STAT 410 | Fall 2020
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Examples for 12/02/2020 (3)

$H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$ Reject H_0 if $\Lambda = \dfrac{L(\theta_0)}{L(\hat{\theta})} \le k$

Example: Inspired by **6.3.8** (7th and 6th edition)

6.3.8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

(a) Show that the likelihood ratio test of $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^{n} X_i$. Obtain the null distribution of Y.

(b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \le 4$ or $Y \ge 17$.

Let $X_1, X_2, \ldots, X_5$ be a random sample of size $n = 5$ from a Poisson distribution with mean $\lambda > 0$. We wish to test $H_0: \lambda = 2$ vs. $H_1: \lambda \ne 2$.

Recall that $Y = \sum_{i=1}^{n=5} X_i$ has Poisson(5λ) distribution.

Recall that $\hat{\lambda} = \overline{X} = \dfrac{Y}{n}$ is the maximum likelihood estimator of λ.

a) Find the values of $\Lambda(y) = \dfrac{L(\lambda_0 = 2; y)}{L(\hat{\lambda}; y)}$ for $y = 0, 1, 2, 3, 4, \ldots, 24, 25$.

($\Lambda(y)$ would only be decreasing for $y > 25$. Note that $\Lambda(y = 10) = 1$.)

$$\Lambda(y) = \frac{L(\lambda_0; y)}{L(\hat{\lambda}; y)} = \frac{\prod_{i=1}^{n} \dfrac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!}}{\prod_{i=1}^{n} \dfrac{\overline{x}^{x_i} e^{-\overline{x}}}{x_i!}} = \left(\frac{n\lambda_0}{y}\right)^{y} e^{y - n\lambda_0} \le k.$$

Excel:

$\Lambda(y)$ = (5*2/y)^y * exp(y-5*2) for $y \neq 0$

$\Lambda(y)$ = exp(-5*2) for $y = 0$

OR

= POISSON(y,5*2,0) / POISSON(y,y,0)

y	Λ(y)	P(Y=y)		
				left tail
0	0.000123	4.54E-05	4.54E-05	
1	0.001234	0.000454	0.000499	
2	0.008387	0.00227	0.002769	
3	0.033773	0.007567	0.010336	
4	0.096826	0.018917	0.029253	
5	0.215614	0.037833	0.067086	
6	0.392568	0.063055	0.130141	
7	0.604547	0.090079	0.220221	
8	0.806661	0.112599		
9	0.949561	0.12511		
10	1	0.12511		
11	0.952741	0.113736		
12	0.828732	0.09478		right tail
13	0.663162	0.072908	0.208444	
14	0.491344	0.052077	0.135536	
15	0.338925	0.034718	0.083458	
16	0.218699	0.021699	0.04874	
17	0.132565	0.012764	0.027042	
18	0.075762	0.007091	0.014278	
19	0.040957	0.003732	0.007187	
20	0.021006	0.001866	0.003454	
21	0.010248	0.000889	0.001588	
22	0.004767	0.000404	0.0007	
23	0.002119	0.000176	0.000296	
24	0.000902	7.32E-05	0.00012	
25	0.000368	2.93E-05	4.69E-05	
…	…	…	…	

b) Likelihood Ratio Test: Reject H_0 if $\Lambda(y) \le k$.

Let $k = 0.20$. Find

(i) the significance level,

(ii) power when $\lambda = 1$,

(iii) power when $\lambda = 3$

for the corresponding rejection region.

If $k = 0.20$, then $\Lambda \le k \quad \Leftrightarrow \quad Y \le 4 \quad \text{or} \quad Y \ge 17$

(i) significance level $= P(Y \le 4) + P(Y \ge 17) = 0.029253 + 0.027042 = \mathbf{0.056295}$.

