STAT 410
Fall 2020

Exam 1

Thursday, October 8, 8:00 – 9:50 pm CDT

There are 3 problems on the exam, with 3, 1, and 5 parts, respectively. The point value of each question is shown in parentheses before the question. The total number of points for the exam is 75.

Make sure that you include everything you wish to submit, and that the submission process has completed. You do not need to include the question statements with your work. However, please label you work clearly. Neatness and organization are appreciated. Please put your final answers at the end of your work and mark them clearly.

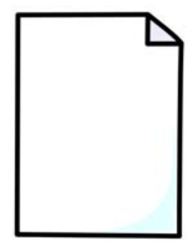
If the answer is a function, specify what kind of function it is (p.d.f., p.m.f., c.d.f., or m.g.f.), and **its support must be included**.

Be sure to show all your work; your partial credit might depend on it.

No credit will be given without supporting work.

The exam is closed book and closed notes.

You are allowed to use a calculator and
one 8½" x 11" sheet (both sides) with notes.



You are allowed to use
https://www.wolframalpha.com/calculators/integral-calculator/
https://www.symbolab.com/solver/definite-integral-calculator
https://www.integral-calculator.com/
https://www.desmos.com/calculator

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x^3}{64}, \qquad 0 \le x \le 4, \qquad \text{zero elsewhere.}$$

a) (8) Find the probability distribution of $Y = \frac{1}{\sqrt{X}}$.

$$0 < x < 4 \qquad 0 < \sqrt{x} < 2 \qquad \infty > \frac{1}{\sqrt{x}} > \frac{1}{2}. \qquad y > \frac{1}{2}.$$

$$F_X(x) = P(X \le x) = \int_0^x \frac{u^3}{64}\,du = \left(\frac{u^4}{256}\right)\Bigg|_0^x = \frac{x^4}{256}, \qquad 0 \le x < 4.$$

$$F_Y(y) = P(Y \le y) = P(\frac{1}{\sqrt{X}} \le y) = P(X \ge \frac{1}{y^2})$$

$$= 1 - F_X(\frac{1}{y^2}) = 1 - \frac{1}{256\,y^8}, \qquad y \ge \frac{1}{2}.$$

$$F_Y(y) = 0, \qquad y < \frac{1}{2}.$$

OR

$$y = \frac{1}{\sqrt{x}} \qquad x = \frac{1}{y^2} = g^{-1}(y) \qquad \frac{dx}{dy} = -\frac{2}{y^3}$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = \frac{1}{64\,y^6} \cdot \left|-\frac{2}{y^3}\right| = \frac{1}{32\,y^9}, \qquad y > \frac{1}{2}.$$

Indeed, $\frac{d}{dy}\left(1 - \frac{1}{256\,y^8}\right) = \frac{1}{32\,y^9}.$ ☺

1. (continued)

Let $X_1, X_2, X_3, X_4, X_5, X_6$ be a random sample (i.i.d.) of size $n = 6$ from a continuous probability distribution with the probability density function

$$f_X(x) = \frac{x^3}{64}, \qquad 0 \le x \le 4, \qquad \text{zero elsewhere.}$$

$$F_X(x) = P(X \le x) = \int_0^x \frac{u^3}{64}\, du = \left(\frac{u^4}{256}\right)\Bigg|_0^x = \frac{x^4}{256} = \left(\frac{x}{4}\right)^4, \qquad 0 \le x < 4.$$

b) (6) Find $P(\min X_i < 2)$.

$$P(\min X_i < 2) = 1 - P(\min X_i \ge 2) = 1 - [P(X \ge 2)]^6$$

$$= 1 - [1 - F_X(2)]^6 = 1 - \left[1 - \frac{1}{16}\right]^6 = 1 - [0.9375]^6 \approx \mathbf{0.321}.$$

c) (6) Find $E(\max X_i)$.

