STAT 410 **Examples for 10/30/2020 (3)** Fall 2020
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Examples for 10/19/2020 (2) and Examples for 10/23/2020 (2) (continued)

1. Let $\beta > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\beta) = \frac{\beta^2 \ln x}{x^{\beta+1}}, \qquad x > 1, \qquad \text{zero otherwise.}$$

Recall: $W = \ln X$ has a Gamma$\left(\alpha = 2, \theta = \frac{1}{\beta}\right)$ distribution.

$\Rightarrow$ $Y = \sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} W_i$ has a Gamma$\left(\alpha = 2n, \theta = \frac{1}{\beta}\right)$ distribution.

k) Suggest a confidence interval for β with $(1-\alpha)\,100\,\%$ confidence level.

① Use $Y = \sum_{i=1}^{n} \ln X_i$.

② If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$Y = \sum_{i=1}^{n} \ln X_i = \sum_{i=1}^{n} W_i$ has a Gamma$\left(\alpha = 2n, \theta = \frac{1}{\beta}\right)$ distribution.

$2\beta \sum_{i=1}^{n} \ln X_i$ has a $\chi^2(2\alpha = 4n)$ distribution.

$$\Rightarrow \quad P\left(\chi^2_{1-\alpha/2}(4n) < 2\beta \sum_{i=1}^{n} \ln X_i < \chi^2_{\alpha/2}(4n)\right) = 1-\alpha.$$

$$\Rightarrow \quad P\left(\frac{\chi^2_{1-\alpha/2}(4n)}{2\sum_{i=1}^{n} \ln X_i} < \beta < \frac{\chi^2_{\alpha/2}(4n)}{2\sum_{i=1}^{n} \ln X_i}\right) = 1-\alpha.$$

A $(1-\alpha)100\,\%$ confidence interval for β: $$\left(\frac{\chi^2_{1-\alpha/2}(4n)}{2\sum\limits_{i=1}^{n} \ln x_i}, \frac{\chi^2_{\alpha/2}(4n)}{2\sum\limits_{i=1}^{n} \ln x_i} \right).$$

l) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2.0$, $x_4 = 3.0$, $x_5 = 5.0$.
Use part (k) to construct a 95% confidence interval for β.

$\chi^2_{0.975}(20) = 9.591$, $\chi^2_{0.025}(20) = 34.17$.

$$\sum_{i=1}^{n} \ln x_i = \ln 1.3 + \ln 1.4 + \ln 2.0 + \ln 3.0 + \ln 5.0 \approx 4.$$

$$\left(\frac{\chi^2_{1-\alpha/2}(4n)}{2\sum\limits_{i=1}^{n} \ln x_i}, \frac{\chi^2_{\alpha/2}(4n)}{2\sum\limits_{i=1}^{n} \ln x_i} \right) = \left(\frac{9.591}{2 \cdot 4}, \frac{34.17}{2 \cdot 4} \right) \approx \mathbf{(1.20, 4.27)}.$$

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> x = c(1.3,1.4,2.0,3.0,5.0)
> y = sum(log(x))
> y
[1] 4.000034
> qchisq(0.025,4*5)
[1] 9.590777
> qchisq(0.025,4*5)/(2*y)
[1] 1.198837
> qchisq(0.975,4*5)
[1] 34.16961
> qchisq(0.975,4*5)/(2*y)
[1] 4.271165
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Recall: $\hat{\beta} = \dfrac{2n}{\sum_{i=1}^{n} \ln X_i}$ is the maximum likelihood estimator for β.

m) Show that $\hat{\beta}$ is asymptotically normally distributed (as $n \to \infty$). Find the parameters.

① By CLT, $\sqrt{n}\left(\overline{W} - \mu_W\right) \xrightarrow{D} N\left(0, \sigma_W^2\right)$.

② If $g(x)$ is differentiable at μ and $g'(\mu) \neq 0$, then

$$\sqrt{n}\left(g\left(\overline{W}\right) - g\left(\mu_W\right)\right) \xrightarrow{D} N\left(0, \left[g'\left(\mu_W\right)\right]^2 \sigma_W^2\right).$$

That is, for large n,

$$g\left(\overline{W}\right) \text{ is approximately } N\left(g(\mu_W), [g'(\mu_W)]^2 \frac{\sigma_W^2}{n}\right).$$

$W = \ln X$ has a $\text{Gamma}\left(\alpha = 2, \theta = \frac{1}{\beta}\right)$ distribution.

By CLT, $\sqrt{n}\left(\overline{W} - \mu_W\right) \xrightarrow{D} N\left(0, \sigma_W^2\right)$.

$$\sqrt{n}\left(\overline{W} - \frac{2}{\beta}\right) \xrightarrow{D} N\left(0, \frac{2}{\beta^2}\right).$$

$g(x) = \dfrac{2}{x}$. $\quad g\left(\overline{W}\right) = \hat{\beta}$. $\quad g\left(\dfrac{2}{\beta}\right) = \beta$.

$g'(x) = -\dfrac{2}{x^2}$. $\quad g'\left(\dfrac{2}{\beta}\right) = -\dfrac{\beta^2}{2}$. $\quad \left(-\dfrac{\beta^2}{2}\right)^2 \cdot \dfrac{2}{\beta^2} = \dfrac{\beta^2}{2}$.

$$\sqrt{n}\left(g\left(\overline{W}\right) - g\left(\frac{2}{\beta}\right)\right) = \sqrt{n}\left(\hat{\beta} - \beta\right) \xrightarrow{D} N\left(0, \frac{\beta^2}{2}\right).$$

For large n, $\hat{\beta} \sim N\left(\beta, \dfrac{\beta^2}{2n}\right)$.