Assiut University

Faculty of Science

Department of Mathematics

Course Title: Mathematics (III)

Course Code: MATH211

Course hours per week:

Lecture	Tutorial / Practical	Total
3	2	5

For

Faculty of Computers & Information

By

Professor

OSAMA RASHED SAYED

Table of Contents

CHAPTER I: FOURIER SERIES

Introduction	5
Piecewise continuity	6
Theory	7
Integrals	13
Useful trig results	14
Alternative notation	15
Full worked solutions	17
Exercise Set (1)	60

CHAPTER II: FOURIER TRANSFORM

Fourier transform	63
Properties of ***Fourier*** transform	70
Exercise Set (2)	106

CHAPTER III: FOURIER INTEGRALS

The Fourier integrals	110
Fourier transform	112
Parseval's identities for Fourier integrals	122
The convolution theorem	125
Supplementary Problems	136

CHAPTER IV: THE LAPLACE TRANSFORM

Laplace transform	140
Sufficient conditions for existence of Laplace transform	149
Properties of Laplace Transform	151
Laplace transform of derivatives	160
Laplace transform of integrals	169
Methods of finding Laplace transforms	178
Some special functions	181
Laplace transform of special functions	207
Exercise Set	209

CHAPTER V: THE INVERSE LAPLACE TRANSFORM

Properties of inverse Laplace transform 222
Inverse Laplace transforms of derivatives 239
Inverse Laplace transforms of integrals 241
Methods of finding inverse Laplace transforms 253
The Heaviside expansion formula 264
The Beta Function 276
Evaluation of integrals 270
Miscellaneous problems 272

CHAPTER IV: THE Z–TRANSFORM AND THE DIFFERENCE EQUATIONS

Difference Equations 279
The z- Transform 283
Poles and Zeros 289
Frequency Response 292
Important Properties of the *z*-Transform 285
Initial Value Theorem 296
Final Value Theorem 296
Convolution 297
Inverse z-Transform 299
Direct Long Division Method 301
Partial Fraction Expansion 303
Distinct Real Poles 303
Distinct complex poles 306
Solving Linear Difference Equation Using z-Transform 310
MATLAB program 312
Pulse Transfer Function and Impulse Response Sequence 314
Discrete-Time Convolution and Impulse Response Function 322
Frequency Response of Discrete-Time Systems 324
Periodicity of discrete-time frequency response function 329
Problems 333

CHAPTER(I)

FOURIR SERIES

CHAPTER (I)

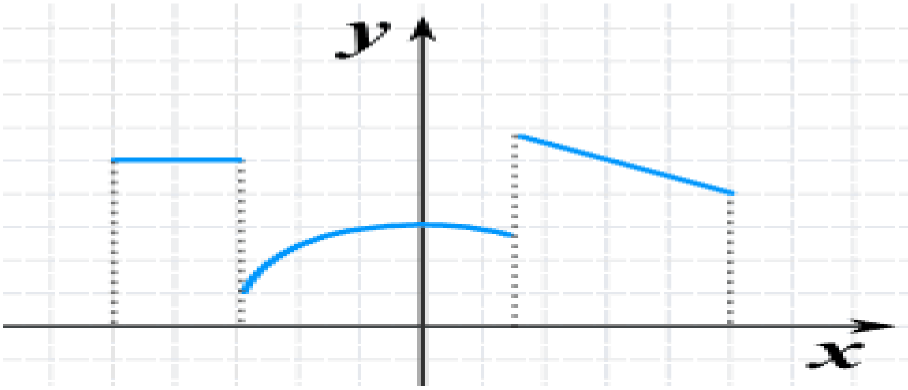
FOURIER SERIES

► INTRODUCTION

Fourier series has long provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. It has spurred generalizations and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis.

● Piecewise continuity

A function $y = F(x)$ is called piecewise continuous in an interval $\alpha \leqq x \leqq \beta$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left-hand limits.

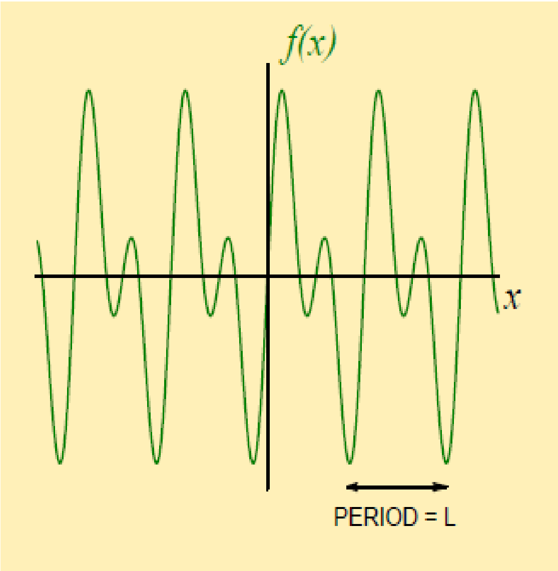


$\alpha \qquad t_1 t_2 \qquad \beta$

An example of a function which is piecewise continuous is shown graphically in the above Figure. This function has discontinuities at t_1 and t_2. Note that the right and left hand limits at t_2, for example, are represented by $\lim_{\epsilon \to 0} F(t_2 + \epsilon) = F(t_2 + 0) = F(t_2^+)$ and $\lim_{\epsilon \to 0} F(t_2 - \epsilon) = F(t_2 - 0) = F(t_2^-)$, respectively, where ϵ is positive.

►Theory

♠ A graph of periodic function $f(x)$that has period L exhibits the same pattern every L units along the x-axis so that $f(x+L)=f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods).



♠ This property of repetition defines a fundamental spatial frequency$k = \frac{2\pi}{L}$ that can be used to give a first approximation to the periodic pattern $f(x)$:

$$f(x) \simeq c_1 \sin(kx + \alpha_1)$$

$$= a_1 \cos(kx) + b_1 \sin(kx),$$

where symbols with subscript 1 are constants that determine the am-plagued and phase of this first approximation.

♠A much **better approximation** of the periodic pattern $f(x)$ can be built up by adding an appropriate combination of harmonics to this fundamental (sine-wave) pattern. For example, adding

$$c_2 \sin(2kx + \alpha_2) = a_2 \cos(2kx) + b_2 \sin(2kx)$$

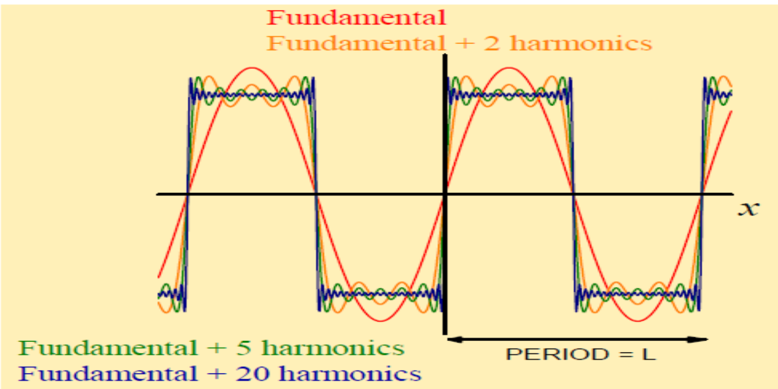
(The 2nd harmonic)

$$c_3 \sin(2kx + \alpha_3) = a_3 \cos(2kx) + b_3 \sin(2kx)$$

(The 3rd harmonic)

Here, symbols with subscripts are constants that determine the am-plagued and phase of each harmonic contribution.

One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine- wave plus a combination of harmonics of this fundamental frequency. This sum is called a Fourier series.



●In this chapter, we consider working out Fourier series for functions $f(x)$ with period $= 2\pi$. Their fundamental frequency is then $k = \frac{2\pi}{L} = 1,$ and their Fourier series representations involve terms like.

$$a_1 \cos x\,,\ \ a_2 \cos 2x, a_3 \cos 3x, \ldots$$

$$b_1 \sin x\ \ , b_2 \sin 2x\,, b_3 \sin 3x, \ldots$$

We also include a constant term $a_0/2$ in the Fourier series.This allows us to represent functions that are, for example, entirely above the x-axis with a sufficient number of harmonics included, our approximate series can exactly represent a given function $f(x)$.

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$

$$+b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

A more compact way of writing the Fourier series of a function$f(x)$ with period $2\pi,$ uses the variable subscript $n = 1{,}2{,}3, \ldots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

●We need to work out the Fourier coefficients (a_0, a_nandb_n) for given function $f(x)$. This process is broken down intothree steps.

STEP ONE $\quad a_0 = \frac{1}{\pi} \int_{2\pi} f(x) dx$

STEP TWO $\quad a_n = \frac{1}{\pi} \int_{2\pi} f(x) \cos nx \, dx$

STEP THREE $\quad b_n = \frac{1}{\pi} \int_{2\pi} f(x) \sin nx \, dx$

Where integrations are over a single interval in x of $L = 2\pi$.

Dirichlet Conditions:

(i) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$.

(ii) $f(x)$ is periodic with period $2\,L$.

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

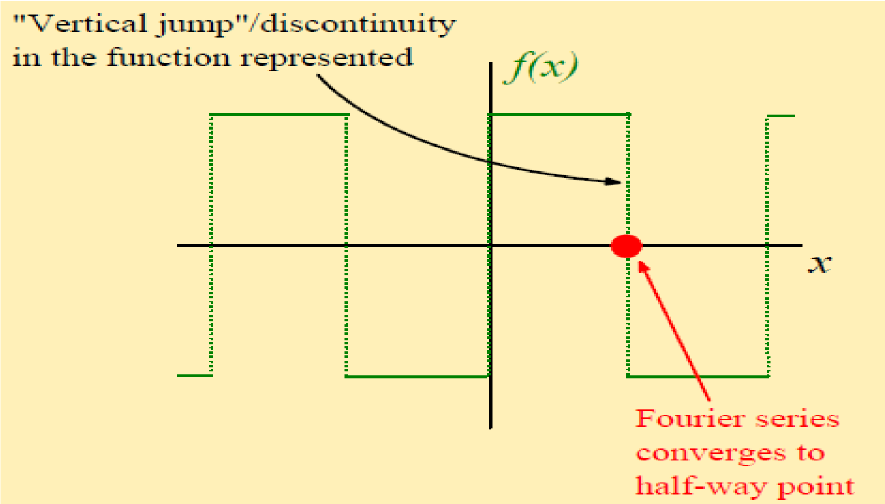
Theorem 1.

If $f(x)$be function satisfies Dirichlet Conditions, then the Fourier series of $f(x)$ with coefficients $(a_0, a_n \text{and} b_n)$ converges to:

(a) $f(x)$ if x is a point continuity.

(b)$\frac{f(x+0)+f(x-0)}{2}$if x is a point of discontinuity.

●Finally, according to Dirichlet Conditions, specifying a particular value of $x = x_1$ in a Fourier series, gives a series of constants that equal $f(x_1)$. However, if $f(x)$ is discontinuous at this value of x, then the series converges to a value that is half- way between the two possible function values.



•Integrals

Formula for integration by parts: x

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b \frac{du}{dx}\,v\,dx$$

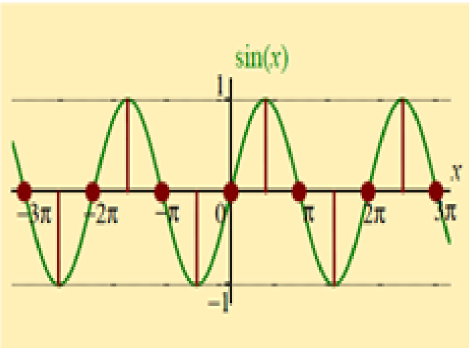
$f(x)$	$\int f(x)\mathrm{d}x$	$f(x)$	$\int f(x)\mathrm{d}x$
x^n	$\frac{x^{n+1}}{n+1}\quad (n\neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1}\quad (n\neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a}\quad (a>0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1}e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2}-\frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4}-\frac{x}{2}$
$\cos^2 x$	$\frac{x}{2}+\frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4}+\frac{x}{2}$

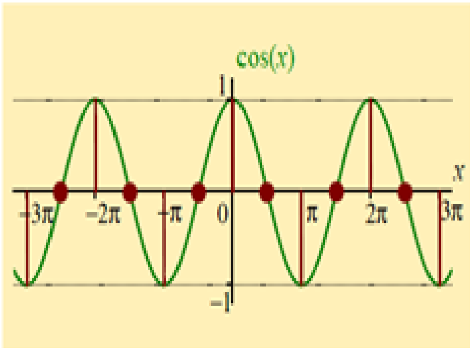
$f(x)$	$\int f(x)\,\mathrm{d}x$	$f(x)$	$\int f(x)\,\mathrm{d}x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert\quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert\quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert\quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert\quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
		$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

• Useful trig results

When calculating the Fourier coefficients a_n and b_n, forwhich $n = 1,2,3, \ldots$, the following trig results are useful. Each of these results, which are also true for$n = 0, -1, -2 - 3, \ldots$ can be deduced from the graph of $\sin x$ or that of $\cos x$.

■ $\sin n\pi = 0$ ■ $\cos n\pi = (-1)^n$



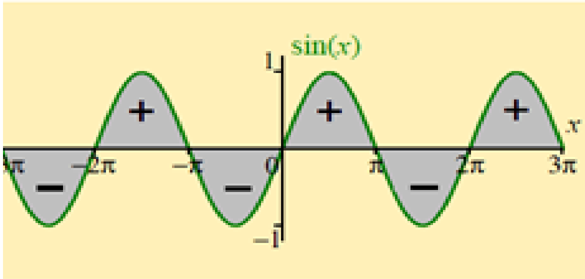


■ $\sin n\dfrac{\pi}{2} = \begin{cases} 0 & , n \text{ } even \\ 1 & , n = 1,5,9, \ldots \\ -1 & , n = 3,7,11, \ldots \end{cases}$

■ $\cos n\dfrac{\pi}{2} = \begin{cases} 0 & , n \text{ } odd \\ 1 & , n = 0,4,8, \ldots \\ -1 & , n = 2,6,10, \ldots \end{cases}$

Areas cancel when integrating over whole periods.

- ■ $\int_{2\pi} \sin nx \;\, dx = 0$
- ■ $\int_{2\pi} \cos nx \;\, dx = 0$



●Alternative notation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nkx + b_n \sin nkx]$$

The corresponding Fourier coefficients are

STEP ONE $a_0 = \frac{2}{L}\int_L \; f(x)\, dx$

STEP TWO$a_n = \frac{2}{L}\int_L \; f(x) \cos nkx \, dx$

STEP THREE$b_n = \frac{2}{L}\int_L \; f(x) \sin nkx$

and integrations are over a single interval in x of L.

♠ For a waveform $f(x)$ with period $2L = \frac{2\pi}{k}$, we have that $k = \frac{2\pi}{2L} = \frac{\pi}{L}$ and $nkx = \frac{n\pi x}{L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

The corresponding Fourier coefficients are.

STEP ONE $a_0 = \frac{1}{L} \int_{2L} f(x)\, dx$

STEP TWO $a_n = \frac{1}{L} \int_{2L} f(x) \cos \frac{n\pi x}{L}\, dx$

STEP THREE $b_n = \frac{1}{L} \int_{2L} f(x) \sin \frac{n\pi x}{L}\, dx$

integrations are over a single interval in x of $2L$.

♠ For a waveform $f(x)$ with period $T = \frac{2\pi}{\omega}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$$

The corresponding Fourier coefficients are

STEP ONE $a_0 = \frac{2}{T} \int_T f(t)\, dt$

STEP TWO $a_n = \frac{2}{T} \int_T f(t) \cos n\omega t\, dt$

STEP THREE $b_n = \frac{2}{T} \int_T f(t) \sin n\omega t\, dt$

integrations are over a single interval in t of T.

Full worked solutions: -

Example 1.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval - $2\pi < x < 2\pi$.

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

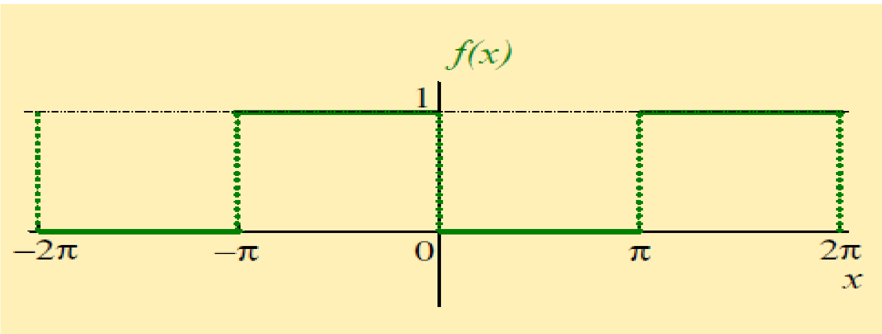
$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

c) By giving an appropriate value to x, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Solution.

a) The graph of $f(x)$ in the interval - $2\pi < x < 2\pi$.



b) <u>Fourier series representation of $f(x)$</u>

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 1.\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0.\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} dx$$

$$= \frac{1}{\pi}[x]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\big(0 - (-\pi)\big)$$

$$= \frac{1}{\pi}\ .(\pi)$$

$i.e.\ \ a_0 = 1.$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 1.\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} 0.\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}[\sin nx]_{-\pi}^{0}$$

$$= \frac{1}{n\pi}(\sin 0 - \sin(-n\pi))$$

$$= \frac{1}{n\pi}\,.(0 + \sin n\pi)$$

$$i.e.\quad a_n = \frac{1}{n\pi}(0+0) = 0.$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 1.\sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} 0.\sin nx dx$$

$$= \int_{-\pi}^{0} \sin nx\, dx = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}$$

$$= -\frac{1}{n\pi}[\cos nx]_{-\pi}^{0} = -\frac{1}{n\pi}(\cos 0 - \cos(-n\pi))$$

$$= -\frac{1}{n\pi}\,.(1 - \cos n\pi) = -\frac{1}{n\pi}(1 - (-1)^n)$$

$$= \begin{cases} 0 & , n\ even \\ -\dfrac{2}{n\pi} & , n\ odd \end{cases} \quad ,$$

We now have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

With the three steps giving

$$a_0 = 1, a_n = 0, \text{ and } \quad b_n = \begin{cases} 0 & , n\ even \\ -\frac{2}{n\pi} & , n\ odd \end{cases}$$

It may be helpful to construct a table of values of b_n

n	1	2	3	4	5
b_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5}\right)$

Substituting our results now gives

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

c) Pick an appropriate value of x, to show that

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

Comparing this series with

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

We need to introduce a minus sign in front of the constants $\frac{1}{3}, \frac{1}{7}, \ldots$

So, we need $\sin x = 1, \sin 3x = -1,$ $\sin 5x = 1,$ $\sin 7x = -1,$ etc. The first condition of $\sin x = 1$ suggeststrying $x = \frac{\pi}{2}$. This choice gives

$$\sin\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} + \frac{1}{7}\sin 7\frac{\pi}{2}$$

i.e. $1 - \quad \frac{1}{3} \quad + \quad \frac{1}{5} \quad - \quad \frac{1}{7}$

Looking at the graph of $f(x)$, we also have that $f\left(\frac{\pi}{2}\right) = 0$. Picking $x = \frac{\pi}{2}$ thus gives

$$0 = \frac{1}{2} - \frac{2}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \cdots\right]$$

$$i.e. \;\; 0 = \frac{1}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

A little manipulation then gives a series representation of $\frac{\pi}{4}$

$$\frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] = \frac{1}{2}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \blacktriangleleft$$

Example 2.

Let $f(x)$ be a function of period 2π such that.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$

b) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$+\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

c) By giving appropriate values to x, show that

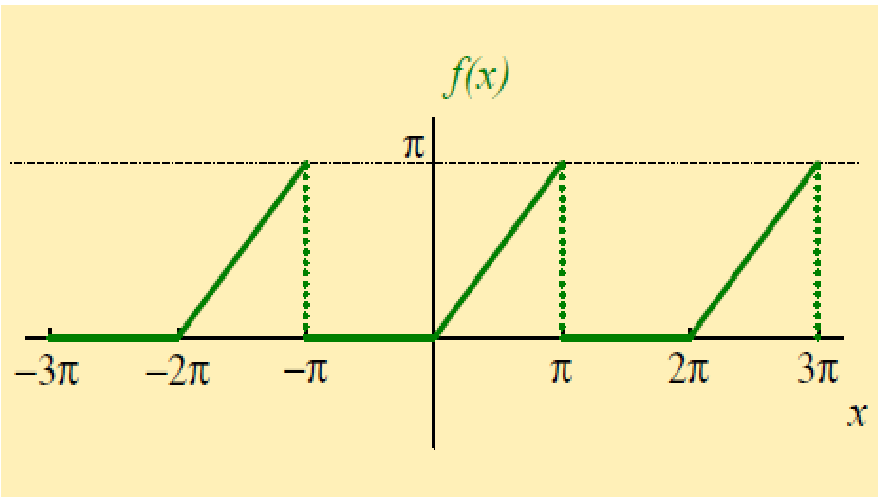
$$(i)\ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and

$$(ii)\ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Solution.

a) The graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,.\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}.$$

i.e. $a_0 = \frac{\pi}{2}$.

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x.\cos nx\, dx$$

$$i.e.\ a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left\{\left[x\ \frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}\ dx\right\}$$

(using integration by parts)

$$i.e.\ a_n \quad = \frac{1}{\pi}\left\{\left(\pi\ \frac{\sin n\pi}{n} - 0\right) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(0-0) + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi n^2}\{\cos n\pi - \cos 0\}$$

$$= \frac{1}{\pi n^2}\{(-1)^n - 1\}$$

i.e. $a_n = \begin{cases} 0 & , n\ even \\ -\frac{2}{\pi\, n^2} & , n\ odd \end{cases}.$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\, dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\sin nx\, dx + \frac{1}{\pi}\int_0^{\pi} x.\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{\left[x \left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right) dx\right\}$$

(using integration by parts)

$$= \frac{1}{\pi}\left\{-\frac{1}{n}[x \cos nx]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(\pi \cos n\pi - 0) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{\pi n^2}(0-0)$$

$$= -\frac{1}{n}(-1)^n$$

i.e. $$b_n = \begin{cases} -\frac{1}{n} & , n \; even \\ +\frac{1}{n} & , n \; odd \end{cases}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{\pi}{2}$, $a_n = \begin{cases} 0 & , n \ even \\ -\frac{2}{\pi n^2} & , n \ odd \end{cases}$,

$$b_n = \begin{cases} -\dfrac{1}{n} & , n \ even \\ \dfrac{1}{n} & , n \ odd \end{cases}$$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi} \cdot \frac{1}{3^2}$	0	$-\frac{2}{\pi} \cdot \frac{1}{5^2}$
b_n	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$

This table of coefficients gives

$$f(x) = \frac{1}{2}\left(\frac{\pi}{2}\right) + \left(-\frac{2}{\pi}\right)\cos x + 0.\cos 2x$$

$$+\left(-\frac{2}{\pi} \cdot \frac{1}{3^2}\right)\cos 3x + 0.\cos 4x$$

$$+\left(-\frac{2}{\pi} \cdot \frac{1}{5^2}\right)\cos 5x + \cdots$$

$$+\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - \cdots$$

$$i.e.\;\; f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$+\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

and we have found the required series!

c) <u>Pick an appropriate value of *x*, to show that</u>

$$\boxed{(i)\,\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

Comparing this series with

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$+\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

The required series of constants does <u>not</u> involve

terms like $\frac{1}{3^2}$, $\frac{1}{5^2}$, $\frac{1}{7^2}$, ... So, we need to pick a value

of x that sets the $\cos nx$ terms to zero.

