STAT 410

Fall 2020
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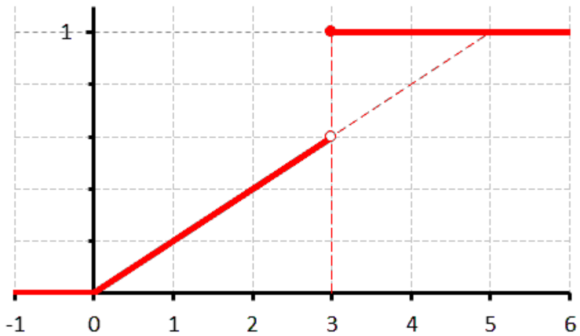
Examples for 09/02/2020 (1)

Mixed Random Variables

1. Consider a random variable X with c.d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{5} & 0 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$



For example, a light bulb's lifetime has a Uniform distribution on $(0, 5)$, and it is replaced at failure or at age 3, whichever occurs first. X is the age of the light bulb at the time of replacement.

a) Find $\mu = E(X)$.

b) Find $\sigma^2 = Var(X)$.

Discrete portion of the probability distribution of X:

$F(x)$ jumps at $x = 3$ from $\frac{3}{5}$ to 1. $\qquad p(3) = 1 - \frac{3}{5} = \frac{2}{5} = 0.4.$

Continuous portion of the probability distribution of X:

$$f(x) = F'(x) = \begin{cases} \dfrac{1}{5} & 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}.$$

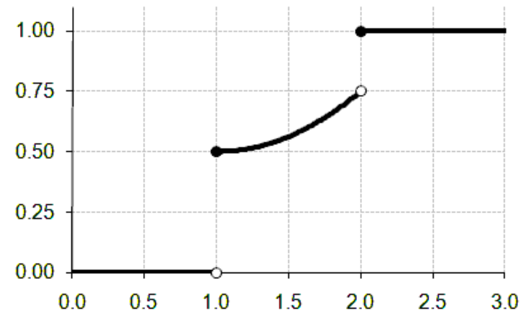
a) $\mu = E(X) = 3 \cdot 0.4 + \int_0^3 x \cdot \frac{1}{5}\, dx = 1.2 + \left.\frac{x^2}{10}\right|_0^3 = 1.2 + 0.9 = \mathbf{2.1}.$

b) $E(X^2) = 3^2 \cdot 0.4 + \int_0^3 x^2 \cdot \frac{1}{5}\, dx = 3.6 + \left.\frac{x^3}{15}\right|_0^3 = 3.6 + 1.8 = 5.4.$

$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = 5.4 - 2.1^2 = \mathbf{0.99}.$

2. Consider a random variable X with c.d.f.

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x^2 - 2x + 3}{4} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



a) Find $\mu = \mathrm{E}(\mathrm{X})$.

b) Find $\sigma^2 = \mathrm{Var}(\mathrm{X})$.

Discrete portion of the probability distribution of X:

$$p(1) = {}^1\!/_2, \qquad p(2) = {}^1\!/_4.$$

Continuous portion of the probability distribution of X:

$$f(x) = \begin{cases} \dfrac{x-1}{2} & 1 < x < 2 \\ 0 & \text{o.w.} \end{cases}.$$

a) $$\mu = \mathrm{E}(\mathrm{X}) = 1 \cdot {}^1\!/_2 + 2 \cdot {}^1\!/_4 + \int_1^2 x \cdot \frac{x-1}{2}\, dx = {}^{17}\!/_{12}.$$

b) $$\mathrm{E}(\mathrm{X}^2) = 1^2 \cdot {}^1\!/_2 + 2^2 \cdot {}^1\!/_4 + \int_1^2 x^2 \cdot \frac{x-1}{2}\, dx = {}^{53}\!/_{24}.$$

$$\sigma^2 = \mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = {}^{29}\!/_{144}.$$

$\mathrm{P}(1 < \mathrm{X} < 1.5) = \mathrm{F}(1.5-) - \mathrm{F}(1) = 0.5625 - 0.5000 = 0.0625.$

