STAT 410 | **Examples for 10/05/2020 (3)** | Fall 2020
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1. Alex accidentally forgot to stock up on toilet paper before the stay-at-home order. Now he has to buy toilet paper on the black market. Though the price of toilet paper on the black market has mostly stabilized, it still varies from day to day. The daily price of a generic brand 12-pack, X, and the daily price of a generic brand 6-pack, Y, (in rubles) jointly follow a bivariate normal distribution with

$$\mu_X = 2{,}470, \qquad \sigma_X = 30, \qquad \mu_Y = 1{,}250, \qquad \sigma_Y = 25, \qquad \rho = 0.60.$$

a) What is the probability that a 12-pack costs more than 2,482 rubles today?

b) Suppose that today's price of a 6-pack is 1,270 rubles. What is the probability that a 12-pack costs more than 2,482 rubles today?

c) What is the probability that a 6-pack costs less than 1,234 rubles today?

d) Suppose that today's price of a 12-pack is 2,460 rubles. What is the probability that a 6-pack costs less than 1,234 rubles today?

e) What is the probability that 2 (two) 6-packs cost more than 1 (one) 12-pack?

f) To ensure that he will not be without toilet paper ever again, Alex buys 7 (seven) 12-packs and 18 (eighteen) 6-packs. What is the probability that he paid more than 40,000 rubles?

1. (continued)

Of course, Alex also forgot to stock up on face masks. Suppose that the daily price of a generic brand 12-pack, X, and the daily price of a generic brand 6-pack, Y, and the daily price of a generic brand washable face mask, W, (in rubles) jointly follow a multivariate normal distribution with mean vector $\vec{\boldsymbol{\mu}}$ and covariance matrix $\boldsymbol{\Sigma}$.

$$\vec{\boldsymbol{\mu}} = \begin{pmatrix} \mu_X \\ \mu_Y \\ \mu_W \end{pmatrix} = \begin{pmatrix} 2{,}470 \\ 1{,}250 \\ 603 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \spadesuit & \clubsuit & 390 \\ \clubsuit & 625 & 140 \\ 390 & 140 & 400 \end{pmatrix}.$$

g) What is the value of $\spadesuit$?

h) What is the value of $\clubsuit$? (We all know that every covariance matrix is symmetric.)

i) Find the correlation between Y and W, ρ_{YW}.

j) What is the probability that a face mask costs less than 587 rubles today?

k) Suppose that today's price of a 6-pack is 1,260 rubles. What is the probability that a face mask costs less than 587 rubles today?

"Hint": (Y, W) jointly follow a bivariate normal distribution.
We have μ_Y, σ_Y^2, μ_W, σ_W^2, ρ_{YW}.

l) Suppose that today's price of a face mask is 587 rubles. What is the probability that a 6-pack costs less than 1,260 rubles today?

m) What is the probability that 1 (one) 6-pack and 2 (two) face masks cost more than 1 (one) 12-pack?

n) Alex buys 7 (seven) 12-packs, 18 (eighteen) 6-packs, and 5 (five) face masks. What is the probability that he paid more than 43,000 rubles?

[1 US dollar is approximately 78 rubles]

Answers:

1. Alex accidentally forgot to stock up on toilet paper before the stay-at-home order. Now he has to buy toilet paper on the black market. Though the price of toilet paper on the black market has mostly stabilized, it still varies from day to day. The daily price of a generic brand 12-pack, X, and the daily price of a generic brand 6-pack, Y, (in rubles) jointly follow a bivariate normal distribution with

$$\mu_X = 2{,}470, \qquad \sigma_X = 30, \qquad \mu_Y = 1{,}250, \qquad \sigma_Y = 25, \qquad \rho = 0.60.$$

a) What is the probability that a 12-pack costs more than 2,482 rubles today?

$$P(X > 2{,}482) = P\left(Z > \frac{2{,}482 - 2{,}470}{30}\right) = P(Z > 0.40) = \mathbf{0.3446}.$$

b) Suppose that today's price of a 6-pack is 1,270 rubles. What is the probability that a 12-pack costs more than 2,482 rubles today?

Need $P(X > 2{,}482 \mid Y = 1{,}270)$.

