Examples for 09/16/2020 (Disc)

Fall 2020
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Examples for 09/09/2020 (Disc) (continued)

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 16 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let the joint probability density function of X and Y be defined by

$$f(x,y) = \frac{x+4y}{9}, \qquad 0 < y < 1, \quad y < x < 3, \qquad \text{zero otherwise.}$$

Recall:

$$f_X(x) = \begin{cases} \dfrac{x^2}{3} & 0 < x < 1 \\ \dfrac{x+2}{9} & 1 < x < 3 \end{cases}$$

$$f_Y(y) = \frac{3+8y-3y^2}{6} = \frac{1}{2} + \frac{4}{3}y - \frac{1}{2}y^2, \qquad 0 < y < 1.$$

f) Find $P(Y > 0.6 \mid X = 0.8)$.

g) Find $P(Y > 0.6 \mid X = 1.2)$.

h) Find $P(X < 1.5 \mid Y = 0.5)$.

i) Find $P(X < 1.5 \mid Y > 0.5)$.

j) Find $E(X \mid Y = y)$.

k) Find $E(Y \mid X = x)$.

2. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = \frac{1}{32}x^2 y, \qquad 0 < x < 4, \;\; 0 < y < \sqrt{x}, \qquad \text{zero elsewhere.}$$

Recall:

$$f_X(x) = \frac{1}{64}x^3, \qquad 0 < x < 4.$$

$$f_Y(y) = \frac{y}{96}\cdot\left(64 - y^6\right) = \frac{2}{3}y - \frac{1}{96}y^7, \qquad 0 < y < 2.$$

f) Find $P(Y > 1 \mid X < 3)$.

g) Find $P(Y > 1 \mid X = 3)$.

h) Find $E(Y \mid X = x)$.

i) Find $P(X < 3 \mid Y > 1)$.

j) Find $P(X < 3 \mid Y = 1)$.

k) Find $E(X \mid Y = y)$.

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} 40\,x\,y^3 & 0 < x < 1, \;\; 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

i) Find $P(X < 0.7 \mid Y = 0.40)$.

j) Find $P(X > 0.8 \mid Y = 0.60)$.

k) Find $E(X \mid Y = y)$.

l) Find $P(Y < 0.2 \mid X = 0.50)$.

m) Find $P(Y > 0.4 \mid X = \frac{2}{3})$.

n) Find $E(Y \mid X = x)$.

4. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{9}{2\,(\,2x+y\,)^3}, \qquad y>1, \quad 0<x<y, \qquad \text{zero elsewhere.}$$

h) Find $P(\,Y<3 \mid X=\frac{1}{2}\,)$.

i) Find $P(\,Y<3 \mid X=2\,)$.

j) Find $E(\,Y \mid X=x\,)$.

k) Find $P(\,X>1 \mid Y=3\,)$.

l) Find $E(\,X \mid Y=y\,)$.

1. Let the joint probability density function of X and Y be defined by

$$f(x,y) = \frac{x+4y}{9}, \qquad 0<y<1, \quad y<x<3, \qquad \text{zero otherwise.}$$

Recall:
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$$f_Y(y) = \frac{3+8y-3y^2}{6} = \frac{1}{2}+\frac{4}{3}y-\frac{1}{2}y^2, \qquad 0<y<1.$$

f) Find $P(Y>0.6 \mid X=0.8)$.

For $0<x<1$, $\quad f_X(x) = \dfrac{x^2}{3}.$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+4y}{3x^2}, \qquad 0<y<x.$$

$$f_{Y|X}(y|0.8) = \frac{0.8+4y}{3\cdot 0.8^2} = \frac{2+10y}{4.8}, \qquad 0<y<0.8.$$

$$P(Y>0.6 \mid X=0.8) = \int_{0.6}^{0.8} \frac{2+10y}{4.8}\,dy = \left(\frac{2y+5y^2}{4.8}\right)\Bigg|_{y=0.6}^{y=0.8} = \mathbf{\frac{3}{8}} = \mathbf{0.375}.$$

g) Find $P(Y>0.6 \mid X=1.2)$.