$\mathrm{P}(1 < \mathrm{X} \le 1.5) = \mathrm{F}(1.5) - \mathrm{F}(1) = 0.5625 - 0.5000 = 0.0625.$

$\mathrm{P}(1 \le \mathrm{X} < 1.5) = \mathrm{F}(1.5-) - \mathrm{F}(1-) = 0.5625 - 0.0000 = 0.5625.$

$\mathrm{P}(1 \le \mathrm{X} \le 1.5) = \mathrm{F}(1.5) - \mathrm{F}(1-) = 0.5625 - 0.0000 = 0.5625.$

3. Let X be a random variable of the mixed type having the distribution function

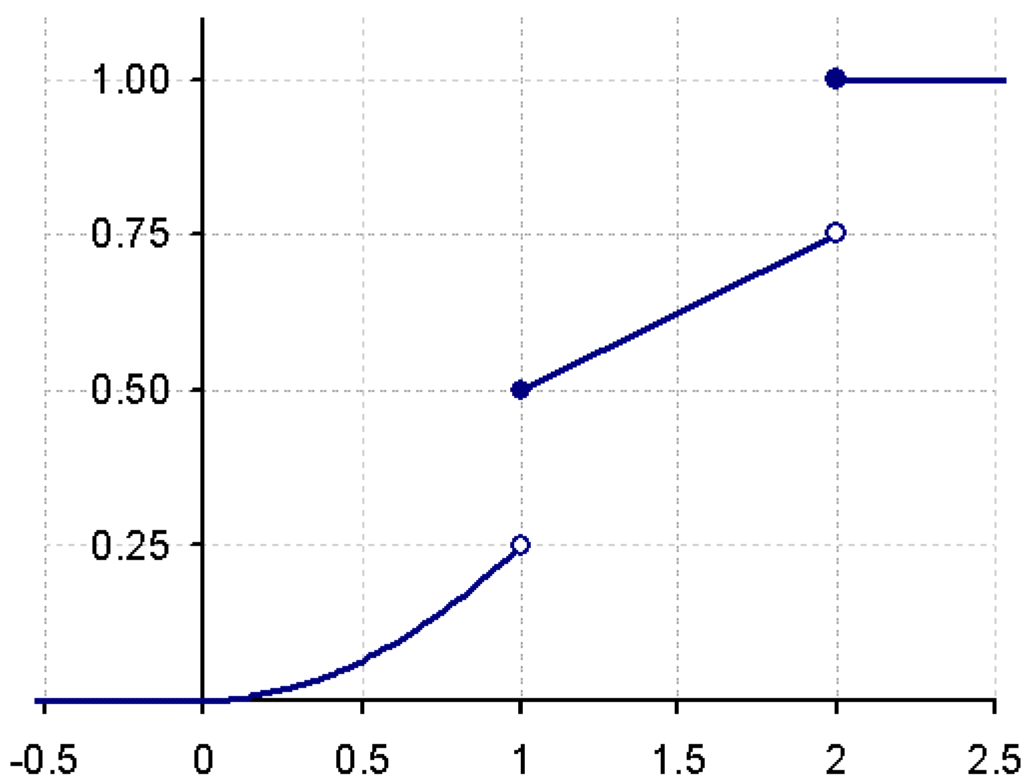
$$F(x) = \begin{cases} 0 & x < 0, \\ \dfrac{x^2}{4} & 0 \le x < 1, \\ \dfrac{x+1}{4} & 1 \le x < 2, \\ 1 & 2 \le x. \end{cases}$$

a) Carefully sketch the graph of $F(x)$.

b) Find the mean and the variance of X.

c) Find $P(X = 1)$, $P(X = 1/2)$, $P(1/4 < X < 1)$, $P(1/4 < X \le 1)$, $P(1/2 \le X < 2)$, $P(1/2 \le X \le 2)$.

a)



Discrete portion of the probability distribution of X:

