Examples for 09/25/2020 (1)

In general, if $X_1, X_2, \ldots, X_n$ is a random sample of size n from a continuous distribution with cumulative distribution function $F(x)$ and probability density function $f(x)$, then

$$F_{\max X_i}(x) = P(\max X_i \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot \ldots \cdot P(X_n \le x) = \left(F(x)\right)^n.$$

$$f_{\max X_i}(x) = F'_{\max X_i}(x) = n \cdot \left(F(x)\right)^{n-1} \cdot f(x).$$

$$1 - F_{\min X_i}(x) = P(\min X_i > x) = P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

$$= P(X_1 > x) \cdot P(X_2 > x) \cdot \ldots \cdot P(X_n > x) = \left(1 - F(x)\right)^n.$$

$$F_{\min X_i}(x) = 1 - \left(1 - F(x)\right)^n.$$

$$f_{\min X_i}(x) = F'_{\min X_i}(x) = n \cdot \left(1 - F(x)\right)^{n-1} \cdot f(x).$$

Let $Y_k = k^{\text{th}}$ smallest of $X_1, X_2, \ldots, X_n$.

$$F_{Y_k}(x) = P(Y_k \le x) = P(k^{\text{th}} \text{ smallest observation} \le x)$$

$$= P(\text{at least } k \text{ observations are} \le x)$$

$$= \sum_{i=k}^{n} \binom{n}{i} \cdot \left(F(x)\right)^i \cdot \left(1 - F(x)\right)^{n-i}.$$

$$f_{Y_k}(x) = F'_{Y_k}(x) = \frac{n!}{(k-1)! \cdot (n-k)!} \cdot \left(F(x)\right)^{k-1} \cdot \left(1 - F(x)\right)^{n-k} \cdot f(x).$$

1. Let X_1, X_2, X_3, X_4 be a random sample (i.i.d.) of size $n = 4$ from a probability distribution with the p.d.f.

$$f(x) = 3/x^4, \qquad x > 1.$$

Let $Y_k = k^{\text{th}}$ smallest of $X_1, X_2, \ldots, X_n$.

For $x \le 1$, $\qquad F(x) = 0.$

For $x > 1$, $\qquad F(x) = \int_1^x \frac{3}{y^4}\,dy = -\frac{1}{y^3}\Big|_1^x = 1 - \frac{1}{x^3}.$

a) Find $P(Y_4 < 1.75) = P(\max X_i < 1.75)$.

$$P(\max X_i < 1.75) = P(X_1 < 1.75, X_2 < 1.75, X_3 < 1.75, X_4 < 1.75)$$
$$= F(1.75)^4 = \left(1 - 1/1.75^3\right)^4 \approx \mathbf{0.4377643}.$$

b) Find $P(Y_4 > 2) = P(\max X_i > 2)$.

$$P(\max X_i > 2) = 1 - P(\max X_i \le 2) = 1 - P(X_1 \le 2, X_2 \le 2, X_3 \le 2, X_4 \le 2)$$
$$= 1 - F(2)^4 = 1 - \left(1 - 1/2^3\right)^4 = 1 - \left(7/8\right)^4 \approx \mathbf{0.41381836}.$$

b½) Find $E(Y_4) = E(\max X_i)$.

$$E(\max X_i) = \int_1^\infty x \cdot 4 \cdot \left(1 - \frac{1}{x^3}\right)^3 \cdot \frac{3}{x^4}\,dx = \ldots$$

c) Find $P(Y_1 > 1.25) = P(\min X_i > 1.25)$.

$$P(\min X_i > 1.25) = P(X_1 > 1.25, X_2 > 1.25, X_3 > 1.25, X_4 > 1.25)$$
$$= (1 - F(1.25))^4 = \left(1/1.25^3\right)^4 = \left(4/5\right)^{12} \approx \mathbf{0.06872}.$$

OR

$$f_{\min X_i}(x) = n\cdot(1-F(x))^{n-1}\cdot f(x) = 4\cdot\left({}^{1}\!/_{x^3}\right)^3\cdot\left({}^{3}\!/_{x^4}\right) = {}^{12}\!/_{x^{13}},$$

$x > 1.$

$$P(\min X_i > 1.25) = \int_{1.25}^{\infty} \frac{12}{x^{13}}\,dx = -\frac{1}{x^{12}}\Bigg|_{1.25}^{\infty} = \frac{1}{1.25^{12}} = \left({}^{4}\!/_{5}\right)^{12} \approx \mathbf{0.06872}.$$

d) Find $P(1.1 < Y_1 < 1.2) = P(1.1 < \min X_i < 1.2)$.

$$P(1.1 < \min X_i < 1.2) = \int_{1.1}^{1.2} \frac{12}{x^{13}}\,dx = -\frac{1}{x^{12}}\Bigg|_{1.1}^{1.2} = \frac{1}{1.1^{12}} - \frac{1}{1.2^{12}} \approx \mathbf{0.206474}.$$

d½) Find $E(Y_1) = E(\min X_i)$.

$$E(\min X_i) = \int_{1}^{\infty} x\cdot\frac{12}{x^{13}}\,dx = \mathbf{\frac{12}{11}}.$$

e) Find $P(1.1 < Y_2 < 1.2)$.

