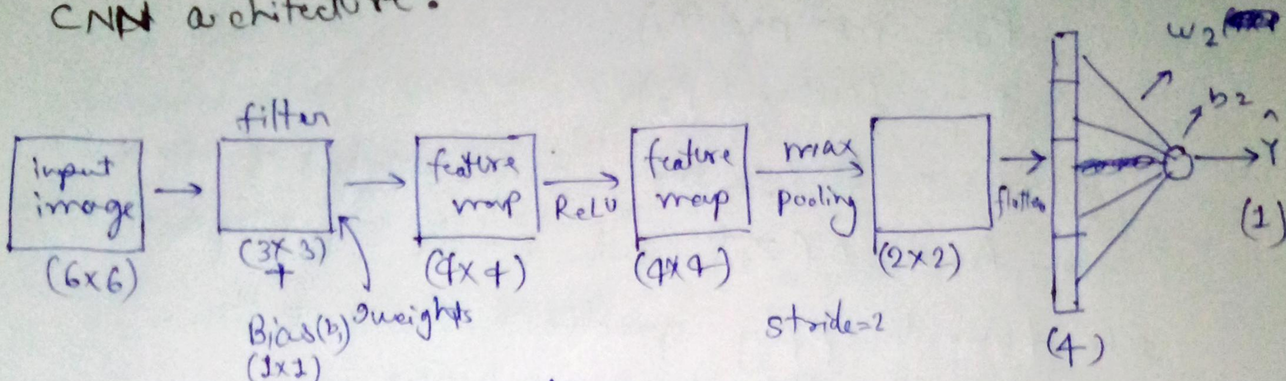
12 Oct -2024

BACKPROPAGATION IN CNN

CNN architecture:



Trainable parameters:

$W_1 - (3\times3)$ $\quad w_2 (1\times4)$

$b_1 = (1,1)$ $\quad b_2 = (1\times1)$

Total 15 trainable parameters

- our aim is to find the optimal value of these is to minimize the final loss
- let us consider it is binary classification problem (cat and dogs classification). Hence loss function is (binary cross entropy)

$$L = -Y_i \log(\hat{Y}_i) - (1-Y_i)\log(1-\hat{Y}_i)$$

- for multi class classification we use the softmax

$$X \rightarrow \circledast \rightarrow Z_1 \xrightarrow[\text{ReLU}]{} A_1 \xrightarrow[\text{pooling}]{\text{max}} P_1 \xrightarrow{\text{flatten}} F \longrightarrow Z_2 \xrightarrow{\text{sigmoid}} A_2 \rightarrow L$$

(W_1, b_1) $\qquad$ (W_2, b_2)

- In the above flow L is loss and. $A_2 \approx \hat{Y}$ (output) forward prop:

$Z_1 = \text{conv}(X, W_1) + b_1$

$A_1 = \text{ReLU}(Z_1)$

$P_1 = \text{max pool}(A_1)$

$F = \text{flatten}(P_1)$

$Z_2 = W_2 \cdot F + b_2$ ∴ dot product of W_2 and F

$A_2 = \sigma(Z_2)$ and $L = \frac{1}{m}\sum_{i=1}^{m}\left[-Y_i \log(A_2) - (1-Y_i)\log(1-A_2)\right]$

we have to apply gradient descent to minimize the loss where,

weight and bias updates

$$\begin{cases} W_1 = W_1 - \eta\frac{\partial L}{\partial W_1} \text{ and } W_2 = W_2 - \eta\frac{\partial L}{\partial W_2} \\ b_1 = b_1 - \eta\frac{\partial L}{\partial b_1} \text{ and } b_2 = b_2 - \eta\frac{\partial L}{\partial b_2} \end{cases}$$

