Examples for 10/05/2020 (1)

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Multivariate Normal Distribution:

$\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \, |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \, \boldsymbol{\Sigma}^{-1} \, (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

$\mathbf{Z} \sim N_n(\mathbf{0}, \mathbf{I}_n)$

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} \exp\left\{ -\frac{1}{2} \mathbf{z}^{\mathrm{T}} \, \mathbf{z} \right\} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} z_i^2 \right\}$$

$$\mathrm{M}_{\mathbf{X}}(\mathbf{t}) = \exp\left\{ \mathbf{t}^{\mathrm{T}} \, \boldsymbol{\mu} \; + \; \frac{1}{2} \mathbf{t}^{\mathrm{T}} \, \boldsymbol{\Sigma} \, \mathbf{t} \right\} \qquad\qquad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \cdots \\ t_n \end{pmatrix} \in \mathbf{R}^n$$

$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ $\qquad\qquad$ $\mathbf{A} - m \times n$ $\qquad\qquad$ $\mathbf{b} \in \mathbf{R}^m$

$\Rightarrow \qquad \mathbf{Y} \sim N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

$\mathbf{Z} \sim N_n(\mathbf{0}, \mathbf{I}_n) \qquad\qquad \Rightarrow \qquad\qquad \mathbf{X} = \mathbf{S}\,\mathbf{Z} + \boldsymbol{\mu} \;\sim\; N_n(\boldsymbol{\mu}, \mathbf{S}\,\mathbf{S}^{\mathrm{T}})$

Example:

$$\mathbf{X} \sim N_3(\mu, \Sigma) \qquad \mu = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix}$$

a) Find $P(X_1 > 6)$.

$X_1 \sim N(5, 4)$

$$P(X_1 > 6) = P\left(Z > \frac{6-5}{\sqrt{4}}\right) = P(Z > 0.5) = \mathbf{0.3085}.$$

b) Find $P(5X_2 + 4X_3 > 70)$.

$$E(5X_2 + 4X_3) = 5 \cdot 3 + 4 \cdot 7 = 43.$$

$$\mathrm{Var}(5X_2 + 4X_3) = 25\,\mathrm{Var}(X_2) + 40\,\mathrm{Cov}(X_2, X_3) + 16\,\mathrm{Var}(X_3)$$

$$= 25 \cdot 4 + 40 \cdot 2 + 16 \cdot 9 = 324.$$

OR

$$\mathrm{Var}(5X_2 + 4X_3) = \begin{pmatrix} 0 & 5 & 4 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 & 28 & 46 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} = 324.$$

$5X_2 + 4X_3 \sim N(43, 324)$

$$P(5X_2 + 4X_3 > 70) = P\left(Z > \frac{70-43}{\sqrt{324}}\right) = P(Z > 1.50) = \mathbf{0.0668}.$$

c) Find $P(3X_3 - 5X_2 < 17)$.

$$E(3X_3 - 5X_2) = 3 \cdot 7 - 5 \cdot 3 = 6.$$

$$\text{Var}(3X_3 - 5X_2) = 9\,\text{Var}(X_3) - 30\,\text{Cov}(X_2, X_3) + 25\,\text{Var}(X_2)$$

$$= 9 \cdot 9 - 30 \cdot 2 + 25 \cdot 4 = 121.$$

OR

$$\text{Var}(3X_3 - 5X_2) = \begin{pmatrix} 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 & -14 & 17 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} = 121.$$

$$3X_3 - 5X_2 \sim N(6, 121)$$

$$P(3X_3 - 5X_2 < 17) = P\left(Z < \frac{17-6}{\sqrt{121}}\right) = P(Z < 1.00) = \mathbf{0.8413}.$$

d) Find $P(4X_1 - 3X_2 + 5X_3 < 80)$.

$$E(4X_1 - 3X_2 + 5X_3) = 4\mu_1 - 3\mu_2 + 5\mu_3 = 4 \cdot 5 - 3 \cdot 3 + 5 \cdot 7 = 46.$$

$$\text{Var}(4X_1 - 3X_2 + 5X_3) = \begin{pmatrix} 4 & -3 & 5 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 19 & -6 & 39 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$$

$$= 289.$$

$$4X_1 - 3X_2 + 5X_3 \sim N(46, 289)$$

$$P(4X_1 - 3X_2 + 5X_3 < 80) = P\left(Z < \frac{80-46}{\sqrt{289}}\right) = P(Z < 2.00) = \mathbf{0.9772}.$$

For fun:

$\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \qquad \begin{array}{l} \mathbf{X}_1 \text{ is of dimension } m < n \\ \mathbf{X}_2 \text{ is of dimension } n - m \end{array} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{X}_1 \mid \mathbf{X}_2 \sim N_m\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)$$

e)* Find $\mathrm{P}(\mathrm{X}_1 > 8 \mid \mathrm{X}_2 = 1, \mathrm{X}_3 = 10)$.

$$\boldsymbol{\Sigma} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} = \begin{pmatrix} 4 & | & -1 & 0 \\ - & + & - & - \\ -1 & | & 4 & 2 \\ 0 & | & 2 & 9 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{22} = \begin{pmatrix} 4 & 2 \\ 2 & 9 \end{pmatrix} \qquad \boldsymbol{\Sigma}_{22}^{-1} = \frac{1}{32}\begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} = \frac{1}{32}\begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix} = \frac{1}{32}\begin{pmatrix} -9 & 2 \end{pmatrix}$$

$$\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2) = 5 + \frac{1}{32}\begin{pmatrix} -9 & 2 \end{pmatrix}\begin{pmatrix} 1-3 \\ 10-7 \end{pmatrix} = 5.75.$$

$$\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} = 4 - \frac{1}{32}\begin{pmatrix} -9 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = 3.71875.$$

$$\mathrm{X}_1 \mid \mathrm{X}_2 = 1, \mathrm{X}_3 = 10 \sim N(5.75, 3.71875)$$

$$\mathrm{P}(\mathrm{X}_1 > 8 \mid \mathrm{X}_2 = 1, \mathrm{X}_3 = 10) = \mathrm{P}\left(\mathrm{Z} > \frac{8 - 5.75}{\sqrt{3.71875}}\right) = \mathrm{P}(\mathrm{Z} > 1.17) = \mathbf{0.1210}.$$