Lecture 2
Background

Introduction to Matrices

Matrix: A matrix is a collection of numbers or functions arranged into rows and columns.

Matrices are denoted by capital letters $A, B, \ldots, Y, Z$. The numbers or functions are called elements of the matrix. The elements of a matrix are denoted by small letters $a, b, \ldots, y, z$.

Rows and Columns: The horizontal and vertical lines in a matrix are, respectively, called the rows and columns of the matrix.

Order of a Matrix: The size (or dimension) of matrix is called as order of matrix. Order of matrix is based on the number of rows and number of columns. It can be written as $r \times c$; r means no. of row and c means no. of columns.

If a matrix has m rows and n columns then we say that the size or order of the matrix is $m \times n$. If A is a matrix having m rows and n columns then the matrix can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

The element, or entry, in the ith row and jth column of a $m \times n$ matrix A is written as a_{ij}

For example: The matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \end{pmatrix}$ has two rows and three columns. So order of A will be 2×3

Square Matrix: A matrix with equal number of rows and columns is called square matrix.

For Example The matrix $A = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ has three rows and three columns. So it is a square matrix of order 3.

Equality of matrices: The two matrices will be equal if they must have

a) The same dimensions (i.e. same number of rows and columns)

b) Corresponding elements must be equal.

Example: The matrices $A = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ equal matrices

(i.e A = B) because they both have same orders and same corresponding elements.

Column Matrix: A column matrix X is any matrix having n rows and only one column. Thus the column matrix X can be written as

$$X = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{pmatrix} = [b_{i1}]_{n\times 1}$$

A column matrix is also called a column vector or simply a vector.

Multiple of matrix: A multiple of a matrix A by a nonzero constant k is defined to be

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Notice that the product kA is same as the product Ak. Therefore, we can write $kA = Ak$.

It implies that if we multiply a matrix by a constant *k*, then each element of the matrix is to be multiplied by *k*.

Example 1:

(a)
$$5\cdot\begin{bmatrix}2 & -3\\ 4 & -1\\ 1/5 & 6\end{bmatrix}=\begin{bmatrix}10 & -15\\ 20 & -5\\ 1 & 30\end{bmatrix}$$

(b)
$$e^{t}\cdot\begin{bmatrix}1\\ -2\\ 4\end{bmatrix}=\begin{bmatrix}e^{t}\\ -2e^{t}\\ 4e^{t}\end{bmatrix}$$

Since we know that $kA = Ak$. Therefore, we can write

$$e^{-3t}\cdot\begin{bmatrix}2\\ 5\end{bmatrix}=\begin{bmatrix}2e^{-3t}\\ 5e^{-3t}\end{bmatrix}=\begin{bmatrix}2\\ 5\end{bmatrix}e^{-3t}$$

Addition of Matrices: Only matrices of the same order may be added by adding corresponding elements.
If $A=[a_{ij}]$ and $B=[b_{ij}]$ are two $m\times n$ matrices then $A+B=[a_{ij}+b_{ij}]$
Obviously order of the matrix A + B is $m\times n$

Example 2: Consider the following two matrices of order 3×3

$$A=\begin{pmatrix}2 & -1 & 3\\ 0 & 4 & 6\\ -6 & 10 & -5\end{pmatrix},\quad B=\begin{pmatrix}4 & 7 & -8\\ 9 & 3 & 5\\ 1 & -1 & 2\end{pmatrix}$$

Since the given matrices have same orders, therefore, these matrices can be added and their sum is given by

$$A+B=\begin{pmatrix}2+4 & -1+7 & 3+(-8)\\ 0+9 & 4+3 & 6+5\\ -6+1 & 10+(-1) & -5+2\end{pmatrix}=\begin{pmatrix}6 & 6 & -5\\ 9 & 7 & 11\\ -5 & 9 & -3\end{pmatrix}$$

Example 3: Write the following single column matrix as the sum of three column vectors

$$\begin{pmatrix}3t^{2}-2e^{t}\\ t^{2}+7t\\ 5t\end{pmatrix}$$

Solution:

$$\begin{pmatrix} 3t^2-2e^t \\ t^2+7t \\ 5t \end{pmatrix} = \begin{pmatrix} 3t^2 \\ t^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 7t \\ 5t \end{pmatrix} + \begin{pmatrix} -2e^t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} t + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} e^t$$

