Math 525, Spring 2018. Homework 7
Due: Thursday, March 15, 2018

(1) Given chain complexes $A_\bullet$ and $B_\bullet$, let $\mathrm{Hom}(A_\bullet, B_\bullet)$ be the set of chain maps between them. Show that 'chain homotopy' is an equivalence relation on this set.

(2) Verify that homotopic maps induce the same homomorphism between reduced homology groups.

(3) For an exact sequence $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$ show that $C = 0$ if and only if the map $A \longrightarrow B$ is surjective and the map $D \longrightarrow E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

(4) Show that if the short exact $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ splits, that is, there is a homomorphism $\gamma : B \longrightarrow A$ such that $\gamma \circ \alpha = \mathrm{id}_A$, then the map

$$\begin{aligned} B &\xrightarrow{\ \phi\ } A \oplus C \\ b &\longmapsto (\gamma(b), \beta(b)) \end{aligned}$$

is an isomorphism.

(5) Show that if a subspace $A \subseteq X$ is the image of a retraction $r : X \longrightarrow A$ then, for every n, $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.
(Hint: show that the long exact sequence of the pair (X, A) breaks up into the short exact sequences $0 \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow 0$, and that these sequences split.)

(6) Prove the 'Snake Lemma': Given a commuting diagram of Abelian groups and homomorphisms,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

in which the rows are short exact sequences, there are induced maps and a connecting homomorphism making the sequence

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow \mathrm{coker}\ \alpha \longrightarrow \mathrm{coker}\ \beta \longrightarrow \mathrm{coker}\ \gamma \longrightarrow 0$$

exact. (Recall that the cokernel of a homomorphism $R \xrightarrow{f} S$, is the group $S/f(R)$.)