STAT 410

Fall 2020
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Examples for 11/18/2020 (3)

Examples for 10/19/2020 (3) & Examples for 11/06/2020 (1) (continued)

6 – 7. The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0$, $\delta > 0$. Consider the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall: $W = X^{\delta}$ has an $\text{Exponential}\left(\theta = \frac{1}{\beta}\right) = \text{Gamma}\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the above probability distribution.

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} X_i^{\delta} = \sum_{i=1}^{n} W_i \quad \text{has a } \text{Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right) \text{ distribution.} \qquad !!!$$

Recall: $Y = \sum_{i=1}^{n} X_i^{\delta}$ is a sufficient statistic for β.

Suppose $\delta = 3$. We wish to test $H_0: \beta = 3$ vs. $H_1: \beta < 3$.

6. p) Suppose $n = 5$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

Hint 1: We have $f(x;\beta) = 3\,\beta\,x^2\,e^{-\beta x^3}$, $\quad x > 0$.

Let $\beta < 3$. Start with

$$\frac{L(H_0;\, x_1, x_2, \ldots, x_n)}{L(H_1;\, x_1, x_2, \ldots, x_n)} = \frac{L(3;\, x_1, x_2, \ldots, x_n)}{L(\beta;\, x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} f(x_i;3)}{\prod_{i=1}^{n} f(x_i;\beta)} \le k.$$

Simplify this. Since $\mathrm{Y} = \sum_{i=1}^{n} \mathrm{X}_i^3$ is a sufficient statistic for β,

and the final form of the "best" rejection region should look like this:

$$\text{“ Reject } \mathrm{H}_0 \text{ if } \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{5} x_i^3 \; [\leq \text{or} \geq] \; c \text{”.}$$

The direction of the inequality sign is what you are trying to determine.

Hint 2: $\mathrm{Y} = \sum_{i=1}^{n} \mathrm{X}_i^3 = \sum_{i=1}^{n} \mathrm{W}_i$ has a $\text{Gamma}(\alpha = n, \theta = \frac{1}{\beta})$ distribution.

Hint 3: Want c such that $0.10 = \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}(\sum_{i=1}^{n} \mathrm{X}_i^3 \; ? \; c \mid \beta = 3)$.

Hint 4: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2\mathrm{T}/\theta = 2\lambda\mathrm{T}$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

q) Suppose $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$.
Find the p-value of this test.

7. Consider the rejection region Reject H_0 if $\sum_{i=1}^{5} x_i^3 \geq 3$.

r) Find the significance level α of this rejection region.

Hint 1: $\alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}(\sum_{i=1}^{5} \mathrm{X}_i^3 \geq 3 \mid \beta = 3)$.

Hint 2: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then
$\mathrm{F_T}(t) = \mathrm{P}(\mathrm{T} \leq t) = \mathrm{P}(\mathrm{Y} \geq \alpha)$ and $\mathrm{P}(\mathrm{T} > t) = \mathrm{P}(\mathrm{Y} \leq \alpha - 1)$,
where Y has a $\text{Poisson}(\lambda t)$ distribution.

s) Find the power of this rejection region if $\beta = 2$ and if $\beta = 1$.

6 – 7. The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0, \delta > 0$. Consider the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall: $\mathrm{W} = \mathrm{X}^{\delta}$ has an Exponential$\left(\theta = \frac{1}{\beta}\right)$ = Gamma$\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample from the above probability distribution.

$\Rightarrow$ $\mathrm{Y} = \sum_{i=1}^{n} \mathrm{X}_i^{\delta} = \sum_{i=1}^{n} \mathrm{W}_i$ has a Gamma$\left(\alpha = n, \theta = \frac{1}{\beta}\right)$ distribution. **! ! !**

Recall: $\mathrm{Y} = \sum_{i=1}^{n} \mathrm{X}_i^{\delta}$ is a sufficient statistic for β.

Suppose $\delta = 3$. We wish to test $\mathrm{H}_0: \beta = 3$ vs. $\mathrm{H}_1: \beta < 3$.

6. p) Suppose $n = 5$. Find the uniformly most powerful rejection region with $\alpha = 0.10$.

Hint 1: We have $f(x;\beta) = 3\,\beta\,x^2\,e^{-\beta x^3}, \qquad x > 0.$

Let $\beta < 3$. Start with

$$\frac{\mathrm{L}(\mathrm{H}_0;\ x_1, x_2, \ldots, x_n)}{\mathrm{L}(\mathrm{H}_1;\ x_1, x_2, \ldots, x_n)} = \frac{\mathrm{L}(3;\ x_1, x_2, \ldots, x_n)}{\mathrm{L}(\beta;\ x_1, x_2, \ldots, x_n)} = \frac{\prod_{i=1}^{n} f(x_i;3)}{\prod_{i=1}^{n} f(x_i;\beta)} \le k.$$

Simplify this. Since $\mathrm{Y} = \sum_{i=1}^{n} \mathrm{X}_i^{3}$ is a sufficient statistic for β,

and the final form of the "best" rejection region should look like this:

$$\text{“ Reject } \mathrm{H}_0 \text{ if } \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{5} x_i^3 \ [\le \text{or} \ge]\ c \text{ ”.}$$

The direction of the inequality sign is what you are trying to determine.

Hint 2: $\quad Y = \sum_{i=1}^{n} X_i^3 = \sum_{i=1}^{n} W_i$ has a Gamma$\left(\alpha = n, \theta = \frac{1}{\beta}\right)$ distribution.

