Examples for 11/18/2020 (5)

1. **4.5.5** (7th edition) **5.5.5** (6th edition)

Let X_1, X_2 be a random sample of size $n = 2$ from the distribution having p.d.f. $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. We reject $H_0: \theta = 2$ in favor of $H_1: \theta = 1$ if the observed values of X_1, X_2, say x_1, x_2, are such that

$$\frac{f(x_1;2) \cdot f(x_2;2)}{f(x_1;1) \cdot f(x_2;1)} \le \frac{1}{2}.$$

Here $\Omega = \{\theta : \theta = 1, 2\}$. Find the significance level of the test and the power of the test when H_0 is false.

That is, we know that the most powerful rejection region for testing $H_0: \theta = 2$ vs $H_1: \theta = 1$ is

$$\text{Reject } H_0 \quad \text{if} \quad \frac{L(2)}{L(1)} = \frac{f(x_1;2) \cdot f(x_2;2)}{f(x_1;1) \cdot f(x_2;1)} \le k. \qquad \text{Let } k = \frac{1}{2}.$$

(That is, reject H_0 if it is more than twice as likely to observe a data set like ours under the assumption that H_1 is true than under the assumption that H_0 is true.)

Find (i) the significance level of the test and (ii) the power of the test.

2. 1. (continued)

Let $X_1, X_2, \ldots, X_{10}$ be a random sample of size $n = 10$ from the distribution having p.d.f. $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. We reject $H_0: \theta = 2$ in favor of $H_1: \theta = 1$ if the observed values of $X_1, X_2, \ldots, X_{10}$, say $x_1, x_2, \ldots, x_{10}$, are such that

$$\frac{f(x_1;\, 2) \cdot f(x_2;\, 2) \cdot \ldots \cdot f(x_{10};\, 2)}{f(x_1;\, 1) \cdot f(x_2;\, 1) \cdot \ldots \cdot f(x_{10};\, 1)} \le \frac{1}{2}.$$

Find the significance level of the test and the power of the test.

"Hint": Technology is wonderful.

3. 2. (continued)

Let $X_1, X_2, \dots, X_{10}$ be a random sample of size $n = 10$ from the distribution having p.d.f. $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. We wish to test $H_0: \theta = 2$ vs $H_1: \theta = 1$ using a 5% level of significance.

a) Find the most powerful rejection region with $\alpha = 0.05$.

b) Find the power of this test. "Hint": Technology is wonderful.

c) Suppose $\sum_{i=1}^{10} x_i = 8.8$. Find the p-value of the test.

1. **4.5.5** (7th edition) **5.5.5** (6th edition)

Let X_1, X_2 be a random sample of size $n = 2$ from the distribution having p.d.f. $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. We reject $H_0: \theta = 2$ in favor of $H_1: \theta = 1$ if the observed values of X_1, X_2, say x_1, x_2, are such that

$$\frac{f(x_1;2)\cdot f(x_2;2)}{f(x_1;1)\cdot f(x_2;1)} \le \frac{1}{2}.$$

Here $\Omega = \{\theta : \theta = 1, 2\}$. Find the significance level of the test and the power of the test when H_0 is false.

That is, we know that the most powerful rejection region for testing

$H_0: \theta = 2$ vs $H_1: \theta = 1$ is

$$\text{Reject } H_0 \quad \text{if} \quad \frac{L(2)}{L(1)} = \frac{f(x_1;2)\cdot f(x_2;2)}{f(x_1;1)\cdot f(x_2;1)} \le k. \qquad \text{Let } k = \frac{1}{2}.$$

(That is, reject H_0 if it is more than twice as likely to observe a data set like ours under the assumption that H_1 is true than under the assumption that H_0 is true.)

