HW5: Principal Component Analysis

Due Oct 27, 2020 by 11:05am **Points** 100 **Submitting** a file upload
Available until Oct 27, 2020 at 11:05am

This assignment was locked Oct 27, 2020 at 11:05am.

Assignment Goals

- Explore Principal Component Analysis (PCA) and the related Python packages
- Make pretty pictures :)

Summary

In this project, you'll be implementing a handwriting analysis program using Principal Component Analysis (PCA).

We'll walk you through the process step-by-step (at a high level).

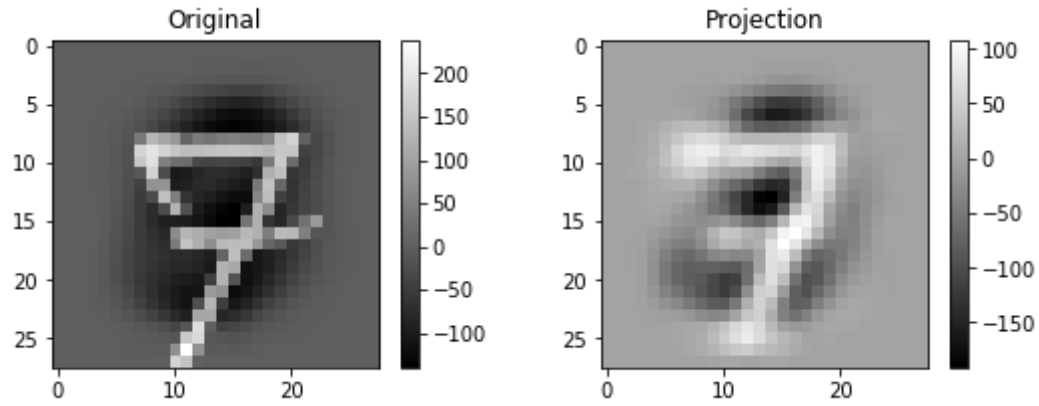
Packages Needed for this Project

In this project, you'll need the following packages:

```
>>> from scipy.linalg import eigh
>>> import numpy as np
>>> import matplotlib.pyplot as plt
```

Dataset

You'll be using part of the MNIST dataset **mnist.npy.** **(https://canvas.wisc.edu/courses/205182/files/15607446/download?download_frd=1)**



The dataset contains 2000 samples images, each of size 28*28. We will use n to refer to number of samples where n=2000. And d to refer to number of features for each sample which is d=28*28=784. We will only test your code using this data set. Here we'll use x_i to refer to the ith sample which would be a d dimension column vector.

Program Specification

Implement these **six (6)** Python functions to do PCA on our provided dataset, in a file called **pca.py**:

1. **load_and_center_dataset(filename)** — load the dataset from a provided .npy file, re-center it around the origin and **return** it as a NumPy array of floats
2. **get_covariance(dataset)** — calculate and **return** the covariance matrix of the dataset as a NumPy matrix (d x d array)
3. **get_eig(S, m)** — perform eigen decomposition on the covariance matrix S and **return** a diagonal matrix (NumPy array) with the largest m eigenvalues on the diagonal, *and* a matrix (NumPy array) with the corresponding eigenvectors as columns
4. **get_eig_perc(S, perc)** — similar to get_eig, but instead of returning the first m, return all eigenvectors that explains more than perc % of variance
5. **project_image(image, U)** — project each image into your m-dimensional space and **return** the new representation as a d x 1 NumPy array
6. **display_image(orig, proj)** — use matplotlib to display a visual representation of the original image and the projected image side-by-side

NOTE: your orig, proj, image and return value of project_image should have np.shape as (784,)

Load and Center the Dataset

First, if you haven't, download our sample dataset to the machine you're working on: **mnist.npy**.

Once you have it, you'll want to use the numpy function `load()` to load the file into Python. (You may need to install NumPy first.)

```
>>> x = np.load(filename)
```

Then, you should reshape the data to n*d (n, the number of images we're analyzing, and d, the dimension of those images). Here we have d=28*28=784 (28 by 28 pixels).

```
>>> x = np.reshape(x,(2000,784))
```

This should give you an *nxd* dataset, where each **row** is an image feature vector of 784 pixels. Note this configuration for future calculations.

Your next step is to re-center this dataset around the origin. Re-centering is simply to subtract the dataset mean $\mu_{x^{orig}}$ from each data point x_i^{orig}. This is defined as $x_i^{cent} = x_i^{orig} - \mu_{x^{orig}}$ where $\mu_{x^{orig}} = \frac{1}{n}\sum_i x_i^{orig}$

From now on, we will just use x_i to refer to x_i^{cent}.

