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THE CLASSIFICATION OF COMPLETE MINIMAL SURFACES IN $\mathbb{R}^3$ WITH TOTAL CURVATURE GREATER THAN -8π

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Introduction. One of the fundamental problems in the classical theory of minimal surfaces is the existence of complete immersed minimal surfaces in three dimensional Euclidean space. Two famous examples of these surfaces are the catenoid and Enneper's surface each of which has total Gaussian curvature equal to -4π. The complete minimal surfaces of finite total curvature have some very special analytic and geometric properties that are not shared by general minimal surfaces. For example, R. Osserman [9] has shown that if the total curvature of a complete minimal surface $f: M \to \mathbb{R}^3$ is finite, then M is "conformally" diffeomorphic to a compact "Riemann surface" $\overline{M}$ punctured in a finite number of points and the Gauss map $g: M \to \mathbb{P}^2$ extends "conformally" to $\overline{M}$. The reason for the quotes is that one needs to make sense of "conformal structure" and "conformal map" when a surface is nonorientable. An immediate corollary to Osserman's theorem is that the total curvature for a complete minimal surface in $\mathbb{R}^3$ is always a multiple of -2π. R. Osserman then used the special properties of the Gauss map to prove that the plane is the unique complete minimal surface with total curvature greater than -4π and that the catenoid and Enneper's surface are the unique such orientable surfaces with total curvature -4π.

In this paper we present an elementary global analysis of the topology of complete nonorientable minimal surfaces $f: \overline{M} - \{p_1, \ldots, p_k\} \to \mathbb{R}^3$ with finite total curvature where $\overline{M}$ is compact. In the first section we prove a formula that states that the total curvature of this surface divided by 2π is congruent modulo 2 to the Euler characteristic of $\overline{M}$. This formula together with some topological results in [4] are then applied to prove that the total curvature of this nonorientable surface must be less than -4π and that if the total curvature of the surface equals -6π, then the surface is "conformally" diffeomorphic to a projective plane punctured in one or two points.

In section 2 we derive a Weierstrass type representation to analytically present nonorientable minimal surfaces which are homeomorphic to subsets of the projective plane. Using this representation, it is shown that the oriented two-sheeted cover of a minimal Möbius strip (with or without boundary) has well-defined associate surfaces and hence it is not rigid. The existence of associate surfaces places strong restrictions on the coordinate functions of the minimal Möbius strip. In section 3, these restrictions on the coordinate functions of a minimal Möbius strip are exploited to prove there exists a unique complete minimal Möbius strip with total curvature -6π.

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In section 4 the results of the previous sections combine with some rather lengthy computations of residues to prove the main result of the paper. This result states that up to a projective transformation of R^3 there are exactly four complete minimal surfaces of finite total curvature greater than -8π. These surfaces are the plane, the catenoid, Enneper's surface and the minimal Möbius strip discussed in section 3.

The work in this paper was motivated in part by the author's joint paper [4] with L. Jorge. The result (unpublished) that there are no nonorientable minimal surfaces with total curvature -4π was first found jointly with L. Jorge by another method. The author would like to thank both L. Jorge and L. Barbosa for helpful discussions with this material.

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Section 1. A topological congruence formula for the total curvature. In this section we derive a topological congruence formula for the total curvature of a complete nonorientable minimally immersed surface $f\colon M \to \mathsf{R}^3$. By the classical existence theorem for isothermal parameters on the oriented two-sheeted covering space of M there always exists a system of coordinates on M so that the change of coordinates is conformal or anticonformal. By abuse of language we will say that such a system of coordinates gives M a "conformal" structure. A map $f\colon M_1 \to M_2$ between two nonorientable "Riemann surfaces" will be called "conformal" if in some local "conformal" coordinates on M_1 and M_2, the map f is conformal.

THEOREM 1. *Let M_1 and M_2 be compact surfaces without boundary and let $f\colon M_1 \to M_2$ be a branched cover of M_2. If the Euler characteristic $\chi(M_2)$ is odd, then $\chi(M_1)$ and the degree of f are either both even or both odd. If $\chi(M_2)$ is even, then $\chi(M_1)$ is even.*

