ge \alpha)$, where Y has a Poisson$(\lambda t)$ distribution.

e) What is the power of the rejection rule "Reject H_0 if $\sum_{i=1}^{3} X_i \le 0.04$" if $\lambda = 30$? If $\lambda = 40$? If $\lambda = 50$? If $\lambda = 100$?

2. Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with mean θ.

a) Find a uniformly most powerful rejection region for testing

$$H_0: \theta = 3 \text{ vs. } H_1: \theta > 3$$

that is based on the statistic $\sum_{i=1}^{n} X_i$.

That is, find a rejection region that is most powerful for testing

$$H_0: \theta = 3 \text{ vs. } H_1: \theta = \theta_1 \quad \text{for all } \theta_1 > 3.$$

b) If $n = 12$, use the fact that $\frac{2}{\theta} \cdot \sum_{i=1}^{12} X_i$ is $\chi^2(24)$ to find a uniformly most powerful rejection region for testing $H_0: \theta = 3$ vs. $H_1: \theta > 3$ of size $\alpha = 0.10$.

c) If $\theta = 7$, what is the power of the rejection region from part (b)?

3. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$ distribution. We are interested in testing $H_0: \sigma = 2$ vs. $H_1: \sigma = 5$.

a) Use the likelihood ratio to show that the best rejection region is

$$C = \{(x_1, x_2, \dots, x_n) : \sum_{i=1}^{n} x_i^2 > c\}.$$

b) If $n = 10$, find the value of c such that $\alpha = 0.10$.

Hint: $\dfrac{\sum (X_i - \mu)^2}{\sigma^2}$ has a $\chi^2(n)$ distribution; here $\mu = 0$.

c) If $n = 10$ and c is from part (b), find the probability of Type II Error.

4. **8.1.8** (7th and 6th edition)

If $X_1, X_2, \dots, X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find the form of the best (most powerful) rejection region for testing $H_0: \theta = 1$ against $H_1: \theta = 2$.

5. Let X_1, X_2, X_3, X_4 be a random sample of size $n = 4$ from a Geometric(p) distribution (the number of independent trials until the first "success"). That is,

$$P(X_1 = k) = (1-p)^{k-1}\, p, \quad k = 1, 2, 3, \dots .$$

We are interested in testing $H_0: p = 0.30$ vs. $H_1: p = 0.40$.

a) Use the likelihood ratio to find the best rejection region in terms of $\sum_{i=1}^{n} x_i$.

b) Find the best rejection region with the significance level closest to 0.07.

c) If the rejection region from part (b) is used, find the power of the test.

Hint: If $X_1, X_2, \dots, X_n$ are independent Geometric(p), then $Y = \sum_{i=1}^{n} X_i$ has Negative Binomial distribution (the number of independent trials until the n^{th} "success"). Then

$$P(Y = y) = \binom{y-1}{n-1} \cdot p^{n} \cdot (1-p)^{y-n}, \quad y = n, n+1, n+2, \dots .$$

EXCEL:	$=$NEGBINOM.DIST$(y-n, n, p, 0)$	gives	$P(Y = y)$
	$=$NEGBINOM.DIST$(y-n, n, p, 1)$	gives	$P(Y \le y)$

6. Consider

$$f_1(x) = \sin x, \quad 0 < x < \pi/2, \quad \text{zero elsewhere},$$

$$f_2(x) = \cos x, \quad 0 < x < \pi/2, \quad \text{zero elsewhere}.$$

You will have just a single observation of X on which to base your choice between

$$H_0: \text{X has p.d.f. } f_1(x) \quad \text{vs.} \quad H_1: \text{X has p.d.f. } f_2(x).$$

Use the likelihood ratio to find the best rejection region with the significance level $\alpha = 0.10$ and find the power of this test.

7. You will have just a single observation of X on which to base your choice between

H_0: X has a Normal distribution with mean $\mu = 5$ and standard deviation $\sigma = 2$

vs.

H_1: X has a Binomial distribution with $n = 25$ and $p = 0.20$.

Consider the rejection rule "Reject H_0 if X is an integer".

Find $\alpha = \mathrm{P}(\text{Type I Error})$ and $\beta = \mathrm{P}(\text{Type II Error})$. *Justify your answer.*

8.* **8.1.6** (7th and 6th edition)

Let $X_1, X_2, \ldots, X_{10}$ be a random sample from a distribution that is $N(\theta_1, \theta_2)$.