$$F_{\max X_i}(x) = [F_X(x)]^n = \left(\frac{x^4}{256}\right)^6 = \frac{x^{24}}{4^{24}}, \qquad 0 \le x < 4.$$

$$f_{\max X_i}(x) = F'_{\max X_i}(x) = \frac{24\, x^{23}}{4^{24}}, \qquad 0 < x < 4.$$

$$E(\max X_i) = \int_0^4 x \cdot \frac{24\, x^{23}}{4^{24}}\, dx = \int_0^4 \frac{24\, x^{24}}{4^{24}}\, dx = \frac{24\, x^{25}}{25 \cdot 4^{24}}\Bigg|_0^4 = \mathbf{3.84}.$$

2. (6) Suppose $P(X=1)=0.20, \quad P(X=2)=0.80,$
$P(Y=1)=0.75, \quad P(Y=2)=0.25,$
X and Y are independent.

What is the probability distribution of $U = X/Y$?

$$p(x,y) = p_X(x) \cdot p_Y(y).$$

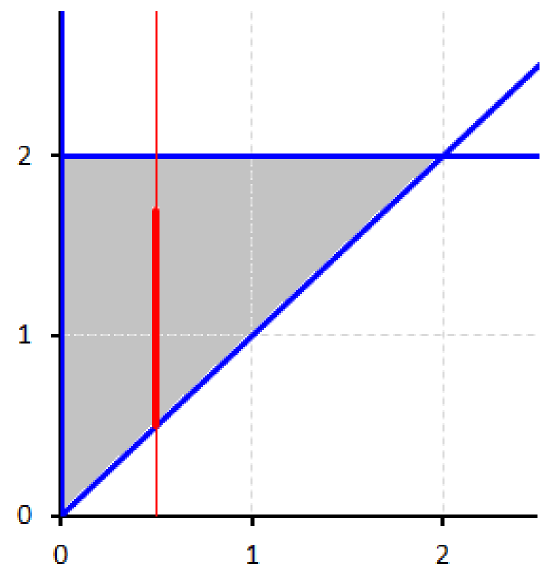
x	y = 1	y = 2	
1	0.20×0.75 0.15 $u = 1$	0.20×0.25 0.05 $u = 0.5$	0.20
2	0.80×0.75 0.60 $u = 2$	0.80×0.25 0.20 $u = 1$	0.80
	0.75	0.25	1.00

u	$p_U(u)$
0.5	0.05
1	0.35
2	0.60

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{5}{16}\,x\,y^2, \qquad 0 < x < y < 2, \qquad \text{zero otherwise.}$$

a) (7) Find $P(Y < 1.7 \mid X = 0.5)$.



$$f_X(x) = \int_x^2 \frac{5}{16}\,x\,y^2\,dy = \frac{5}{48}\,x\,y^3 \Big|_x^2$$

$$= \frac{5}{48}\,x\left(8 - x^3\right), \qquad 0 < x < 2.$$

$$f_{Y|X}(y \mid x) = \frac{\frac{5}{16}\,x\,y^2}{\frac{5}{48}\,x\left(8 - x^3\right)} = \frac{3\,y^2}{8 - x^3}, \qquad x < y < 2.$$

$$f_{Y|X}(y \mid 0.5) = \frac{3\,y^2}{7.875}, \qquad 0.5 < y < 2.$$

$$P(Y < 1.7 \mid X = 0.5) = \int_{0.5}^{1.7} \frac{3\,y^2}{7.875}\,dy = \frac{y^3}{7.875}\Big|_{0.5}^{1.7} = \frac{1.7^3 - 0.5^3}{7.875} = \mathbf{0.608}.$$

