STAT 410

Examples for 09/11/2020 (2)

Fall 2020
A. Stepanov

3. Let the joint probability density function for (X, Y) be

$$f(x, y) = C\,x\,y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0 < y < 1, \quad y < x < 2 \}$.

b) Find the value of C that would make this a valid joint probability distribution.

c) Find the marginal probability density function of X, $f_X(x)$.

"Hint": $$f_X(x) = \begin{cases} \clubsuit(x) & 0 < x < 1 \\ \spadesuit(x) & 1 < x < 2 \end{cases}$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Are X and Y independent? *Justify your answer.*
If not, find $\mathrm{Cov}(X, Y)$.

f) Find the probability $P(X + Y \le 2)$.

g) Find the probability $P(X \cdot Y \le 1)$.

h) Find the probability $P\left(\frac{Y}{X} \le \frac{1}{2}\right)$.

i) Find $P(Y > 0.6 \mid X = 0.7)$.

j) Find $P(Y > 0.6 \mid X = 1.7)$.

k) Find $E(Y \mid X = x)$.

l) Find $P(X < 1.7 \mid Y = 0.6)$.

m) Find $P(X < 1.7 \mid Y > 0.6)$.

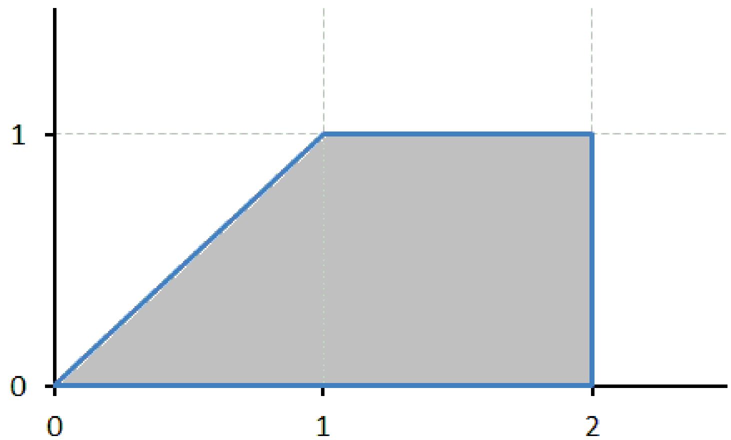
n) Find $E(X \mid Y = y)$.

Answers:

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0 < y < 1, \quad y < x < 2 \}$.



b) Find the value of C that would make this a valid joint probability distribution.

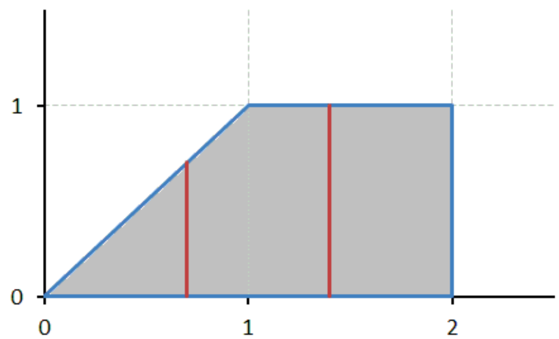
$$\int_0^1 \left(\int_y^2 C\,x\,y^3\,dx \right) dy = \int_0^1 \left(\frac{C}{2}\,x^2\,y^3 \right) \Bigg|_{x=y}^{x=2} dy = \int_0^1 \left(2\,C\,y^3 - \frac{C}{2}\,y^5 \right) dy$$

$$= \left(\frac{C}{2}\,y^4 - \frac{C}{12}\,y^6 \right) \Bigg|_{y=0}^{y=1} = \frac{C}{2} - \frac{C}{12} = \frac{5\,C}{12} = 1.$$

$$\Rightarrow \qquad C = \mathbf{\frac{12}{5}} = \mathbf{2.4}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

"Hint": $$f_X(x) = \begin{cases} \clubsuit(x) & 0 < x < 1 \\ \spadesuit(x) & 1 < x < 2 \end{cases}$$