Find a best test of the simple hypothesis H_0: $\theta_1 = 0$, $\theta_2 = 1$ against the alternative simple hypothesis H_1: $\theta_1 = 1$, $\theta_2 = 4$.

Answers:

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with the p.d.f.

$$f(x;\lambda) = \lambda e^{-\lambda x}, \qquad x > 0.$$

We wish to test $H_0: \lambda = 25$ vs. $H_1: \lambda > 25$.

a) If $n = 3$, find a uniformly most powerful rejection region with the significance level $\alpha = 0.05$ that is based on the statistic $\sum_{i=1}^{3} X_i$.

Hint: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(H_0; x_1, x_2, \ldots, x_n)}{L(H_1; x_1, x_2, \ldots, x_n)} = \frac{L(25; x_1, x_2, \ldots, x_n)}{L(\lambda; x_1, x_2, \ldots, x_n)}$$

$$= \frac{\prod_{i=1}^{n} 25 e^{-25 x_i}}{\prod_{i=1}^{n} \lambda e^{-\lambda x_i}} = \left(\frac{25}{\lambda}\right)^n \exp\left\{(\lambda - 25)\sum_{i=1}^{n} x_i\right\}.$$

Since $\lambda > 25$, $\qquad \lambda(x_1, x_2, \ldots, x_n) \le k \qquad \Leftrightarrow \qquad \sum_{i=1}^{n} x_i \le c.$

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{3} X_i \le c \mid \lambda = 25\right)$$

$$= P\left(50 \sum_{i=1}^{3} X_i \le 50\, c \mid \lambda = 25\right) = P(\chi^2(6) \le 50\, c).$$

$\Rightarrow \qquad 50\, c = \chi^2_{0.95}(6) = 1.635. \qquad \Rightarrow \qquad c = \mathbf{0.0327}.$

Reject H_0 if $\sum_{i=1}^{3} x_i \le 0.0327.$

b) Find the power or the test in part (a) if $\lambda = 33.7$.

$$\text{Power} = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P\Big(\sum_{i=1}^{3} X_i \le 0.0327 \mid \lambda = 33.7\Big)$$

$$= P\Big(67.4 \sum_{i=1}^{3} X_i \le 2.204 \mid \lambda = 33.7\Big) = P\big(\chi^2(6) \le 2.204\big) = \mathbf{0.10}.$$

c) Suppose $\sum_{i=1}^{3} x_i = 0.06$. Find the p-value.

Hint: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_T(t) = P(T \le t) = P(Y \ge \alpha)$, where Y has a $\text{Poisson}(\lambda t)$ distribution.

$$\text{p-value} = P\Big(\sum_{i=1}^{3} X_i \text{ as extreme or more extreme than } \big(\sum_{i=1}^{3} x_i\big)_{\text{observed}} \mid H_0 \text{ true}\Big)$$

$$= P\Big(\sum_{i=1}^{3} X_i \le 0.06 \mid \lambda = 25\Big) = P(Y \ge 3)$$

where Y has a $\text{Poisson}(25 \times 0.06 = 1.5)$ distribution

$$= 1 - P(Y \le 2) = 1 - 0.809 = \mathbf{0.191}.$$

d) Find the significance level of the rejection rule "Reject H_0 if $\sum_{i=1}^{3} X_i \le 0.04$".

Hint: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_T(t) = P(T \le t) = P(Y \ge \alpha)$, where Y has a $\text{Poisson}(\lambda t)$ distribution.

$$\text{significance level} = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\Big(\sum_{i=1}^{3} X_i \le 0.04 \mid \lambda = 25\Big)$$

$= P(Y \ge 3)$, where Y has a $\text{Poisson}(25 \times 0.04 = 1.0)$ distribution.

$$= 1 - 0.920 = \mathbf{0.080}.$$

e) What is the power of the rejection rule "Reject H_0 if $\sum_{i=1}^{3} X_i \le 0.04$" if $\lambda = 30$? If $\lambda = 40$? If $\lambda = 50$? If $\lambda = 100$?

$$\text{Power}(\lambda) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P\left(\sum_{i=1}^{3} X_i \le 0.04 \mid \lambda\right)$$

$$= P(Y \ge 3), \quad \text{where } Y \text{ has a Poisson}(\lambda \times 0.04) \text{ distribution.}$$

$\text{Power}(\lambda = 30) = P(\text{Poisson}(1.2) \ge 3) = 1 - 0.879 = \mathbf{0.121}$.

