STAT 410

Practice Problems 11

Fall 2020
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1. As Alex is leaving for college, his parents give him a car, but warn him that they would take the car away if Alex gets 6 speeding tickets. Suppose that Alex receives speeding tickets according to Poisson process with the average rate of one ticket per six months.

a) Find the probability that it would take Alex longer than two years to get his sixth speeding ticket.

b) Find the probability that it would take Alex less than four years to get his sixth speeding ticket.

c) Find the probability that Alex would get his sixth speeding ticket during the fourth year.

d) Find the probability that Alex would get his sixth speeding ticket during the third year.

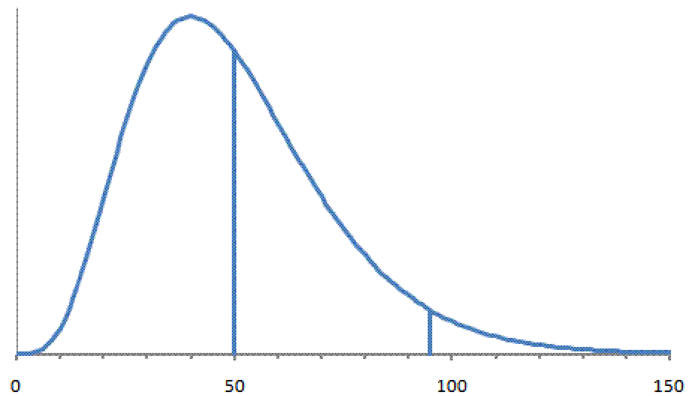
2. Alex purchased a laptop computer at *Joe's Discount Store*. He also purchased "Lucky 7" warranty plan that would replace the laptop at no cost if it needs 7 or more repairs in 3 years. Suppose the laptop requires repairs according to Poisson process with the average rate of one repair per 4 month.

a) Find the probability that the laptop would not need to be replaced. That is, find the probability that the seventh time the laptop needs repair will be after 3 years, when the warranty expires.

b) Find the probability that the seventh time the laptop needs repair will be during the second year of warranty.

c) Find the probability that the seventh time the laptop needs repair will be during the last 6 month of the warranty period.

3. In Neverland, annual income (in \$) is distributed according to Gamma distribution with $\alpha = 5$ and $\theta = 10{,}000$. Every year, the IRS audits 1% of the individuals with income below \$50,000, 3% of the individuals with income between \$50,000 and \$95,000, and 6% of the individuals with income above \$95,000. Suppose that the individuals to be audited at selected at random.



in thousands of \$

a) What proportion of Neverland's population falls into each of the three income groups? That is, find $P(X < \$50{,}000)$, $P(\$50{,}000 < X < \$95{,}000)$, and $P(X > \$95{,}000)$. [“Hint”: The answers should add up to 100%.]

b) You have overheard Mr. Statman complain about being audited. What is the probability that Mr. Statman’s income is below \$50,000? Between \$50,000 and \$95,000? Above \$95,000? [“Hint”: The answers should add up to 100%.]

c) Find the salary that would place an individual in the top 1%.

(Neverland $\neq$ USA.)

Answers:

1. As Alex is leaving for college, his parents give him a car, but warn him that they would take the car away if Alex gets 6 speeding tickets. Suppose that Alex receives speeding tickets according to Poisson process with the average rate of one ticket per six months.

X_t = number of speeding tickets in t years. Poisson(λt)

T_k = time of the kth speeding ticket. Gamma, $\alpha = k$.

one ticket per six months $\Rightarrow$ $\lambda = 2$.

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$P(T_\alpha \le t) = P(X_t \ge \alpha) \quad \text{and} \quad P(T_\alpha > t) = P(X_t \le \alpha - 1),$$

where X_t has a Poisson(λt) distribution.

a) Find the probability that it would take Alex longer than two years to get his sixth speeding ticket.

$$P(T_6 > 2) = P(X_2 \le 5) = P(\text{Poisson}(4) \le 5) = \mathbf{0.785}.$$

OR

$$P(T_6 > 2) = \int_2^\infty \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_2^\infty \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

b) Find the probability that it would take Alex less than four years to get his sixth speeding ticket.

$$P(T_6 < 4) = P(X_4 \geq 6) = 1 - P(X_4 \leq 5)$$

$$= 1 - P(\text{Poisson}(8) \leq 5) = 1 - 0.191 = \mathbf{0.809}.$$

OR

$$P(T_6 < 4) = \int_0^4 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_0^4 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

c) Find the probability that Alex would get his sixth speeding ticket during the fourth year.