Find (i) the significance level of the test and (ii) the power of the test.

$$\frac{f(x_1;2)\cdot f(x_2;2)}{f(x_1;1)\cdot f(x_2;1)} \le \frac{1}{2} \quad \Rightarrow \quad \frac{\frac{1}{2}e^{-x_1/2}\cdot\frac{1}{2}e^{-x_2/2}}{e^{-x_1}\cdot e^{-x_2}} \le \frac{1}{2}$$

$$\Rightarrow \quad x_1 + x_2 \le 2\ln 2 \approx 1.3863.$$

$\alpha = P(X_1 + X_2 \le 2\ln 2 \mid \theta = 2)$

Power $= P(X_1 + X_2 \le 2\ln 2 \mid \theta = 1)$

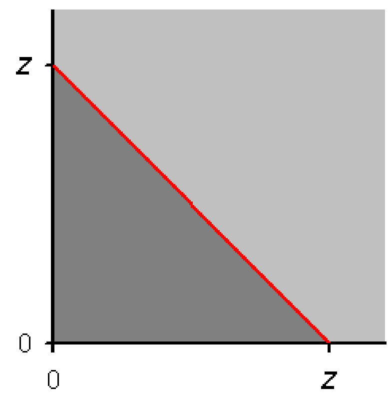
Consider $P(X_1 + X_2 \le z \mid \theta)$, $\qquad z > 0$.

$$P(X_1 + X_2 \le z) = P(X_2 \le z - X_1)$$

$$= \int_0^z \left(\int_0^{z-x} \frac{1}{\theta} e^{-x/\theta} \frac{1}{\theta} e^{-y/\theta} \, dy \right) dx$$

$$= \int_0^z \frac{1}{\theta} e^{-x/\theta} \left(1 - e^{-z/\theta + x/\theta} \right) dx$$

$$= 1 - e^{-z/\theta} - \frac{z}{\theta} e^{-z/\theta}, \qquad z > 0.$$



Significance level: $\theta = 2$ $z = 2 \ln 2$

$$\alpha = 1 - e^{-2\ln 2/2} - \frac{2\ln 2}{2} e^{-2\ln 2/2} = 1 - \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2} - \frac{1}{2}\ln 2 \approx \mathbf{0.1534}.$$

Power: $\theta = 1$ $z = 2 \ln 2$

$$\text{Power} = 1 - e^{-2\ln 2/1} - \frac{2\ln 2}{1} e^{-2\ln 2/1} = 1 - \frac{1}{4} - \frac{1}{2}\ln 2 = \frac{3}{4} - \frac{1}{2}\ln 2 \approx \mathbf{0.4034}.$$

OR

$X_1 + X_2$ has a Gamma distribution with $\alpha = 2$ and θ.

$$P(X_1 + X_2 \le z \mid \theta) = P(\text{Poisson}(\tfrac{z}{\theta}) \ge 2) = 1 - e^{-z/\theta} - \frac{z}{\theta} e^{-z/\theta}.$$

OR

$\frac{2}{\theta} \times (X_1 + X_2)$ has a $\chi^2(2\alpha = 4 \text{ d.f.})$ distribution.

$$\alpha = P(X_1 + X_2 \le 2 \ln 2 \mid \theta = 2) = P(\chi^2(4) \le 2 \ln 2).$$

$$\text{Power} = P(X_1 + X_2 \le 2 \ln 2 \mid \theta = 1) = P(\chi^2(4) \le 4 \ln 2).$$

2. 2. (continued)

Let $X_1, X_2, \ldots, X_{10}$ be a random sample of size $n = 10$ from the distribution having p.d.f. $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. We reject $H_0: \theta = 2$ in favor of $H_1: \theta = 1$ if the observed values of $X_1, X_2, \ldots, X_{10}$, say $x_1, x_2, \ldots, x_{10}$, are such that

$$\frac{f(x_1;\,2)\cdot f(x_2;\,2)\cdot\ldots\cdot f(x_{10};\,2)}{f(x_1;\,1)\cdot f(x_2;\,1)\cdot\ldots\cdot f(x_{10};\,1)} \le \frac{1}{2}.$$

Find the significance level of the test and the power of the test.

"Hint": Technology is wonderful.

$$\frac{f(x_1;\,2)\cdot f(x_2;\,2)\cdot\ldots\cdot f(x_{10};\,2)}{f(x_1;\,1)\cdot f(x_2;\,1)\cdot\ldots\cdot f(x_{10};\,1)} \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{\frac{1}{2}\,e^{-x_1/2}\cdot\frac{1}{2}\,e^{-x_2/2}\cdot\ldots\cdot\frac{1}{2}\,e^{-x_{10}/2}}{e^{-x_1}\cdot e^{-x_2}\cdot\ldots\cdot e^{-x_{10}}} \le \frac{1}{2}$$

$$\Rightarrow \quad e^{x_1/2}\cdot e^{x_2/2}\cdot\ldots\cdot e^{x_{10}/2} \le 2^9$$

$$\Rightarrow \quad x_1 + x_2 + \ldots + x_{10} \le 18\ln 2 \approx 12.47665.$$

$X_1 + X_2 + \ldots + X_{10}$ has a Gamma distribution with $\alpha = 10$ and θ.