$\text{Power}(\lambda = 40) = P(\text{Poisson}(1.6) \ge 3) = 1 - 0.783 = \mathbf{0.217}$.

$\text{Power}(\lambda = 50) = P(\text{Poisson}(2.0) \ge 3) = 1 - 0.677 = \mathbf{0.323}$.

$\text{Power}(\lambda = 100) = P(\text{Poisson}(4.0) \ge 3) = 1 - 0.238 = \mathbf{0.762}$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with mean θ.

$$f_X(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty.$$

a) Find a uniformly most powerful rejection region for testing

$$H_0: \theta = 3 \text{ vs. } H_1: \theta > 3$$

that is based on the statistic $\sum_{i=1}^{n} X_i$.

That is, find a rejection region that is most powerful for testing

$$H_0: \theta = 3 \text{ vs. } H_1: \theta = \theta_1 \quad \text{for all } \theta_1 > 3.$$

Let $\theta_1 > 3$.

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(\theta = 3; x_1, x_2, \ldots, x_n)}{L(\theta = \theta_1; x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{3} e^{-x_i/3}}{\prod_{i=1}^{n} \frac{1}{\theta_1} e^{-x_i/\theta_1}}.$$

$$= \left(\frac{\theta_1}{3}\right)^n \exp\left\{\left(-\frac{1}{3}+\frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i\right\} = \left(\frac{\theta_1}{3}\right)^n \exp\left\{-\frac{\theta_1 - 3}{3\theta_1}\sum_{i=1}^{n} x_i\right\}.$$

If $\theta_1 > 3$, $\qquad \lambda(x_1, x_2, \ldots, x_n) < k \qquad \Leftrightarrow \qquad \sum_{i=1}^{n} x_i > c.$

$\Rightarrow$ Same rejection region for all $\theta_1 > 3$.

$\Rightarrow$ Uniformly most powerful rejection region for $\mathrm{H}_0 : \theta = 3$ vs. $\mathrm{H}_1 : \theta > 3$.

b) If $n = 12$, use the fact that $\frac{2}{\theta} \cdot \sum_{i=1}^{12} \mathrm{X}_i$ is $\chi^2(24)$ to find a uniformly most powerful rejection region for testing $\mathrm{H}_0 : \theta = 3$ vs. $\mathrm{H}_1 : \theta > 3$ of size $\alpha = 0.10$.

$$\mathbf{0.10} = \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}\left(\sum_{i=1}^{12} x_i > c \mid \theta = 3\right)$$

$$= \mathrm{P}\left(\frac{2}{3}\sum_{i=1}^{n} x_i > \frac{2}{3}\, c \mid \theta = 3\right) = \mathrm{P}\left(\chi^2(24) > \frac{2}{3}\, c\right).$$

$\Rightarrow \qquad \frac{2}{3}\, c = \chi^2_{0.10}(24) = 33.20. \qquad \Rightarrow \qquad c = \mathbf{49.8}.$

Reject H_0 if $\sum_{i=1}^{n} x_i > \mathbf{49.8}$. $\qquad (\Leftrightarrow \quad \bar{x} > 4.15 \,)$

c) If $\theta = 7$, what is the power of the rejection region from part (b)?

$$\text{Power}(\theta = 7) = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \theta = 7) = \mathrm{P}\left(\sum_{i=1}^{12} x_i > 49.8 \mid \theta = 7\right)$$

$$= \mathrm{P}\left(\frac{2}{7}\sum_{i=1}^{n} x_i > \frac{2}{7}\, 49.8 \mid \theta = 3\right) \approx \mathrm{P}(\chi^2(24) > 14.23)$$

is **between 0.90 and 0.95**.

EXCEL: =CHISQ.DIST.RT(49.8*2/7,24) $\Rightarrow$ **0.94132**.

3. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$ distribution. We are interested in testing $H_0: \sigma = 2$ vs. $H_1: \sigma = 5$.

a) Use the likelihood ratio to show that the best rejection region is

$$C = \{ (x_1, x_2, \dots, x_n) : \textstyle\sum_{i=1}^{n} x_i^2 > c \}.$$

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(\sigma = 2; x_1, x_2, \dots, x_n)}{L(\sigma = 5; x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,2} \exp\left\{ -x_i^2 / 8 \right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,5} \exp\left\{ -x_i^2 / 50 \right\}}$$

$$= \left(\frac{5}{2} \right)^n \exp\left\{ \left(\frac{1}{50} - \frac{1}{8} \right) \sum_{i=1}^{n} x_i^2 \right\} = \left(\frac{5}{2} \right)^n \exp\left\{ -\frac{21}{200} \sum_{i=1}^{n} x_i^2 \right\}.$$

$$\lambda(x_1, x_2, \dots, x_n) < k \qquad \Leftrightarrow \qquad \sum_{i=1}^{n} x_i^2 > c.$$

b) If $n = 10$, find the value of c such that $\alpha = 0.10$.

