9

DIFFERENTIAL EQUATIONS

We have looked at a variety of models for the growth of a single species that lives alone in an environment.

9.6

Predator-Prey Systems

In this section, we will learn about:

Models that take into account the interaction of two species in the same habitat.

PREDATOR-PREY SYSTEMS

We will see that these models take the form of a pair of linked differential equations.

We first consider the following situation.

- One species, the prey, has an ample food supply.
- The second, the predator, feeds on the prey.

Examples of prey and predators include:

- Rabbits and wolves in an isolated forest
- Food fish and sharks
- Aphids and ladybugs
- Bacteria and amoebas

PREDATOR-PREY SYSTEMS

Our model will have two dependent variables, and both are functions of time.

We let $R(t)$ be the number of prey (R for rabbits) and $W(t)$ be the number of predators (W for wolves) at time t.

ABSENCE OF PREDATORS

In the absence of predators, the ample food supply would support exponential growth of the prey, that is,

$$\frac{dR}{dt} = kR$$

where *k* is a positive constant.

ABSENCE OF PREY

In the absence of prey, we assume that the predator population would decline at a rate proportional to itself, that is,

$$\frac{dW}{dt} = -rW$$

where *r* is a positive constant.

PREDATOR-PREY SYSTEMS

With both species present, we assume that:

- The principal cause of death among the prey is being eaten by a predator.
- The birth and survival rates of the predators depend on their available food supply—namely, the prey.

PREDATOR-PREY SYSTEMS

We also assume that the two species encounter each other at a rate that is proportional to both populations and is, therefore, proportional to the product *RW*.

- The more there are of either population, the more encounters there are likely to be.

Equation 1

A system of two differential equations that incorporates these assumptions is

$$\frac{dR}{dt} = kR - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

where *k*, *r*, *a*, and *b* are positive constants.

Notice that:

- The term $-aRW$ decreases the natural growth rate of the prey.
- The term bRW increases the natural growth rate of the predators.

LOTKA-VOLTERRA EQUATIONS

The equations in (1) are known as the predator-prey equations, or the Lotka-Volterra equations.

- They were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).

A solution of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time.

- As the system is coupled (R and W occur in both equations), we can't solve one equation and then the other.

- We have to solve them simultaneously.

PREDATOR-PREY SYSTEMS

Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t.

- However, we can use graphical methods to analyze the equations.

Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations with:

$k = 0.08$, $a = 0.001$, $r = 0.02$, $b = 0.00002$

The time t is measured in months.

a. Find the constant solutions (called the equilibrium solutions) and interpret the answer.

b. Use the system of differential equations to find an expression for dW/dR.

c. Draw a direction field for the resulting differential equation in the *RW*-plane. Then, use that direction field to sketch some solution curves.

d. Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

e. Use (d) to make sketches of R and W as functions of t.

With the given values of *k*, *a*, *r*, and *b*, the Lotka-Volterra equations become:

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Both *R* and *W* will be constant if both derivatives are 0.

That is,

$$R' = R(0.08 - 0.001W) = 0$$

$$W' = W(-0.02 + 0.00002R) = 0$$

One solution is given by:

$$R = 0 \text{ and } W = 0$$

- This makes sense.
- If there are no rabbits or wolves, the populations are certainly not going to increase.

The other constant solution is:

$$W = \frac{0.08}{0.001} = 80 \qquad R = \frac{0.02}{0.00002} = 1000$$

- So, the equilibrium populations consist of 80 wolves and 1000 rabbits.

This means that 1000 rabbits are just enough to support a constant wolf population of 80.

- The wolves aren’t too many—which would result in fewer rabbits.
- They aren’t too few—which would result in more rabbits.

We use the Chain Rule

to eliminate *t* :

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt}$$

Hence,

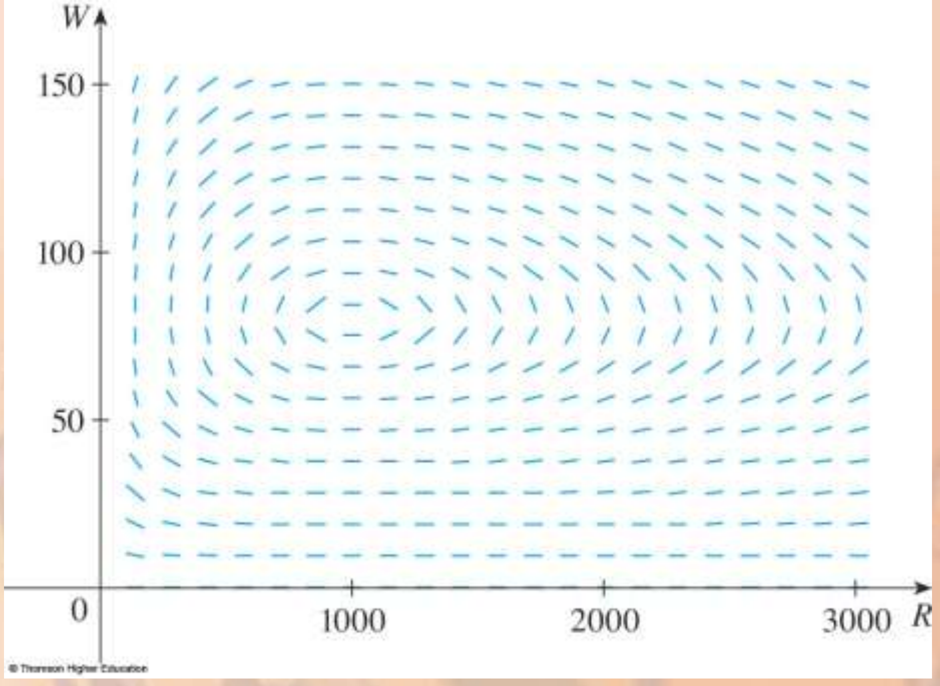
$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

If we think of *W* as a function of *R*,

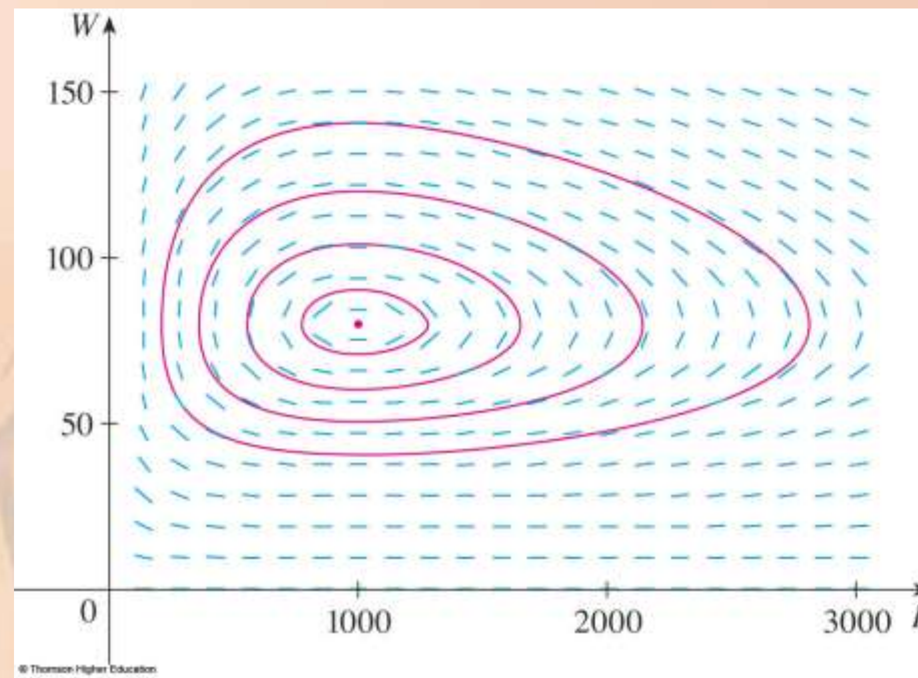
we have the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

We draw the direction field for the differential equation.