For $1<x<3$, $\quad f_X(x) = \dfrac{x+2}{9}.$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+4y}{x+2}, \qquad 0<y<1.$$

$$f_{Y|X}(y|1.2) = \frac{1.2+4y}{1.2+2} = \frac{3+10y}{8}, \qquad 0 < y < 1.$$

$$P(Y > 0.6 \mid X = 1.2) = \int_{0.6}^{1} \frac{3+10y}{8}\,dy = \left(\frac{3y+5y^2}{8}\right)\Bigg|_{y=0.6}^{y=1} = \mathbf{0.55}.$$

h) Find $P(X < 1.5 \mid Y = 0.5)$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2x+8y}{9+24y-9y^2}, \qquad y < x < 3.$$

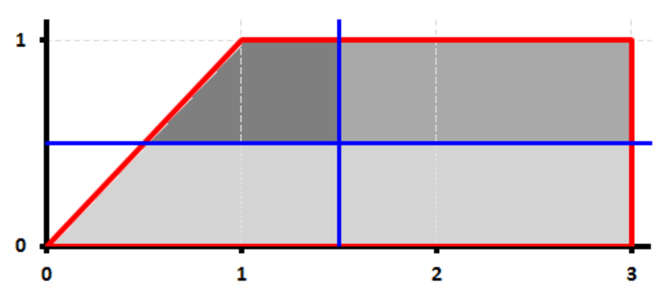
$$f_{X|Y}(x|0.5) = \frac{2x+4}{18.75}, \qquad 0.5 < x < 3.$$

$$P(X < 1.5 \mid Y = 0.5) = \int_{0.5}^{1.5} \frac{2x+4}{18.75}\,dx = \left(\frac{x^2+4x}{18.75}\right)\Bigg|_{x=0.5}^{x=1.5} = \mathbf{0.32}.$$

i) Find $P(X < 1.5 \mid Y > 0.5)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

provided $P(B) > 0$.



$$P(B) = P(Y > 0.5)$$

$$= \int_{0.5}^{1} \frac{3+8y-3y^2}{6}\,dy$$

$$= \left(\frac{3y+4y^2-y^3}{6}\right)\Bigg|_{y=0.5}^{y=1} = \frac{29}{48}.$$

$$\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{X} < 1.5 \cap \mathrm{Y} > 0.5) = \int_{0.5}^{1} \left(\int_{y}^{1.5} \frac{x+4y}{9}\, dx \right) dy$$

$$= \int_{0.5}^{1} \left(\frac{x^2+8xy}{18} \right) \Bigg|_{x=y}^{x=1.5} dy = \int_{0.5}^{1} \frac{9+48y-36y^2}{72}\, dy$$

$$= \left(\frac{9y+24y^2-12y^3}{72} \right) \Bigg|_{y=0.5}^{y=1} = \frac{1}{6}.$$

$$\mathrm{P}(\mathrm{X} < 1.5 \mid \mathrm{Y} > 0.5) = \frac{\mathrm{P}(\mathrm{A} \cap \mathrm{B})}{\mathrm{P}(\mathrm{B})} = \frac{\frac{1}{6}}{\frac{29}{48}} = \mathbf{\frac{8}{29}} \approx 0.275862.$$

j) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y)$.

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y} = y) = \int_{y}^{3} x \cdot \frac{2x+8y}{9+24y-9y^2}\, dx = \frac{\frac{2}{3}x^3+4x^2y}{9+24y-9y^2} \Bigg|_{x=y}^{x=3}$$

$$= \frac{18+36y-\frac{14}{3}y^3}{9+24y-9y^2} = \frac{54+108y-14y^3}{27+72y-27y^2}$$

$$= \frac{18+42y+14y^2}{9+27y}, \qquad 0 < y < 1.$$

k) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$.