Proof of Theorem 1. In the case that M_2 is orientable or the case f has no branch points, then the theorem is well known and follows from the Riemann-Hurwitz formula. Suppose now that M_2 has odd Euler characteristic. By the classification of surfaces M_1 is the connected sum of a projective plane and an orientable surface. Therefore, there exists a thin open Möbius strip T disjoint from the branch locus of f such that $X = (M_2 - T)$ is a compact orientable surface with boundary. Let $Y = f^{-1}(X)$ and $f|\, Y\colon Y \to X$ be the associated branched covering space. It is standard that Y is an orientable surface and $\chi(Y) = 2(1-g) - k$ where g is the genus of Y and k is the number of boundary components of Y. Since each component of a finite cover of $\overline{T}$ is a cylinder or Möbius strip which has zero Euler characteristic, $\chi(Y) = \chi(M_1)$. Let $W = f^{-1}(T)$ and $f|\, W\colon W \to T$. It is simple to show that the number k of boundary components of W is congruent mod 2 to the degree of $f|\, W$ which equals the degree of f. This k is the same as the number of boundary components of Y. The first part of the theorem now follows from the equation

$$\chi(M_1) = \chi(Y) = 2(1 - g) - k.$$

If $\chi(M_2)$ is even, then the classification of compact nonorientable surfaces implies that M_2 is a Klein bottle K connected sum with an orientable surface M'. We may consider K to be a cylinder glued in a twisted way along one of its boundary curves. These boundary curves give rise to a Jordan curve γ in K whose regular neighborhood T is a cylinder and $K - T$ is also a cylinder. Now consider γ to be contained under inclusion in $M_2 = K \# M'$ and suppose that γ is disjoint from the branch set of f. Using the same type argument as in the proof of the first statement of the theorem one checks that $\chi(M_1)$ is even. This completes the proof of the theorem.

COROLLARY 1. *Suppose that $\overline{M}$ is a compact surface and that $f: \overline{M} - \{p_1, \dots, p_k\} \to \mathbf{R}^3$ is a complete minimal immersion. If the total curvature $c(M)$ is finite, then $c(M)/2\pi \equiv \chi(\overline{M})$ mod 2.*

Proof of Corollary 1. By Osserman it follows that the Gauss map $g: \overline{M} - \{p_1, \dots, p_k\} \to \mathbf{P}^2$ extends to a branched covering space $\bar{g}: \overline{M} \to \mathbf{P}^2$. Since $c(M) = 2\pi$ degree $(\bar{g})$, the theorem implies $c(M)/2\pi \equiv \chi(\overline{M})$ mod 2 as was to be shown.

COROLLARY 2. *The only complete minimally immersed surfaces $f: M \to \mathbf{R}^3$ of total curvature greater than -6π are the plane, the catenoid and Enneper's surface. Furthermore, if the total curvature of f is -6π, then M is conformally diffeomorphic to a projective plane punctured in one or two points.*

Proof of Corollary 2. From Osserman [9] we know that the catenoid and Enneper's surface are the unique orientable complete minimal surfaces with total curvature -4π and hence there are no examples of nonorientable surfaces with total curvature -2π. Let $f: \overline{M} - \{p_1, \dots, p_k\} \to \mathbf{R}^3$ be a complete nonorientable minimal surface with finite total curvature -4π. Corollary 1 implies that the Euler characteristic of $\overline{M}$ is even and the classification of nonorientable surfaces implies $\chi(\overline{M})$ is nonpositive.

Theorem 4 in [4] states that the total curvature of a complete minimal immersion $F: M = (\overline{M} - \{q_1, \dots, q_k\}) \to \mathbf{R}^3$ has total curvature not greater than $2\pi(\chi(\overline{M}) - 2k)$ with equality if and only if the ends of the surface are embedded. Since the surface under consideration has nonpositive Euler characteristic and total curvature -4π, $\chi(\overline{M}) = 0$ and $F(M)$ has one end which is embedded. Another result in [4] states that a complete minimal surface in $\mathbf{R}^3$ which has one end which is embedded must be a flat plane or have infinite total curvature. Thus there are no nonorientable examples with total curvature -4π.

If $M = \overline{M} - \{p_1, \dots, p_k\}$ and $f: M \to \mathbf{R}^3$ is a complete minimal surface with total curvature $c(M) = -6\pi$, then as before $c(M) \leqslant 2\pi(\chi(\overline{M}) - 2k)$ with equality if and only if the ends of M are embedded. By Corollary 1 $\chi(\overline{M}) = +1$ with $k < 2$ or $\chi(\overline{M}) = -1$ with $k = 1$. However, the argument given in the previous paragraph eliminates the possibility of $\chi(\overline{M}) = -1$ with $k = 1$. Thus M must be the projective plane punctured in one or two points and this fact completes the proof of the corollary.

Remark. Corollary 1 continues to be true in the case the minimal immersion $f: \overline{M} - \{p_1, \ldots, p_k\} \to \mathbb{R}^3$ has a finite number of branch points. The classical Henneberg's surface which is a "complete" branched minimal Möbius strip with two branch points has total curvature -2π. Therefore, Corollary 2 is only valid with the restriction that the surface is immersed.

