Math 525, Spring 2018. Homework 2
Due: Thursday, February 1, 2018

(1) Show that there are no retractions $r : X \longrightarrow A$ in the following cases:
a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to $\mathbb{S}^1$.
b) $X = \mathbb{S}^1 \times \mathbb{D}^2$ with A its boundary torus $\mathbb{S}^1 \times \mathbb{S}^1$.
c) (Given two space Z and Y, let $Z \vee Y$ be the space obtained by identifying two points $z_0 \in Z$ and $y_0 \in Y$ in the spaces $Z \sqcup Y$. Thus for example $\mathbb{S}^1 \vee \mathbb{S}^1$ is the figure eight.) $X = \mathbb{D}^2 \vee \mathbb{D}^2$ with $A = \mathbb{S}^1 \vee \mathbb{S}^1$.
d) X a closed disk with two points on its boundary identified and A its boundary, $\mathbb{S}^1 \vee \mathbb{S}^1$.
e) X the Möbius band and A its boundary circle.

(2) Let X be a topological space and $\gamma : \mathbb{S}^1 \longrightarrow X$. Show that $[\gamma] \in \pi_1(X)$ is the identity iff γ extends to a map $\mathbb{D}^2 \longrightarrow X$. (That is, iff there exists a map $\widetilde{\gamma} : \mathbb{D}^2 \longrightarrow X$ such that $\widetilde{\gamma}\big|_{\mathbb{S}^1} = \gamma$.)

(3) What if we defined π_1 without taking a base point into account? Given a topological space X, let $[\mathbb{S}^1, X]$ be the homotopy classes of maps $\mathbb{S}^1 \longrightarrow X$ with no conditions on basepoints. Given $x_0 \in X$, there is a natural map

$$\Phi : \pi_1(X, x_0) \longrightarrow [\mathbb{S}^1, X].$$

Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X)$. Hence Φ can be used to identify $[\mathbb{S}^1, X]$ with the set of conjugacy classes in $\pi_1(X)$, when X is path connected.

(4) Let A_1, A_2, A_3 be compact subsets of $\mathbb{R}^3$. Use the Borsuk-Ulam theorem to show that there is one plane $P \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.
This is often referred to as the *ham sandwich theorem,* as we can facetiously rephrase it as: 'given a ham sandwich, it is always possible to slice it so that in both resulting sandwiches the top bread, the ham, and the bottom bread each have the same size'.

(5) From the isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.