Difference of Matrices: The difference of two matrices A and B of same order $m\times n$ is defined to be the matrix $A-B=A+(-B)$

The matrix $-B$ is obtained by multiplying the matrix B with -1. So that $-B=(-1)B$

Multiplication of Matrices: We can multiply two matrices if and only if, the number of columns in the first matrix equals the number of rows in the second matrix.
Otherwise, the product of two matrices is not possible.
OR
If the order of the matrix A is $m\times n$ then to make the product AB possible order of the matrix B must be $n\times p$. Then the order of the product matrix AB is $m\times p$. Thus

$$A_{m\times n}\cdot B_{n\times p}=C_{m\times p}$$

If the matrices A and B are given by

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

Then

$$AB=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1} & \cdots & a_{11}b_{1p}+a_{12}b_{2p}+\cdots+a_{1n}b_{np} \\ a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2n}b_{n1} & \cdots & a_{21}b_{1p}+a_{22}b_{2p}+\cdots+a_{2n}b_{np} \\ \vdots & \vdots & \vdots \\ a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{mn}b_{n1} & \cdots & a_{m1}b_{1p}+a_{m2}b_{2p}+\cdots+a_{mn}b_{np} \end{bmatrix}$$

$$= \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{n \times p}$$

Example 4: If possible, find the products AB and BA, when

(a) $$A = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$$

(b) $$A = \begin{pmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix}$$

Solution: (a) The matrices A and B are square matrices of order 2. Therefore, both of the products AB and BA are possible.

$$AB = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 4 \cdot 9 + 7 \cdot 6 & 4 \cdot (-2) + 7 \cdot 8 \\ 3 \cdot 9 + 5 \cdot 6 & 3 \cdot (-2) + 5 \cdot 8 \end{pmatrix} = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}$$

Similarly $$BA = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 9 \cdot 4 + (-2) \cdot 3 & 9 \cdot 7 + (-2) \cdot 5 \\ 6 \cdot 4 + 8 \cdot 3 & 6 \cdot 7 + 8 \cdot 5 \end{pmatrix} = \begin{pmatrix} 30 & 53 \\ 48 & 82 \end{pmatrix}$$

Note: From above example it is clear that generally a matrix multiplication is not commutative i.e. $AB \neq BA$.

(b) The product AB is possible as the number of columns in the matrix A and the number of rows in B is 2. However, the product BA is not possible because the number of column in the matrix B and the number of rows in A is not same.

$$AB = \begin{pmatrix} 5 & 8 \\ 1 & 0 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \cdot (-4) + 8 \cdot 2 & 5 \cdot (-3) + 8 \cdot 0 \\ 1 \cdot (-4) + 0 \cdot 2 & 1 \cdot (-3) + 0 \cdot 0 \\ 2 \cdot (-4) + 7 \cdot 2 & 2 \cdot (-3) + 7 \cdot 0 \end{pmatrix} = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}, \quad BA = \begin{pmatrix} 30 & 53 \\ 48 & 82 \end{pmatrix}$$

Clearly $AB \neq BA$.

$$AB = \begin{pmatrix} -4 & -15 \\ -4 & -3 \\ 6 & -6 \end{pmatrix}$$

However**,** the product BA is not possible.

Example 5:

(a) $$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ 1 & -7 & 9 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot(-3)+(-1)\cdot 6+3\cdot 4 \\ 0\cdot(-3)+4\cdot 6+5\cdot 6 \\ 1\cdot(-3)+(-7)\cdot 6+9\cdot 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 44 \\ -9 \end{pmatrix}$$

(b) $$\begin{pmatrix} -4 & 2 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x+2y \\ 3x+8y \end{pmatrix}$$

Multiplicative Identity: For a given any integer n, the $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the multiplicative identity matrix. If A is a matrix of order $n \times n$, then it can be verified that $I \cdot A = A \cdot I = A$

Example: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$**,** $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are identity matrices of orders 2 x 2 and 3 x 3 respectively and If $B = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$ then we can easily prove that BI = IB = B

Zero Matrix or Null matrix: A matrix whose all entries are zero is called zero matrix or null matrix and it is denoted by O.

