STAT 410

Examples for 09/30/2020 (2)

Fall 2020
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1. In Anytown, the price of a gallon of milk (X) and the price of a package of Oreo cookies (Y) vary from day to day according to a bivariate normal distribution with

$$\mu_X = \$3.38, \quad \sigma_X = \$0.16,$$

$$\mu_Y = \$3.17, \quad \sigma_Y = \$0.10, \quad \rho = 0.50.$$

a) Find the probability that the price of a package of Oreo cookies is above \$3.33. That is, find $P(Y > 3.33)$.

b) Find the probability that the price of a package of Oreo cookies is above \$3.33, if the price of a gallon of milk is \$3.54. That is, find $P(Y > 3.33 \mid X = 3.54)$.

c) Find the probability that on a given day, the price of a package of Oreo cookies is higher than the price of a gallon of milk. That is, find $P(Y > X)$.

d) Alex is planning a Milk-and-Oreos party for his imaginary friends. He buys 5 gallons of milk and 7 packages of Oreo cookies. Find the probability that he paid less than \$40. That is, find $P(5X + 7Y < 40)$.

2. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Assume X and Y jointly follow a bivariate normal distribution with parameters

$$\mu_X = 370, \quad \sigma_X = 50, \quad \mu_Y = 270, \quad \sigma_Y = 40, \quad \rho = -0.80.$$

a) Find the probability that more than \$250 million was spent on butter during a particular month. That is, find $P(Y > 250)$.

b) Suppose the government of Neverland spent \$450 million dollars on guns during a particular month. Find the probability that more than \$250 million was spent on butter during the same month. That is, find $P(Y > 250 \mid X = 450)$.

c) Suppose the government of Neverland spent \$250 million dollars on butter during a particular month. Find the probability that more than \$450 million was spent on guns during the same month. That is, find $P(X > 450 \mid Y = 250)$.

d) Find the probability that the government of Neverland spends more on guns than on butter during a given month. That is, find $P(X > Y)$.

e) Find the probability that the government of Neverland spends more on guns than twice the amount it spends on butter during a given month. That is, find $P(X > 2Y)$.

f) Find the probability that the government of Neverland exceeds the \$700 million spending limit during a given month. That is, find $P(X + Y > 700)$.

3. a) At Anytown College, the heights of female students are normally distributed with mean 66 inches and standard deviation 1.5 inches. The heights of male students are also normally distributed with mean 69 inches and standard deviation 2 inches. For Homecoming, a male student and a female student are selected independently at random to be the King and the Queen. What is the probability that the female student selected to be the Queen is taller than the male student selected to be the King?

b) Suppose that a population of married couples in Anytown have heights in inches, X for the wife, and Y for the husband. Suppose that (X, Y) has a bivariate normal distribution with parameters $\mu_X = 66$, $\sigma_X = 1.5$, $\mu_Y = 69$, $\sigma_Y = 2$, $\rho = 0.44$. What is the probability that the wife is taller than her husband?

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1. In Anytown, the price of a gallon of milk (X) and the price of a package of Oreo cookies (Y) vary from day to day according to a bivariate normal distribution with

$$\mu_X = \$3.38, \quad \sigma_X = \$0.16, \quad \mu_Y = \$3.17, \quad \sigma_Y = \$0.10, \quad \rho = 0.50.$$

a) Find the probability that the price of a package of Oreo cookies is above \$3.33. That is, find $P(Y > 3.33)$.

Y has Normal distribution with mean $\mu_Y = 3.17$ and standard deviation $\sigma_Y = 0.10$.

$$P(Y > 3.33) = P\left(Z > \frac{3.33 - 3.17}{0.10}\right) = P(Z > 1.60) = \mathbf{0.0548}.$$

b) Find the probability that the price of a package of Oreo cookies is above \$3.33, if the price of a gallon of milk is \$3.54. That is, find $P(Y > 3.33 \mid X = 3.54)$.

