Examples for 12/02/2020 (1)

Example 6:

Let X have a Binomial distribution with the number of trials $n = 15$ and with probability of "success" p. We wish to test $\mathrm{H}_0 : p = 0.30$ vs. $\mathrm{H}_1 : p \neq 0.30$.

Recall that $\hat{p} = \dfrac{\mathrm{X}}{n}$ is the maximum likelihood estimator of p.

a) Find the values of $\Lambda(x) = \dfrac{\mathrm{L}(p_0 = 0.30; x)}{\mathrm{L}(\hat{p}; x)}$ for $x = 0, 1, 2, 3, 4, \ldots, 14, 15$.

Reject H_0 if

$$\Lambda = \frac{\mathrm{L}(p_0)}{\mathrm{L}(\hat{p})} = \frac{\binom{n}{\mathrm{X}}(p_0)^{\mathrm{X}}(1-p_0)^{n-\mathrm{X}}}{\binom{n}{\mathrm{X}}\left(\frac{\mathrm{X}}{n}\right)^{\mathrm{X}}\left(1-\frac{\mathrm{X}}{n}\right)^{n-\mathrm{X}}} = \frac{(n\,p_0)^{\mathrm{X}}(n-n\,p_0)^{n-\mathrm{X}}}{(\mathrm{X})^{\mathrm{X}}(n-\mathrm{X})^{n-\mathrm{X}}} \le k$$

EXCEL:

$\Lambda(x)$ = ((15*0.30)^x * (15*0.30)^(15-x)) / (x^x * (15-x)^(15-x)) $x = 1, 2, 3, 4, \ldots, 14,$

$\Lambda(x)$ = 0.70^15 $x = 0,$

$\Lambda(x)$ = 0.30^15 $x = 15.$

OR

= BINOMDIST(x,15,0.30,0) / BINOMDIST(x,15,x/15,0)

If H_0 is true, "in the perfect world" $\mathrm{X} = 15 \times 0.30 = 4.5$.

Note that Λ takes its largest values "around" 4.5.

x		Λ(k)		P(X = x)			
0		0.004748		0.004748		0.004747562	left tail
1		0.080181		0.03052		0.0352676	
2		0.315183		0.09156		0.126827715	
3		0.679783		0.17004		0.296867928	
4		0.960225		0.218623			
5		0.961846		0.20613			
6		0.71267		0.147236		0.27837856	
7		0.399573		0.08113		0.131142573	
8		0.171245		0.03477		0.05001254	
9		0.056099		0.01159		0.015242526	
10		0.013907		0.00298		0.003652521	
11		0.00255		0.000581		0.000672234	
12		0.000332		8.29E-05		9.16587E-05	
13		2.82E-05		8.2E-06		8.71935E-06	
14		1.32E-06		5.02E-07		5.16561E-07	
15		1.43E-08		1.43E-08		1.43489E-08	right tail

b) Likelihood Ratio Test: Reject H_0 if $\Lambda(x) \le k$.