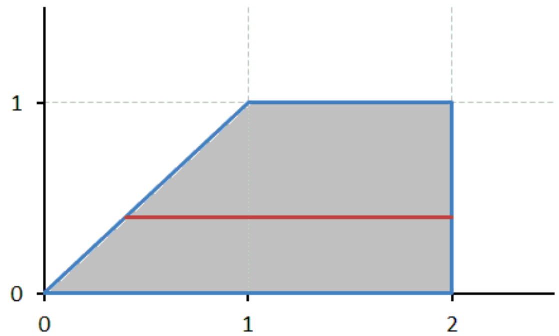
For $0 < x < 1$, $$f_X(x) = \int_0^x \frac{12}{5} x y^3 \, dy = \left(\frac{3}{5} x y^4 \right) \Bigg|_{y=0}^{y=x} = \frac{3}{5} x^5.$$

For $1 < x < 2$, $$f_X(x) = \int_0^1 \frac{12}{5} x y^3 \, dy = \left(\frac{3}{5} x y^4 \right) \Bigg|_{y=0}^{y=1} = \frac{3}{5} x.$$

$$f_X(x) = \begin{cases} \frac{3}{5} x^5 & 0 < x < 1 \\ \frac{3}{5} x & 1 < x < 2 \end{cases}$$

Check: $$\int_0^1 \frac{3}{5} x^5 \, dx + \int_1^2 \frac{3}{5} x \, dx = \left(\frac{1}{10} x^6 \right) \Bigg|_{x=0}^{x=1} + \left(\frac{3}{10} x^2 \right) \Bigg|_{x=1}^{x=2} = \frac{1}{10} + \frac{9}{10} = 1.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.



For $0 < y < 1$, $f_Y(y) = \int_y^2 \frac{12}{5} x y^3 \, dx = \left(\frac{6}{5} x^2 y^3 \right) \Big|_{x=y}^{x=2} = \frac{24}{5} y^3 - \frac{6}{5} y^5.$

Check: $\int_0^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy = \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1.$

e) Are X and Y independent? *Justify your answer.*

If not, find $\mathrm{Cov}(X, Y)$.

The support of (X, Y) is NOT a rectangle. X and Y are **NOT independent**.

OR

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are **NOT independent**.

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot \frac{3}{5} x^5\,dx + \int_1^2 x \cdot \frac{3}{5} x\,dx$$

$$= \left(\frac{3}{35} x^7 \right) \Bigg|_{x=0}^{x=1} + \left(\frac{1}{5} x^3 \right) \Bigg|_{x=1}^{x=2} = \frac{3}{35} + \frac{7}{5} = \frac{52}{35}.$$

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy$$

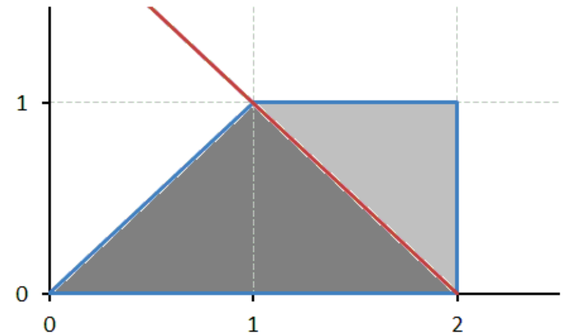
$$= \left(\frac{24}{25} y^5 - \frac{6}{35} y^7 \right) \Bigg|_{y=0}^{y=1} = \frac{138}{175}.$$

$$\mathrm{E}(XY) = \int_0^1 \left(\int_y^2 xy \cdot \frac{12}{5} x y^3\,dx \right) dy = \int_0^1 \left(\frac{4}{5} x^3 y^4 \right) \Bigg|_{x=y}^{x=2} dy$$

$$= \int_0^1 \left(\frac{32}{5} y^4 - \frac{4}{5} y^7 \right) dy = \left(\frac{32}{25} y^5 - \frac{1}{10} y^8 \right) \Bigg|_{y=0}^{y=1} = \frac{59}{50}.$$

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X) \cdot \mathrm{E}(Y) = \frac{59}{50} - \frac{52}{35} \cdot \frac{138}{175} = \mathbf{\frac{103}{12{,}250}} \approx 0.0084.$$

f) Find the probability $P(X+Y \le 2)$.



