hw1

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Problem 2. Show that a space X is contractible iff every map $f : X \longrightarrow Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \longrightarrow X$ is nullhomotopic.

solution. Let X be contractible. Then, $\mathbb{1} \simeq x_0$ for some $x_0 \in X$. Now consider any $f : X \longrightarrow Y$. Then,

$$f \simeq fx_0 = y_0$$

for some $y_0 = f(x_0) \in Y$. Thus f is nullhomotopic.

Let let $f : X \longrightarrow Y$ be an arbitrary map between X and some arbitrary space Y. Let f be nullhomotopic. In particular let $Y = X$ with $f = \mathbb{1}$. Then, $\mathbb{1}$ is nullhomotopic and hence X is contractible.

Second part can be solved in a similar fashion but with right composotion instead of left. □

Problem 3. Show that $f : X \longrightarrow Y$ is a homotopy equivalence if there exist maps $g, h : Y \longrightarrow X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that $f : X \longrightarrow Y$ is a homotopy equivalence if there exist $g, h : Y \longrightarrow X$ such that fg and hf are homotopy equivalences.

solution. Observe the following

$$\begin{aligned} g &= \mathbb{1}g \\ &\simeq (hf)g \\ &= h(fg) \\ &\simeq h. \end{aligned}$$

In other words, g and h are homotopic.

Now let fg and hf be homotopy equivalences with i_g and i_h as inverses. Notice That $(fg)i_g = f(gi_g) \simeq \mathbb{1}$. Also, $gi_gfg \simeq g$ therefore $(gi_g)f$. Thus gi_g is a homotopy inverse to f. □

Problem 4. Show that the number of path components is a homotopy invariant. (That is, show that if X and Y are homotopy equivalent spaces then they have the same number of path components.)

solution. To show that the number of path components remains the same, it suffices to show that the homotopy equivalence induces a bijection between the sets of path components. Let X and Y be homotopically equivalent with homotopy equivalence f and inverse g. Further, let $\sim_X$ be the equivalence relation between two points in X to be path connected. Similarly, define $\sim_Y$. Let

$$P : X/{\sim_X} \longrightarrow Y/{\sim_Y}$$

such that for any $X/{\sim_X} \ni [x_0] \mapsto [f(x_0)]$. This map is well defined since if $\hat{x} \in [x_0]$ and $\gamma : I \longrightarrow X$ is a path between $\hat{x}$ and x_0, then by the continuity of f, the map $f \circ \gamma : I \longrightarrow Y$ is path between $f(\hat{x})$ and $f(x_0)$. Therefore, $f(\hat{x}) \in [f(x_0)]$. This shows that P is well defined.

Next, we show that P is injective. Let x_0 and x_1 be such that $[f(x_0)] = [f(x_1)]$ with $x_0 \neq x_1$. There is a path $\gamma : I \longrightarrow Y$ connecting $f(x_0)$ and $f(x_1)$. By the continuity of g, the map $g \circ \gamma$ is a path that connects x_0 and x_1. Hence $[x_0] = [x_1]$. This shows that P is injective.

Lastly, we prove the surjectivity of P. Let $[y] \in Y/{\sim_Y}$. Since f is a homotopy equivalence with homotopy inverse g, we have that $fg \simeq \mathbb{1}$. Let $H : Y \times I \longrightarrow Y$ denote the homotopy between fg and $\mathbb{1}$. The map $H(y, \cdot) : I \longrightarrow Y$ is a path between y and $f(g(y))$. Therefore, $P([g(y)]) = [y]$. This proves that P is surjective.

The homotopy equivalence f induces a bijection between the sets of path components and hence the number of path components is homotopy invariant. □

Problem 6. Given a space X, a path connected subspace A, and a point $x_0 \in A$, show that the map $\pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic, relative to its endpoints, to a path in A.

solution. Let $\iota : A \hookrightarrow X$ be the inclusion map and let $\iota_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ denote the homomorphism induced by said inclusion. Suppose ι_* is surjective. Then, for any equivalnce class of loops $[f] \in \pi_1(X, x_0)$, one can find one or more equivalence classes of loops $[g] \in \pi_1(A, x_0)$ such that

$$\begin{aligned}
\iota_*([g]) &= [\iota g] && \text{(def. of } \iota_*) \\
&= [g] && \text{(loops } [g] \text{ are in } A) \\
&= [f] && \text{(surjectivity assertion).}
\end{aligned}$$

In other words, and loop f around x_0 is homotopic to at least one other loop $[g]$ around x_0 that lies entirely within A. Let $\phi : I \longrightarrow X$ be a path with endpoints $x_0, x_1 \in A$. Since A is path connected, all of its fundamental groups are isomorphic. Thus, WLOG, let Φ be a path from x_0 to some $x_1 \in A$ that lies in X. The path $\bar{\phi}\Phi$ is a loop with basepoint x_0. By the surjectivity of ι_*, there exists some $[\gamma] \in \pi_1(A, x_0)$ such that $\phi\Phi \simeq \gamma$. Concatenating with $\bar{\phi}$ on both sides, we see that $\Phi \simeq \bar{\phi}\gamma$, a path that lies within A. □