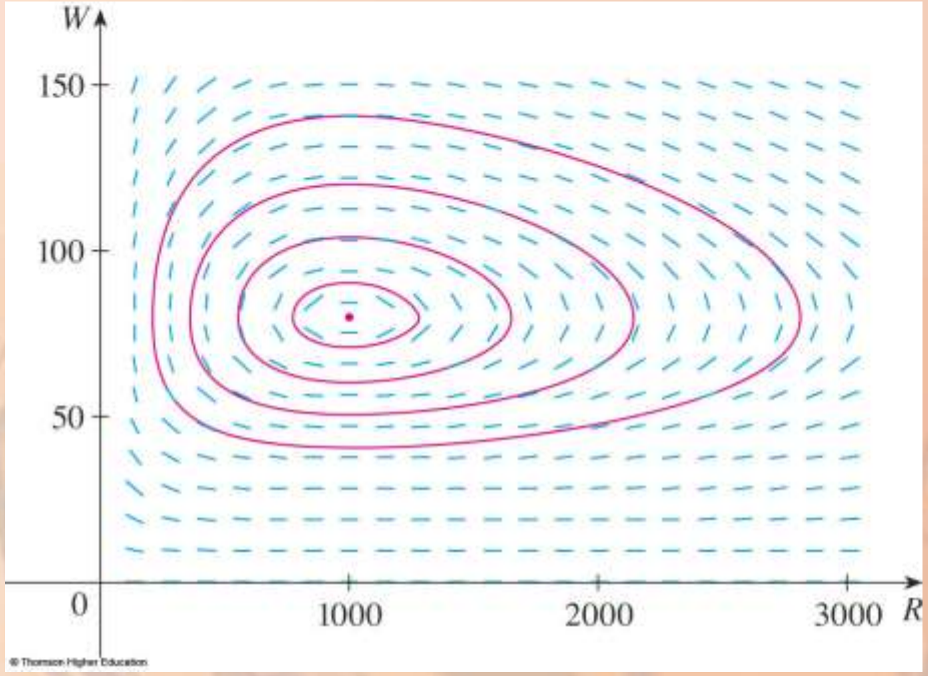


Then, we use the field to sketch several solution curves.

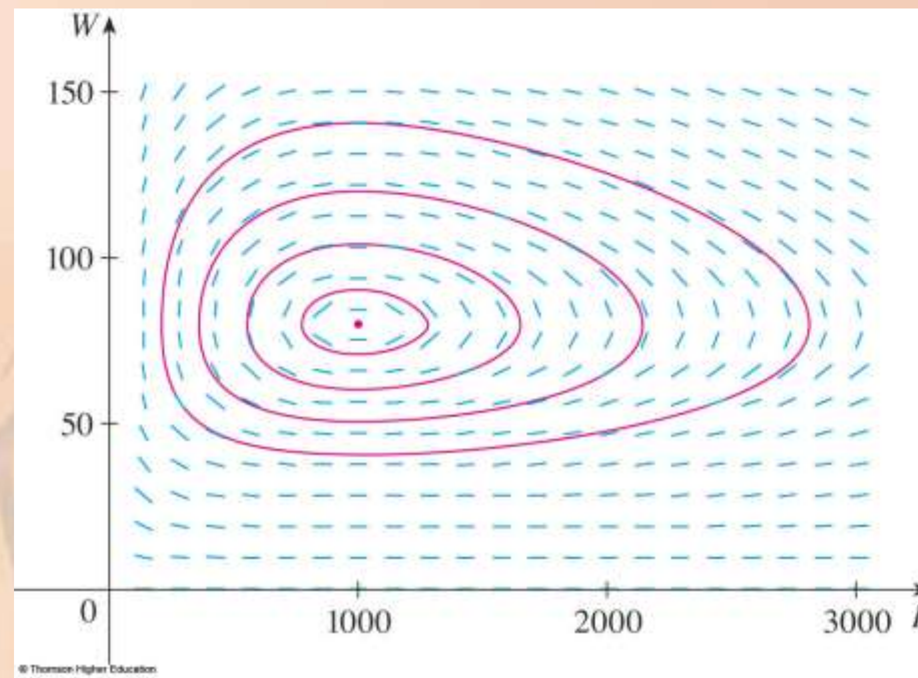


PREDATOR-PREY SYSTEMS Example 1 c

If we move along a solution curve, we observe how the relationship between *R* and *W* changes as time passes.



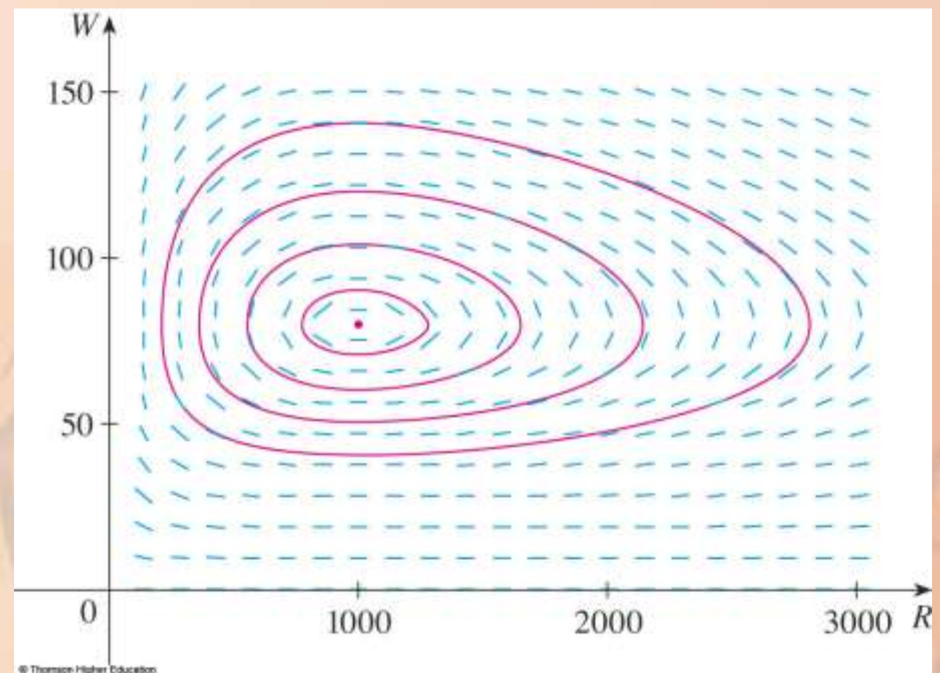
Notice that the curves appear to be closed in the sense that, if we travel along a curve, we always return to the same point.



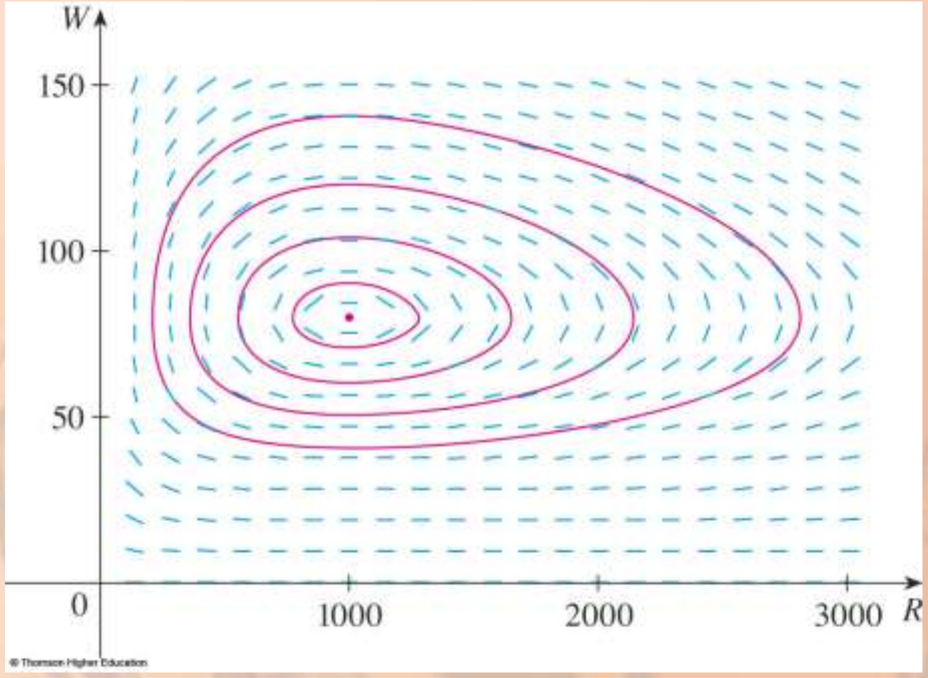
EQUILIBRIUM POINT Example 1 c

Notice also that the point (1000, 80) is inside all the solution curves.