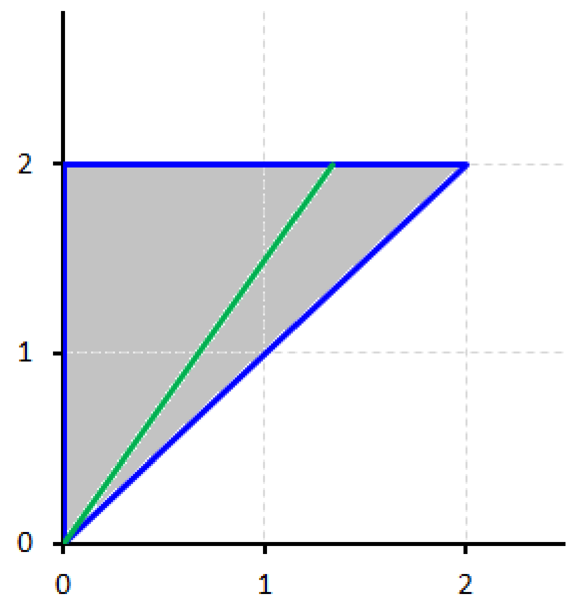
b) (7) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y})$.

$$f_{\mathrm{Y}}(y) = \int_0^y \frac{5}{16} x y^2 \, dx = \frac{5}{32} x^2 y^2 \Big|_0^y = \frac{5}{32} y^4, \qquad 0 < y < 2.$$

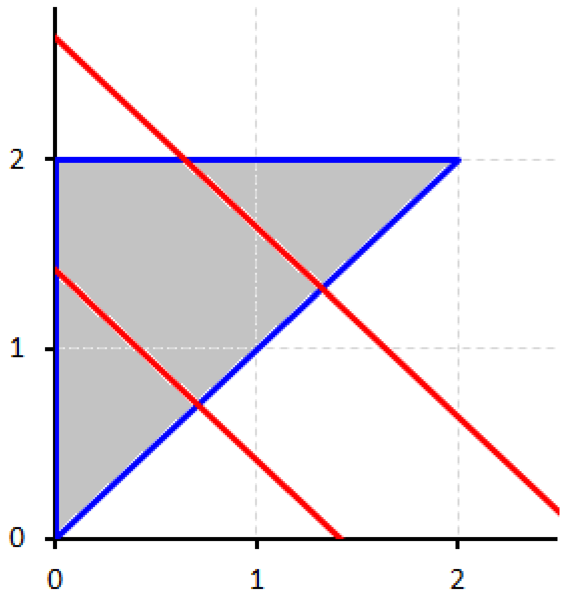
$$f_{\mathrm{X}\mid\mathrm{Y}}(x \mid y) = \frac{\frac{5}{16} x y^2}{\frac{5}{32} y^4} = \frac{2x}{y^2}, \qquad 0 < x < y.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y) = \int_0^y x \cdot \frac{2x}{y^2} \, dx = \int_0^y \frac{2x^2}{y^2} \, dx = \frac{2x^3}{3 \cdot y^2} \Big|_0^y = \frac{2}{3} y, \qquad 0 < y < 2.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}) = \frac{2}{3} \mathrm{Y}.$$



c) (14) Find the probability distribution of W = X + Y.



$F_W(w) = P(X + Y \le w) = \ldots$

There are 2 cases:

$0 < w < 2, \qquad 2 < w < 4.$

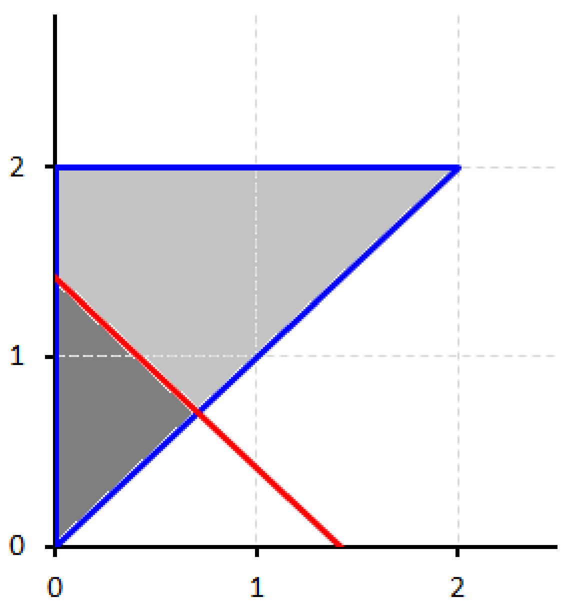
Technically, there are 4 cases:

$w < 0,$

$0 < w < 2, \qquad 2 < w < 4,$

$w > 4,$

but $w < 0$ and $w > 4$ are boring.