You can take advantage of the fact that `x` (as defined above) is a NumPy array and as such, has this convenient behavior:

```
>>> x = np.array([[1,2,5],[3,4,7]])
>>> np.mean(x, axis=0)
=> array([2., 3., 6.])
>>> x - np.mean(x, axis=0)
=> array([[-1., -1., -1.],
          [ 1.,  1.,  1.])
```

After you've implemented this function, it should work like this:

```
>>> x = load_and_center_dataset('mnist.npy')
>>> len(x)
=> 2000
>>> len(x[0])
=> 784
>>> np.average(x)
=> -2.5057507089662307e-16
```

Its center isn't *exactly* zero, but taking into account precision errors over 2000 arrays of 784 floats, it's what we call Close Enough. (np.average(x) have absolute value below 10^{-10})

Find Covariance Matrix

As given in the notes, the covariance matrix is defined as (if you consider ith sample x_i as a column vector)

$$S = \frac{1}{n-1}\sum_i x_i x_i^T$$

(if you consider ith sample $x_{row\ i}$ as a row vector, then it is)

$$S = \frac{1}{n-1} \sum_i x_{row\ i}^T x_{row\ i}$$

To calculate this, you'll need a couple of tools from NumPy again:

```
>>> x = np.array([[1,2,5],[3,4,7]])
>>> np.transpose(x)
=> array([[1, 3],
          [2, 4],
          [5, 7]])
>>> np.dot(x, np.transpose(x))
=> array([[30, 46],
          [46, 74]])
>>> np.dot(np.transpose(x), x)
=> array([[10, 14, 26],
          [14, 20, 38],
          [26, 38, 74]])
```

The result of this function for our sample dataset should be a d*d (784x784) matrix.

Get m Largest Eigenvalues/Eigenvectors

You'll never have to do eigen decomposition by hand again (https://docs.scipy.org/doc/scipy/reference/generated/scipy.linalg.eigh.html) ! Use `scipy.linalg.eigh()` to help, particularly the optional `eigvals` argument.

We want the *largest m eigenvalues* of S.

Return the eigenvalues as a diagonal matrix, in descending order, and the corresponding eigenvectors as columns in a matrix.

To return more than one thing from a function in Python:

```
def multi_return():
    return "a string", 5
mystring, myint = multi_return()
```

Make sure to return the diagonal matrix of eigenvalues FIRST, then the eigenvectors in corresponding columns. You may have to rearrange the output of eigh() to get the eigenvalues in decreasing order -- and *make sure to keep the eigenvectors in the corresponding columns* after that rearrangement.

```
>>> Lambda, U = get_eig(S, 2)
>>> print(Lambda)
[[350880.76329673       0.        ]
 [     0.         245632.27295307]]
>>> np.sum(U)
12.594115166027581
```

Get all Eigenvectors that Explain More than Certain % of Variance

We want all the *eigenvalues* that explain more than a certain percentage of variance. Let u_i and λ_i be an eigenvector and corresponding eigenvalue.

Then the percentage of variance explained is:

$$\frac{\lambda_i}{\sum_j \lambda_j}$$

Return the eigenvalues as a diagonal matrix, in descending order, and the corresponding eigenvectors as columns in a matrix.

Make sure to return the diagonal matrix of eigenvalues FIRST, then the eigenvectors in corresponding columns. You may have to rearrange the output of eigh() to get the eigenvalues in decreasing order -- and *make sure to keep the eigenvectors in the corresponding columns* after that rearrangement.

```
>>> Lambda, U = get_eig_perc(S, 0.07)
>>> print(Lambda)
[[350880.76329673       0.        ]
 [     0.         245632.27295307]]
>>> np.sum(U)
12.594115166027581
```

