STAT 410

Practice Problems 10

Fall 2020
A. Stepanov

1. In a college health fitness program, let X denote the weight in kilograms of a male freshman at the beginning of the program and let Y denote his weight change during a semester. Assume that X and Y have a bivariate normal distribution with $\mu_X = 75$, $\sigma_X = 9$, $\mu_Y = 2.5$, $\sigma_Y = 1.5$, $\rho = -0.6$. (The lighter students tend to gain weight, while the heavier students tend to lose weight.)

a) What proportion of the students that weigh 85 kg end up losing weight during the semester? That is, find $P(Y < 0 \mid X = 85)$.

b) What proportion of the students that weigh over 87 kg at the end of the semester? That is, find $P(X + Y > 87)$.

2. Suppose that prices of the stocks for company U, company V, and company W follow a multivariate normal distribution with means $\mu_U = \$53$, $\mu_V = \$95$, and $\mu_W = \$62$ per share, respectively, and the covariance matrix (in dollars squared)

$$\begin{array}{c} \\ U \\ V \\ W \end{array} \begin{array}{c} \begin{array}{ccc} U & V & W \end{array} \\ \begin{bmatrix} 16 & -4 & 2 \\ -4 & 9 & 5 \\ 2 & 5 & 25 \end{bmatrix} \end{array}$$

Bob has 5 shares of company U and 3 shares of company V.
Carl has 4 shares of company V and 2 shares of company W.

a) What is the probability that the value of Bob's portfolio exceeds \$531?

b) What is the probability that the value of Carl's portfolio exceeds \$531?

c) What is the probability that Bob's portfolio is worth more than Carl's portfolio?

d) What is the probability that the two portfolios together are worth less than \$1025?

3. Suppose $\mathbf{X}$ follows a 3-dimensional multivariate normal distribution with

mean $\boldsymbol{\mu} = \begin{pmatrix} 44 \\ 40 \\ 30 \end{pmatrix}$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} 9 & -6 & 8 \\ -6 & 25 & -6 \\ 8 & -6 & 12 \end{pmatrix}$.

a) Find $P(X_1 > 38)$.

b) Find $P(X_2 > 38)$.

c) Find $P(X_2 + X_3 > 66)$.

d) Find $P(X_2 - X_3 > 3)$.

e) Find $P(X_1 + 2X_2 + 3X_3 > 240)$.

4. Suppose $\mathbf{X}$ has a multivariate normal distribution with mean

$\boldsymbol{\mu} = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} 9 & 4 & 0 & -2 \\ 4 & 16 & 10 & 2 \\ 0 & 10 & 25 & -5 \\ -2 & 2 & -5 & 4 \end{pmatrix}$.

a) Find $P(X_1 > 8)$.

b) Find $P(3X_1 - 5X_2 + 4X_3 > -2)$.

c)* Find $P(X_1 > 8 \mid X_2 = 5, X_3 = 8, X_4 = 3)$.

5.* Let Z_1 and Z_2 be independent standard normal random variables $N(0, 1)$. What is the distribution of $Y_1 = Z_1 / Z_2$?

Hint: Let $Y_2 = Z_2$. Obtain the joint p.d.f. $f_{Y_1, Y_2}(y_1, y_2)$ of Y_1 and Y_2 first, then find the marginal p.d.f. $f_{Y_1}(y_1)$ of Y_1.

Hint: $$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$$
$$= \int_{-\infty}^{0} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 + \int_{0}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 .$$

6.* Let X and Y be independent random variables with common moment generating function $M(t) = e^{t^2/2}$. Let $W = X + Y$, $V = X - Y$. Determine the joint moment generating function, $M(t_1, t_2)$, of W and V. Are W and V independent?

7. **3.5.1** (7th and 6th edition)

Let X and Y have a bivariate normal distribution with respective parameters $\mu_X = 2.8$, $\mu_Y = 110$, $\sigma_X^2 = 0.16$, $\sigma_Y^2 = 100$, and $\rho = 0.6$. Compute:

(a) $P(106 < Y < 124)$. (b) $P(106 < Y < 124 \mid X = 3.2)$.