Hint: $\dfrac{\sum (X_i - \mu)^2}{\sigma^2}$ has a $\chi^2(n)$ distribution; here $\mu = 0$.

Want $$0.10 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{n} X_i^2 > c \mid \sigma = 2 \right)$$

$$= P\left(\frac{1}{4} \cdot \sum_{i=1}^{n} X_i^2 > \frac{c}{4} \mid \sigma = 2 \right) = P\left(\chi^2(10) > \frac{c}{4} \right).$$

$$\frac{c}{4} = \chi^2_{0.10}(10) = 15.99. \qquad \Rightarrow \qquad c = 4\chi^2_{0.10}(10) = 4 \times 15.99 = \mathbf{63.96}.$$

c) If $n = 10$ and c is from part (b), find the probability of Type II Error.

$$P(\text{Type II Error}) = P(\text{Accept } H_0 \mid H_0 \text{ is not true}) = P\left(\sum_{i=1}^{n} X_i^2 \le c \mid \sigma = 5 \right)$$

$$= P\left(\sum_{i=1}^{n} X_i^2 \le 63.96 \mid \sigma = 5 \right) = P\left(\frac{1}{25} \sum_{i=1}^{n} X_i^2 \le \frac{64.96}{25} \mid \sigma = 5 \right)$$

$$= P\left(\chi^2(10) \le 2.5584 \right) \approx \mathbf{0.01}.$$

4. **8.1.8** (7th and 6th edition)

If $X_1, X_2, \ldots, X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find the form of the best (most powerful) rejection region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$.

H_0: Beta(1, 1) vs. H_1: Beta(2, 2).

$$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(H_0; x_1, x_2, \ldots, x_n)}{L(H_1; x_1, x_2, \ldots, x_n)} = \frac{1}{\prod_{i=1}^{n} \left[6 x_i (1 - x_i)\right]}$$

$$\lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad \prod_{i=1}^{n} \left[x_i (1 - x_i)\right] \ge c.$$

5. Let X_1, X_2, X_3, X_4 be a random sample of size $n = 4$ from a Geometric(p) distribution (the number of independent trials until the first "success"). That is,

$$P(X_1 = k) = (1-p)^{k-1}\, p, \quad k = 1, 2, 3, \ldots .$$

We are interested in testing $H_0 : p = 0.30$ vs. $H_1 : p = 0.40$.

a) Use the likelihood ratio to find the best rejection region in terms of $\sum_{i=1}^{n} x_i$.

b) Find the best rejection region with the significance level closest to 0.07.

c) If the rejection region from part (b) is used, find the power of the test.

Hint: If $X_1, X_2, \ldots, X_n$ are independent Geometric(p), then $Y = \sum_{i=1}^{n} X_i$ has Negative Binomial distribution (the number of independent trials until the n^{th} "success"). Then

$$P(Y = y) = \binom{y-1}{n-1} \cdot p^{n} \cdot (1-p)^{y-n}, \quad y = n, n+1, n+2, \ldots .$$

EXCEL:	=NEGBINOM.DIST$(y-n, n, p, 0)$	gives	$P(Y = y)$
	=NEGBINOM.DIST$(y-n, n, p, 1)$	gives	$P(Y \le y)$

a) $$\lambda(x_1, x_2, \ldots, x_n) = \frac{L(p = 0.30;\, x_1, x_2, \ldots, x_n)}{L(p = 0.40;\, x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \left(0.70^{\,x_i - 1} \cdot 0.30 \right)}{\prod_{i=1}^{n} \left(0.60^{\,x_i - 1} \cdot 0.40 \right)}$$

$$= \left(\frac{0.70}{0.60} \right)^{\sum_{i=1}^{n} x_i} \cdot \left(\frac{0.60 \cdot 0.30}{0.70 \cdot 0.40} \right)^{n} = \left(\frac{7}{6} \right)^{\sum_{i=1}^{n} x_i} \cdot \left(\frac{18}{28} \right)^{n} .$$

$$\lambda(x_1, x_2, \ldots, x_n) \le k \quad \Leftrightarrow \quad \sum_{i=1}^{n} x_i \cdot \ln\left(\frac{7}{6} \right) + n \cdot \ln\left(\frac{18}{28} \right) \le \ln k$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} x_i \le c.$$

