STAT 410 **Examples for 09/11/2020 (1)** Fall 2020

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2.3 Conditional Distributions and Expectations. (continued)

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{otherwise}\end{cases}$$

Recall: $f_X(x) = x+\frac{1}{2},\quad 0<x<1.$ $\qquad f_Y(y) = y+\frac{1}{2},\quad 0<y<1.$

a) Find the conditional p.d.f. $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$ of Y given $X = x$, $0<x<1$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}, \qquad 0<y<1.$$

b) Find $P(Y<{}^1\!/_2 \mid X={}^3\!/_4)$.

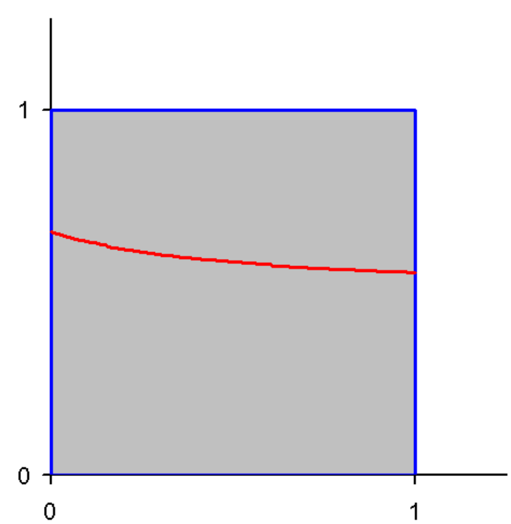
$$P(Y<{}^1\!/_2 \mid X={}^3\!/_4) = \int_0^{1/2} \frac{\frac{3}{4}+y}{\frac{3}{4}+\frac{1}{2}}\,dy = \left(\frac{3y+2y^2}{5}\right)\Bigg|_0^{1/2} = 0.40.$$

c) Find $E(Y|X=x)$.

$$E(Y|X=x) = \int_0^1 y\cdot\frac{x+y}{x+\frac{1}{2}}\,dy$$

$$= \frac{\frac{1}{2}x+\frac{1}{3}}{x+\frac{1}{2}} = \frac{3x+2}{6x+3},$$

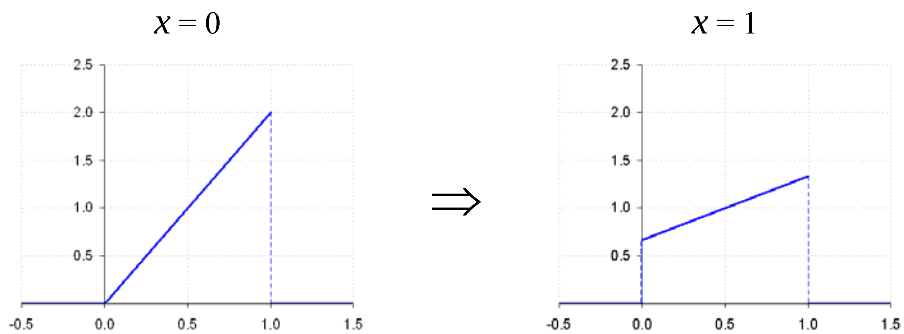
$$0<x<1.$$



Recall: $\mathrm{Cov}(X,Y) = -\dfrac{1}{144},$

$$\rho_{XY} = -\frac{1}{11}.$$

$f_{Y|X}(y|x)$:



4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 12\,x(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 6\,x(1-x),\ \ 0 < x < 1,$ $\quad E(X) = \frac{1}{2},$

$f_Y(y) = 2\,e^{-2y},\ \ y > 0,$ $\quad E(Y) = \frac{1}{2}.$ $\quad$ X and Y are independent.

Find $f_{X|Y}(x|y),\ E(X|Y=y),\ f_{Y|X}(y|x),\ E(Y|X=x).$

Since X and Y are independent, and $f(x,y) = f_X(x) \cdot f_Y(y),$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = f_X(x) = 6\,x(1-x), \qquad 0 < x < 1,$$

$$E(X|Y=y) = E(X) = \frac{1}{2},$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = f_Y(y) = 2\,e^{-2y}, \qquad y > 0,$$

$$E(Y|X=x) = E(Y) = \frac{1}{2}.$$

5. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = x_1 \exp\{-x_2\}$, for $0 < x_1 < x_2 < \infty$, zero elsewhere.

a) Find the conditional p.d.f. $f_{1|2}(x_1|x_2)$ of X_1 given $X_2 = x_2$, $\quad 0 < x_2 < \infty$.