$$P(3 < T_6 < 4) = P(T_6 > 3) - P(T_6 > 4) = P(X_3 \leq 5) - P(X_4 \leq 5)$$

$$= P(\text{Poisson}(6) \leq 5) - P(\text{Poisson}(8) \leq 5) = 0.446 - 0.191 = \mathbf{0.255}.$$

OR

$$P(3 < T_6 < 4) = \int_3^4 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_3^4 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

d) Find the probability that Alex would get his sixth speeding ticket during the third year.

$$P(2 < T_6 < 3) = P(T_6 > 2) - P(T_6 > 3) = P(X_2 \leq 5) - P(X_3 \leq 5)$$

$$= P(\text{Poisson}(4) \leq 5) - P(\text{Poisson}(6) \leq 5) = 0.785 - 0.446 = \mathbf{0.339}.$$

OR

$$P(2 < T_6 < 3) = \int_2^3 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_2^3 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

2. Alex purchased a laptop computer at *Joe's Discount Store*. He also purchased "Lucky 7" warranty plan that would replace the laptop at no cost if it needs 7 or more repairs in 3 years. Suppose the laptop requires repairs according to Poisson process with the average rate of one repair per 4 month.

X_t = number of repairs in t years. Poisson(λt)

T_k = time of the kth repair. Gamma, $\alpha = k$.

one repair per 4 month $\Rightarrow$ $\lambda = 3$ per year.

a) Find the probability that the laptop would not need to be replaced. That is, find the probability that the seventh time the laptop needs repair will be after 3 years, when the warranty expires.

$$P(T_7 > 3) = P(X_3 \le 6) = P(\text{Poisson}(9) \le 6) = \mathbf{0.207}.$$

OR

$$P(T_7 > 3) = \int_3^\infty \frac{3^7}{\Gamma(7)}\, t^{7-1} e^{-3t}\, dt = \int_3^\infty \frac{3^7}{6!}\, t^6 e^{-3t}\, dt = \ldots$$

b) Find the probability that the seventh time the laptop needs repair will be during the second year of warranty.

$$P(1 < T_7 < 2) = P(T_7 > 1) - P(T_7 > 2) = P(X_1 \le 6) - P(X_2 \le 6)$$

$$= P(\text{Poisson}(3) \le 6) - P(\text{Poisson}(6) \le 6) = 0.966 - 0.606 = \mathbf{0.360}.$$

OR

$$P(1 < T_7 < 2) = \int_1^2 \frac{3^7}{\Gamma(7)}\, t^{7-1} e^{-3t}\, dt = \int_1^2 \frac{3^7}{6!}\, t^6 e^{-3t}\, dt = \ldots$$

c) Find the probability that the seventh time the laptop needs repair will be during the last 6 month of the warranty period.

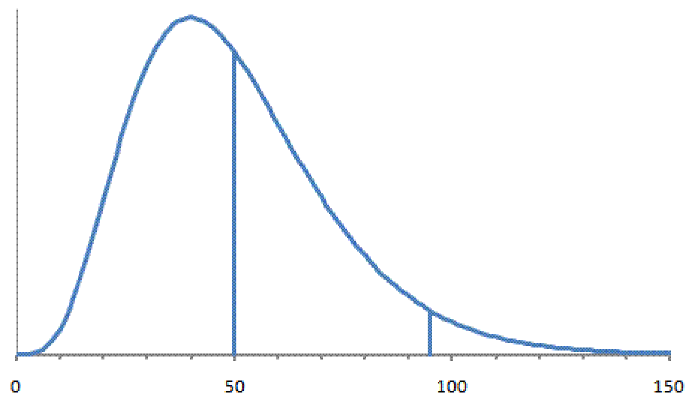
$$P(2.5 < T_7 < 3) = P(T_7 > 2.5) - P(T_7 > 3) = P(X_{2.5} \le 6) - P(X_3 \le 6)$$

$$= P(\text{Poisson}(7.5) \le 6) - P(\text{Poisson}(9) \le 6) = 0.378 - 0.207 = \mathbf{0.171}.$$

OR

$$P(2.5 < T_7 < 3) = \int_{2.5}^{3} \frac{3^7}{\Gamma(7)} t^{7-1} e^{-3t}\, dt = \int_{2.5}^{3} \frac{3^7}{6!} t^6 e^{-3t}\, dt = \ldots$$

3. In Neverland, annual income (in \$) is distributed according to Gamma distribution with $\alpha = 5$ and $\theta = 10{,}000$. Every year, the IRS audits 1% of the individuals with income below \$50,000, 3% of the individuals with income between \$50,000 and \$95,000, and 6% of the individuals with income above \$95,000. Suppose that the individuals to be audited at selected at random.



in thousands of \$

a) What proportion of Neverland's population falls into each of the three income groups? That is, find $P(X < \$50{,}000)$, $P(\$50{,}000 < X < \$95{,}000)$, and $P(X > \$95{,}000)$. [“Hint”: The answers should add up to 100%.]