Case 1: $\qquad 0 \le w < 2.$

$$F_W(w) = P(X + Y \le w) = \ldots$$

$$= \int_0^{\frac{w}{2}} \left(\int_0^{y} \frac{5}{16}\, x\, y^2\, dx \right) dy + \int_{\frac{w}{2}}^{w} \left(\int_0^{w-y} \frac{5}{16}\, x\, y^2\, dx \right) dy$$

$$= \int_0^{\frac{w}{2}} \frac{5}{32}\, y^4\, dy + \int_{\frac{w}{2}}^{w} \frac{5}{32}\, (w-y)^2\, y^2\, dy$$

$x = y \quad \& \quad x + y = w \quad \Rightarrow \quad x = y = \dfrac{w}{2}.$

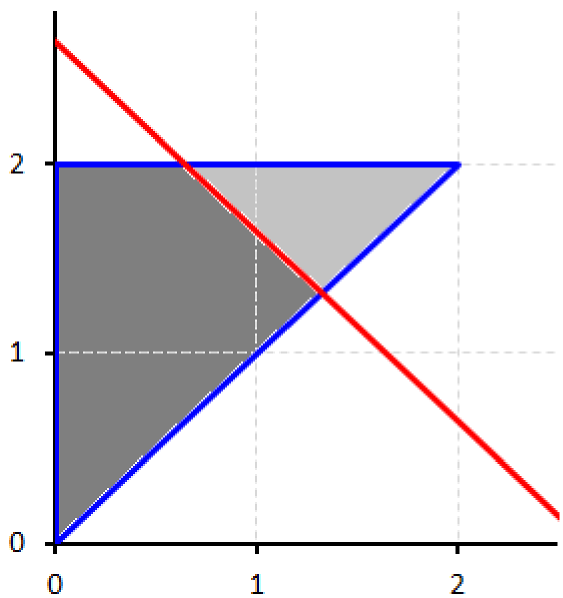
$$= \left.\frac{y^5}{32}\right|_0^{\frac{w}{2}} + \frac{5}{32}\left.\left(\frac{y^3}{3}w^2 - \frac{y^4}{2}w + \frac{y^5}{5}\right)\right|_{\frac{w}{2}}^{w}$$

$$= \frac{w^5}{1{,}024} + \frac{w^5}{384} = \frac{11w^5}{3{,}072}, \qquad 0 \le w < 2.$$

$$\ldots = \int_0^{\frac{w}{2}} \left(\int_x^{w-x} \frac{5}{16}\,x\,y^2\,dy\right) dx = \int_0^{\frac{w}{2}} \frac{5}{48}\,x\left((w-x)^3 - x^3\right) dx$$

$$= \frac{5}{48}\left.\left(\frac{x^2}{2}w^3 - x^3w^2 + \frac{3x^4}{4}w - \frac{2x^5}{5}\right)\right|_0^{\frac{w}{2}}$$

$$= \frac{11w^5}{3{,}072}, \qquad 0 \le w < 2.$$



Case 2: $\quad 2 \le w < 4.$

$$F_W(w) = P(X + Y \le w) = \ldots$$

$$= 1 - \int_{\frac{w}{2}}^{2} \left(\int_{w-y}^{y} \frac{5}{16}\,x\,y^2\,dx\right) dy$$

$$= 1 - \int_{\frac{w}{2}}^{2} \frac{5y^2}{32}\left(2yw - w^2\right) dy$$

$$= 1 - \frac{5}{32}\left.\left(\frac{y^4}{2}w - \frac{y^3}{3}w^2\right)\right|_{\frac{w}{2}}^{2}$$