$$f_2(x_2) = \int_0^{x_2} x_1 e^{-x_2} \, dx_1 = \frac{x_2^2}{2} e^{-x_2}, \qquad 0 < x_2 < \infty.$$

X_2 has a Gamma distribution with $\alpha = 3$, $\theta = 1$.

$$f_{1|2}(x_1|x_2) = \frac{x_1 e^{-x_2}}{\frac{x_2^2}{2} e^{-x_2}} = \frac{2x_1}{x_2^2}, \qquad 0 < x_1 < x_2.$$

For example, $P(X_1 > 3 \mid X_2 = 5) = \int_3^5 \frac{2x_1}{25} \, dx_1 = \frac{16}{25} = 0.64.$

$$P(X_1 < 2 \mid X_2 = 5) = \int_0^2 \frac{2x_1}{25} \, dx_1 = \frac{4}{25} = 0.16.$$

b) Find the conditional p.d.f. $f_{2|1}(x_2|x_1)$ of X_2 given $X_1 = x_1$, $\quad 0 < x_1 < \infty$.

$$f_1(x_1) = \int_{x_1}^{\infty} x_1 e^{-x_2} \, dx_2 = x_1 e^{-x_1}, \qquad 0 < x_1 < \infty.$$

X_1 has a Gamma distribution with $\alpha = 2$, $\theta = 1$.

$$f_{2|1}(x_2|x_1) = \frac{x_1 e^{-x_2}}{x_1 e^{-x_1}} = e^{x_1 - x_2}, \qquad x_1 < x_2 < \infty.$$

For example, $P(X_2 < 8 \mid X_1 = 5) = \int_5^8 e^{5-x_2} \, dx_2 = 1 - e^{-3} \approx 0.9502.$

$$P(X_2 > 6 \mid X_1 = 5) = \int_6^{\infty} e^{5-x_2} \, dx_2 = e^{-1} \approx 0.3679.$$

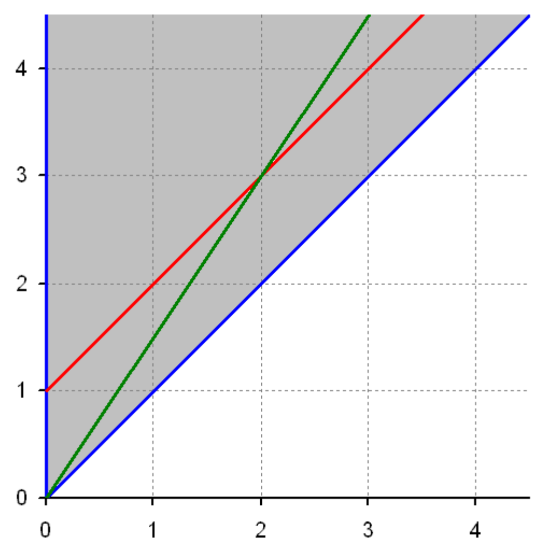
c) Find $E(X_1 | X_2 = x_2)$, $E(X_2 | X_1 = x_1)$.

$$E(X_1 | X_2 = x_2) = \int_0^{x_2} x_1 \cdot \frac{2x_1}{x_2^2} dx_1 = \frac{2}{3} x_2, \qquad 0 < x_2 < \infty.$$

$$E(X_2 | X_1 = x_1) = \int_{x_1}^{\infty} x_2 \cdot e^{x_1 - x_2} dx_2 = x_1 + 1, \qquad 0 < x_1 < \infty.$$

If $E(Y | X = x)$ is linear in x, then

$$E(Y | X = x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X).$$



$\mu_1 = 2,\quad \sigma_1^2 = 2,\qquad \mu_2 = 3,\quad \sigma_2^2 = 3.$

$$\Rightarrow \qquad \rho = \frac{\sqrt{2}}{\sqrt{3}}.$$

OR

$$E(X_1 X_2) = \int_0^{\infty}\left(\int_0^{x_2} x_1^2\, x_2\, e^{-x_2}\, dx_1\right) dx_2 = \int_0^{\infty} \frac{x_2^4}{3} e^{-x_2}\, dx_2 = \frac{\Gamma(5)}{3} = 8.$$

$$\operatorname{Cov}(X_1, X_2) = 8 - 2 \cdot 3 = 2. \qquad \rho = \frac{2}{\sqrt{2} \cdot \sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}.$$