We not that $\cos n\frac{\pi}{2} = 0$ when n is odd, and note also that $\cos nx$ terms in the Fourier series all have odd n .

i.e. $\cos x = \cos 3x = \cos 5x = \cdots = 0$

when $x = \frac{\pi}{2}$,

i.e. $\cos\frac{\pi}{2} = \cos 3\frac{\pi}{2} = \cos 5\frac{\pi}{2} = \cdots = 0$

Setting $x = \frac{\pi}{2}$ in the series for $f(x)$ gives

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos\frac{\pi}{2} + \frac{1}{3^2}\cos\frac{3\pi}{2} + \frac{1}{5^2}\cos\frac{5\pi}{2} + \cdots\right]$$

$$+\left[\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2}\right.$$

$$\left. + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} - \cdots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}[0 + 0 + 0 + \cdots]$$

$$+\left[1 - \frac{1}{2}\underbrace{\sin\pi}_{=0} + \frac{1}{3}\,.(-1) - \frac{1}{4}\underbrace{\sin 2\pi}_{=0} + \frac{1}{5}.(1) - \cdots\right]$$

The graph of $f(x)$ shows that $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, so that

$$\frac{\pi}{2} = \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

i.e.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

<u>Pick an appropriate value of x , to show that</u>

$$(ii)\,\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Compare this series with

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$+\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

This time, we want to use coefficients of the $\cos nx$ terms,and the same choice of x needs to set the $\sin nx$ terms to zero. Picking $x = 0$ gives

$\sin x = \sin 2x = \sin 3x = 0$ and $\cos x = \cos 3x = \cos 5x = 1$.

Note also that the graph of $f(x)$ gives $f(x) = 0$ when $x = 0$.So, picking $x = 0$ gives

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \frac{1}{7^2}\cos 0 + \cdots\right] + \sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} - \cdots$$

i.e.

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] + 0 - 0 + 0 - \cdots$$

We then find that

$$\frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] = \frac{\pi}{4}$$

and $$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}. \blacktriangleleft$$

Example 3.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$

b) Show that the Fourier series for $f(x)$ in the interval $0 < x < 2\pi$ is

$$\frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$-\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

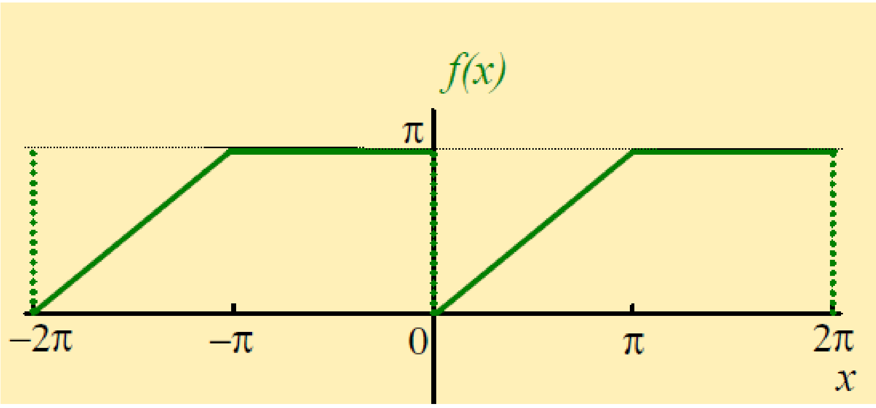
c) By giving appropriate values to x, show that

$$(i)\, \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$(ii)\, \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Solution.

a) <u>The graph of $f(x)$ in the interval $-2\pi < x < 2\pi$.</u>



b) <u>Fourier series representation of $f(x)$</u>

STEP ONE

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} xdx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,.\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}[x]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + (2\pi - \pi) \quad = \frac{\pi}{2} + \pi$$

i.e. $a_0 = \frac{3\pi}{2}$

STEP TWO

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx \, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi \, . \cos nx \, dx$$

$$= \frac{1}{\pi}\underbrace{\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\; dx\right]}_{using\ integration\ by\ parts}$$

$$+ \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(\pi\ \sin nx - 0 \, . \sin n0) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$+ \frac{1}{n}(\sin 2n\pi - \sin n\pi)$$

$$\text{i.e.}\, a_n = \frac{1}{\pi}\left[\frac{1}{n}(0-0) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right.$$

$$\left. + \frac{1}{n}(0-0)\right]$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1) = \frac{1}{n^2\pi}((-1)^n - 1)$$

$$i.e. \quad a_n = \begin{cases} -\dfrac{2}{n^2\pi} & , n\ odd \\ 0 & , n\ even \end{cases}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,.\sin nx\, dx$$

$$= \frac{1}{\pi}\underbrace{\left[\left[\left(x\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)\, dx\right]}_{using\ integration\ by\ parts}$$

$$+ \frac{\pi}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right]$$

$$- \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right]$$

$$-\frac{1}{n}(1-(-1)^n)$$

$$= -\frac{1}{n}(-1)^n + \;\; 0 \;\; -\frac{1}{n}(1-(-1)^n)$$

$$\text{i. e.}\, b_n = -\frac{1}{n}(-1)^n - \frac{1}{n} + \frac{1}{n}(-1)^n = \quad -\frac{1}{n}$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Where $a_0 = \frac{3\pi}{2}, a_n = \begin{cases} 0 & , n\ even \\ -\frac{2}{n^2\pi} & , n\ odd \end{cases}, b_n = -\frac{1}{n}$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3^2}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5^2}\right)$
b_n	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$

his table of coefficients

$$f(x) = \frac{1}{2}\left(\frac{3\pi}{2}\right) + \left(-\frac{2}{\pi}\right)\left[\cos x + 0.\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right] + (-1)\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$- \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

and we have found the required series!

a) <u>Pick an appropriate value of *x*, to show that</u>

(i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Compare this series with

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$- \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

Here, we want to set the $\cos nx$ terms to zero (since their coefficients are $1, \frac{1}{3^2}, \frac{1}{5^2}, \ldots$) .

Since $\cos n\frac{\pi}{2} = 0$ when *n i*s odd, we will try setting $x = \frac{\pi}{2}$ in the series.

Note also that $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ .This gives:

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos\frac{\pi}{2} + \frac{1}{3^2}\cos 3\frac{\pi}{2} + \frac{1}{5^2}\cos 5\frac{\pi}{2} + \cdots\right]$$

$$-\left[\sin\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{4}\sin 4\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} \ldots\right]$$

and

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}[0 + 0 + 0 + \cdots]$$

$$-\left[(1) + \frac{1}{2}(0) + \frac{1}{3}.(-1) + \frac{1}{4}.(0) + \frac{1}{5}.(1) + \cdots\right]$$

Then $\frac{\pi}{2} = \frac{3\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

Or, $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{3\pi}{4}-\frac{\pi}{2}$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4},$ as required.

To show that

$$(ii)\,\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

We want zero $\sin nx$ terms and to use the coefficients of $\cos nx$.

Setting $x=0$ eliminates the $\sin nx$ terms from the series,and also gives

$$\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\frac{1}{7^2}\cos 7x+\cdots$$

$=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}$ (i.e. the desires series)

The graph of $f(x)$ shows a discontinuity (a "vertical jump") at $x=0$.The Fourier series converges to a value that is half-way between the two values of $f(x)$ around this discontinuity.

That is the series will converge to $\frac{\pi}{2}$ at $x = 0$.i.e.,

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \frac{1}{7^2}\cos 0 \ldots\right]$$

$$- \left[\sin 0 + \frac{1}{2}\sin 0 + \frac{1}{3}\sin 0 + \cdots\right]$$

and

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$- [0 + 0 + 0 + \cdots]$$

Finally, this gives

$$-\frac{\pi}{4} = -\frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

And

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots ◄$$

Example 4.

Let $f(x)$ be a function of of period 2π such that

$f(x) = \frac{x}{2}$ over the interval $0 < x < 2\pi$.

a) Sketch a graph of $f(x)$ in the interval $0 < x < 4\pi$.

b) Show that the Fourier series for $f(x)$ in the interval$0 < x < 2\pi$ is

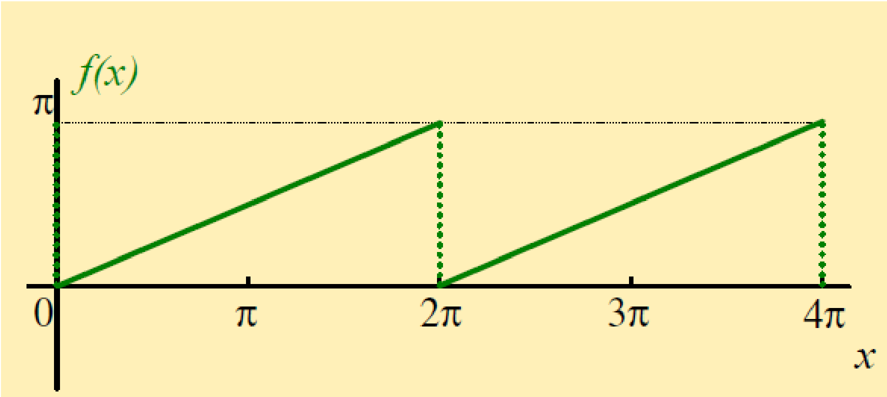
$$\frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right\}$$

c) By giving an appropriate value to x , show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Solution.

a) <u>The graph of $f(x)$ in the interval $0 < x < 4\pi$</u>



b) <u>Fourier series representation of $f(x)$</u>

STEP ONE

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \quad = \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\, dx \quad = \frac{1}{\pi}\left[\frac{x^2}{4}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^2}{4} - 0\right] \qquad i.e.\ a_0 = \pi.$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\cos nx\, dx$$

$$= \frac{1}{2\pi}\underbrace{\left\{\left[x\ \frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} \sin nx\, dx\right\}}_{\text{using integration by parts}}$$

$$= \frac{1}{2\pi}\left\{\left(2\pi\frac{\sin n2\pi}{n} - 0.\frac{\sin n.0}{n}\right) - \frac{1}{n}.0\right\}$$

$$= \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}.0\right\},\ i.e.\ \ a_n = 0\,.$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\underbrace{\left[\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi}\left(\frac{-\cos nx}{n}\right)\,dx\right]}_{\text{using integration by parts}}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n}(-2\pi\cos 2n\pi + 0) + \frac{1}{2}.0\right\}\quad ,$$

$$= \frac{-2\pi}{2\pi n}\cos(2n\pi) \ = -\frac{1}{n}\cos(2n\pi)$$

$i.e.\, b_n = -\frac{1}{n}$, since $2n$ is even. We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

where $a_0 = \pi,\ a_n = 0,\ b_n = -\frac{1}{n}$

These Fourier coefficients give

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(0 - \frac{1}{n} \sin nx\right)$$

$$i.e.\ \ f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right\}$$

c) <u>Pick an appropriate value of x, to show that</u>

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots}$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{4}$ and

$$\frac{\pi}{4} = \frac{\pi}{2} - \left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \cdots\right]$$

$$\frac{\pi}{4} = \frac{\pi}{2} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]$$

$$\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right] = \frac{\pi}{4}$$

$$i.e.\ \ 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots = \tfrac{\pi}{4}. \blacktriangleleft$$

Example 5.

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

a) Sketch a graph of $f(x)$ in the interval

$$-2\pi < x < 2\pi.$$

b) Show that the Fourier series for $f(x)$ in the

interval$0 < x < 2\pi$ is

$$\frac{\pi}{4} + \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$+ \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots$$

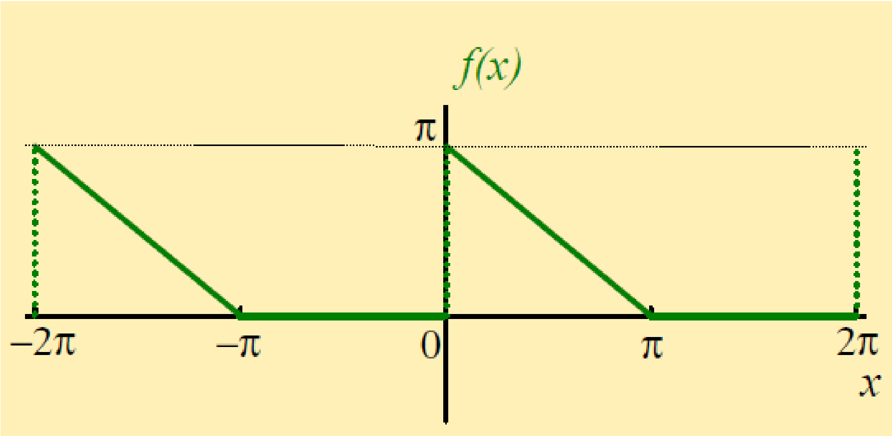
c) By giving an appropriate value to x , show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Solution.

$$\boxed{f(x) = \begin{cases} \pi - x & , 0 < x < \pi \\ 0 & , \pi < x < 2\pi, \text{ and has period } 2\pi \end{cases}}$$

a) The graph of $f(x)$ in the interval $-2\pi < x < 2\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0.\ dx$$

$$= \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} + 0 = \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0\right]$$

$$i.e.\ a_0 = \frac{\pi}{2}.$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0.\, dx$$

$$= \frac{1}{\pi}\underbrace{\left[\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} (-1).\frac{\sin nx}{n}\, dx\right]}_{\text{using integration by parts}} + 0$$

$$= \frac{1}{\pi}\left\{(0-0) + \int_0^{\pi} \frac{\sin nx}{n}\, dx\right\}$$

$$= \frac{1}{\pi n}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = -\frac{1}{\pi n^2}(\cos n\pi - \cos 0)$$

$$i.e.\, a_n = -\frac{1}{\pi n^2}((-1)^n - 1)$$

$$i.e. \quad a_n = \begin{cases} 0 & , n\ even \\ \dfrac{1}{\pi n^2} & , n\ odd \end{cases}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx + \int_{\pi}^{2\pi} 0.\,dx$$

$$= \frac{1}{\pi}\left\{\left[\left[(\pi - x)\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}\right.\right.$$

$$\left.\left. - \int_0^{\pi}(-1).\left(-\frac{\cos nx}{n}\right)\,dx\right]\right\} + 0$$

$$= \frac{1}{\pi}\left\{\left(0 - \left(-\frac{\pi}{n}\right)\right) - \frac{1}{n}\,.0\right\}$$

$\text{i.}\,e.\,b_n = \frac{1}{n}$

In summary, $a_0 = \frac{\pi}{2}$ and a table of another Fourier coefficients are

n	1	2	3	4	5
$a_n = \frac{2}{\pi n^2}$ (when n is odd)	$\frac{2}{\pi}$	0	$\frac{2}{\pi}\frac{1}{3^2}$	0	$\frac{2}{\pi}\frac{1}{5^2}$
$b_n = \frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

$$\therefore\ \ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\cos x + \frac{2}{\pi}\frac{1}{3^2}\cos 3x + \frac{2}{\pi}\frac{1}{5^2}\cos 5x + \cdots$$

$$+ \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots$$

$$i.e.\quad f(x) = \frac{\pi}{4}$$

$$+ \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

$$+ \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots$$

d) <u>To show that</u>

$$\boxed{\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots},$$

note that, as $x \to 0$, the series converges to the

half-way of $\frac{\pi}{2}$,and then

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left(\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \cdots\right)$$

$$+ \sin 0 + \tfrac{1}{2}\sin 0 + \tfrac{1}{3}\sin 0 + \cdots$$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + 0$$

$$\frac{\pi}{2} = \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

giving $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ ◀

Example 6.

Let $f(x)$ be a function of period 2π such

that$f(x) = x$ in the range $-\pi < x < \pi$.

a) Sketch a graph of $f(x)$ in the interval

$-3\pi < x < 3\pi$.

b) Show that the Fourier series for $f(x)$ in the

interval$-\pi < x < \pi$is

$$2\left[\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - \cdots\right]$$

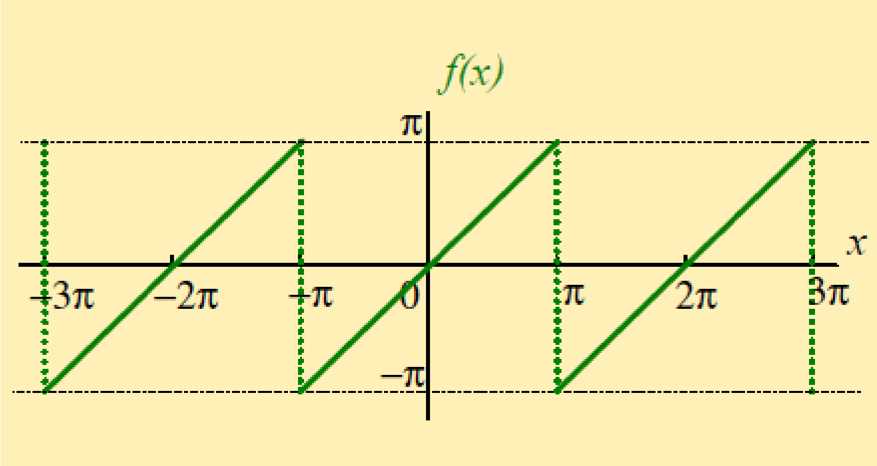
c) By giving an appropriate value to x , show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Solution.

$f(x) = x$, over the interval $-\pi < x < \pi$ and has period 2π

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



b) Fourier series representation of $f(x)$

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} \quad = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0.$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\underbrace{\left[\left[x\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{\sin nx}{n}\right)\,dx\right]}_{\text{using integration by parts}}$$

$$i.e.\ a_n \ = \frac{1}{\pi}\left\{\frac{1}{n}(\pi \sin n\pi - (-\pi)\sin(-n\pi)) -\right.$$

$$\left.\frac{1}{n}\int_{-\pi}^{\pi} \sin nx\, dx\right\} \ = \frac{1}{\pi}\left\{\frac{1}{n}(0-0) - \frac{1}{n}\ .0\right\} = 0.$$

Since $\sin n\pi = 0$ and $\int_{2\pi} \sin nx\, dx = 0.$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}[x\cos nx]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(\pi\cos n\pi - (-\pi)\cos(-n\pi)) + \frac{1}{n}\ .0\right\}$$

$$= -\frac{\pi}{n\pi}\ .(\cos n\pi + \cos n\pi) = -\frac{1}{n}(2\ \cos n\pi)$$

$$i.e.\ \ b_n = \frac{-2}{n}(-1)^n.$$

We thus have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

with $a_0 = 0,\ a_n = 0,\ b_n = -\frac{2}{n}(-1)^n$ and

n	1	2	3
b_n	2	-1	$\frac{2}{3}$

Therefore

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots.$$

$$i.e.\ f(x) = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right].$$

c) <u>Pick an appropriate value of *x*, to show that</u>

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{2}$ and

$$\frac{\pi}{2} = 2\left[\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} - \cdots\right]$$

$$\frac{\pi}{2} = 2\left[1 + 0 - \frac{1}{3}.(-1) - 0 + \frac{1}{5}.(1) - 0 + \frac{1}{7}.(-1) + \cdots\right]$$

$$\frac{\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$i.e.\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots ◀$$

Example 7.

Let $f(x)$ be a function of period 2π such that

$f(x) = x^2$ over the interval $-\pi < x < \pi$.

a) Sketch a graph of $f(x)$ in the interval

$$-3\pi < x < 3\pi$$

b) Show that the Fourier series for $f(x)$ in the

interval $-\pi < x < \pi$ is

$$\frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right]$$

c) By giving an appropriate value to x , show that

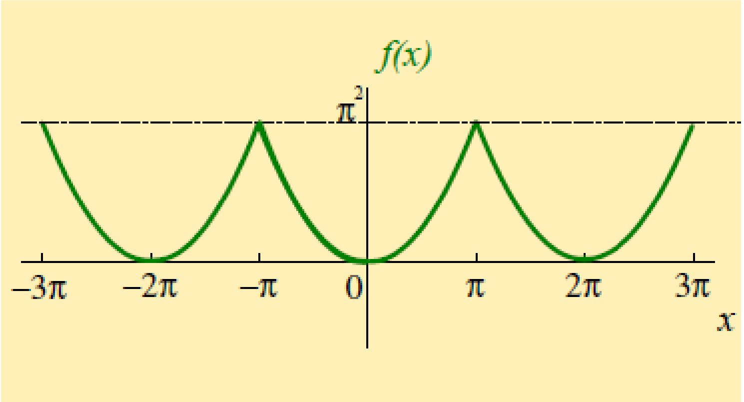
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \ldots$$

Solution.

$\boxed{f(x) = x^2\text{, over the interval } -\pi < x < \pi \text{ and has period } 2\pi}$

a) <u>Sketch a graph of $f(x)$ in the interval</u>

<u>$-3\pi < x < 3\pi$</u>



b) <u>Fourier series representation of $f(x)$</u>

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3} - \left(-\frac{\pi^3}{3}\right)\right) = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) \ i.e.\ a_0 = \frac{2\pi^2}{3}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\underbrace{\left[\left[x^2\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\left(\frac{\sin nx}{n}\right) dx\right]}_{\text{using integration by parts}}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}\left(\pi^2 \sin n\pi - \pi^2 \sin(-n\pi)\right) - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx\, dx\right\}$$

$$= \frac{-2}{n\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{-2}{n\pi}\underbrace{\left[\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{-\cos nx}{n}\right)\, dx\right]}_{\text{using integration by parts}}$$

$$= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx \, dx\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left(\pi\cos n\pi - (-\pi)\cos(-n\pi)\right) + \frac{1}{n}.0\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left(\pi(-1)^n + \pi(-1)^n\right)\right\}$$

$$= \frac{-2}{n\pi}\left\{\frac{-2\pi}{n}(-1)^n\right\}$$

$$= \frac{-2}{n\pi}\left\{\frac{-2\pi}{n}(-1)^n\right\} = \frac{4\pi}{\pi n^2}(-1)^n = \frac{4}{n^2}(-1)^n$$

$$i.e. \quad a_n = \begin{cases} \dfrac{4}{n^2} & , n \ even \\ \dfrac{-4}{n^2} & , n \ odd \end{cases}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi} \underbrace{\left[\left[x^2 \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \left(\frac{-\cos nx}{n} \right) dx \right]}_{\text{using integration by parts}}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} [x^2 \cos nx]_{-\pi}^{\pi} + \frac{2}{n} \int_{-\pi}^{\pi} x \cos nx \, dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi^2 \cos n\pi - \pi^2 \cos(-n\pi)) + \frac{2}{n} \int_{-\pi}^{\pi} x \cos nx \, dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} \left(\underbrace{(\pi^2 \cos n\pi - \pi^2 \cos(n\pi))}_{=0} \right) + \right.$$

$$\left. \frac{2}{n} \int_{-\pi}^{\pi} x \cos nx \, dx \right\} = \frac{2}{n\pi} \int_{-\pi}^{\pi} x \cos nx \, dx$$

$$i.e. \quad b_n = \frac{2}{n\pi} \underbrace{\left[\left[x \left(\frac{\sin nx}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\frac{\sin nx}{n} \right) dx \right]}_{\text{using integration by parts}}$$

$$= \frac{2}{n\pi} \left\{ \frac{1}{n} (\pi \sin n\pi - (-\pi) \sin(-n\pi)) - \right.$$

$$\left. \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, dx \right\}$$

$$= \frac{2}{n\pi} \left\{ \frac{1}{n} (0 + 0) - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, dx \right\}$$

$$= \frac{2}{\pi n^2} \int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $\quad a_0 = \frac{2\pi^2}{3}, a_n = \begin{cases} \frac{4}{n^2} & , n\ even \\ \frac{-4}{n^2} & , n\ odd \end{cases}$

n	1	2	3	4
a_n	$-4(1)$	$4\left(\frac{1}{2^2}\right)$	$-4\left(\frac{1}{3^2}\right)$	$4\left(\frac{1}{4^2}\right)$

$i.e.\ f(x) = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x \ldots\right] + [0 + 0 + \cdots]$

$i.e.\ f(x) = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x \ldots\right]$

c) <u>To show that</u>

$$\boxed{\ldots\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}}$$

use the fact that $\quad \cos n\pi = \begin{cases} 1 & , n\ even \\ -1 & , n\ odd \end{cases}$

i.e. $\quad \cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \cdots$

with $x = \pi$ gives

$$\cos\pi - \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi - \frac{1}{4^2}\cos 4\pi + \cdots$$

$$i.e. \quad (-1) - \frac{1}{2^2}\,.(1) + \frac{1}{3^2}\,.(-1) - \frac{1}{4^2}\,.(1) + \cdots$$

$$= -1.\underbrace{\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)}_{\text{(the desired series)}}$$

The graph of $f(x)$ gives that $f(\pi) = \pi^2$ and the series converges to this value.