For example $O=\begin{pmatrix}0\\0\end{pmatrix}$; $O=\begin{pmatrix}0&0\\0&0\end{pmatrix}$; $O=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}$

and so on. If A and O are the matrices of same orders, then $A+O=O+A=A$

Associative Law: The matrix multiplication is associative. This means that if A, B and C are $m\times p$, $p\times r$ and $r\times n$ matrices, then $A(BC)=(AB)C$
The result is an $m\times n$ matrix. This result can be verified by taking any three matrices which are conformable for multiplication.

Distributive Law: If B and C are matrices of order $r\times n$ and A is a matrix of order $m\times r$, then the distributive law states that

$$A(B+C)=AB+AC$$

Furthermore, if the product $(A+B)C$ is defined, then

$$(A+B)C=AC+BC$$

Determinant of a Matrix: Associated with every square matrix A of constants, there is a number called the determinant of the matrix, which is denoted by $\det(A)$ or $|A|$

Example 6: Find the determinant of the following matrix $A=\begin{pmatrix}3&6&2\\2&5&1\\-1&2&4\end{pmatrix}$

Solution: The determinant of the matrix A is given by

$$\det(A)=\begin{vmatrix}3&6&2\\2&5&1\\-1&2&4\end{vmatrix}$$

We expand the $\det(A)$ by first row, we obtain

$$\det(A)=\begin{vmatrix}3&6&2\\2&5&1\\-1&2&4\end{vmatrix}=3\begin{vmatrix}5&1\\2&4\end{vmatrix}-6\begin{vmatrix}2&1\\-1&4\end{vmatrix}+2\begin{vmatrix}2&5\\-1&2\end{vmatrix}$$

or $$\det(A)=3(20-2)-6(8+1)+2(4+5)=18$$

Transpose of a Matrix: The transpose of $m \times n$ matrix A is denoted by A^{tr} and it is obtained by interchanging rows of A into its columns. In other words, rows of A become the columns of A^{tr}. Clearly A^{tr} is $n \times m$ matrix.

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, then $A^{tr} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$

Since order of the matrix A is $m \times n$, the order of the transpose matrix A^{tr} is $n \times m$.

Properties of the Transpose:

The following properties are valid for the transpose;

- The transpose of the transpose of a matrix is the matrix itself: $(\underline{A}^T)^T = \underline{A}$
- The transpose of a matrix times a scalar (k) is equal to the constant times the transpose of the matrix: $(k\underline{A})^T = k\underline{A}^T$
- The transpose of the sum of two matrices is equivalent to the sum of their transposes: $(\underline{A} + \underline{B})^T = \underline{A}^T + \underline{B}^T$
- The transpose of the product of two matrices is equivalent to the product of their transposes in reversed order: $(\underline{A}\underline{B})^T = \underline{B}^T\underline{A}^T$
- The same is true for the product of multiple matrices: $(\underline{A}\underline{B}\underline{C})^T = \underline{C}^T\underline{B}^T\underline{A}^T$

Example 7: (a) The transpose of matrix $A = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$ is $A^T = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

(b) If $X = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$, then $X^T = [5 \quad 0 \quad 3]$

Multiplicative Inverse: Suppose that A is a square matrix of order $n \times n$. If there exists an $n \times n$ matrix B such that $AB = BA = I$, then B is said to be the multiplicative inverse of the matrix A and is denoted by $B = A^{-1}$.

For example: If $A=\begin{pmatrix}1 & 4\\ 2 & 10\end{pmatrix}$ then the matrix $B=\begin{pmatrix}5 & -2\\ -1 & 1/2\end{pmatrix}$ is multiplicative inverse of A because AB = $\begin{pmatrix}1 & 4\\ 2 & 10\end{pmatrix}\begin{pmatrix}5 & -2\\ -1 & 1/2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ =I

Similarly we can check that BA = I

Singular and Non-Singular Matrices: A square matrix A is said to be a **non-singular** matrix if $\det(A)\neq 0$, otherwise the square matrix A is said to be **singular**. Thus for a singular matrix A we must have $\det(A)=0$

Example: $A=\begin{bmatrix}2 & 3 & -1\\ 1 & 1 & 0\\ 2 & -3 & 5\end{bmatrix}$

$$|A| = 2(5-0)-3(5-0)-1(-3-2)$$
$$=10-15+5=0$$

which means that A is singular.

