STAT 410

Examples for 11/11/2020 (Disc)

Fall 2020
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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Suppose a random sample of size $n = 16$ is taken from a normal distribution with $\sigma = 8$ for the purpose of testing $H_0: \mu \ge 60$ vs. $H_1: \mu < 60$ at a 5% level of significance. What is the power of this test if $\mu = 55.55$?

① Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

② $\text{Power}(\mu = 55.55) = P(\text{Reject } H_0 \mid \mu = 55.55)$.

2. Suppose a random sample of size $n = 16$ is taken from a normal distribution with $\sigma = 8$ for the purpose of testing $H_0: \mu = 60$ vs. $H_1: \mu \ne 60$ at a 5% level of significance. What is the power of this test if $\mu = 56.78$?

① Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

② $\text{Power}(\mu = 56.78) = P(\text{Reject } H_0 \mid \mu = 56.78)$.

3. Let $X_1, X_2, \ldots, X_{25}$ be a random sample of size $n = 25$ from a $N(\mu, \sigma^2 = 100)$ population, and suppose the null hypothesis $H_0: \mu = 100$ is to be tested.

a) If the alternative hypothesis is $H_1: \mu > 100$, compute the power of the appropriate test at $\mu = 101$ and at $\mu = 102$. Use $\alpha = 0.05$.

b) For the test in (a) compute the p-value associated with $\overline{X} = 103.5$.

4. Let $X_1, X_2, \ldots, X_{25}$ be a random sample of size $n = 25$ from a $N(\mu, \sigma^2 = 100)$ population, and suppose the null hypothesis $H_0: \mu = 100$ is to be tested.

a) If the alternative hypothesis is $H_1: \mu \neq 100$, compute the power of the appropriate test at $\mu = 101$ and at $\mu = 102$. Use $\alpha = 0.05$.

b) For the test in (a) compute the p-value associated with $\overline{X} = 103.5$.

Answers:

1. Suppose a random sample of size $n = 16$ is taken from a normal distribution with $\sigma = 8$ for the purpose of testing $H_0: \mu \geq 60$ vs. $H_1: \mu < 60$ at a 5% level of significance. What is the power of this test if $\mu = 55.55$?

① Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

② $\text{Power}(\mu = 55.55) = P(\text{Reject } H_0 \mid \mu = 55.55)$.

① Reject H_0 if $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} < -z_\alpha$. $\qquad Z = \dfrac{\overline{X} - 60}{8 / \sqrt{16}} < -1.645$.

$$\overline{X} < 60 - 1.645 \cdot \frac{8}{\sqrt{16}} = \mathbf{56.71}.$$

② $\text{Power}(\mu = 55.55) = P(\text{Reject } H_0 \mid \mu = 55.55) = P(\overline{X} < 56.71 \mid \mu = 55.55)$

$$= P\left(Z < \frac{56.71 - 55.55}{8 / \sqrt{16}} \right) = P(Z < 0.58) = \mathbf{0.7190}.$$

2. Suppose a random sample of size $n = 16$ is taken from a normal distribution with $\sigma = 8$ for the purpose of testing $H_0: \mu = 60$ vs. $H_1: \mu \neq 60$ at a 5% level of significance. What is the power of this test if $\mu = 56.78$?

① Find the Rejection Region for the test at $\alpha = 0.05$. That is, for which values of the sample mean $\overline{X}$ should we reject H_0, if a 5% level of significance is used?

② $\text{Power}(\mu = 56.78) = P(\text{Reject } H_0 \mid \mu = 56.78)$.

① Rejection Region: Reject H_0 if

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2} \quad \text{or} \quad Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2}$$

$$\Rightarrow \quad \overline{X} < \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \overline{X} > \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \quad \overline{X} < 60 - 1.96 \cdot \frac{8}{\sqrt{16}} \quad \text{or} \quad \overline{X} > 60 + 1.96 \cdot \frac{8}{\sqrt{16}}$$

$$\Rightarrow \quad \overline{X} < \mathbf{56.08} \quad \text{or} \quad \overline{X} > \mathbf{63.92}$$