$$= 1 - \frac{5}{32}\left(8w - \frac{8w^2}{3} + \frac{w^5}{96}\right) = 1 - \frac{5w}{4} + \frac{5w^2}{12} - \frac{5w^5}{3{,}072}$$

$$= \frac{3{,}072 - 3{,}840\,w + 1{,}280\,w^2 - 5w^5}{3{,}072}, \qquad 2 \le w < 4.$$

$$\ldots = \int_0^{\frac{w}{2}} \left(\int_0^{y} \frac{5}{16}\, x\, y^2\, dx \right) dy \;+\; \int_{\frac{w}{2}}^{2} \left(\int_0^{w-y} \frac{5}{16}\, x\, y^2\, dx \right) dy$$

$$= \int_0^{\frac{w}{2}} \frac{5}{32}\, y^4\, dy \;+\; \int_{\frac{w}{2}}^{2} \frac{5}{32}\, (w-y)^2\, y^2\, dy$$

$$= \left. \frac{y^5}{32} \right|_0^{\frac{w}{2}} + \frac{5}{32} \left. \left(\frac{y^3}{3}\, w^2 - \frac{y^4}{2}\, w + \frac{y^5}{5} \right) \right|_{\frac{w}{2}}^{2}$$

$$= \frac{w^5}{1{,}024} + \frac{5}{32} \left(\frac{8}{3}\, w^2 - 8\, w + \frac{32}{5} - \frac{1}{60}\, w^5 \right)$$

$$= 1 - \frac{5\, w}{4} + \frac{5\, w^2}{12} - \frac{5\, w^5}{3{,}072}, \qquad 2 \le w < 4.$$

$$\ldots = \int_0^{w-2} \left(\int_x^{2} \frac{5}{16}\, x\, y^2\, dy \right) dx \;+\; \int_{w-2}^{\frac{w}{2}} \left(\int_x^{w-x} \frac{5}{16}\, x\, y^2\, dy \right) dx = \ldots$$

OR

$$f_{\mathrm{W}}(w) = f_{\mathrm{X+Y}}(w) = \int_{-\infty}^{\infty} f(w-y, y)\, dy. \qquad f(w-y, y) = \frac{5}{16}\, (w-y)\, y^2.$$

$0 < x$ $\qquad$ $0 < w - y$ $\qquad$ $y < w$

$x < y$ $\qquad$ $w - y < y$ $\qquad$ $y > \frac{w}{2}$

$y < 2$

$y > \frac{w}{2}$ & $y < w, \;\; y < 2.$

Case 1: $0 < w < 2.$

$$f_W(w) = \int_{\frac{w}{2}}^{w} \frac{5}{16}(w-y)\,y^2\,dy = \frac{5}{16}\left(\frac{y^3}{3}w - \frac{y^4}{4}\right)\Bigg|_{\frac{w}{2}}^{w} = \frac{55\,w^4}{3{,}072}, \qquad 0 < w < 2.$$

Case 2: $2 \le w < 4.$

$$f_W(w) = \int_{\frac{w}{2}}^{2} \frac{5}{16}(w-y)\,y^2\,dy = \frac{5}{16}\left(\frac{y^3}{3}w - \frac{y^4}{4}\right)\Bigg|_{\frac{w}{2}}^{2} = \frac{5}{6}w - \frac{5}{4} - \frac{25}{3{,}072}w^4$$

$$2 < w < 4.$$

OR

$$f_W(w) = f_{X+Y}(w) = \int_{-\infty}^{\infty} f(x, w-x)\,dx.$$

$$f(x, w-x) = \frac{5}{16}x\,(w-x)^2.$$

$0 < x$

$x < y \qquad x < w - x \qquad x < \frac{w}{2}$

$y < 2 \qquad w - x < 2 \qquad x > w - 2$

$x > 0, \;\; x > w - 2$ & $x < \frac{w}{2}.$

Case 1: $0 < w < 2$.

$$f_W(w) = \int_0^{\frac{w}{2}} \frac{5}{16}\, x\,(w-x)^2\, dx = \frac{5}{16}\left(\frac{x^2}{2} w^2 - \frac{2x^3}{3} w + \frac{x^4}{4} \right) \Bigg|_0^{\frac{w}{2}} = \frac{55\,w^4}{3{,}072},$$

$0 < w < 2$.