Minor of an element of a matrix:

Let A be a square matrix of order n x n. Then minor M_{ij} of the element $a_{ij}\in A$ is the determinant of $(n-1)\times(n-1)$ matrix obtained by deleting the *ith* row and *jth* column from A.

Example: If $A=\begin{bmatrix}2 & 3 & -1\\ 1 & 1 & 0\\ 2 & -3 & 5\end{bmatrix}$ is a square matrix. The Minor of $3\in A$ is denoted by M_{12} and is defined to be $M_{12}=\begin{vmatrix}1 & 0\\ 2 & 5\end{vmatrix}$ = 5-0 = 5

Cofactor of an element of a matrix:

Let A be a non singular matrix of order $n\times n$ and let C_{ij} denote the cofactor (signed minor) of the corresponding entry $a_{ij}\in A$, then it is defined to be $C_{ij}=(-1)^{i+j}M_{ij}$

Example: If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ is a square matrix. The cofactor of $3 \in A$ is denoted by C_{12} and is defined to be $C_{12} = = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} =$ - (5 - 0) = -5

Theorem: If A is a square matrix of order $n \times n$ then the matrix has a multiplicative inverse A^{-1} if and only if the matrix A is non-singular.

Theorem: Then inverse of the matrix A is given by $A^{-1} = \frac{1}{\det(A)}(C_{ij})^{tr}$

1. For further reference we take $n = 2$ so that A is a 2×2 non-singular matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Therefore $C_{11} = a_{22}$, $C_{12} = -a_{21}$, $C_{21} = -a_{12}$ and $C_{22} = a_{11}$. So that

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}^{tr} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

2. For a 3×3 non-singular matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ and so on.}$$

Therefore, inverse of the matrix A is given by $A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$.

Example 8: Find, if possible, the multiplicative inverse for the matrix $A=\begin{pmatrix}1 & 4\\ 2 & 10\end{pmatrix}$.

Solution: The matrix A is non-singular because $\det(A)=\begin{vmatrix}1 & 4\\ 2 & 10\end{vmatrix}=10-8=2$

Therefore, A^{-1} exists and is given by $A^{-1}=\frac{1}{2}\begin{pmatrix}10 & -4\\ -2 & 1\end{pmatrix}=\begin{pmatrix}5 & -2\\ -1 & 1/2\end{pmatrix}$

Check: $$AA^{-1}=\begin{pmatrix}1 & 4\\ 2 & 10\end{pmatrix}\begin{pmatrix}5 & -2\\ -1 & 1/2\end{pmatrix}=\begin{pmatrix}5-4 & -2+2\\ 10-10 & -4+5\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

$$AA^{-1}=\begin{pmatrix}5 & -2\\ -1 & 1/2\end{pmatrix}\begin{pmatrix}1 & 4\\ 2 & 10\end{pmatrix}=\begin{pmatrix}5-4 & 20-20\\ -1+1 & -4+5\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

Example 9: Find, if possible, the multiplicative inverse of the following matrix

$$A=\begin{pmatrix}2 & 2\\ 3 & 3\end{pmatrix}$$

Solution: The matrix is singular because

$$\det(A)=\begin{vmatrix}2 & 2\\ 3 & 3\end{vmatrix}=2\cdot 3-2\cdot 3=0$$

Therefore, the multiplicative inverse A^{-1} of the matrix does not exist.

Example 10: Find the multiplicative inverse for the following matrix

$$A=\begin{pmatrix}2 & 2 & 0\\ -2 & 1 & 1\\ 3 & 0 & 1\end{pmatrix}.$$

Solution: Since $\det(A)=\begin{vmatrix}2 & 2 & 0\\ -2 & 1 & 1\\ 3 & 0 & 1\end{vmatrix}=2(1-0)-2(-2-3)+0(0-3)=12\neq 0$

