STAT 410 | **Examples for 12/04/2020 (1)** | Fall 2020, A. Stepanov

– 1. In Anytownville, 10% of the people leave their keys in the ignition of their cars. Anytownville's police records indicate that 4.2% of the cars with keys left in the ignition are stolen. On the other hand, only 0.2% of the cars without keys left in the ignition are stolen. Suppose a car in Anytownville is stolen. What is the probability that the keys were left in the ignition?

$P(\text{ Keys }) = 0.10,$ $\qquad$ $P(\text{ Keys}'\,) = 1 - 0.10 = 0.90.$

$P(\text{ Stolen } \mid \text{ Keys }) = 0.042.$ $\qquad$ $P(\text{ Stolen } \mid \text{ Keys}'\,) = 0.002.$

Need $P(\text{ Keys } \mid \text{ Stolen }) = ?$

Bayes' Theorem:

$$P(\text{Keys} \mid \text{Stolen}) = \frac{P(\text{Keys}) \times P(\text{Stolen} \mid \text{Keys})}{P(\text{Keys}) \times P(\text{Stolen} \mid \text{Keys}) + P(\text{Keys}') \times P(\text{Stolen} \mid \text{Keys}')}$$

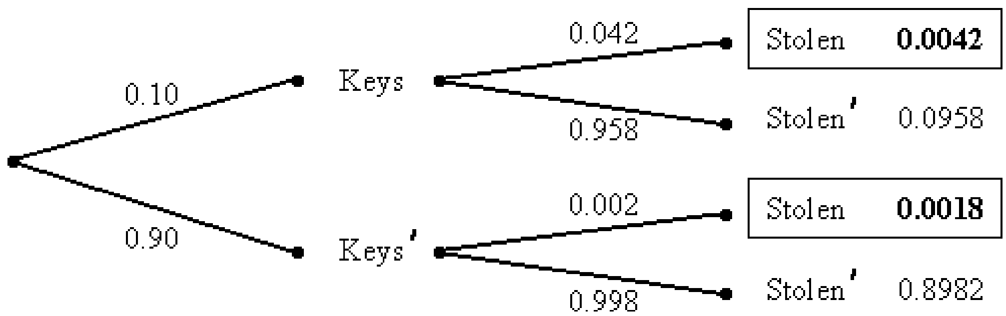
$$= \frac{0.10 \times 0.042}{0.10 \times 0.042 + 0.90 \times 0.002} = \mathbf{0.70}.$$

OR

	Stolen	Stolen′	
Keys	*0.042 · 0.10* 0.0042	0.0958	0.10
Keys′	*0.002 · 0.90* 0.0018	0.8982	0.90
	0.0060	0.9940	1.00

$$P(\text{ Keys } \mid \text{ Stolen }) = \frac{P(\text{ Keys } \cap \text{ Stolen })}{P(\text{ Stolen })} = \frac{0.0042}{0.0060} = \mathbf{0.70}.$$

OR



P(Stolen) = 0.0042 + 0.0018 = 0.0060.

$$P(\text{Keys} \mid \text{Stolen}) = \frac{P(\text{Keys} \cap \text{Stolen})}{P(\text{Stolen})} = \frac{0.0042}{0.0060} = \mathbf{0.70}.$$

0. When correctly adjusted, a machine that makes widgets operates with a 5% defective rate. However, there is a 10% chance that a disgruntled employee kicks the machine, in which case the defective rate jumps up to 30%.

a) Suppose that a widget made by this machine is selected at random and is found to be defective. What is the probability that the machine had been kicked?

$$P(D) = 0.90 \times 0.05 + 0.10 \times 0.30 = 0.075.$$

$$P(K \mid D) = \frac{0.10 \times 0.30}{0.90 \times 0.05 + 0.10 \times 0.30} = \frac{0.030}{0.075} = \mathbf{0.40}.$$

b) A random sample of 20 widgets was examined, 4 widgets out of these 20 are found to be defective. What is the probability that the machine had been kicked?

$$P(X = 4 \mid K') = \binom{20}{4}(0.05)^4(0.95)^{16} \approx 0.0133,$$

$$P(X = 4 \mid K) = \binom{20}{4}(0.30)^4(0.70)^{16} \approx 0.1304.$$

$$P(K \mid X = 4) = \frac{0.10 \times 0.1304}{0.90 \times 0.0133 + 0.10 \times 0.1304} = \frac{0.01304}{0.02501} \approx \mathbf{0.52}.$$

$f(x;\theta) = f(x \mid \theta)$ – p.d.f. (or p.m.f.) of x for given θ.