Project the Images

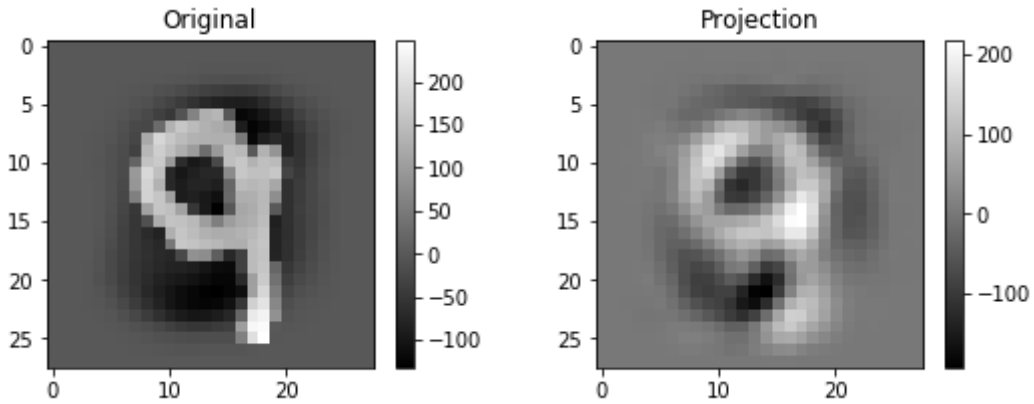
Given one of the images from your dataset and the results of your `get_eig()` function, create and return the *projection* of that image.

For any image $\mathbf{x}_i$, we project it into the m dimensional space span eigenvectors by:

$x_i^{proj} = \sum_{j=1}^{m} \alpha_{ij} u_j$ where $\alpha_{ij} = u_j^T x_i$, $\alpha_i \in R^m$ and $x_i^{proj} \in R^d$.

Visualize

We'll be using **matplotlib's imshow (https://matplotlib.org/3.1.3/api/_as_gen/matplotlib.pyplot.imshow.html)**. First, make sure you have the **matplotlib library (https://matplotlib.org/3.1.1/users/installing.html)**, for creating figures.



Follow these steps to visualize your images:

1. Reshape the images to be 28x28 (you should have calculated them as 1d vectors of 784 numbers).

2. Create a figure with one row of two **subplots** (https://matplotlib.org/api/_as_gen/matplotlib.pyplot.subplots.html).
3. The first subplot (on the left) should be **titled** (https://matplotlib.org/3.1.3/gallery/subplots_axes_and_figures/figure_title.html) "Original", and the second (on the right) should be titled "Projection".
4. Use `imshow()` with optional argument `aspect='equal'` and `cmap='gray'` to render the original image in the first subplot and the projection in the second subplot.
5. Use the return value of `imshow()` to create a **colorbar** (https://matplotlib.org/3.1.0/gallery/color/colorbar_basics.html) for each image.
6. **Render** (https://matplotlib.org/api/_as_gen/matplotlib.pyplot.show.html) your plots!

NOTE: to plot image on CSL machines, you will need to save the plot to a file (cause it doesn't have graphic interface)

1. add the following lines to the beginning of your code.

```
>>> import matplotlib
>>> matplotlib.use('Agg')
```

2. indead of using plt.show() use plt.savefig("filename.png").

(You should use plt.show() in your submission.)

Submission Notes

Please submit your files in a zip file named **hw5_<netid>.zip**, where you replace <netid> with your netID (your wisc.edu login). Inside your zip file, there should be **only** one file named: **pca.py**. Do not submit a Jupyter notebook .ipynb file.

Be sure to **remove all debugging output** before submission; your functions should run silently (except for the image rendering window). Failure to remove debugging output will be **penalized (10pts)**.

If a regrading request isn't justifiable (the initial grade is correct and clear, subject to instructors' judgment), the request for regrading will be **penalized (10 pts)**.

This assignment due at 10/27/2020 11:00am. Submitting right at 11:00am will result in a late submission. It is preferred to first submit a version well before the deadline (at least one hour before) and check the content/format of the submission to make

sure it's the right version. Then, later update the submission until the deadline if needed.

Changelog

See below for updates to the assignment since the initial release

Added explanation for n and d in dataset section.

Updated to correct the diagonal flipped sample image.

Elaborate that $\boldsymbol{x_i}$ is column vector.

Elaborate on the definition of covariance matrix as row vectors.

Add clarification that students will only be tested on mnist.npy

Added clarification on np.shape() of vectors

Added guide about how to test plot on CSL machines

Changed the CSL guide from jpg to png

Rubric

Criteria	Ratings		Pts
load_and_center_dataset	**20 pts** **Full Marks**	**0 pts** **No Marks**	20 pts
get_covariance	**15 pts** **Full Marks**	**0 pts** **No Marks**	15 pts
get_eig & get_eig_perc	**25 pts** **Full Marks**	**0 pts** **No Marks**	25 pts
project_image	**15 pts** **Full Marks**	**0 pts** **No Marks**	15 pts
display_image	**15 pts** **Full Marks**	**0 pts** **No Marks**	15 pts
image correctly formated	**10 pts** **Full Marks**	**0 pts** **No Marks**	10 pts
			Total Points: 100