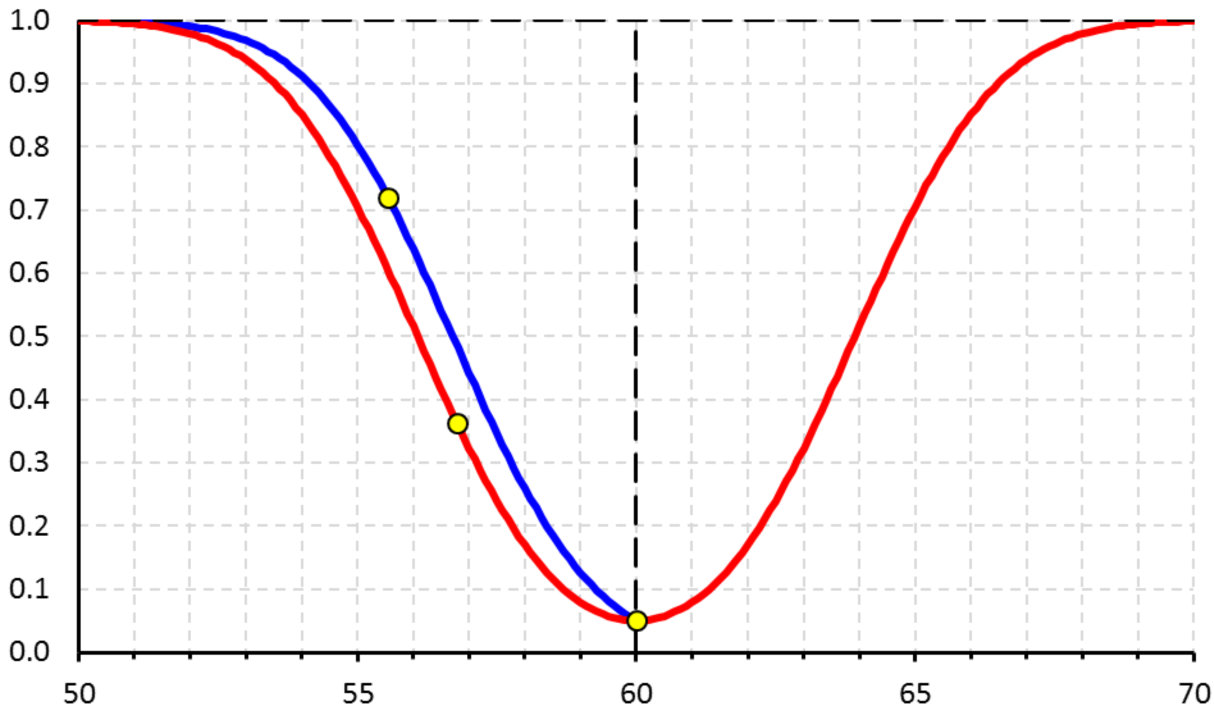
② $\text{Power}(\mu = 56.78) = P(\text{Reject } H_0 \mid \mu = 56.78)$

$$= P(\overline{X} < 56.08 \mid \mu = 56.78) + P(\overline{X} > 63.92 \mid \mu = 56.78)$$

$$= P\left(Z < \frac{56.08 - 56.78}{8/\sqrt{16}}\right) + P\left(Z > \frac{63.92 - 56.78}{8/\sqrt{16}}\right)$$

$$= P(Z < -0.35) + P(Z > 3.57) = 0.3632 + 0.0002 = \mathbf{0.3634}.$$

For fun:



Problem 1 **Problem 2**

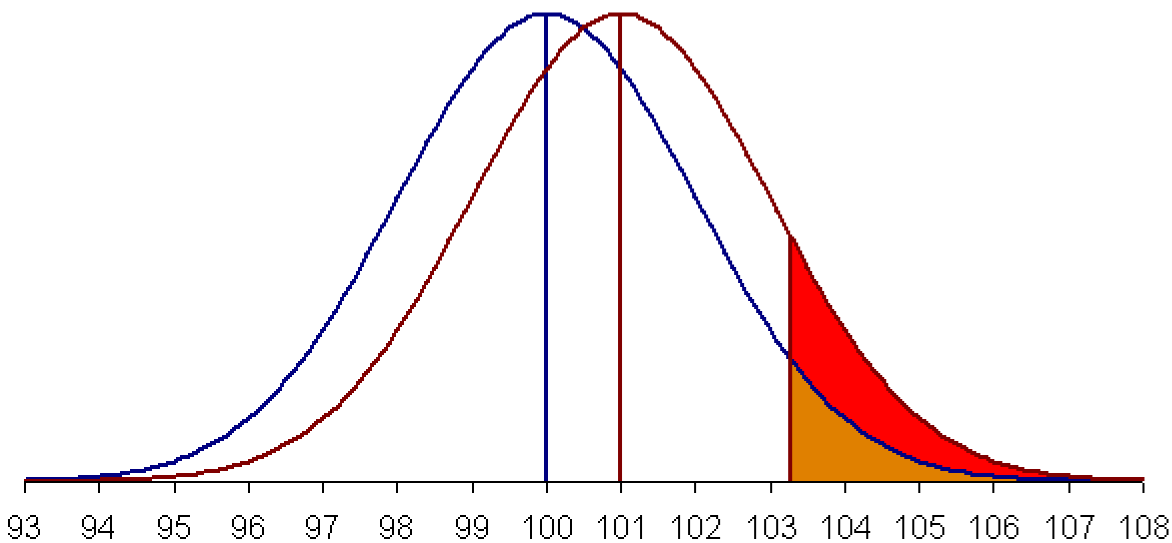
3. Let $X_1, X_2, \dots, X_{25}$ be a random sample of size $n = 25$ from a $N(\mu, \sigma^2 = 100)$ population, and suppose the null hypothesis $H_0: \mu = 100$ is to be tested.

