虚功原理—张量形式证明：$\tilde{\sigma}$ 为任意二阶张量，$\tilde{v}$ 为一阶张量。

高斯散度定理：

$$\int_\Omega \nabla \cdot \tilde{v} \, d\Omega = \oint_{\partial\Omega} \tilde{v} \cdot \tilde{n} \, ds .$$

$$\nabla = \frac{\partial}{\partial x_j} \tilde{e}_j \, , \qquad \tilde{e}_i \cdot \tilde{e}_j = \delta_{ij} .$$

$$\sigma : (\nabla \otimes v) = \sigma_{ij} \tilde{e}_i \otimes \tilde{e}_j : \left(\frac{\partial v_m}{\partial x_n} \tilde{e}_n \otimes \tilde{e}_m \right)$$

$$= \sigma_{ij} \frac{\partial v_j}{\partial x_i}$$

$$\Rightarrow \nabla \cdot (\tilde{\sigma} \cdot \tilde{v}) = \frac{\partial (\sigma_{ij} v_j)}{\partial x_k} \tilde{e}_k \cdot \tilde{e}_i = \frac{\partial (\sigma_{ij} v_j)}{\partial x_i}$$

$$= \sigma_{ij,i} v_j + \sigma_{ij} v_{j,k} = \nabla \cdot \tilde{\sigma} \cdot \tilde{v} + \tilde{\sigma} : (\nabla \otimes \tilde{v}).$$

取 $\tilde{\sigma}$ 为应力，$\tilde{v}$ 为虚位移，两边积分并利用高斯散度定理：

$$\int_\Omega \nabla \cdot (\tilde{\sigma} \cdot \tilde{v}) \, d\Omega \overset{\text{Gauss}}{=\!=} \oint_{\partial\Omega} (\tilde{\sigma} \cdot \tilde{v}) \cdot \tilde{n} \, ds$$

$$\Rightarrow \int_\Omega [\underset{①}{\nabla\cdot\tilde{\sigma}\cdot\tilde{v}} + \underset{②}{\tilde{\sigma} : (\nabla\otimes\tilde{v})}]\, d\Omega = \oint_{\partial\Omega} \underset{③}{(\tilde{\sigma}\cdot\tilde{v})\cdot\tilde{n}\, dS}$$

① $\int_\Omega \nabla\cdot\tilde{\sigma}\cdot\tilde{v}\, d\Omega$ 由 $\nabla\cdot\tilde{\sigma} + \tilde{f} = 0$ (体力平衡方程)

$\Rightarrow -\int_\Omega \tilde{f}\cdot\tilde{v}\, d\Omega$ ，此即为体力虚功，属于外力虚功.

② $\int_\Omega \tilde{\sigma} : (\nabla\otimes\tilde{v})\, d\Omega$

$$\tilde{\sigma} : (\nabla\otimes\tilde{v}) = \sigma_{ij}\, \vec{e}_i\otimes\vec{e}_j : \overset{v_{m,n}}{\frac{\partial v_m}{\partial x_n}}\, \vec{e}_n\otimes\vec{e}_m$$

$$= \sigma_{ij} v_{j,i} = \sigma_{ij}\cdot\left[\frac{1}{2}(v_{j,i}+v_{i,j}) + \frac{1}{2}(v_{j,i}-v_{i,j})\right]$$

又 $\sigma_{ij}\cdot(v_{j,i}-v_{i,j}) \xlongequal{\sigma_{ij}=\sigma_{ji}} \sigma_{ji}(v_{j,i}-v_{i,j})$

$$= \sigma_{ij}(v_{i,j}-v_{j,i}) = -\sigma_{ij}(v_{j,i}-v_{i,j})$$

$\Rightarrow \sigma_{ij}\cdot\frac{1}{2}(v_{j,i}-v_{i,j}) = 0,\quad \Rightarrow \sigma_{ij} v_{j,i} = \sigma_{ij}\cdot\frac{1}{2}(v_{j,i}+v_{i,j})$

$$\Rightarrow \sigma_{ij} v_{j,i} = \sigma_{ij}\cdot\frac{1}{2}(v_{j,i}+v_{i,j}) = \sigma_{ij}\cdot\varepsilon_{ij}$$

故 $\int_\Omega \sigma_{ij}\varepsilon_{ij}\,d\Omega$ 为内力虚功.

③ $\oint_{\partial\Omega}(\tilde{\sigma}\cdot\hat{v})\cdot\hat{n}\,ds = \oint_{\partial\Omega}(\tilde{\sigma}\cdot\tilde{n})\cdot\tilde{v}\,ds$

$\tilde{\sigma}\cdot\tilde{n}+\hat{p}=0$ (面力平衡方程).

$\Rightarrow \oint_{\partial\Omega} -\hat{p}\cdot\tilde{v}\,ds$ 称为面力虚功,属于外力虚功.

$\Rightarrow \int_\Omega -\tilde{f}\cdot\hat{v}\,d\Omega + \int_\Omega \sigma_{ij}\varepsilon_{ij}\,ds = \oint_{\partial\Omega} -\hat{p}\,\hat{v}\,ds$

$\Rightarrow \int_\Omega \sigma_{ij}\varepsilon_{ij}\,ds = \int_\Omega \tilde{f}\cdot\hat{v}\,d\Omega - \oint_{\partial\Omega}\hat{p}\cdot\tilde{v}\,ds$

即 内虚功 = 外虚功,虚功原理得证.