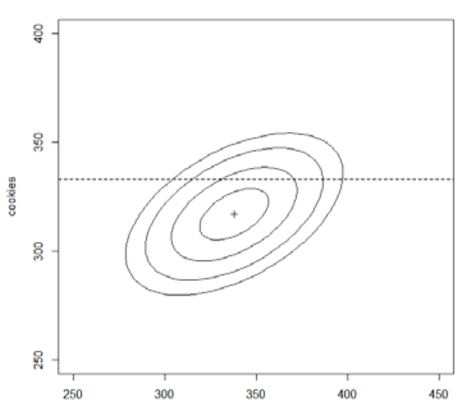
Given $X = 3.54$, Y has Normal distribution

with mean $\mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X) = 3.17 + 0.5 \cdot \dfrac{0.10}{0.16} \cdot (3.54 - 3.38) = 3.22$

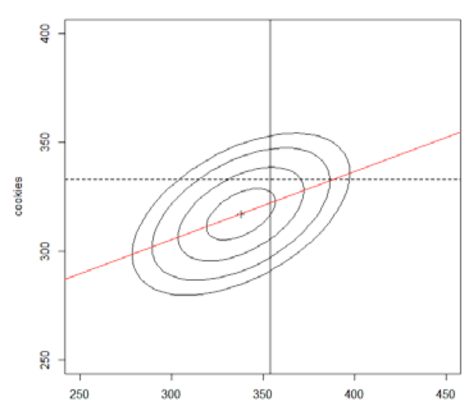
and variance $\left(1 - \rho^2\right) \cdot \sigma_Y^2 = \left(1 - 0.5^2\right) \cdot 0.10^2 = 0.0075.$

$$P(Y > 3.33 \mid X = 3.54) = P\left(Z > \frac{3.33 - 3.22}{\sqrt{0.0075}}\right) \approx P(Z > 1.27) = \mathbf{0.1020}.$$

a)



b)



c) Find the probability that on a given day, the price of a package of Oreo cookies is higher than the price of a gallon of milk. That is, find $P(Y > X)$.

$P(Y > X) = P(X - Y < 0)$.

$X - Y$ has Normal distribution,

$$E(X - Y) = \mu_X - \mu_Y = 3.38 - 3.17 = 0.21,$$

$$\text{Var}(X - Y) = \sigma_X^2 - 2\,\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 - 2\,\rho\,\sigma_X\,\sigma_Y + \sigma_Y^2$$

$$= 0.16^2 - 2 \cdot 0.50 \cdot 0.16 \cdot 0.10 + 0.10^2 = 0.0196.$$

$$P(X - Y < 0) = P\left(Z < \frac{0 - 0.21}{\sqrt{0.0196}}\right) = P(Z < -1.50) = \mathbf{0.0668}.$$

d) Alex is planning a Milk-and-Oreos party for his imaginary friends. He buys 5 gallons of milk and 7 packages of Oreo cookies. Find the probability that he paid less than \$40. That is, find $P(5X + 7Y < 40)$.

$5X + 7Y$ has Normal distribution,

$$E(5X + 7Y) = 5\,\mu_X + 7\,\mu_Y = 5 \cdot 3.38 + 7 \cdot 3.17 = 39.09,$$

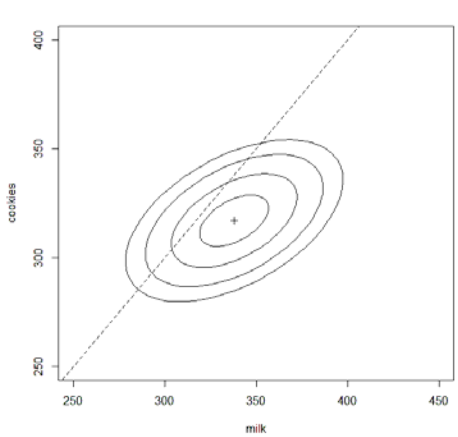
$$\text{Var}(5X + 7Y) = 25\,\sigma_X^2 + 70\,\sigma_{XY} + 49\,\sigma_Y^2$$

$$= 25\,\sigma_X^2 + 70\,\rho\,\sigma_X\,\sigma_Y + 49\,\sigma_Y^2$$

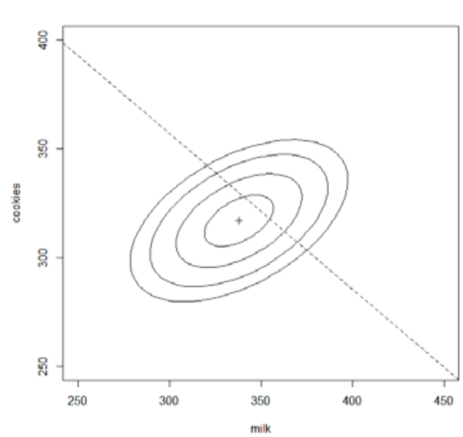
$$= 25 \cdot 0.16^2 + 70 \cdot 0.5 \cdot 0.16 \cdot 0.10 + 49 \cdot 0.10^2 = 1.69.$$

