STAT 410

Examples for 11/06/2020 (4)

Fall 2020
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1. The Inverse Gamma distribution has applications in Bayesian statistics, machine learning, reliability engineering, and survival analysis.

Let $\alpha > 0$, $\beta > 0$. Consider the probability density function

$$f(x; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}, \qquad 0 < x < \infty.$$

a) Show that $\mathrm{E}(\mathrm{X}^k) = \dfrac{\beta^k \, \Gamma(\alpha - k)}{\Gamma(\alpha)}$, $\quad k < \alpha$.

b) Show that $\mathrm{W} = \dfrac{1}{\mathrm{X}}$ has a Gamma distribution with parameters α and $\theta = \dfrac{1}{\beta}$.

1. (continued)

Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be a random sample from an Inverse Gamma distribution.

Suppose α is known.

c) Find the sufficient statistic $\mathrm{Y} = u(\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n)$ for β.

d) (i) Find the maximum likelihood estimator $\hat{\beta}$ of β.

(ii) Suppose $\alpha = 3$, $n = 4$, $x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$.
Find the maximum likelihood estimate of β.

e) (i) Suppose $\alpha > 1$. Find the method of moments estimator $\tilde{\beta}$ of β.

(ii) Suppose $\alpha = 3$, $n = 4$, $x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$.
Find the method of moments estimate of β.

f) Suppose $\alpha = 3$. Construct a consistent estimator of β based on $\sum\limits_{i=1}^{n} \mathrm{X}_i^2$.

g) (i) Suggest a $(1-\alpha)\,100\,\%$ confidence interval for β based on $\sum_{i=1}^{n} \frac{1}{\mathrm{X}_i}$.

(ii) Suppose $\alpha = 3,\quad n = 4,\quad x_1 = 5,\ x_2 = 10,\ x_3 = 4,\ x_4 = 20.$

Construct a 90% confidence interval for β.

h) Suppose $\alpha = 3,\quad \beta = 25,\quad n = 4.$ Find $\mathrm{P}\big(\sum_{i=1}^{4} \frac{1}{\mathrm{X}_i} \le 0.50\big)$.

i) Suppose $n > \frac{1}{\alpha}$. The maximum likelihood estimator of β, $\hat{\beta}$, is NOT an unbiased estimator of β. Use $\hat{\beta}$ to construct an unbiased estimator of β, $\hat{\hat{\beta}}$.

j) Find the Fisher information $\mathrm{I}(\beta)$.

k) Suppose $n > \frac{2}{\alpha}$. Is $\hat{\hat{\beta}}$ and efficient estimator of β? If not, find its efficiency.

l) Suppose $\alpha > 2$. The method of moments estimator of β, $\tilde{\beta}$, is an unbiased estimator of β. Is $\tilde{\beta}$ and efficient estimator of β? If not, find its efficiency.

1. The Inverse Gamma distribution has applications in Bayesian statistics, machine learning, reliability engineering, and survival analysis.

Let $\alpha > 0$, $\beta > 0$. Consider the probability density function

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}, \qquad 0 < x < \infty.$$

a) Show that $\mathrm{E}(X^k) = \dfrac{\beta^k \, \Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad k < \alpha.$

$$\mathrm{E}(X^k) = \int_0^{\infty} x^k \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x} \, dx$$

$$= \frac{\beta^k \, \Gamma(\alpha-k)}{\Gamma(\alpha)} \cdot \int_0^{\infty} \frac{\beta^{\alpha-k}}{\Gamma(\alpha-k)} x^{-\alpha+k-1} e^{-\beta/x} \, dx$$

$$= \frac{\beta^k \, \Gamma(\alpha-k)}{\Gamma(\alpha)}, \qquad \text{since } \frac{\beta^{\alpha-k}}{\Gamma(\alpha-k)} x^{-\alpha+k-1} e^{-\beta/x} \text{ is the p.d.f.}$$

of Inverse Gamma distribution with parameters $\alpha' = \alpha - k$ and β.