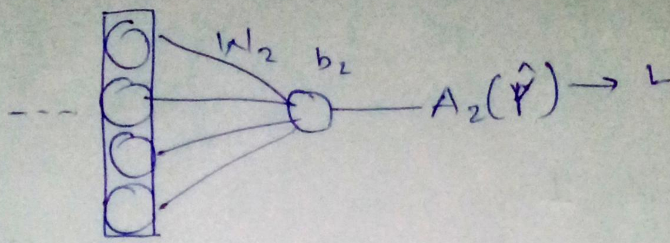
Here W_1, W_2 and all derivative is matrix

$$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial W_2} \quad \text{——— (i)}$$

$$\frac{\partial L}{\partial b_2} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial b_2} \quad \text{——— (ii)}$$

$$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial F} \times \frac{\partial F}{\partial P_1} \times \frac{\partial P_1}{\partial A_1} \times \frac{\partial A_1}{\partial Z_1} \times \frac{\partial Z_1}{\partial W_1} \quad \text{——— (iii)}$$

$$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial A_2} \times \frac{\partial A_2}{\partial Z_2} \times \frac{\partial Z_2}{\partial F} \times \frac{\partial F}{\partial P_1} \times \frac{\partial P_1}{\partial A_1} \times \frac{\partial A_1}{\partial Z_1} \times \frac{\partial Z_1}{\partial b_1} \quad \text{——— (iv)}$$



forward propagation equation

$$Z_2 = W_2 F + b_2$$

$$A_2 = \sigma(Z_2)$$

currently now focus on equation (i) and (ii)

$$\frac{\partial L}{\partial a_2} = \frac{\partial}{\partial a_2}\left[-Y_i \log(a_2) - (1-Y_i)\log(1-a_2)\right]$$

$$= -\frac{Y_i}{a_2} + \frac{(1-Y_i)}{(1-a_2)}$$

$$= \frac{-Y_i(1-a_2) + a_2(1-Y_i)}{a_2(1-a_2)}$$

$$= \frac{-Y_i + \cancel{Y_i a_2} + a_2 - \cancel{a_2 Y_i}}{a_2(1-a_2)}$$

$$\frac{\partial L}{\partial a_2} = \frac{(a_2 - Y_i)}{a_2(1-a_2)}$$

$$\frac{\partial a_2}{\partial Z_2} = \sigma(Z_2)\left[1 - \sigma(Z_2)\right]$$

$$\frac{\partial a_2}{\partial Z_2} = a_2\left[1 - a_2\right]$$

// $\frac{\partial Z_2}{\partial W_2} = F$

// $\frac{\partial Z_2}{\partial b_2} = 1$

Now put all values in equation (i)

$$\frac{\partial L}{\partial W_2} = \frac{(a_2 - Y_i)}{\cancel{a_2(1-a_2)}} \times \cancel{a_2(1-a_2)} \times F$$

$$\frac{\partial L}{\partial W_2} = (a_2 - Y_i) \times F$$

$$\left(\frac{\partial L}{\partial W_2}\right)_{(1,4)} = (A_2 - Y).F^T \text{ ——— (i)}$$

and also put values in equation (ii)

$$\frac{\partial L}{\partial b_2} = \frac{(a_2 - Y_i)}{a_2(1-a_2)} \times a_2(1-a_2)$$

$$\frac{\partial L}{\partial b_2} = (a_2 - Y_i)$$

$$\frac{\partial L}{\partial b_2} = (A_2 - Y) \text{ ——— (ii)}$$

Now apply back propagation in:

→ Convolution

→ Flatten

→ Max Pooling

from equation (iii) and (iv)

$$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial A_2} \cdot \frac{\partial A_2}{\partial Z_2} \cdot \frac{\partial Z_2}{\partial F} \cdot \frac{\partial F}{\partial P_1} \cdot \frac{\partial P_1}{\partial A_1} \cdot \frac{\partial A_1}{\partial Z_1} \cdot \frac{\partial Z_1}{\partial w_1}$$

$$\frac{\partial L}{\partial b_1} = \underbrace{\frac{\partial L}{\partial A_2} \cdot \frac{\partial A_2}{\partial Z_2} \cdot \frac{\partial Z_2}{\partial F} \cdot \frac{\partial F}{\partial P_1} \cdot \frac{\partial P_1}{\partial A_1}}_{\frac{\partial L}{\partial A_1}} \cdot \frac{\partial A_1}{\partial Z_1} \cdot \frac{\partial Z_1}{\partial b_1}$$

we already calculated

$$\frac{\partial L}{\partial A_2} \cdot \frac{\partial A_2}{\partial Z_2} = (A_2 - Y)$$