$$= P(\text{Poisson}(10) \le 4) + P(\text{Poisson}(10) \ge 17)$$

$$= P(\text{Poisson}(10) \le 4) + [1 - P(\text{Poisson}(10) \le 16)]$$

$$= 0.029 + [1 - 0.973] = 0.029 + 0.027 = \mathbf{0.056}.$$

(ii) $\text{Power}(\lambda = 1) = P(Y \le 4) + P(Y \ge 17)$

$$= P(\text{Poisson}(5) \le 4) + P(\text{Poisson}(5) \ge 17)$$

$$= P(\text{Poisson}(5) \le 4) + [1 - P(\text{Poisson}(5) \le 16)]$$

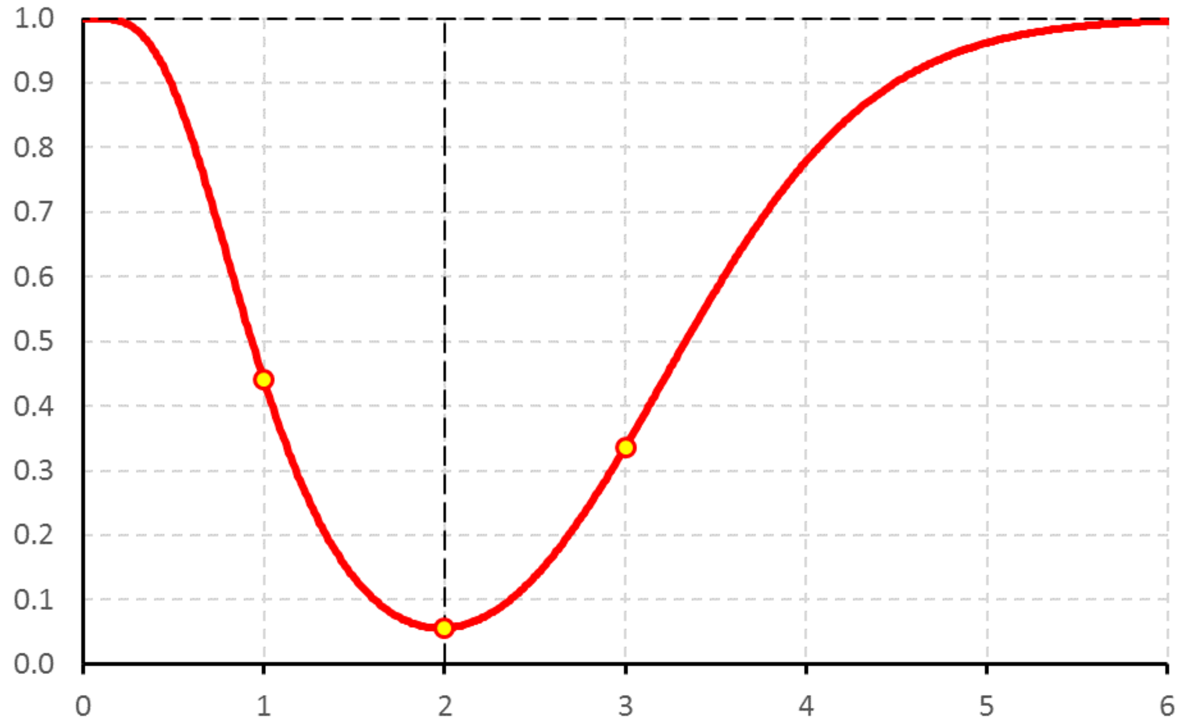
$$= 0.440 + [1 - 1.000] = 0.440 + 0.000 = \mathbf{0.440}.$$

(iii) $\text{Power}(\lambda = 3) = P(Y \le 4) + P(Y \ge 17)$

$$= P(\text{Poisson}(15) \le 4) + P(\text{Poisson}(15) \ge 17)$$

$$= P(\text{Poisson}(15) \le 4) + [1 - P(\text{Poisson}(15) \le 16)]$$

$$= 0.001 + [1 - 0.664] = 0.001 + 0.336 = \mathbf{0.337}.$$



c) Suppose we observe $y = \sum_{i=1}^{n=5} x_i = 19$. Find the p-value.

Suppose we observe $y = 19$. $\Lambda(19) = 0.040957$.

as extreme or more extreme $\Leftrightarrow$ $\Lambda \le 0.040957$ $\Leftrightarrow$ $Y \le 3$ or $Y \ge 19$

p-value = 0.010336 + 0.007187 = **0.017523**.

d) Suppose we observe $y = \sum_{i=1}^{n=5} x_i = 6$. Find the p-value.

Suppose we observe $y = 6$. $\Lambda(6) = 0.392568$.

as extreme or more extreme $\Leftrightarrow$ $\Lambda \le 0.392568$ $\Leftrightarrow$ $X \le 6$ or $X \ge 15$

p-value = 0.130141 + 0.083458 = **0.213599**.