Section 2. The Weierstrass representation for a nonorientable surface and some applications. The classical Weierstrass representation gives an analytic representation for the coordinate functions of a branched minimally immersed orientable surface $h: M \to \mathbb{R}^3$. Recall that this representation is based on the existence of a meromorphic function $g: M \to S^2$ which is the Gauss map for the surface and a meromorphic 1-form α which can be expressed in local coordinates by $\alpha = f(z)\,dz$. The representation theorem states that after a translation of $h(M)$, then

$$h(p) = Re \int_{p_0}^{p} \begin{matrix} w_1 = (1 - g^2)\alpha \\ w_2 = i(1 + g^2)\alpha \\ w_3 = 2g\alpha \end{matrix}$$

We shall call the meromorphic 1-forms w_i the Weierstrass 1-forms for h and in the case when M is a planar domain and dz is globally defined we shall call f and g the Weierstrass functions in the Weierstrass representation. Conversely, if $h: M \to \mathbb{R}^3$ is well defined by the above formula for some g and α, then h is a branched minimal immersion.

In the next proposition we give the "Weierstrass formulae" for a nonorientable minimal surface which is covered by a planar domain.

PROPOSITION 1. *Let $h: M \to \mathbb{R}^3$ be a branched minimal immersion of a nonorientable surface. Let $\pi: \tilde{M} \to M$ denote the oriented two-sheeted covering of M and $\tilde{h} = h \circ \pi: \tilde{M} \to \mathbb{R}^3$. If $\tilde{M}$ is a planar domain, then*

(1) *There exists a conformal embedding of M in $\mathbb{P}^2$ so that $\tilde{M} \subset S^2$ is invariant under the inversion $I(z) = -1/\bar{z}$ and I induces the natural order two covering transformation for the covering space $\pi: \tilde{M} \to M$.*

(2) $g(I(z)) = \overline{I(g(z))}$.

(3) $f(z) = -\overline{f(I(z))}/(zg(z))^2$.

Conversely, if $\tilde{M} \subset S^2$ is connected and invariant under the involution I, $f, g: S^2 \to S^2$ are meromorphic functions which satisfy (2) *and* (3), *and the Weierstrass representation with the functions f and g gives rise to a well-defined map $\tilde{h}: \tilde{M} \to \mathbb{R}^3$, then $\tilde{h} = h \circ \pi$ where $\pi: \tilde{M} \to M$ is the oriented two-sheeted covering space of the orbit space M for the action of I on $\tilde{M}$.*

Proof of Proposition 1. Let $\pi' : S^2 \to \mathsf{P}^2$ be the natural two-sheeted covering space of the projective plane where one considers P^2 to be the orbit space of the action of antipodal map $I(z) = -1/\bar{z}$ on S^2. The uniformization theorem for conformal structures on planar domains states that such surfaces embed conformally in S^2. Actually this theorem implies that M "conformally" embeds in the "conformal" structure on P^2 induced from S^2 and π'. After defining $\tilde{M}$ by $\tilde{M} = (\pi')^{-1}(M)$, the map $\pi = \pi' \mid \tilde{M} : \tilde{M} \to M$ is the required covering space. Now let $\tilde{h} = h \circ \pi : \tilde{M} \to \mathsf{R}^3$.

Property (2) follows from the observation that g is the Gauss map for $\tilde{h} : \tilde{M} \to \mathsf{R}^3$ and that the unit normal at $\tilde{h}(z)$ is parallel and in the opposite direction of $\tilde{h}(I(z))$.

Now consider the action of I on the Weierstrass forms. For example, let

$$\delta = I^*(w_3) = I^*(2fgdz) = 2f(-1/\bar{z})\, g(-1/\bar{z}) \frac{1}{\bar{z}^2}\, \overline{dz}\,.$$

By property (2), we have

$$\bar{\delta} = -2\,\overline{f(-1/\bar{z})}\,/\big(z^2 g(z)\big)\, dz.$$

Since I extends to the identity map in R^3, $\mathrm{Re}(I^*(w_3)) = \mathrm{Re}(\bar{w}_3)$ which implies that $I^*(w_3) = \bar{w}_3$. This last equation shows $\bar{\delta} = w_3$ which proves property (3).