$p(1) = 1/4, \qquad p(2) = 1/4.$

Continuous portion of the probability distribution of X:

$$f(x) = F'(x) = \begin{cases} x/2 & 0 < x < 1 \\ 1/4 & 1 < x < 2. \\ 0 & \text{o.w.} \end{cases}$$

b) $\mu = E(X) = 1 \cdot 1/4 + 2 \cdot 1/4 + \int_0^1 x \cdot \frac{x}{2}\,dx + \int_1^2 x \cdot \frac{1}{4}\,dx = \mathbf{31/24}.$

$$E(X^2) = 1^2 \cdot 1/4 + 2^2 \cdot 1/4 + \int_0^1 x^2 \cdot \frac{x}{2}\,dx + \int_1^2 x^2 \cdot \frac{1}{4}\,dx = 47/24.$$

$$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = 47/24 - \left(31/24\right)^2 = \mathbf{167/576}.$$

c) $P(X=1) = F(1) - F(1-) = 1/2 - 1/4 = \mathbf{1/4}.$

$$P(X = 1/2) = F(1/2) - F(1/2-) = 1/16 - 1/16 = \mathbf{0}.$$

$$P(1/4 < X < 1) = F(1-) - F(1/4) = 1/4 - 1/64 = \mathbf{15/64}.$$

OR

$$P(1/4 < X < 1) = \int_{1/4}^1 \frac{x}{2}\,dx = \mathbf{15/64}.$$

$$P(1/4 < X \le 1) = F(1) - F(1/4) = 1/2 - 1/64 = \mathbf{31/64}.$$

$$P(1/2 \le X < 2) = F(2-) - F(1/2-) = 3/4 - 1/16 = \mathbf{11/16}.$$

$$P(1/2 \le X \le 2) = F(2) - F(1/2-) = 1 - 1/16 = \mathbf{15/16}.$$

4.* An insurance policy reimburses a loss up to a benefit limit of C and has a deductible of d. The policyholder's loss, X, follows a distribution with density function $f(x)$. Find the expected value of the benefit paid under the insurance policy?

$$\text{Benefit Paid} = \begin{cases} 0 & x < d \\ x - d & d \le x < C + d \\ C & x \ge C + d \end{cases}$$

$$\mathrm{E}(\text{Benefit Paid}) = \int_0^d 0 \cdot f_X(x)\,dx + \int_d^{C+d} (x-d) \cdot f_X(x)\,dx + \int_{C+d}^{\infty} C \cdot f_X(x)\,dx.$$

For example, if X has an Exponential distribution with mean θ,

$$\mathrm{E}(\text{Benefit Paid}) = \int_d^{C+d} (x-d) \cdot f_X(x)\,dx + \int_{C+d}^{\infty} C \cdot f_X(x)\,dx$$

$$= \int_d^{C+d} (x-d) \cdot \frac{1}{\theta} e^{-x/\theta}\,dx + \int_{C+d}^{\infty} C \cdot \frac{1}{\theta} e^{-x/\theta}\,dx$$

$$= \left(-(x-d) \cdot e^{-x/\theta} - \theta\, e^{-x/\theta} \right) \Big|_d^{C+d} + \left(-C\, e^{-x/\theta} \right) \Big|_{C+d}^{\infty}$$

$$= \theta\, e^{-d/\theta} - \theta\, e^{-(C+d)/\theta}.$$

For example, if $d = 2$, $C = 10$, X has an Exponential distribution with mean $\theta = 5$,

$$\mathrm{E}(\text{Benefit Paid}) = \int_2^{12} (x-2) \cdot f_X(x)\,dx + \int_{12}^{\infty} 10 \cdot f_X(x)\,dx$$

$$= \int_2^{12} (x-2) \cdot \frac{1}{5} e^{-x/5}\,dx + \int_{12}^{\infty} 10 \cdot \frac{1}{5} e^{-x/5}\,dx$$

$$= \left(-(x-2) \cdot e^{-x/5} - 5\, e^{-x/5} \right) \Big|_2^{12} + \left(-10\, e^{-x/5} \right) \Big|_{12}^{\infty}$$

$$= 5\, e^{-2/5} - 5\, e^{-12/5} = 5\, e^{-0.4} - 5\, e^{-2.4} \approx \mathbf{2.898}.$$