- It is called an equilibrium point.
- It corresponds to the equilibrium solution $R = 1000$, $W = 80$.



When we represent solutions of a system of differential equations as here, we refer to the *RW*-plane as the phase plane.

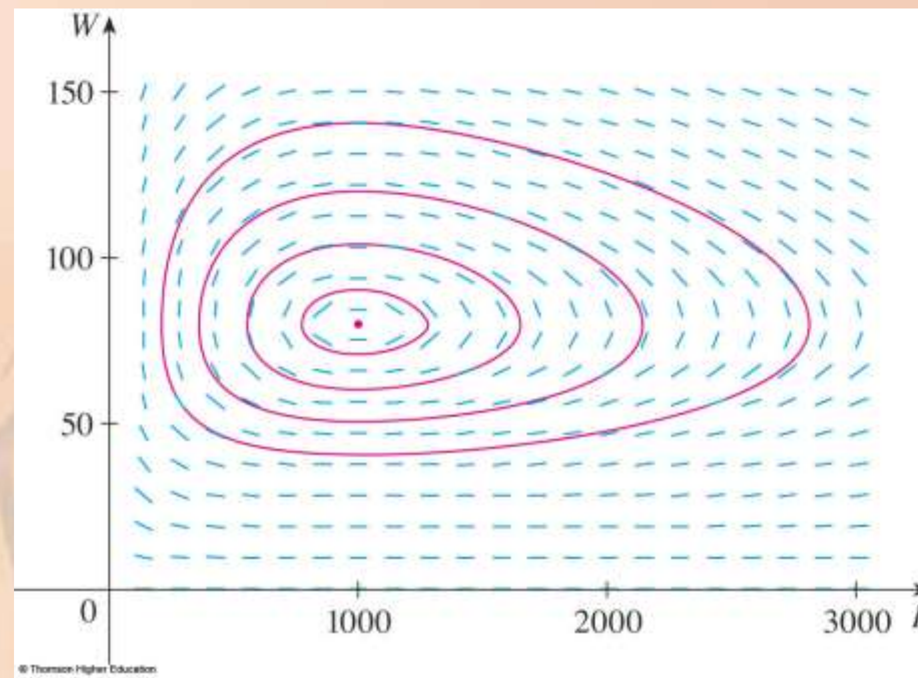


PHASE TRAJECTORIES

Example 1 c

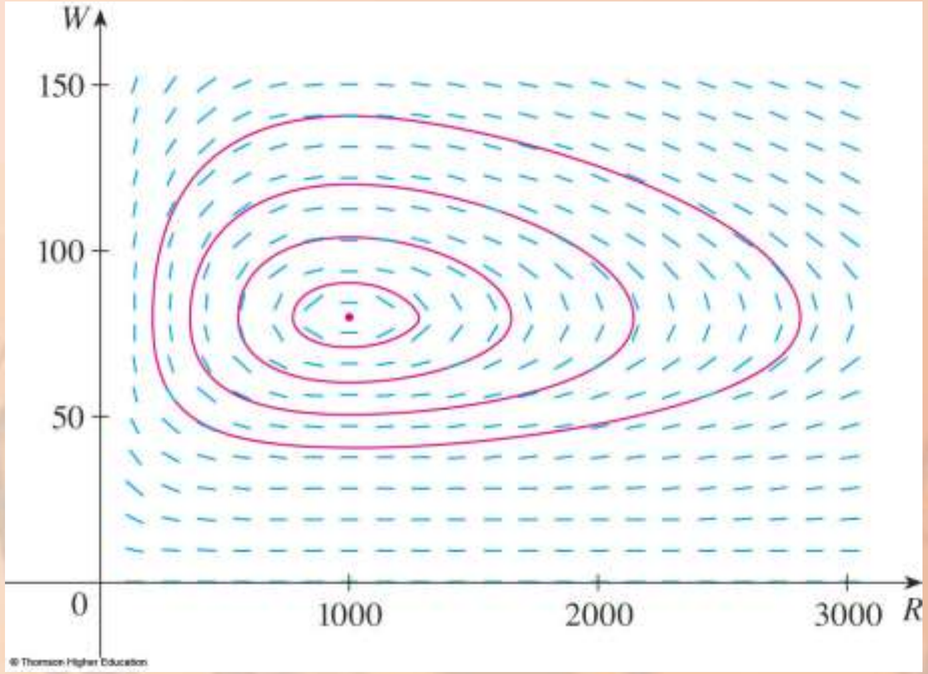
Then, we call the solution curves phase trajectories.

- So, a phase trajectory is a path traced out by solutions (*R*, *W*) as time goes by.



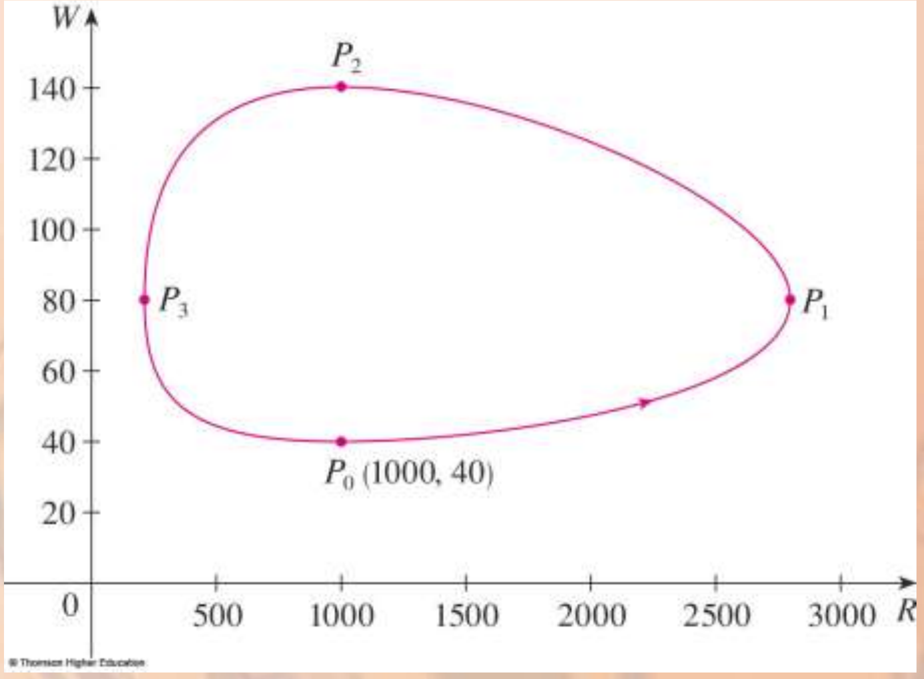
A phase portrait, as shown, consists of:

- Equilibrium points
- Typical phase trajectories

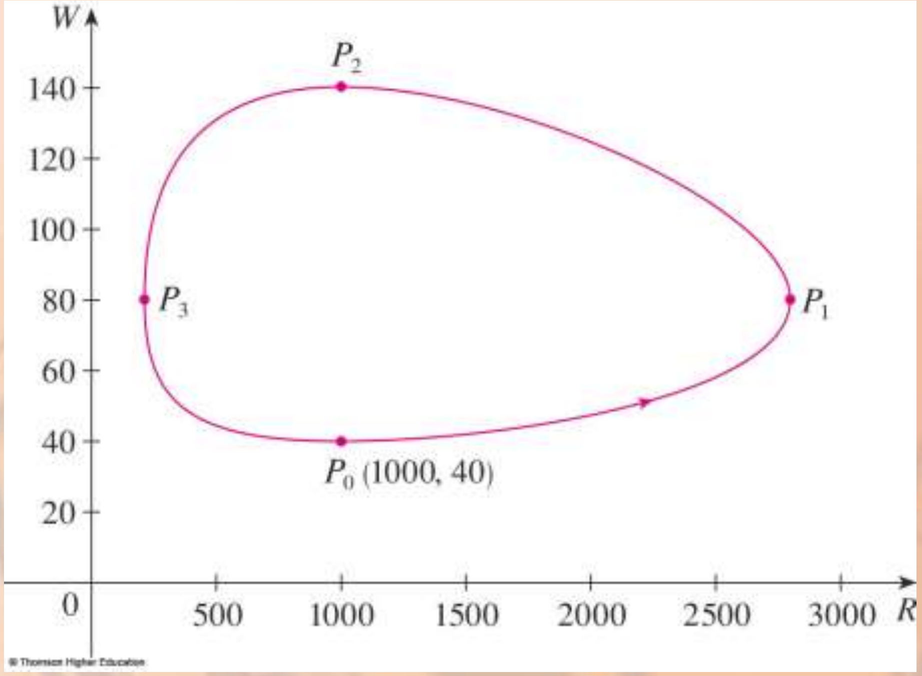


Starting with 1000 rabbits and 40 wolves corresponds to drawing the solution curve through the point $P_0(1000, 40)$.

- The figure shows the phase trajectory with the direction field removed.



Starting at the point P_0 at time $t = 0$ and letting t increase, do we move clockwise or counterclockwise around the phase trajectory?



If we put R = 1000 and W = 40 in the first differential equation, we get:

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40)$$

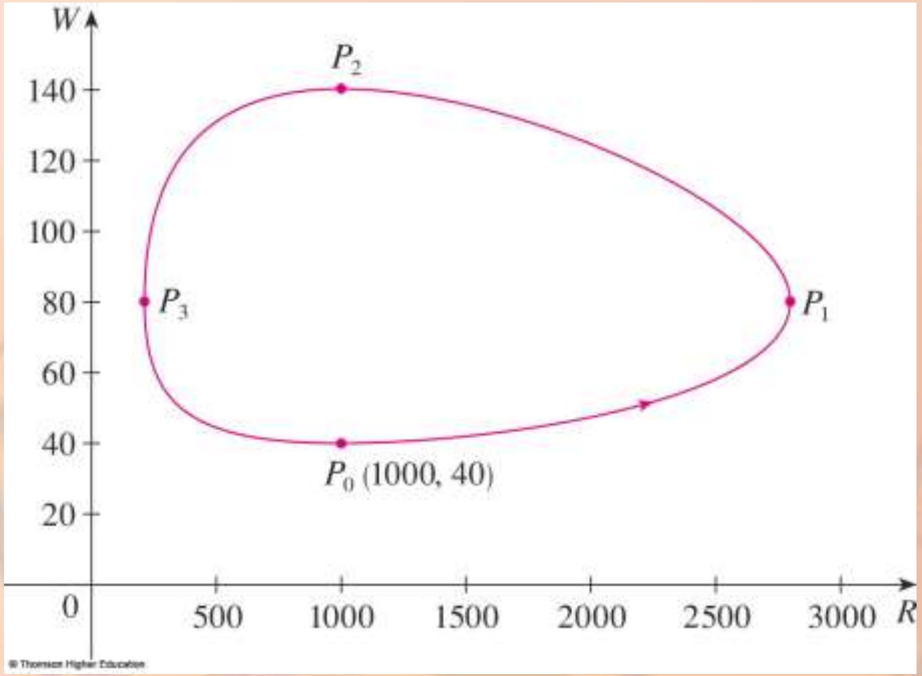
$$= 80 - 40 = 40$$

Since $dR/dt > 0$, we conclude that R is increasing at P_0.

So, we move counterclockwise around the phase trajectory.

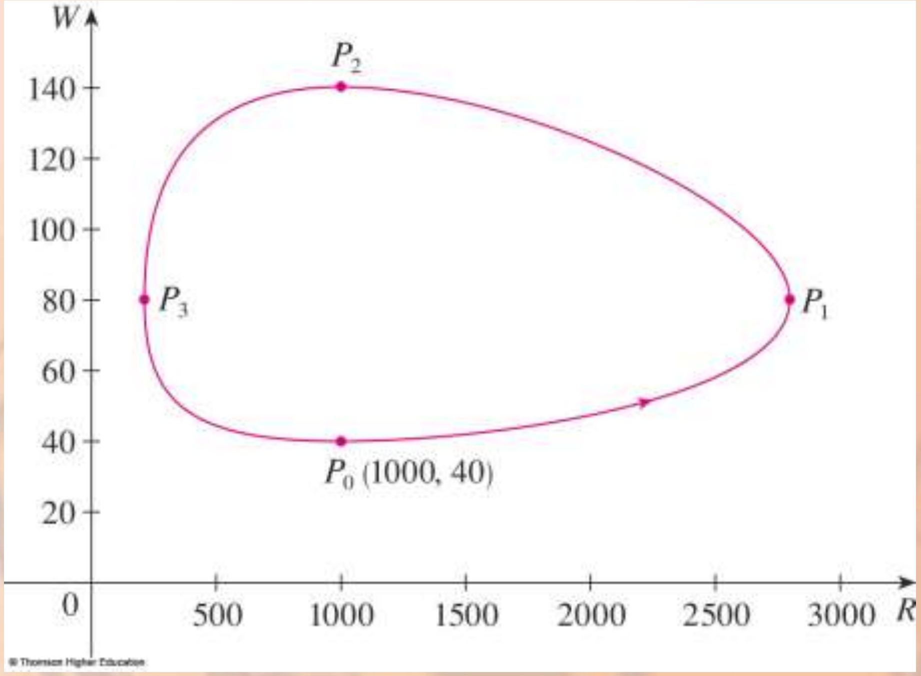
We see that, at P_0, there aren't enough wolves to maintain a balance between the populations.

- So, the rabbit population increases.



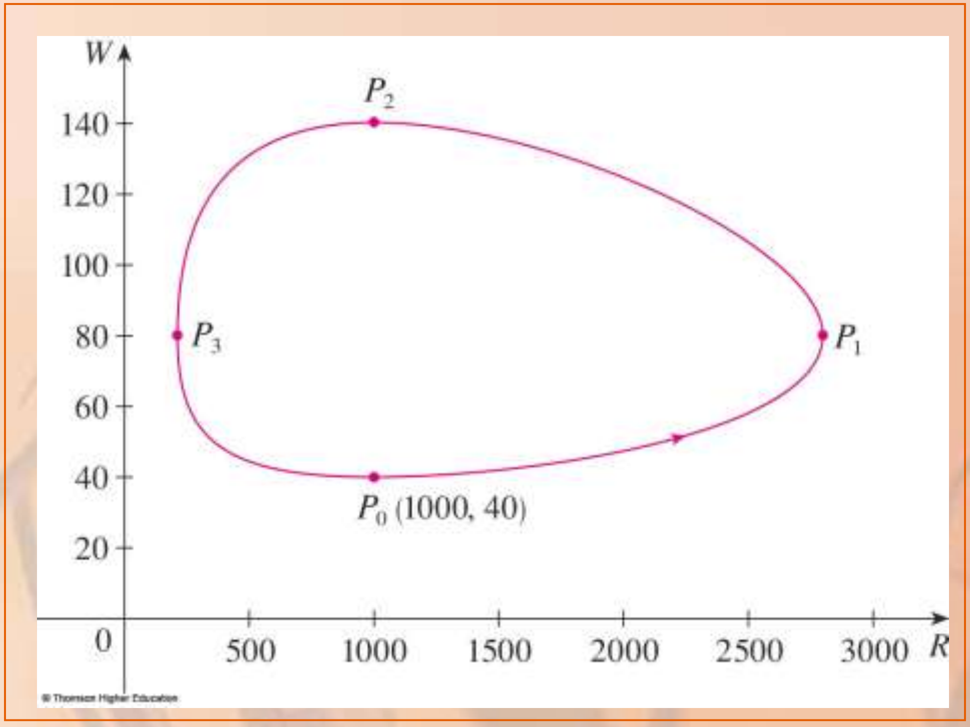
That results in more wolves.