We now prove the converse to the first part of the proposition. If $\tilde{M}$ is invariant by the inversion I, g satisfies (2) and f satisfies (3), then for each Weierstrass 1-form w_i, one calculates that $I^*(w_i) = \bar{w}_i$. Suppose now that f and g give rise to a well-defined map $\tilde{h} : \tilde{M} \to \mathsf{R}^3$ where

$$\tilde{h}(p) = \mathrm{Re} \int_{p_0}^{p} (w_1, w_2, w_3).$$

Then

$$\begin{aligned}
\tilde{h}(I(p)) &= \mathrm{Re} \int_{p_0}^{I(p)} (w_1, w_2, w_3) = \mathrm{Re}\left(\int_{p_0}^{I(p_0)} (w_1, w_2, w_3) + \int_{I(p_0)}^{I(p)} (w_1, w_2, w_3) \right) \\
&= v_0 + \mathrm{Re} \int_{p_0}^{p} (I^*(w_1), I^*(w_2), I^*(w_3)) \\
&= v_0 + \mathrm{Re} \int_{p_0}^{p} (\bar{w}_1, \bar{w}_2, \bar{w}_3) = v_0 + \tilde{h}(p).
\end{aligned}$$

Now $\tilde{h}(p) = \tilde{h}(I(I(p))) = v_0 + \tilde{h}(I(p)) = 2v_0 + \tilde{h}(p)$. Therefore $v_0 = 0$ and $\tilde{h}$ factors through the orbit space M of the action by I which proves the converse to the proposition.

Remark. The arguments in Proposition 1 naturally generalize to give a Weierstrass type representation for every nonorientable minimal surface $h : M \to \mathsf{R}^3$. In the general case let $\pi : \tilde{M} \to M$ be the oriented two-sheeted cover of M and let $\tilde{I}: \tilde{M} \to \tilde{M}$ be the order two deck transformation for this cover. As before let $I: S^2 \to S^2$ be the map $I(z) = -1/\bar{z}$. In this general case $g(\tilde{I}(z)) = I(g(z))$ and $\tilde{I}^*(\alpha) = -g^2\bar{\alpha}$ where g and α appear in the Weierstrass representation of $\tilde{h} = h \circ \pi: \tilde{M} \to \mathbf{R}^3$. Conversely, if g and α satisfy these properties on the oriented two-sheeted cover $\pi: \tilde{M} \to M$ of M and if they give rise by the Weierstrass representation to a branched minimal immersion $\tilde{h}: \tilde{M} \to \mathbf{R}^3$, then $\tilde{h} = h \circ \pi$ for some $h: M \to \mathbf{R}^3$.

The following technical lemma shows the Weierstrass 1-forms on the oriented two-sheeted cover of a nonorientable minimal surface contain period relations.

LEMMA 1. *Let $\tilde{h}: \tilde{M} \to \mathbf{R}^3$ be a branched minimal immersion of a connected oriented planar domain with $2k$ annular ends $\{A_1, A_2, \ldots, A_{2k}\}$. Suppose that $\tilde{h}$ factors as $h \circ \pi$ where $\pi: \tilde{M} \to M$ is the oriented two-sheeted covering space of a nonorientable surface M and $h: M \to \mathbf{R}^3$ is a branched minimal immersion. Suppose that $I(A_i) = A_{i+k}$ where I is the associated order two deck transformation of the covering space $\pi: \tilde{M} \to M$. Let γ_i denote a generator of the fundamental group of A_i. If w_j denotes a Weierstrass 1-form and $\operatorname{Im} \int_{\gamma_i} w_j = 0$ for $1 \leqslant i < k$, then w_j is an exact differential.*

Proof of Lemma 1. The lemma will be proved by showing that the periods of w_j are all zero. Let γ_i be as above with the orientation induced from the orientation of the boundary annulus A_i. We first show that $\int_{\gamma_k} w_j = 0$.

If we let $[\gamma_i]$ denote the one dimensional integer homology class of γ_i, then the class $\sum_{i=1}^{k} [\gamma_i]$ can be represented by a closed oriented curve γ invariant under I in an orientation preserving way. Therefore $I_*(\delta = \sum_{i=1}^{k}[\gamma_i]) = \delta$. Since the form w_j is closed, we get

$$\operatorname{Re} \int_\delta w_j + i \operatorname{Im} \int_\delta w_j = \int_{\delta \circ I} w_j = \int_\delta I^* w_j = \int_\delta \bar{w}_j = \operatorname{Re} \int_\delta w_j - i \operatorname{Im} \int_\delta w_j.$$

In the next to last equation we used the fact that $I^*(w_j) = \bar{w}_j$ as discussed in the proof of Proposition 1. Since the surface $\tilde{h}: \tilde{M} \to \mathbf{R}^3$ exists, the real periods of w_j are all zero. From the above equation and our hypothesis concerning the imaginary periods of γ_i for $1 \leqslant i < k$ we get

$$\operatorname{Im} \int_{\gamma_k} w_j = \operatorname{Im} \int_\delta w_j = -\operatorname{Im} \int_\delta w_j = -\operatorname{Im} \int_{\gamma_k} w_j.$$

This equation forces the period of w_j for γ_k to be zero.