a) If the alternative hypothesis is $H_1: \mu > 100$, compute the power of the appropriate test at $\mu = 101$ and at $\mu = 102$. Use $\alpha = 0.05$.

$H_0: \mu = 100$ vs. $H_1: \mu > 100.$ Right - tailed.

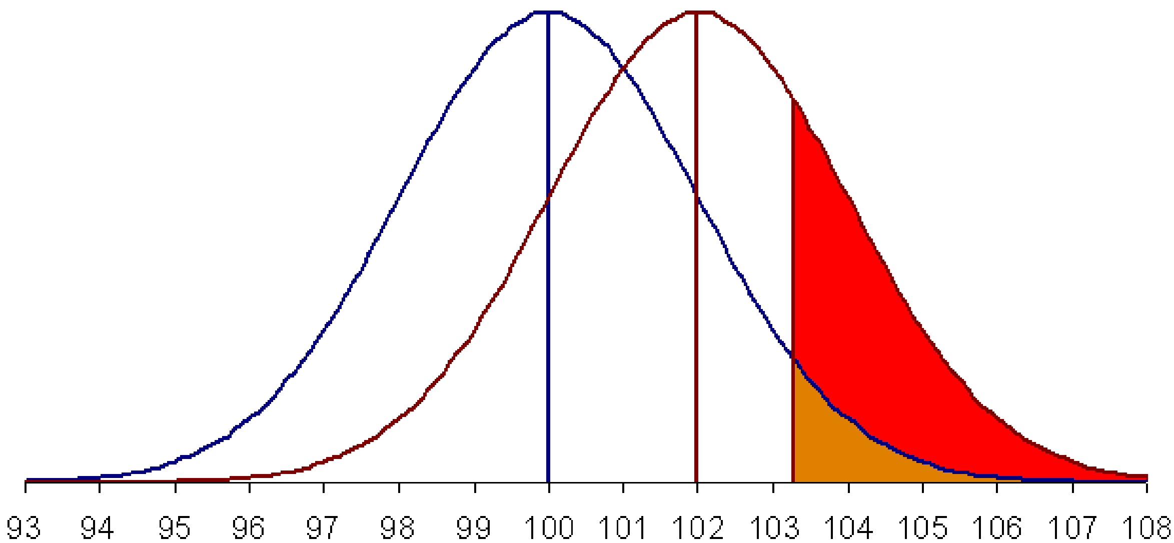
Rejection Region: Reject H_0 if $Z > z_\alpha = 1.645.$

$$\overline{X} > 100 + 1.645 \cdot \frac{10}{\sqrt{25}} = 103.29.$$



$$\text{Power}(101) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(\overline{X} > 103.29 \mid \mu = 101)$$

$$= P\left(Z > \frac{103.29 - 101}{10 / \sqrt{25}} \right) = P(Z > 1.145) \approx \frac{0.1271 + 0.1251}{2} = \mathbf{0.1261}.$$



$$\text{Power}(102) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true}) = P(\overline{X} > 103.29 \mid \mu = 102)$$

$$= P\left(Z > \frac{103.29 - 102}{10/\sqrt{25}} \right) = P(Z > 0.645) \approx \frac{0.2611 + 0.2578}{2} = \mathbf{0.25945}.$$

b) For the test in (a) compute the p-value associated with $\overline{X} = 103.5$.

Test Statistic: $$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{103.5 - 100}{10/\sqrt{25}} = 1.75.$$

P-value = right tail = $P(Z > 1.75) = \mathbf{0.0401}$.