- Eventually, there are so many wolves that the rabbits have a hard time avoiding them.



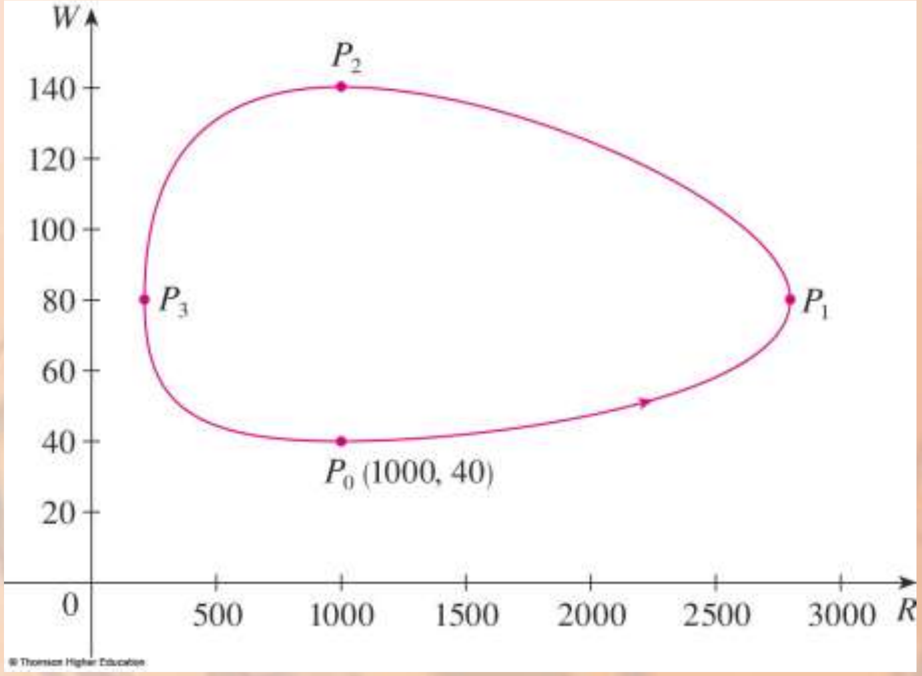
Hence, the number of rabbits begins to decline.

- This is at P_1, where we estimate that R reaches its maximum population of about 2800.



This means that, at some later time, the wolf population starts to fall.

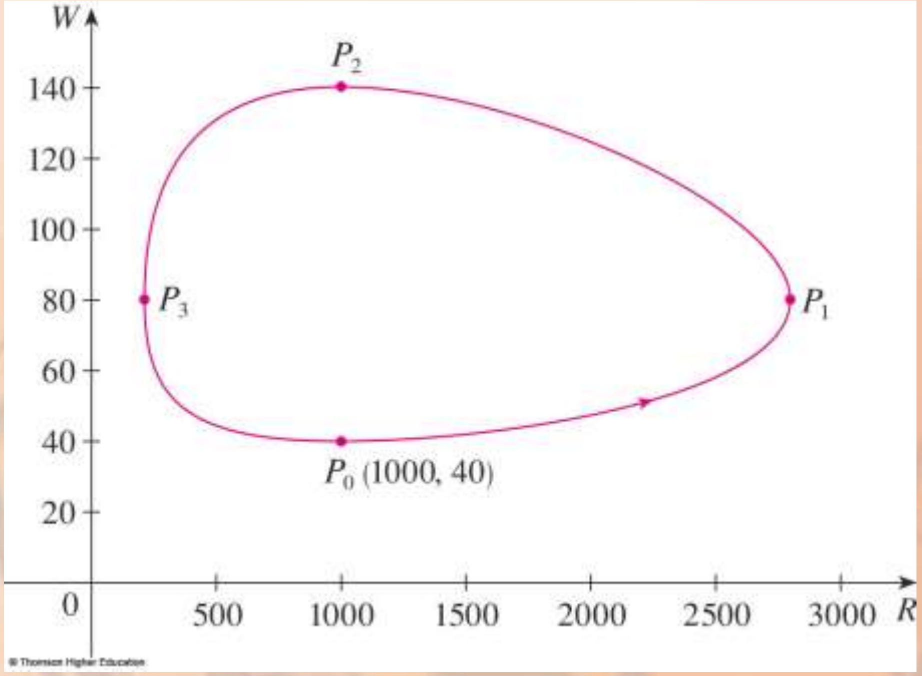
- This is at P_2, where R = 1000 and $W \approx 140$.



However, this benefits the rabbits.

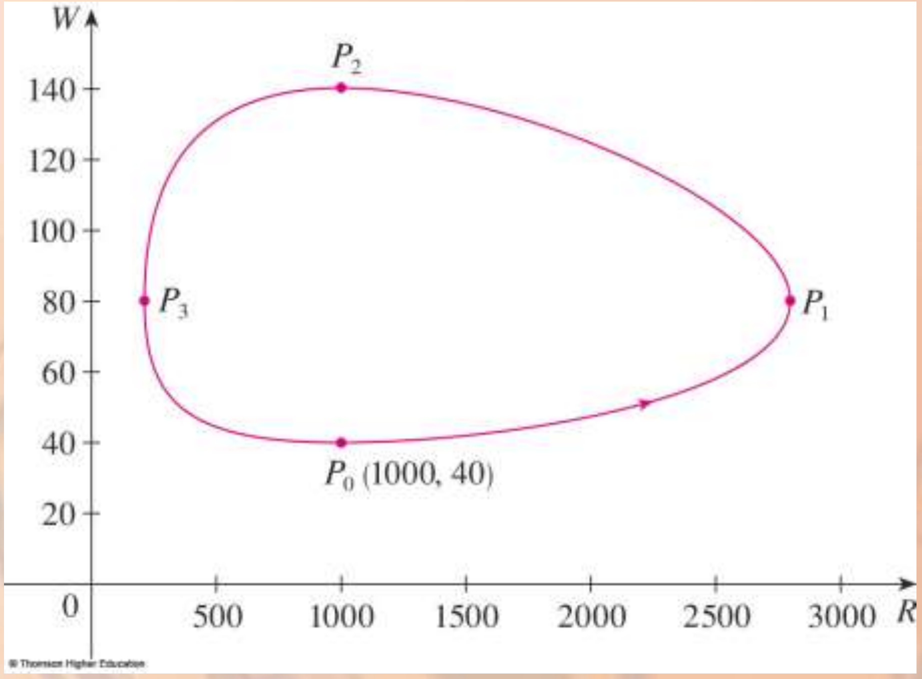
So, their population later starts to increase.

- This is at P_3, where $W = 80$ and $R \approx 210$.