$$P(5X + 7Y < 40) = P\left(Z < \frac{40 - 39.09}{\sqrt{1.69}}\right) = P(Z < 0.70) = \mathbf{0.7580}.$$

c)



d)



```
> mux = 3.38
> sigx = 0.16
> muy = 3.17
> sigy = 0.10
> rho = 0.50
>
> ## part (a)
> 1 - pnorm(3.33,muy,sigy)
[1] 0.05479929
>
> ## part (b)
> 1 - pnorm(3.33,muy+rho*(sigy/sigx)*(3.54-mux),sigy*sqrt(1-rho^2))
[1] 0.1020119
>
> ## part (c)
> pnorm(0,mux-muy,sqrt(sigx^2-2*rho*sigx*sigy+sigy^2))
[1] 0.0668072
>
> ## part (d)
> pnorm(40,5*mux+7*muy,sqrt(25*sigx^2+70*rho*sigx*sigy+49*sigy^2))
[1] 0.7580363
```

2. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Assume X and Y jointly follow a bivariate normal distribution with parameters

$$\mu_X = 370, \quad \sigma_X = 50, \quad \mu_Y = 270, \quad \sigma_Y = 40, \quad \rho = -0.80.$$

a) Find the probability that more than \$250 million was spent on butter during a particular month. That is, find $P(Y > 250)$.

Y has Normal distribution with mean $\mu_Y = 270$ and standard deviation $\sigma_Y = 40$.

$$P(Y > 250) = P\left(Z > \frac{250 - 270}{40}\right) = P(Z > -0.50) = \mathbf{0.6915}.$$

b) Suppose the government of Neverland spent \$450 million dollars on guns during a particular month. Find the probability that more than \$250 million was spent on butter during the same month. That is, find $P(Y > 250 \mid X = 450)$.

Given $X = 450$, Y has Normal distribution

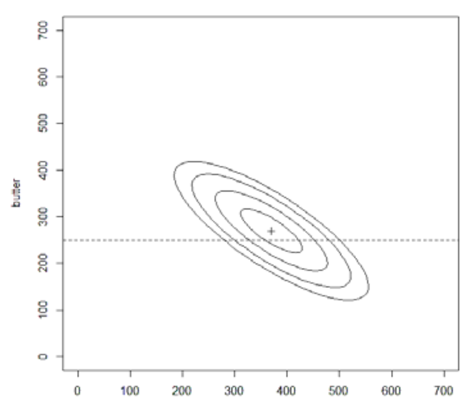
with mean $\mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X) = 270 - 0.80 \cdot \dfrac{40}{50} \cdot (450 - 370) = 218.8$

and variance $\left(1 - \rho^2\right) \cdot \sigma_Y^2 = \left(1 - (-0.80)^2\right) \cdot 40^2 = 576$

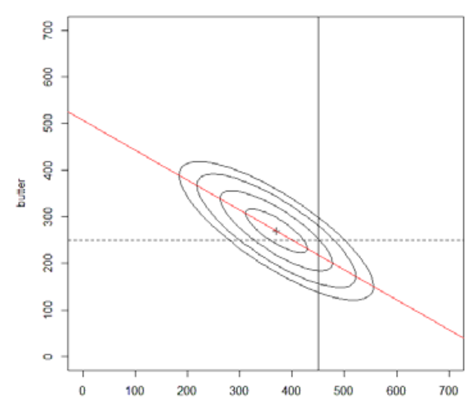
(standard deviation = 24).

$$P(Y > 250 \mid X = 450) = P\left(Z > \frac{250 - 218.8}{24}\right) = P(Z > 1.30) = \mathbf{0.0968}.$$

a)



b)



c) Suppose the government of Neverland spent \$250 million dollars on butter during a particular month. Find the probability that more than \$450 million was spent on guns during the same month. That is, find $P(X > 450 \mid Y = 250)$.

Given $Y = 250$, X has Normal distribution

with mean $\mu_X + \rho \dfrac{\sigma_X}{\sigma_Y}(y - \mu_Y) = 370 - 0.80 \cdot \dfrac{50}{40} \cdot (250 - 270) = 390$

and variance $\left(1 - \rho^2\right) \cdot \sigma_X^2 = \left(1 - (-0.80)^2\right) \cdot 50^2 = 900$

(standard deviation = 30).

$$P(X > 450 \mid Y = 250) = P\left(Z > \frac{450 - 390}{30}\right) = P(Z > 2.00) = \mathbf{0.0228}.$$

d) Find the probability that the government of Neverland spends more on guns than on butter during a given month. That is, find $P(X > Y)$.