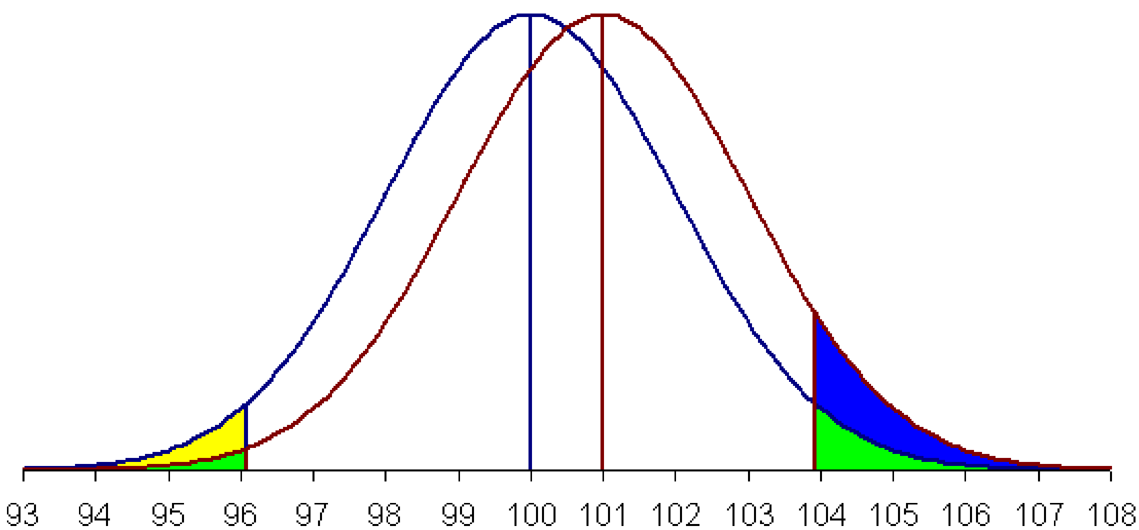
4. Let $X_1, X_2, \dots, X_{25}$ be a random sample of size $n = 25$ from a $N(\mu, \sigma^2 = 100)$ population, and suppose the null hypothesis $H_0: \mu = 100$ is to be tested.

a) If the alternative hypothesis is $H_1: \mu \neq 100$, compute the power of the appropriate test at $\mu = 101$ and at $\mu = 102$. Use $\alpha = 0.05$.

$H_0: \mu = 100$ vs. $H_1: \mu \neq 100.$ Two - tailed.

Rejection Region: Reject H_0 if $Z < -z_{\alpha/2} = -1.960$ or $Z > z_{\alpha/2} = 1.960.$

$$\overline{X} < 100 - 1.960 \cdot \frac{10}{\sqrt{25}} = 96.08 \quad \text{or} \quad \overline{X} > 100 + 1.960 \cdot \frac{10}{\sqrt{25}} = 103.92.$$

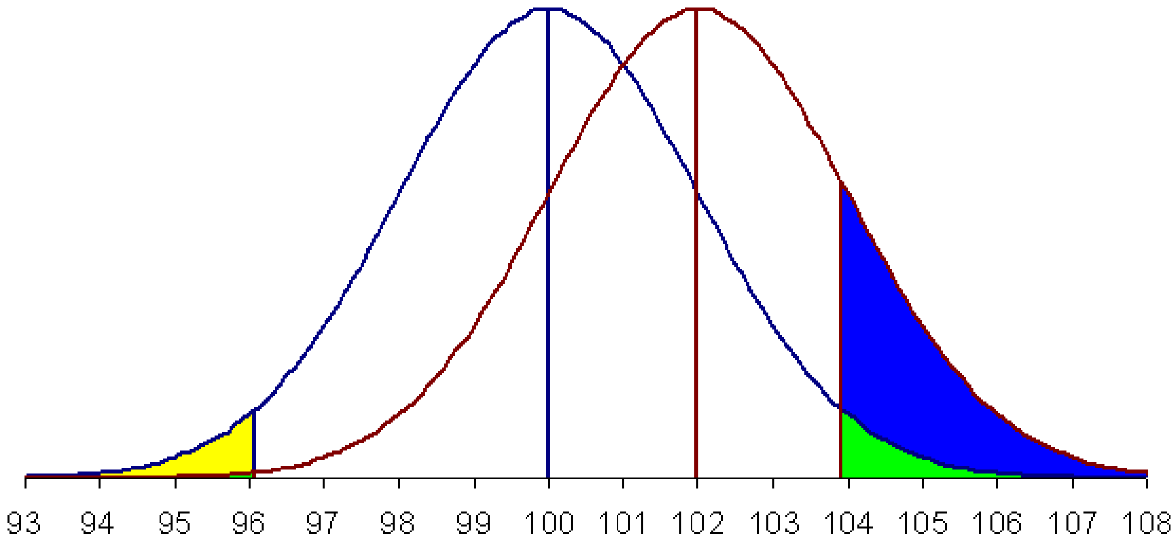


$$\text{Power}(101) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true})$$

$$= P(\overline{X} < 96.08 \mid \mu = 101) + P(\overline{X} > 103.92 \mid \mu = 101)$$