Given $Y = 1{,}270$, X has Normal distribution

with mean $\mu_X + \rho \dfrac{\sigma_X}{\sigma_Y}(y - \mu_Y) = 2{,}470 + 0.60 \cdot \dfrac{30}{25} \cdot (1{,}270 - 1{,}250) = 2{,}484.4$

and variance $\left(1 - \rho^2\right) \cdot \sigma_X^2 = \left(1 - 0.60^2\right) \cdot 30^2 = 576$

(standard deviation = 22).

$$P(X > 2{,}482 \mid Y = 1{,}270) = P\left(Z > \frac{2{,}482 - 2{,}484.4}{24}\right) = P(Z > -0.10) = \mathbf{0.5398}.$$

c) What is the probability that a 6-pack costs less than 1,234 rubles today?

$$P(Y < 1{,}234) = P\left(Z < \frac{1{,}234 - 1{,}250}{25}\right) = P(Z < -0.64) = \mathbf{0.2611}.$$

d) Suppose that today's price of a 12-pack is 2,460 rubles. What is the probability that a 6-pack costs less than 1,234 rubles today?

Need $P(Y < 1{,}234 \mid X = 2{,}460)$.

Given $X = 2{,}460$, Y has Normal distribution

with mean $\mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X) = 1{,}250 + 0.60 \cdot \dfrac{25}{30} \cdot (2{,}460 - 2{,}470) = 1{,}245$

and variance $\left(1 - \rho^2\right) \cdot \sigma_Y^2 = \left(1 - 0.60^2\right) \cdot 25^2 = 400$

(standard deviation = 20).

$$P(Y < 1{,}234 \mid X = 2{,}460) = P\left(Z < \frac{1{,}234 - 1{,}245}{20}\right) = P(Z < -0.55) = \mathbf{0.2912}.$$

e) What is the probability that 2 (two) 6-packs cost more than 1 (one) 12-pack?

Need $P(2Y > X) = P(X - 2Y < 0)$.

$X - 2Y$ has Normal distribution,

$$E(X - 2Y) = \mu_X - 2\mu_Y = 2{,}470 - 2 \cdot 1{,}250 = -30,$$

$$\mathrm{Var}(X - 2Y) = \sigma_X^2 - 4\sigma_{XY} + 4\sigma_Y^2$$

$$= \sigma_X^2 - 4\rho\sigma_X\sigma_Y + 4\sigma_Y^2$$

$$= 30^2 - 4 \cdot 0.60 \cdot 30 \cdot 25 + 4 \cdot 25^2 = 1{,}600$$

(standard deviation = 40).

$$P(X - 2Y < 0) = P\left(Z < \frac{0-(-30)}{40}\right) = P(Z < 0.75) = \mathbf{0.7734}.$$

f) To ensure that he will not be without toilet paper ever again, Alex buys 7 (seven) 12-packs and 18 (eighteen) 6-packs. What is the probability that he paid more than 40,000 rubles?

Need $P(7X + 18Y > 40{,}000)$.

$7X + 18Y$ has Normal distribution,

$$E(7X + 18Y) = 7\mu_X + 18\mu_Y = 7 \cdot 2{,}470 + 18 \cdot 1{,}250 = 39{,}790,$$

$$\mathrm{Var}(7X + 18Y) = 7^2\sigma_X^2 + 2 \cdot 7 \cdot 18\,\sigma_{XY} + 18^2\sigma_Y^2$$

$$= 49\,\sigma_X^2 + 252\,\rho\sigma_X\sigma_Y + 324\,\sigma_Y^2$$

$$= 49 \cdot 30^2 + 252 \cdot 0.60 \cdot 30 \cdot 25 + 324 \cdot 25^2 = 360{,}000$$

(standard deviation = 600).

$$P(7X + 18Y > 40{,}000) = P\left(Z > \frac{40{,}000 - 39{,}790}{600}\right) = P(Z > 0.35) = \mathbf{0.3632}.$$

1. (continued)

Of course, Alex also forgot to stock up on face masks. Suppose that the daily price of a generic brand 12-pack, X, and the daily price of a generic brand 6-pack, Y, and the daily price of a generic brand washable face mask, W, (in rubles) jointly follow a multivariate normal distribution with mean vector $\vec{\boldsymbol{\mu}}$ and covariance matrix $\boldsymbol{\Sigma}$.

$$\vec{\boldsymbol{\mu}} = \begin{pmatrix} \mu_X \\ \mu_Y \\ \mu_W \end{pmatrix} = \begin{pmatrix} 2,470 \\ 1,250 \\ 603 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \spadesuit & \clubsuit & 390 \\ \clubsuit & 625 & 140 \\ 390 & 140 & 400 \end{pmatrix}.$$

g) What is the value of $\spadesuit$?