For $0 < x < 1$, $\qquad f_{\mathrm{Y}|\mathrm{X}}(y|x) = \dfrac{x+4y}{3x^2}, \qquad 0 < y < x.$

$$E(Y \mid X = x) = \int_0^x y \cdot \frac{x+4y}{3x^2}\, dy = \frac{\frac{1}{2}xy^2 + \frac{4}{3}y^3}{3x^2} \Bigg|_{y=0}^{y=x}$$

$$= \frac{11}{18}x, \qquad 0 < x < 1.$$

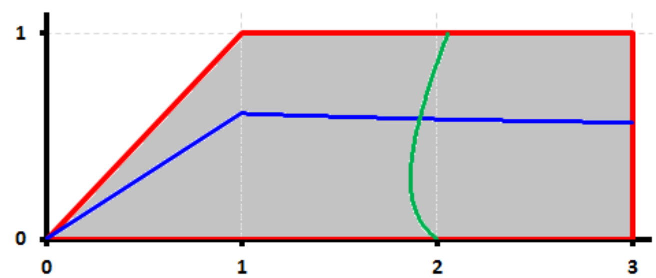
For $1 < x < 3$, $\qquad f_{Y|X}(y|x) = \frac{x+4y}{x+2}, \qquad 0 < y < 1.$

$$E(Y \mid X = x) = \int_0^1 y \cdot \frac{x+4y}{x+2}\, dy = \frac{\frac{1}{2}xy^2 + \frac{4}{3}y^3}{x+2} \Bigg|_{y=0}^{y=1}$$

$$= \frac{3x+8}{6x+12}, \qquad 1 < x < 3.$$

$E(X \mid Y = y)$

$E(Y \mid X = x)$



2. Let X and Y have the joint p.d.f.

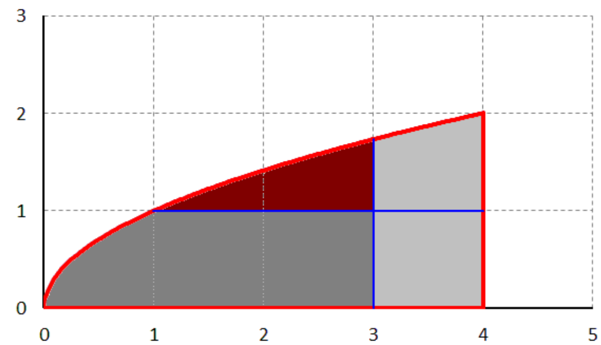
$$f_{X,Y}(x,y) = \frac{1}{32}x^2 y, \qquad 0<x<4, \quad 0<y<\sqrt{x}, \qquad \text{zero elsewhere.}$$

Recall:

$$f_X(x) = \frac{1}{64}x^3, \qquad 0<x<4.$$

$$f_Y(y) = \frac{y}{96}\cdot\left(64-y^6\right) = \frac{2}{3}y - \frac{1}{96}y^7, \qquad 0<y<2.$$

f) Find $P(Y>1 \mid X<3)$.



$$P(A \mid B) = \frac{P(A\cap B)}{P(B)},$$

provided $P(B)>0$.

$$P(B) = P(X<3) = \int_0^3 \frac{1}{64}x^3\,dx = \frac{81}{256} \approx 0.3164.$$

$$P(A\cap B) = P(Y>1 \cap X<3) = \int_1^3\left(\int_1^{\sqrt{x}} \frac{1}{32}x^2 y\,dy\right)dx = \frac{17}{96} \approx 0.1771.$$

$$P(Y>1 \mid X<3) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{17}{96}}{\frac{81}{256}} = \mathbf{\frac{136}{243}} \approx 0.5597.$$

g) Find $P(Y>1 \mid X=3)$.

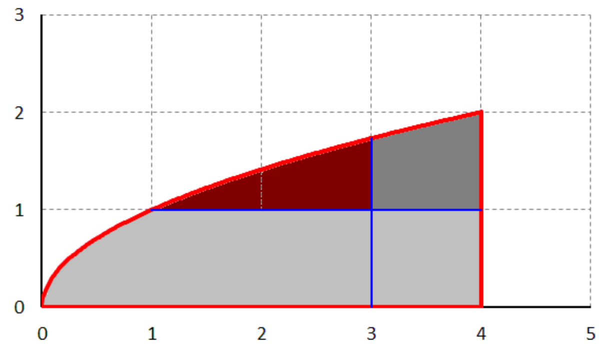
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2y}{x}, \qquad 0<y<\sqrt{x}.$$

$$P(Y>1 \mid X=3) = \int_1^{\sqrt{3}} \frac{2y}{3}\,dy = \mathbf{\frac{2}{3}}.$$

h) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \int_0^{\sqrt{x}} y \cdot \frac{2y}{x}\, dy = \frac{2}{3}\sqrt{x}, \qquad 0 < x < 4.$$

i) Find $P(X < 3 \mid Y > 1)$.