Since $I_*(\sum_{i=2}^{k+1} [\gamma_i]) = \sum_{i=2}^{k+1} [\gamma_i]$ and the periods of w_j for γ_i with $2 \leqslant i < k+1$ are zero, the argument given above shows that w_j has no period along γ_{k+1}. A simple induction argument then proves that $\int_{\gamma_i} w_j = 0$ for all i. Since $\tilde{M}$ is a planar domain, its first homology group is generated the curves γ_i for $1 \leqslant i < 2k$. Therefore w_j has no periods and that means that w_j is exact. This completes the proof of the lemma.

The simplest case of the above lemma is when the surface M is a Möbius strip. If $h: M \to \mathbf{R}^3$ is a minimal Möbius strip, then for the oriented two sheeted cover

$\tilde{M}$ we get an induced surface $\tilde{h}: \tilde{M} \to \mathbf{R}^3$. In this case $\tilde{M}$ is an annulus and by the previous lemma the Weierstrass 1-forms are exact. This observation yields the following proposition.

PROPOSITION 2. *Let $h: M \to \mathbf{R}^3$ be a branched minimal immersion of a closed or open Möbius strip and let $\pi: \tilde{M} \to M$ be the oriented two-sheeted cover of M by the annulus. Then the associate surfaces to $\tilde{h} = h \circ \pi: \tilde{M} \to \mathbf{R}^3$ are well defined on $\tilde{M}$.*

Section 3. The existence and uniqueness of a complete minimal Möbius strip with total curvature -6π. In section 1 it was shown that if a complete minimal surface has total curvature -6π, then this surface $h: M \to \mathbf{R}^3$ is conformally diffeomorphic to the projective plane punctured in one or two points. We now show that there is a unique such h in the case M is a once punctured projective plane. Here unique means unique up to a projective or similarity transformation of $\mathbf{R}^3$.

THEOREM 2. *There is a unique complete minimally immersed Möbius strip in $\mathbf{R}^3$ with finite total curvature -6π. The Weierstrass representation for the oriented two-sheeted covering annulus is found by letting $g(z) = z^2[(z+1)/(z-1)]$ and $f(z) = i[(z-1)^2/z^4]$. Furthermore, the image Möbius strip is invariant under a rotation by $180°$ around a straight line contained in the surface.*

Proof. By Proposition 1, we may assume that the g and f in the Weierstrass representation satisfy the properties (2) and (3) given in that proposition. The proof will be a case by case study of the possible examples that arise from the possible Gauss maps $g(z)$. First, after a rotation in $\mathbf{R}^3$ of a possible surface $\tilde{h}(\tilde{M})$, we may assume that the Gauss map has a zero at one end of $\tilde{M}$ and a pole at the other end of $\tilde{M}$. After a rotation of coordinates on $\tilde{M} \subset S^2$ we may also assume that $g(0) = 0$ and $g(\infty) = \infty$. Recall that g is a meromorphic function on S^2 of degree three because $c(M) = -6\pi$.

Case 1. The extended map g to S^2 has a branch point of order two at the ends of $\tilde{M}$.

Since g has degree three, $g(z) = cz^3$ is the only possibility for some nonzero complex number c. Since the map $\tilde{h}$ is an immersion, $f(z) = d/z^4$ for some non-zero complex number d. The form $w_3 = 2fgdz$ in the Weierstrass representation has a nonzero period in contradiction to Proposition 2. Thus this case cannot occur.

Case 2. The extended map g to S^2 has a branch point of order one at the ends of $\tilde{M}$.

In this case $g(z) = cz^2[(z-b)/(z-a)]$ for nonzero complex numbers a, b and c. After a rotation of the coordinates of $\tilde{M} \subset S^2$ around the z-axis, we may assume that a is a positive real number. After a rotation of $\tilde{h}(\tilde{M})$ around the z-axis, we may assume that c is a positive real number. Note that this last movement does not change the previous value of the positive constant a.

Since $g(-1/\bar{z}) = -1/\overline{g(z)}$, $b = -1/a$ and $c = a$. Proposition 1 implies that up to multiplication by a real scalar that $f(z) = [i(z-a)^2/z^4]$ and one checks that such an f satisfies property (3) in that proposition.

Recall that Proposition 2 states that the Weierstrass 1-forms are exact. After simple algebraic manipulations the 1-forms $\alpha_1 = fdz$, $\alpha_2 = fg^2dz$ and $\alpha_3 = \frac{1}{2}w_3 = fgdz$ are also seen to be exact. Now $\mathrm{Res}(\alpha_3)(0) = c(1 - a^2/a)i = 0$ which implies that $a = 1$. Since for each i $\mathrm{Res}(\alpha_i)(0) = 0$ when $a = 1$, this value of a gives rise by the Weierstrass representation to a well-defined map $\tilde{h}: \tilde{M} \to \mathbf{R}^3$. Proposition 1 now shows that $\tilde{h}$ factors through a map of the Möbius strip into $\mathbf{R}^3$.