$\pi(\theta)$ – prior distribution of θ.

$f(x,\theta) = f(x \mid \theta) \times \pi(\theta)$ – joint p.d.f. of x and θ.

$f(x)$ – marginal p.d.f. of x.

$$\pi(\theta \mid x) = \frac{f(x,\theta)}{f(x)} = \frac{f(x \mid \theta) \times \pi(\theta)}{f(x)}$$ – posterior distribution of θ, given x.

1. Alex has a special coin he uses to make decisions in class. We do not know whether the coin is fair or not, so we assign the following prior probability distribution on the probability of "tails" p:

$$\mathrm{P}(p = 0.40) = 0.25, \qquad \mathrm{P}(p = 0.50) = 0.50, \qquad \mathrm{P}(p = 0.60) = 0.25.$$

Find the posterior distribution of p, given that we observe $x = 2$ "tails" in $n = 10$ coin tosses.

$$\mathrm{P}(\mathrm{X} = 2 \mid p = 0.40) = \binom{10}{2}(0.40)^2(0.60)^8 \approx 0.121,$$

$$\mathrm{P}(\mathrm{X} = 2 \mid p = 0.50) = \binom{10}{2}(0.50)^2(0.50)^8 \approx 0.044,$$

$$\mathrm{P}(\mathrm{X} = 2 \mid p = 0.60) = \binom{10}{2}(0.60)^2(0.40)^8 \approx 0.011.$$

$$\mathrm{P}(\mathrm{X} = 2) = 0.25 \times 0.121 + 0.50 \times 0.044 + 0.25 \times 0.011 = 0.055.$$

$$\mathrm{P}(p = 0.40 \mid \mathrm{X} = 2) = \frac{0.25 \times 0.121}{0.25 \times 0.121 + 0.50 \times 0.044 + 0.25 \times 0.011} = \mathbf{0.55}.$$

$$\mathrm{P}(p = 0.50 \mid \mathrm{X} = 2) = \frac{0.50 \times 0.044}{0.25 \times 0.121 + 0.50 \times 0.044 + 0.25 \times 0.011} = \mathbf{0.40}.$$

$$\mathrm{P}(p = 0.60 \mid \mathrm{X} = 2) = \frac{0.25 \times 0.011}{0.25 \times 0.121 + 0.50 \times 0.044 + 0.25 \times 0.011} = \mathbf{0.05}.$$

2. Let X have a Poisson distribution with mean λ. The prior probability distribution of λ is

$$P(\lambda = 1) = 0.40, \qquad P(\lambda = 2) = 0.60.$$

Find the posterior distribution of λ, given that we observe $x = 4$.

$$P(X = 4 \mid \lambda = 1) = \frac{1^4 e^{-1}}{4!} \approx 0.015,$$

$$P(X = 4 \mid \lambda = 2) = \frac{2^4 e^{-2}}{4!} \approx 0.090.$$

$$P(X = 4) = 0.40 \times 0.015 + 0.60 \times 0.090 = 0.060.$$

$$P(\lambda = 1 \mid X = 4) = \frac{0.40 \times 0.015}{0.40 \times 0.015 + 0.60 \times 0.090} = \frac{0.006}{0.060} = \mathbf{0.10}.$$

$$P(\lambda = 2 \mid X = 4) = \frac{0.60 \times 0.090}{0.40 \times 0.015 + 0.60 \times 0.090} = \frac{0.054}{0.060} = \mathbf{0.90}.$$

3. Let X have a Exponential distribution with mean θ. The prior probability distribution of θ is

$$P(\theta = 1) = 0.70, \qquad P(\theta = 2) = 0.30.$$

Find the posterior distribution of θ, given that the observed X is between 2 and 3.