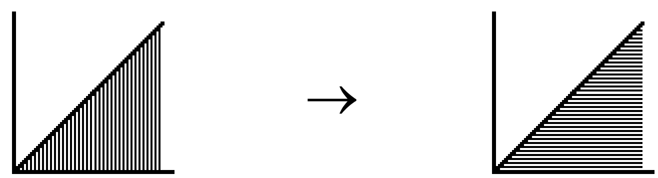
Fact: Let X be a nonnegative continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} (1 - F(x))\,dx.$$

~ **1.9.20** (7th edition) ~ **1.9.19** (6th edition)

Proof:

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} x\, f(x)\,dx = \int_0^{\infty} \left(\int_0^{x} dy \right) f(x)\,dx = \int_0^{\infty} \left(\int_0^{x} f(x)\,dy \right) dx$$



$$\int_0^{\infty} \left(\int_0^{x} f(x)\,dy \right) dx = \int_0^{\infty} \left(\int_y^{\infty} f(x)\,dx \right) dy$$

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \int_0^{\infty} \left(\int_y^{\infty} f(x)\,dx \right) dy = \int_0^{\infty} \mathrm{P}(\mathrm{X} > y)\,dy = \int_0^{\infty} (1 - F(y))\,dy.$$

Example: Find the expected value of an Exponential (θ) distribution.

For Exponential (θ), $\quad 1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = e^{-x/\theta}, \quad t > 0.$

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} (1 - \mathrm{F}(x))\,dx = \int_0^{\infty} e^{-x/\theta}\,dx = \theta.$$

Fact: Let X be a random variable of the discrete type with pmf $p(x)$ that is positive on the nonnegative integers and is equal to zero elsewhere. Then

$$\mathrm{E}(\mathrm{X}) = \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)],$$

where $\mathrm{F}(x)$ is the cdf of X.

~ **1.9.21** (7th edition) ~ **1.9.20** (6th edition)

Proof:

$$1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = p(x+1) + p(x+2) + p(x+3) + p(x+4) + \ldots$$

$$\begin{array}{ll}
1-\mathrm{F}(0) & p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \ldots \\
1-\mathrm{F}(1) & \phantom{p(1) + {}} p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \ldots \\
1-\mathrm{F}(2) & \phantom{p(1) + p(2) + {}} p(3) + p(4) + p(5) + p(6) + p(7) + \ldots \\
1-\mathrm{F}(3) & \phantom{p(1) + p(2) + p(3) + {}} p(4) + p(5) + p(6) + p(7) + \ldots \\
1-\mathrm{F}(4) & \phantom{p(1) + p(2) + p(3) + p(4) + {}} p(5) + p(6) + p(7) + \ldots \\
\ldots & \qquad\qquad\qquad \ldots
\end{array}$$

$$\Rightarrow \quad \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)] = 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + \ldots = \mathrm{E}(\mathrm{X}).$$

Example: Find the expected value of a Geometric(p) distribution.

For Geometric(p), $\quad 1 - \mathrm{F}(x) = \mathrm{P}(\mathrm{X} > x) = (1-p)^x, \quad x = 0, 1, 2, \ldots$.

$$\mathrm{E}(\mathrm{X}) = \sum_{x=0}^{\infty} [1 - \mathrm{F}(x)] = \sum_{x=0}^{\infty} [1 - p]^x = \frac{1}{1 - [1 - p]} = \frac{1}{p}.$$

1.

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{5} & 0 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

$$E(X) = \int_0^{\infty} (1 - F(x))\,dx = \int_0^{3} \left(1 - \frac{x}{5}\right) dx = \left(x - \frac{x^2}{10}\right)\Bigg|_0^3 = \mathbf{2.1}.$$

2.

$$F(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x^2 - 2x + 3}{4} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$$E(X) = \int_0^{\infty} (1 - F(x))\,dx = \int_0^{1} (1)\,dx + \int_1^{2} \left(1 - \frac{x^2 - 2x + 3}{4}\right) dx$$

$$= 1 + \int_1^{2} \left(\frac{1}{4} + \frac{x}{2} - \frac{x^2}{4}\right) dx = 1 + \left(\frac{x}{4} + \frac{x^2}{4} - \frac{x^3}{12}\right)\Bigg|_1^2 = \frac{17}{12}.$$