Rejection Region: Reject H_0 if $\sum_{i=1}^{n} X_i \le c$. (left tail)

Intuition: If p is larger, the first "success" will tend to occur sooner.

b) Want $0.07 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(Y \le c \mid p = 0.30) = \text{CDF @ } c.$

	A	B	C
1	y	CDF @ y	
2	4	=NEGBINOM.DIST(A2–4,4,0.30,1)	
3	=A2+1	=NEGBINOM.DIST(A3–4,4,0.30,1)	
4	=A3+1	=NEGBINOM.DIST(A4–4,4,0.30,1)	
5	=A4+1	=NEGBINOM.DIST(A5–4,4,0.30,1)	
6	=A5+1	=NEGBINOM.DIST(A6–4,4,0.30,1)	
7	=A6+1	=NEGBINOM.DIST(A7–4,4,0.30,1)	
…	…	…	

$\Rightarrow$

	A	B	C
1	y	CDF @ y	
2	4	0.0081	
3	5	0.03078	
4	6	**0.07047**	
5	7	0.126036	
6	8	0.194104	
7	9	0.270341	
…	…	…	

CDF @ $6 = P(Y \le 6 \mid p = 0.30) \approx 0.07.$

OR

$$P(Y = 4 \mid p = 0.30) = \binom{4-1}{4-1}(0.30)^4 (0.70)^{4-4} = 0.0081,$$

$$P(Y = 5 \mid p = 0.30) = \binom{5-1}{4-1}(0.30)^4 (0.70)^{5-4} = 0.02268,$$

$$P(Y = 6 \mid p = 0.30) = \binom{6-1}{4-1}(0.30)^4 (0.70)^{6-4} = 0.03969.$$

$$P(Y \le 6 \mid p = 0.30) = 0.0081 + 0.02268 + 0.03969 = 0.07047 \approx 0.07 = \alpha.$$

Rejection Region: Reject H_0 if $Y = \sum_{i=1}^{n} X_i \le \mathbf{6}$.

c) Power = $P(\text{Reject } H_0 \mid H_0 \text{ is not true}) = P(Y \le 6 \mid p = 0.40)$ = CDF @ 6.

	A	B	C
1	y	CDF @ y	
2	4	=NEGBINOM.DIST(A2–4,4,0.40,1)	
3	=A2+1	=NEGBINOM.DIST(A3–4,4,0.40,1)	
4	=A3+1	=NEGBINOM.DIST(A4–4,4,0.40,1)	
5	=A4+1	=NEGBINOM.DIST(A5–4,4,0.40,1)	
6	=A5+1	=NEGBINOM.DIST(A6–4,4,0.40,1)	
7	=A6+1	=NEGBINOM.DIST(A7–4,4,0.40,1)	
…	…	…	

$\Rightarrow$

	A	B	C
1	y	CDF @ y	
2	4	0.0256	
3	5	0.08704	
4	6	**0.1792**	
5	7	0.289792	
6	8	0.405914	
7	9	0.51739	
…	…	…	

Power = $P(Y \le 6 \mid p = 0.40)$ = CDF @ 6 = **0.1792**.

OR

$$P(Y = 4 \mid p = 0.40) = \binom{4-1}{4-1}(0.40)^4(0.60)^{4-4} = 0.0256,$$

$$P(Y = 5 \mid p = 0.40) = \binom{5-1}{4-1}(0.40)^4(0.60)^{5-4} = 0.06144,$$

$$P(Y = 6 \mid p = 0.40) = \binom{6-1}{4-1}(0.40)^4(0.60)^{6-4} = 0.09216.$$

Power = $P(Y \le 6 \mid p = 0.40) = 0.0256 + 0.06144 + 0.09216 =$ **0.1792**.

