STAT 410 **Examples for 10/19/2020 (1)** Fall 2020
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1. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

That is, find $\hat{\lambda} = \arg\max \mathrm{L}(\lambda) = \arg\max \ln \mathrm{L}(\lambda)$, where $\mathrm{L}(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$.

Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.

Find the maximum likelihood estimate of λ.

b) What is the probability distribution of $W = X^2$?

c) Suppose $n = 5$ and $\lambda = 0.2$. Find $P(\sum_{i=1}^{n} X_i^2 < 35)$.

Hint: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$F_T(t) = P(T \le t) = P(Y \ge \alpha)$ and $P(T > t) = P(Y \le \alpha - 1)$,

where Y has a $\text{Poisson}(\lambda t)$ distribution.

d) Suppose $n = 5$ and $\lambda = 0.2$. Find c such that $P(\sum_{i=1}^{n} X_i^2 < c) = 0.01$.

Hint: If T has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

e) Is the maximum likelihood estimator of λ, $\hat{\lambda}$, an unbiased estimator of λ?

If $\hat{\lambda}$ is not an unbiased estimator of λ, construct an unbiased estimator of λ based on $\hat{\lambda}$.

f) Find $\mathrm{MSE}(\hat{\lambda}) = \mathrm{E}\left[(\hat{\lambda} - \lambda)^2\right] = \left(\mathrm{bias}(\hat{\lambda})\right)^2 + \mathrm{Var}(\hat{\lambda})$.

g) Find $\mathrm{E}(\mathrm{X}^k)$, $k > -4$.

Hint 1: Consider $u = \lambda x^2$ or $u = x^2$.

Hint 2: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du$, $a > 0$.

Hint 3: $\dfrac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}$ is the p.d.f. of $\mathrm{Gamma}(\alpha,\ \theta = \frac{1}{\lambda})$ distribution

Hint 4: If T_α has a $\mathrm{Gamma}(\alpha, \theta = {}^1\!/_\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^k) = \frac{\theta^k\, \Gamma(\alpha + k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha + k)}{\lambda^k\, \Gamma(\alpha)}, \qquad k > -\alpha.$$

h) Obtain a method of moments estimator of λ, $\tilde{\lambda}$.

That is, if $\mathrm{E}(\mathrm{X}) = h(\lambda)$, solve $\overline{\mathrm{X}} = h(\tilde{\lambda})$ for $\tilde{\lambda}$.

Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.

Find a method of moments estimate of λ.

Answers:

1. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

That is, find $\hat{\lambda} = \arg\max \mathrm{L}(\lambda) = \arg\max \ln \mathrm{L}(\lambda)$, where $\mathrm{L}(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$.

Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.

Find the maximum likelihood estimate of λ.

$$\mathrm{L}(\lambda) = \prod_{i=1}^{n}\left(2\lambda^2 x_i^3 e^{-\lambda x_i^2}\right) = 2^n \lambda^{2n}\left(\prod_{i=1}^{n} x_i^3\right) e^{-\lambda \sum_{i=1}^{n} x_i^2}.$$

$$\ln \mathrm{L}(\lambda) = n \cdot \ln 2 + 2n \cdot \ln\lambda + \sum_{i=1}^{n} \ln\left(x_i^3\right) - \lambda \cdot \sum_{i=1}^{n} x_i^2.$$

$$(\ln \mathrm{L}(\lambda))' = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} X_i^2}.$$

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\qquad \sum_{i=1}^{n} x_i^2 = 40.$

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} x_i^2} = \mathbf{0.25}.$$

b) What is the probability distribution of $W = X^2$?

Why $W = X^2$? Because the maximum likelihood estimator $\hat{\lambda}$ is made out of them. If we want to know more about $\hat{\lambda}$, we may want to know more about the distribution of $W = X^2$.

Let $W = X^2$ $\qquad X = \sqrt{W} = v(W)$ $\qquad v'(w) = \dfrac{1}{2\sqrt{w}}$

$$f_W(w) = f_X(v(w)) \cdot |v'(w)| = 2\lambda^2 w^{3/2} e^{-\lambda w} \cdot \frac{1}{2\sqrt{w}} = \lambda^2 w e^{-\lambda w}$$

$$= \frac{\lambda^2}{\Gamma(2)} w^{2-1} e^{-\lambda w}, \qquad w > 0.$$

$\Rightarrow$ W has Gamma$\left(\alpha = 2, \theta = \dfrac{1}{\lambda}\right)$ distribution.