Case 3. The extended map g to S^2 has no branch point at the ends of M.

In this case we may assume after a rotation of the coordinates and of the surface around the z-axis that

$$g(z) = cz\frac{(z + 1/a)(z + 1/\bar{b})}{(z-a)(z-b)},$$

where a and c are positive real numbers and b is a nonzero complex number. In this case

$$f(z) = r(z-a)^2(z-b)^2/z^4.$$

Recall from previous discussion in Case 2 that if the map $\tilde{h}$ exists, then the 1-forms $\alpha_1 = fg^2dz$, $\alpha_2 = fdz$ and $\alpha_3 = \frac{1}{2}w_3 = fgdz$ are exact. Calculating the residues, one gets $\mathrm{Res}(\alpha_2)(0) = -2r(a+b)$ and so $a = -b$. Since $\mathrm{Res}(\alpha_3)(0) = -cr(a^2 + 1/a^2) \neq 0$, Case 3 cannot occur.

It remains to show that there is a straight line contained on the example given in Case 2 above. Consider the map $A: S^2 \to S^2$ defined by $A(z) = \bar{z}$. After a translation of the point $\tilde{h}(i)$ to the origin one checks that this automorphism has as fixed point set the real axis on S^2 and that A extends to an isometry of $\mathbf{R}^3$ which is a rotation around the y-axis. It is interesting to note that this automorphism also has an isolated fixed point on the Möbius strip at the point corresponding to $h(i)$.

Remark. In [6] the author showed that the flat three dimensional torus $T^3 = \mathbf{R}^3/Z^3$ contains for every k greater than 2 an immersed nonorientable compact minimal surface M with $\chi(M) = -k$. Furthermore for odd integers k these surfaces lift to $\mathbf{R}^3$ to give proper nonorientable minimally immersed surfaces. By an application of Alexander duality a proper embedded surface in $\mathbf{R}^3$ disconnects $\mathbf{R}^3$ and hence is orientable. Thus all proper immersed nonorientable minimal surfaces are not embeddings. Whether there existed a nonorientable minimal surface of finite total curvature has been a frequently mentioned problem which now has a positive solution by Theorem 2.

Section 4. The classification theorem. In this section, we shall prove that the minimal Möbius strip constructed in the previous section is the unique example of a complete minimal surface of finite total curvature -6π. Once this has been proved, the Corollary 2 to Theorem 1 will imply the next theorem.

Classification Theorem. If $f: M \to \mathbb{R}^3$ is a complete minimal surface of finite total curvature greater than -8π, then up to a projective transformation of $\mathbb{R}^3$ this surface is a plane, the catenoid, Enneper's surface or the minimal Möbius strip described in Theorem 2.

By Corollary 2 to Theorem 1 the classification theorem will be proved once we have shown that there does not exist a complete minimal surface of finite total curvature -6π which is conformally a projective plane punctured in two points. The proof of the nonexistence of such an example is a long computation with the periods of the Weierstrass 1-forms. To simplify these computations we recall the following lemma from [4] whose proof is a straightforward computation.

LEMMA 2. *If*

$$\alpha = \frac{\prod_{j=1}^{m} (z-\delta_j)^{m_j}}{\prod_{i=1}^{n} (z-\beta_i)^2}\, dz, \quad \delta_j \neq \beta_i \quad \textit{for all } i \textit{ and } j,$$

is an exact differential, then for each k

$$\sum_{j=1}^{m} \frac{m_j}{\beta_k - \delta_j} = \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2}{\beta_k - \beta_j}.$$

LEMMA 3. *There does not exist a complete minimal immersion $h: M \to \mathbb{R}^3$ with total curvature -6π where M is "conformally" diffeomorphic to the projective plane punctured in two points.*

Proof of Lemma 3. We shall assume that the oriented two-sheeted cover $\pi: \tilde{M} \to M$ satisfies the conditions stated in Proposition 1. Let $\tilde{h} = h \circ \pi : \tilde{M} \to \mathbb{R}^3$. As in the calculations in Theorem 2 we take advantage of the large symmetry of the sphere to make the Gauss map more cononical. Let $g: S^2 \to S^2$ denote the Gauss map of $\tilde{h}$ extended as a meromorphic function to S^2. First of all rotate the coordinates of $\tilde{M}$ so that two ends of $\tilde{M}$ correspond to $z = 0$ and $z = a$ where a is a positive constant. Furthermore in our choice of our rotation we make sure that the order of branching of g at $z = 0$ is greater than or equal to the order of branching at $z = a$. Next rotate the surface $\tilde{h}(\tilde{M})$ so that $g(0) = 0$. From Proposition 1 the other two ends of $\tilde{M}$ are at $z = -1/a, \infty$ with $g(\infty) = \infty$ and $g(-1/a) = -1/\overline{g(a)}$.