Case 2: $2 \le w < 4$.

$$f_W(w) = \int_{w-2}^{\frac{w}{2}} \frac{5}{16}\, x\,(w-x)^2\, dx = \frac{5}{16}\left(\frac{x^2}{2} w^2 - \frac{2x^3}{3} w + \frac{x^4}{4} \right) \Bigg|_{w-2}^{\frac{w}{2}}$$

$$= \frac{55\,w^4}{3{,}072} - \frac{5\,(w-2)^2\, w^2}{32} + \frac{5\,(w-2)^3\, w}{24} - \frac{5\,(w-2)^4}{64},$$

$2 < w < 4$.

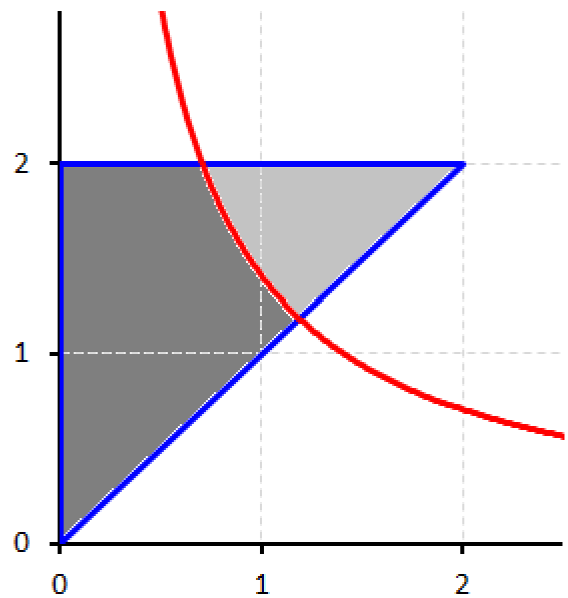
c.d.f.

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \dfrac{11\,w^5}{3{,}072} & 0 \le w < 2 \\ 1 - \dfrac{5\,w}{4} + \dfrac{5\,w^2}{12} - \dfrac{5\,w^5}{3{,}072} & 2 \le w < 4 \\ 1 & w \ge 4 \end{cases}$$

p.d.f.

$$f_W(w) = \begin{cases} \dfrac{55\,w^4}{3{,}072} & 0 < w < 2 \\ \dfrac{5}{6}\,w - \dfrac{5}{4} - \dfrac{25}{3{,}072}\,w^4 & 2 < w < 4 \\ 0 & \text{otherwise} \end{cases}$$

d) (9) Find the probability distribution of $V = X \cdot Y$.



$F_V(v) = P(X \cdot Y \le v) = \ldots$

There is 1 case:

$0 < v < 4.$

Technically, there are 3 cases:

$v < 0,$

$0 < v < 4,$

$v > 4,$

but $v < 0$ and $v > 4$ are boring.

$x = y \quad \& \quad x \cdot y = v \quad \Rightarrow \quad x = y = \sqrt{v}\,.$

$$\ldots = 1 - \int_{\sqrt{v}}^{2} \left(\int_{\frac{v}{y}}^{y} \frac{5}{16} x y^2 \, dx \right) dy = 1 - \int_{\sqrt{v}}^{2} \frac{5}{32} \left(y^2 - \frac{v^2}{y^2} \right) y^2 \, dy$$

$$= 1 - \int_{\sqrt{v}}^{2} \left(\frac{5}{32} y^4 - \frac{5}{32} v^2 \right) dy = 1 - \left(\frac{1}{32} y^5 - \frac{5}{32} y v^2 \right) \Bigg|_{\sqrt{v}}^{2}$$