OR

$$\mathrm{E}(X^k) = \int_0^{\infty} x^k \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x} \, dx \qquad w = \frac{1}{x} \qquad dx = -\frac{1}{w^2} dw$$

$$= \int_0^{\infty} w^{-k} \frac{\beta^{\alpha}}{\Gamma(\alpha)} w^{\alpha+1} e^{-\beta w} \frac{1}{w^2} dw = \int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} w^{\alpha-k-1} e^{-\beta w} \, dw$$

$$= \frac{\beta^k \, \Gamma(\alpha-k)}{\Gamma(\alpha)} \cdot \int_0^{\infty} \frac{\beta^{\alpha-k}}{\Gamma(\alpha-k)} w^{\alpha-k-1} e^{-\beta w} \, dw$$

$$= \frac{\beta^k \, \Gamma(\alpha-k)}{\Gamma(\alpha)}, \qquad \text{since } \frac{\beta^{\alpha-k}}{\Gamma(\alpha-k)} w^{\alpha-k-1} e^{-\beta w} \text{ is the p.d.f.}$$

of Gamma distribution with parameters $\alpha' = \alpha - k$ and $\theta = \dfrac{1}{\beta}$.

$\mathrm{E}(X^k)$ does NOT exist for $k \geq \alpha$.

b) Show that $W = \dfrac{1}{X}$ has a Gamma distribution with parameters α and $\theta = \dfrac{1}{\beta}$.

$$w = g(x) = \frac{1}{x} \qquad x = g^{-1}(w) = \frac{1}{w} \qquad \frac{dx}{dw} = -\frac{1}{w^2}$$

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha+1} e^{-\beta w} \times \frac{1}{w^2}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\beta w}, \qquad w > 0.$$

$\Rightarrow$ $W = \dfrac{1}{X}$ has a Gamma distribution with parameters α and $\theta = \dfrac{1}{\beta}$.

1. (continued)

Let $X_1, X_2, \ldots, X_n$ be a random sample from an Inverse Gamma distribution.

Suppose α is known.

c) Find the sufficient statistic $Y = u(X_1, X_2, \ldots, X_n)$ for β.

$$\prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} x_i^{-\alpha-1} e^{-\beta/x_i} = \left[\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \exp\left\{ -\beta \sum_{i=1}^{n} \frac{1}{x_i} \right\} \right] \left(\prod_{i=1}^{n} x_i \right)^{-\alpha-1}.$$

By Factorization Theorem, $Y = \displaystyle\sum_{i=1}^{n} \frac{1}{X_i}$ is a sufficient statistic for β.

OR

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x} = \exp\left\{ -\beta \frac{1}{x} + \alpha \ln \beta - \ln \Gamma(\alpha) - (\alpha+1) \ln x \right\}.$$

$K(x) = \dfrac{1}{x}$. $\qquad \Rightarrow \qquad Y = \displaystyle\sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} \frac{1}{X_i}$ is a sufficient statistic for β.

d) (i) Find the maximum likelihood estimator $\hat{\beta}$ of β.

(ii) Suppose $\alpha = 3$, $n = 4$, $x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$.

Find the maximum likelihood estimate of β.

$$L(\beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x_i^{-\alpha-1} e^{-\beta / x_i} = \frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \left(\prod_{i=1}^{n} x_i \right)^{-\alpha-1} \exp\left\{ -\beta \sum_{i=1}^{n} \frac{1}{x_i} \right\}.$$

$$\ln L(\beta) = n\alpha \ln\beta - n \ln\Gamma(\alpha) - (\alpha+1) \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} \frac{1}{x_i}.$$

$$\frac{d}{d\beta} \ln L(\beta) = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} \frac{1}{x_i} = 0. \qquad \hat{\beta} = \frac{n\alpha}{\sum_{i=1}^{n} \frac{1}{X_i}}.$$

$x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$. $\qquad \displaystyle\sum_{i=1}^{n} \frac{1}{x_i} = 0.60.$

$$\hat{\beta} = \frac{12}{0.60} = \mathbf{20}.$$

e) (i) Suppose $\alpha > 1$. Find the method of moments estimator $\tilde{\beta}$ of β.

(ii) Suppose $\alpha = 3$, $n = 4$, $x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$.