For later use we define some auxilary 1-forms α_1, α_2, α_3. These forms are defined by $\alpha_1 = fg^2dz$, $\alpha_2 = fdz$ and $\alpha_3 = 1/2w_3 = fg\,dz$. A straightforward application of Lemma 1 together with the Weierstrass representation expressed in terms of α_1, α_2, and α_3 shows that if the form α_1 or α_2 has a zero residue at $z = 0$, then both α_1 and α_2 are exact. Similarly, if α_3 has a zero residue at $z = 0$, then α_3 is exact. These exactness properties of α_1, α_2 and α_3 will be used repeatedly in the analysis of the various possibilities for the Gauss map.

Case 1. The map g has a branch point of order two at $z = 0$.

After a rotation of $\tilde{h}(\tilde{M})$ around the z-axis we may assume that $g(z) = z^3$. In this form for $g(z)$, Proposition 1 implies that

$$f(z) = \frac{d}{(z(z-a)(z+1/a))^2}$$

where d is some purely imaginary number. Since α_1 has no residue at $z = 0$, α_1 and α_2 must both be exact. However, $\mathrm{Res}(\alpha_2)(0) = 2d(1/a - a) = 0$ which implies $a = 1$. Then $\mathrm{Res}(\alpha_2)(1) = -3d/4 \neq 0$ which contradicts that α_2 is exact. Thus Case 1 cannot occur.

Case 2. The map g has a branch point of order one at $z = 0$.

In this case we may assume, after a rotation of $\tilde{h}(\tilde{M})$ and an application of Proposition 1, that

$$g(z) = cz^2(z + 1/\bar{b})/(z - b)$$

where $c = |b|$. If $a = b$, then $f(z) = d/z^2(z + 1/a)^2$ where d is purely imaginary. Since $\mathrm{Res}(\alpha_1)(0) = 0$, α_2 is exact. However, $\mathrm{Res}(\alpha_2)(0) = -2a^3d \neq 0$ which is impossible. Therefore, a is different from b.

Since $a \neq b$, we get

$$f(z) = d\left[\frac{(z-b)}{z(z-a)(z+1/a)}\right]^2.$$

We see that $\mathrm{Res}(\alpha_1)(0) = 0$ and so both α_1 and α_2 are exact. Since α_2 is exact, Lemma 2 applied to α_2 for $\beta_k = 0$ yields $a \neq 1$ and $b = a/(1 - a^2)$. Now Lemma 2 applied to α_2 for $\beta_k = a$ gives the equations

$$a - b = \frac{a^3 + a}{2a^2 + 1} \quad \text{or} \quad a - \frac{a}{1 - a^2} = -\frac{a^3}{1 - a^2} = \frac{a^3 + a}{2a^2 + 1}.$$

The right hand side of the second equation is positive for $z = a$ positive. The left hand side of the second equation is negative for $z = a$ where $0 < a < 1$. Since after a possible rotation of the coordinates of $\tilde{M}$ around the z-axis by 180° we may assume that $0 < a < 1$, we arrive at a contradiction and so Case 2 cannot occur.

Case 3. The map g has no branch points at the ends of M.

In this case, we may assume that

$$g(z) = \frac{cz(z + 1/\bar{b})(z + 1/\bar{t})}{(z - b)(z - t)}$$

where $c = |bt|$. If $a = b$, then by Case 2 we may assume that $a \neq t$ and that $f(z) = [d(z - t)^2/z^2(z + 1/a)^2]$. Since $\operatorname{Res}(\alpha_1)(0) = 0$, the form α_2 is exact. An application of Lemma 2 for α_2 and $\beta_k = 0$ shows $t = -1/a$ and hence g has degree one rather than degree three. This contradiction shows we may assume that $a \neq b$.