8. **3.5.6** (7th and 6th edition)

Let X and Y have a bivariate normal distribution with parameters $\mu_X = 20$, $\mu_Y = 40$, $\sigma_X^2 = 9$, $\sigma_Y^2 = 4$, and $\rho = 0.6$. Find the shortest interval for which 0.90 is the conditional probability that Y is in the interval, given that $X = 22$.

9. **3.5.14** (7th and 6th edition)

Let $\mathbf{X} = (X_1, X_2, X_3)$ have a multivariate normal distribution with mean vector $\mathbf{0}$ and variance-covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find $P(X_1 > X_2 + X_3 + 2)$.

Hint: Find the vector $\mathbf{a}$ so that $\mathbf{a}\mathbf{X} = X_1 - X_2 - X_3$ and make use of Theorem 3.5.1.

Theorem 3.5.1. Suppose $\mathbf{X}$ has a $N_n(\mu, \Sigma)$ distribution. Let $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b} \in \mathbf{R}^m$. Then $\mathbf{Y}$ has a $N_m(\mathbf{A}\mu + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}')$ distribution.

10.* **3.5.4** (7th and 6th edition)

Let U and V be independent random variables, each having a standard normal distribution. Show that the mgf $E(e^{t(UV)})$ of the random variable UV is $(1-t^2)^{-1/2}$, $-1 < t < 1$.

Hint: Compare $E(e^{tUV})$ with the integral of a bivariate normal pdf that has means equal to zero.

1. In a college health fitness program, let X denote the weight in kilograms of a male freshman at the beginning of the program and let Y denote his weight change during a semester. Assume that X and Y have a bivariate normal distribution with $\mu_X = 75$, $\sigma_X = 9$, $\mu_Y = 2.5$, $\sigma_Y = 1.5$, $\rho = -0.6$. (The lighter students tend to gain weight, while the heavier students tend to lose weight.)

$\mu_X = 75$, $\sigma_X = 9$, $\mu_Y = 2.5$, $\sigma_Y = 1.5$, $\rho = -0.6$.

a) What proportion of the students that weigh 85 kg end up losing weight during the semester? That is, find $P(Y < 0 \mid X = 85)$.

Given $X = 85$, Y has Normal distribution

with mean $\mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X) = 2.5 + (-0.6) \cdot \dfrac{1.5}{9} \cdot (85 - 75) = 1.5$

and variance $\left(1 - \rho^2\right) \cdot \sigma_Y^2 = \left(1 - (-0.6)^2\right) \cdot 1.5^2 = 1.44$

(standard deviation = 1.2).

$$P(Y < 0 \mid X = 85) = P\left(Z < \frac{0 - 1.5}{1.2}\right) = P(Z < -1.25) = \Phi(-1.25) = \mathbf{0.1056}.$$

b) What proportion of the students that weigh over 87 kg at the end of the semester? That is, find $P(X + Y > 87)$.

X + Y has Normal distribution,

$E(X + Y) = \mu_X + \mu_Y = 75 + 2.5 = 77.5,$

$\operatorname{Var}(X + Y) = \sigma_X^2 + 2\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2$

$= 9^2 + 2 \cdot (-0.6) \cdot 9 \cdot 1.5 + 1.5^2 = 67.05.$

$SD(X + Y) \approx 8.1884.$

$$P(X + Y > 87) = P\left(Z > \frac{87 - 77.5}{8.1884}\right) = P(Z > 1.16) = \mathbf{0.1230}.$$

2. Suppose that prices of the stocks for company U, company V, and company W follow a multivariate normal distribution with means $\mu_U = \$53$, $\mu_V = \$95$, and $\mu_W = \$62$ per share, respectively, and the covariance matrix (in dollars squared)

$$\begin{array}{c|ccc|} & U & V & W \\ \hline U & 16 & -4 & 2 \\ V & -4 & 9 & 5 \\ W & 2 & 5 & 25 \\ \hline \end{array}$$

Bob has 5 shares of company U and 3 shares of company V.
Carl has 4 shares of company V and 2 shares of company W.

a) What is the probability that the value of Bob's portfolio exceeds \$531?