Suppose X and Y are independent, X is Gamma(α_1, θ), Y is Gamma(α_2, θ).

If random variables X and Y are independent, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

$$\Rightarrow \quad M_{X+Y}(t) = \frac{1}{(1-\theta t)^{\alpha_1}} \cdot \frac{1}{(1-\theta t)^{\alpha_2}} = \frac{1}{(1-\theta t)^{\alpha_1+\alpha_2}}, \qquad t < \frac{1}{\theta}.$$

$\Rightarrow$ X + Y is Gamma$(\alpha_1 + \alpha_2, \theta)$;

$\Rightarrow$ $\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} W_i$ has Gamma$\left(\alpha = 2n, \theta = \dfrac{1}{\lambda}\right)$ distribution.

c) Suppose $n = 5$ and $\lambda = 0.2$. Find $P\left(\sum_{i=1}^{n} X_i^2 < 35\right)$.

Hint: If T has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$F_T(t) = P(T \le t) = P(Y \ge \alpha)$ and $P(T > t) = P(Y \le \alpha - 1)$,

where Y has a Poisson(λt) distribution.

$\sum_{i=1}^{n} X_i^2$ has Gamma$\left(\alpha = 2n, \theta = \dfrac{1}{\lambda}\right)$ distribution.

$\Rightarrow \qquad \sum_{i=1}^{n} X_i^2$ has Gamma($\alpha = 10, \theta = 5$) distribution.

$$P\left(\sum_{i=1}^{n} X_i^2 < 35 \right) = P\left(\text{Gamma}(\alpha = 10, \theta = 5) < 35 \right)$$

$$= P\left(\text{Poisson}(0.2 \cdot 35) \geq 10 \right) = 1 - P\left(\text{Poisson}(7) \leq 9 \right)$$

$$= 1 - 0.830 = \mathbf{0.170}.$$

d) Suppose $n = 5$ and $\lambda = 0.2$. Find c such that $P\left(\sum_{i=1}^{n} X_i^2 < c \right) = 0.01$.

Hint: If T has a Gamma($\alpha, \theta = 1/\lambda$) distribution, where α is an integer, then $2T/\theta = 2\lambda T$ has a $\chi^2(2\alpha)$ distribution (a chi-square distribution with 2α degrees of freedom).

$\sum_{i=1}^{n} X_i^2$ has Gamma($\alpha = 2n, \theta = \frac{1}{\lambda}$) distribution.

$\Rightarrow \qquad \sum_{i=1}^{n} X_i^2$ has Gamma($\alpha = 10, \theta = 5$) distribution.

$$P\left(\sum_{i=1}^{n} X_i^2 < c \right) = P\left(2 \cdot 0.2 \cdot \sum_{i=1}^{n} X_i^2 < 2 \cdot 0.2 \cdot c \right) = P\left(\chi^2(20) < 0.4 \cdot c \right) = 0.01.$$

$\Rightarrow \qquad 0.4\, c = \chi^2_{0.99}(20) = 8.26.$

$\Rightarrow \qquad c = \mathbf{20.65}.$

e) Is the maximum likelihood estimator of λ, $\hat{\lambda}$, an unbiased estimator of λ? If $\hat{\lambda}$ is not an unbiased estimator of λ, construct an unbiased estimator of λ based on $\hat{\lambda}$.

$Y = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} W_i$ has Gamma$(\alpha = 2n, \theta = \frac{1}{\lambda})$ distribution.

$$\hat{\lambda} = \frac{2n}{Y}. \qquad E(\hat{\lambda}) = E\left(\frac{2n}{Y}\right) = 2n \cdot E\left(\frac{1}{Y}\right).$$

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^k) = \frac{\theta^k \, \Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{\lambda^k \, \Gamma(\alpha)}, \qquad k > -\alpha.$$

$$\Rightarrow \quad E\left(\frac{1}{Y}\right) = E(Y^{-1}) = \frac{\lambda}{(\alpha-1)} = \frac{\lambda}{2n-1}.$$

$$E(\hat{\lambda}) = E\left(\frac{2n}{Y}\right) = \frac{2n}{2n-1} \cdot \lambda = \lambda + \frac{\lambda}{2n-1} \neq \lambda.$$

$\hat{\lambda}$ is NOT an unbiased estimator of λ.