Now,

// $\frac{\partial Z_2}{\partial F} = W_2$

// $\frac{\partial F}{\partial P_1} =$ reshape(P_1.shape)
because there is no trainable parameter in flatten layer

// $\frac{\partial P_1}{\partial A_1} =$? (back propagation in max pooling)
there there is also there is no trainable parameter
hence do the reverse order operation

$$\frac{\partial L}{\partial A_1} = \begin{cases} \frac{\partial L}{\partial P_{1xy}} & \text{, if } A_{mn} \text{ is the max element} \\ 0 & \text{, otherwise.} \end{cases}$$

Here $\frac{\partial L}{\partial A_1} = \frac{\partial L}{\partial A_2} \cdot \frac{\partial A_2}{\partial Z_2} \cdot \frac{\partial Z_2}{\partial F} \cdot \frac{\partial F}{\partial P_1} \cdot \frac{\partial P_1}{\partial A_1}$

(x,y) dimention of P and (m,n) is dimension of A

$$= \frac{\partial A_1}{\partial Z_1} = \begin{cases} 1 & \text{if } Z_{1_{xy}} > 0 \\ 0 & \text{if } Z_{1_{xy}} < 0 \end{cases}$$

Now backpropogation in convolution:

$$\underset{(6,6)}{X} \longrightarrow \circledast \longrightarrow Z_1 \qquad \left(\frac{\partial L}{\partial Z_1}\right)_{(4,4)}$$

$$\uparrow$$

$$\underset{(3,3)}{w_1 \;\; b_1}$$

let us assume

$$\underset{(3,3)}{X} \longrightarrow \circledast \longrightarrow Z_1 \qquad \left(\frac{\partial L}{\partial Z_1}\right)_{(2,2)}$$

$$\uparrow$$

$$\underset{(2,2)}{w_1 \;\; b_1}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \qquad W_1 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

after convolution operation :

$$Z_{11} = X_{11}W_{11} + X_{12} \cdot W_{12} + X_{21} W_{21} + X_{22} W_{22} + b_1$$

$$Z_{12} = X_{12}W_{11} + X_{13}W_{12} + X_{22} W_{21} + X_{23}W_{22} + b_1$$

$$Z_{21} = X_{21}W_{11} + X_{22}W_{12} + X_{31}W_{21} + X_{32} W_{22} + b_1$$

$$Z_{22} = X_{22}W_{11} + X_{23}W_{12} + X_{32} W_{21} + X_{33}W_{22} + b_1$$

$$\frac{\partial L}{\partial Z_1} = \begin{bmatrix} \frac{\partial L}{\partial Z_{11}} & \frac{\partial L}{\partial Z_{12}} \\ \frac{\partial L}{\partial Z_{21}} & \frac{\partial L}{\partial Z_{22}} \end{bmatrix}$$

// $$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial Z_1} \times \frac{\partial Z_1}{\partial b_1}$$

$$= \frac{\partial L}{\partial Z_{11}} \times \frac{\partial Z_{11}}{\partial b_1} + \frac{\partial L}{\partial Z_{12}} \times \frac{\partial Z_{12}}{\partial b_1} + \frac{\partial L}{\partial Z_{21}} \times \frac{\partial Z_{21}}{\partial b_1} + \frac{\partial L}{\partial Z_{22}} \times \frac{\partial Z_{22}}{\partial b_1}$$

$$\frac{\partial L}{\partial b_1} = \frac{\partial L}{\partial Z_{11}} + \frac{\partial L}{\partial Z_{12}} + \frac{\partial L}{\partial Z_{21}} + \frac{\partial L}{\partial Z_2}$$

$$\frac{\partial L}{\partial b_1} = \text{sum}\left(\frac{\partial L}{\partial Z_1}\right) \rightarrow \text{scalar value.}$$