Let $k = 0.15$. Find

(i) the significance level,

(ii) power when $p = 0.20$,

(iii) power when $p = 0.40$

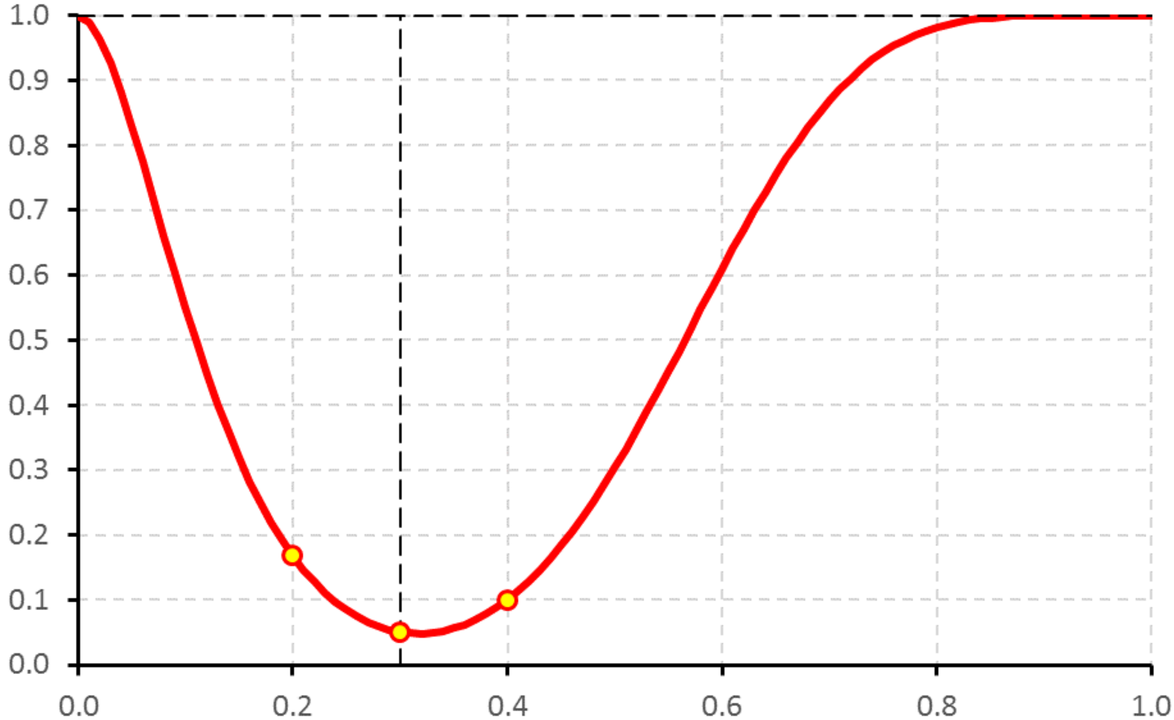
for the corresponding rejection region.

If $k = 0.15$, then $\Lambda \le k \quad \Leftrightarrow \quad X \le 1 \text{ or } X \ge 9$

(i) significance level $= 0.035 + 0.015 = \mathbf{0.050}$.

(ii) $\text{Power}(p = 0.20) = 0.167 + 0.001 = \mathbf{0.168}$.

(iii) $\text{Power}(p = 0.40) = 0.005 + 0.095 = \mathbf{0.100}$.



c) Suppose we observe $X = 7$. Find the p-value of this test.

$\Lambda(7) = 0.399573$.

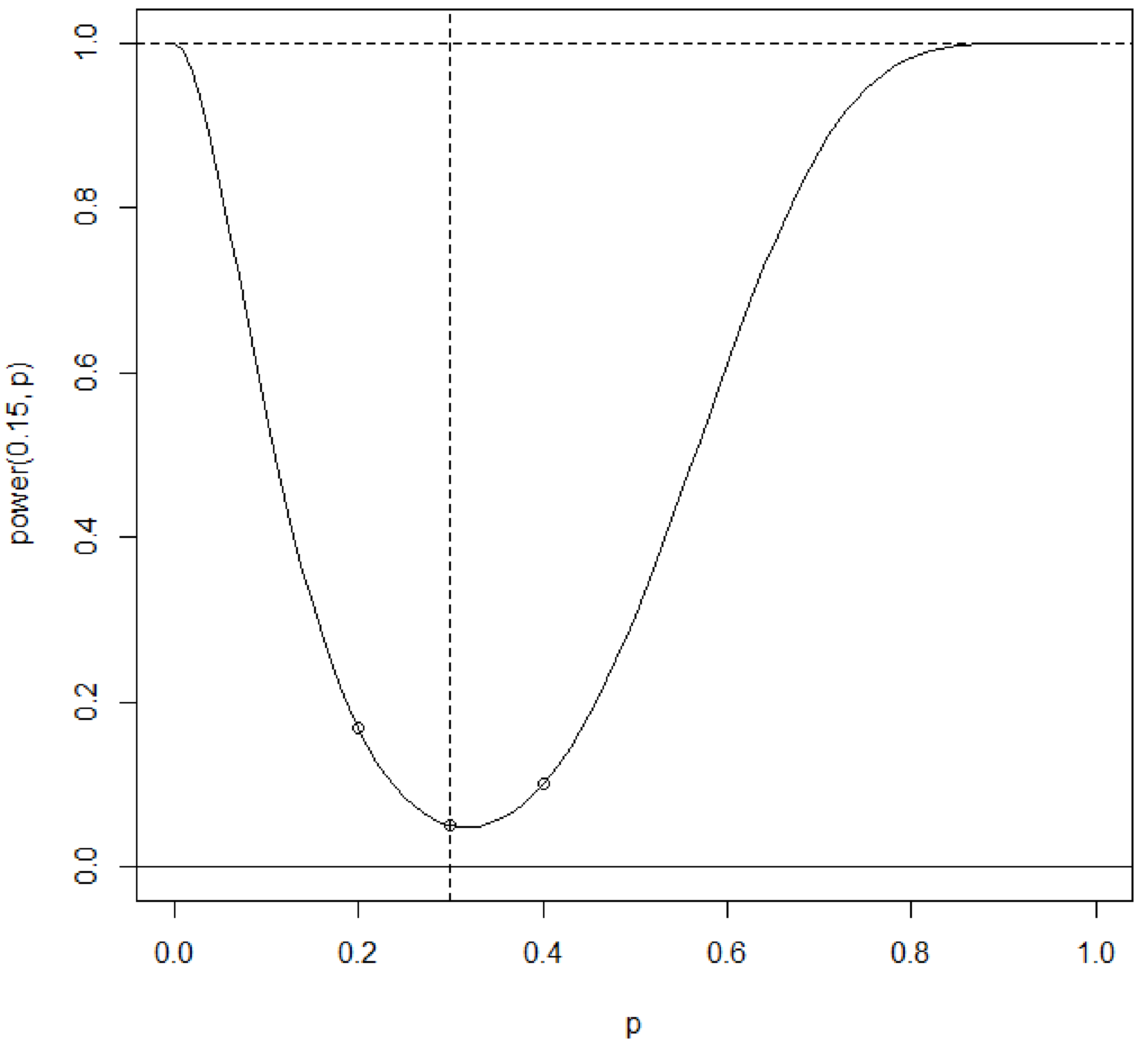
as extreme or more extreme $\Leftrightarrow$ $\Lambda \le 0.399573$ $\Leftrightarrow$ $X \le 2$ or $X \ge 7$