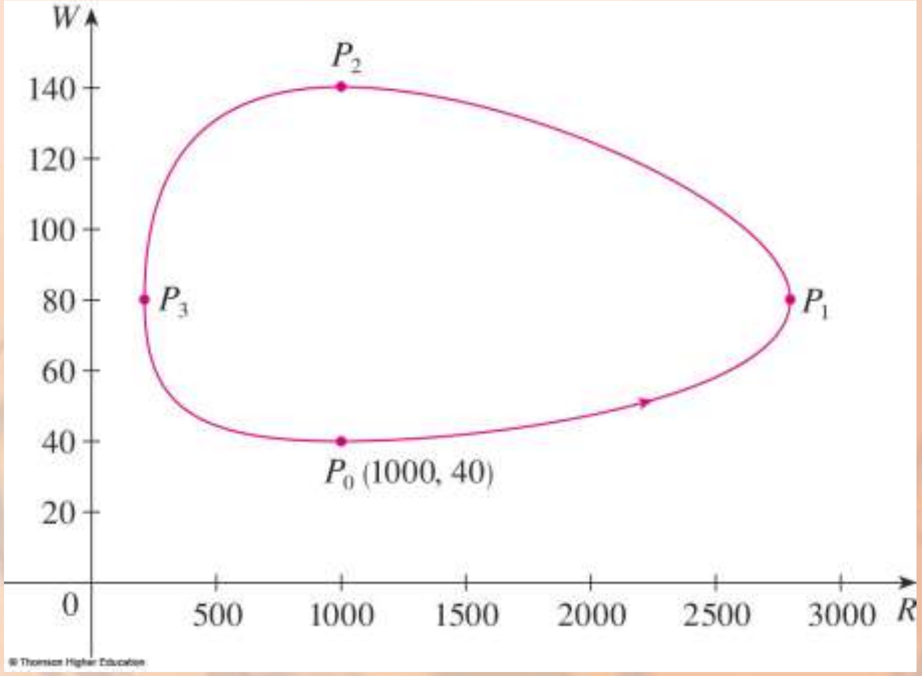
Consequently, the wolf population eventually starts to increase as well.

- This happens when the populations return to their initial values (R = 1000, W = 40), and the entire cycle begins again.



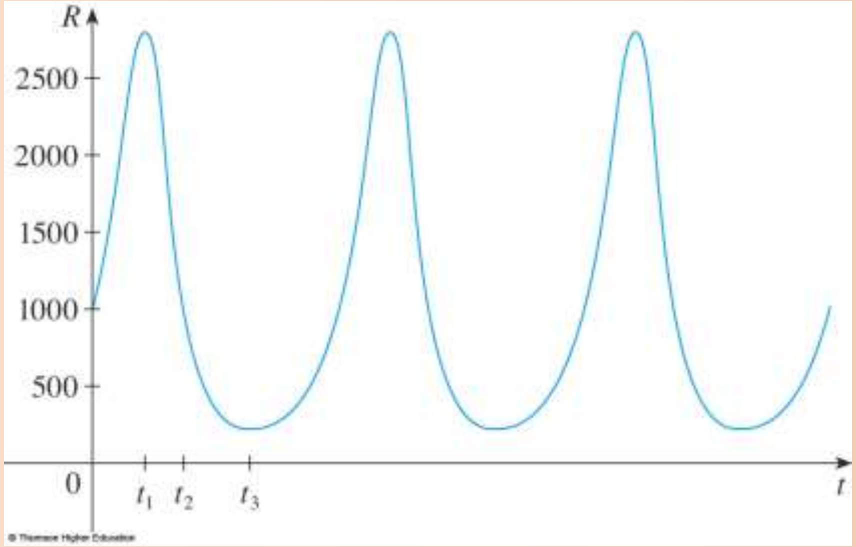
From the description in (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of $R(t)$ and $W(t)$.

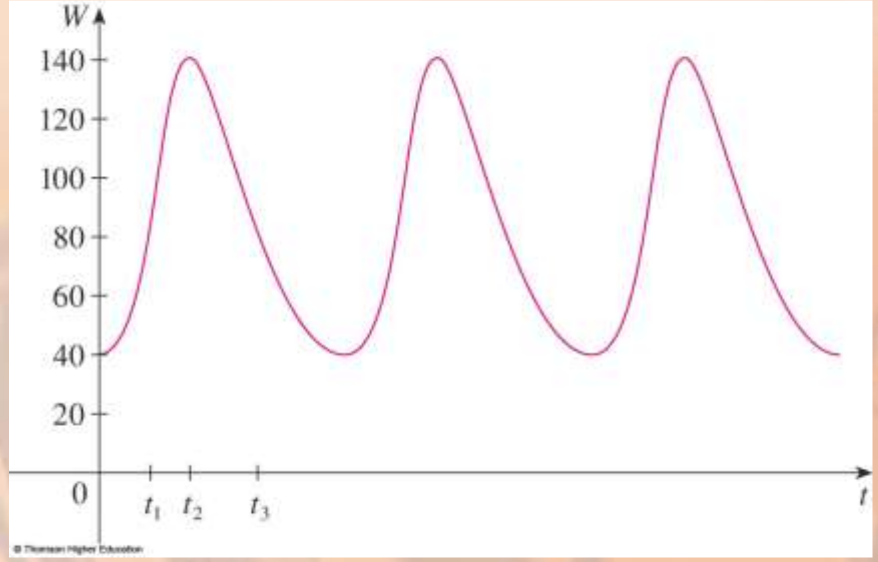
Suppose the points P_1, P_2, and P_3 are reached at times t_1, t_2, and t_3 .



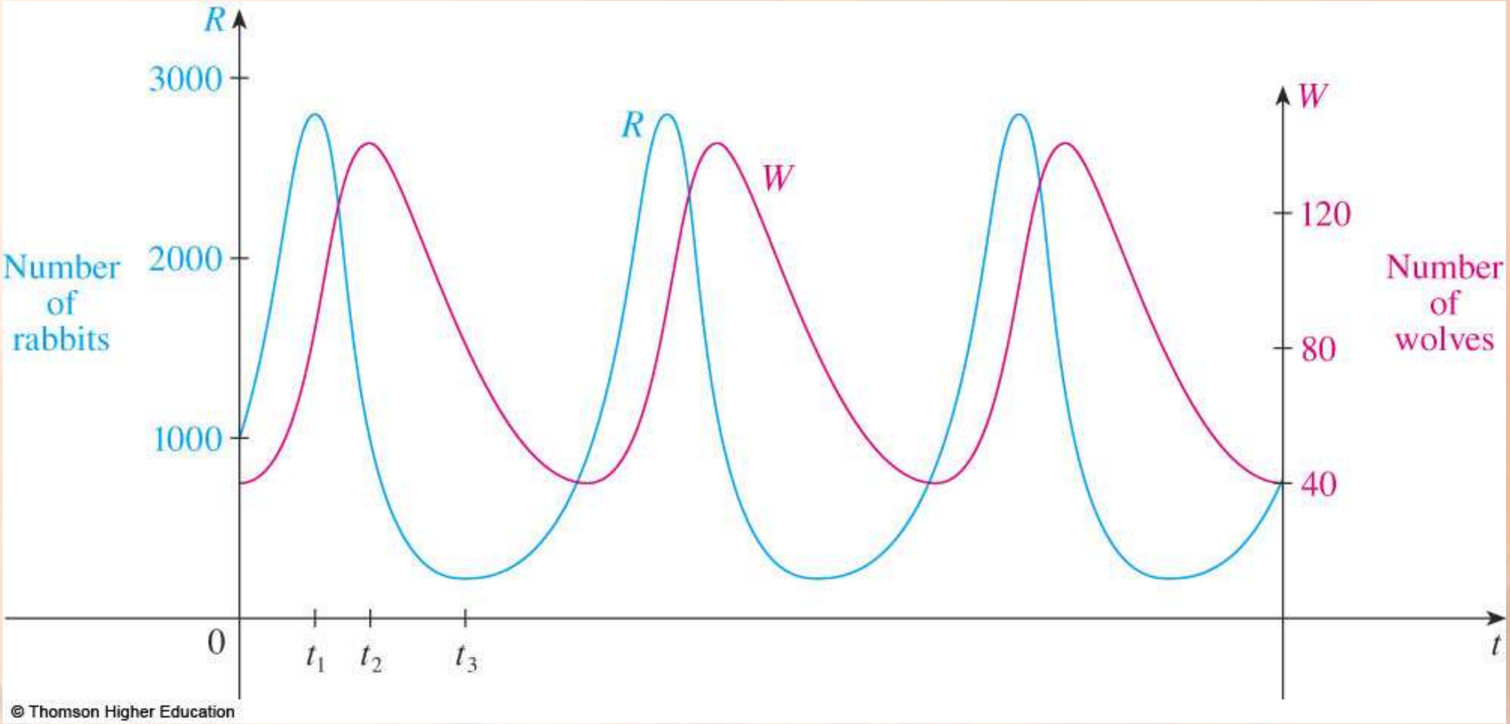
PREDATOR-PREY SYSTEMS Example 1 e

Then, we can sketch graphs of *R* and *W*, as shown.