$P(X > Y) = P(X - Y > 0)$.

$X - Y$ has Normal distribution,

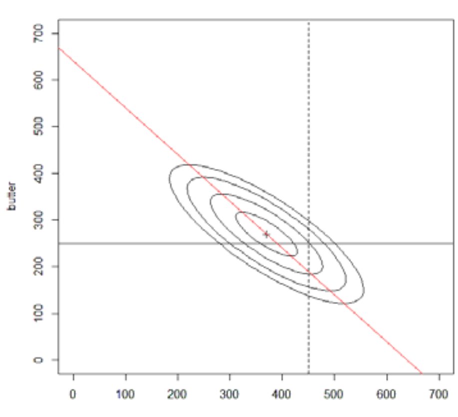
$E(X - Y) = \mu_X - \mu_Y = 370 - 270 = 100,$

$\mathrm{Var}(X - Y) = \sigma_X^2 - 2\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2$

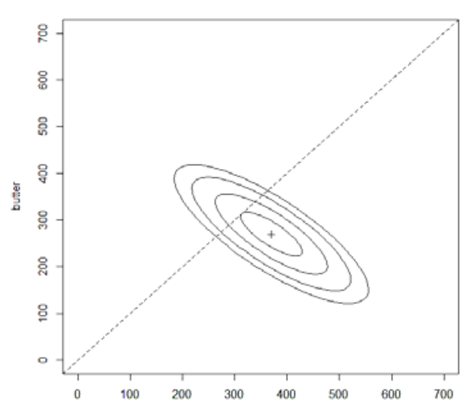
$= 50^2 - 2 \cdot (-0.80) \cdot 50 \cdot 40 + 40^2 = 7300.$

$$P(X - Y > 0) = P\left(Z > \frac{0 - 100}{\sqrt{7300}}\right) \approx P(Z > -1.17) = \mathbf{0.8790}.$$

c)



d)



e) Find the probability that the government of Neverland spends more on guns than twice the amount it spends on butter during a given month. That is, find $P(X > 2Y)$.

$P(X > 2Y) = P(X - 2Y > 0)$.

$X - 2Y$ has Normal distribution,

$$E(X - 2Y) = \mu_X - 2\mu_Y = 370 - 2 \cdot 270 = -170,$$

$$\mathrm{Var}(X - 2Y) = \sigma_X^2 - 4\sigma_{XY} + 4\sigma_Y^2 = \sigma_X^2 - 4\rho\sigma_X\sigma_Y + 4\sigma_Y^2$$

$$= 50^2 - 4 \cdot (-0.80) \cdot 50 \cdot 40 + 4 \cdot 40^2 = 15300.$$

$$P(X - 2Y > 0) = P(Z > \frac{0 + 170}{\sqrt{15300}}) \approx P(Z > 1.37) = \mathbf{0.0853}.$$

f) Find the probability that the government of Neverland exceeds the \$700 million spending limit during a given month. That is, find $P(X + Y > 700)$.

$X + Y$ has Normal distribution,

$$E(X + Y) = \mu_X + \mu_Y = 370 + 270 = 640,$$

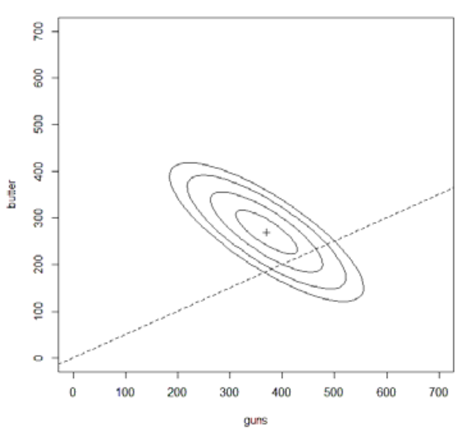
$$\mathrm{Var}(X + Y) = \sigma_X^2 + 2\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2$$