Consider $\quad \hat{\hat{\lambda}} = \frac{2n-1}{2n} \cdot \hat{\lambda} = \frac{2n-1}{\sum_{i=1}^{n} X_i^2}.$

Then $\quad E(\hat{\hat{\lambda}}) = \frac{2n-1}{2n} \cdot E(\hat{\lambda}) = \lambda.$

$\hat{\hat{\lambda}}$ is an unbiased estimator of λ.

f) Find $\operatorname{MSE}(\hat{\lambda}) = \mathrm{E}\left[(\hat{\lambda}-\lambda)^2\right] = \left(\operatorname{bias}(\hat{\lambda})\right)^2 + \operatorname{Var}(\hat{\lambda})$.

$$\operatorname{Var}(\hat{\lambda}) = 4n^2 \operatorname{Var}\left(\frac{1}{\mathrm{Y}}\right).$$

$$\operatorname{Var}\left(\frac{1}{\mathrm{Y}}\right) = \mathrm{E}\left(\frac{1}{\mathrm{Y}^2}\right) - \left[\mathrm{E}\left(\frac{1}{\mathrm{Y}}\right)\right]^2.$$

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^k) = \frac{\theta^k\, \Gamma(\alpha+k)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+k)}{\lambda^k\, \Gamma(\alpha)}, \qquad k > -\alpha.$$

$$\Rightarrow \qquad \mathrm{E}\left(\frac{1}{\mathrm{Y}^2}\right) = \mathrm{E}(\mathrm{Y}^{-2}) = \frac{\lambda^2}{(\alpha-1)(\alpha-2)} = \frac{\lambda^2}{(2n-1)(2n-2)}.$$

$$\operatorname{Var}(\hat{\lambda}) = 4n^2 \operatorname{Var}\left(\frac{1}{\mathrm{Y}}\right) = 4n^2 \left[\frac{\lambda^2}{(2n-1)(2n-2)} - \frac{\lambda^2}{(2n-1)^2}\right]$$

$$= \frac{4n^2\lambda^2}{(2n-1)^2\,(2n-2)}.$$

$$\operatorname{MSE}(\hat{\lambda}) = (\operatorname{bias}(\hat{\lambda}))^2 + \operatorname{Var}(\hat{\lambda}) = \frac{\lambda^2}{(2n-1)^2} + \frac{4n^2\lambda^2}{(2n-1)^2\,(2n-2)}$$

$$= \frac{\left(4n^2+2n-2\right)\lambda^2}{(2n-1)^2\,(2n-2)} = \frac{(2n+2)\lambda^2}{(2n-1)(2n-2)}.$$

g) Find $E(X^k)$, $k > -4$. Hint 1: Consider $u = \lambda x^2$ or $u = x^2$.

Hint 2: $\Gamma(a) = \int\limits_0^\infty u^{a-1} e^{-u}\, du, \quad a > 0.$

Hint 3: $\dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}$ is the p.d.f. of Gamma$(\alpha,\ \theta = \frac{1}{\lambda})$ distribution

Hint 4: If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, then

$$E(T_\alpha^m) = \frac{\theta^m\, \Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\, \Gamma(\alpha)}, \qquad m > -\alpha.$$

$$E(X^k) = \int\limits_0^\infty x^k \cdot 2\lambda^2 x^3 e^{-\lambda x^2}\, dx \qquad u = \lambda x^2 \qquad du = 2\lambda x\, dx$$

$$= \lambda \cdot \int\limits_0^\infty \left(\frac{u}{\lambda}\right)^{\frac{k}{2}+1} e^{-u}\, du = \lambda^{-k/2} \cdot \int\limits_0^\infty u^{\frac{k}{2}+1} e^{-u}\, du$$