6. Consider

$$f_1(x) = \sin x, \quad 0 < x < {}^{\pi}/_{2}, \quad \text{zero elsewhere,}$$

$$f_2(x) = \cos x, \quad 0 < x < {}^{\pi}/_{2}, \quad \text{zero elsewhere.}$$

You will have just a single observation of X on which to base your choice between

$$\text{H}_0 : \text{X has p.d.f. } f_1(x) \quad \text{vs.} \quad \text{H}_1 : \text{X has p.d.f. } f_2(x).$$

Use the likelihood ratio to find the best rejection region with the significance level $\alpha = 0.10$ and find the power of this test.

Best rejection region is:

$$\text{Rejects H}_0 \text{ if } \frac{\text{L}(\text{H}_0; x)}{\text{L}(\text{H}_1; x)} = \frac{\sin(x)}{\cos(x)} = \tan(x) < k.$$

$$\tan(x) < k \qquad \Leftrightarrow \qquad x < c = \tan^{-1}(k).$$

$$0.10 = \alpha = \text{P}(\text{Reject H}_0 \mid \text{H}_0 \text{ is true}) = \text{P}(\text{X} < c \mid \text{H}_0 \text{ is true})$$

$$= \int_0^c \sin(x)\,dx = 1 - \cos(c).$$

$$\Rightarrow \qquad c = \cos^{-1}(0.90) \approx \mathbf{0.451}.$$

$$\text{Power} = \text{P}(\text{Reject H}_0 \mid \text{H}_0 \text{ is not true}) = \text{P}(\text{X} < c \mid \text{H}_0 \text{ is not true})$$

$$= \int_0^c \cos(x)\,dx = \sin(c) = \sqrt{1 - 0.90^2} = \sqrt{0.19} \approx \mathbf{0.43589}.$$

7. You will have just a single observation of X on which to base your choice between

H_0 : X has a Normal distribution with mean $\mu = 5$ and standard deviation $\sigma = 2$

vs.

H_1 : X has a Binomial distribution with $n = 25$ and $p = 0.20$.

Consider the rejection rule "Reject H_0 if X is an integer".

Find $\alpha = \mathrm{P}(\text{Type I Error})$ and $\beta = \mathrm{P}(\text{Type II Error})$. *Justify your answer.*

$$\alpha = \mathrm{P}(\text{Type I Error}) = \mathrm{P}(\text{Reject } H_0 \mid H_0 \text{ is true})$$

$$= \mathrm{P}(\text{X is an integer} \mid \text{X has a Normal distribution}) = \mathbf{0},$$

since Normal distribution is a continuous distribution.

$$\beta = \mathrm{P}(\text{Type II Error}) = \mathrm{P}(\text{Do NOT Reject } H_0 \mid H_0 \text{ is false})$$

$$= \mathrm{P}(\text{X is NOT an integer} \mid \text{X has a Binomial distribution}) = \mathbf{0}.$$

since Binomial distribution is a discrete integer-valued distribution.

8.* **8.1.6** (7th and 6th edition)

Let $X_1, X_2, \dots, X_{10}$ be a random sample from a distribution that is $N(\theta_1, \theta_2)$. Find a best test of the simple hypothesis $H_0: \theta_1 = 0, \ \theta_2 = 1$ against the alternative simple hypothesis $H_1: \theta_1 = 1, \ \theta_2 = 4$.

$H_0: N(0, 1)$ vs. $H_1: N(1, 4)$.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(H_0; x_1, x_2, \dots, x_n)}{L(H_1; x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \cdot x_i^2\right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,2} \exp\left\{-\frac{1}{8}(x_i - 1)^2\right\}}$$

$$= \left(\frac{1}{2^n}\right) \cdot \exp\left\{\sum_{i=1}^{n} \left[\frac{1}{8} \cdot (x_i - 1)^2 - \frac{1}{2} \cdot x_i^2\right]\right\}$$

$$= \left(\frac{1}{2^n}\right) \cdot \exp\left\{\sum_{i=1}^{n} \left[-\frac{3}{8} \cdot x_i^2 - \frac{2}{8} \cdot x_i + \frac{1}{8}\right]\right\}$$

$$= \left(\frac{1}{2^n}\right) \cdot \exp\left\{-\frac{1}{8}\left(3 \cdot \sum_{i=1}^{n} x_i^2 + 2 \cdot \sum_{i=1}^{n} x_i\right) + \frac{n}{8}\right\}$$

$n = 10.$

$$\lambda(x_1, x_2, \dots, x_n) \le k \qquad \Leftrightarrow \qquad 3 \cdot \sum_{i=1}^{10} x_i^2 + 2 \cdot \sum_{i=1}^{10} x_i \ge c.$$