$5U + 3V$ has Normal distribution,

$$E(5U + 3V) = 5\mu_U + 3\mu_V = 5 \cdot 53 + 3 \cdot 95 = 550,$$

$$\mathrm{Var}(5U + 3V) = 25\,\sigma_U^2 + 30\,\sigma_{UV} + 9\,\sigma_V^2$$

$$= 25 \cdot 16 + 30 \cdot (-4) + 9 \cdot 9 = 361,$$

OR

$$\mathrm{Var}(5U + 3V) = \begin{pmatrix} 5 & 3 & 0 \end{pmatrix} \begin{pmatrix} 16 & -4 & 2 \\ -4 & 9 & 5 \\ 2 & 5 & 25 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 68 & 7 & 25 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = 361.$$

$$\mathrm{SD}(5U + 3V) = 19.$$

$$P(5U + 3V > 531) = P(Z > \frac{531 - 550}{19}) = P(Z > -1.00) = \mathbf{0.8413}.$$

b) What is the probability that the value of Carl's portfolio exceeds \$531?

$4V + 2W$ has Normal distribution,

$$E(4V + 2W) = 4\mu_V + 2\mu_W = 4 \cdot 95 + 2 \cdot 62 = 504,$$

$$\mathrm{Var}(4V+2W) = 16\,\sigma_V^2 + 16\,\sigma_{VW} + 4\,\sigma_W^2$$

$$= 16\cdot 9 + 16\cdot 5 + 4\cdot 25 = 324,$$

OR

$$\mathrm{Var}(4V+2W) = \begin{pmatrix} 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 16 & -4 & 2 \\ -4 & 9 & 5 \\ 2 & 5 & 25 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 & 46 & 70 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = 324.$$

$$\mathrm{SD}(4V+2W) = 18.$$

$$P(4V+2W > 531) = P\left(Z > \frac{531-504}{18}\right) = P(Z > 1.50) = \mathbf{0.0668}.$$

c) What is the probability that Bob's portfolio is worth more than Carl's portfolio?

$$P(5U+3V > 4V+2W) = P(5U-V-2W > 0) = \ ?$$

$5U-V-2W$ has Normal distribution,

$$E(5U-V-2W) = 5\,\mu_U - \mu_V - 2\,\mu_W = 5\cdot 53 - 95 - 2\cdot 62 = 46,$$

$$\mathrm{Var}(5U-V-2W) = \begin{pmatrix} 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} 16 & -4 & 2 \\ -4 & 9 & 5 \\ 2 & 5 & 25 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 80 & -39 & -45 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} = 529.$$

$$\mathrm{SD}(5U-V-2W) = 23.$$

$$P(5U-V-2W > 0) = P\left(Z > \frac{0-46}{23}\right) = P(Z > -2.00) = \mathbf{0.9772}.$$

d) What is the probability that the two portfolios together are worth less than \$1025?

$$P(5U + 3V + 4V + 2W < 1025) = P(5U + 7V + 2W < 1025) = ?$$

$5U + 7V + 2W$ has Normal distribution,

$$E(5U + 7V + 2W) = 5\mu_U + 7\mu_V + 2\mu_W = 5 \cdot 53 + 7 \cdot 95 + 2 \cdot 62 = 1054,$$

$$\mathrm{Var}(5U + 7V + 2W) = \begin{pmatrix} 5 & 7 & 2 \end{pmatrix} \begin{pmatrix} 16 & -4 & 2 \\ -4 & 9 & 5 \\ 2 & 5 & 25 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 56 & 53 & 95 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix} = 841.$$

$$SD(5U + 7V + 2W) = 29.$$

$$P(5U + 7V + 2W < 1025) = P\left(Z < \frac{1025 - 1054}{29}\right) = P(Z < -1.00) = \mathbf{0.1587}.$$

3. Suppose $\mathbf{X}$ follows a 3-dimensional multivariate normal distribution with

mean $\boldsymbol{\mu} = \begin{pmatrix} 44 \\ 40 \\ 30 \end{pmatrix}$ and covariance matrix $\Sigma = \begin{pmatrix} 9 & -6 & 8 \\ -6 & 25 & -6 \\ 8 & -6 & 12 \end{pmatrix}$.

a) Find $P(X_1 > 38)$.

