Examples for 12/04/2020 (2)

A. Stepanov

Examples for 10/19/2020 (2) and 10/23/2020 (2) and 10/30/2020 (3) and 11/04/2020 (3) and 11/18/2020 (2) (continued)

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x \mid \beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \beta > 0, \qquad \text{zero otherwise.}$$

Let the prior p.d.f. of β be $\text{Gamma}(\alpha, \theta)$. That is,

$$\pi(\beta) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,\beta^{\alpha-1} e^{-\beta/\theta}, \qquad \beta > 0.$$

Recall (Examples for 10/19/2020 (2)):

$$\hat{\beta} = \frac{2n}{\sum\limits_{i=1}^{n} \ln X_i} \quad \text{is the maximum likelihood estimator for } \beta.$$

a) Find the posterior distribution of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

HINT: $\quad \dfrac{1}{x} = e^{-\ln x}$.

b) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. Show that it is a weighted average of the maximum likelihood estimate $\hat{\beta}$ and the prior mean $\alpha\,\theta$. (What are the weights?)

c) Use part (a) to construct a $(1-\gamma)100\%$ credible interval for β, given $X_1 = x_1$, $X_2 = x_2, \ldots, X_n = x_n$.

d) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.
Let $\alpha = 4$, $\theta = 0.50$.

(i) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

(ii) Construct a 90% credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

1. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x \mid \beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \beta > 0, \qquad \text{zero otherwise.}$$

Let the prior p.d.f. of β be $\text{Gamma}(\alpha, \theta)$. That is,

$$\pi(\beta) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,\beta^{\alpha-1}\,e^{-\beta/\theta}, \qquad \beta > 0.$$

Recall (Examples for 10/19/2020 (2)):

$$\hat{\beta} = \frac{2n}{\sum_{i=1}^{n} \ln X_i} \quad \text{is the maximum likelihood estimator for } \beta.$$

a) Find the posterior distribution of β, given $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$.

HINT: $\frac{1}{x} = e^{-\ln x}$.

$$f(x_1, x_2, \dots x_n, \lambda) = \prod_{i=1}^{n} \frac{\beta^2 \ln x_i}{x_i^{\beta+1}} \times \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,\beta^{\alpha-1}\,e^{-\beta/\theta}$$

$$= \dots\ \beta^{2n+\alpha-1}\,e^{-\beta\left(\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}\right)}.$$

$\Rightarrow$ the posterior distribution of β, given $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$,

is **Gamma** with New $\alpha = 2n + \alpha$ and New $\theta = \dfrac{1}{\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}}$.

b) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$. Show that it is a weighted average of the maximum likelihood estimate $\hat{\beta}$ and the prior mean $\alpha\,\theta$. (What are the weights?)

(conditional mean of λ given $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$)

$$= (\text{New } \alpha) \times (\text{New } \theta) = \frac{2n+\alpha}{\sum\limits_{i=1}^{n} \ln x_i + \frac{1}{\theta}}$$

$$= \frac{2n}{\sum\limits_{i=1}^{n} \ln x_i} \cdot \frac{\sum\limits_{i=1}^{n} \ln x_i}{\sum\limits_{i=1}^{n} \ln x_i + \frac{1}{\theta}} + \alpha\,\theta \cdot \frac{\frac{1}{\theta}}{\sum\limits_{i=1}^{n} \ln x_i + \frac{1}{\theta}}.$$

c) Use part (a) to construct a $(1-\gamma)\,100\%$ credible interval for β, given $X_1 = x_1$, $X_2 = x_2, \dots, X_n = x_n$.

$(\beta \mid x_1, x_2, \dots, x_n)$ has a

$$\text{Gamma}\left(\text{New } \alpha = 2n + \alpha\,,\ \text{New } \theta = \frac{1}{\sum\limits_{i=1}^{n} \ln x_i + \frac{1}{\theta}}\right) \text{ distribution.}$$

$\dfrac{2}{\text{New } \theta}\,(\beta \mid x_1, x_2, \dots, x_n)$ has a $\chi^2(2 \times \text{New } \theta)$ distribution.

$2\left(\sum\limits_{i=1}^{n} \ln x_i + \dfrac{1}{\theta}\right)(\beta \mid x_1, x_2, \dots, x_n)$ has a $\chi^2(4n + 2\alpha)$ distribution.

$$P\left(\chi^2_{1-\gamma/2}(4n+2\alpha) < 2\left(\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}\right)(\beta \mid x_1, x_2, \ldots, x_n) < \chi^2_{\gamma/2}(4n+2\alpha) \right)$$

$$= 1-\gamma.$$

$$P\left(\frac{\chi^2_{1-\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}\right)} < (\beta \mid x_1, x_2, \ldots, x_n) < \frac{\chi^2_{\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}\right)} \right) = 1-\gamma.$$

$$\left(\frac{\chi^2_{1-\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}\right)}, \frac{\chi^2_{\gamma/2}(4n+2\alpha)}{2\left(\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}\right)} \right)$$

is a $(1-\gamma)\,100\%$ credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

d) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.
Let $\alpha = 4$, $\theta = 0.50$.

$$\sum_{i=1}^{5} \ln x_i = \ln 1.3 + \ln 1.4 + \ln 2.0 + \ln 3.0 + \ln 5.0 \approx 4.$$

$$\hat{\beta} \approx \frac{2 \cdot 5}{4} = 2.5. \qquad \text{Prior mean} = \alpha\,\theta = 2.$$

(i) Find the conditional mean of β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$$\frac{2n+\alpha}{\sum_{i=1}^{n} \ln x_i + \frac{1}{\theta}} \approx \frac{10+4}{4+2} = \mathbf{\frac{7}{3}} \approx \mathbf{2.33333}.$$

(ii) Construct a 90% credible interval for β, given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

$\chi^2_{0.95}(28) = 16.93$, $\chi^2_{0.05}(28) = 41.34$.

$\left(\frac{16.93}{2\cdot 6}, \frac{41.34}{2\cdot 6}\right)$ $\mathbf{(1.411, 3.445)}$