# 傅里叶变换.

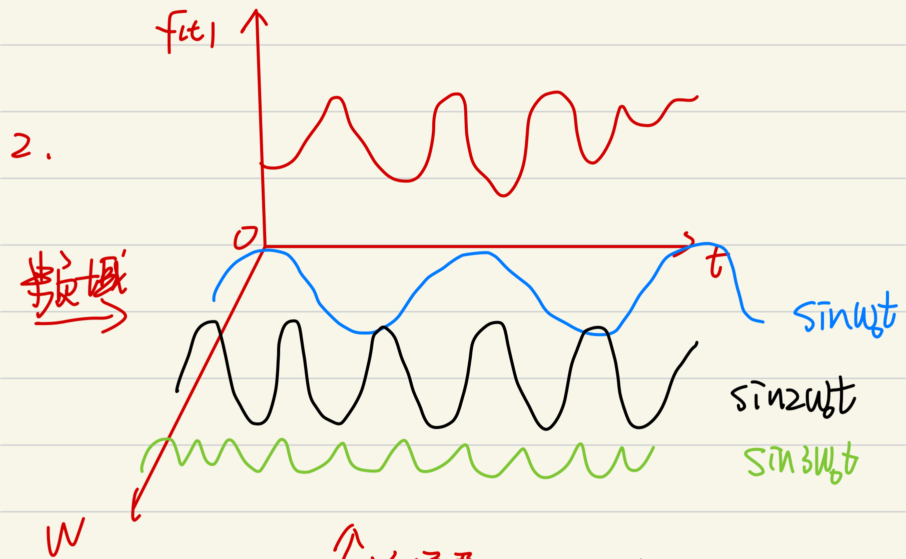
频率为w. 也称为基频率

1. 任何周期函数 $f(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nwt) + b_n\sin(nwt)]$

(傅里叶级数)

$T = \frac{2\pi}{w}$

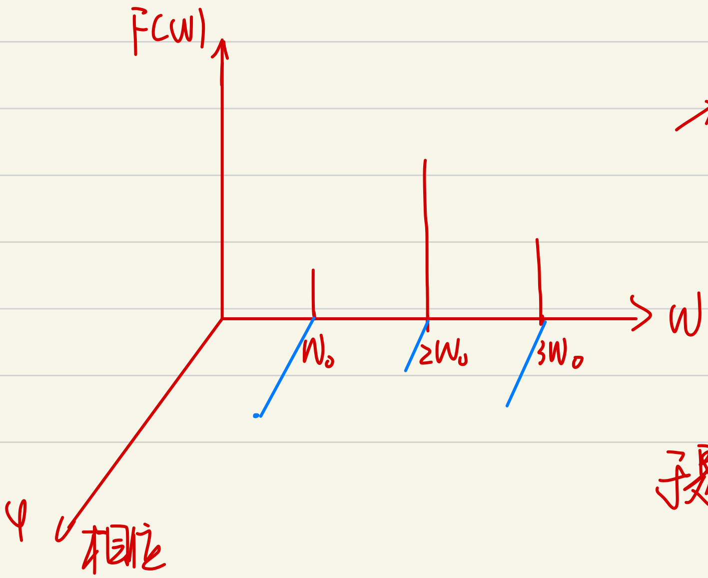
2.




这样看即为时域

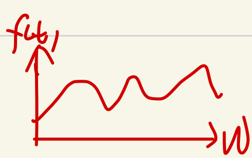
为了和坐标轴区分, W改为$W_0$.

此外W是f(t)的周期函数




见周期函数的频率是离散的点, 而对于非周期函数, 可看成是周期为无穷大.

于是 $W_0 \to 0$, ⇒



1、$\sin\omega t$、$\cos\omega t$、… $\sin n\omega t$、$\cos n\omega t$ … 构成一组基.

是正交的. 但 $\overrightarrow{\sin n\omega t} = (\sin n\omega t_1, \sin n\omega t_2, \cdots)$

故 $\overrightarrow{\sin n\omega t} \cdot \overrightarrow{\cos n\omega t} = \sum_{n=1}^{\infty} \sin n\omega t_n \cos n\omega t_n = \int_0^{2\pi} \sin n\omega t \cos n\omega t \, dt$

$= 0.$

(形象理解, 不严谨)

其余同理.

对于 $f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$

$$\underset{e^{ix}=\cos x + i\sin x}{\overset{a_0\text{纳入其中}}{=\!=\!=}} \sum_{n=0}^{\infty} \left[ a_n \cdot \frac{e^{in\omega t} + e^{-in\omega t}}{2} + b_n \cdot \frac{e^{in\omega t} - e^{-in\omega t}}{2i} \right] \quad (b_0 = 0)$$

$$= \sum_{n=0}^{\infty} \left[ \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{in\omega t} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-in\omega t} \right]$$

$$\underset{\text{对称}}{\overset{n\omega\text{与}-n\omega}{=\!=\!=}} \sum_{n=-\infty}^{\infty} C_n e^{in\omega t}. \quad \left(\text{例如 } C_{-1} = \frac{a_1}{2} - \frac{b_1}{2i}\right).$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} \underset{\text{坐标}}{C_n} \, \underset{\text{基}}{e^{in\omega t}}$$

$$C_m e^{im\omega t} = f(t) - \sum_{n\neq m} C_n e^{in\omega t},$$

$$\Rightarrow \quad C_m = f(t)\, e^{-im\omega t} - \left(\sum_{n\neq m} C_n e^{in\omega t}\right)\cdot e^{-im\omega t}$$

在f(t)周期上积分有

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} C_m = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-im\omega t}\, dt$$

$$- \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{n\neq m} C_n e^{in\omega t}\right)\cdot e^{-im\omega t}\, dt$$

三角函数在[0,T]上正交

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-im\omega t}\, dt$$

$$\Rightarrow \quad C_m = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-im\omega t}\, dt.$$

对于非周期函数. $\Delta\omega = (n+1)\omega - n\omega = \omega = \frac{2\pi}{T}$,

$$C_m = \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-im\omega t}\, dt$$

，记 $\omega_n = n\cdot\omega$.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega t} = \sum_{n=-\infty}^{\infty} \frac{\Delta \omega}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_n t} dt \cdot$$

$$e^{i\omega_n t}$$

当$T \to \infty$时. $\Delta\omega \to 0$. $(\omega \to 0)$

$$\Rightarrow f(t) = \int_{-\infty}^{-\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \; e^{i\omega t} d\omega.$$

$$\left( \lim_{\substack{n \to \infty \\ T \to \infty}} \sum_{i=1}^{n} \frac{\Delta\omega}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_i t} dt \; e^{i\omega_i t} \right.$$

$\omega_i = i \cdot \omega \in (-\infty, +\infty)$

$\Delta\omega = \omega_{i+1} - \omega_i$

$$\left. \int_{-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \, e^{i\omega t} d\omega \right)$$

举例说明: $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \frac{1}{n} = \int_0^1 \frac{1}{1+x} dx.$

$$\Rightarrow f(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} f(t) e^{-i\omega t} dt \; e^{i\omega t} d\omega.$$

令 $F(\omega) = \int_{-\infty}^{+\infty} \frac{1}{2\pi} f(t) e^{-i\omega t} dt$. 此称为傅里叶变换.

而 $f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} dt$，称为傅里叶逆变换.