Therefore, the given matrix is non singular. So, the multiplicative inverse A^{-1} of the matrix A exists. The cofactors corresponding to the entries in each row are

$$C_{11}=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}=1,\qquad C_{12}=-\begin{vmatrix}-2 & 1\\ 3 & 1\end{vmatrix}=5,\quad C_{13}=\begin{vmatrix}-2 & 1\\ 3 & 0\end{vmatrix}=-3$$

$$C_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2, \quad C_{23} = -\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2, \quad C_{32} = -\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2, \quad C_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6$$

Hence $$\mathrm{A}^{-1} = \frac{1}{12}\begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix} = \begin{pmatrix} 1/12 & -1/6 & 1/6 \\ 5/12 & 1/6 & -1/6 \\ -1/4 & 1/2 & 1/2 \end{pmatrix}$$

We can also verify that $A \cdot A^{-1} = A^{-1} \cdot A = I$

Derivative of a Matrix of functions:

Suppose that

$$A(t) = \left[a_{ij}(t) \right]_{m \times n}$$

is a matrix whose entries are functions those are differentiable on a common interval, then derivative of the matrix $A(t)$ is a matrix whose entries are derivatives of the corresponding entries of the matrix $A(t)$. Thus

$$\frac{dA}{dt} = \left[\frac{da_{ij}}{dt} \right]_{m \times n}$$

The derivative of a matrix is also denoted by $A'(t)$.

Integral of a Matrix of Functions:

Suppose that $A(t) = \left(a_{ij}(t)\right)_{m \times n}$ is a matrix whose entries are functions those are continuous on a common interval containing t, then integral of the matrix $A(t)$ is a matrix whose entries are integrals of the corresponding entries of the matrix $A(t)$. Thus

$$\int_{t_0}^{t} A(s)ds = \left(\int_{t_0}^{t} a_{ij}(s)ds \right)_{m \times n}$$

Example 11: Find the derivative and the integral of the following matrix $X(t) = \begin{pmatrix} \sin 2t \\ e^{3t} \\ 8t - 1 \end{pmatrix}$

Solution: The derivative and integral of the given matrix are, respectively, given by

$$X'(t) = \begin{pmatrix} \frac{d}{dt}(\sin 2t) \\ \frac{d}{dt}(e^{3t}) \\ \frac{d}{dt}(8t-1) \end{pmatrix} = \begin{pmatrix} 2\cos 2t \\ 3e^{3t} \\ 8 \end{pmatrix} \quad \text{and} \int_0^t X(s)ds = \begin{pmatrix} \int_0^t \sin 2sds \\ \int_0^t e^{3s}ds \\ \int_0^t 8s-1ds \end{pmatrix} = \begin{pmatrix} -1/2\cos 2t + 1/2 \\ 1/3e^{3t} - 1/3 \\ 4t^2 - t \end{pmatrix}$$

Exercise:

Write the given sum as a single column matrix

1. $3t\begin{pmatrix} 2 \\ t \\ -1 \end{pmatrix} + (t-1)\begin{pmatrix} -1 \\ -t \\ 3 \end{pmatrix} - 2\begin{pmatrix} 3t \\ 4 \\ -5t \end{pmatrix}$

2. $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 5 & -1 \\ 0 & -4 & -2 \end{pmatrix}\begin{pmatrix} t \\ 2t-1 \\ -t \end{pmatrix} + \begin{pmatrix} -t \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ -6 \end{pmatrix}$

Determine whether the given matrix is singular or non-singular. If singular, find A^{-1}.

3. $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 0 \\ -2 & 5 & -1 \end{pmatrix}$

4. $A = \begin{pmatrix} 4 & 1 & -1 \\ 6 & 2 & -3 \\ -2 & -1 & 2 \end{pmatrix}$

Find $\dfrac{dX}{dt}$

5. $X = \begin{pmatrix} \frac{1}{2}\sin 2t - 4\cos 2t \\ -3\sin 2t + 5\cos 2t \end{pmatrix}$

6. If $A(t) = \begin{pmatrix} e^{4t} & \cos \pi t \\ 2t & 3t^2 - 1 \end{pmatrix}$ then find (a) $\int_0^2 A(t)dt$, (b) $\int_0^t A(s)ds.$

7. Find the integral $\int_1^2 B(t)dt$ if $B(t) = \begin{pmatrix} 6t & 2 \\ 1/t & 4t \end{pmatrix}$