$$P(B \mid A) = \frac{P(A \cap B)}{P(A)},$$

provided $P(A) > 0$.

$$P(A) = P(Y < 1) = \int_1^2 \left(\frac{2}{3}y - \frac{1}{96}y^7\right) dy = \frac{171}{256} \approx 0.6680.$$

$$P(A \cap B) = P(Y > 1 \cap X < 3) = \int_1^3 \left(\int_1^{\sqrt{x}} \frac{1}{32} x^2 y\, dy \right) dx = \frac{17}{96} \approx 0.1771.$$

$$P(X < 3 \mid Y > 1) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{17}{96}}{\frac{171}{256}} = \mathbf{\frac{136}{513}} \approx 0.2651.$$

j) Find $P(X < 3 \mid Y = 1)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{3x^2}{64 - y^6}, \qquad y^2 < x < 4.$$

$$P(X < 3 \mid Y = 1) = \int_1^3 \frac{3x^2}{64 - 1^6}\, dx = \mathbf{\frac{26}{63}} \approx 0.4127.$$

k) Find $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \int_{y^2}^{4} x \cdot \frac{3x^2}{64 - y^6}\, dx = \frac{3}{4} \cdot \frac{256 - y^8}{64 - y^6} = \frac{3}{4} \cdot \frac{64 + 16y^2 + 4y^4 + y^6}{16 + 4y^2 + y^4},$$

$$0 < y < 2.$$

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} 40\,x\,y^3 & 0 < x < 1,\ \ 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

i) Find $P(X < 0.7 \mid Y = 0.40)$.

$$f_Y(y) = \int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx = 20\,y^3 - 20\,y^4, \qquad 0 < y < 1.$$

$$f_{X|Y}(x \mid y) = \frac{2\,x}{1-y}, \qquad \sqrt{y} < x < 1, \qquad 0 < y < 1.$$

$$P(X < 0.7 \mid Y = 0.40) = \int_{\sqrt{0.40}}^{0.70} \frac{2\,x}{1-0.40}\,dx = \frac{0.49 - 0.40}{0.60} = \mathbf{0.15}.$$

j) Find $P(X > 0.8 \mid Y = 0.60)$.

$$P(X > 0.8 \mid Y = 0.60) = \int_{0.80}^{1} \frac{2\,x}{1-0.60}\,dx = \frac{1-0.64}{0.40} = \mathbf{0.90}.$$

k) Find $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \int_{\sqrt{y}}^{1} x \cdot \frac{2\,x}{1-y}\,dx = \frac{2}{3} \cdot \frac{1 - y^{3/2}}{1-y}, \qquad 0 < y < 1.$$

l) Find $P(Y < 0.2 \mid X = 0.50)$.

$$f_X(x) = \int_0^{x^2} 40\,x\,y^3\,dy = 10\,x^9, \qquad 0 < x < 1.$$

$$f_{Y|X}(y \mid x) = \frac{4\,y^3}{x^8}, \qquad 0 < y < x^2, \qquad 0 < x < 1.$$

$$P(Y < 0.2 \mid X = 0.50) = \int_0^{0.2} \frac{4\,y^3}{0.5^8}\,dy = \frac{0.2^4}{0.5^8} = \mathbf{0.4096}.$$

m) Find $P(Y > 0.4 \mid X = \frac{2}{3})$.

$$P(Y > 0.4 \mid X = \tfrac{2}{3}) = \int_{0.4}^{4/9} \frac{4\,y^3}{(2/3)^8}\,dy = \frac{(4/9)^4 - 0.4^4}{(2/3)^8} = \mathbf{0.3439}.$$

n) Find $E(Y \mid X = x)$.

$$E(Y \mid X = x) = \int_0^{x^2} y \cdot \frac{4\,y^3}{x^8}\,dy = \frac{4\,x^2}{5}, \qquad 0 < x < 1.$$

4. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{9}{2\,(\,2x+y\,)^3}, \qquad y>1, \quad 0<x<y, \qquad \text{zero elsewhere.}$$

Recall:

$$f_X(x) = \begin{cases} \dfrac{9}{4\,(\,2x+1\,)^2}, & 0<x<1 \\[2ex] \dfrac{1}{4x^2}, & 1<x<\infty \end{cases}$$

$$f_Y(y) = \frac{1}{y^2}, \qquad 1<y<\infty.$$

h) Find $P(\,Y<3 \mid X=\frac{1}{2}\,)$.

If $0<x<1$,
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2\,(\,2x+1\,)^2}{(\,2x+y\,)^3}, \qquad y>1.$$

$$P(\,Y<3 \mid X=\tfrac{1}{2}\,) = \int_1^3 \frac{8}{(\,1+y\,)^3}\,dy = -\frac{4}{(\,1+y\,)^2}\Bigg|_1^3 = \mathbf{\frac{3}{4}} = \mathbf{0.75}.$$

i) Find $P(\,Y<3 \mid X=2\,)$.

If $x>1$,
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{18x^2}{(\,2x+y\,)^3}, \qquad y>x.$$

$$P(\,Y<3 \mid X=2\,) = \int_2^3 \frac{72}{(\,4+y\,)^3}\,dy = -\frac{36}{(\,4+y\,)^2}\Bigg|_2^3 = \mathbf{\frac{13}{49}} \approx 0.2653.$$

j) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x)$.

If $0<x<1$,

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x) = \int_1^{\infty} y \cdot \frac{2(2x+1)^2}{(2x+y)^3}\, dy$$

$$= 2(2x+1)^2 \cdot \left[\frac{x}{(2x+y)^2} - \frac{1}{(2x+y)} \right] \Bigg|_1^{\infty} = 2(x+1).$$

If $x>1$,

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x) = \int_x^{\infty} y \cdot \frac{18x^2}{(2x+y)^3}\, dy$$

$$= 18x^2 \cdot \left[\frac{x}{(2x+y)^2} - \frac{1}{(2x+y)} \right] \Bigg|_x^{\infty} = 4x.$$

k) Find $\mathrm{P}(\mathrm{X}>1 \mid \mathrm{Y}=3)$.

$$f_{\mathrm{X}|\mathrm{Y}}(x|y) = \frac{f_{\mathrm{X,Y}}(x,y)}{f_{\mathrm{Y}}(y)} = \frac{9y^2}{2(2x+y)^3}, \quad 0<x<y, \quad y>1.$$

$$\mathrm{P}(\mathrm{X}>1 \mid \mathrm{Y}=3) = \int_1^3 \frac{81}{2(2x+3)^3}\, dx = -\frac{81}{8(2x+3)^2} \Bigg|_1^3 = \mathbf{\frac{7}{25}} = \mathbf{0.28}.$$

l) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y)$.

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \int_0^y x \cdot \frac{9y^2}{2(2x+y)^3}\, dx = \ldots$$

$$u = 2x+y \qquad x = \frac{u-y}{2} \qquad dx = \frac{1}{2}\, du$$

$$\ldots = \int_y^{3y} \frac{9y^2(u-y)}{8u^3}\, dx = \frac{9y^2}{8} \cdot \left[-\frac{1}{u} + \frac{y}{u^2} \right] \Bigg|_y^{3y} = \frac{y}{4}, \quad y>1.$$

OR

$$E(2X+Y \mid Y=y) = \int_0^y (2x+y) \cdot \frac{9y^2}{2(2x+y)^3}\,dx = \int_0^y \frac{9y^2}{2(2x+y)^2}\,dx$$

$$= -\frac{9y^2}{4(2x+y)}\Bigg|_0^y = \frac{3y}{2}.$$

Also, $\quad E(2X+Y \mid Y=y) = 2\,E(X \mid Y=y)+y.$

$$2\,E(X \mid Y=y)+y = \frac{3y}{2}. \qquad \Rightarrow \qquad E(X \mid Y=y) = \frac{y}{4}.$$