$$\int_0^1 \left(\int_y^{2-y} \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=y}^{x=2-y} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{24}{5} y^4 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{24}{25} y^5 \right) \Bigg|_{y=0}^{y=1} = \mathbf{\frac{6}{25}} = \mathbf{0.24}.$$

OR

$$1 - \int_0^1 \left(\int_{2-y}^{2} \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=2-y}^{x=2} dy$$

$$= 1 - \int_0^1 \left(\frac{24}{5} y^4 - \frac{6}{5} y^5 \right) dy = 1 - \left(\frac{24}{25} y^5 - \frac{1}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \mathbf{\frac{6}{25}}.$$

OR

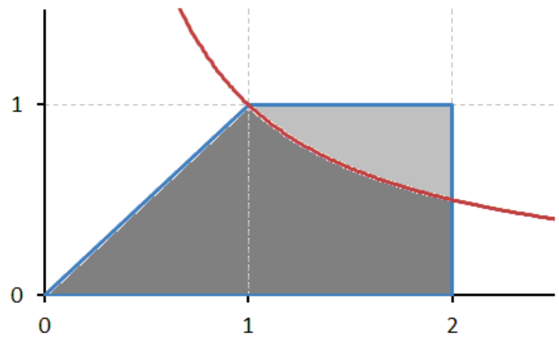
$$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{2-x} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= \int_0^1 \frac{3}{5} x^5 \, dx + \int_1^2 \frac{3}{5} x (2-x)^4 \, dx = \frac{1}{10} + \int_1^2 \frac{3}{5} x (2-x)^4 \, dx = \ldots$$

OR

$$1 - \int_1^2 \left(\int_{2-x}^{1} \frac{12}{5} x y^3 \, dy \right) dx = 1 - \int_1^2 \left(\frac{3}{5} x - \frac{3}{5} x (2-x)^4 \right) dx = \ldots$$

g) Find the probability $P(X \cdot Y \leq 1)$.



$$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{1/x} \frac{12}{5} x y^3 \, dy \right) dx = \int_0^1 \frac{3}{5} x^5 \, dx + \int_1^2 \frac{3}{5} \frac{1}{x^3} \, dx$$

$$= \left(\frac{1}{10} x^6 \right) \Big|_{x=0}^{x=1} + \left(-\frac{3}{10} \frac{1}{x^2} \right) \Big|_{x=1}^{x=2} = \frac{1}{10} - \frac{3}{40} + \frac{3}{10} = \mathbf{\frac{13}{40}} = \mathbf{0.325}.$$

OR

$$\int_0^{1/2} \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy + \int_{1/2}^1 \left(\int_y^{1/y} \frac{12}{5} x y^3 \, dx \right) dy$$

$$= \int_0^{1/2} \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy + \int_{1/2}^1 \left(\frac{6}{5} y - \frac{6}{5} y^5 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_0^{1/2} + \left(\frac{3}{5} y^2 - \frac{1}{5} y^6 \right) \Big|_{1/2}^1$$

$$= \frac{3}{40} - \frac{1}{320} + \frac{3}{5} - \frac{1}{5} - \frac{3}{20} + \frac{1}{320} = \frac{2}{5} - \frac{3}{40} = \mathbf{\frac{13}{40}}.$$

OR

$$1 - \int_1^2 \left(\int_{1/x}^1 \frac{12}{5} x y^3 \, dy \right) dx = 1 - \int_1^2 \left(\frac{3}{5} x - \frac{3}{5} \frac{1}{x^3} \right) dx = 1 - \left(\frac{3}{10} x^2 + \frac{3}{10} \frac{1}{x^2} \right) \Big|_{x=1}^{x=2}$$