Setting $x = \pi$ in the Fourier series thus gives.

$$\pi^2 = \frac{\pi^2}{3} -$$

$$4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \cdots\right)$$

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$\frac{2\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$i.e. \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots. \blacktriangleleft$$

Exercise Set (1)

1-Let the function $f(x)$ be 2π-periodic and suppose that it is presented by the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Calculate the coefficients a_0, a_n and b_n.

2-Find the Fourier series for the triangle wave

$$f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi < x < 0 \\ \frac{\pi}{2} + x, & 0 < x < \pi \end{cases}$$

defined on the interval $[-\pi, \pi]$.

3-Find the Fourier series for the function

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi \leq x < \frac{-\pi}{2} \\ 0 & \text{if} \quad \frac{-\pi}{2} \leq x < \frac{\pi}{2} \\ -1 & \text{if} \quad \frac{\pi}{2} \leq x < \pi \end{cases}$$

defined on the interval $[-\pi, \pi]$.

4- Find Fourier coefficient $a_0$in the Fourier series of $f(x) = e^{-x}, 0 \leq x \leq 2\pi$ and $f(x) = f(x + 2\pi)$.

5- Compute the Fourier Series of

$$f(x) = x \sin x,$$

$0 \le x \le 2\pi$ and $f(x) = f(x + 2\pi)$.

6-Fourier series representation of periodic function

$f(x) = \pi^2 - x^2, -\pi \le x \le \pi$ is

$\pi^2 - x^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos n\, x,$

then the value of

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots =$$

4- Find The value of a_n in the Fourier series

$$\text{of} f(x) = \begin{cases} \cos x : -\pi < x < 0 \\ -\cos x : \quad 0 < x < \pi \end{cases}$$

6- Find Fourier coefficient a_0 in the Fourier series

of$f(x) = \sqrt{2}\, sin \frac{x}{2}$ and $f(x + 2\pi) = f(x)$.

7- Find Fourier coefficient a_0 in the Fourier series

of $f(x) = \begin{cases} -\pi & 0 < x < \pi \\ (x - \pi) & \pi < x < 2\pi \end{cases}$

and $f(x + 2\pi) = f(x)$.

CHAPTER (II)

FOURIER TRANSFORMS

CHAPTER (II)

FOURIER TRANSFORMS

1. Fourier Transforms

We have seen that Fourier series are powerful tools in treating various problems involving periodic functions. Since many practical problems do not involve periodic functions, we need to develop a method of Fourier analysis that includes no periodic functions. In this chapter, we shall discuss a frequency representation of no periodic functions by means of Fourier transforms.

Definition 1.

The Fourier transform of $f(t)$(symbolized by$\mathfrak{I}$) is defined by

$$F(\omega) = \mathfrak{I}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad (2.1)$$

The inverse Fourier transform of $F(\omega)$ (symbolized by $\mathfrak{I}^{-1}$) is defined by

$f(t) = \mathfrak{I}^{-1}[F(\omega)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad (2.2)$$

Equations (2.1) and (2.2) are often called the Fourier transform pair. The condition for the existence of $F(\omega)$ is usually given by

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty \qquad (2.3)$$

Note that condition (2.3) is a sufficient but not a necessary for the existence of $\Im[f(t)]$.Functions that do not satisfy (2.3) may have Fourier transforms. The function$F(\omega) = \Im[f(t)]$is, in general, complex, and

$$F(\omega) = R(\omega) + jX(\omega) = |F(\omega)|e^{j\phi(\omega)} \qquad (2.4)$$

where $|F(\omega)| = \sqrt{[R(\omega)]^2 + [X(\omega)]^2}$ is called the **magnitude spectrum** of $f(t)$ and$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)}$ is the **phase spectrum** of $f(t)$.

Example 1.

Find the Fourier transform of $f(t)$ defined by

$$f(t) = \begin{cases} e^{-\alpha t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

where $\alpha > 0$ (Fig2-1 a). Also plot the magnitude spectrum and phase spectrum of $f(t)$.

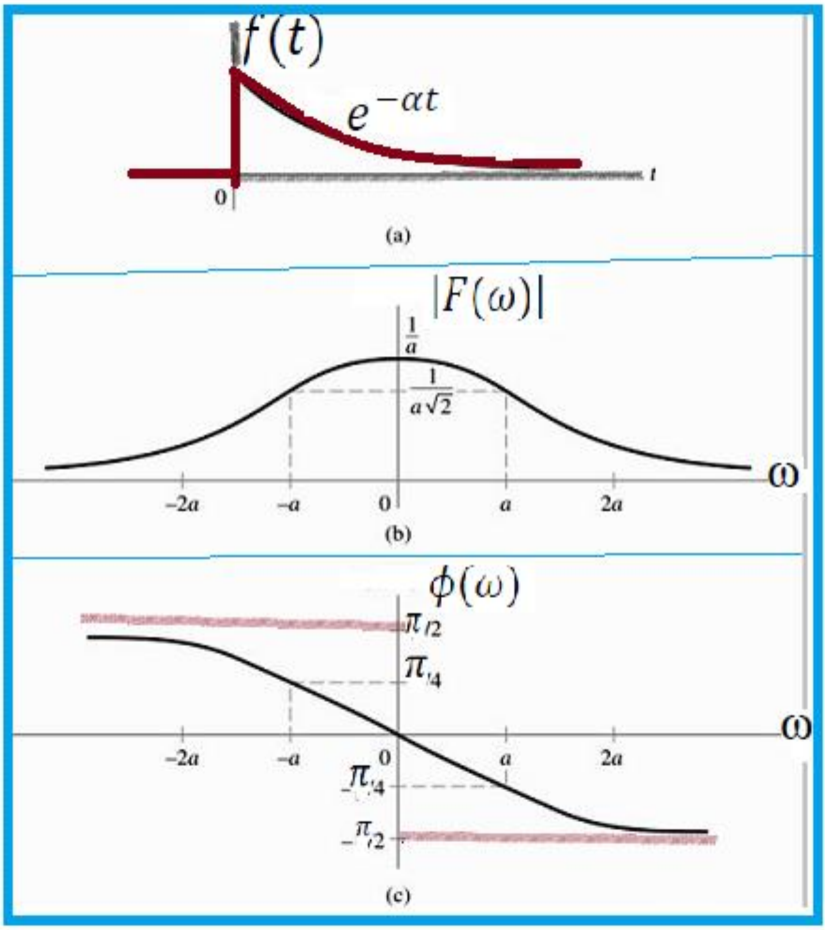


Fig 2-1 (a) Exponential function; (b) Magnitude spectrum; (c) Phase spectrum.

Solution. By the definition of Fourier transform of $f(t)$, we have

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-\alpha t}e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(\alpha+j\omega)t}dt = \frac{1}{-(\alpha+j\omega)}e^{-(\alpha+j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha+j\omega} = \frac{1}{\sqrt{\alpha^2+\omega^2}}e^{-j\tan^{-1}\left(\frac{\omega}{\alpha}\right)}$$

$$= |F(\omega)|e^{j\phi(\omega)},$$

where $|F(\omega)| = \frac{1}{\sqrt{\alpha^2+\omega^2}}$,$\phi(\omega) = -\tan^{-1}(\omega/\alpha)$.The magnitude spectrum $|F(\omega)|$ and the phase spectrum $\phi(\omega)$of $f(t)$ are plotted in Fig 2-1(b) and 2-1(c), respectively. ◄

Example 2.

Show that $\int_{-\infty}^{\infty}|f(t)|dt < \infty$ (2.3) is a sufficient condition for the existence of the Fourier transform of $f(t)$.

Solution.

Since$e^{-j\omega t} = \cos\omega t - j\sin\omega t$, then we have

$\left|e^{-j\omega t}\right| = \sqrt{\cos^2\omega t + \sin^2\omega t} = 1$. Therefore

$$\left|f(t)e^{-j\omega t}\right| = |f(t)|$$

It follows that if$\int_{-\infty}^{\infty}|f(t)|dt = \int_{-\infty}^{\infty}\left|f(t)e^{-j\omega t}\right|dt$is finite, then $\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$is finite, that is,

$\Im[f(t)]$exists. ◄

Example 3.

Find the Fourier transform of the rectangular pulse (Box function)$p_d(t)$(shown in Fig.2-2) defined by:

$$p_d(t) = \begin{cases} 1, & |t| < \dfrac{1}{2}d \\ 0, & |t| > \dfrac{1}{2}d \end{cases}$$

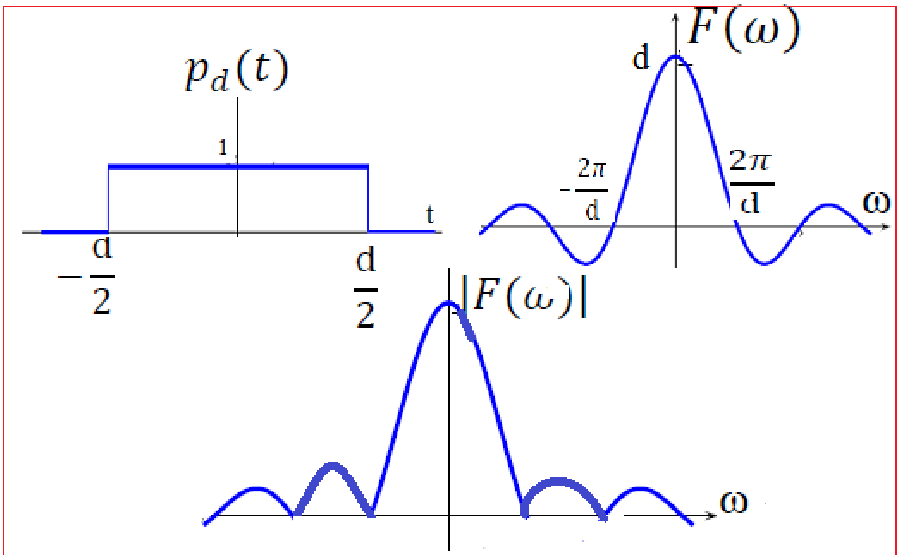


Fig. 2-2

Solution.

From the definition of the Fourier transform,

$$F(\omega) = \Im[p_d(t)] = \int_{-\infty}^{\infty} p_d(t) e^{-j\omega t} dt$$

$$= \int_{-d/2}^{d/2} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-d/2}^{d/2}$$

$$= \frac{1}{j\omega}\left[e^{j\omega d/2} - e^{-j\omega d/2}\right] = \frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$= d\,\frac{\sin(\omega d/2)}{(\omega d/2)}. \blacktriangleleft$$

Example 4. Find the Fourier transform of $f(t) = e^{-a|t|}$, where $a > 0$.

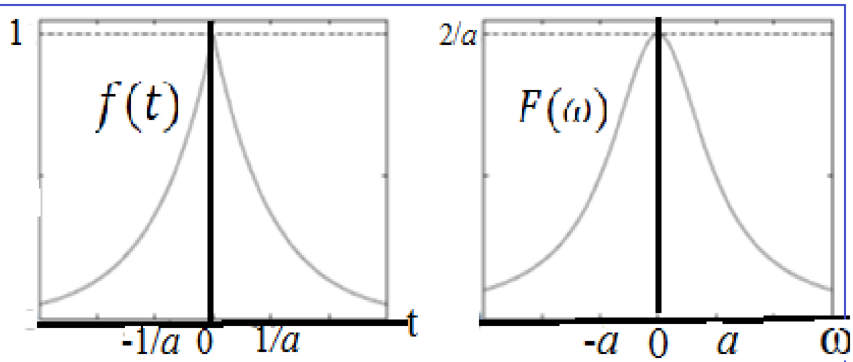


Fig. 2-3

Solution. $f(t) = e^{-a|t|} = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2} . \blacktriangleleft$$

2. Properties of Fourier Transforms

We use the notation $f(t) \leftrightarrow F(\omega)$ to denote a transform pair (repeated here):

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt,$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}dt.$$

Theorem 1.

(Elementary properties of Fourier transform)

If $f(t) \leftrightarrow F(\omega)$, $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$, then

a. Linearity:

$$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) \qquad (2.5)$$

where a_1 and a_2 are constants.

b. Time Shifting:

$$f(t - t_0) \leftrightarrow F(\omega)e^{-j\omega t_0} \qquad (2.6)$$

c. Frequency Shifting:

$$f(t)e^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0) \qquad (2.7)$$

d. Scaling: for a real constant *a*

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \qquad (2.8)$$

e. Time-reversal:

$$f(-t) \leftrightarrow F(-\omega) \qquad (2.9)$$

f. Symmetry:

$$F(t) \leftrightarrow 2\pi f(-\omega) \qquad (2.10)$$

Proof.

a. Linearity:

$$\Im[a_1 f_1(t) + a_2 f_2(t)]$$

$$= \int_{-\infty}^{\infty} [a_1 f_1(t) + a_2 f_2(t)] e^{-j\omega t} dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1(t)\, e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2(t)\, e^{-j\omega t} dt$$

$$= a_1 F_1(\omega) + a_2 F_2(\omega).$$

b. Time Shifting:

$$\Im[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-j\omega t} dt$$

Letting $t - t_0 = x$, then $dt = dx$; hence,

$$\Im[f(t-t_0)] = \int_{-\infty}^{\infty} f(x) e^{-j\omega(t_0+x)} dx$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

$$= e^{-j\omega t_0} F(\omega).$$

c. Frequency Shifting:

$$\Im\left[f(t) e^{j\omega_0 t}\right] = \int_{-\infty}^{\infty} \left[f(t) e^{j\omega_0 t}\right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t) e^{-j(\omega-\omega_0)t} dt = F(\omega - \omega_0).$$

d. Scaling:

For $a > 0$, $\Im[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t} dt$

Let $at = x$; then

$$\Im[f(at)] = \frac{1}{a} \int_{-\infty}^{\infty} f(x)\, e^{-j(\omega/a)x} dx$$

The dummy variable can be represented by any symbol,

$$\Im[f(at)] = \frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega/a)t} dt = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

For $\quad a < 0, \quad \Im[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t} dt$

Again, let $at = x$; then

$$\Im[f(at)] = \frac{1}{a}\int_{\infty}^{-\infty} f(x)\, e^{-j(\omega/a)x} dx$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega/a)t} dt = \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

Consequently, combining these two results, we obtain$\Im[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$.

e. Time-reversal:

Letting $a = -1$ in the scaling property (2.8), we have$\Im[f(-t)] = F(-\omega)$.

f. Symmetry:

From the Definition 1, we have

$$2\pi f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

Changing *t* to-*t*, we obtain

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t} d\omega$$

Now interchanging *t* and ω, we obtain

$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t} dt = \Im[F(t)]$.◄

Example 5.

Give an interpretation of the scaling property of the

Fourier transform$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

Solution.

The function $f(at)$ represents the function $f(t)$ compressed in the time scale by a factor of a. Similarly, the function $F(\omega/a)$represents the function $F(\omega)$ expanded in the frequency scaleby the same factor a. The scaling property therefore states that compression in the time domain is equivalent to expansion in the frequency domain, and vice versa.◄

Example 6.

Find the Fourier transform of the function

$$f(t) = \frac{\sin at}{\pi t}$$

Solution.

From the result of Example3, we have

$$\Im[p_d(t)] = \frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right) \qquad (a)$$

Now from the symmetry property of the Fourier transforms (2.10), we have

$$\Im\left[\frac{2}{t}\sin\left(\frac{dt}{2}\right)\right] = 2\pi p_d(-\omega) \qquad (b)$$

Or $\quad \Im\left[\frac{\sin\left(\frac{1}{2}dt\right)}{\pi t}\right] = p_d(-\omega) \qquad (c)$

Since $p_d(\omega)$ is defined by (see Example3)

$$p_d(\omega) = \begin{cases} 1 \quad for \ |\omega| < \frac{1}{2}d \\ 0 \quad for \ |\omega| > \frac{1}{2}d \end{cases}$$

it is an even function of ω. Hence,

$$p_d(-\omega) = p_d(\omega)$$

Letting $\frac{1}{2}d = a$ in equation (c), we obtain

$$\Im\left(\frac{\sin at}{\pi t}\right) = p_{2a}(-\omega) \qquad (d)$$

where $\quad p_{2a}(-\omega) = \begin{cases} 1 & for\ |\omega| < a \\ 0 & for\ |\omega| < a \end{cases}$

Plots of $f(t) = \sin at/\pi t$ and its Fourier transform $F(\omega)$ areshown in Fig. 2-4a and 2-4b, respectively.

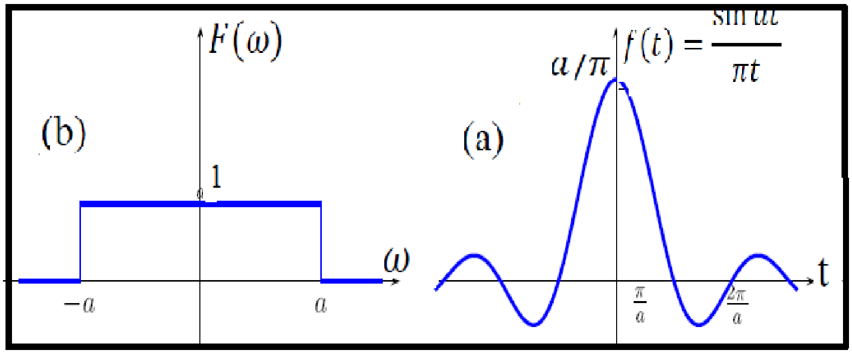


Figure 2-4 (a) Function $\sin at/\pi t$; (b) Fourier transform.◄

Example7.

Find the Fourier transform of $f(t) = 1/(a^2 + t^2)$.

Solution. From the result of Example 4, we have

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2 + \omega^2}$$

Thus, applying the symmetry property, we obtain

$$\frac{2a}{a^2+t^2} \leftrightarrow 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

Dividing both sides by $2a$ (linearity property),

$$\frac{1}{a^2 + t^2} \leftrightarrow \frac{\pi}{a} e^{-a|\omega|}$$

Plots of $f(t)$ and $F(\omega)$ are shown in Fig. 2-5a and 2-5b, respectively.

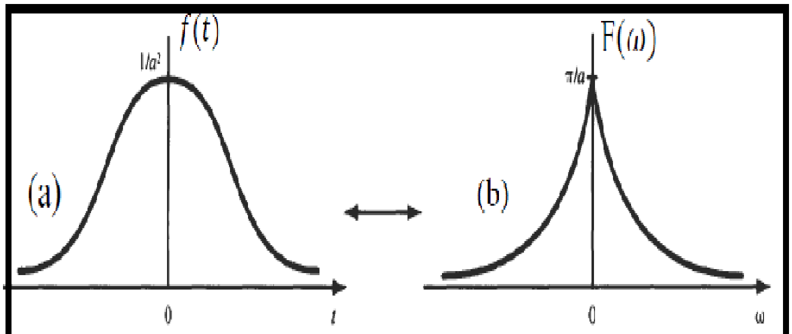


Fig.2-5 $1/(a^2+t^2)$ and its Fourier transform.

Theorem 2. (The modulation theorem)

If $f(t) \leftrightarrow F(\omega)$, then

$$f(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}F(\omega-\omega_0)+\frac{1}{2}F(\omega+\omega_0) \quad (2.11)$$

Proof. Since $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$ and

$e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$, we have

$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)$.

So, from frequency shifting property (2.7) and the linearity property (2.5), we obtain

$$\Im[f(t)\cos\omega_0 t] = \Im\left[\frac{1}{2}f(t)e^{j\omega_0 t}+\frac{1}{2}f(t)e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2}\Im\left[f(t)e^{j\omega_0 t}\right]+\frac{1}{2}\Im\left[f(t)e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2}F(\omega-\omega_0)+\frac{1}{2}F(\omega+\omega_0). \blacktriangleleft$$

Example 8.

If $F(\omega) = \Im[f(t)]$, find $\Im[f(t)\sin\omega_0 t]$.

Solution:-

With the identity $\sin\omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t}-e^{-j\omega_0 t}\right)$

and using the frequency-shifting property (2.7) and the linearity property (2.5), we obtain

$\Im[f(t)\sin\omega_0 t]$

$$= \Im\left[\frac{1}{2j}\, f(t)e^{j\omega_0 t} - \frac{1}{2j} f(t)e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2j} F(\omega - \omega_0) - \frac{1}{2j} F(\omega + \omega_0)$$

$$= \frac{1}{2j} [F(\omega - \omega_0) - F(\omega + \omega_0)]. \blacktriangleleft$$

Example 9.

Find the Fourier transform of the cosine function of finite duration *d*.

Solution.

The cosine function of finite duration *d* (see Figure 2-6a) can be expressed as a pulse-modulated function, i.e.,$f(t) = p_d(t) \cos \omega_0 t$, where

$$p_d(t) = \begin{cases} 1 \quad for \quad |t| < \dfrac{1}{2} d \\ \\ 0 \quad for \quad |t| > \dfrac{1}{2} d \end{cases}$$

Now from the result of Example3, we have

$$\Im[p_d(t)] = \frac{2}{\omega} \sin\left(\frac{\omega d}{2}\right)$$

Then applying the modulation theorem (2.11),

$$F(\omega) = \mathfrak{I}[p_d(t)\cos\omega_0 t]$$

$$= \frac{\sin\frac{1}{2}d(\omega-\omega_0)}{\omega-\omega_0} + \frac{\sin\frac{1}{2}d(\omega+\omega_0)}{\omega+\omega_0}$$

The Fourier transform $F(\omega)$ is plotted in Fig. 2-6b.

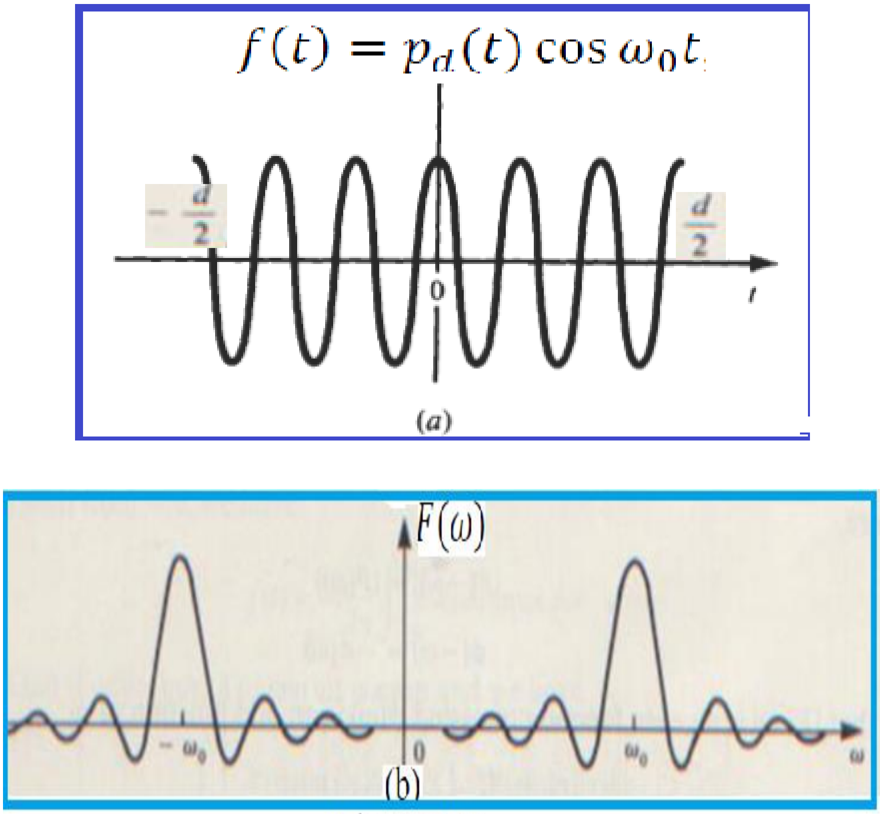


Figure 2-6

Theorem 3.

