1) Master Theorem

On a given recurrence relation, $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ which a represents number of brunches on each iteration, $\left(\frac{n}{b}\right)$ represents input size on each iteration.

If $a \geq 1$, $b > 1$ and $f(n)$ is asymtotic positive function Master theorem could be applied with 3 cases.

case 1 If $f(n) \in O\left(n^{\log_b a - \epsilon}\right)$, $\epsilon > 0$. Then $T(n) = \Theta\left(n^{\log_b a}\right)$

case 2 If $f(n) \in \Theta\left(n^{\log_b a}\right)$ Then $T(n) = \Theta\left(n^{\log_b a} \log n\right)$

case 3 $\left[\text{If } f(n) \in \Omega\left(n^{\log_b a + \epsilon}\right), \epsilon > 0\right]$ AND $\left[a f\left(\frac{n}{b}\right) \leq c \cdot f(n), \ c < 1\right]$
Then $T(n) \in \Theta(f(n))$

ⓐ $T(n) = 16T\left(\frac{n}{4}\right) + n!$

$a = 16$, $b = 4$, $f(n) = n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ (Stirling's Formula)

Compare $f(n)$ complexity: $f(n) = n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ n

$$\lim_{n \to \infty} \frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}}{n^{\log_4 4^2}} = \lim_{n \to \infty} \frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}}{n^2} = \lim_{n \to \infty} \sqrt{2\pi n}\,\frac{n^{n-2}}{e^n} = \infty$$

$*$ So $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \in \Omega\left(n^{\log_4 16 + \epsilon}\right)$ $\epsilon = 0.1$

$**$ $a f\left(\frac{n}{b}\right) \leq c \cdot f(n)$, $c < 1$ lets say $c = \frac{1}{2}$

$\Rightarrow 16\left(\frac{n}{4}\right)! \leq \frac{1}{2} n! \Rightarrow \boxed{\left(\frac{n}{4}\right)! \leq \frac{n!}{32}}$ for large n its true ✓

From $*$ and $**$ it is Case III. $T(n) = \Theta(f(n)) = \Theta(n!)$

(b) $T(n) = \sqrt{2}\, T\left(\frac{n}{4}\right) + \log n$

$a = \sqrt{2} \approx 1.4,\ b = 4,\ f(n) = \log n$

$a \geq 1$, $b > 1$, $f(n)$ is asymtotically increasing

* compare $f(n)$: $\lim_{n \to \infty} \frac{f(n)}{n^{\log_b a}} \Rightarrow \lim_{n \to \infty} \frac{\log n}{n^{\log_4 \sqrt{2}}} \approx \lim_{n \to \infty} \frac{\ln(n)}{n^{0.25}}$

take derivative of both side $\to \lim_{n \to \infty} \frac{1/n}{\frac{1}{4 n^{3/4}}} \Rightarrow \lim_{n \to \infty} \frac{4 n^{3/4}}{n}$

divide by $n^{3/4}$ $\Rightarrow \lim_{n \to \infty} \frac{4}{\frac{n}{n^{3/4}}} \Rightarrow \lim_{n \to \infty} \frac{4}{n^{1/4}} = 0$

So, $f(n)$ grows much slower than $n^{\log_b a}$. Then $f(n) \in O(n^{\log_b a - \epsilon})$

$\log n \in O(n^{\log_4 \sqrt{2} - \epsilon})$ case I

$$\boxed{T(n) = \Theta(n^{\log_4 \sqrt{2}})}$$

(c) $\boxed{T(n) = 8T\left(\frac{n}{2}\right) + 4n^3}$ $a = 8,\ b = 2,\ f(n) = 4n^3$

$a \geq 1$, $b > 1$, $f(n)$ asymtotically increasing

$4n^3 \in \Theta(n^{\log_2 2^3}) \Rightarrow 4n^3 \in \Theta(n^3)$ Case II

$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^{\log_2 8} \log n)$

$$\boxed{\Rightarrow T(n) = \Theta(n^3 \log n)}$$

(d) $T(n) = 64T\left(\frac{n}{8}\right) - n^2 \log n$

Master theorem could not be applied because $f(n)$ is not asymtotically increasing.