If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $P(T > t) = P(Y \le \alpha - 1)$, where Y has a $\text{Poisson}(\lambda t = t/\theta)$ distribution.

$$P(X > \$50{,}000) = P(\text{Poisson}(5.0) \le 4) = 0.440.$$

$$P(X > \$95{,}000) = P(\text{Poisson}(9.5) \le 4) = 0.040.$$

$$\Rightarrow \quad P(X < \$50{,}000) = 1 - 0.440 = \mathbf{0.56}.$$

$$P(\$50{,}000 < X < \$95{,}000) = 0.440 - 0.040 = \mathbf{0.40}.$$

$$P(X > \$95{,}000) = \mathbf{0.04}.$$

b) You have overheard Mr. Statman complain about being audited. What is the probability that Mr. Statman's income is below \$50,000? Between \$50,000 and \$95,000? Above \$95,000? ["Hint": The answers should add up to 100%.]

$$P(\text{audit} \mid X < \$50{,}000) = 0.01,$$

$$P(\text{audit} \mid \$50{,}000 < X < \$95{,}000) = 0.03,$$

$$P(\text{audit} \mid X > \$95{,}000) = 0.06.$$

$$P(\text{audit}) = 0.56 \times 0.01 + 0.40 \times 0.03 + 0.04 \times 0.06 = 0.0056 + 0.0120 + 0.0024 = 0.02.$$

$$P(X < \$50{,}000 \mid \text{audit}) = \frac{0.56 \times 0.01}{0.56 \times 0.01 + 0.40 \times 0.03 + 0.04 \times 0.06} = \frac{0.0056}{0.0200} = \mathbf{0.28}.$$

$$P(\$50{,}000 < X < \$95{,}000 \mid \text{audit}) = \frac{0.40 \times 0.03}{0.56 \times 0.01 + 0.40 \times 0.03 + 0.04 \times 0.06} = \frac{0.0120}{0.0200} = \mathbf{0.60}.$$

$$P(X > \$95{,}000 \mid \text{audit}) = \frac{0.04 \times 0.06}{0.56 \times 0.01 + 0.40 \times 0.03 + 0.04 \times 0.06} = \frac{0.0024}{0.0200} = \mathbf{0.12}.$$

	audit	not audit	Total
$X < \$50{,}000$	0.0056	0.5544	0.56
$\$50{,}000 < X < \$95{,}000$	0.0120	0.3880	0.40
$X > \$95{,}000$	0.0024	0.0376	0.04
Total	0.0200	0.9800	1.00

c) Find the salary that would place an individual in the top 1%. $(\text{Neverland} \neq \text{USA.})$

$$\frac{2\,T_5}{\theta} = \frac{2\,T_5}{10{,}000} \text{ has a } \chi^2(2\alpha = 10) \text{ distribution.}$$

$\chi^2_{0.01}(10) = 23.21.$ $\qquad P(\chi^2(10) > 23.21) = 0.01.$

$$0.01 = P(T_5 > a) = P\left(\frac{2\,T_5}{10{,}000} > \frac{2\,a}{10{,}000}\right) = P\left(\chi^2(10) > \frac{2\,a}{10{,}000}\right).$$

$\dfrac{2\,a}{10{,}000} = 23.21.$ $\qquad \Rightarrow \qquad a =$ **$116,050**. $\qquad$ At least $116,050.

For fun:

Top 5%:

$\chi^2_{0.05}(10) = 18.31.$ $\qquad P(\chi^2(10) > 18.31) = 0.05.$

$$0.05 = P(T_5 > b) = P\left(\frac{2\,T_5}{10{,}000} > \frac{2\,b}{10{,}000}\right) = P\left(\chi^2(10) > \frac{2\,b}{10{,}000}\right).$$

$\dfrac{2\,b}{10{,}000} = 18.31.$ $\qquad \Rightarrow \qquad b =$ **$91,550**. $\qquad$ At least $91,550.

Top 10%:

$\chi^2_{0.10}(10) = 15.99.$ $\qquad P(\chi^2(10) > 15.99) = 0.10.$

$$0.10 = P(T_5 > c) = P\left(\frac{2\,T_5}{10{,}000} > \frac{2\,c}{10{,}000}\right) = P\left(\chi^2(10) > \frac{2\,c}{10{,}000}\right).$$

$\dfrac{2\,c}{10{,}000} = 15.99.$ $\qquad \Rightarrow \qquad c =$ **$79,950**. $\qquad$ At least $79,950.