Let $f(t) \leftrightarrow F(\omega)$. If $f(t)$ is real, then let

$$f(t) = f_e(t) + f_0(t)$$

where $f_e(t)$ and $f_0(t)$ are the even and odd components of $f(t)$,respectively.

Let $F(\omega) = R(\omega) + jX(\omega)$, where $R(\omega)$ and $jX(\omega)$ are the real and imaginary parts of $F(\omega)$, respectively. Then

$$R(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt \qquad (2.12)$$

$$X(\omega) = -\int_{-\infty}^{\infty} f(t) \sin \omega t dt \qquad (2.13)$$

$$R(\omega) = R(-\omega) \qquad (2.14)$$

$$X(\omega) = -X(-\omega) \qquad (2.15)$$

The condition for a real function is

$$F(-\omega) = F^*(\omega) \qquad (2.16)$$

The even component is

$$f_e(t) \leftrightarrow R(\omega) \qquad (2.17)$$

The odd component is

$$f_0(t) \leftrightarrow jX(\omega) \qquad (2.18)$$

where $F^*(\omega)$ is the complex conjugate of $F(\omega)$.

Proof.

If $f(t)$ is real, then using the identity

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

we can rewrite the Fourier transform of $f(t)$as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

But $F(\omega) = R(\omega) + jX(\omega)$

Thus, equating the real and imaginary parts, we have

(2.12) and (2.13):

$R(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt$, and

$X(\omega) = -\int_{-\infty}^{\infty} f(t) \sin \omega t dt$

Next, since $f(t)$ is real,

$$R(-\omega) = \int_{-\infty}^{\infty} f(t) \cos(-\omega t) \, dt$$

$= \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt = R(\omega).$

$$X(-\omega) = -\int_{-\infty}^{\infty} f(t) \sin(-\omega t) \, dt$$

$= \int_{-\infty}^{\infty} f(t) \sin \omega t dt = -X(\omega).$

Hence, $R(\omega)$ is an even function of ω and $X(\omega)$ is an odd function of ω((2.14) and (2.15)). From (2.14) and (2.15), we see that (2.16) holds:

$F(-\omega) = R(-\omega) + jX(-\omega)$

$= R(\omega) - jX(\omega) = F^*(\omega).$

Thus, (2.16) is a necessary condition for $f(t)$ to be real.

To verify (2.17) and (2.18), let $f(t) = f_e(t) + f_0(t)$

Since any function can be expressed as the sum of an even and an odd component (exercise), we have

$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$,

$$f_0(t) = \frac{1}{2}[f(t) - f(-t)]$$

Now, if $f(t)$ is real, then from (2.9) (time-reversal property) and (2.16) we have

$\mathfrak{I}[f(t)] = F(\omega) = R(\omega) + jX(\omega)$

$\mathfrak{I}[f(-t)] = F(-\omega) = F^*(\omega) = R(\omega) - jX(\omega)$

Thus, by the linearity property we obtain (2.17):

$$\mathfrak{I}[f_e(t)] = \frac{1}{2}F(\omega) + \frac{1}{2}F^*(\omega)$$

$$= \frac{1}{2}[R(\omega) + jX(\omega)] + \frac{1}{2}[R(\omega) - jX(\omega)] = R(\omega)$$

and (2.18):

$$\Im[f_0(t)] = \frac{1}{2}F(\omega) - \frac{1}{2}F^*(\omega)$$

$$= \frac{1}{2}[R(\omega) + jX(\omega)] - \frac{1}{2}[R(\omega) - jX(\omega)]$$

$$= jX(\omega).◄$$

Theorem 4.

Equation (2.16) is a necessary and sufficient condition for $f(t)$ to be real.

Proof:

We have already shown that $F(-\omega) = F^*(\omega)$ (2.16) is a necessary condition for$f(t)$ to be real.

Now we can prove that (2.16) is also a sufficient condition for $f(t)$ to be real as follows.

Suppose that $F(-\omega) = F^*(\omega)$. We want to prove that $f(t)$ is real function. So, by contrary, let$f(t) =$

$f_1(t) + jf_2(t)$,where $f_1(t)$ and $f_2(t)$ are real functions. Then from the inverseFourier transform,

$$f(t) = f_1(t) + jf_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [R(\omega) + jX(\omega)](\cos \omega t + j \sin \omega t\ ) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [R(\omega) \cos \omega t - X(\omega) \sin \omega t\ ] d\omega$$

$$+ j\frac{1}{2\pi}\int_{-\infty}^{\infty} [R(\omega) \sin \omega t\ + X(\omega) \cos \omega t] d\omega$$

Hence,

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [R(\omega) \cos \omega t - X(\omega) \sin \omega t\ ] d\omega \quad (2.19)$$

$$f_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [R(\omega) \sin \omega t\ + X(\omega) \cos \omega t] d\omega \quad (2.20)$$

Now if $F(-\omega) = F^*(\omega)$, then

$$R(-\omega) = R(\omega) \ \text{ and } \ X(-\omega) = -X(\omega)$$

Consequently, since the product of an even and an odd function is an odd function, $R(\omega)\sin\omega t$ and $X(\omega)\cos\omega t$ are odd functions of ω, and the integrand in(2.20) is an odd function of ω. Hence, we conclude $f_2(t) = 0$.That is, $f(t)$ is real. ◄

Example 9.

Show that if the Fourier transform of a real function $f(t)$ is real,then $f(t)$ is an even function of t, and if the Fourier transform of a real function $f(t)$ is purely imaginary, then $f(t)$ is an odd function of t.

Solution.

If $f(t) \leftrightarrow F(\omega) = R(\omega) + jX(\omega)$ and $f(t) = f_e(t) + f_0(t)$, where $f_e(t)$ and $f_0(t)$ are the even and odd components of $f(t)$, respectively, then by (2.17) and (2.18), we have $f_e(t) \leftrightarrow R(\omega)$

and $f_0(t) \leftrightarrow jX(\omega)$. Thus, if $F(\omega) = R(\omega)$, then $X(\omega) = 0$, which implies that$f_0(t) = 0$; that is, $f(t) = f_e(t)$.

Next, if $F(\omega) = jX(\omega)$, then $R(\omega) = 0$, which implies that $f_e(t) = 0$; that is$f(t) = f_e(t)$.◄

Example 10.

If $f(t)$ is real, show that its magnitude spectrum $|F(\omega)|$ is aneven function of ω and its phase spectrum $\phi(\omega)$ is an odd function of ω.

Solution:-

If $f(t)$ is real, from the condition for a real function (2.16)$F(-\omega) = F^*(\omega)$. Now, From (2.4),$F^*(\omega) = |F(\omega)|e^{-j\phi(\omega)}$ and $F(-\omega) = |F(-\omega)|e^{j\phi(-\omega)}$.

Hence, $|F(-\omega)|e^{j\phi(-\omega)} = |F(\omega)|e^{-j\phi(\omega)}$.

Therefore,$|F(-\omega)| = |F(\omega)|$, $\phi(-\omega) = -\phi(\omega)$, indicating that $|F(\omega)|$ is an even function of ω and $\phi(\omega)$ is anodd function of ω.◄

Example 11.

If $f(t)$ is real and even, then show that

$F(\omega) = R(\omega) = 2\int_0^\infty f(t)\cos\omega t\,dt$ and $f(t) = \frac{1}{\pi}\int_0^\infty R(\omega)\cos\omega t\,d\omega$

Solution.

From the result of Example 9, if $f(t)$ is real and even, then we have $X(\omega) = 0$ and

$$F(\omega) = R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

Now if $f(t)$ is even, $\cos\omega t$ is even; thus, $f(t)\cos\omega t$ is alsoeven and we have

$$\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = 2\int_0^{\infty} f(t)\cos\omega t\,dt$$

Thus, $F(\omega) = R(\omega) = 2\int_0^{\infty} f(t)\cos\omega t\,dt$

Next, from (2.19) with $X(\omega) = 0$, we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)\cos\omega t\,d\omega$$

Now, by (2.14), $R(\omega)$ is even; thus, $R(\omega)\cos\omega t$ is even andwe have

$\int_{-\infty}^{\infty} R(\omega)\cos\omega t\,d\omega = 2\int_0^{\infty} R(\omega)\cos\omega t\,d\omega$

Thus, we obtain $f(t) = \frac{1}{\pi}\int_0^{\infty} R(\omega)\cos\omega t\,d\omega$.◀

Example 12.

If $f(t)$ is real and odd, then show that

$F(\omega) = jX(\omega) = -2j\int_0^{\infty} f(t)\sin\omega t\,dt$ and

$f(t) = -\frac{1}{\pi}\int_0^{\infty} X(\omega)\sin\omega t\,d\omega$.

Solution.

From the result of Example 9, if $f(t)$ is real and odd, then wehave $R(\omega) = 0$ and

$F(\omega) = jX(\omega) = -j\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt.$

Now $f(t)$ is odd and $\sin\omega t$ is also odd; thus,

$f(t)\sin\omega t$ is even, and we

have $\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt = 2\int_{0}^{\infty} f(t)\sin\omega t\, dt.$

Thus, $F(\omega) = jX(\omega) = -2j\int_{0}^{\infty} f(t)\sin\omega t\, dt$

Next, from (2.19) with $R(\omega) = 0$, we have

$$f(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\sin\omega t\, d\omega$$

Now, by (2.15), $X(\omega)$ is odd; thus, $X(\omega)\sin\omega t$ is

even and we have

$$\int_{-\infty}^{\infty} X(\omega)\sin\omega t\, d\omega = 2\int_{0}^{\infty} X(\omega)\sin\omega t\, d\omega$$

Thus, we obtain $f(t) = -\frac{1}{\pi}\int_{0}^{\infty} X(\omega)\sin\omega t\, d\omega.$◄

Theorems 5. (Differentiation theorems)

If $f(t) \leftrightarrow F(\omega)$, then

$$f'(t) \leftrightarrow j\omega F(\omega) \qquad (2.21)$$

provided that $f(t) \to 0$ as $t \to \pm\infty$, where the equation (2. 21) is called the time domain of the differentiation theorem. Also,

$$(-jt)f(t) \leftrightarrow F'(\omega) \qquad (2.22)$$

where the equation (2. 22) is called the frequency domain of the differentiation theorem.

Proof.

On integration by parts, we have

$$\Im[f'(t)] = \int_{-\infty}^{\infty} f'(t)\, e^{-j\omega t} dt$$

$$= f(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt \quad (2.23)$$

Since $f(t) \to 0$ as $t \to \pm\infty$, we obtain

$$\Im[f'(t)] = j\omega \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt = j\omega\, F(\omega)$$

Equation (2.21) shows that differentiation in the time domain corresponds to multiplication of the Fourier transform by $j\omega$, provided that $f(t) \to 0$ as $t \to \pm\infty$. Note that if $f(t)$ has a finite number of jump discontinuities, $f'(t)$ contains impulses. Then the Fourier transform of $f'(t)$ for this case must contain the Fourier transform of the impulses in $f'(t)$, which we'll discuss in later. By repeated application of (2.21), we have for $n = 1,2,$ ...

$$\Im\left[f^{(n)}(t)\right] = (j\omega)^n F(\omega)$$

$$= (j\omega)^n \Im[f(t)] \qquad (2.24)$$

Note that (2.24) does not guarantee the existence of the Fourier transform of $f^{(n)}(t)$- it only indicates that if the transform exists, then it is given by $(j\omega)^n F(\omega)$.We now prove the frequency domain differentiation theorem (2.22).

Since$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$, we have

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

Changing the order of differentiation and integration, we obtain

$\frac{dF(\omega)}{d\omega} = \int_{-\infty}^{\infty} f(t) \frac{\partial}{\partial\omega}\left(e^{-j\omega t}\right) dt$

$= \int_{-\infty}^{\infty} [-jt\, f(t)]\, e^{-j\omega t} dt = \Im[-jt\, f(t)]$.◄

Example 13.

Show that if $\Im[f(t)] = F(\omega)$, then

$|F(\omega)| \leq \int_{-\infty}^{\infty} |f(t)|\, dt,$ $|F(\omega)| \leq \frac{1}{|\omega|}\int_{-\infty}^{\infty} \left|\frac{df(t)}{dt}\right| dt,$ and

$$|F(\omega)| \leq \frac{1}{\omega^2}\int_{-\infty}^{\infty} \left|\frac{d^2 f(t)}{dt^2}\right| dt.$$

These inequalities determine the upper bounds of

$|F(\omega)|$.

Solution:-

By definition (2.1),$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$

Hence, by elementary calculus, we have

$|F(\omega)| = \left|\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt\right| \leq$

$\int_{-\infty}^{\infty} \left|f(t) e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |f(t)| dt.$

Since

$\left|e^{-j\omega t}\right| = |\cos \omega t - j \sin \omega t| =$

$\sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1,\ \left|f(t) e^{-j\omega t}\right| = |f(t)|.$

Next, by the differentiation theorem (2.21),

$$j\omega F(\omega) = \Im[f'(t)] = \int_{-\infty}^{\infty} f'(t) e^{-j\omega t} dt$$

Hence, in a similar fashion, we obtain.

$$|j\omega F(\omega)| = \left|\int_{-\infty}^{\infty} f'(t) e^{-j\omega t} dt\right| \leq \int_{-\infty}^{\infty} |f'(t)| dt$$

Since

$$|j| = 1, \quad |j\omega F(\omega)| = |\omega||F(\omega)|$$

Thus,

$$|F(\omega)| \leq \frac{1}{|\omega|}\int_{-\infty}^{\infty}|f'(t)|dt$$

Next, by repeated application of (2.21), or by

(2.24), we have

$$(j\omega)^2 F(\omega) = \Im\left[f^{(2)}(t)\right] = \int_{-\infty}^{\infty}\frac{d^2 f(t)}{dt^2}e^{-j\omega t}dt$$

Hence,$|(j\omega)^2 F(\omega)| = \left|\int_{-\infty}^{\infty}\frac{d^2 f(t)}{dt^2}e^{-j\omega t}dt\right| \leq$

$\int_{-\infty}^{\infty}\left|\frac{d^2 f(t)}{dt^2}\right|dt$or $|F(\omega)| \leq \frac{1}{|\omega^2|}\int_{-\infty}^{\infty}\left|\frac{d^2 f(t)}{dt^2}\right|dt$.◀

EXAMPLE 14.

If $f(t) \leftrightarrow F(\omega)$and $F(\omega)$ can be differentiated everywhere *n* times, show that

$t^p f(t) \leftrightarrow \frac{1}{(-j)^p}\frac{d^p F(\omega)}{d\omega^p}$ for every $p \leq n$.

Solution:-

From (2.22), $(-jt)f(t) \leftrightarrow \frac{dF(\omega)}{d\omega}$

By repeated application of (2.22), we obtain

$$(-jt)^2 f(t) \leftrightarrow \frac{d^2F(\omega)}{d\omega^2}$$

$$\vdots$$

$$(-jt)^p f(t) \leftrightarrow \frac{d^pF(\omega)}{d\omega^p}, \quad p \leq n$$

Dividing by $(-j)^p$, we obtain

$$t^p\ f(t) \leftrightarrow \frac{1}{(-j)^p}\frac{d^pF(\omega)}{d\omega^p}$$

for $p \leq n$.◄

Theorem 6. (Integration theorem)

If $f(t) \leftrightarrow F(\omega)$and

$\int_{-\infty}^{\infty} f(t)\, dt = F(0) = 0$(2.25)then

$$\left[\int_{-\infty}^{t} f(x)\, dx\right] \leftrightarrow \frac{1}{j\omega} F(\omega)(2.26)$$

Proof. Consider the function$\phi(t) = \int_{-\infty}^{t} f(x)\, dx$

Then $\phi'(t) = f(t)$. Hence, if $\Im[\phi(t)] = \boldsymbol{\Phi}(\omega)$,

then from (2.21) we have

$$\Im[\phi'(t)] = \Im[f(t)] = j\omega\boldsymbol{\Phi}(\omega)$$

provided that

$$\lim_{t\to\infty} \phi(t) = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} f(t)\, dt = F(0) = 0$$

Therefore$\boldsymbol{\Phi}(\omega) = \frac{1}{j\omega}\Im[f(t)] = \frac{1}{j\omega}F(\omega)$. That is,

$\Im\left[\int_{-\infty}^{t} f(x)\, dx\right] = \frac{1}{j\omega}F(\omega) = \frac{1}{j\omega}\Im[f(t)]$.◄

EXAMPLE 15.

(a) Find the Fourier transform of the pulse $f_1(t)$ shown in Figure 2-7a.

(b) The pulse $f_2(t)$ shown in Figure 2-7b is the integral of $f_1(t)$. Use the result of part (a) to obtain the Fourier transform of $f_2(t)$.

(c) Check the result by direct integration.

Solution:-

(a) $$f_1(t) = \begin{cases} A/T, & -T < t < 0 \\ -A/T, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

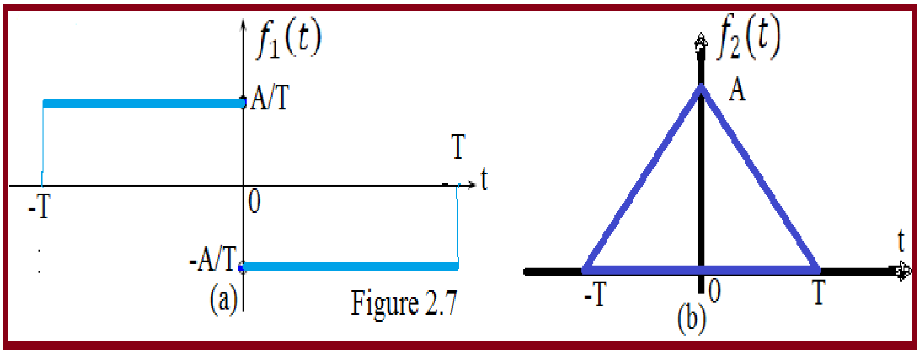


Figure 2.7

By Definition 1, we have

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)\, e^{-j\omega t} dt$$

$$= \int_{-T}^{0} (A/T)\, e^{-j\omega t} dt + \int_{0}^{T} (-A/T)\, e^{-j\omega t} dt$$

$$= \frac{A}{T}\left(\frac{1}{-j\omega}\right)\left(1 - e^{j\omega T}\right) - \frac{A}{T}\left(\frac{1}{-j\omega}\right)\left(e^{-j\omega T} - 1\right)$$

$$= \frac{A}{j\omega T}\left(e^{j\omega T} + e^{-j\omega T} - 2\right)$$

$$= \frac{2A}{j\omega T}\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) - 1\right]$$

$$= \frac{2A}{j\omega T}(\cos\omega T - 1) = -\frac{4A}{j\omega T}\sin^2(\omega T/2) \quad (a)$$

(b) Now

$$f_2(t) = \int_{-\infty}^{t} f_1(x) dx \quad \text{and} \quad \int_{-\infty}^{\infty} f_1(t) dt = 0$$

Thus, by the integration theorem (2.26), we have

$$F_2(\omega) = \frac{1}{j\omega} F_1(\omega) = \frac{1}{j\omega}\left(-\frac{4A}{j\omega T}\right)\sin^2(\omega T/2)$$

$$= \frac{4A}{\omega^2 T}\sin^2(\omega T/2) = AT\,\frac{\sin^2(\omega T/2)}{(\omega T/2)^2} \qquad (b)$$

(c) $f_2(t) = \begin{cases} (A/T)t + A, & -T < t < 0 \\ (-A/T)t + A, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$

Thus, by Definition (2.1), we have

$$F_2(\omega) = \int_{-\infty}^{\infty} f_2(t)\, e^{-j\omega t} dt$$

$$= \int_{-T}^{0} \left(\frac{A}{T}t + A\right) e^{-j\omega t} dt$$

$$+ \int_{0}^{T} \left(-\frac{A}{T}t + A\right) e^{-j\omega t} dt$$

$$= \frac{A}{T}\left[\int_{-T}^{0} te^{-j\omega t} dt - \int_{0}^{T} te^{-j\omega t} dt\right]$$

$$+ A\int_{-T}^{T} e^{-j\omega t} dt$$

$$= \frac{A}{T}\left[\frac{t}{-j\omega} e^{-j\omega t}\Big|_{-T}^{0} + \frac{1}{j\omega}\int_{-T}^{0} e^{-j\omega t} dt\right.$$

$$\left. - \frac{t}{-j\omega} e^{-j\omega t}\Big|_{0}^{T} - \frac{1}{j\omega}\int_{0}^{T} e^{-j\omega t} dt\right]$$

$$+ A\left(\frac{1}{-j\omega} e^{-j\omega t}\Big|_{-T}^{T}\right)$$

$$= \frac{A}{T}\left[\frac{-T}{j\omega} e^{j\omega T} - \frac{1}{(j\omega)^2} e^{-j\omega t}\Big|_{-T}^{0} + \frac{T}{j\omega} e^{-j\omega T}\right.$$

$$\left. + \frac{1}{(j\omega)^2} e^{-j\omega t}\Big|_{0}^{T}\right] + \frac{A}{j\omega}\left(e^{j\omega T} - e^{-j\omega T}\right)$$

$$= \frac{A}{T\omega^2}\left(2 - e^{j\omega T} - e^{-j\omega T}\right)$$

$$= \frac{2A}{T\omega^2}\left[1 - \frac{1}{2}\left(e^{j\omega T} + e^{-j\omega T}\right)\right]$$

$$= \frac{2A}{T\omega^2}(1 - \cos \omega T)$$

$$= \frac{4A}{T\omega^2}\sin^2(\omega T/2) = AT\frac{\sin^2(\omega T/2)}{(\omega T/2)^2},$$

which is the same as the result of eq.(b).◄

♣Parseval's Formula

Let $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$

Then Parseval's formula is given by:

$$\int_{-\infty}^{\infty} f(x)\, G(x) dx = \int_{-\infty}^{\infty} F(x)\, g(x) dx \quad (2.27)$$

Equation (2.27) can be written as

$$\int_{-\infty}^{\infty} f(\omega) \Im[g(t)] d\omega = \int_{-\infty}^{\infty} \Im[f(t)] g(\omega) d\omega \ (2.28)$$

Since $f(t) = \Im^{-1}[F(\omega)]$ and $g(t) = \Im^{-1}[G(\omega)]$,

eq. (2.27) can also be written as.

$$\int_{-\infty}^{\infty} \Im^{-1}[F(\omega)] G(t) dt = \int_{-\infty}^{\infty} F(t)\, \Im^{-1}[G(\omega)] dt$$

(2.29)

Theorem 7.

Derive Parseval's formula (2.27).

Solution.

From the definition of the Fourier transform,

$$F(y) = \int_{-\infty}^{\infty} f(x)\, e^{-jxy} dx$$

$$G(x) = \int_{-\infty}^{\infty} g(y)\, e^{-jxy} dy$$

Then

$$\int_{-\infty}^{\infty} f(x)\, G(x) dx = \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} g(y)\, e^{-jxy} dy \right] dx$$

Interchanging the order of integration,

$$\int_{-\infty}^{\infty} f(x)\, G(x) dx = \int_{-\infty}^{\infty} g(y) \left[\int_{-\infty}^{\infty} f(x)\, e^{-jxy} dx \right] dy$$

$$= \int_{-\infty}^{\infty} g(y)\, F(y) dy$$

and because we can change the dummy variable's symbol, we obtain (2.27):

$$\int_{-\infty}^{\infty} f(x)\, G(x) dx = \int_{-\infty}^{\infty} F(x)\, g(x) dx. \blacktriangleleft$$

EXAMPLE 16.

Using Parseval's formula (2.27) or (2.28) prove that

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)G(-x)dx$$

where $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$

Proof.