(e) $T(n) = 3T\left(\frac{n}{3}\right) + \sqrt{n}$

$a=3,\ b=3,\ f(n) = n^{1/2}$

$a \geq 1,\ b > 1,\ f(n)$ is asymtotically inc.

$n^{1/2} \in O\left(n^{\log_3 3 - \epsilon}\right)$ where $\epsilon = 0.01$ Case 1

$T(n) = \Theta\left(n^{\log_b a}\right) \Rightarrow \boxed{T(n) = \Theta\left(n^{\log_3 3}\right) = \Theta(n)}$

(f) $T(n) = 2^n T\left(\frac{n}{2}\right) - n^n$ $\boxed{a = 2^n,\ b = 2,\ f(n) = -n^n}$

master theorem could not be applied because $f(n)$ is not asymtotically increasing.

(g) $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{\log n}$ $a=3,\ b=3,\ f(n) = \frac{n}{\log n}$

Because $f(n)$ is not polinomial master theorem could not be applied here.

② ⓐ

9 subproblem indicates $\boxed{a = 9}$

$\frac{n}{3}$ size indicates $\boxed{b = 3}$

quadradic time to combine solutions indicates $\boxed{f(n) = (n^2 + c)}$

$$T(n) = 9T\left(\frac{n}{3}\right) + n^2 + c$$

$n^2 + c \in \Theta\left(n^{\log_3 3^2}\right) = \Theta(n^2)$ case II of Master The

$T(n) = \Theta\left(n^{\log_b a} \log n\right) = \Theta\left(n^{\log_3 3^2} \log n\right) = \Theta(n^2 \log n)$

ⓑ

8 subproblem $\Rightarrow a = 8$

half size of problem size $\Rightarrow b = 2$

$f(n) \in O(n^3)$

$$T(n) = 8T\left(\frac{n}{2}\right) + n^3 + c$$

by another version of master theorem, $a = 8, b = 2, d = 3$

[if $a = b^d$ then $T(n) \in \Theta(n^d \log n) \rightarrow$ (case II)]

$8 = 2^3$ then $T(n) \in \Theta(n^3 \log n)$

ⓒ

dividing into two subproblems $\Rightarrow a = 2$

quarter of size $\Rightarrow b = 4$

combining solutions $= f(n) \in O(\sqrt{n})$

$$T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + c$$

$\sqrt{n} + c \in \Theta\left(n^{\log_4 2}\right) \Rightarrow \sqrt{n} + c \in \Theta(n^{1/2})$

case II of Master Theorem

$T(n) = \Theta\left(n^{\log_b a} \log n\right) = \Theta\left(\sqrt{n} \log n\right)$

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(4) For the given binary search algorithm,

∝ $a=1$, bcs iteration continues on one branch

∝ $b=2$, size of list is divided by two each time

∝ $f(n)=1$, There is no work to gather solutions since there is just one return statement.

1. step $T(n) = T\left(\frac{n}{2}\right)+1$

2 step $T\left(\frac{n}{2}\right) = T\left(\frac{n}{4}\right)+1$

3. step $T\left(\frac{n}{4}\right) = T\left(\frac{n}{8}\right)+1$

⋮

k. step $T\left(\frac{n}{2^{k-1}}\right) = T\left(\frac{n}{2^k}\right)+1$

sum all of the equations

Final sum= $\boxed{T(n) = T\left(\frac{n}{2^k}\right)+k}$

(*) At the final step we know there is only 1 element on the list. So $\boxed{T\left(\frac{n}{2^k}\right) = T(1)}$

From here

$\boxed{n = 2^k \Longleftrightarrow \log n = k}$

Reconfigure Final Sum $\Rightarrow$ $\boxed{T(n) = T(1) + \log n}$

$\Rightarrow$ $\boxed{T(n) = O(\log n)}$