$Y_k = k^{\text{th}}$ smallest of $X_1, X_2, \ldots, X_n$.

$$f_{Y_k}(x) = \frac{n!}{(k-1)!\cdot(n-k)!}\cdot(F(x))^{k-1}\cdot(1-F(x))^{n-k}\cdot f(x).$$

$$f_{Y_2}(x) = \frac{4!}{1!\cdot 2!}\cdot\left(1-\frac{1}{x^3}\right)\cdot\left(\frac{1}{x^3}\right)^2\cdot\frac{3}{x^4} = 36\cdot\left(\frac{1}{x^{10}}-\frac{1}{x^{13}}\right), \qquad x > 1.$$

$$P(1.1 < Y_2 < 1.2) = \int_{1.1}^{1.2} 36\cdot\left(\frac{1}{x^{10}}-\frac{1}{x^{13}}\right)dx \approx \mathbf{0.301741}.$$

e½) Find $E(Y_2)$.

$$E(Y_2) = \int_{1}^{\infty} x\cdot 36\left(\frac{1}{x^{10}}-\frac{1}{x^{13}}\right)dx = \mathbf{\frac{27}{22}}.$$

1½. Suppose the size of largemouth bass in a particular lake is uniformly distributed over the interval 0 to 8 pounds. A fisherman catches (a random sample of) 5 fish.

X_1, X_2, X_3, X_4, X_5 $\qquad\qquad$ $Y_k = k^{\text{th}}$ smallest.

First find $F_X(x) = P(X \le x)$ $\qquad\qquad$ $F_X(x) = \int_0^x \frac{1}{8}\,dy = \frac{x}{8}, \quad 0 < x < 8.$

a) What is the probability that the smallest fish weighs less than 2 pounds?

$P(Y_1 < 2) = 1 - P(Y_1 > 2).$

$P(Y_1 > 2) = P(X_1 > 2, X_2 > 2, X_3 > 2, X_4 > 2, X_5 > 2) = \left({}^6\!/_8\right)^5 \approx 0.2373.$

$P(Y_1 < 2) = 1 - P(Y_1 > 2) = 1 - \left({}^6\!/_8\right)^5 \approx \mathbf{0.7627}.$

OR

$$f_{\min X_i}(x) = n \cdot (1 - F(x))^{n-1} \cdot f(x) = 5 \cdot \left(1 - {}^x\!/_8\right)^4 \cdot \left({}^1\!/_8\right), \qquad 0 < x < 8.$$

$$P(Y_1 < 2) = \int_0^2 f_{\min X_i}(x)\,dx.$$

b) What is the probability that the largest fish weighs over 7 pounds?

$P(Y_5 > 7) = 1 - P(Y_5 < 7).$

$P(Y_5 < 7) = P(X_1 < 7, X_2 < 7, X_3 < 7, X_4 < 7, X_5 < 7) = \left({}^7\!/_8\right)^5 \approx 0.5129.$

$P(Y_5 > 7) = 1 - P(Y_5 < 7) = 1 - \left({}^7\!/_8\right)^5 \approx \mathbf{0.4871}.$

OR

$$f_{\max X_i}(x) = n \cdot (F(x))^{n-1} \cdot f(x) = 5 \cdot \left({}^x\!/_8\right)^4 \cdot \left({}^1\!/_8\right) = {}^{5x^4}\!/_{8^5}, \qquad 0 < x < 8.$$

$$P(Y_5 > 7) = \int_7^8 f_{\max X_i}(x)\,dx = \int_7^8 \frac{5x^4}{8^5}\,dx = \left.\frac{x^5}{8^5}\right|_7^8 = 1 - \left({}^7\!/_8\right)^5 \approx \mathbf{0.4871}.$$

c) What is the probability that the largest fish weighs between 6 and 7 pounds?

$$P(6 < Y_5 < 7) = \int_6^7 f_{\max X_i}(x)\,dx = \int_6^7 \frac{5x^4}{8^5}\,dx = \left.\frac{x^5}{8^5}\right|_6^7$$

$$= \left(7/8\right)^5 - \left(6/8\right)^5 \approx \mathbf{0.2756}.$$

d) What is the probability that the second largest (fourth smallest) fish weighs between 4 and 6 pounds?

Second largest = Fourth smallest

$$P(4 < Y_4 < 6) = \int_4^6 \frac{5!}{(4-1)!\cdot(5-4)!}\cdot\left(\frac{y}{8}\right)^{4-1}\cdot\left(1-\frac{y}{8}\right)^{5-4}\cdot\frac{1}{8}\,dy$$

$$= \left(\frac{1}{8}\right)^5\cdot\int_4^6 20\cdot y^3\cdot(8-y)\,dy = \left(\frac{1}{8}\right)^5\cdot\int_4^6 \left(160\,y^3 - 20\,y^4\right)dy$$

$$= \left(\frac{1}{8}\right)^5\cdot\left.\left(40\,y^4 - 4\,y^5\right)\right|_4^6 \approx \mathbf{0.4453}.$$

OR

Let W_6 = number of fish (out of 5) that weigh less than 6 pounds.

W_6 has Binomial distribution, $n = 5$, $p = 6/8 = 0.75$.

$$P(Y_4 < 6) = P(W_6 \ge 4) = {}_5C_4\, 0.75^4\, 0.25^1 + {}_5C_5\, 0.75^5\, 0.25^0$$

$= 0.3955 + 0.2373 = 0.6328.$

Let W_4 = number of fish (out of 5) that weigh less than 4 pounds.

W_4 has Binomial distribution, $n = 5$, $p = 4/8 = 0.50$.

$$P(Y_4 < 4) = P(W_4 \ge 4) = {}_5C_4\, 0.50^4\, 0.50^1 + {}_5C_5\, 0.50^5\, 0.50^0$$

$= 0.15625 + 0.03125 = 0.1875.$

$$P(4 < Y_4 < 6) = P(Y_4 < 6) - P(Y_4 < 4) = 0.6328 - 0.1875 = \mathbf{0.4453}.$$