By (2.28),

$\int_{-\infty}^{\infty} f(\omega)\Im[g(t)]d\omega = \int_{-\infty}^{\infty} \Im[f(t)]g(\omega)d\omega$

Now, by the symmetry property (2.10), we have

$$G(t) \leftrightarrow 2\pi g(-\omega)$$

Using the time-reversal property (2.9), we have

$$G(-t) \leftrightarrow 2\pi g(\omega)$$

Thus, by (2.28), we obtain

$\int_{-\infty}^{\infty} f(\omega)\Im[G(-t)]d\omega = \int_{-\infty}^{\infty} \Im[f(t)]\, G(-\omega)d\omega$

Or$\int_{-\infty}^{\infty} f(\omega)2\pi g(\omega)d\omega = \int_{-\infty}^{\infty} F(\omega)G(-\omega)d\omega$

Dividing both sides by 2π and changing the

dummy variable, we obtain

$\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)G(-x)dx.$◄

Exercise Set (2)

1- If $f(t)$ is purely imaginary, that is, $f(t) = jg(t)$, where $g(t)$is real, show that the real and imaginary parts of $F(\omega)$ are

$$R(\omega) = \int_{-\infty}^{\infty} g(t) \sin \omega t \, dt,$$

$$X(\omega) = \int_{-\infty}^{\infty} g(t) \cos \omega t \, dt$$

Also show that $R(\omega)$ and $X(\omega)$ are odd and even functions ofω;that is,

$R(-\omega) = -R(\omega),\;\; X(-\omega) = X(\omega), F(-\omega) = -F^*(\omega)$.

2- If $\Im[f(t)] = F(\omega)$, show that $\Im[f^*(t)] = F^*(-\omega)$, where $f^*(t)$is the conjugate of $f(t)$ and $F^*(-\omega)$ is the conjugate of $F(-\omega)$.

3- If $F(\omega) = \Im[f(t)]$, show that

$$\Im\left[f(at)e^{j\omega_0 t}\right] = \frac{1}{|a|}F\left(\frac{\omega-\omega_0}{a}\right)$$

4-The *n*th moment m_n of a function $f(t)$ is defined by.

$$m_n = \int_{-\infty}^{\infty} t^n f(t)dt \quad \text{for } n = 0,1,2,\ldots$$

Show that

$$m_n = (j)^n \frac{d^n F(0)}{d\omega^n} \quad \text{for } n = 0,1,2,\ldots$$

where

$$\frac{d^n F(0)}{d\omega^n} = \left.\frac{d^n F(\omega)}{d\omega^n}\right|_{\omega=0} \quad \text{and } F(\omega) = \Im[f(t)]$$

[Hint: Use the differentiation theorem (2.22).]

5-Use the result of Problem 4 to show that $F(\omega) = \Im[f(t)]$can be expressed as

$$F(\omega) = \sum_{n=0}^{\infty}(-j)^n m_n \frac{\omega^n}{n!}$$

[Hint: Substitute $e^{-j\omega t} = \sum_{n=0}^{\infty}(-j\omega t)^n/n!$ in (2.1) and integrate term wise.]

6-Let $F(\omega)$ be the Fourier transform of $f(t)$and $f_k(t)$ be defined by

$$f_k(t) = \frac{1}{2\pi}\int_{-k}^{k} F(\omega)e^{-j\omega t}d\omega$$

Show that

$$f_k(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t-x)\frac{\sin kx}{x}dx$$

7- Let $F(\omega) = \Im[f(t)]$ and $G(\omega) = \mathcal{g}[g(t)]$.

Prove that

$$(a) \int_{-\infty}^{\infty} f(t)g(-t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)d\omega$$

$$(b) \int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$

where the asterisk denotes complex conjugation.

[Hint: Use Parseval's formula (2.27)].

Chapter III

Fourier Integrals

CHAPTER III

FOURIER INTEGRALS

1. The FOURIER INTEGRAL

Let us assume the following conditions on $f(x)$

1- $f(x)$ satisfies the Dirichlet conditions in every finite interval $(-L, L)$.

2- $\int_{-\infty}^{\infty}|f(x)|dx$converges, i.e. $f(x)$ is absolutely integrable in $(-\infty, \infty)$.

Then **Fourier's integral theorem** states that

$$f(x) = \int_0^{\infty}\{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}d\omega \quad (1)$$

where

$$\begin{cases} A(\omega) &= \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos\omega x dx \\ B(\omega) &= \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin\omega x dx \end{cases} \qquad (2)$$

The result (1) holds if x is a point of continuity of $f(x)$. If x is a point of discontinuity, we must replace $f(x)$ by $\frac{f(x+0)+f(x-0)}{2}$ as in the case of Fourier series.

Note that the above conditions are sufficient but not necessary.

The similarity of (1) and (2) with corresponding results for Fourier series is apparent.

The right-hand side of (1) is sometimes called a Fourier integral expansion of $f(x)$.

2. EQUIVALENT FORMS OF FOURIER'S INTEGRAL THEOREM

Fourier's integral theorem can also be written in the forms

$$f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\omega(x-u)\,du\,d\omega \qquad (3)$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega x}d\omega\int_{-\infty}^{\infty}f(u)e^{i\omega u}du \qquad (4)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)e^{i\omega(u-x)}dud\omega$$

where it is understood that if $f(x)$ is not cotinuous at x the left side must be replaced by $\frac{f(x+0)+f(x-0)}{2}$.

These results can be simplified somewhat if $f(x)$ is either an odd or an even function, and we have

$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos\omega x d\omega\int_0^{\infty}f(u)\cos\omega u du$,(5)

if $f(x)$ is even.

$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\omega x d\omega\int_0^{\infty}f(u)\sin\omega u du$, (6)

if $f(x)$ is odd.

3. FOURIER TRANSFORMS

From (4) it follows that if

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)e^{i\omega u}du \qquad (7)$$

Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega \qquad (8)$$

The function $F(\omega)$ is called the **Fourier transform** of $f(x)$ and is sometimes written $F(\omega) = \Im\{f(x)\}$.

The function $f(x)$ is the **inverse Fourier transform** of $F(\omega)$ and is written $f(x) = \Im^{-1}\{F(\omega)\}$.

Note: The constants preceding the integral signs in (7) and (8) were here taken as equal to $1/\sqrt{2\pi}$. However, they can be any constants different from zero so long as their product is $1/2\pi$. The above is called the symmetric form.

If $f(x)$ is an even function, equation (5) yields

$$\begin{cases} F_c(\omega) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\infty} f(u) \cos \omega u du \\ f(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\infty} F_c(\omega) \cos \omega x d\omega \end{cases} \qquad (9)$$

and we call $F_c(\omega)$ and $f(x)$ Fourier cosine transforms of each other.

If $f(x)$ is an odd function, equation (6) yields

$$\begin{cases} F_s(\omega) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\infty} f(u) \sin \omega u du \\ f(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\infty} F_s(\omega) \sin \omega x d\omega \end{cases} \qquad (10)$$

and we call $F_s(\omega)$ and $f(x)$ Fourier sine transforms of each other.

Example 1.

(a)Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

(b) Graph $f(x)$ its Fourier transform for $a = 3$.

Solution.

(a) The Fourier transform of $f(x)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{i\omega u} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1) e^{i\omega u} du$$

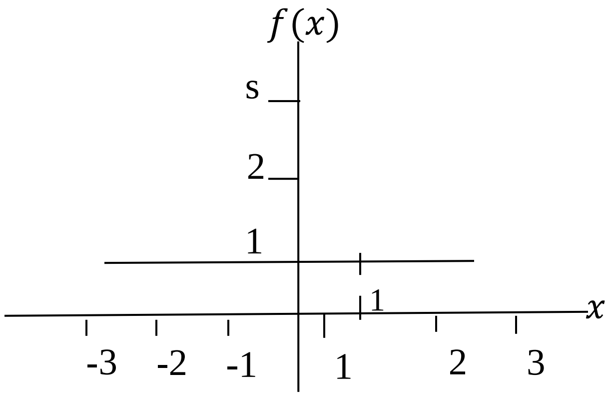
$$= \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega u}}{i\omega} \bigg|_{-a}^{a}$$

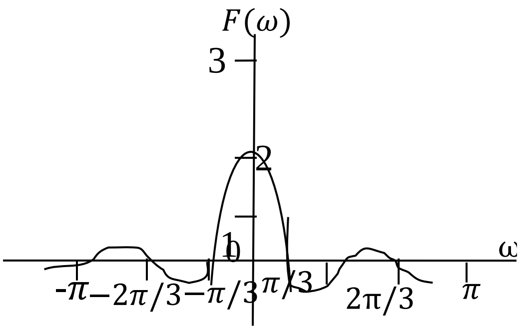
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\omega a} - e^{-i\omega a}}{i\omega} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}\ , \omega \neq 0$$

For $\omega = 0$,we obtain $F(\omega) = \sqrt{2/\pi} a$.

(b) The graphs of $f(x)$ and $F(\omega)$ for $\omega = 3$ are shown in the following figures.





Example 2.

(a) Use the result of Example 1 to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} d\omega.$$

(b) Deduce the value of $\int_0^{\infty} \frac{\sin u}{u} du$

Solution.

(a) From Fourier's integral theorem, if

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{i\omega u} du$$

then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

Then from Example 1,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega} e^{-i\omega x} \, d\omega = \begin{cases} 1 & |x| < a \\ 1/2 & |x| = a \\ 0 & |x| > a \end{cases} \qquad (1)$$

The left side of (1) is equal to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} \, d\omega$$
$$- \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega a \sin \omega x}{\omega} \, d\omega \qquad (2)$$

The integrand in the second integral of (2) is odd and so the integral is zero. Then from (1) and (2),

$$\int_{-\infty}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} \, d\omega = \begin{cases} \pi & |x| < a \\ \pi/2 & |x| = a \\ 0 & |x| > a \end{cases} \qquad (3)$$

(b) If $x = 0$ and $a = 1$ in the result of (a),

$$\int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} \, d\omega = \pi \qquad or \qquad \int_{0}^{\infty} \frac{\sin \omega}{\omega} \, d\omega = \frac{\pi}{2}$$

since the integrand is even.

Example 3.

If $f(x)$ is an even function show that:

$$(a)\ F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\omega u\, du\,,$$

$$(b)\ f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(\omega)\cos\omega x\, d\omega.$$

Solution.

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\omega u}\, du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\cos\omega u\, du + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\sin\omega u\, du \quad (1)$$

(a) If $f(u)$ is even, $f(u)\cos\lambda u$ is even and $f(u)\sin\lambda u$ is odd. Then the second integral on the right of (1) is zero and the result can be written

$$F(\omega) = \frac{2}{\sqrt{2\pi}}\int_0^\infty f(u)\cos\omega u\, du$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\omega u\, du$$

(b) From (a),$F(-\omega) = F(\omega)$ so that $F(\omega)$ is an even function. Then by using a proof exactly analogous to that in (a), the required result follows. A similar result holds for odd functions and can be obtained by replacing the cosine by the sine.

Example 4.

Solve the integral equation

$$\int_0^\infty f(x)\cos\omega x\, dx = \begin{cases} 1-\omega & 0 \le \omega \le 1 \\ 0 & \omega > 1 \end{cases}$$

Solution.

Let $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\, dx = F(\omega)$ and choose

$$F(\omega) = \begin{cases} \sqrt{2/\pi}(1-\omega) & 0 \le \omega \le 1 \\ 0 & \omega > 1 \end{cases} .$$

Then by Example 3.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(\omega)\cos\omega x\, d\omega$$

$$\sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}\ (1-\omega)\cos\omega x\, d\omega$$

$$= \frac{2}{\pi}\int_0^1 (1-\omega)\cos\omega x\, d\omega = \frac{2(1-\cos x)}{\pi x^2}. \blacktriangleleft$$

Example 5.

Use Example 4 to show that $\int_0^\infty \frac{\sin^2 u}{u^2}\ du = \frac{\pi}{2}$.

Solution.

As obtained in Example 4,

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\cos\omega x\, dx = \begin{cases} 1-\omega & 0 \le \omega \le 1 \\ 0 & \omega > 1 \end{cases}$$

Taking the limit as $\omega^{\to 0+}$, we find

$$\int_0^\infty \frac{1-\cos x}{x^2}\ dx = \frac{\pi}{2}$$

But this integral can be written as $\int_0^\infty \frac{2\ \sin^2(x/2)}{x^2}\ dx$

which becomes $\int_0^\infty \frac{\sin^2 u}{u^2}\ du$ on letting $x = 2u$, so

that the required result follows.$\blacktriangleleft$

Example 6.

Show that $\int_0^\infty \frac{\cos\omega x}{\omega^2+1}\, d\omega = \frac{\pi}{2}e^{-x}\,, \quad x \geq 0.$

Solution.

Let $f(x) = e^{-x}$ in the Fourier integral theorem

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \, d\omega \int_0^\infty f(u)\cos\omega u \, du$$

Then

$$\frac{2}{\pi}\int_0^\infty \cos\omega x \, d\omega \int_0^\infty e^{-u}\cos\omega u \, du = e^{-x}$$

Since $\int_0^\infty e^{-u}\cos\omega u \, du = \frac{1}{\omega^2+1}$, we have

$\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+1}\, d\omega = e^{-x}$ *or* $\int_0^\infty \frac{\cos\omega x}{\omega^2+1}\, d\omega =$

$\frac{\pi}{2}e^{-x}$.◄

4. PARSEVAL'S IDENTITIES FOR FOURIER INTEGRALS

If $F_s(\omega)$ and $G_s(\omega)$ are Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

$$\int_0^{\infty} F_s(\omega)G_s(\omega)d\omega = \int_0^{\infty} f(x)g(x)\,dx \qquad (11)$$

Similarly, if $F_c(\omega)$ and $G_c(\omega)$are Fourier cosine transforms of $f(x)$ and (x) , then

$$\int_0^{\infty} F_c(\omega)G_c(\omega)d\omega = \int_0^{\infty} f(x)g(x)\,dx \qquad (12)$$

In the special case where $f(x) = g(x)$, (11) and (12) become respectively

$$\int_0^{\infty} \{F_c(\omega)\}^2 d\omega = \int_0^{\infty} \{f(x)\}^2 dx \qquad (13)$$

$$\int_0^{\infty} \{F_c(\omega)\}^2 d\omega = \int_0^{\infty} \{f(x)\}^2 dx \qquad (14)$$

The above relations are known as **Parseval's identities** for integrals. Similar relations hold for General Fourier transforms. Thus if $F(\omega)$ and $G(\omega)$ are Fourier transforms of $f(x)$ and $g(x)$respectively, we can prove that

$$\int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}\,d\omega = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \quad (15)$$

Where the bar signifies the complex conjugate obtained by replacing *i* by -*i*.

Example 7.

Verify Parseval's identity for Fourier integrals for the Fourier transforms of Example 1.

Solution.

We must show that

$\int_{-\infty}^{\infty}\{f(x)\}^2\,dx = \int_{-\infty}^{\infty}\{F(\omega)\}^2\,d\omega$, where

$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$ and $F(\omega)\sqrt{\frac{2}{\pi}}\frac{\sin \omega a}{\omega}$.

This is equivalent to

$$\int_{-a}^{a} (1)^2 \, dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \frac{\sin^2 a\omega}{\omega^2} \, d\omega$$

Or

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega a}{\omega^2} \, d\omega = 2\int_{0}^{\infty} \frac{\sin^2 \omega a}{\omega^2} d\omega = \pi a$$

$i.e.,$

$$\int_{0}^{\infty} \frac{\sin^2 \omega a}{\omega^2} \, d\omega = \frac{\pi a}{2}$$

By letting $a\omega = u$ and using Example 5, it is seen that this is correct. The method can also be used to find $\int_0^\infty \frac{\sin^2 u}{u^2} \, du$directly. ◄

5. THE CONVOLUTION THEOREM

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$\int_{-\infty}^{\infty} F(\omega)G(\omega)e^{-i\omega x}d\omega$$

$$= \int_{-\infty}^{\infty} f(u)g(x-u)du \quad (16)$$

If we define the convolution, denoted by $f * g,$ of the functions f and g to be

$$f * g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(x-u)du \qquad (17)$$

Then (16) can be written

$$\mathfrak{I}\{f^{*}g\} = \mathfrak{I}(f)\mathfrak{I}(g) \qquad (18)$$

Or in words, the Fourier transform of the convolution of two functions is equal to the product of their Fourier transforms. This is called the convolution theorem for Fourier transforms.

PROOF OF THE FOURIER INTEGRAL THEOREM.

Present a heuristic demonstration of Fourier's integral theorem by use of a limiting form of Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad (1)$$

Where $a_n = \frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{n\pi u}{L}\, du$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(u)\sin\frac{n\pi u}{L}\, du.$

Then by substitution,

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)du$$

$$+\frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi}{L}(u-x)du \quad (2)$$

If we assume that $\int_{-\infty}^{\infty} |f(u)|\, du$ converges, the first term on the right of (2) approaches zero as $L \to \infty$, while the remaining part appears to approach.

$$\lim_{L\to\infty} \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \frac{n\pi}{L} (u - x)\, du \quad (3)$$

This last step is not rigorous and makes the demonstration heuristic. Calling $\Delta\omega = \pi/L$, (3) can be written.

$$f(x) = \lim_{\Delta\omega\to 0} \sum_{n=1}^{\infty} \Delta\omega\, F(n\Delta\omega) \qquad (4)$$

Where we have written.

$$F(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos \omega(u - x) du \qquad (5)$$

But the limit (4) is equal to.

$$f(x) = \int_0^\infty F(\omega)d\omega$$

$$= \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty f(u)\cos\omega(u-x)du$$

Which is Fourier's integral formula.

This demonstration serves only to provide a possible result. To be rigorous, we start with integral.

$$\frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty f(u)\cos\omega(u-x)dx$$

and examine the convergence.

This method is considered in Examples 8-11.

Examples 8.

Prove that:

$$(a)\ \lim_{\omega\to\infty}\int_0^L \frac{\sin\omega v}{v}\,dv = \frac{\pi}{2},$$

$$(b)\ \lim_{\omega\to\infty}\int_{-L}^{0}\frac{\sin\omega\nu}{\nu}\,d\nu=\frac{\pi}{2}$$

Proof.

(a) Let $\omega\nu=y$. Then

$$\lim_{\omega\to\infty}\int_{0}^{L}\frac{\sin\omega v}{v}\,dv=\lim_{\omega\to\infty}\int_{0}^{\omega L}\frac{\sin y}{y}\,dy$$

$$=\int_{0}^{\infty}\frac{\sin y}{y}\,dy=\frac{\pi}{2}$$

(b)Let $\omega\nu=-$y .

Then

$$\lim_{\omega\to\infty}\int_{-L}^{0}\frac{\sin\omega v}{v}\,dv=\lim_{\omega\to\infty}\int_{0}^{\infty}\frac{\sin y}{y}\,dy=\frac{\pi}{2}.\blacktriangleleft$$

Example 9.

Riemann's theorem states that if $F(x)$ is sectionally continuous in $(a\,,b)$, then

$$\lim_{\omega\to\infty}\int_{a}^{b}F(x)\sin\omega x\,dx=0$$

with a similar result for the cosine. Use this to prove that

$$(a)\ \lim_{\omega\to\infty}\int_0^L f(x+v)\frac{\sin\omega v}{v}dv=\frac{\pi}{2}f(x+0)$$

$$(b)\ \lim_{\omega\to\infty}\int_{-L}^0 f(x+v)\frac{\sin\omega v}{v}dv=\frac{\pi}{2}f(x-0)$$

Where $f(x)$ and $f'(x)$ are assumedsectionally continuous in $(0\,,L)$ and $(-L\,,0)$, respectively.

(a) Using Example 8 (a) , it is seen that a proof of the given result amounts to proving that.

$$\lim_{\omega\to\infty}\int_0^L\{f(x+v)-f(x+0)\}\frac{\sin\omega v}{v}\ dv=0$$

This follows at once from Riemann's theorem, because

$F(v) = \frac{f(x+v)-f(x+0)}{v}$ is sectionally continuous in $(0\,,L)$ since $\lim_{v\to 0+} F(v)$ exists and $f(x)$ is sectionally continuous.

(b) A proof of this is analogous to that in part (a) if we make use of Example 8(b).

Example 10.

If $f(x)$ satisfies the additional condition that $\int_{-\infty}^{\infty} |f(x)| dx$ converges, prove that

$$(a)\ \lim_{\omega\to\infty} \int_0^{\infty} f(x+v)\frac{\sin \omega v}{v} dv = \frac{\pi}{2} f(x+0),$$

$$(b)\ \lim_{\omega\to\infty} \int_{-\infty}^{0} f(x+v)\frac{\sin \omega v}{v} dv = \frac{\pi}{2} f(x-0).$$

Proof.

We have

$$\int_0^{\infty} f(x+v)\frac{\sin \omega v}{v} dv =$$

$$\int_0^L f(x+v)\frac{\sin\omega v}{v}dv$$

$$+\int_L^\infty f(x+v)\frac{\sin\omega v}{v}dv \quad (1)$$

$$\int_0^\infty f(x+0)\frac{\sin\omega v}{v}dv =$$

$$\int_0^L f(x+0)\frac{\sin\omega v}{v}dv$$

$$+\int_L^\infty f(x+0)\frac{\sin\omega v}{v}dv \quad (2)$$

Subtracting,

$$\int_0^x \{f(x+v)-f(x+0)\}\frac{\sin\omega v}{v}\ dv \qquad (3)$$

$$=\int_0^L \{f(x+v)-f(x+0)\}\frac{\sin\omega v}{v}\ dv$$

$$+\int_L^\infty f(x+v)\frac{\sin\omega v}{v}\ dv$$

$$-\int_L^\infty f(x+0)\frac{\sin\omega v}{v}\ dv$$

Denoting the integrals in (3) by I , I_1 , I_2 and I_3 respectively, we have $I = I_1 + I_2 + I_3$ so that

$$|I| \leqq |I_1| + |I_2| + |I_3| \qquad (4)$$

Now $|I_2| \leq \int_L^\infty \left| f(x+v) \frac{\sin \omega v}{v} \right| dv$

$$\leq \frac{1}{L} \int_L^\infty |f(x+v)|\, dv$$

Also $\qquad |I_3| \leqq |f(x+0)| \left| \int_L^\infty \frac{\sin \omega v}{v}\, dv \right|$

Since $\int_0^\infty |f(x)| dx$ and $\int_0^\infty \frac{\sin \omega v}{v}\, dv$ both converge, we can choose L so large that $|I_2| \leq \epsilon/3$, sufficently large, so that the required result follows.

This result follows by reasoning exactly analogous to that in part (a).◄

Example 11.

Prove Fourier's integral formula where $f(x)$satisfies the conditions stated on first page of this chapter.

Proof.

We must prove that

$$\lim_{L\to\infty}\frac{1}{\pi}\int_{\omega=0}^{L}\int_{u=-\infty}^{\infty} f(u)\cos\omega(x-u)du\,d\omega$$

$$=\frac{f(x+0)+f(x-0)}{2}$$

Since $\left|\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)du\right| \le \int_{-\infty}^{\infty}|f(u)|du$

which converges, it follows by the Weierstrass test that $\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)du$ converges absolutely and uniformly for all ω. Thus, we can reverse the order of integration to obtain

$$\frac{1}{\pi}\int_{\omega=0}^{L} d\omega\int_{u=-\infty}^{\infty} f(u)\cos\omega(x-u)du$$

$$= \frac{1}{\pi} \int_{u=-\infty}^{\infty} f(u)du \int_{\omega=0}^{L} \cos \omega(x-u)du$$

$$= \frac{1}{\pi} \int_{u=-\infty}^{\infty} f(u) \frac{\sin L(u-x)}{u-x} du$$

$$= \frac{1}{\pi} \int_{v=-\infty}^{\infty} f(x+v) \frac{\sin Lv}{v} dv$$

$$= \tfrac{1}{\pi} f(x+v) \tfrac{\sin Lv}{v} dv + \tfrac{1}{\pi} \int_0^{\infty} f(x+v) \tfrac{\sin Lv}{v} dv,$$

where we have let $u = x + v$.

Letting $L \to \infty$, we see by Example 10 that the given integral converges to $\frac{f(x+0)+f(x-0)}{2}$ as required.◄

Supplementary Problems

1- (a) Find the Fourier transform of

$$f(x) = \begin{cases} 1/2\epsilon & |x| \leqq \epsilon \\ 0 & |x| > \epsilon \end{cases}$$

(b)Determine the limit of this transform as $\epsilon \to 0+$

and discuss the result.