Find the method of moments estimate of β.

$$E(X) = \frac{\beta^{1}\,\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{\beta}{(\alpha-1)}.$$

$$\overline{X} = \frac{\tilde{\beta}}{(\alpha-1)}. \qquad \Rightarrow \qquad \tilde{\beta} = (\alpha-1)\,\overline{X}.$$

$x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$. $\qquad \bar{x} = \dfrac{39}{4}.$

$$\tilde{\beta} = 2 \cdot \frac{39}{4} = \mathbf{\frac{39}{2}} = \mathbf{19.5}.$$

f) Suppose $\alpha = 3$. Construct a consistent estimator of β based on $\sum_{i=1}^{n} X_i^2$.

$$E(X^2) = \frac{\beta^2\,\Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{\beta^2}{(\alpha-1)(\alpha-2)} = \frac{\beta^2}{2}.$$

By WLLN, $$\overline{X^2} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E(X^2) = \frac{\beta^2}{2}.$$

Consider $$\tilde{\tilde{\beta}} = \sqrt{2\,\overline{X^2}} = \sqrt{\frac{2}{n}\sum_{i=1}^{n} X_i^2}.$$

$$X_n \xrightarrow{P} a,\ g \text{ is continuous at } a \ \Rightarrow\ g(X_n) \xrightarrow{P} g(a)$$

Since $g(x) = \sqrt{2x}$ is continuous at $\frac{\beta^2}{2}$, $\quad \tilde{\tilde{\beta}} = g(\overline{X^2}) \xrightarrow{P} g\left(\frac{\beta^2}{2}\right) = \beta.$

g) (i) Suggest a $(1-\alpha)\,100\,\%$ confidence interval for β based on $\sum_{i=1}^{n}\frac{1}{X_i}$.

(ii) Suppose $\alpha = 3$, $n = 4$, $x_1 = 5$, $x_2 = 10$, $x_3 = 4$, $x_4 = 20$.
Construct a 90% confidence interval for β.

$W = \frac{1}{X}$ has a Gamma distribution with parameters α and $\theta = \frac{1}{\beta}$.

$\Rightarrow \quad \sum_{i=1}^{n}\frac{1}{X_i} = \sum_{i=1}^{n} W_i$ has a Gamma distribution with parameters $n\alpha$ and $\theta = \frac{1}{\beta}$.

$\Rightarrow \quad \dfrac{2\sum_{i=1}^{n}\frac{1}{X_i}}{\theta} = 2\beta\sum_{i=1}^{n}\frac{1}{X_i}$ has a $\chi^2(2n\alpha)$ distribution.

$$\Rightarrow \quad P\left(\chi^2_{1-\alpha/2}(2n\alpha) < 2\beta \sum_{i=1}^{n} \frac{1}{X_i} < \chi^2_{\alpha/2}(2n\alpha) \right) = 1-\alpha.$$

$$\Rightarrow \quad P\left(\frac{\chi^2_{1-\alpha/2}(2n\alpha)}{2\sum_{i=1}^{n} \frac{1}{X_i}} < \beta < \frac{\chi^2_{\alpha/2}(2n\alpha)}{2\sum_{i=1}^{n} \frac{1}{X_i}} \right) = 1-\alpha.$$

A $(1-\alpha)\,100\,\%$ confidence interval for β: $\left(\frac{\chi^2_{1-\alpha/2}(2n\alpha)}{2\sum_{i=1}^{n} \frac{1}{X_i}}, \frac{\chi^2_{\alpha/2}(2n\alpha)}{2\sum_{i=1}^{n} \frac{1}{X_i}} \right)$

$x_1 = 5,\ \ x_2 = 10,\ \ x_3 = 4,\ \ x_4 = 20.$ $\qquad \sum_{i=1}^{n} \frac{1}{x_i} = 0.60.$

$\chi^2_{0.95}(24) = 13.85, \qquad \chi^2_{0.05}(24) = 36.42.$

$\left(\frac{13.85}{2 \cdot 0.60}, \frac{36.42}{2 \cdot 0.60} \right)$ $\qquad$ $\mathbf{(11.54, 30.35)}$

h) Suppose $\alpha = 3,\quad \beta = 25,\quad n = 4.$ Find $P\left(\sum_{i=1}^{4} \frac{1}{X_i} \le 0.50 \right)$.

$\sum_{i=1}^{4} \frac{1}{X_i}$ has a Gamma distribution with parameters "α" = 12 and "θ" = $\frac{1}{25}$.

$$P\left(\sum_{i=1}^{4} \frac{1}{X_i} \le 0.50 \right) = P(\text{Poisson}(25 \cdot 0.50) \ge 12) = 1 - P(\text{Poisson}(12.5) \le 11)$$

$$= 1 - 0.406 = \mathbf{0.594}.$$

OR $\quad \int_0^{0.50} \frac{25^{12}}{\Gamma(12)}\, w^{12-1} e^{-25w}\, dw = \ldots \qquad$ OR $\quad P(\chi^2(24) \le 25) = \ldots$

i) Suppose $n > \frac{1}{\alpha}$. The maximum likelihood estimator of β, $\hat{\beta}$, is NOT an unbiased estimator of β. Use $\hat{\beta}$ to construct an unbiased estimator of β, $\hat{\hat{\beta}}$.