p-value $= 0.127 + 0.131 =$ **0.258**.

OR

```
> x = 0:15
> x
 [1]  0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
> probHo = dbinom(x,15,0.3)
> probHo
 [1] 4.747562e-03 3.052004e-02 9.156011e-02 1.700402e-01 2.186231e-01
 [6] 2.061304e-01 1.472360e-01 8.113003e-02 3.477001e-02 1.159000e-02
[11] 2.980287e-03 5.805754e-04 8.293934e-05 8.202792e-06 5.022117e-07
[16] 1.434891e-08
> Lambda = probHo/dbinom(x,15,x/15)
> Lambda
 [1] 4.747562e-03 8.018077e-02 3.151827e-01 6.797832e-01 9.602246e-01
 [6] 9.618460e-01 7.126703e-01 3.995727e-01 1.712454e-01 5.609941e-02
[11] 1.390662e-02 2.549972e-03 3.315731e-04 2.823694e-05 1.319386e-06
[16] 1.434891e-08
>
> power = function(k,p) {
+ pw = 0
+ for (i in 0:15) {
+    if (Lambda[i+1]<=k) {pw = pw + dbinom(i,15,p)}
+ }
+ pw
+ }
> power(0.15,0.30)
[1] 0.05051013
> power(0.15,0.20)
[1] 0.1679108
> power(0.15,0.40)
[1] 0.1002194
>
> pvalue = function(x) {
+ pv = 0
+ for (i in 0:15) {
+    if (Lambda[i+1]<=Lambda[x+1]) {pv = pv + probHo[i+1]}
+ }
+ pv
+ }
> pvalue(7)
[1] 0.2579703
```

```
> p = seq(0,1, by=0.01)
> plot(p,power(0.15,p), type="l", xlim=c(0,1), ylim=c(0,1))
> abline(h=0)
> abline(h=1, lty=2)
> abline(v=0.3, lty=2)
> points(0.30,power(0.15,0.30))
> points(0.20,power(0.15,0.20))
> points(0.40,power(0.15,0.40))
```



Example 7:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Geometric(p) distribution. That is,

$$p_X(k) = p \cdot (1-p)^{k-1}, \qquad k = 1, 2, 3, \ldots .$$

Consider $\quad H_0: p = p_0 \quad$ vs. $\quad H_1: p \neq p_0$

Recall $\quad \hat{p} = 1 \big/ \overline{X} = n \big/ \sum_{i=1}^{n} X_i$.