$$P(X_1 + X_2 + \ldots + X_{10} \le z \mid \theta) = P(\text{Poisson}(\tfrac{z}{\theta}) \ge 10) = 1 - P(\text{Poisson}(\tfrac{z}{\theta}) \le 9).$$

```
> 1-ppois(9,9*log(2))
[1] 0.1013059
>
> 1-ppois(9,18*log(2))
[1] 0.7967764
```

OR

$\frac{2}{\theta} \times (X_1 + X_2 + \ldots + X_{10})$ has a $\chi^2(2\alpha = 20 \text{ d.f.})$ distribution.

$\alpha = P(X_1 + X_2 + \ldots + X_{10} \le 18 \ln 2 \mid \theta = 2) = P(\chi^2(20) \le 18 \ln 2)$.

Power $= P(X_1 + X_2 + \ldots + X_{10} \le 18 \ln 2 \mid \theta = 1) = P(\chi^2(20) \le 36 \ln 2)$.

```
> pchisq(18*log(2),20)
[1] 0.1013059
>
> pchisq(36*log(2),20)
[1] 0.7967764
```

3. 2. (continued)

Let $X_1, X_2, \ldots, X_{10}$ be a random sample of size $n = 10$ from the distribution having p.d.f. $f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. We wish to test $H_0: \theta = 2$ vs $H_1: \theta = 1$ using a 5% level of significance.

a) Find the most powerful rejection region with $\alpha = 0.05$.

The most powerful rejection region: Reject H_0 if $\sum_{i=1}^{n} x_i \le c$.

Intuition: θ is "θ". Small θ $\Rightarrow$ small X.

The sign is the same as the sign in H_1.

$X_1 + X_2 + \ldots + X_{10}$ has a Gamma distribution with $\alpha = 10$ and θ.

$$0.05 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{10} X_i \le c \mid \theta = 2\right)$$

$$= P\left(\frac{2}{2} \sum_{i=1}^{10} X_i \le \frac{2}{2} c \mid \theta = 2\right) = P(\chi^2(20) \le c).$$

$\Rightarrow$ $c = \chi^2_{0.95}(20) = \mathbf{10.85}$. Reject H_0 if $\sum_{i=1}^{10} x_i \le 10.85$.

b) Find the power of this test. "Hint": Technology is wonderful.

$$P(X_1 + X_2 + \ldots + X_{10} \le 10.85 \mid \theta = 1) = P(\text{Poisson}(10.85) \ge 10)$$

$$= 1 - P(\text{Poisson}(10.85) \le 9).$$

```
> 1-ppois(9,10.85)
[1] 0.6429912
```

OR

$\frac{2}{\theta} \times (X_1 + X_2 + \ldots + X_{10})$ has a $\chi^2(2\alpha = 20 \text{ d.f.})$ distribution.

Power $= P(X_1 + X_2 + \ldots + X_{10} \le 10.85 \mid \theta = 1) = P(\chi^2(20) \le 21.70)$.

```
> pchisq(21.70,20)
[1] 0.6429912
```

c) Suppose $\sum_{i=1}^{10} x_i = 8.8$. Find the p-value of the test.

P-value $= P(\text{value of } \sum_{i=1}^{10} X_i \text{ as extreme or more extreme than } 8.8 \mid H_0 \text{ true})$

$= P(\sum_{i=1}^{10} X_i \le 8.8 \mid \theta = 2) = P(Y \ge \alpha = 10) = 1 - P(Y \le 9)$.

where Y has a Poisson$(\frac{8.8}{2} = 4.4)$ distribution

$= 1 - 0.985 = \mathbf{0.015}$.

OR

P-value $= P(X_1 + X_2 + \ldots + X_{10} \le 8.8 \mid \theta = 2) = P(\chi^2(20) \le 8.8)$.

```
> pchisq(8.8,20)
[1] 0.01488994
```