$X_1 \sim N(44, 9)$

$$P(X_1 > 38) = P\left(Z > \frac{38-44}{3}\right) = P(Z > -2.00) = \mathbf{0.9772}.$$

b) Find $P(X_2 > 38)$.

$X_2 \sim N(40, 25)$

$$P(X_2 > 38) = P\left(Z > \frac{38-40}{5}\right) = P(Z > -0.40) = \mathbf{0.6554}.$$

c) Find $P(X_2 + X_3 > 66)$.

$$E(X_2 + X_3) = 40 + 30 = 70.$$

$$\text{Var}(X_2 + X_3) = \text{Var}(X_2) + 2\,\text{Cov}(X_2, X_3) + \text{Var}(X_3) = 25 - 12 + 12 = 25.$$

OR

$$\text{Var}(X_2 + X_3) = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 & -6 & 8 \\ -6 & 25 & -6 \\ 8 & -6 & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 19 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 25.$$

$$P(X_2 + X_3 > 66) = P\left(Z > \frac{66-70}{5}\right) = P(Z > -0.80) = \mathbf{0.7881}.$$

d) Find $P(X_2 - X_3 > 3)$.

$$E(X_2 - X_3) = 40 - 30 = 10.$$

$$\operatorname{Var}(X_2 - X_3) = \operatorname{Var}(X_2) - 2\operatorname{Cov}(X_2, X_3) + \operatorname{Var}(X_3) = 25 + 12 + 12 = 49.$$

OR

$$\operatorname{Var}(X_2 - X_3) = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 9 & -6 & 8 \\ -6 & 25 & -6 \\ 8 & -6 & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -14 & 31 & -18 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 49.$$

$$P(X_2 - X_3 > 3) = P\left(Z > \frac{3-10}{7}\right) = P(Z > -1.00) = \mathbf{0.8413}.$$

e) Find $P(X_1 + 2X_2 + 3X_3 > 240)$.

$$E(X_1 + 2X_2 + 3X_3) = 44 + 2 \cdot 40 + 3 \cdot 30 = 214.$$

$$\operatorname{Var}(X_1 + 2X_2 + 3X_3) = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 9 & -6 & 8 \\ -6 & 25 & -6 \\ 8 & -6 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 & 26 & 32 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 169.$$

$$P(X_1 + 2X_2 + 3X_3 > 240) = P\left(Z > \frac{240-214}{13}\right) = P(Z > 2.00) = \mathbf{0.0228}.$$

4. Suppose $\mathbf{X}$ has a multivariate normal distribution with mean

$$\boldsymbol{\mu} = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and covariance matrix } \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 4 & 0 & -2 \\ 4 & 16 & 10 & 2 \\ 0 & 10 & 25 & -5 \\ -2 & 2 & -5 & 4 \end{pmatrix}.$$

a) Find $P(X_1 > 8)$.