Reject H_0 if $\quad \Lambda = \dfrac{L(p_0)}{L(\hat{p})} \leq k$

$$\Lambda = \frac{L(p_0)}{L(\hat{p})} = \frac{(1-p_0)^{\sum_{i=1}^{n} X_i - n} (p_0)^n}{\left(1 - \dfrac{n}{\sum_{i=1}^{n} X_i}\right)^{\sum_{i=1}^{n} X_i - n} \left(\dfrac{n}{\sum_{i=1}^{n} X_i}\right)^n} \qquad \text{if } \sum_{i=1}^{n} X_i > n,$$

$$\Lambda = \frac{L(p_0)}{L(\hat{p})} = \frac{(p_0)^n}{1} = (p_0)^n \qquad \text{if } \sum_{i=1}^{n} X_i = n.$$

Suppose $\quad n = 5, \; p_0 = 0.30.$

If H_0 is true, "in the perfect world" $\quad \sum_{i=1}^{n} X_i = \dfrac{5}{0.30} \approx 16.66667.$

Note that Λ takes its largest values "around" 16.66667.

	n	Po					
	5	0.3					
ΣXi							
y		Λ(y)		P(ΣXi = y)			left tail
5		0.00243		0.00243		0.00243	
6		0.025396		0.008505		0.010935	
7		0.078447		0.017861		0.028796	
8		0.165732		0.029172		0.057968	
9		0.282547		0.040841		0.098809	
10		0.418212		0.05146		0.150268	
11		0.559444		0.060036		0.210305	
12		0.693316		0.06604		0.276345	
13		0.809253		0.069342		0.345686	
14		0.900012		0.070112			
15		0.961846		0.06871			
16		0.994111		0.065587			
17		0.998595		0.061214			
18		0.978751		0.056035			
19		0.938977		0.050431			right tail
20		0.884011		0.044716		0.282224	
21		0.818484		0.039126		0.237508	
22		0.746604		0.033833		0.198381	
23		0.67198		0.028946		0.164549	
24		0.597545		0.024528		0.135603	
25		0.525554		0.020603		0.111075	
26		0.457633		0.017169		0.090472	
27		0.394853		0.014204		0.073302	
28		0.337828		0.011672		0.059099	
29		0.286801		0.009532		0.047427	
30		0.241738		0.00774		0.037895	
31		0.2024		0.006252		0.030155	
32		0.168415		0.005024		0.023903	
33		0.139327		0.004019		0.018879	
34		0.114641		0.003202		0.01486	
35		0.093852		0.00254		0.011658	
36		0.076467		0.002007		0.009118	
37		0.062024		0.001581		0.00711	
38		0.050097		0.001241		0.005529	
39		0.040302		0.000971		0.004289	
40		0.0323		0.000757		0.003318	

If $k = 0.30$, then $\Lambda \le k \iff \sum_{i=1}^{n} X_i \le 9$ or $\sum_{i=1}^{n} X_i \ge 29$

significance level = 0.098809 + 0.047427 = 0.146236.

Power$(p = 0.20)$ = 0.019581 + 0.314887 = 0.334468.

Power$(p = 0.40)$ = 0.266568 + 0.003195 = 0.269763.

If $k = 0.20$, then $\Lambda \le k \iff \sum_{i=1}^{n} X_i \le 8$ or $\sum_{i=1}^{n} X_i \ge 32$

significance level = 0.057968 + 0.023903 = 0.081871.

Power$(p = 0.20)$ = 0.010406 + 0.228729 = 0.239135.

Power$(p = 0.40)$ = 0.173670 + 0.001031 = 0.174701.

If $k = 0.15$, then $\Lambda \le k \iff \sum_{i=1}^{n} X_i \le 7$ or $\sum_{i=1}^{n} X_i \ge 33$

significance level = 0.028796 + 0.018879 = 0.047675.

Power$(p = 0.20)$ = 0.004672 + 0.204384 = 0.209056.

Power$(p = 0.40)$ = 0.096256 + 0.000702 = 0.096958.

Suppose we observe $\sum_{i=1}^{n} X_i = 35$. $\Lambda(35) = 0.093852$.

as extreme or more extreme $\iff \Lambda \le 0.093852 \iff \sum_{i=1}^{n} X_i \le 7$ or $\sum_{i=1}^{n} X_i \ge 35$

p-value = 0.028796 + 0.011658 = 0.040454.

Suppose we observe $\sum_{i=1}^{n} X_i = 12$. $\Lambda(12) = 0.693316$.

as extreme or more extreme $\iff \Lambda \le 0.693316 \iff \sum_{i=1}^{n} X_i \le 12$ or $\sum_{i=1}^{n} X_i \ge 23$

p-value = 0.276345 + 0.164549 = 0.440894.