$Y = \sum_{i=1}^{n} \frac{1}{X_i}$ has a Gamma distribution with parameters $n\alpha$ and $\theta = \frac{1}{\beta}$.

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} \cdot \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} y^{n\alpha-1} e^{-\beta y}\, dy = \frac{\beta}{n\alpha-1}.$$

Indeed, $\hat{\beta} = \frac{n\alpha}{Y}$ is NOT an unbiased estimator of β, $E(\hat{\beta}) = \frac{n\alpha}{n\alpha-1}\beta$.

$\hat{\hat{\beta}} = \frac{n\alpha-1}{Y} = \frac{n\alpha-1}{\sum_{i=1}^{n} \frac{1}{X_i}}$ is an unbiased estimator of β.

j) Find the Fisher information $I(\beta)$.

$$\ln f(x;\alpha,\beta) = -\beta\frac{1}{x} + \alpha\ln\beta - \ln\Gamma(\alpha) - (\alpha+1)\ln x.$$

$$\frac{\partial}{\partial\beta}\ln f(x;\alpha,\beta) = -\frac{1}{x} + \frac{\alpha}{\beta}. \qquad \frac{\partial^2}{\partial\beta^2}\ln f(x;\alpha,\beta) = -\frac{\alpha}{\beta^2}.$$

$$I(\beta) = -E\left[\frac{\partial^2}{\partial\beta^2}\ln f(X;\alpha,\beta)\right] = \frac{\alpha}{\beta^2}.$$

OR

$$I(\beta) = \mathrm{Var}\left[\frac{\partial}{\partial\beta}\ln f(X;\alpha,\beta)\right] = \mathrm{Var}\left[\frac{1}{X}\right] = \alpha\,\theta^2 = \frac{\alpha}{\beta^2}.$$

k) Suppose $n > \dfrac{2}{\alpha}$. Is $\hat{\hat{\beta}}$ and efficient estimator of β? If not, find its efficiency.

$$\mathrm{E}\left[\left(\frac{1}{\mathrm{Y}}\right)^2\right] = \int_0^\infty \frac{1}{y^2} \cdot \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} y^{n\alpha-1} e^{-\beta y}\, dy = \frac{\beta^2}{(n\alpha-1)(n\alpha-2)}.$$

$$\mathrm{Var}(\hat{\hat{\beta}}) = (n\alpha-1)^2 \left[\frac{\beta^2}{(n\alpha-1)(n\alpha-2)} - \frac{\beta^2}{(n\alpha-1)^2}\right] = \frac{\beta^2}{n\alpha-2}.$$

Rao-Cramer Lower Bound: $\qquad \dfrac{1}{n \cdot \mathrm{I}(\beta)} = \dfrac{\beta^2}{n\alpha}.$

$\hat{\hat{\beta}}$ is NOT an efficient estimator of β. $\qquad$ $(\text{efficiency of } \hat{\hat{\beta}}) = \dfrac{n\alpha-2}{n\alpha}.$

Note that $(\text{efficiency of } \hat{\hat{\beta}}) \to 1$ as $n \to \infty$.

l) Suppose $\alpha > 2$. The method of moments estimator of β, $\tilde{\beta}$, is an unbiased estimator of β. Is $\tilde{\beta}$ and efficient estimator of β? If not, find its efficiency.

$$\mathrm{E}(\mathrm{X}^2) = \frac{\beta^k\, \Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

$$\mathrm{Var}(\tilde{\beta}) = (\alpha-1)^2 \cdot \frac{\mathrm{Var}(\mathrm{X})}{n} = (\alpha-1)^2 \left[\frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2}\right] \cdot \frac{1}{n}$$

$$= \frac{\beta^2}{n(\alpha-2)}.$$

Rao-Cramer Lower Bound: $\qquad \dfrac{1}{n \cdot \mathrm{I}(\beta)} = \dfrac{\beta^2}{n\alpha}.$

$\tilde{\beta}$ is NOT an efficient estimator of β. $\qquad$ $(\text{efficiency of } \tilde{\beta}) = \dfrac{\alpha-2}{\alpha}.$

Note that $(\text{efficiency of } \tilde{\beta}) \not\to 1$ as $n \to \infty$.