$X_1 \sim N(5, 9)$

$$P(X_1 > 8) = P\left(Z > \frac{8-5}{3}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$

b) Find $P(3X_1 - 5X_2 + 4X_3 > -2)$.

$$3\mu_1 - 5\mu_2 + 4\mu_3 = 3 \cdot 5 - 5 \cdot 2 + 4 \cdot 3 = 17.$$

$$\begin{pmatrix} 3 & -5 & 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 9 & 4 & 0 & -2 \\ 4 & 16 & 10 & 2 \\ 0 & 10 & 25 & -5 \\ -2 & 2 & -5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 & -28 & 50 & -36 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \\ 0 \end{pmatrix} = 361.$$

$3X_1 - 5X_2 + 4X_3 \sim N(17, 361)$

$$P(3X_1 - 5X_2 + 4X_3 > -2) = P\left(Z > \frac{-2-17}{\sqrt{361}}\right) = P(Z > -1.00) = \mathbf{0.8413}.$$

c)* Find $P(X_1 > 8 \mid X_2 = 5, X_3 = 8, X_4 = 3)$.

$$\boldsymbol{\Sigma}_{22} = \begin{pmatrix} 16 & 10 & 2 \\ 10 & 25 & -5 \\ 2 & -5 & 4 \end{pmatrix} \qquad \boldsymbol{\Sigma}_{22}^{-1} = \frac{1}{500}\begin{pmatrix} 75 & -50 & -100 \\ -50 & 60 & 100 \\ -100 & 100 & 300 \end{pmatrix} = \begin{pmatrix} 0.15 & -0.1 & -0.2 \\ -0.1 & 0.12 & 0.2 \\ -0.2 & 0.2 & 0.6 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{12}\,\boldsymbol{\Sigma}_{22}^{-1} = \begin{pmatrix} 4 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0.15 & -0.1 & -0.2 \\ -0.1 & 0.12 & 0.2 \\ -0.2 & 0.2 & 0.6 \end{pmatrix} = \begin{pmatrix} 1 & -0.8 & -2 \end{pmatrix}$$

$$\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\,\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2) = 5 + \begin{pmatrix} 1 & -0.8 & -2 \end{pmatrix} \cdot \begin{pmatrix} 5-2 \\ 8-3 \\ 3-4 \end{pmatrix} = 6.$$

$$\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\,\boldsymbol{\Sigma}_{22}^{-1}\,\boldsymbol{\Sigma}_{21} = 9 - \begin{pmatrix} 1 & -0.8 & -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} = 1.$$

$X_1 \mid X_2 = 5, X_3 = 8, X_4 = 3 \sim N(6, 1)$

$P(X_1 > 8 \mid X_2 = 5, X_3 = 8, X_4 = 3) = P\left(Z > \frac{8-6}{1}\right) = P(Z > 2.00) = \mathbf{0.0228}.$

5.* Let Z_1 and Z_2 be independent standard normal random variables $N(0, 1)$.

What is the distribution of $Y_1 = Z_1 / Z_2$?

Hint: Let $Y_2 = Z_2$. Obtain the joint p.d.f. $f_{Y_1,Y_2}(y_1, y_2)$ of Y_1 and Y_2 first, then find the marginal p.d.f. $f_{Y_1}(y_1)$ of Y_1.

Hint: $$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\,dy_2$$
$$= \int_{-\infty}^{0} f_{Y_1,Y_2}(y_1, y_2)\,dy_2 + \int_{0}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\,dy_2 .$$

$$Z_1 = Y_1 \times Y_2, \qquad Z_2 = Y_2 .$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2 . \qquad |J| = |y_2| .$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_1^2 y_2^2}{2} \right) \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_2^2}{2} \right) \times |y_2|$$
$$= \frac{1}{2\pi} \exp\left(-\frac{y_2^2}{2} \cdot \left(1 + y_1^2\right) \right) \times |y_2|, \qquad -\infty < y_1 < \infty,\ -\infty < y_2 < \infty.$$

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\,dy_2 = 2 \times \int_{0}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\,dy_2$$
$$= \int_{0}^{\infty} \frac{1}{\pi} \cdot \exp\left(-\frac{y_2^2}{2} \cdot \left(1 + y_1^2\right) \right) \cdot y_2\,dy_2$$

$$u = \frac{y_2^2}{2} \qquad du = y_2\,dy_2$$

$$= \int_{0}^{\infty} \frac{1}{\pi} \cdot \exp\left(-u \cdot \left(1 + y_1^2\right) \right) du = \frac{1}{\pi\left(1 + y_1^2\right)}, \qquad -\infty < y_1 < \infty.$$

