STAT 410 — **Examples for 11/06/2020 (1)** — Fall 2020, A. Stepanov

Examples for 10/19/2020 (3) (continued)

5. The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0$, $\delta > 0$. Consider the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall: $W = X^{\delta}$ has an $\text{Exponential}\left(\theta = \frac{1}{\beta}\right) = \text{Gamma}\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the above probability distribution.

$$\Rightarrow \qquad Y = \sum_{i=1}^{n} X_i^{\delta} = \sum_{i=1}^{n} W_i \text{ has a } \text{Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right) \text{ distribution.} \qquad !!!$$

Suppose δ is known.

k) Suggest a confidence interval for β with $(1-\alpha)100\%$ confidence level.

l) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$. Use part (k) to construct a 90% confidence interval for β.

m) Find a sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for β.

n) Find the Fisher information $I(\beta)$.

o) Recall that $\hat{\hat{\beta}} = \dfrac{n-1}{\sum_{i=1}^{n} X_i^{\delta}}$ is an unbiased estimator of β.

Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not an efficient estimator of β, find its efficiency. “Hint”: Recall Examples for 10/19/2020 (3) **2** (g).

Answers:

5. The Weibull distribution has many applications in reliability engineering, survival analysis, and general insurance. Let $\beta > 0$, $\delta > 0$. Consider the probability density function

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

Recall: $W = X^{\delta}$ has an $\text{Exponential}\left(\theta = \frac{1}{\beta}\right) = \text{Gamma}\left(\alpha = 1, \theta = \frac{1}{\beta}\right)$ distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the above probability distribution.

$\Rightarrow \quad Y = \sum\limits_{i=1}^{n} X_i^{\delta} = \sum\limits_{i=1}^{n} W_i$ has a $\text{Gamma}\left(\alpha = n, \theta = \frac{1}{\beta}\right)$ distribution. **! ! !**

Suppose δ is known.

k) Suggest a confidence interval for β with $(1-\alpha)100\%$ confidence level.

$Y = \sum\limits_{i=1}^{n} X_i^{\delta} = \sum\limits_{i=1}^{n} W_i$ has a Gamma distribution with $\alpha = n$ and $\theta = \frac{1}{\beta}$ $(\lambda = \beta)$.

If T_α has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T_\alpha/\theta = 2\lambda T_\alpha$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$2Y/\theta = 2\beta \sum\limits_{i=1}^{n} X_i^{\delta}$ has a chi-square distribution with $r = 2\alpha = 2n$ d.f.

$$\Rightarrow \quad P\left(\chi^2_{1-\alpha/2}(2n) < 2\beta \sum_{i=1}^{n} X_i^{\delta} < \chi^2_{\alpha/2}(2n) \right) = 1-\alpha.$$

$$\Rightarrow \quad P\left(\frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i^{\delta}} < \beta < \frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i^{\delta}} \right) = 1-\alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for β: $\left(\dfrac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i^{\delta}}, \dfrac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^{n} X_i^{\delta}} \right)$.

l) Suppose $\delta = 3$, $n = 5$, and $x_1 = 0.2$, $x_2 = 1.2$, $x_3 = 0.2$, $x_4 = 0.9$, $x_5 = 0.3$. Use part (k) to construct a 90% confidence interval for β.

$$\sum_{i=1}^{n} x_i^3 = 2.5.$$

$\chi^2_{0.95}(10) = 3.940$, $\qquad \chi^2_{0.05}(10) = 18.31$.

$\left(\dfrac{3.940}{2 \cdot 2.5}, \dfrac{18.31}{2 \cdot 2.5} \right)$ $\qquad$ $\mathbf{(0.788, 3.662)}$

m) Find a sufficient statistic $Y = u(X_1, X_2, \dots, X_n)$ for β.

$$f(x_1, x_2, \dots x_n; \beta) = f(x_1; \beta)\, f(x_2; \beta)\, \dots\, f(x_n; \beta)$$

$$= \prod_{i=1}^{n} \left(\beta\, \delta\, x_i^{\delta-1}\, e^{-\beta x_i^{\delta}} \right) = \left[\beta^n\, e^{-\beta \Sigma x_i^{\delta}} \right] \left(\delta^n \prod_{i=1}^{n} x_i^{\delta-1} \right).$$

By Factorization Theorem, $Y = \sum_{i=1}^{n} X_i^{\delta}$ is a sufficient statistic for β.

OR

$$f(x; \beta) = \exp\{ -\beta x^{\delta} + \ln \beta + \ln \delta + (\delta - 1) \ln x \}. \qquad \Rightarrow \qquad K(x) = x^{\delta}.$$

$\Rightarrow \qquad Y = \sum_{i=1}^{n} X_i^{\delta}$ is a sufficient statistic for β.

n) Find the Fisher information $I(\beta)$.

$$\ln f(x; \beta) = -\beta x^{\delta} + \ln \beta + \ln \delta + (\delta - 1) \ln x.$$

$$\frac{\partial}{\partial \beta} \ln f(x; \beta) = -x^{\delta} + \frac{1}{\beta}. \qquad \frac{\partial^2}{\partial \beta^2} \ln f(x; \beta) = -\frac{1}{\beta^2}.$$

$$I(\beta) = \operatorname{Var}\left[\frac{\partial}{\partial \beta} \ln f(X; \beta) \right] = \operatorname{Var}\left[-X^{\delta} + \frac{1}{\beta} \right] = \operatorname{Var}(W) = \alpha\, \theta^2 = \frac{1}{\beta^2}.$$

$$I(\beta) = -E\left[\frac{\partial^2}{\partial \beta^2} \ln f(X; \beta) \right] = -E\left[-\frac{1}{\beta^2} \right] = \frac{1}{\beta^2}.$$

o) Recall that $\hat{\hat{\beta}} = \dfrac{n-1}{\sum_{i=1}^{n} X_i^{\delta}}$ is an unbiased estimator of β.

Is $\hat{\hat{\beta}}$ an efficient estimator of β? If $\hat{\hat{\beta}}$ is not an efficient estimator of β, find its efficiency. "Hint": Recall Examples for 10/19/2020 (3) **2** (g).

Recall: $$\operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{(n-1)(n-2)} - \left(\frac{\beta}{n-1}\right)^2 = \frac{\beta^2}{(n-1)^2(n-2)}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \operatorname{Var}\left(\frac{n-1}{Y}\right) = (n-1)^2 \operatorname{Var}\left(\frac{1}{Y}\right) = \frac{\beta^2}{n-2}.$$

$$\text{Rao-Cramer lower bound} = \frac{1}{n \cdot I(\beta)} = \frac{\beta^2}{n}.$$

$$\operatorname{Var}(\hat{\hat{\beta}}) = \frac{\beta^2}{n-2} > \frac{\beta^2}{n}.$$

$\operatorname{Var}(\hat{\hat{\beta}})$ does NOT attain its Rao-Cramer lower bound.

$\Rightarrow$ $\hat{\hat{\beta}}$ is NOT an efficient estimator of β,

its efficiency $= \dfrac{n-2}{n} \to 1$ as $n \to \infty$.