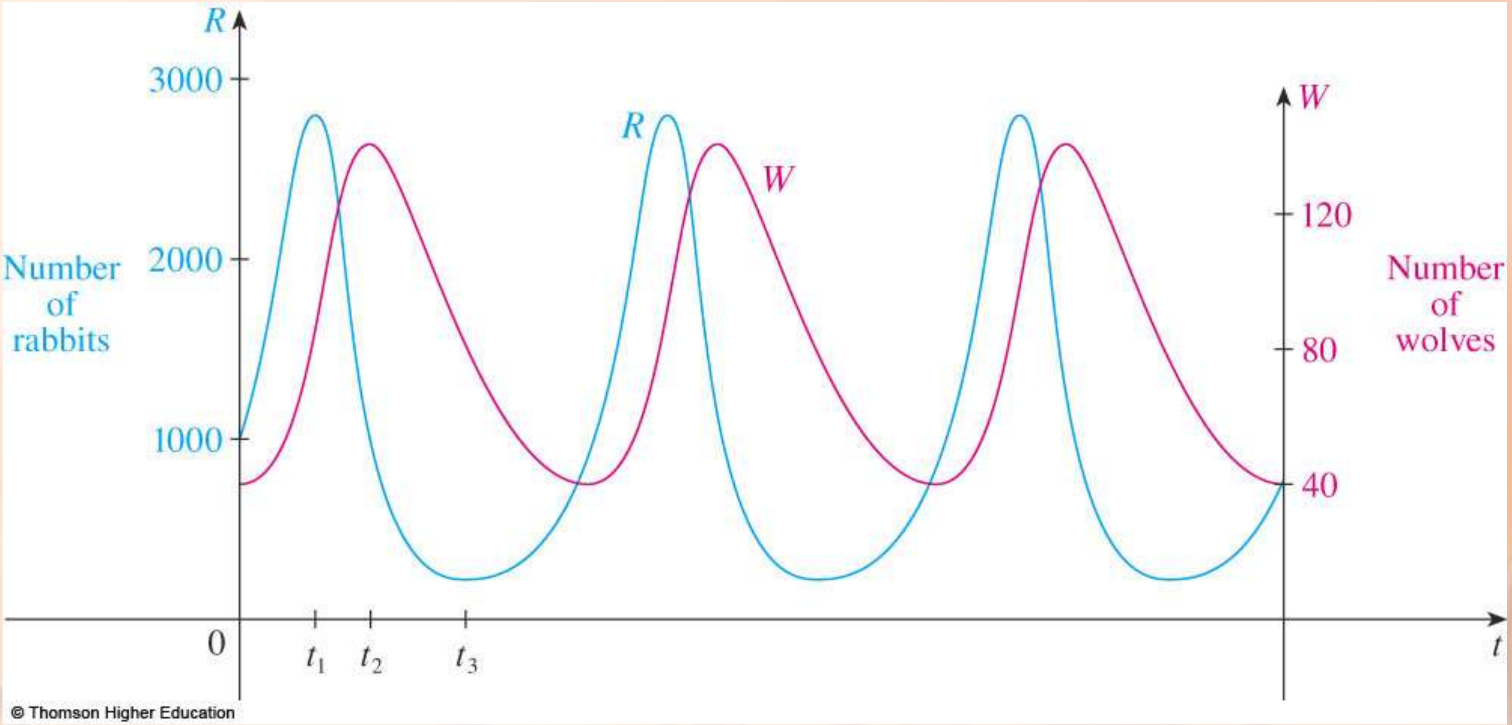




To make the graphs easier to compare, we draw them on the same axes, but with different scales for *R* and *W*.



Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves.



REAL-WORLD PREDICTIONS

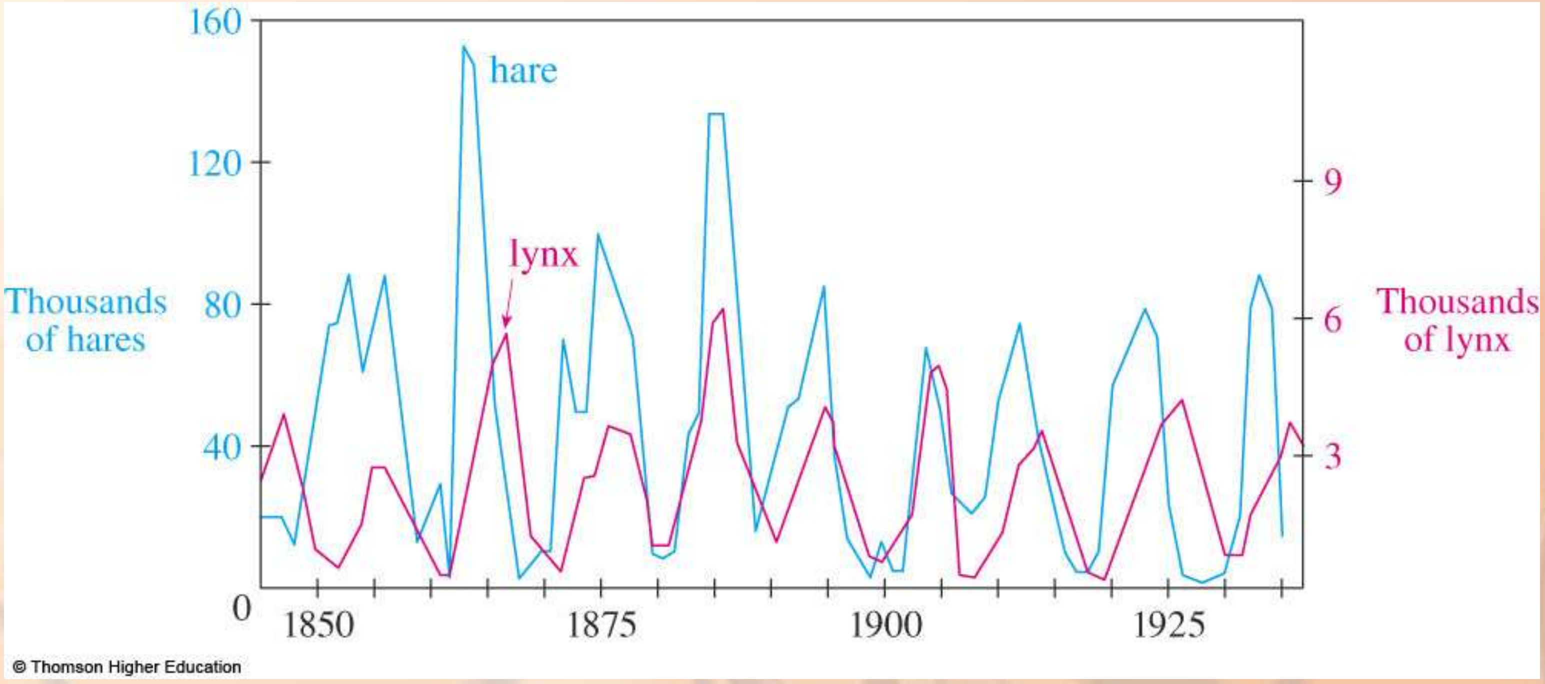
An important part of the modeling process, as discussed in Section 1.2, is to interpret our mathematical conclusions as real-world predictions and test them against real data.

REAL-WORLD PREDICTIONS

For instance, the Hudson's Bay Company, which started trading in animal furs in Canada in 1670, has kept records that date back to the 1840s.