Recall: (Standard) Cauchy distribution: $$f_X(x) = \frac{1}{\pi\left(1 + x^2\right)}, \qquad -\infty < x < \infty.$$

6.* Let X and Y be independent random variables with common moment generating function $\mathrm{M}(t) = e^{t^2/2}$. Let $\mathrm{W} = \mathrm{X} + \mathrm{Y}$, $\mathrm{V} = \mathrm{X} - \mathrm{Y}$. Determine the joint moment generating function, $\mathrm{M}(t_1, t_2)$, of W and V. Are W and V independent?

$$
\begin{aligned}
\mathrm{M}(t_1, t_2) &= \mathrm{E}[\exp\{t_1 \mathrm{W} + t_2 \mathrm{V}\}] = \mathrm{E}[\exp\{t_1(\mathrm{X}+\mathrm{Y}) + t_2(\mathrm{X}-\mathrm{Y})\}] \\
&= \mathrm{E}[\exp\{(t_1+t_2)\mathrm{X} + (t_1-t_2)\mathrm{Y}\}] \\
&= \mathrm{E}[\exp\{(t_1+t_2)\mathrm{X}\} \cdot \exp\{(t_1-t_2)\mathrm{Y}\}] \\
&\qquad \text{since X and Y are independent} \\
&= \mathrm{E}[\exp\{(t_1+t_2)\mathrm{X}\}] \cdot \mathrm{E}[\exp\{(t_1-t_2)\mathrm{Y}\}] \\
&= \mathrm{M_X}(t_1+t_2) \cdot \mathrm{M_Y}(t_1-t_2) \\
&= \exp\{\tfrac{1}{2}(t_1+t_2)^2\} \cdot \exp\{\tfrac{1}{2}(t_1-t_2)^2\} = e^{t_1^2+t_2^2}.
\end{aligned}
$$

$$\mathrm{M}(t_1, t_2) = e^{t_1^2} \cdot e^{t_2^2} = \mathrm{M_W}(t_1) \cdot \mathrm{M_V}(t_2),$$

where $\qquad \mathrm{M_W}(t_1) = e^{t_1^2}, \qquad \mathrm{M_V}(t_2) = e^{t_2^2}.$

$\Rightarrow$ W and V are independent $\mathrm{N}(0, 2)$ random variables.

7. **3.5.1** (7th and 6th edition)

Let X and Y have a bivariate normal distribution with respective parameters $\mu_X = 2.8$, $\mu_Y = 110$, $\sigma_X^2 = 0.16$, $\sigma_Y^2 = 100$, and $\rho = 0.6$. Compute:

(a) $P(106 < Y < 124)$.

Y has Normal distribution with mean $\mu_Y = 110$ and standard deviation $\sigma_Y = 10$.

$$P(106 < Y < 124) = P\left(\frac{106-110}{10} < Z < \frac{124-110}{10}\right) = P(-0.40 < Z < 1.40)$$

$$= \Phi(1.40) - \Phi(-0.40) = 0.9192 - 0.3446 = \mathbf{0.5746}.$$

(b) $P(106 < Y < 124 \mid X = 3.2)$.

Given $X = 3.2$, Y has Normal distribution

with mean $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 110 + 0.6 \cdot \frac{10}{0.4} \cdot (3.2 - 2.8) = 116$

and variance $(1-\rho^2)\cdot\sigma_Y^2 = (1-0.6^2)\cdot 10^2 = 64$

(standard deviation = 8).

$$P(106 < Y < 124 \mid X = 3.2) = P\left(\frac{106-116}{8} < Z < \frac{124-116}{8}\right) = P(-1.25 < Z < 1.00)$$

$$= \Phi(1.00) - \Phi(-1.25) = 0.8413 - 0.1056 = \mathbf{0.7357}.$$

8. **3.5.6** (7th and 6th edition)

Let X and Y have a bivariate normal distribution with parameters $\mu_X = 20$, $\mu_Y = 40$, $\sigma_X^2 = 9$, $\sigma_Y^2 = 4$, and $\rho = 0.6$. Find the shortest interval for which 0.90 is the conditional probability that Y is in the interval, given that $X = 22$.