Hint 3: $\quad$ Want c such that $0.10 = \alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{n} X_i^3 \ ?\ c \mid \beta = 3\right)$.

Hint 4: $\quad$ If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

Let $\beta < 3$.

$$\lambda(x_1, x_2, \dots, x_n) = \frac{L(H_0;\, x_1, x_2, \dots, x_n)}{L(H_1;\, x_1, x_2, \dots, x_n)} = \frac{\prod_{i=1}^{n}\left(9\, x_i^2\, e^{-3 x_i^3}\right)}{\prod_{i=1}^{n}\left(3\,\beta\, x_i^2\, e^{-\beta x_i^3}\right)}$$

$$= \left(\frac{3}{\beta}\right)^n \exp\left\{(\beta - 3) \sum_{i=1}^{n} x_i^3\right\}.$$

$$\lambda(x_1, x_2, \dots, x_n) \le k \quad \Leftrightarrow \quad (\beta - 3) \sum_{i=1}^{n} x_i^3 \le k_1$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} x_i^3 \ge c \qquad (\text{since } \beta < 3).$$

Reject H_0 if $\sum_{i=1}^{n} x_i^3 \ge c$.

$\sum_{i=1}^{n} X_i^3$ has a Gamma distribution with $\alpha = n = 5$ and $\theta = \frac{1}{\beta}$.

Then $\frac{2}{\theta} \sum_{i=1}^{n} X_i^3 = 2\beta \sum_{i=1}^{n} X_i^3$ has a $\chi^2(2\alpha = 10 \text{ degrees of freedom})$ distribution.

$$0.10 = \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}(\sum_{i=1}^{n} \mathrm{X}_i^3 \ge c \mid \beta = 3)$$

$$= \mathrm{P}(2 \cdot 3 \cdot \sum_{i=1}^{5} \mathrm{X}_i^3 \ge 2 \cdot 3 \cdot c \mid \beta = 3) = \mathrm{P}(\chi^2(10) \ge 6c).$$

$\Rightarrow \quad 6c = \chi^2_{0.10}(10) = 15.99. \qquad \Rightarrow \qquad c = 2.665.$

Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge \mathbf{2.665}$.

OR

$6c = \chi^2_{0.10}(10) = 15.98718. \qquad \Rightarrow \qquad c = 2.66453.$

Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge \mathbf{2.66453}$.

OR

$$0.10 = \alpha = \mathrm{P}(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \mathrm{P}(\sum_{i=1}^{n} \mathrm{X}_i^3 \ge c \mid \beta = 3)$$

$$= \mathrm{P}(\text{Poisson}(c \times 3) \le 5 - 1) = \mathrm{P}(\text{Poisson}(3c) \le 4).$$

$\mathrm{P}(\text{Poisson}(8.0) \le 4) = 0.100. \qquad 3c = 8. \qquad c = 2.66666\ldots$.

Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge \mathbf{2.66666\ldots}$.

q) Suppose $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$.
Find the p-value of this test.

$$\sum_{i=1}^{n} x_i^3 = 2.5.$$

$$\text{p-value} = P\left(\sum_{i=1}^{5} X_i^3 \ge 2.5 \mid \beta = 3\right) = P(\text{Poisson}(2.5 \times 3) \le 5 - 1)$$

$$= P(\text{Poisson}(7.5) \le 4) = \mathbf{0.132}.$$

OR

$$\text{p-value} = P\left(\sum_{i=1}^{5} X_i^3 \ge 2.5 \mid \beta = 3\right) = P\left(2 \cdot 3 \cdot \sum_{i=1}^{5} X_i^3 \ge 2 \cdot 3 \cdot 2.5 \mid \beta = 3\right)$$

$$= P(\chi^2(10) \ge 15) = 0.132062.$$

7. Consider the rejection region Reject H_0 if $\sum_{i=1}^{5} x_i^3 \ge 3$.

r) Find the significance level α of this rejection region.

Hint 1: $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{5} X_i^3 \ge 3 \mid \beta = 3\right)$.

Hint 2: If T has a Gamma$(\alpha, \theta = {}^1\!/_\lambda)$ distribution, where α is an integer, then

$F_T(t) = P(T \le t) = P(Y \ge \alpha)$ and $P(T > t) = P(Y \le \alpha - 1)$,

where Y has a Poisson(λt) distribution.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P\left(\sum_{i=1}^{5} X_i^3 \ge 3 \mid \beta = 3\right)$$

$$= P(\text{Poisson}(3 \times 3) \le 5 - 1) = P(\text{Poisson}(9) \le 4) = \mathbf{0.055}.$$

OR $$= P(\chi^2(10) \ge 18) = 0.054964.$$

s) Find the power of this rejection region if $\beta = 2$ and if $\beta = 1$.

$$\text{Power}(\beta) = P(\text{Reject } H_0 \mid \beta) = P\left(\sum_{i=1}^{5} X_i^3 \geq 3 \mid \beta\right)$$

$$= P\left(\text{Poisson}(3 \times \beta) \leq 5 - 1\right) = P\left(\text{Poisson}(3\beta) \leq 4\right).$$

OR $$= P(\chi^2(10) \geq 6\beta).$$

$\text{Power}(2) = P\left(\text{Poisson}(6) \leq 4\right) =$ **0.285**. 0.285057.

$\text{Power}(1) = P\left(\text{Poisson}(3) \leq 4\right) =$ **0.815**. 0.815263.