$$= P\left(Z < \frac{96.08 - 101}{10/\sqrt{25}}\right) + P\left(Z > \frac{103.92 - 101}{10/\sqrt{25}}\right)$$

$$= P(Z < -2.46) + P(Z > 1.46) = 0.0069 + 0.0721 = \mathbf{0.0790}.$$



$$\text{Power}(102) = P(\text{Reject } H_0 \mid H_0 \text{ is NOT true})$$

$$= P(\overline{X} < 96.08 \mid \mu = 102) + P(\overline{X} > 103.92 \mid \mu = 102)$$

$$= P\left(Z < \frac{96.08 - 102}{10/\sqrt{25}}\right) + P\left(Z > \frac{103.92 - 102}{10/\sqrt{25}}\right)$$

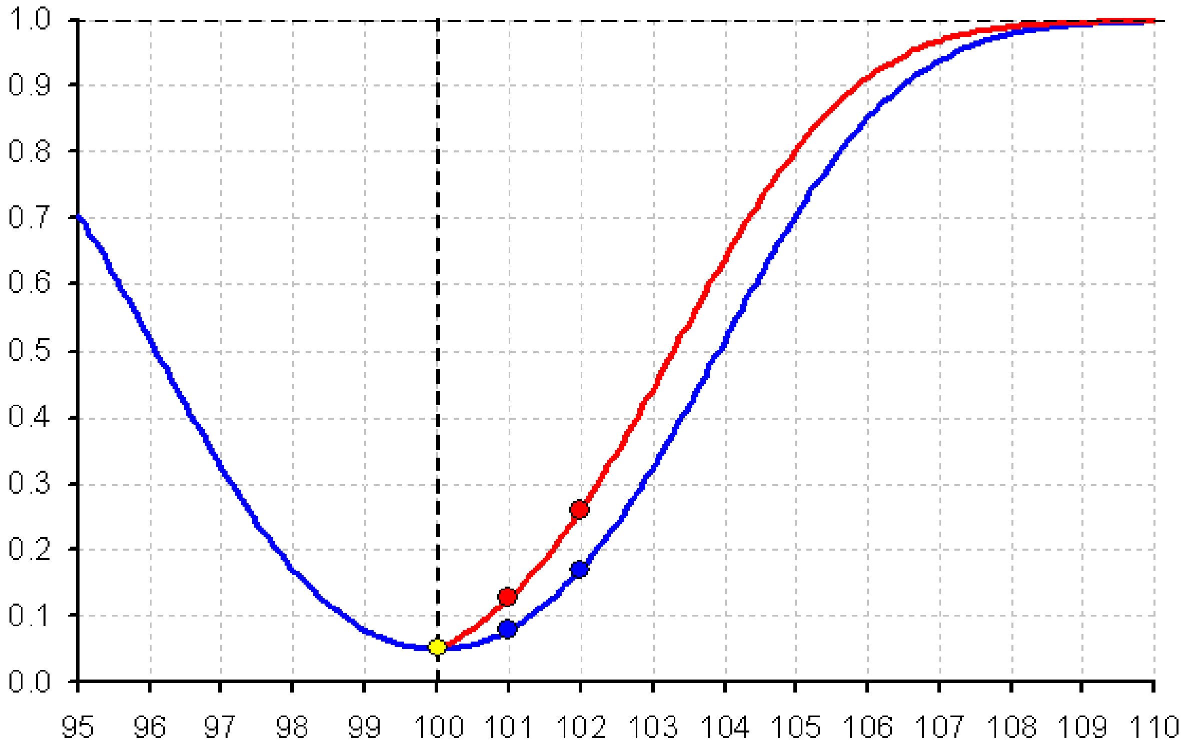
$$= P(Z < -2.96) + P(Z > 0.96) = 0.0015 + 0.1685 = \mathbf{0.1700}.$$

b) For the test in (a) compute the p-value associated with $\overline{X} = 103.5$.

Test Statistic: $$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{103.5 - 100}{10/\sqrt{25}} = 1.75.$$

P-value = 2 tails = $2 \times P(Z > 1.75) = 2 \times 0.0401 = \mathbf{0.0802}$.

For fun:



Problem 3 **Problem 4**