STAT 410

Examples for 09/21/2020 (1)

Fall 2020
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0. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

a) What is the probability distribution of $U = X + Y$?

	y			
x	0	1	2	$p_X(x)$
1	0.15 $u = 1$	0.10 $u = 2$	0	0.25
2	0.25 $u = 2$	0.30 $u = 3$	0.20 $u = 4$	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

u	$p_U(u)$
1	0.15
2	0.35
3	0.30
4	0.20

b) What is the probability distribution of $V = XY$?

	y			
x	0	1	2	$p_X(x)$
1	0.15 $v = 0$	0.10 $v = 1$	0	0.25
2	0.25 $v = 0$	0.30 $v = 2$	0.20 $v = 4$	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

v	$p_V(v)$
0	0.40
1	0.10
2	0.30
4	0.20

c) What is the joint probability distribution of $U = X + Y$ and $V = XY$?

	y			
x	0	1	2	$p_X(x)$
1	0.15 $u=1$ $v=0$	0.10 $u=2$ $v=1$	0	0.25
2	0.25 $u=2$ $v=0$	0.30 $u=3$ $v=2$	0.20 $u=4$ $v=4$	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

	v				
u	0	1	2	4	$p_U(u)$
1	0.15	0	0	0	0.15
2	0.25	0.10	0	0	0.35
3	0	0	0.30	0	0.30
4	0	0	0	0.20	0.20
$p_V(v)$	0.40	0.10	0.30	0.20	1.00

Let X_1 and X_2 have joint p.d.f. $f(x_1, x_2)$.

$\mathcal{S} = \{(x_1, x_2) : f(x_1, x_2) > 0\}$ – support of (X_1, X_2).

Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$.

$y_1 = u_1(x_1, x_2)$ one-to-one transformation

$y_2 = u_2(x_1, x_2)$ maps $\mathcal{S}$ onto $\mathcal{T}$ – support of (Y_1, Y_2).

$x_1 = w_1(y_1, y_2)$

$x_2 = w_2(y_1, y_2)$

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

The joint p.d.f. $f_{Y_1, Y_2}(y_1, y_2)$ of (Y_1, Y_2) is

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)) \cdot |J| & (y_1, y_2) \in \mathcal{T} \\ 0 & \text{elsewhere.} \end{cases}$$

1. Let X_1 and X_2 have joint p.d.f. $f(x_1, x_2) = 2\,e^{-(x_1 + x_2)}$, $0 < x_1 < x_2$.

a) Find the joint p.d.f. $f_{Y_1, Y_2}(y_1, y_2)$ of the variables

$Y_1 = X_2 - X_1$ and $Y_2 = X_1$.

$Y_2 = X_1 \quad \Rightarrow \quad X_1 = Y_2$

$Y_1 = X_2 - X_1 \quad \Rightarrow \quad X_2 = Y_1 + X_1 = Y_1 + Y_2$

$$J = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$x_1 > 0 \quad \Rightarrow \quad y_2 > 0$

$x_2 > x_1 \quad \Rightarrow \quad y_1 > 0$

$$f_{Y_1, Y_2}(y_1, y_2) = 2\,e^{-(y_2 + y_1 + y_2)} \times |-1| = 2\,e^{-(y_1 + 2y_2)}, \qquad y_1 > 0, \quad y_2 > 0.$$

Note that $\quad f_{Y_1}(y_1) = e^{-y_1}, \qquad y_1 > 0,$

$f_{Y_2}(y_2) = 2\,e^{-2\,y_2}, \qquad y_2 > 0,$

Y_1 and Y_2 are independent.

b) Find the joint p.d.f. $f_{Z_1,Z_2}(z_1, z_2)$ of the variables

$Z_1 = X_1 + X_2$ and $Z_2 = X_2 / X_1$.