$$= \lambda^{-k/2}\, \Gamma\left(\frac{k}{2}+2\right).$$

OR

$$E(X^k) = \int\limits_0^\infty x^k \cdot 2\lambda^2 x^3 e^{-\lambda x^2}\, dx \qquad u = x^2 \qquad du = 2x\, dx$$

$$= \lambda^2 \cdot \int\limits_0^\infty u^{\frac{k}{2}+1} e^{-\lambda u}\, du$$

$$= \lambda^{-k/2}\, \Gamma\left(\frac{k}{2}+2\right) \cdot \int\limits_0^\infty \frac{1}{\Gamma\left(\frac{k}{2}+2\right)} \lambda^{\frac{k}{2}+2} u^{\frac{k}{2}+1} e^{-\lambda u}\, du = \lambda^{-k/2}\, \Gamma\left(\frac{k}{2}+2\right),$$

since $\dfrac{1}{\Gamma\left(\frac{k}{2}+2\right)} \lambda^{\frac{k}{2}+2} u^{\frac{k}{2}+1} e^{-\lambda u}$ is the p.d.f. of Gamma$\left(\alpha = \frac{k}{2}+2, \theta = \frac{1}{\lambda}\right)$.

OR

$$\mathrm{E}(\mathrm{X}^k) = \mathrm{E}(\mathrm{W}^{k/2}) = \ldots$$

W has Gamma($\alpha = 2, \theta = \frac{1}{\lambda}$) distribution. $\mathrm{W} = \mathrm{T}_2$.

If T_α has a Gamma($\alpha, \theta = 1/\lambda$) distribution, then

$$\mathrm{E}(\mathrm{T}_\alpha^m) = \frac{\theta^m\,\Gamma(\alpha+m)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+m)}{\lambda^m\,\Gamma(\alpha)}, \qquad m > -\alpha.$$

$$\ldots = \frac{\Gamma\left(2+\frac{k}{2}\right)}{\lambda^{k/2}\,\Gamma(2)} = \lambda^{-k/2}\,\Gamma\left(\frac{k}{2}+2\right).$$

h) Obtain a method of moments estimator of λ, $\tilde{\lambda}$.

That is, if $\mathrm{E}(\mathrm{X}) = h(\lambda)$, solve $\overline{\mathrm{X}} = h(\tilde{\lambda})$ for $\tilde{\lambda}$.

Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.

Find a method of moments estimate of λ.

$$\mathrm{E}(\mathrm{X}) = \lambda^{-1/2}\Gamma\left(\frac{1}{2}+2\right) = \lambda^{-1/2}\cdot\Gamma\left(\frac{5}{2}\right) = \lambda^{-1/2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)$$

$$= \lambda^{-1/2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \lambda^{-1/2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{3}{4}\cdot\sqrt{\frac{\pi}{\lambda}}.$$

$$\frac{3}{4}\cdot\sqrt{\frac{\pi}{\lambda}} = \overline{\mathrm{X}} \qquad \Rightarrow \qquad \tilde{\lambda}_1 = \frac{9\pi}{16\left(\overline{\mathrm{X}}\right)^2}.$$

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\qquad \bar{x} = \frac{12.2}{5} = 2.44.$

$$\tilde{\lambda}_1 = \frac{9\pi}{16\left(\bar{x}\right)^2} \approx 0.09448\,\pi \approx 0.29682.$$

OR

$$E(X^2) = \lambda^{-2/2}\Gamma\left(\frac{2}{2}+2\right) = \lambda^{-1}\cdot\Gamma(3) = \lambda^{-1}\cdot 2! = \frac{2}{\lambda}.$$

$$\frac{2}{\lambda} = \overline{X^2} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i^2 \qquad \Rightarrow \qquad \tilde{\lambda}_2 = \frac{2}{\overline{X^2}} = \frac{2n}{\sum_{i=1}^{n} X_i^2}.$$

$x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$. $\qquad \sum_{i=1}^{n} x_i^2 = 40.$

$$\tilde{\lambda}_2 = \frac{2n}{\sum_{i=1}^{n} x_i^2} = 0.25.$$