$$P(2 < X < 3 \mid \theta = 1) = e^{-2} - e^{-3} \approx 0.08555.$$

$$P(2 < X < 3 \mid \theta = 2) = e^{-1} - e^{-1.5} \approx 0.14475.$$

$$P(2 < X < 3) = 0.70 \times 0.08555 + 0.30 \times 0.14475 = 0.10331.$$

$$P(\theta = 1 \mid 2 < X < 3) = \frac{0.70 \times 0.08555}{0.10331} = \mathbf{0.58}.$$

$$P(\theta = 2 \mid 2 < X < 3) = \frac{0.30 \times 0.14475}{0.10331} = \mathbf{0.42}.$$

4. ~ 6.8-1 ~ 9.2-1 ~ 7.2-1 (STAT 400 textbook)

Let Y be the sum of the observations of a random sample from a Poisson distribution with mean λ. Let the prior p.d.f. of λ be $\text{Gamma}(\alpha, \theta)$.

Recall: The maximum likelihood estimator of λ is $\hat{\lambda} = \dfrac{Y}{n}$.

a) Find the posterior p.d.f. of λ, given that $Y = y$.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Poisson distribution with mean λ. Then $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $n\lambda$.

$$f(y \mid \lambda) = \frac{(n\lambda)^y \cdot e^{-n\lambda}}{y!}.$$

$$f(y, \lambda) = f(y \mid \lambda) \times \pi(\lambda)$$

$$= \frac{(n\lambda)^y \cdot e^{-n\lambda}}{y!} \times \frac{1}{\Gamma(\alpha)\, \theta^{\alpha}}\, \lambda^{\alpha-1} e^{-\lambda/\theta}$$

$$= \ldots\ \lambda^{y+\alpha-1} e^{-\lambda\left(n+\frac{1}{\theta}\right)}.$$

$\Rightarrow$ the posterior distribution of λ, given $Y = y$, is **Gamma** with

New $\alpha = y + \alpha$ and New $\theta = \dfrac{1}{n + \frac{1}{\theta}}$.

b) Find the conditional mean of λ, given that $Y = y$. Show that it is a weighted average of the maximum likelihood estimate $\hat{\lambda}$ and the prior mean $\alpha\,\theta$.

$$(\text{conditional mean of } \lambda, \text{ given } Y = y) = (\text{New } \alpha) \times (\text{New } \theta) = \frac{y+\alpha}{n+\frac{1}{\theta}}.$$

$$\frac{y+\alpha}{n+\frac{1}{\theta}} = \frac{y}{n}\cdot\frac{n}{n+\frac{1}{\theta}} + \alpha\,\theta\cdot\frac{\frac{1}{\theta}}{n+\frac{1}{\theta}} = \hat{\lambda}\cdot\frac{n}{n+\frac{1}{\theta}} + \alpha\,\theta\cdot\frac{\frac{1}{\theta}}{n+\frac{1}{\theta}}.$$

c) Use part (a) to construct a $(1-\gamma)\,100\,\%$ credible interval for λ, given that $Y = y$. That is, construct an interval estimate for λ with posterior probability $(1-\gamma)$.

$2\left(n+\frac{1}{\theta}\right)(\lambda\,|\,y)$ has a $\chi^2(2y+2\alpha)$ distribution.

$$\Rightarrow \quad P\left(\chi^2_{1-\gamma/2}(2y+2\alpha) < 2\left(n+\frac{1}{\theta}\right)(\lambda\,|\,y) < \chi^2_{\gamma/2}(2y+2\alpha)\right) = 1-\gamma.$$

$$\Rightarrow \quad P\left(\frac{\chi^2_{1-\gamma/2}(2y+2\alpha)}{2\left(n+\frac{1}{\theta}\right)} < (\lambda\,|\,y) < \frac{\chi^2_{\gamma/2}(2y+2\alpha)}{2\left(n+\frac{1}{\theta}\right)}\right) = 1-\gamma.$$

$$\Rightarrow \quad \left(\frac{\chi^2_{1-\gamma/2}(2y+2\alpha)}{2\left(n+\frac{1}{\theta}\right)},\ \frac{\chi^2_{\gamma/2}(2y+2\alpha)}{2\left(n+\frac{1}{\theta}\right)}\right)$$

is a $(1-\gamma)\,100\%$ credible interval for λ, given $Y = y$.