Ans. $(a)\,\frac{1}{\sqrt{2\pi}}\frac{\sin a\epsilon}{a\epsilon}$, $(b)\,\frac{1}{\sqrt{2\pi}}$

2- (a) Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2 & x < 1 \\ 0 & x > 1 \end{cases}$$

(b) Evaluate $\int_a^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx.$

Ans. $(a)\ 2\sqrt{\frac{2}{\pi}}\left(\frac{\omega\cos\omega - \sin\omega}{\omega^3}\right)$, $(b)\,\frac{3\pi}{16}$

3- If $f(x) = \begin{cases} 1 & 0 \leqq x < 1 \\ 0 & x \geqq 1 \end{cases}$

Find:

(a) Fourier sine transform,

(b) Fourier cosine transform of $f(x)$.

In Each case obtain the graph of $f(x)$ and its

transform.

Ans. $(a)\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos\omega}{\omega}\right)$, $(b)\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$

3- (a) Find the Fourier sine transform of

$$e^{-x}, x \geqq 0.$$

(b) Show that $\int_0^x \frac{x\sin mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0$

By using the result in (a).

Ans. $(a)\sqrt{2/\pi}[\omega/(1+\omega^2)]$

5- Solve for $Y(x)$ the integral equation

$$\int_0^\infty Y(x)\sin xt\,dx = \begin{cases} 1 & 0 \leqq t < 1 \\ 2 & 1 \leqq t < 2 \\ 0 & t \geqq 2 \end{cases}$$

and verify the solution by direct substitution.

Ans .$Y(x) = (2 + 2\cos x - 4\cos 2x)/\pi x$

6- Evaluate $(a)\int_0^x \frac{dx}{(x^2+1)^2}$, $(b)\int_0^x \frac{x^2\,dx}{(x^2+1)^2}$

by use of Parseval's identity.[Hint: Use the Fourier sine and cosine transforms of $e^{-x}\,, x > 0$] Ans.

$(a)\ \pi/4, \quad (b)\ \pi/4$

7- Use Problem 3 to show that

$$(a)\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx = \frac{\pi}{2}\,,$$

$$(b)\int_0^\infty \frac{\sin^4 x}{x^2}\,dx = \frac{\pi}{2}$$

8- Show that $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\,dx = \frac{\pi}{15}$

Chapter IV

LAPLACE TRANSFORM

CHAPTER IV

LAPLACE TRANSFORM

The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, the Laplace method is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive.

Definition 1. Let $F(t)$ be a function of t specified for $t > 0$. Then the Laplace transform of $F(t)$ is

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st}\, F(t)dt. \qquad (1)$$

Where we assume at present that the parameter s is real. Later it will be found useful to consider $s$is complex.

The Laplace transform of $F(t)$ is said to be exist if the integral (1) converges for some value of s; otherwise it does not exist.

► **Laplace transforms of some elementary functions**

Example 1. Prove that

(a) $\mathcal{L}\{1\} = \frac{1}{s},\ s > 0;$

(b) $\mathcal{L}\{t\} = \frac{1}{s^2}, s > 0;$

(c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a.$

Solution.

(a) $\mathcal{L}\,\{1\} = \int_0^\infty e^{-st}dt$

$$= \lim_{P\to\infty} \int_0^P e^{-st}\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-st}}{-s}\bigg|_0^P = \lim_{P\to\infty} \frac{1-e^{-sP}}{s} = \frac{1}{s}, \text{ where } \ s > 0.$$

(b) $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt = \lim_{P\to\infty} \int_0^P te^{-st}\, dt$

$$= \lim_{P\to\infty} \left\{ t\left(\frac{e^{-st}}{-s}\right)\Big|_0^P - \int_0^P \left(\frac{e^{-st}}{-s}\right) dt \right\}$$

$$= \lim_{P\to\infty} \left\{ t\left(\frac{e^{-st}}{-s}\right)\Big|_0^P - \left(\frac{e^{-st}}{s^2}\right)\Big|_0^P \right\}$$

$$= \lim_{P\to\infty} \left(\frac{1}{s^2} - \frac{e^{-sP}}{s^2} - \frac{Pe^{-sP}}{s} \right)$$

$$= \frac{1}{s^2} \quad \text{if } s > 0,$$

where we have used integration by parts.

(c) $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\, dt$

$$= \lim_{P\to\infty} \int_0^P e^{-(s-a)t}\, dt$$

$$= \lim_{P\to\infty} \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^P$$

$$= \lim_{P\to\infty} \{\frac{e^{-(s-a)P}}{-(s-a)} + \frac{1}{s-a}\} = \frac{1}{s-a} \quad if \;\; s > a. \blacktriangleleft$$

Example 2.

Prove that

(a) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$;

(b) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \quad if \;\; s > 0.$

Solution.

(a) $\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \;\, dt$

$$= \lim_{P\to\infty} \int_0^P e^{-st} \sin at \;\, dt$$

$$= \lim_{P\to\infty} \left. \frac{e^{-st}(-\mathrm{s}\sin at - a\;\cos at)}{s^2+a^2} \right|_0^P$$

$$= \lim_{P\to\infty} \left\{ \frac{a}{s^2+a^2} - \frac{e^{-sP}(\mathrm{s}\sin at + a\;\cos at)}{s^2+a^2} \right\}$$

$$= \frac{a}{s^2+a^2} \quad if \;\; s > 0$$

(b) $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \;\, dt$

$$= \lim_{P\to\infty} \int_0^P e^{-st} \cos at \;\, dt$$

$$= \lim_{P\to\infty} \left. \frac{e^{-st}(\mathrm{s}\cos at - a\;\sin at)}{s^2+a^2} \right|_0^P$$

$$= \lim_{P\to\infty}\left\{\frac{a}{s^2+a^2} - \frac{e^{-sP}(\mathrm{s}\cos at - a\ \sin at)}{s^2+a^2}\right\}$$

$$= \frac{s}{s^2+a^2} \quad if\ \ s > 0.$$

We have used here the results

$$\int e^{at}\sin\beta t\,dt = \frac{e^{at}(a\sin\beta t - \beta\cos\beta t)}{a^2+\beta^2}$$

$$\int e^{at}\cos\beta t\,dt = \frac{e^{at}(a\cos\beta t + \beta\sin\beta t)}{a^2+\beta^2}$$

Another method.

Assuming that the result of Example 1(c) holds for complex numbers (which can by proved), we have

$$\mathcal{L}\left\{e^{iat}\right\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \qquad (*)$$

But $e^{iat} = \cos at + i\sin at$.Hence

$$\mathcal{L}\left\{e^{iat}\right\} = \int_0^\infty e^{-st}(\cos at + i\sin at) \qquad (**)$$

$$= \int_0^\infty e^{-st}\cos at\,dt + \int_0^\infty e^{-st}\sin at\,dt$$

$= \mathcal{L}\,\{\cos st\} + i\,\mathcal{L}\,\{\sin at\}.$

From (*) and (**) we have by equating real and imaginary parts, $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, and

$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ ◄

Example 3.

Prove for $s > |a|$ that:

(a) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$;

(b) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2+a^2}$.

Solution.

(a) $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}e^{at}\, dt - \frac{1}{2}\int_0^\infty e^{-st}e^{-at} dt$$

$$= \frac{1}{2}\,\mathcal{L}\{e^{at}\} - \frac{1}{2}\,\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2}$$

for $s > |a|$.

(b) Exercise. ◄

Example 4.

Find $\mathcal{L}\,\{F(t)\}$ if

$$F(t) = \begin{cases} 5, & 0 < t < 3 \\ 0, & t > 3 \end{cases}.$$

Solution.

By definition,

$$\mathcal{L}\,\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

$$= \int_0^3 e^{-st}(5)dt + \int_3^\infty e^{-st}(0)dt$$

$$= 5\int_0^3 e^{-st}\,dt$$

$$= 5\left.\frac{e^{-st}}{-s}\right|_0^3$$

$$= \frac{5\left(1-e^{-3s}\right)}{s}. \blacktriangleleft$$

The adjacent table shows Laplace transforms of various elementary functions.

	$F(t)$	$\mathcal{L}\{F(t)\}$
1.	1	$\frac{1}{s} \quad s > 0$
2.	t	$\frac{1}{s^2} \quad s > 0$
3.	t^n $n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}} \quad s > 0$ Factorial $n = n! = 1.2 \ldots n$ Also, by definition $0! = 1$.
4.	e^{at}	$\frac{1}{s-a}, \quad s > a$
5.	$\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$

●Functions of exponential order

If real constants $M > 0$ and γ exist such that for all $t > 0$, $|e^{-\gamma t}F(t)| < M \quad or \quad |F(t)| < Me^{\gamma t}$
We say that $F(t)$ is a function of exponential order $\gamma\ as\ t \to \infty$ or, briefly, is of exponential order.

Example 5.

$F(t) = t^2$ is of exponential order 3(for example), since $|t^2| = t^2 < e^{3t}$ for all $t > 0$.

But$F(t) = e^{t^3}$ is not of exponential order since $\left|e^{-\gamma t}e^{t^3}\right| = e^{t^{3-\gamma t}}$can be made larger than any givenconstant by increasing t.◄

Intuitively, functions of exponential order cannot "grow" in absolute value more rapidly than $Me^{\gamma t}$as t increases. In practice, however, this is no restriction since $M$and γ can be as large as desired. Bounded functions, such as $\sin at$or $\cos at$, are of exponential order.

●Sufficient conditions for existence of Laplace transform.

Theorem 1.

If $F(t)$ is piecewise continuous in every finite interval $0 \leq t \leq N$ and of exponential order γ for $t > 0$, then its Laplacetransform $f(s)$ exists for all $s > \gamma$.

Proof.

We have for any positive number N,

$$\int_0^\infty e^{-st}F(t)dt = \int_0^N e^{-st}F(t)dt + \int_N^\infty e^{-st}F(t)dt$$

Since $F(t)$ is piecewise continuous in every finite interval$0 \leq t \leq N$, the first and the second integral on the right exist, since $F(t)$ is of exponential order $\gamma$for $t > 0$. To see this we have only to observe that in such case.

$$\left|\int_N^\infty e^{-st}F(t)dt\right| \leq \int_N^\infty |e^{-st}F(t)|\ dt$$

$$\leq \int_0^\infty e^{-st}|F(t)|\ dt \leq \int_0^\infty e^{-st}Me^{\gamma t}dt = \frac{M}{s-\gamma}.$$

Thus, the Laplace transform exists for $s > \gamma$.◄

It must be emphasized that the stated conditions are sufficient to guarantee the existence of the Laplace transform. If the conditions are not satisfied, however, the Laplace transform may or may not exist. Thus, the conditions are not necessary for the existence of the Laplace transform. For other sufficient conditions, see the following exercise.

Exercise.

Suppose that $F(t)$ is unbounded as $t \to 0$. Prove that $\mathcal{L}\{F(t)\}$ exists if the following conditions are satisfied;

(a) $F(t)$ is piecewise continuous in any interval $N_1 \leq t \leq N,$ where $N_1 > 0$.

(b) $\lim_{t \to 0} t^n F(t) = 0$ for some constant *n* such that $0 < n < 1$.

(c)$F(t)$ is of exponential order γ for $t > 0$.

● In the following list of theorems, we assume, unless otherwise stated, that all functions satisfy the conditions of Theorem 1 so that their Laplace Transforms exist.

Properties of Laplace Transform

● Linearity property

Theorem 2. If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ arefunctions with Laplace transforms $f_1(s)$ and $f_2(s)$, respectively.Then

$$\mathcal{L}\,\{c_1 F_1(t) + c_2 F_2(t)\} = c_1\,\mathcal{L}\,[F_1(t)] + c_2\,\mathcal{L}\,\{F_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s)$$

The result is easily extended to more than two functions.

Proof. Let $\mathcal{L}\,\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)dt$ and $\mathcal{L}\,\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)dt$. Then if c_1 and c_2 are any constants,

$$\mathcal{L}\,\{c_1 F_1(t) + c_2 F_2(t)\}$$
$$= \int_0^\infty e^{-st}\{c_1 F_1(t) + c_2 F_2(t)\}dt$$
$$= c_1 \int_0^\infty e^{-st} F_1(t)dt + c_2 \int_0^\infty e^{-st} F_2(t)dt$$

$= c_1\mathcal{L}\{F_1(t)\} + c_2\,\mathcal{L}\{F_2(t)\} = c_1 f_1(s) + c_2 f_2(s).$

◀

Example 6.

Find $\mathcal{L}\,\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$.

Solution.

By the linearity property we have

$$\mathcal{L}\,\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$$

$$= 4\mathcal{L}\,\{e^{5t}\} + 6\mathcal{L}\{t^3\} - 3\mathcal{L}\{\sin 4t\} + 2\{\cos 2t\}$$

$$= 4\left(\frac{1}{s-5}\right) + 6\left(\frac{3!}{s^4}\right) - 3\left(\frac{4}{s^2+16}\right) + 2\left(\frac{s}{s^2+4}\right)$$

$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$,

where $s > 5$.◀

Exercise. Find $\mathcal{L}\,\{4t^2 - 3\ \cos 2t + 5e^{-t}\}$

Example 7.

Using the linearity property of the Laplace

transformation, find $\mathcal{L}\{\sinh at\}$ and $\mathcal{L}\,\{\cosh at\}$.

Solution.

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\ \mathcal{L}\ (e^{at}) - \frac{1}{2}\ \mathcal{L}\,\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{a}{s^2-a^2}\ ,$$

for $s > |a|$.

$$\mathcal{L}\,\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\ \mathcal{L}\,\{e^{at}\} + \frac{1}{2}\ \ \mathcal{L}\,\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{s}{s^2-a^2}\,,$$

for $s > |a|$. ◄

The symbol $\mathcal{L}$, which transforms $F(t)$ into $f(s)$, is often called the Laplace transformation operator.

Because of the property of $\mathcal{L}$ expressed in this theorem, we say that $\mathcal{L}$ is a linear operator or that it has the linearityProperty.

●First translation or shifting property

Theorem 3.

If $\mathcal{L}\{F(t)\} = f(s)$ then$\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

Proof.

Since$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = f(s)$, we have

$$\mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}F(t)dt = f(s-a). \blacktriangleleft$$

Example 8. Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$ we have

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}. \blacktriangleleft$$

Example9. Find

(a) $\mathcal{L}\{t^2e^{3t}\}$;

(b) $\mathcal{L}\{e^{2t}\sin 4t\}$;

(c) $\mathcal{L}\{e^{4t}\cosh 5t\}$;

(d) $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$.

Solution.

(a) $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$. Then $\mathcal{L}\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}$.

(b) $\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}$. Then

$$\mathcal{L}\{e^{2t}\sin 4t\} = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}.$$

(c)$\mathcal{L}\{\cosh 5t\} = \frac{s}{s^2-25}$. Then

$$\mathcal{L}\{e^{4t}\cosh 5t\} = \frac{s-4}{(s-4)^2-25} = \frac{s-4}{s^2-8s-9}$$

Another method

$$\mathcal{L}\{e^{4t}\cosh 5t\} = \mathcal{L}\left\{e^{4t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\right\}$$

$= \frac{1}{2}\mathcal{L}\{e^{9t}+e^{-t}\} = \frac{1}{2}\left\{\frac{1}{s-9}+\frac{1}{s+1}\right\} = \frac{s-4}{s^2-8s-9}$.

(c) $\mathcal{L}\{3\cos 6t - 5\sin 6t\} = 3\mathcal{L}\{\cos 6t\} -$

$5\mathcal{L}\{\sin 6t\}$

$$= 3\left(\frac{s}{s^2+36}\right) - 5\left(\frac{6}{s^2+36}\right) = \frac{3s-30}{s^2+36}.$$

Then

$$\mathcal{L}\,\{e^{2t}(3\cos 6t - 5\sin 6t)\} = \frac{3(s+2)-30}{(s+2)^2+36}$$

$$= \frac{3s-30}{s^2+4s+40}. \blacktriangleleft$$

● Second translation or shifting property

Theorem 4.

If $\mathcal{L}\,\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$,

Then$\mathcal{L}\,\{G(t)\} = e^{-as}\, f(s)$.

Proof.

$$\mathcal{L}\,\{G(t)\} = \int_0^\infty e^{-st} G(t)dt$$

$$= \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st} F(t-a)dt$$

$$= \int_a^\infty e^{-st} F(t-a)dt$$

$$= \int_0^\infty e^{-s(u+a)} F(u)du$$

$$= e^{-as}\int_0^{\infty} e^{-su}F(u)du = e^{-as}f(s).$$

Where we have used the substitution $t = u + a$.◄

Example 10. Since $\mathcal{L}\,\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$, the Laplace

transform of the function

$G(t) = \begin{cases} (t-2)^3, & t > 2 \\ 0, & t < 2 \end{cases}$ is $\frac{6\,e^{-2s}}{s^4}$. ◄

Example 11. Find$\mathcal{L}$ $\{F(t)\}$ if

$$F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > 2\pi/3 \\ 0, & t < 2\pi/3. \end{cases}$$

Solution.

Method 1.

$$\mathcal{L}\,\{F(t)\} = \int_0^{2\pi/3} e^{-st}(0)dt$$

$$+\int_{2\pi/3}^{\infty} e^{-st}cos\left(t - \frac{2\pi}{3}\right)dt$$

$$= \int_0^\infty e^{-s(u+\frac{2\pi}{3})} \cos u \, du$$

$$= e^{-2\pi s/3} \int_0^\infty e^{-su} \cos u \, du = \frac{se^{-2\pi s/3}}{s^2+1}.$$

Method 2.Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$ it follows from

Theorem 4, with$a = \frac{2\pi}{3}$, that

$$\mathcal{L}\{F(t)\} = \frac{s\, e^{-2\pi s/3}}{s^2+1}. \blacktriangleleft$$

● Change of scale property

Theorem 5.

If $\mathcal{L}\{F(t)\} = f(s)$,then$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof.

$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at) dt$

$$= \int_0^\infty e^{-s(u/a)} F(u) d(u/a)$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} F(u) du \;\; = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Using the transformation $t = u/a$. ◀

Example 12.

Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we have

$\mathcal{L}\{\sin 3t\} = \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2+1} = \frac{3}{s^2+9}$. ◄

Example 13.

Given $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$.

Find:$\mathcal{L}\left\{\frac{\sin at}{t}\right\}$.

Solution.

$$\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\mathcal{L}\left\{\frac{\sin at}{t}\right\}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{1}{\left(\frac{s}{a}\right)}\right)$$

$= \frac{1}{a}\tan^{-1}(a/s)$.

Then $\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}(a/s)$. ◄

●Laplace transforms of derivatives

Theorem 6.

If $\mathcal{L}\{F(t)\} = f(s)$, then for $n = 1,2,3, \ldots$

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s),$$

Proof.

We have $f(s) = \int_0^\infty e^{-st} F(t) dt$. By Leibnitz's rule for differentiating under the integral sign,

$$\frac{df}{ds} = f'(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t) dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} F(t) dt$$

$$= \int_0^\infty -t e^{-st} F(t) dt$$

$$= -\int_0^\infty e^{-st} \{tF(t)\}\, dt$$

$$= -\mathcal{L}\{tF(t)\}$$

Thus $\mathcal{L}\{tF(t)\} = -\frac{df}{ds} = -f'(s)$ (1)

This proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Assume the theorem true for $n = k,\ i.e.$ assume.

$$\int_0^\infty e^{-st}\{t^k F(t)\}\, dt = (-1)^k f^{(k)}(s)(2)$$

Then$\frac{d}{ds}\int_0^\infty e^{-st}\{t^k F(t)\}\, dt = (-1)^k f^{(k+1)}(s)$

or by Leibnitz's rule,

$$-\int_0^\infty e^{-st} t\{t^k F(t)\}\, dt = (-1)^k f^{(k+1)}(s)$$

i.e.$\int_0^\infty e^{-st}\{t^{k+1} F(t)\}\, dt) =$

$$(-1)^{k+1} f^{(k+1)}(s) \qquad (3)$$

It follows that if (2) is true, i.e. if the theorem holds for $n = k,$ (3)is true, i.e. the theorem holds for $n = k + 1.$ But by (1) the theorem istrue for $= 1$. Hence it is true for $n = 1 + 1 = 2$ and n =2+1=3, etc., and thus for all positive integer values of n. ◄

Exercise. Prove that Leibnitz's rule.

Example 14.

Find $(a)\ \mathcal{L}\,\{t \sin at\}, \quad (b)\ \mathcal{L}\,\{t^2 \cos at\}$.

Solution.

(a)$\mathcal{L}\,\{\sin at\} = \frac{a}{s^2+a^2}$, we have by Theorem 6

$$\mathcal{L}\,\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}.$$

(d) $\mathcal{L}\,\{\cos at\} = \frac{s}{s^2+a^2}$, we have by Theorem 6

$$\mathcal{L}\,\{\mathrm{t}^2 \cos \mathrm{at}\} = \frac{\mathrm{d}^2}{\mathrm{ds}^2}\left(\frac{\mathrm{s}}{\mathrm{s}^2+\mathrm{a}^2}\right) = \frac{2a^3-6a^2s}{(s^2+a^2)^3}. \blacktriangleleft$$

Example 15.

Since $\mathcal{L}\,\{e^{2t}\} = \frac{1}{s-2}$, we have

$$\mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}.$$

$$\mathcal{L}\,\{t^2\, e^{2t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}. \blacktriangleleft$$

Theorem 7.

If $\mathcal{L}\,\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$.

Proof. Using integration by parts, we have

$$\mathcal{L}\,\{F'(t)\} = \int_0^\infty e^{-st}F'(t)dt$$

$$= \lim_{P\to\infty}\int_0^P e^{-st}F'(t)dt$$

$$= \lim_{P\to\infty}\left\{e^{-st}F(t)|_0^P + s\int_0^P e^{-st}F(t)dt\right\}$$

$$= \lim_{P\to\infty}\{e^{-st}F(P) - F(0)\} + s\int_0^\infty e^{-st}F(t)dt$$

$$= s\int_0^\infty e^{-st}F(t)\ dt\ \ - F(0)$$

$$= sf(s) - F(0).$$

Using the fact that $F(t)$is of exponential order γ as $t\to\infty$ so that $\lim_{P\to\infty} e^{-sP}F(P) = 0$ for $s>\gamma$. ◄

Theorem 8.

If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0).$$

Proof.

$$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0) = sg(s) - G(0)$$

Let $G(t) = F'(t)$. Then

$$\mathcal{L}\{F''(t)\} = s\mathcal{L}\{F'(t)\} - F'(0)$$

$$= s[s\mathcal{L}\{F(t)\} - F(0)] - F'(0)$$

$$= s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0)$$

$$= s^2 f(s) - sF(0) - F'(0).$$ ◄

Note.

The generalization to higher order derivatives can by proved by using mathematical induction.

Example 16.

Use Theorem 7 to derive each of the following

Laplace transforms:

(a) $\mathcal{L}\{1\} = \frac{1}{s}$, (b) $\mathcal{L}\{t\} = \frac{1}{s^2}$, (c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

Solution.

Theorem 7 states, under suitable conditions, that

$$\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} - F(0) \qquad (*)$$

(a) Let $F(t) = 1$. Then $F'(t) = 0, F(0) = 1$, and

(*) becomes $\mathcal{L}\{0\} = s\mathcal{L}\{1\} - 1$ or $\mathcal{L}\{1\} = \frac{1}{s}$.

(b) Let $F(t) = t$. Then $F'(t) = 1, F(0) = 0$,

and (*)becomes using part (a)

$\mathcal{L}\{1\} = \frac{1}{s} = s\mathcal{L}\{t\} - 0$ or $\mathcal{L}\{t\} = \frac{1}{s^2}$.

By using mathematical induction, we can similarly show $\mathcal{L}\{t^n\} = n!/s^{n+1}$ for any positive integer *n*.