Since a is different from b, we get

$$f(z) = \frac{d(z - b)^2(z - t)^2}{z^2(z - a)^2(z + 1/a)^2}.$$

A simple calculation with Proposition 1 part (3) shows that d satisfies the equation

$$\bar{d} = -\frac{db^2t^2}{|bt|^2}. \tag{1}$$

Since $\operatorname{Res}(\alpha_1)(0) = 0$, both α_1 and α_2 are exact. An application of Lemma 2 to α_2 for the values $\beta_k = 0,\ a,\ -1/a$ gives rise by direct substitution to the following three equations:

$$\frac{1}{b} + \frac{1}{t} = \frac{1 - a^2}{a} \tag{2}$$

$$\frac{a - b}{|a - b|^2} + \frac{a - t}{|a - t|^2} = \frac{2a^2 + 1}{a^3 + a} \tag{3}$$

$$\frac{ab + 1}{|ab + 1|^2} + \frac{at + 1}{|at + 1|^2} = \frac{a^2 + 2}{a^2 + 1}. \tag{4}$$

Calculating residues we get $\operatorname{Res}(\alpha_3)(0) = (cdb^2t^2/|bt|^2)$. Since the imaginary part of $\operatorname{Res}(\alpha_3)(0)$ is zero and c is real, then db^2t^2 must be real. Now equation (1) shows that d is real and that bt is purely imaginary.

We are now going to use the fact that bt is purely imaginary and $z = a$ is real to eventually derive a contradiction. First we rewrite equation (2) as

$$b + t = bt(1 - a^2)/a \tag{5}$$

and then take the real parts of each side of the equation to get that $\operatorname{Re}(b) = -\operatorname{Re}(t)$. Now we may write

$$b = b_1 + ib_2 \quad \text{and} \quad t = -b_1 + it_2.$$

After taking the imaginary parts of equations (2), (3) and (4) and inverting the corresponding fractions we get three new equations. (Note since bt is imaginary, equation (2) shows that neither b_2 nor t_2 are zero and that we may invert these fractions.):

$$\frac{b_1^2 + b_2^2}{b_2} = \frac{b_1^2 + t_2^2}{-t_2} \tag{6}$$

$$\frac{a^2 - 2ab_1 + b_1^2 + b_2^2}{b_2} = \frac{a^2 + 2ab_1 + b_1^2 + t_2^2}{-t_2} \tag{7}$$

$$\frac{2ab_1 + 1 + a^2(b_1^2 + b_2^2)}{b_2} = \frac{-2ab_1 + 1 + a^2(b_1^2 + t_2^2)}{-t_2}. \tag{8}$$

Apply (6) to simplify (7) to get

$$\frac{a^2 - 2ab_1}{b_2} = \frac{a^2 + 2ab_1}{-t_2} \quad \text{or} \quad \frac{a - 2b_1}{b_2} = \frac{a + 2b_1}{-t_2}. \tag{9}$$

Then apply (6) to simplify (8) to get

$$\frac{2ab_1 + 1}{b_2} = \frac{-2ab_1 + 1}{-t_2}. \tag{10}$$

Equations (9) and (10) yield the equation

$$\frac{2ab_1 + 1}{-2ab_1 + 1} = \frac{a - 2b_1}{a + 2b_1}. \tag{11}$$

After cross multiplying denominators and numerators of the fractions in equation (11) and cancelling terms, we get the equations

$$a^2 + 1 = 0 \quad \text{or} \quad b_1 = 0. \tag{12}$$

Since a is positive, the first equation in (12) cannot occur. If $b_1 = 0$, then b and t are both purely imaginary. Therefore bt is real which contradicts the discussion following equation (4). This contradiction shows that Case 3 does not occur. This completes the proof of the lemma.

As remarked at the beginning of this section, Lemma 3 together with Corollary 2 to Theorem 1 and Theorem 2 implies the classification theorem.

Remark. Recently many interesting new orientable examples of complete minimal surfaces of small total Gaussian curvature have been found. The most notable examples appear in [1], [2], [4], [7], and [8].

REFERENCES

1. C. C. CHEN AND P. A. Q. SIMÕES, *Superficies minimas do* R^n, Escola de geometria diferencial, Universidade Estadual de Campinas, São Paulo, Brazil, 1980.
2. F. GACKSTATTER, *Topics on minimal surfaces*, Departamento de matemática, USP, São Paulo, Brazil, 1980.
3. D. HOFFMAN AND R. OSSERMAN, *The Geometry of the generalized Gauss map*, Memoirs of Amer. Math. Soc. Vol. 28, Number 236, November, 1980.
4. L. P. JORGE AND W. H. MEEKS, III, *The topology of complete minimal surfaces of finite total curvature*, to appear in Topology.
5. W. H. MEEKS, III, *Lectures on Plateau's problem*, IMPA, Rio de Janeiro, Brazil, 1978.
6. ———, *The conformal structure and geometry of triply periodic minimal surfaces in* R^3, Thesis (revised), University of California at Berkeley, 1976.
7. ———, *A survey of the geometric results in the classical theory of minimal surfaces*, Bulletin of the Brazilian Math. Soc. (1981).
8. ———, *The conjugate surface construction of symmetric complete minimal surfaces of small finite total curvature*, in preparation.
9. R. OSSERMANN, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. **80** (1964), 340–364.

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