$$= 50^2 + 2 \cdot (-0.80) \cdot 50 \cdot 40 + 40^2 = 900,$$

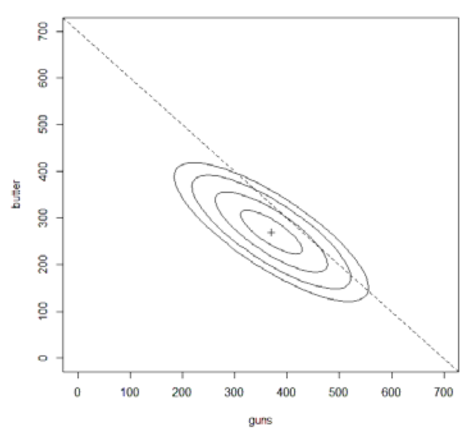
$$SD(X + Y) = 30.$$

$$P(X + Y > 700) = P(Z > \frac{700 - 640}{30}) = P(Z > 2.00) = \mathbf{0.0228}.$$

e)



f)



```
> mux = 370
> sigx = 50
> muy = 270
> sigy = 40
> rho = -0.80
>
> ## part (a)
> 1 - pnorm(250,muy,sigy)
[1] 0.6914625
>
> ## part (b)
> 1 - pnorm(250,muy+rho*(sigy/sigx)*(450-mux),sigy*sqrt(1-rho^2))
[1] 0.09680048
>
> ## part (c)
> 1 - pnorm(450,mux+rho*(sigx/sigy)*(250-muy),sigx*sqrt(1-rho^2))
[1] 0.02275013
>
> ## part (d)
> 1 - pnorm(0,mux-muy,sqrt(sigx^2-2*rho*sigx*sigy+sigy^2))
[1] 0.8790823
>
> ## part (e)
> 1 - pnorm(0,mux-2*muy,sqrt(sigx^2-4*rho*sigx*sigy+4*sigy^2))
[1] 0.08466365
>
> ## part (f)
> 1 - pnorm(700,mux+muy,sqrt(sigx^2+2*rho*sigx*sigy+sigy^2))
[1] 0.02275013
```

3. a) At Anytown College, the heights of female students are normally distributed with mean 66 inches and standard deviation 1.5 inches. The heights of male students are also normally distributed with mean 69 inches and standard deviation 2 inches. For Homecoming, a male student and a female student are selected independently at random to be the King and the Queen. What is the probability that the female student selected to be the Queen is taller than the male student selected to be the King?

Need $P(\text{Queen} - \text{King} > 0) = ?$ $\qquad E(\text{Queen} - \text{King}) = 66 - 69 = -3.$

$\text{Var}(\text{Queen} - \text{King}) = 1.5^2 + 2^2 = 6.25.$ $\qquad \text{SD}(\text{Queen} - \text{King}) = 2.5.$

$(\text{Queen} - \text{King})$ is normally distributed.

$$P(\text{Queen} - \text{King} > 0) = P\left(Z > \frac{0-(-3)}{2.5}\right) = P(Z > 1.20) = 1 - \Phi(1.20) = \mathbf{0.1151}.$$

b) Suppose that a population of married couples in Anytown have heights in inches, X for the wife, and Y for the husband. Suppose that (X, Y) has a bivariate normal distribution with parameters $\mu_X = 66,\ \sigma_X = 1.5,\ \mu_Y = 69,\ \sigma_Y = 2,\ \rho = 0.44.$ What is the probability that the wife is taller than her husband?

Need $P(\text{Wife} - \text{Husband} > 0) = ?$ $\qquad E(\text{Wife} - \text{Husband}) = 66 - 69 = -3.$

$\text{Var}(\text{Wife} - \text{Husband}) = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2 = 1.5^2 - 2 \cdot 0.44 \cdot 1.5 \cdot 2 + 2^2 = 3.61.$

$\text{SD}(\text{Wife} - \text{Husband}) = 1.9.$

$(\text{Wife} - \text{Husband})$ is normally distributed.

$$P(\text{Wife} - \text{Husband} > 0) = P\left(Z > \frac{0-(-3)}{1.9}\right) \approx P(Z > 1.58) = 1 - \Phi(1.58) = \mathbf{0.0571}.$$

```
> mux = 66
> sigx = 1.5
> muy = 69
> sigy = 2
> rhoa = 0
> rhob = 0.44
>
> ## part (a)
> 1 - pnorm(0, mux-muy, sqrt(sigx^2-2*rhoa*sigx*sigy+sigy^2))
[1] 0.1150697
>
> ## part (b)
> 1 - pnorm(0, mux-muy, sqrt(sigx^2-2*rhob*sigx*sigy+sigy^2))
[1] 0.05717406
```