Let $F(t) = e^{at}$. Then $F'(t) = ae^{at}, F(0) = 1$, and (*) becomes

$$\mathcal{L}\{ae^{at}\} = s\mathcal{L}\{e^{at}\} - 1,$$

i.e.,

$$a\mathcal{L}\{e^{at}\} = s\mathcal{L}\,\{e^{at}\} - 1$$

Or

$\mathcal{L}\{e^{at}\} = 1/(s-a)$. ◀

Example 16.

Show that $\mathcal{L}\,\{\sin at\} = \frac{a}{s^2+a^2}$.

Solution.

Let $F(t) = \sin at$. Then $F'(t) = a\cos at, F''(t) = -a^2 \sin at, F(0) = a$.Hence from the result of Theorem 8.

$\mathcal{L}\,\{F''(t)\} = s^2\mathcal{L}\,\{F(t)\} - sF(0) - F'(0)$

Then $\mathcal{L}\,\{-a^2 \sin at\} = s^2\mathcal{L}\,\{\sin at\} - s.0 - a$

i.e. $-a^2\mathcal{L}\,\{\sin at\} = s^2\mathcal{L}\,\{\sin at\} - a$

or $\mathcal{L}\,\{\sin at\} = \frac{a}{s^2+a^2}$. ◀

●Division by t.

Theorem 9.

If $\mathcal{L}\{F(t)\} = f(s)$, then$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$

Provided $\lim_{t\to 0} F(t)/t$ exists.

Proof.

Let $G(t) = \frac{F(t)}{t}$. then $F(t) = t\, G(t)$. Taking the Laplace transform ofBoth sides and by Theorem 6, we have

$$\mathcal{L}\{F(t)\} = -\frac{d}{ds}\mathcal{L}\{G(t)\} \quad or \quad f(s) = -\frac{dg}{ds}$$

Then integrating, we have

$$g(s) = -\int_\infty^s f(u)du = \int_s^\infty f(u)du$$

i.e.$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$

Note that in (1) we have chosen the "constant of integration" so that $\lim_{s\to\infty} g(s) = 0$.

Corollary 1.

(a) $\int_0^\infty \frac{F(t)}{t}\, dt = \int_0^\infty f(u)du$ provided that the integrals converge.

(b) $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Proof.

(a) From Theorem 9,

$$\int_0^\infty e^{-st}\frac{F(t)}{t}\, dt = \int_s^\infty f(u)du$$

Then taking the limit as $s \to 0^+$, assuming the integrals converge,the required result is obtained.

(b) Let $F(t) = \sin t$.So$f(s) = 1/(s^2 + 1)$in (a).

Then$\int_0^\infty \frac{\sin t}{t}\, dt = \int_0^\infty \frac{du}{u^2+1} = \tan^{-1}u|_0^\infty = \frac{\pi}{2}$

Example 17.

Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ *and* $\lim_{t\to 0} \frac{\sin t}{t} = 1$, we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{du}{u^2+1} = \tan^{-1}(1/s). \blacktriangleleft$$

●Laplace transforms of integrals.

Theorem 10.

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\int_0^t F(u)du\right\} = f(s)/s$.

Proof.

Let $G(t) = \int_0^t F(u)du$. Then $G'(t) = F(t)$and

$G(0) = 0$. Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) = s\mathcal{L}\{G(t)\} = f(s)$$

Thus $\mathcal{L}\{G(t)\} = \frac{f(s)}{s}$or $\mathcal{L}\left\{\int_0^t F(u)du\right\} = \frac{f(s)}{s}$. ◀

Example 18.

Find $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\}$.

Solution.

By Example 17, $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$.

By Theorem 10, $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$. ◀

● Periodic functions

Theorem 11. Let $F(t)$ have period $T > 0$ so that $F(t+T) = F(t)$. Then $\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st}F(t)dt}{1-e^{-sT}}$.

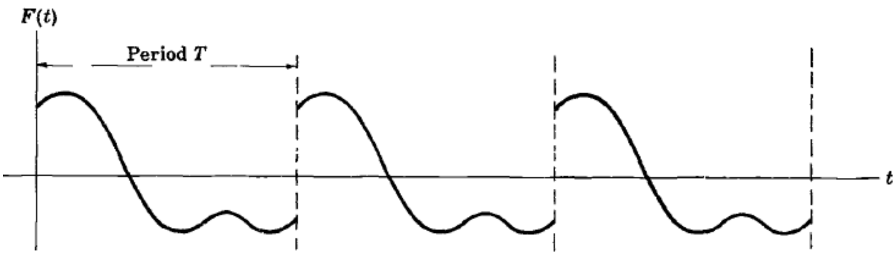


Fig. 2

Proof. We have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)dt$$

$$= \int_0^T e^{-st}F(t)dt + \int_T^{2T} e^{-st}F(t)dt +$$

$$\int_{2T}^{3T} e^{-st}F(t)dt + \cdots$$

In the second integral let $t = u + T$, in the third integral let $t = u + 2T$, etc. Then

$$\mathcal{L}\{F(t)\} = \int_0^T e^{-su}F(u)\,du + \int_0^T e^{-s(u+T)}F(u+T)du$$

$$+\int_0^T e^{-s(u+2T)}F(u+2T)du + \cdots$$

$$= \int_0^T e^{-su}F(u)du + e^{-sT}\int_0^T e^{-su}F(u)du$$

$$+e^{-2sT}\int_0^T e^{-su}F(u)du + \cdots$$

$$= (1 + e^{-sT} + e^{-2sT} + \cdots)\int_0^T e^{-su}F(u)du$$

$$= \frac{\int_0^T e^{-su}F(u)du}{1 - e^{-sT}}.$$

Where we have used the periodicity to write

$$F(u+T) = F(u), F(u+2T) = F(u), \ldots$$

and the fact that

$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ $, |r| < 1.$ ◀

Example 19.

(a) Graph the function

$$F(t) = \begin{cases} \sin t\,, & 0 < t < \pi \\ 0 \quad , & \pi < t < 2\pi \end{cases}$$

Extended periodically with period 2π.

(b) Find $\mathcal{L}\,\{F(t)\}$.

Solution. (a) The graph appears in Fig. 3.

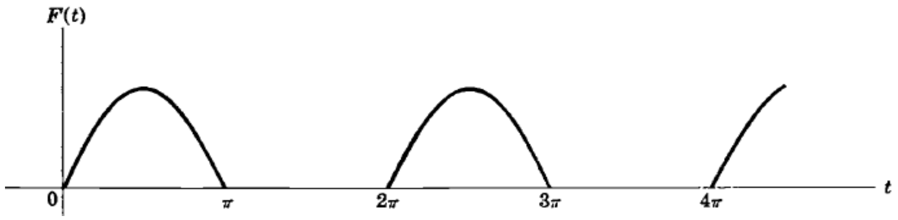


Fig. 3.

(b) By Theorem 11, since $T = 2\pi$, we have

$$\mathcal{L}\,\{F(t)\} = \frac{1}{1 - e^{2\pi s}} \int_0^{2\pi} e^{-st} F(t) dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \; dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{e^{-st}(-s\sin t - \cos t)}{s^2 + 1} \right\} \Bigg|_0^{\pi}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{1 + e^{-\pi s}}{s^2 + 1} \right\}$$

$$= \frac{1}{(1 - e^{-\pi s})(s^2 + 1)} \ .$$

The graph of the function $F(t)$ is often called a half wave rectified sine curve. ◀

● Behavior of $\boldsymbol{f(s)}$ as $\boldsymbol{s \to \infty}$

Theorem 9.

If $\mathcal{L}\{F(t)\} = f(s)$, then $\lim_{s\to\infty} f(s) = 0$.

Proof. Exercise.

● Initial – value theorem

Theorem 10. If the indicated limits exist, then

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sf(s).$$

Proof. We have

$\mathcal{L}\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t)dt = sf(s) - F(0)$. (*)

But if $F'(t)$ is piecewise continuous and of exponential order, we have

$$\lim_{s\to\infty}\int_0^\infty e^{-st}F'(t)dt = 0$$

Then taking the limit as $s \to \infty$ in (*), assuming $F(t)$continuous at $t = 0$, we find that

$$0 = \lim_{s\to\infty} sf(s) - F(0)$$

Or

$$\lim_{s\to\infty} sf(s) = F(0) = \lim_{t\to 0} F(t). \blacktriangleleft$$

If $F(t)$ is not continuous at $t = 0$, the required result still holds but we must use Theorem 7.

● Final – value theorem

Theorem 11.

If the indicated limits exist, then

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sf(s)$$

Proof.

We have

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}F'(t)dt = sf(s) - F(0)$$

The limit of the left-hand side as $s \to 0$ is

$$\lim_{s \to 0} \int_0^\infty e^{-st} F'(t)dt = \int_0^\infty F'(t)dt$$

$$= \lim_{P \to \infty} \int_0^P F'(t)dt$$

$$= \lim_{P \to \infty} \{F(P) - F(0)\}$$

$$= \lim_{t \to \infty} F(t) - F(0)$$

The limit of the right-hand side as $s \to 0$ is

$$\lim_{s \to 0} sf(s) - F(0)$$

Thus $\lim_{t \to \infty} F(t) - F(0) = \lim_{s \to 0} sf(s) - F(0)$

Or, as required, $\lim_{t \to \infty} F(t) = \lim_{s \to 0} sf(t)$. ◄

If $F(t)$ is not continuous, the result still holds but we must use Theorem 7.

Example 20.

Illustrate Theorems 9 and 10 for the function $F(t) = 3e^{-2t}$.

Solution.

We have $F(t) = 3e^{-2t}, f(s) = \mathcal{L}\{F(t)\} = \frac{3}{s+2}$

By the initial – value theorem

$$\lim_{t\to 0} 3e^{-2t} = \lim_{s\to\infty} \frac{3s}{s+2}$$

or $3 = 3$, which illustrates the theorem.

By the final –value theorem,

$$\lim_{t\to\infty} 3e^{-2t} = \lim_{s\to 0} \frac{3s}{s+2}$$

Or $0 = 0$, which illustrates the theorem. ◀

● Generalization of initial – value theorem

If $\lim\limits_{t\to 0} F(t) \,/\, G(t) = 1,$ then we say that for values of t near $t = 0$[small t], $F(t)$ is close to $G(t)$ and we write $F(t) \sim G(t)$ as $t = 0$.

Similarly, if $\lim\limits_{s\to\infty} f(s) \,/\, g(s) = 1$,then we say that for large values of s , $f(s)$ isclose to $g(s)$ and we write $f(s) \sim g(s)$ as $s \to \infty$. With this notation we have the following generalization of Theorem 10

Theorem 12.

If $F(t) \sim G(t)$ as $t = 0$, then $f(s) \sim g(s)$ as $s \to \infty$, where $f(s) = \mathcal{L}\,\{F(t)\}$ and $g(s) = \mathcal{L}\,\{G(t)\}$.

● **Generalization of final – value theorem**

If $\lim_{t\to\infty} F(t)\ / \ G(t) = 1,$ we write $F(t) \sim G(t)$ as $t = \infty$. Similarly, if $\lim_{s\to 0} f(s)\ /\ g(s)\ = 1,$ we write $f(s) \sim g(s)$ as $s \to 0$. Then we have the following generalization of Theorem11.

Theorem 13.

If $F(t) \sim G(t)$ as $= \infty$, then $f(s) \sim g(s)$ as $s \to 0$, where $f(s) = \mathcal{L}\,\{F(t)\}$ and $g(s) = \mathcal{L}\,\{G(t)\}$.

►Methods of finding Laplace transforms

1. Direct method.

This involves direct use of Definition 1.

2. Series method. If $F(t)$has a power series expansion given by

$F(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{n=0}^{\infty} a_n t^n.$

Its Laplace transform can be obtained by taking the sum of the Laplace transforms of each term in the series. Thus

$$\mathcal{L}\,\{F(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!\,a_2}{s^3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{n!\,a_n}{s^n + 1} \qquad (*)$$

A condition under which the result is valid is that the series (*) be convergent for $s > \gamma$.

3. Method of differential equations. This involves finding a differential Equation sates – feed by $F(t)$ and then using the above theorems.

4. Differentiation with respect to a parameter

5. Miscellaneous methods. Involving special devices such as indicated in the above Theorems.

6. Use of Tables.

►Evaluation of integrals

If $f(s) = \mathcal{L}\{F(t)\}$, then

$$\int_0^\infty e^{-st} F(t)dt = f(s) \qquad (\#)$$

Taking the limit as $s \to 0$, we have

$$\int_0^\infty F(t)dt = f(0) \; (\#\#)$$

Assuming the integral to be convergent. The results (#) and (##) are often Useful in evaluating various integrals.

Example21.

Evaluate

$(a) \int_0^\infty te^{-2t} \cos t \;\, dt;$

$(b) \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}.$

Solution.

$$\mathcal{L}\,\{t\cos t\} = \int_0^\infty te^{-st}\cos t\,dt$$

$$= -\frac{d}{ds}\mathcal{L}\,\{\cos t\}$$

$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}.$

Letting $s = 2$, we find $\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$.

(a) If $F(t) = e^{-t} - e^{-3t}$, then

$$f(s) = \mathcal{L}\,\{F(t)\} = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\mathcal{L}\,\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^\infty \left\{\frac{1}{u+1} - \frac{1}{u+3}\right\} du$$

Or

$$\int_0^\infty e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \ln\left(\frac{s+3}{s+1}\right).$$

As $s \to 0^+$, we find $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \ln 3.$◀

2. Some special functions

●The Gamma function

If $n > 0$, we define the gamma function by

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u}\, du$$

The following are some important properties of the gamma function

1. $\Gamma(n+1) = n\,\Gamma(n), \quad n > 0$

Thus since $\Gamma(1) = 1$, we have $\Gamma(2) = 1, \Gamma(3) = 2! = 2$, $\Gamma(4) = 3!$and in general$\Gamma(n+1) = n!$if n is a positive integer. For this the function is some – times called the factorial function.

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. $\Gamma(p)\,\Gamma(1-\mathrm{p}) = \frac{\pi}{\sin \mathrm{p}\pi}\ , \quad 0 < p < 1$

4. For large n , $\Gamma(n+1) \sim \sqrt{2\pi\, n}\, n^n e^{-n}$

[Here ~ means "approximately equal to for large "

More exactly $F(n) \sim G(n)$ if $\lim_{n\to\infty} F(n) / G(n) = 1$]

This is called **Sterling's Formula**.

5. For $n > 0$ we can define $\Gamma(n)$,

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}.$$

Example 1. Prove:

(a) $\Gamma(n+1) = n\Gamma(n),\ n > 0;$

(b) $\Gamma(n+1) = n!,\ \ n = 1,2,3, \ldots.$

Proof.

(a) $\Gamma(n+1) = \int_0^\infty u^n e^{-u} du = \lim_{P\to\infty} \int_0^P u^n e^{-u} du$

$$= \lim_{P\to\infty} \left\{ (u^n)(-e^u)|_0^P - \int_0^P (-e^{-u})(nu^{n-1})\, du \right\}$$

$$= \lim_{P\to\infty} \left\{ -P^n e^{-P} + n \int_0^P u^{n-1} e^{-u} du \right\}$$

$= n \int_0^\infty u^{n-1} e^{-u} du = n\,\Gamma(n)$ if $n > 0$.

(b) $\Gamma(1) = \int_0^\infty e^{-u}\,du = \lim_{P\to\infty} \int_0^P e^{-u}\,du =$

$\lim_{P\to\infty}(1 - e^{-P}) = 1.$

Put $n = 1,2,3, \ldots$ in$\Gamma(n+1) = n\,\Gamma(n)$. Then

$\Gamma(2) = 1 \cdot \Gamma(1) = 1, \Gamma(3) = 2 \cdot \Gamma(2) = 2.1 = 2!,$

$\Gamma(4) = 3 \cdot \Gamma(3) = 3.2! = 3!.$

In general, $\Gamma(n+1) = n!$ if n is a positive integer.◄

Example 2.

Prove: $\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$.

Proof.

Let $I_P = \int_0^P e^{-x^2}\,dx = \int_0^P e^{-y^2}\,dy$ and $\lim_{P\to\infty} I_P =$ I the required value of the integral. Then

$$I_P^2 = \left(\int_0^P e^{-x^2}\,dx\right)\left(\int_0^P e^{-y^2}\,dy\right)$$

$$= \int_0^P \int_0^P e^{-(x^2+y^2)}\, dx\, dy \;= \iint_{R_P} e^{-(x^2+y^2)}\, dx\, dy\,.$$

Where R_P is the square OACE of side P. See Fig.4.

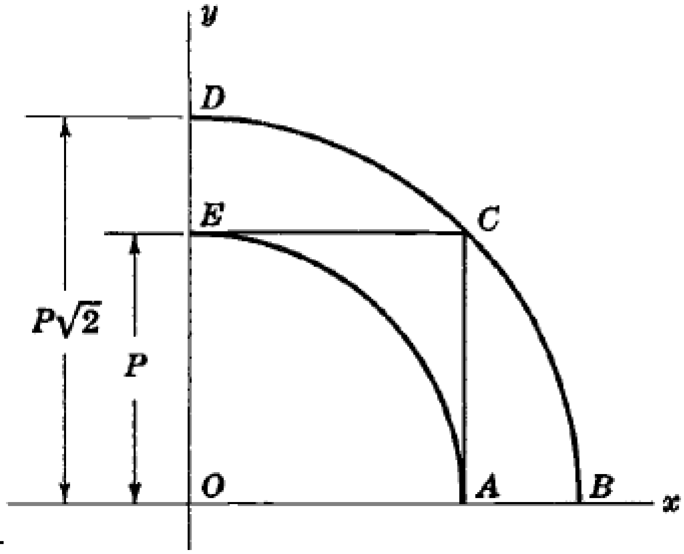


Fig. 4

Since the integrand is positive, we have

$$\iint_{R_1} e^{-(x^2+y^2)}\, dx\, du \leqq I_P^2$$

$$\leqq \iint_{R_2} e^{-(x^2+y^2)}\, dx\, du \quad (*)$$

Where R_1 and R_2 are the regions in the first quadrant bounded by the circles having radius $P$and $P\sqrt{2}$ respectively.Using polar coordinates r, θ we have from (*),

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{P} e^{-r^2} r\, dr\, d\theta \leqq I_P^2$$

$$\leqq \int_{\theta=0}^{\pi/2} \int_{r=0}^{P\sqrt{2}} e^{-r^2}\, r\, dr\, d\theta$$

Or

$$\frac{\pi}{4}\left(1 - e^{-P^2}\right) \leqq I_P^2 \leqq \frac{\pi}{4}\left(1 - e^{-2P^2}\right) (**)$$

Then taking the limit as $P \to \infty$ in $(**)$, we find

$\lim_{P\to\infty} I_P^2 = I^2 = \pi/4$ and $I = \sqrt{\pi}/2$. ◄

Example 3.

Prove: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof.$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-1/2} e^{-u}\, du$

Letting the $u = v^2$ the integral becomes (see the previous example) $2\int_0^\infty e^{-v^2}\, dv = \sqrt{\pi}$. ◄

Example 4.

Prove $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$ if $n > -1,\ \ s > 0$.

Proof.

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n\, dt$. Letting $st = u$, assuming $s > 0$ becomes

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n\, d\left(\frac{u}{s}\right)$$

$$= \frac{1}{s^{n+1}} \int_0^\infty u^n e^{-u} du = \frac{\Gamma(n+1)}{s^{n+1}}. \quad ◄$$

Example 5.

Prove $\mathcal{L}\left\{t^{-1/2}\right\} = \sqrt{\pi/s},\quad s > 0$.

Proof. Let $n = -1/2$ in Example 4. Then

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}. ◄$$

Example 6.

By assuming $\Gamma(n+1) = n\,\Gamma(n)$ holds for all n,

find:

$$\Gamma\left(-\frac{1}{2}\right), \Gamma\left(-\frac{3}{2}\right), \ \Gamma\left(-\frac{5}{2}\right), \Gamma(0), \Gamma(-1), \Gamma(-2).$$

Solution.

Letting$n = -\frac{1}{2}, \Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$. Then

$\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$.

Letting $n = -\frac{3}{2}, \Gamma\left(-\frac{1}{2}\right) = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right)$. Then

$$\Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = (2)\left(\frac{2}{3}\right)\sqrt{\pi} = \frac{4}{3}\sqrt{\pi}\,.$$

Letting $n = -\frac{5}{2}$ $, \Gamma\left(-\frac{3}{2}\right) = -\frac{5}{2}\Gamma\left(-\frac{5}{2}\right)$. Then

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right) = -2\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\sqrt{\pi} = \frac{8}{15}\sqrt{\pi}\,.$$

Letting $\mathrm{n} = 0, \Gamma(1) = 0.\Gamma(0)$and it follows

that $\Gamma(0)$must be infinite since $\Gamma(1) = 1$.

Letting $n = -1, \Gamma(0) = -1 \cdot \Gamma(-1)$ and it follows that $\Gamma(-1)$must be infinite.

Letting $n = -2, \Gamma(-1) = -2 \cdot \Gamma(-2)$ and it follows that $\Gamma(-2)$must be infinite.

$$\Gamma\left(-P - \frac{1}{2}\right) = (-1)^{p+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\dots\left(\frac{2}{2p+1}\right)\sqrt{\pi} \blacktriangleleft$$

●Bessel function

We define a Bessel function of order n by

$$J_n(t) = \frac{t^n}{2^n\Gamma(n+1)}\left\{1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - \cdots\right\} \quad (*)$$

Some important properties are:-

1. $J_{-n}(t) = (-1)^n J_n$ if n is a positive integer.

2. $J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t)$

3. $\frac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t)$ for all n. If $n > 0$, we have $J'_1(t)$.

4. $e^{1/2 t(1/u)} =$

This is called the generating function for the Bessel functions.

5.$J_n(t)$satisfies Bessel's differential equation.

$$t^2Y''(t) + t\,Y'(t) + (t^2 - n^2)Y(t) = 0$$

It is convenient to define $J_n(it) = i^{-n}I_n(t)$ where $I_n(t)$ is called the modified Bessel function of order n.

Example 7.

(a) Find $\mathcal{L}\,\{J_0(t)\}$ where $J_0(t)$ is the Bessel function of order zero

(b) Use the result of (a) to find $\mathcal{L}\,\{J_0(at)\}$.

Solution.

(a) Method 1, using series. Letting $n = 0$ in equation (*), we find

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$$

Then

$$\mathcal{L}\,\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 4^2}\frac{4!}{s^5} - \frac{1}{2^2 4^2 6^2}\frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\,.3}{2\,.4}\left(\frac{1}{s^4}\right) - \frac{1\,.3.\,5}{2\,.4\,.6}\left(\frac{1}{s^6}\right) + \cdots\right\}$$

$$= \frac{1}{s}\left\{\left(1 + \frac{1}{s^2}\right)^{-1/2}\right\} = \frac{1}{\sqrt{s^2+1}}$$

Using the binomial theorem.

Method 2, using differential equations.

The function $J_0(t)$ satisfies thedifferential equation.

$$tJ'_0(t) + J'_0(t) + tJ_0(t) = 0 \qquad (£)$$

[see Property 5 for Bessel function with $n = 0$].

Taking the Laplace transform of both sides of (£) and using Theorems 6 and 9, 12 together with$J_0(0) = 1, J'_0(0) = 0,\ y = \mathcal{L}\{J_0(t)\}$ we have

$$-\frac{d}{ds}\{s^2 y - s(1) - 0\} + \{sy - 1\} - \frac{dy}{ds} = 0$$

from which $\dfrac{dy}{ds} = -\dfrac{sy}{s^2+1}$

Thus $\dfrac{dy}{y} = -\dfrac{s\,ds}{s^2+1}$

and by integration $y = \dfrac{c}{\sqrt{s^2+1}}$

Now $\lim\limits_{s\to\infty} s\,y(s) = \dfrac{cs}{\sqrt{s^2+1}}$

$= c$ and $\lim\limits_{t\to 0} J_0(t) = 1$. Thus by the

we have $c = 1$ and so $\mathcal{L}\{J_0(t)\} = 1/\sqrt{s^2+1}$

(c) $\mathcal{L}\,\{J_0(at)\} = \frac{1}{a}\frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{\sqrt{s^2+a^2}}$ ◀

Example 8.