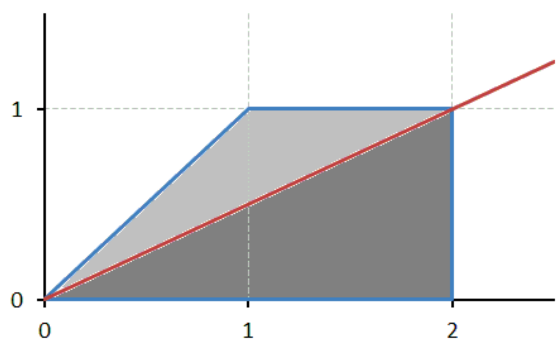
$$= 1 - \left(\frac{6}{5} + \frac{3}{40} - \frac{3}{10} - \frac{3}{10} \right) = 1 - \left(\frac{3}{5} + \frac{3}{40} \right) = 1 - \frac{27}{40} = \mathbf{\frac{13}{40}}.$$

OR

$$1-\int_{1/2}^{1}\left(\int_{1/y}^{2}\frac{12}{5}\,x\,y^{3}\,dx\right)dy = 1-\int_{1/2}^{1}\left(\frac{24}{5}\,y^{3}-\frac{6}{5}\,y\right)dy = 1-\left(\frac{6}{5}\,y^{4}-\frac{3}{5}\,y^{2}\right)\Big|_{1/2}^{1}$$

$$= 1-\left(\frac{6}{5}-\frac{3}{5}-\frac{3}{40}+\frac{3}{20}\right) = 1-\left(\frac{3}{5}+\frac{3}{40}\right) = 1-\frac{27}{40} = \mathbf{\frac{13}{40}}.$$

h) Find the probability $P\left(\frac{Y}{X}\le\frac{1}{2}\right)$.



$$\int_{0}^{2}\left(\int_{0}^{x/2}\frac{12}{5}\,x\,y^{3}\,dy\right)dx = \int_{0}^{2}\frac{3}{80}\,x^{5}\,dx = \left(\frac{1}{160}\,x^{6}\right)\Big|_{x=0}^{x=2} = \mathbf{\frac{2}{5}} = \mathbf{0.40}.$$

OR

$$\int_{0}^{1}\left(\int_{2y}^{2}\frac{12}{5}\,x\,y^{3}\,dx\right)dy = \int_{0}^{1}\left(\frac{24}{5}\,y^{3}-\frac{24}{5}\,y^{5}\right)dy = \left(\frac{6}{5}\,y^{4}-\frac{4}{5}\,y^{6}\right)\Big|_{y=0}^{y=1} = \mathbf{\frac{2}{5}}.$$

OR

$$1-\int_0^1\left(\int_y^{2y}\frac{12}{5}\,x\,y^3\,dx\right)dy \;=\; 1-\int_0^1\left(\frac{24}{5}y^5-\frac{6}{5}y^5\right)dy \;=\; 1-\int_0^1\frac{18}{5}y^5\,dy$$

$$=\;1-\left(\frac{3}{5}y^6\right)\Bigg|_{y=0}^{y=1} \;=\; 1-\frac{3}{5} \;=\; \mathbf{\frac{2}{5}}\,.$$

OR

$$1\;-\;\int_0^1\left(\int_{x/2}^{x}\frac{12}{5}\,x\,y^3\,dy\right)dx\;-\;\int_1^2\left(\int_{x/2}^{1}\frac{12}{5}\,x\,y^3\,dy\right)dx$$

$$=\;1\;-\;\int_0^1\left(\frac{3}{5}x^5-\frac{3}{80}x^5\right)dx\;-\;\int_1^2\left(\frac{3}{5}x-\frac{3}{80}x^5\right)dx$$

$$=\;1\;-\;\left(\frac{1}{10}x^6-\frac{1}{160}x^6\right)\Bigg|_{x=0}^{x=1}\;-\;\left(\frac{3}{10}x^2-\frac{1}{160}x^6\right)\Bigg|_{x=1}^{x=2}$$

$$=\;1-\left(\frac{1}{10}-\frac{1}{160}\right)-\left(\frac{6}{5}-\frac{2}{5}\right)+\left(\frac{3}{10}-\frac{1}{160}\right)\;=\;\mathbf{\frac{2}{5}}\,.$$

i) Find $P(Y > 0.6 \mid X = 0.7)$.