// $$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial Z_1} \cdot \frac{\partial Z_1}{\partial W_1}$$

$$\frac{\partial L}{\partial W_1} = \begin{bmatrix} \frac{\partial L}{\partial W_{11}} & \frac{\partial L}{\partial W_{12}} \\ \frac{\partial L}{\partial W_{21}} & \frac{\partial L}{\partial W_{22}} \end{bmatrix} \qquad \frac{\partial L}{\partial Z_1} = \begin{bmatrix} \frac{\partial L}{\partial Z_{11}} & \frac{\partial L}{\partial Z_{12}} \\ \frac{\partial L}{\partial Z_{21}} & \frac{\partial L}{\partial Z_{22}} \end{bmatrix}$$

but,

$$\frac{\partial L}{\partial W_{11}} = \frac{\partial L}{\partial Z_{11}} \times \frac{\partial Z_{11}}{\partial W_{11}} + \frac{\partial L}{\partial Z_{12}} \times \frac{\partial Z_{12}}{\partial W_{11}} + \frac{\partial L}{\partial Z_{21}} \times \frac{\partial Z_{21}}{\partial W_{11}} + \frac{\partial L}{\partial Z_{22}} \times \frac{\partial Z_{22}}{\partial W_{11}}$$

$$\frac{\partial L}{\partial W_{12}} = \frac{\partial L}{\partial Z_{11}} \times \frac{\partial Z_{11}}{\partial W_{12}} + \frac{\partial L}{\partial Z_{12}} \times \frac{\partial Z_{12}}{\partial W_{12}} + \frac{\partial L}{\partial Z_{21}} \times \frac{\partial Z_{21}}{\partial W_{12}} + \frac{\partial L}{\partial Z_{22}} \times \frac{\partial Z_{22}}{\partial W_{12}}$$

$$\frac{\partial L}{\partial W_{21}} = \frac{\partial L}{\partial Z_{11}} \times \frac{\partial Z_{11}}{\partial W_{21}} + \frac{\partial L}{\partial Z_{12}} \times \frac{\partial Z_{12}}{\partial W_{21}} + \frac{\partial L}{\partial Z_{21}} \times \frac{\partial Z_{21}}{\partial W_{21}} + \frac{\partial L}{\partial Z_{22}} \times \frac{\partial Z_{22}}{\partial W_{21}}$$

$$\frac{\partial L}{\partial W_{22}} = \frac{\partial L}{\partial Z_{11}} \times \frac{\partial Z_{11}}{\partial W_{22}} + \frac{\partial L}{\partial Z_{12}} \times \frac{\partial Z_{12}}{\partial W_{22}} + \frac{\partial L}{\partial Z_{21}} \times \frac{\partial Z_{21}}{\partial W_{22}} + \frac{\partial L}{\partial Z_{22}} \times \frac{\partial Z_{22}}{\partial W_{22}}$$

$$\left\{ \begin{aligned} \frac{\partial L}{\partial W_{11}} &= \frac{\partial L}{\partial Z_{11}} X_{11} + \frac{\partial L}{\partial Z_{12}} X_{12} + \frac{\partial L}{\partial Z_{21}} X_{21} + \frac{\partial L}{\partial Z_{22}} X_{22} \\ \frac{\partial L}{\partial W_{12}} &= \frac{\partial L}{\partial Z_{11}} X_{12} + \frac{\partial L}{\partial Z_{12}} X_{13} + \frac{\partial L}{\partial Z_{21}} X_{22} + \frac{\partial L}{\partial Z_{22}} X_{23} \\ \frac{\partial L}{\partial W_{21}} &= \frac{\partial L}{\partial Z_{11}} X_{21} + \frac{\partial L}{\partial Z_{12}} X_{22} + \frac{\partial L}{\partial Z_{21}} X_{31} + \frac{\partial L}{\partial Z_{22}} X_{32} \\ \frac{\partial L}{\partial W_{22}} &= \frac{\partial L}{\partial Z_{11}} X_{22} + \frac{\partial L}{\partial Z_{12}} X_{23} + \frac{\partial L}{\partial Z_{21}} X_{32} + \frac{\partial L}{\partial Z_{22}} X_{33} \end{aligned} \right.$$

Hence equation (iii) becomes

$$\frac{\partial L}{\partial W_1} = \mathrm{conv}\left(X, \frac{\partial L}{\partial Z_1}\right)$$

and the (iv) becomes.

$$\frac{\partial L}{\partial b_1} = \mathrm{sum}\left(\frac{\partial L}{\partial Z_1}\right)$$