$$= \frac{5}{16} v^2 - \frac{1}{8} v^2 \sqrt{v} = \frac{5}{16} v^2 - \frac{1}{8} v^{2.5}, \qquad 0 \le v < 4$$

$$\ldots = \int_{0}^{\sqrt{v}} \left(\int_{0}^{y} \frac{5}{16} x y^2 \, dx \right) dy + \int_{\sqrt{v}}^{2} \left(\int_{0}^{\frac{v}{y}} \frac{5}{16} x y^2 \, dx \right) dy$$

$$= \int_0^{\sqrt{v}} \frac{5}{32} y^4\, dy \;+\; \int_{\sqrt{v}}^{2} \frac{5}{32} v^2\, dy \;=\; \frac{1}{32} v^2 \sqrt{v} \;+\; \frac{5}{32} v^2 \left(2-\sqrt{v}\right)$$

$$= \frac{5}{16} v^2 - \frac{1}{8} v^2 \sqrt{v} \;=\; \frac{5}{16} v^2 - \frac{1}{8} v^{2.5}, \qquad\qquad 0 \le v < 4$$

$$\ldots \;=\; \int_0^{\frac{v}{2}} \left(\int_x^{2} \frac{5}{16} x y^2\, dy \right) dx \;+\; \int_{\frac{v}{2}}^{\sqrt{v}} \left(\int_x^{\frac{v}{x}} \frac{5}{16} x y^2\, dy \right) dx$$

$$= \frac{5}{48} \left(v^2 - \frac{v^5}{160} \right) \;+\; \frac{5}{48} v^2 \left(2-\sqrt{v}\right) - \frac{v^{2.5}}{48} + \frac{v^5}{1{,}536} \;=\; \ldots$$

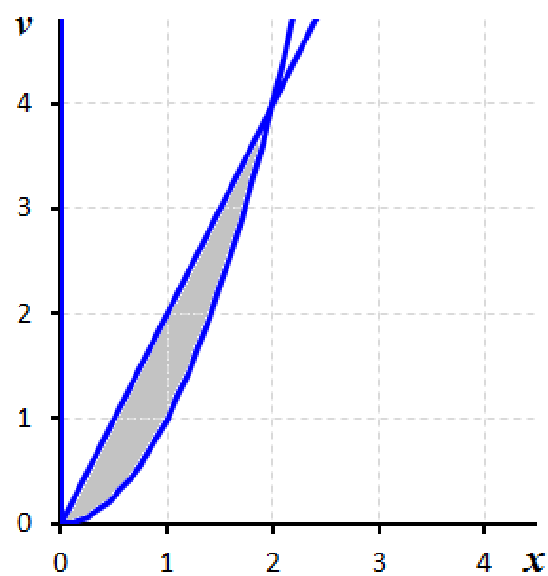
OR

$$X = X, \qquad Y = \frac{V}{X}.$$

$0 < x$

$x < y$	$x < \dfrac{v}{x}$	$x < \sqrt{v}$
$y < 2$	$\dfrac{v}{x} < 2$	$x > \dfrac{v}{2}$

$$J = \begin{vmatrix} 1 & 0 \\ -\dfrac{v}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{1}{x}.$$