For $0 < x < 1$, $\quad f_X(x) = \frac{3}{5} x^5$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{4y^3}{x^4}, \qquad 0 < y < x.$$

$$f_{Y|X}(y|0.7) = \frac{4y^3}{0.7^4}, \qquad 0 < y < 0.7.$$

$$P(Y > 0.6 \mid X = 0.7) = \int_{0.6}^{0.7} \frac{4y^3}{0.7^4}\, dy = \left(\frac{y^4}{0.7^4}\right)\Bigg|_{y=0.6}^{y=0.7} = \frac{0.7^4 - 0.6^4}{0.7^4} \approx 0.46.$$

j) Find $P(Y > 0.6 \mid X = 1.7)$.

For $1 < x < 2$, $\quad f_X(x) = \frac{3}{5} x$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = 4y^3, \qquad 0 < y < 1.$$

$$f_{Y|X}(y|1.7) = 4y^3, \qquad 0 < y < 1.$$

$$P(Y > 0.6 \mid X = 1.7) = \int_{0.6}^{1} 4y^3\, dy = \left(y^4\right)\Big|_{y=0.6}^{y=1} = 1 - 0.6^4 \approx 0.87.$$

k) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$.

For $0 < x < 1$, $$f_{\mathrm{Y}|\mathrm{X}}(y|x) = \frac{4y^3}{x^4}, \qquad 0 < y < x.$$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x) = \int_0^x y \cdot \frac{4y^3}{x^4}\, dy = \frac{4y^5}{5x^4} \Bigg|_{y=0}^{y=x} = \frac{4}{5}x, \qquad 0 < x < 1.$$

For $1 < x < 2$, $$f_{\mathrm{Y}|\mathrm{X}}(y|x) = 4y^3, \qquad 0 < y < 1.$$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x) = \int_0^1 y \cdot 4y^3\, dy = \frac{4y^5}{5} \Bigg|_{y=0}^{y=1} = \frac{4}{5}, \qquad 1 < x < 2.$$

l) Find $\mathrm{P}(\mathrm{X} < 1.7 \mid \mathrm{Y} = 0.6)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x|y) = \frac{f(x,y)}{f_{\mathrm{Y}}(y)} = \frac{2x}{4 - y^2}, \qquad y < x < 2.$$

$$f_{\mathrm{X}|\mathrm{Y}}(x|0.6) = \frac{2x}{4 - 0.6^2}, \qquad 0.6 < x < 2.$$

$$\mathrm{P}(\mathrm{X} < 1.7 \mid \mathrm{Y} = 0.6) = \int_{0.6}^{1.7} \frac{2x}{4 - 0.6^2}\, dx = \left(\frac{x^2}{4 - 0.6^2} \right) \Bigg|_{x=0.6}^{x=1.7}$$

$$= \frac{1.7^2 - 0.6^2}{4 - 0.6^2} \approx 0.695.$$

m) Find $P(X < 1.7 \mid Y > 0.6)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{provided } P(B) > 0.$$

$$P(X < 1.7 \mid Y > 0.6) = \frac{P(X < 1.7 \cap Y > 0.6)}{P(Y > 0.6)} = \frac{\int_{0.6}^{1} \left(\int_{y}^{1.7} \frac{12}{5} x y^3 \, dx \right) dy}{\int_{0.6}^{1} \left(\int_{y}^{2} \frac{12}{5} x y^3 \, dx \right) dy}$$

$$= \frac{\int_{0.6}^{1} \left(\int_{y}^{1.7} \frac{12}{5} x y^3 \, dx \right) dy}{\int_{0.6}^{1} \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy} = \frac{\int_{0.6}^{1} \left(\frac{17.34}{5} y^3 - \frac{6}{5} y^5 \right) dy}{\int_{0.6}^{1} \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy}$$

$$= \frac{\left(\frac{4.335}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_{0.6}^{1}}{\left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_{0.6}^{1}} = \frac{0.563968}{0.8538112} \approx 0.66.$$

n) Find $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \int_{y}^{2} x \cdot \frac{2x}{4 - y^2} \, dx = \frac{\frac{2}{3} x^3}{4 - y^2} \Bigg|_{x=y}^{x=2}$$

$$= \frac{16 - 2y^3}{12 - 3y^2} = \frac{8 + 4y + 2y^2}{6 + 3y}, \qquad 0 < y < 1.$$