$$\spadesuit = \sigma_X^2 = 30^2 = \mathbf{900}.$$

h) What is the value of $\clubsuit$? (We all know that every covariance matrix is symmetric.)

$$\clubsuit = \sigma_{XY} = 0.6 \cdot 30 \cdot 25 = \mathbf{450}.$$

i) Find the correlation between Y and W, ρ_{YW}.

$$\rho_{YW} = \frac{\sigma_{YW}}{\sigma_Y \, \sigma_W} = \frac{140}{25 \cdot \sqrt{400}} = \mathbf{0.28}.$$

j) What is the probability that a face mask costs less than 587 rubles today?

W has Normal distribution, $\mu_W = 603$, $\sigma_W = \sqrt{400} = 20$.

$$P(W < 587) = P\left(Z < \frac{587 - 603}{20}\right) = P(Z < -0.80) = \mathbf{0.2119}.$$

k) Suppose that today's price of a 6-pack is 1,260 rubles. What is the probability that a face mask costs less than 587 rubles today?

"Hint": (Y, W) jointly follow a bivariate normal distribution.
We have μ_Y, σ_Y^2, μ_W, σ_W^2, ρ_{YW}.

Given Y = 1,260, W has Normal distribution

with mean $603 + 0.28 \cdot \frac{20}{25} \cdot (1{,}260 - 1{,}250) = 605.24$

and variance $\left(1 - 0.28^2\right) \cdot 20^2 = 368.64$ (standard deviation 19.2).

$$P(W < 587 \mid Y = 1{,}260) = P\left(Z < \frac{587 - 605.24}{19.2}\right) = P(Z < -0.95) = \mathbf{0.1711}.$$

l) Suppose that today's price of a face mask is 587 rubles. What is the probability that a 6-pack costs less than 1,260 rubles today?

Given W = 587, Y has Normal distribution

with mean $1{,}250 + 0.28 \cdot \frac{25}{20} \cdot (587 - 603) = 1{,}244.4$

and variance $\left(1 - 0.28^2\right) \cdot 25^2 = 576$ (standard deviation 24).

$$P(Y < 1{,}260 \mid W = 587) = P\left(Z < \frac{1{,}260 - 1{,}244.4}{24}\right) = P(Z < 0.65) = \mathbf{0.7422}.$$

m) What is the probability that 1 (one) 6-pack and 2 (two) face masks cost more than 1 (one) 12-pack?

$$P(Y+2W>X) = P(X-Y-2W<0).$$

$X-Y-2W$ has Normal distribution,

$$E(X-Y-2W) = \mu_X - \mu_Y - 2\mu_W = 2{,}470 - 1{,}250 - 2\cdot 603 = 14,$$

$$\text{Var}(X-Y-2W) = \begin{pmatrix} 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 900 & 450 & 390 \\ 450 & 625 & 140 \\ 390 & 140 & 400 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -330 & -455 & -550 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 1{,}225$$

(standard deviation 35).

$$P(X-Y-2W<0) = P(Z<\frac{0-14}{35}) = P(Z<-0.40) = \mathbf{0.3446}.$$

n) Alex buys 7 (seven) 12-packs, 18 (eighteen) 6-packs, and 5 (five) face masks. What is the probability that he paid more than 43,000 rubles?

$7X + 18Y + 5W$ has Normal distribution,

$$E(7X + 18Y + 5W) = 7\mu_X + 18\mu_Y + 5\mu_W$$

$$= 7 \cdot 2{,}470 + 18 \cdot 1{,}250 + 5 \cdot 603 = 42{,}805,$$

$$\mathrm{Var}(7X + 18Y + 5W) = \begin{pmatrix} 7 & 18 & 5 \end{pmatrix} \begin{pmatrix} 900 & 450 & 390 \\ 450 & 625 & 140 \\ 390 & 140 & 400 \end{pmatrix} \begin{pmatrix} 7 \\ 18 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 16{,}350 & 15{,}100 & 7{,}250 \end{pmatrix} \begin{pmatrix} 7 \\ 18 \\ 5 \end{pmatrix} = 422{,}500$$

(standard deviation 650).

$$P(7X + 18Y + 5W > 43{,}000) = P\left(Z > \frac{43{,}000 - 42{,}805}{650}\right) = P(Z > 0.30) = \mathbf{0.3821}.$$