$$Z_2 = X_2 / X_1 \qquad \Rightarrow \qquad X_2 = Z_2 X_1$$

$$Z_1 = X_1 + X_2 = X_1 + Z_2 X_1 \qquad \Rightarrow \qquad X_1 = Z_1 / (1 + Z_2)$$

$$\Rightarrow \qquad X_2 = Z_1 Z_2 / (1 + Z_2)$$

$$J = \begin{vmatrix} \dfrac{1}{1+z_2} & -\dfrac{z_1}{(1+z_2)^2} \\ \dfrac{z_2}{1+z_2} & \dfrac{z_1(1+z_2) - z_1 z_2}{(1+z_2)^2} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{1+z_2} & -\dfrac{z_1}{(1+z_2)^2} \\ \dfrac{z_2}{1+z_2} & \dfrac{z_1}{(1+z_2)^2} \end{vmatrix} = \frac{z_1 + z_1 z_2}{(1+z_2)^3} = \frac{z_1}{(1+z_2)^2}.$$

$$x_1 > 0 \qquad \Rightarrow \qquad z_1 > 0$$

$$x_2 > x_1 \qquad \Rightarrow \qquad z_2 > 1$$

$$f_{Z_1,Z_2}(z_1, z_2) = 2\,e^{-z_1} \times \frac{z_1}{(1+z_2)^2}, \qquad z_1 > 0, \quad z_2 > 1.$$

Note that $\quad f_{Z_1}(z_1) = z_1\,e^{-z_1}, \qquad z_1 > 0,$

$f_{Z_2}(z_2) = \dfrac{2}{(1+z_2)^2}, \qquad z_2 > 1,$

Z_1 and Z_2 are independent.

2. Let X_1 and X_2 have the joint probability density function

$$f_{X_1,X_2}(x_1,x_2) = 15\,x_1 x_2^2, \qquad 0 < x_2 < x_1 < 1, \qquad \text{zero elsewhere.}$$

a) Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1$.

Find the joint probability density function of (Y_1, Y_2), $f_{Y_1,Y_2}(y_1,y_2)$.

Sketch the support of (Y_1, Y_2).

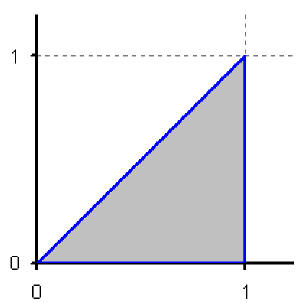
$$X_1 = Y_2 \qquad\qquad J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

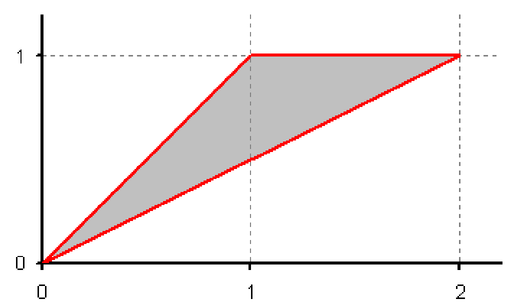
$$X_2 = Y_1 - Y_2$$

$x_2 > 0 \quad \Rightarrow \quad y_1 > y_2$

$x_1 > x_2 \quad \Rightarrow \quad y_1 < 2\,y_2$

$x_1 < 1 \quad \Rightarrow \quad y_2 < 1$





$$f_{Y_1,Y_2}(y_1,y_2) = 15\,y_2\,(y_1-y_2)^2 \times |-1| = 15\,y_2\,(y_1-y_2)^2,$$

$$0 < y_2 < 1, \quad y_2 < y_1 < 2\,y_2.$$

b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = X_1 / X_2$.

Find the joint probability density function of (Y_1, Y_2), $f_{Y_1,Y_2}(y_1, y_2)$.

Sketch the support of (Y_1, Y_2).

$$Y_2 = X_1 / X_2 \quad \Rightarrow \quad X_1 = Y_2 X_2$$

$$Y_1 = X_1 \cdot X_2 = Y_2 X_2 \cdot X_2 \quad \Rightarrow \quad X_2 = (Y_1 / Y_2)^{½} = Y_1^{½} Y_2^{-½}$$

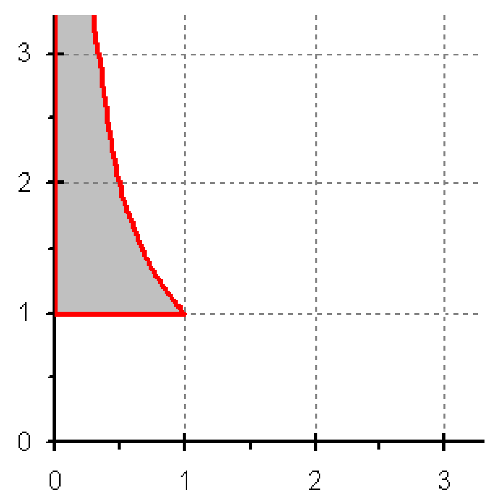
$$\Rightarrow \quad X_1 = (Y_1 Y_2)^{½} = Y_1^{½} Y_2^{½}$$