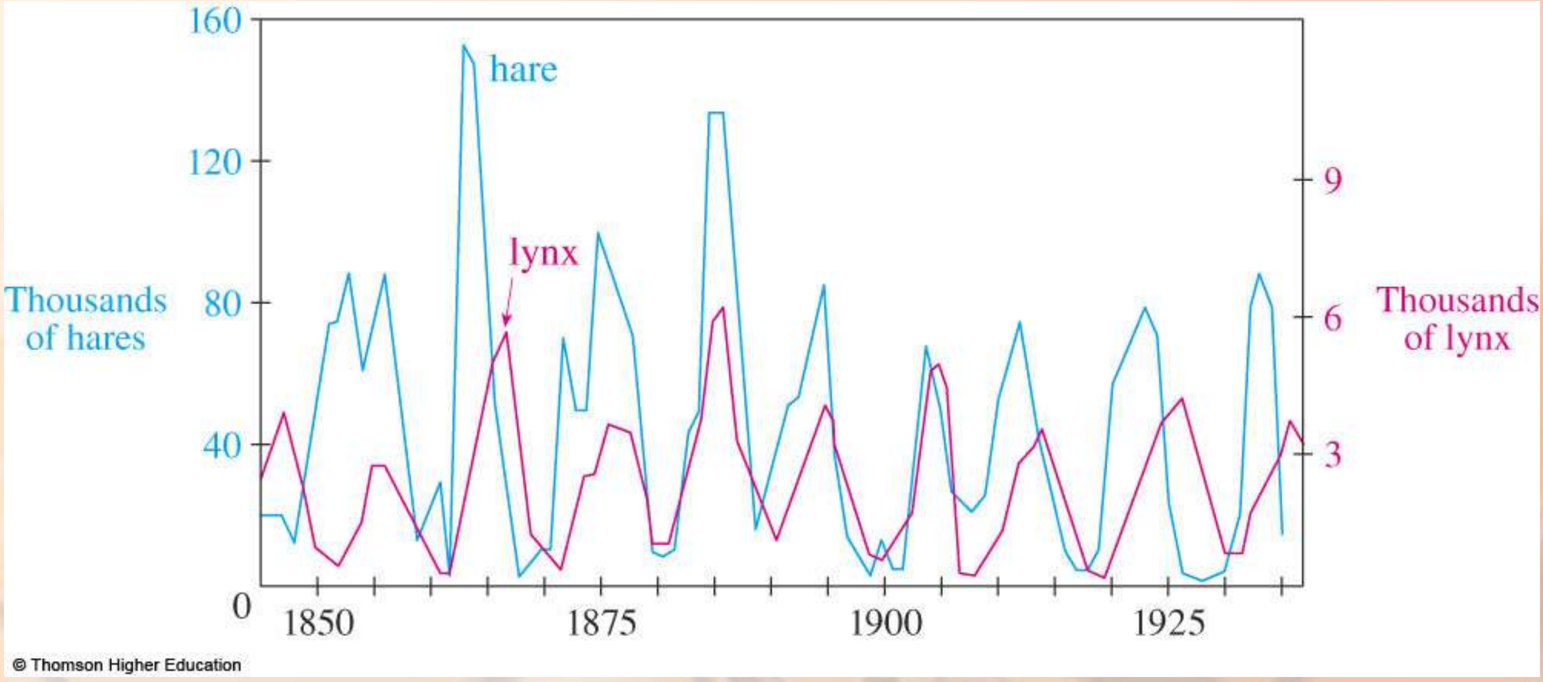
REAL-WORLD PREDICTIONS

The graphs show the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded over a 90-year period.



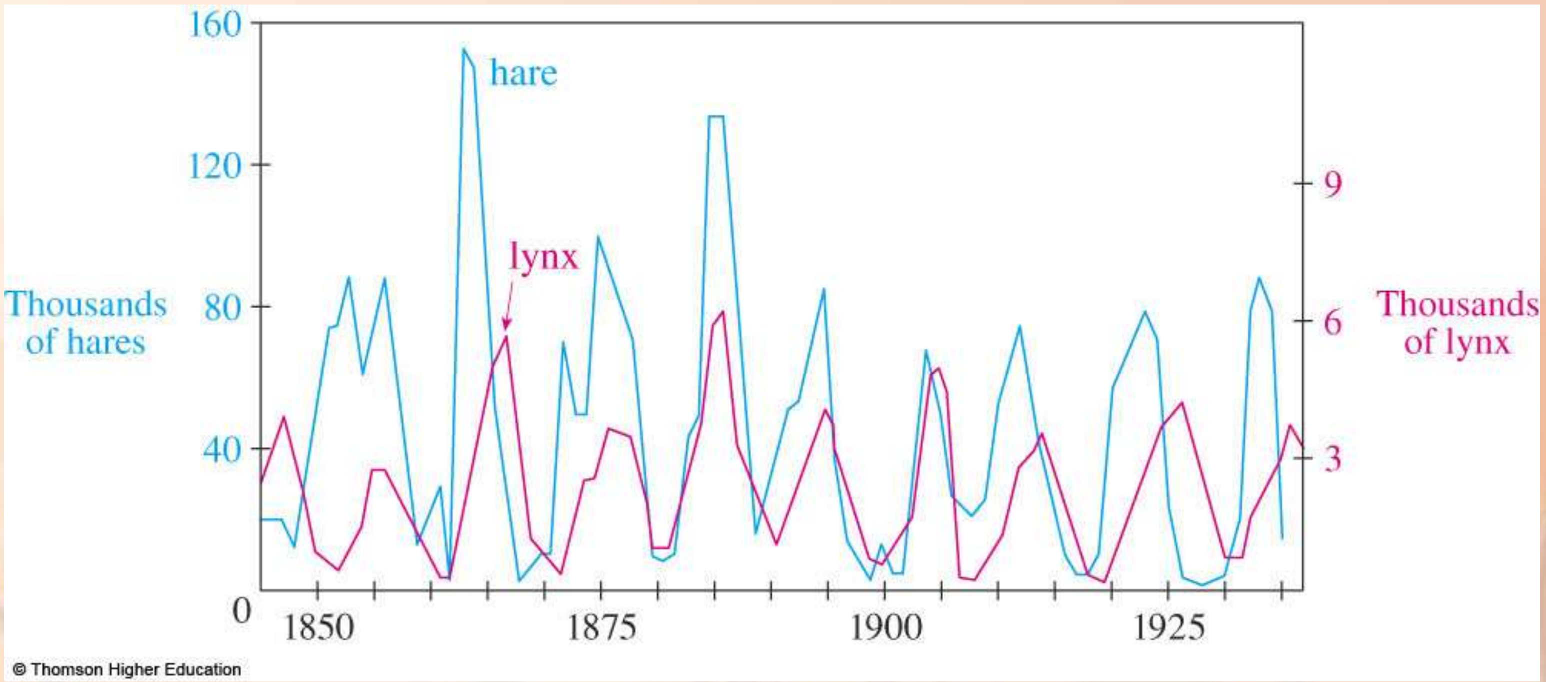
REAL-WORLD PREDICTIONS

You can see that the coupled oscillations in the hare and lynx populations predicted by the Lotka-Volterra model do actually occur.



REAL-WORLD PREDICTIONS

The period of these cycles is roughly 10 years.



SOPHISTICATED MODELS

Though the relatively simple Lotka-Volterra model has had some success in explaining and predicting coupled populations, more sophisticated models have also been proposed.

MODIFYING LOTKA-VOLTERRA EQUATIONS

One way to possibly modify the Lotka-Volterra equations is to assume that, in the absence of predators, the prey grow according to a logistic model with carrying capacity K.

MODIFYING LOTKA-VOLTERRA EQUATIONS

Then, the Lotka-Volterra equations are replaced by the system of differential equations

$$\frac{dR}{dt} = kR\left(1 - \frac{R}{K}\right) - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

- This model is investigated in Exercises 9 and 10.

SOPHISTICATED MODELS

Models have also been proposed to describe and predict population levels of two species that compete for the same resources or cooperate for mutual benefit.

- Such models are explored in Exercise 2.