Given $X = 22$, Y has Normal distribution

with mean $\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 40 + 0.6 \cdot \frac{2}{3} \cdot (22 - 20) = 40.8$

and variance $(1 - \rho^2) \cdot \sigma_Y^2 = (1 - 0.6^2) \cdot 2^2 = 2.56$

(standard deviation = 1.6).

$P(-1.645 < Z < 1.645) = 0.90.$

$40.8 \pm 1.645 \cdot 1.6$ **40.8 ± 2.632** **$(38.168, 43.432)$**

9. **3.5.14** (7th and 6th edition)

Let $\mathbf{X} = (X_1, X_2, X_3)$ have a multivariate normal distribution with mean vector $\mathbf{0}$ and variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find $P(X_1 > X_2 + X_3 + 2)$.

Hint: Find the vector $\mathbf{a}$ so that $\mathbf{a}\mathbf{X} = X_1 - X_2 - X_3$ and make use of Theorem 3.5.1.

Theorem 3.5.1. Suppose $\mathbf{X}$ has a $N_n(\boldsymbol{\mu}, \Sigma)$ distribution. Let $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b} \in \mathbf{R}^m$. Then $\mathbf{Y}$ has a $N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}')$ distribution.

$$E(X_1 - X_2 - X_3) = \mu_1 - \mu_2 - \mu_3 = 0.$$

$$\operatorname{Var}(X_1 - X_2 - X_3) = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 7.$$

$$X_1 - X_2 - X_3 \sim N(0, 7)$$

$$P(X_1 > X_2 + X_3 + 2) = P(X_1 - X_2 - X_3 > 2) = P\left(Z > \frac{2-0}{\sqrt{7}}\right)$$

$$= P(Z > 0.76) = \mathbf{0.2236}.$$

10.* **3.5.4** (7th and 6th edition)

Let U and V be independent random variables, each having a standard normal distribution. Show that the mgf $\mathrm{E}(e^{t(\mathrm{UV})})$ of the random variable UV is $(1-t^2)^{-1/2}$, $-1<t<1$.

Hint: Compare $\mathrm{E}(e^{t\mathrm{UV}})$ with the integral of a bivariate normal pdf that has means equal to zero.

Let $-1<t<1$.

$$\mathrm{E}(e^{t\mathrm{UV}}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tuv}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}\,du\,dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(u^2-2tuv+v^2\right)\right\}du\,dv = \ldots$$

Let $\sigma_1=\sigma_2=\dfrac{1}{\sqrt{1-t^2}}$, $\rho=t$.

$$\ldots = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\exp\left\{-\frac{1}{2\left(1-t^2\right)}\left(\frac{u^2}{\sigma_1^2}-2t\frac{u}{\sigma_1}\frac{v}{\sigma_2}+\frac{v^2}{\sigma_2^2}\right)\right\}du\,dv$$

$$= \frac{1}{\sqrt{1-t^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-t^2}}\exp\left\{-\frac{1}{2\left(1-t^2\right)}\left(\frac{u^2}{\sigma_1^2}-2t\frac{u}{\sigma_1}\frac{v}{\sigma_2}+\frac{v^2}{\sigma_2^2}\right)\right\}du\,dv$$

$$= \frac{1}{\sqrt{1-t^2}},$$

since $\dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-t^2}}\exp\left\{-\dfrac{1}{2\left(1-t^2\right)}\left(\dfrac{u^2}{\sigma_1^2}-2t\dfrac{u}{\sigma_1}\dfrac{v}{\sigma_2}+\dfrac{v^2}{\sigma_2^2}\right)\right\}$ is the p.d.f. of a bivariate normal distribution with $\mu_1=\mu_2=0$, and $\rho=t$.