$$J = \begin{vmatrix} \frac{1}{2} y_1^{-1/2} y_2^{1/2} & \frac{1}{2} y_1^{1/2} y_2^{-1/2} \\ \frac{1}{2} y_1^{-1/2} y_2^{-1/2} & -\frac{1}{2} y_1^{1/2} y_2^{-3/2} \end{vmatrix} = -\frac{1}{4} y_2^{-1} - \frac{1}{4} y_2^{-1} = -\frac{1}{2 y_2}.$$

$$0 < x_2 \quad \Rightarrow \quad 0 < y_1$$

$$x_2 < x_1 \quad \Rightarrow \quad y_2 > 1$$

$$x_1 < 1 \quad \Rightarrow \quad y_2 < 1/y_1$$



$$f_{Y_1,Y_2}(y_1, y_2) = 15 \cdot \sqrt{y_1 y_2} \cdot \left(\sqrt{\frac{y_1}{y_2}} \right)^2 \times \left| -\frac{1}{2 y_2} \right|$$

$$= 7.5 \cdot \sqrt{\frac{y_1^3}{y_2^3}} = 7.5\, y_1^{3/2} y_2^{-3/2}, \qquad y_2 > 1, \; 0 < y_1 < 1/y_2.$$

c) Let $Y_1 = X_2/X_1$ and $Y_2 = X_1 + X_2$.

Find the joint probability density function of (Y_1, Y_2), $f_{Y_1,Y_2}(y_1,y_2)$.

Sketch the support of (Y_1, Y_2).

$$Y_1 = X_2/X_1 \qquad \Rightarrow \qquad X_2 = Y_1 X_1$$

$$Y_2 = X_1 + X_2 = X_1 + Y_1 X_1 \qquad \Rightarrow \qquad X_1 = Y_2/(1+Y_1)$$

$$\Rightarrow \qquad X_2 = Y_1 Y_2/(1+Y_1)$$

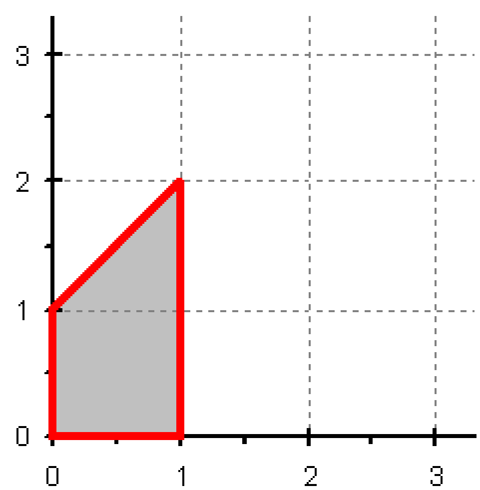
$$J = \begin{vmatrix} -\dfrac{y_2}{(1+y_1)^2} & \dfrac{1}{1+y_1} \\ \dfrac{y_2(1+y_1) - y_1 y_2}{(1+y_1)^2} & \dfrac{y_1}{1+y_1} \end{vmatrix} = \begin{vmatrix} -\dfrac{y_2}{(1+y_1)^2} & \dfrac{1}{1+y_1} \\ \dfrac{y_2}{(1+y_1)^2} & \dfrac{y_1}{1+y_1} \end{vmatrix}$$

$$= -\frac{y_1 y_2 + y_2}{(1+y_1)^3} = -\frac{y_2}{(1+y_1)^2}.$$

$0 < x_2 \quad \Rightarrow \quad y_1 > 0,\ y_2 > 0$

$x_2 < x_1 \quad \Rightarrow \quad y_1 < 1$

$x_1 < 1 \quad \Rightarrow \quad y_2 < y_1 + 1$



$$f_{Y_1,Y_2}(y_1,y_2) = 15 \frac{y_2}{1+y_1} \left(\frac{y_1 y_2}{1+y_1} \right)^2 \times \left| -\frac{y_2}{(1+y_1)^2} \right|$$

$$= 15 \frac{y_1^2 y_2^4}{(1+y_1)^5}, \qquad 0 < y_1 < 1,\ \ 0 < y_2 < y_1 + 1.$$