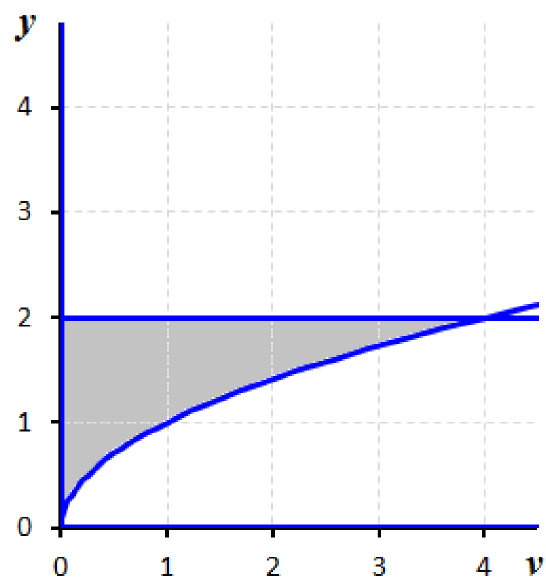
$$f_{X,V}(x,v) = f_{X,Y}\left(x, \frac{v}{x}\right) \cdot |J| = \frac{5}{16}\, x \left(\frac{v}{x}\right)^2 \cdot \frac{1}{x} = \frac{5v^2}{16x^2}.$$

$$f_V(v) = \int_{\frac{v}{2}}^{\sqrt{v}} \frac{5v^2}{16x^2}\, dx = \left(-\frac{5v^2}{16x}\right)\Bigg|_{\frac{v}{2}}^{\sqrt{v}} = \frac{5}{8}v - \frac{5}{16}v\sqrt{v} = \frac{5}{8}v - \frac{5}{16}v^{1.5},$$

$0 < v < 4.$

OR

$X = \dfrac{V}{Y}, \quad Y = Y.$



$0 < x \qquad 0 < \dfrac{v}{y}$

$x < y \qquad \dfrac{v}{y} < y \qquad y > \sqrt{v}$

$y < 2$

$$J = \begin{vmatrix} \frac{1}{y} & -\frac{v}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}.$$

$$f_{Y,V}(y,v) = f_{X,Y}\left(\frac{v}{y}, y\right) \cdot |J| = \frac{5}{16}\left(\frac{v}{y}\right) y^2 \cdot \frac{1}{y} = \frac{5}{16}v.$$

$$f_V(v) = \int_{\sqrt{v}}^{2} \frac{5}{16}v\, dy = \frac{5}{16}v\left(2 - \sqrt{v}\right) = \frac{5}{8}v - \frac{5}{16}v\sqrt{v} = \frac{5}{8}v - \frac{5}{16}v^{1.5},$$

$0 < v < 4.$

e) (12) Let $U = \dfrac{Y}{X}$. Find the joint probability density function of (X, U), $f_{X,U}(x, u)$.

Sketch the support of (X, U).

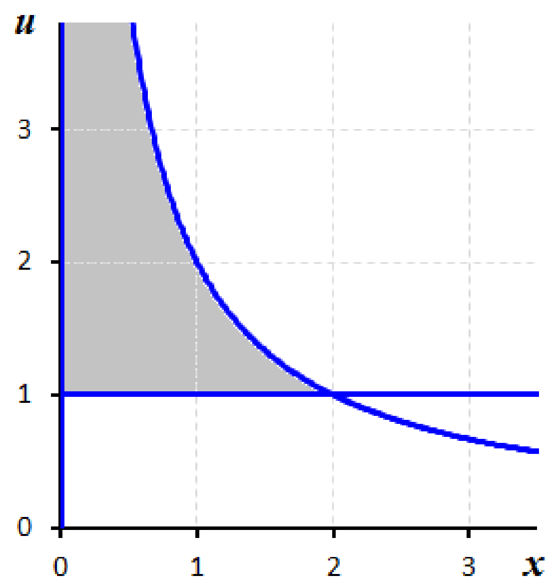
$X = X, \quad Y = U X.$

$0 < x$

$x < y \qquad x < u\,x \qquad u > 1$

$y < 2 \qquad u\,x < 2$

$$J = \begin{vmatrix} 1 & 0 \\ u & x \end{vmatrix} = x.$$



$$f_{X,U}(x, u) = f_{X,Y}(x, u\,x) \cdot |\,J\,| = \frac{5}{16}\,x\,(u\,x)^2 \cdot x = \frac{5}{16}\,x^4\,u^2,$$

$$u > 1, \quad 0 < x < \frac{2}{u}.$$