Find $\mathcal{L}\,\{J_1(t)\}$, where $J_1(t)$ is Bessel's function of order one.

Solution.

From Property 3 for Bessel functions, we have

$J'_0(t) = -J_1(t)$. Hence

$$\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J'_0(t)\} = -[s\,\mathcal{L}\{J_0(t)\} - 1]$$

$$= 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$$

The methods of infinite series and differential

equations can also be used.◀

● **The Error function** is defined as

$$\text{erf}\,(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\,du$$

● **The Complementary Error function** is defined

as

$$\text{erfc}(t) = 1 - \text{erf}(t)$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^1 e^{-u^2}\,du = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\,du$$

Example 9.

Prove: $\mathcal{L}\left\{\text{erf}\sqrt{t}\right\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{1}} e^{-u^2}\,du\right\} = \frac{1}{s\sqrt{s+1}}$.

Solution.

Using infinite series, we have.

$$\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\}$$

$$=\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1-u^2+\frac{u^4}{2!}-\frac{u^6}{3!}+\cdots\right)du\right.$$

$$=\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\left(t^{1/2}-\frac{t^{3/2}}{3}+\frac{t^{5/2}}{5\,.2!}-\frac{t^{7/2}}{7\,.3!}+\cdots\right)\right\}$$

$$=\frac{2}{\sqrt{\pi}}\left\{\frac{\Gamma(3/2)}{s^{3/2}}-\frac{\Gamma(5/2)}{3s^{5/2}}+\frac{\Gamma(7/2)}{5\,.2!\,s^{7/2}}-\frac{\Gamma(9/2)}{7\,.3!\,s^{9/2}}+\cdots\right\}$$

$$=\frac{1}{s^{3/2}}-\frac{1}{2}\frac{1}{s^{5/2}}+\frac{1\,.3}{2\,.4}\frac{1}{s^{7/2}}-\frac{1\,.3\,.5}{2\,.4\,.6}\frac{1}{s^{9/2}}+\cdots$$

$$=\frac{1}{s^{3/2}}\left\{1-\frac{1}{2}\frac{1}{s}+\frac{1\,.3}{2\,.4}\frac{1}{s^2}-\frac{1\,.3\,.5}{2\,.4\,.6}+\cdots\right\}$$

$$=\frac{1}{s^{3/2}}\left(1+\frac{1}{s}\right)^{-1/2}=\frac{1}{s\sqrt{s+1}}\blacktriangleleft$$

Example 10. Show that

(a) $\int_0^\infty J_0(t)dt = 1,$

(b) (b) $\int_0^\infty e^{-t}\,\mathrm{erf}\sqrt{t}\;dt = \sqrt{2}/2.$

Solution.

(a)$\int_0^\infty e^{-st}J_0(t)dt = \frac{1}{\sqrt{s^2+1}}$. Then letting $s \to 0^+$ we find $\int_0^\infty J_0(t)dt = 1$

(b) $\int_0^\infty e^{-st}\,\mathrm{erf}\sqrt{t}\;dt = \frac{1}{s\sqrt{s+1}}$. Then letting $s \to 1$, we find $\int_0^\infty e^{-t}\,\mathrm{erf}\sqrt{t}\;\;dt = \sqrt{2}/2$. ◀

● **The Sine and Cosine integrals** are defined by

$$Si\ (t) = \int_0^t \frac{\sin u}{u}\,du$$

$$Ci\ (t) = \int_1^\infty \frac{\cos u}{u}\;\,du$$

●**The Exponential integral** is defined as

$$Ei\ (t) = \int_t^\infty \frac{e^{-u}}{u}\,du$$

Example 11.

Prove:

$$\mathcal{L}\,\{Si(t)\} = \mathcal{L}\,\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Solution.

Method 1.

Let $F(t) = \int_0^t \frac{\sin u}{u}\,du$. Then $F(0) = 0$ and $F'(t) = \frac{\sin t}{t}$ or $tF'(t) = \sin \mathrm{t}$. Taking the Laplace transform $\mathcal{L}\,\{t\,F'(t)\} = \mathcal{L}\,\{\sin t\}$ or

$$-\frac{d}{ds}\{s\,f(s) - F(0)\} = \frac{1}{s^2+1}$$

i.e.,$\frac{d}{ds}\{s\,f(s)\} = \frac{-1}{s^2+1}$.

Integrating, $s\,f(s) = -\tan^{-1}s + c$. By the initial value theorem, $\lim_{s\to\infty} s\,f(s) = \lim_{t\to 0} F(t) = F(0) = 0$.

So that $= \pi/2$. Thus

$$s\, f(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\frac{1}{s} \quad or$$

$f(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$.

Method 2.

Using infinite series, we have

$$\int_0^t \frac{\sin u}{u}\, du = \int_0^t \frac{1}{u}\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots\right) du$$

$$= t - \frac{t^3}{3\,.\,3!} + \frac{t^5}{5\,.5!} - \frac{t^7}{7\,.7!} + \cdots$$

Then

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\right\} = \mathcal{L}\left\{t - \frac{t^2}{3\,.3!} + \frac{t^5}{5\,.5!} - \frac{t^7}{7\,.7!} + \cdots\right\}$$

$$= \frac{1}{s^2} - \frac{1}{3\,.3!}\,.\frac{3!}{s^4} + \frac{1}{5\,.5!}\,.\frac{5!}{s^6} - \frac{1}{7\,.7!}\,.\frac{7!}{s^8} + \cdots$$

$$= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \cdots$$

$$= \frac{1}{s}\left\{\frac{(1/s)}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \cdots\right\}$$

$= \frac{1}{s}\tan^{-1}\frac{1}{s}$.

Using the series

$\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + \cdots, |x| < 1.$

Method 3.

Letting $u = tv$, $\int_0^t \frac{\sin u}{u}\, du = \int_0^1 \frac{\sin tv}{v}\, dv$

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \mathcal{L}\left\{\int_0^1 \frac{\sin tv}{v}\, dv\right\}$$

$$= \int_0^\infty e^{-st}\left\{\int_0^1 \frac{\sin tv}{v}\, dv\right\} dt$$

$$= \int_0^1 \frac{1}{v}\left\{\int_0^\infty e^{-st}\sin tv\, dt\right\} dv$$

$$= \int_0^1 \frac{\mathcal{L}\{\sin tv\}}{v}\, dv = \int_0^1 \frac{dv}{s^2 + v^2}$$

$$= \frac{1}{s}\tan^{-1}\frac{v}{s}\Big|_0^1 = \frac{1}{s}\tan^{-1}\frac{1}{s}.$$

Where we have assumed permissibility of change of order of integration. ◄

Example 12.

Prove:

$$\mathcal{L}\,\{Ci\ (t)\} = \mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\ du\right\} = \frac{\ln(s^2+1)}{2s}$$

Proof.

Let $F(t) = \int_t^\infty \frac{\cos u}{u}$ du so that $F'(t) = -\frac{\cos\ t}{t}$

and $t\,F'(t) = -\cos t$.

Taking the Laplace transform, we have

$$-\frac{d}{ds}\{s\,f(s) - F(0)\} = \frac{-s}{s^2+1}$$

$$or \quad \frac{d}{ds}\{s\,f(s)\} = \frac{s}{s^2+1}.$$

Then by integration, $s\,f(s) = \frac{1}{2}\ln(s^2+1) + c$

By the final value theorem

$\lim_{s\to 0} s\,f(s) = \lim_{t\to\infty} F(t) = 0$ so that $c = 0$.

Thuss $f(s) = \frac{1}{2}\ln(s^2+1) \quad or \quad f(s) = \frac{\ln(s^2+1)}{2s}$ ◀

Example 13.

Prove: $\mathcal{L}\,\{Ei\ (t)\} = \mathcal{L}\ \left\{\int_t^\infty \frac{e^{-u}}{u}\ du\right\} = \frac{\ln\,(s+1)}{s}$

Proof.

Let $F(t) = \int_t^\infty \frac{e^{-u}}{u}\ du$. Then $t\ F'(t) = -e^{-t}$.

Taking the Laplace transform, we find

$$-\frac{d}{ds}\{s\ f(s) - F(0)\} = \frac{-1}{s+1}$$

$$or \quad \frac{d}{ds}\{s\ f(s)\} = \frac{1}{s+1}.$$

Then by integration,

$$s\ f(s) = In\ (s+1) + c$$

Applying the final – value theorem, we find $c = 0$

and so $f(s) = \frac{\ln\,(s+1)}{s}$. ◄

●**The Unit Step function,** also called Heaviside's

unit function (see Fig. 5), is defined as.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t < a \end{cases}$$

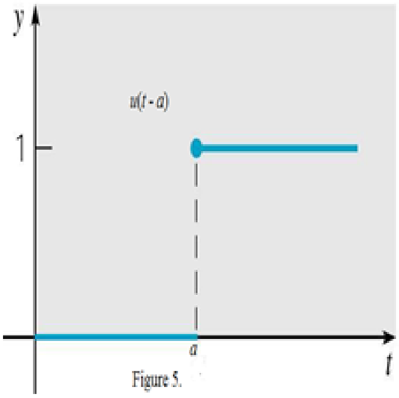


Figure 5.

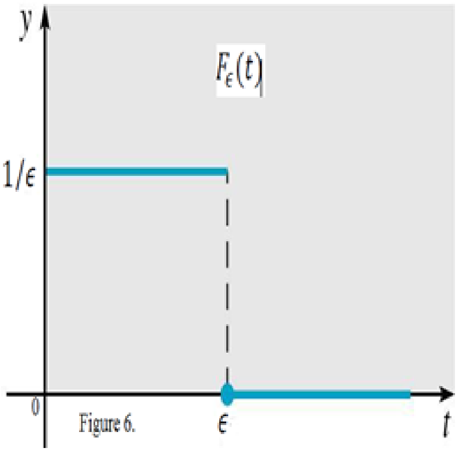


Figure 6.

●The Unit Impulse function or Dirac delta function:-

Consider the function

$$F_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \leqq t \ \leqq \epsilon \\ 0 & \quad t > \ \epsilon \end{cases}$$

where $\epsilon > 0$, whose graph appears in Fig. 6.

It is geometrically evident that as $\epsilon > 0$ the height of the rectangularshaded region increases indefinitely and the width decreases in such a waythat the area is always equal to 1,

$$i.e. \int_0^\infty F_\epsilon(t)\, dt = 1.$$

This idea has led some engineers and physicists to think of a limiting function, denoted by $\delta(t)$, approached by $F_\epsilon(t)\ as\ \epsilon \rightarrow 0$.This limitingfunction they have called the unit impulse function or Dirac delta function

Some of its properties are.

1. $\int_0^\infty \delta(t)dt = 1$

2. $\int_0^\infty \delta(t)G(t)dt = G(0)$ for any continuous function $G(t)$.

3.$\int_0^\infty \delta(t-a)G(t)dt = G(a)$ for any continuous function $G(t)$.

Although mathematically speaking such a function does not exist, manipulations or operations using it can be made rigorous.

Example 13.

Prove that $\mathcal{L}\,\{u(t-a)\} = \frac{e^{-as}}{s}$,where $u(t-a)$ is Heaviside's unit step function

Proof.

We have $u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$

Then

$$\mathcal{L}\{u(t-a)\} = \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}(1)dt$$

$$= \lim_{P\to\infty} \int_a^P e^{-st}\, dt = \lim_{P\to\infty} \frac{e^{-st}}{-s}\bigg|_a^P$$

$$= \lim_{P\to\infty} \frac{e^{-as} - e^{-sP}}{s} = \frac{e^{-as}}{s}.$$

Another method:-

Since $\mathcal{L}\,\{1\} = 1/s,$ we have by Theorem 4 in Section 1,

$\mathcal{L}\,\{u(t-a\} = e^{-as/s}$.◄

Example 14.

Find $\mathcal{L}\{F_\epsilon(t)\}$, where $F_\epsilon(t)$ is the unit impulse function.

Solution.

We have $F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \leqq t \leqq \epsilon \\ 0\ , & t > \epsilon \end{cases}$. Then

$$\mathcal{L}\{F_\epsilon(t)\} = \int_0^\infty e^{-st} F_\epsilon(t) dt$$

$$= \int_0^\epsilon e^{-st}(1/\epsilon)\, dt + \int_\epsilon^\infty e^{st}(0) dt$$

$$= \frac{1}{\epsilon}\int_0^\epsilon e^{st}\, dt = \frac{1-e^{-st}}{\epsilon s}. \blacktriangleleft$$

Example 15.

(a) Show that $\lim_{\epsilon \to 0} \mathcal{L}\{F_\epsilon(t)\} = 1$ in Example 14

(b) Is the result in (a)the same as $\mathcal{L}\left\{\lim_{\epsilon \to 0} F_\epsilon(t)\right\}$?

Explain.

Solution.

(a) This follows at once since

$$\lim_{\epsilon\to 0}\frac{1-e^{st}}{s\epsilon}=\lim_{\epsilon\to 0}\frac{1-(1-s\epsilon+\frac{s^2\epsilon^2}{2!}-\cdots}{s\epsilon}$$

$$=\lim_{\epsilon\to 0}\left(1-\frac{s\epsilon}{2!}+\cdots\right)=1$$

It also follows by use of L' Hospital's rule.

(c) Mathematically speaking, $\lim\limits_{\epsilon\to 0}F_\epsilon(t)$ does not exist, so that $\mathcal{L}\left\{\lim\limits_{\epsilon\to 0}F_\epsilon t)\right\}$ is not defined.

Nevertheless, it proves useful to consider $\delta\ (t)=\lim\limits_{\epsilon\to 0}F_\epsilon(t)$ to be such $\mathcal{L}\ \{\delta(t)\}=1$. We call$\delta\ (t)$ the Dirac delta function or impulse function.◄

Exercise.

Show that $\mathcal{L}\ \{\delta(t-a)\}=e^{as}$, where $\delta\ (t)$ is the Dirac delta function.

●Null functions.

If $\varkappa(t)$ is a function of t such that for all $t > 0$

$$\int_0^t \varkappa(u)du = 0$$

We call $\varkappa(t)$ a null function. For example the

function $F(t)\begin{cases} 1, & t = 1/2 \\ -1, & t = 1 \\ 0\,, & otherwise \end{cases}$

is a null function. In general, any function which is

zero at all but a countable set of points [i. e. a set of

points which can be put into one – to one

correspondence with. The natural numbers 1, 2, 3,

… is a null function.

Example 16.

Which of the following are null functions?

(a) $F(t) = \begin{cases} 1 & t = 1 \\ 0 & otherwise \end{cases}$,

(b) $F(t) = \begin{cases} 1 & 1 \leqq t \leqq 2 \\ 0 & otherwise \end{cases}$

(c) $F(t) = \delta\,(t)$.

Solution.

(a) $F(t)$is a null function, since $\int_0^t F(u)du = 0$

for all $t > 0$.

(b) If $t < 1,$ we have $\int_0^t F(u)du = 0$

If $1 \leqq t \leqq 2, \displaystyle\int_0^t F(u)du = \int_1^t (1)du = t - 1$

If $t > 2,$ we have $\displaystyle\int_0^t F(u)du = \int_1^2 (1)du = 1$

Since $\int_0^t F(u)du \neq 0$ for all $t > 0, F(t)$is not a null function.

(c) Since $\int_0^t \delta(u)du = 1$ for all $t > 0, \delta\,(t)$is not a null function. ◄

●Laplace transforms of special functions

In the following table we have listed Laplace transforms of various special functions.

	$F(t)$	$f(s) = \mathcal{L}\{F(t)\}$
1.	t^n	$\frac{\Gamma(n+1)}{s^n + 1}$
2.	$J_0(at)$	$\frac{1}{\sqrt{s^2 + a^2}}$
3.	$J_n(at)$	$\frac{\left(\sqrt{s^2 + a^2} - s\right)^n}{a^n\sqrt{s^2 + a^2}}$
4.	$\sin\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$
5.	$\frac{\cos\sqrt{t}}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

	$F(t)$	$f(s)=\mathcal{L}\{F(t)\}$
6.	$\text{erf}\,(t)$	$\frac{e^{s^{2/4}}}{s}\ \text{erfc}(s/2)$
7.	$\text{erf}\left(\sqrt{t}\right)$	$\frac{1}{s\sqrt{s+1}}$
8.	$Si\ (t)$	$\frac{1}{s}\ tan^{-1}\frac{1}{s}$
9.	$Ci\ (t)$	$\frac{\text{In}(s^2+1)}{2s}$
10.	$Ei\ (t)$	$\frac{\text{In}(s+1)}{s}$
11.	$u(t-a)$	$\frac{e^{-as}}{s}$
12.	$\delta(t)$	1
13.	$\delta(t-a)$	e^{-as}
14.	$\varkappa(t)$	0

Exercise Set (1)

1. Find the Laplace transforms of each of the following functions. In each case specify the values of s for which the Laplace transform exists

(a) $2e^{4t}$. (b) $3e^{-st}$. (c) $5t-3$.

(d) $2t^2-e^{-t}$. (e) $3\cos 5t$.

(f) $10\sin 6t$; (g) $6\sin 2t-5\cos 2t$.

$(h)(t^2+1)^2$. (i) $(\sin t-\cos t)^2$.

(j) $3\cosh 5t-4\sinh 5t$.

2. Evaluate

(a) $\mathcal{L}\{(5e^{2t}-3)^2\}$, (b) $\mathcal{L}\{4\,cos^2 2t\}$.

3.Find $\mathcal{L}\,\{cosh^2 4t\}$.

4. Find $\mathcal{L}\,\{F(t)\}$ if

(a) $F(t)=\begin{cases}0 & 0<t<2\\ 4 & t>2\end{cases}$,

$(b)\ F(t) = \begin{cases} 2t & 0 \leqq t \leqq 5 \\ 1 & t > 5 \end{cases}$

5. Prove that $\mathcal{L}\{t^n\} = \dfrac{n!}{s^n + 1}$, $\qquad n = 1,2,3, \ldots$

6. Investigate the existence of the Laplace transform of each of the following functions.

$$(a)\frac{1}{t+1}, \quad (b) e^{t^{2-t}}, \qquad (c)\ cot^2 t$$

7. Find

$$\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$$

8. Evaluate each of the following.

$(a)\ \mathcal{L}\{t^3 e^{-3t}\}$;

$(b)\ \mathcal{L}\{e^{-t}\cos 2t\}$;

$(c)\ \mathcal{L}\{2e^{3t}\sin 4t\}$;

$(d)\ \mathcal{L}\{(t+2)^2 e^t\}$;

$(e)\ \mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\}$;

$(f)\ \mathcal{L}\{e^{2t}\cosh 2t\}$;

$(g)\ \mathcal{L}\,\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$;

9. Find $(a)\ \mathcal{L}\,\{e^{-t}sin^2 t\}$. $(b)\ \mathcal{L}\,\{(1+te^{-1})^3\}$

10. Find $\mathcal{L}\,\{F(t)\}$if $F(t) = \begin{cases}(t-1)^2 & t > 1\\ 0 & 0 < t < 1\end{cases}$

11. If $F_1(t), F_2(t), \dots, F_n(t)$ have Laplace tansforms $f_1(s), f_2(s), \dots, f_n(s)$ respectively and $c_1, c_2, \dots, c_n$ are any constants, prove that.

$$\mathcal{L}\,\{c_1F_1(t) + c_2F_2(t) + \cdots + c_nF_n(t)\}$$

$$= c_1f_1(s) + c_2f_2(s) + c_nf_n(s)$$

12. If $\mathcal{L}\,\{F(t)\} = \dfrac{s^2 - s + 1}{(2s+1)^2(s-1)}$,

find $\mathcal{L}\{F(2t)\}$

13. If $\mathcal{L}\{F(t)\} = \dfrac{e^{-\frac{1}{s}}}{s}$,find $\mathcal{L}\{e^{-1}F(3t)\}$

14. If $f(s) = \mathcal{L}\,\{F(t)\}$, prove that for $r > 0$.

$$\mathcal{L}\,\{r^2F(at)\} = \frac{1}{s - \ln r}\, f\left(\frac{s - \ln r}{a}\right)$$

15. (a)If $\mathcal{L}\{F(t)\} = f(s)$, prove that

$\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - s F'(0) - F''(0)$

Stating appropriate conditions on $F(t)$.

(b) Generalize the result of (a) and prove by use of mathematical induction.

16. Given $F(t) = \begin{cases} 2t & 0 \leqq t \leqq 1 \\ t & t > 1 \end{cases}$.

(a)Find $\mathcal{L}\{F(t)\}$

(c)Does the result

$$\mathcal{L}\{F'(t)\} = s\,\mathcal{L}\{F(t)\} - F(0)$$

hold for this case? Explain.

17. Verify directly that

$\mathcal{L}\left\{\int_0^t (u^2 - u + e^{-u})\, du\right\} = \frac{1}{s}\,\mathcal{L}\{t^2 - t + e^{-t}\}.$

17. If $f(s)$=$\mathcal{L}\{F(t)\}$, show that

$$\mathcal{L}\left\{\int_0^t dt_1 \int_0^{t_1} F(u)du\right\} = \frac{f(s)}{s^2}$$

18. Generalize the result of Problem 17.

19. Show that $\mathcal{L}\left\{\int_0^t \frac{1-e^u}{u}\,du\right\} = \frac{1}{s}\ln\left(1+\frac{1}{s}\right)$

20. Show that $\int_{t=0}^{\infty}\int_{u=0}^{t} \frac{e^{-t}\sin u}{u}\,du\,dt = \frac{\pi}{4}$

21. Prove that (a) $\mathcal{L}\{t\ cos\ at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$

(b) $\mathcal{L}\,\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

22. Find $\mathcal{L}\,\{t(3\sin 2t - 2\cos 2t)\}$.

23. Show that $\mathcal{L}\{t^2\sin t\} = \frac{6s^2-2}{(s^2+1)^3}$

24. Evaluate(a) $\mathcal{L}\,\{t\cosh 3t\}$, (b) $\mathcal{L}\{t\sinh 2t\}$.

25. Find(a) $\mathcal{L}\,\{t^2\cos t\}$, (b) $\mathcal{L}\,\{(t^2-3t+2)\sin 3t\}$

26. Find $\mathcal{L}\,\{t^3\cos t\}$Ans.$\frac{6s^4-36s^2+6}{(s^2+1)^4}$

27. Show that $\int_0^{\infty} te^{-st}\sin t\,dt = \frac{3}{50}$

28. Show that $\mathcal{L}\left\{\dfrac{e^{-at}-e^{-bt}}{t}\right\} = \ln\left(\dfrac{s+b}{s++a}\right)$

29. Show that $\mathcal{L}\left\{\dfrac{\cos at - \cos bt}{t}\right\}$

$$= \frac{1}{2}\ln\left\{\frac{s^2+b^2}{s^2+a^2}\right\}$$

30. Find $\mathcal{L}\left\{\dfrac{\sinh t}{t}\right\}$. Ans. $\dfrac{1}{2}\ln\left(\dfrac{s+1}{s-1}\right)$

31. Show that $\displaystyle\int_0^{\infty}\frac{e^{-3t}-e^{6t}}{t}\,dt = \ln 2.$

[Hint. Use Problem 28.]

32. Evaluate $\displaystyle\int_0^{\infty}\frac{\cos 6t - \cos 4t}{t}\,dt$ Ans. $\ln(3/2)$

33. Show that $\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$.

34. Find $\mathcal{L}\{F(t)\}$, where $F(t)$ is the periodic function shown graphically in Fig. 7 below.

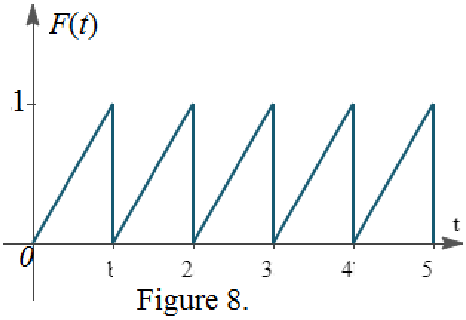


Figure 8.

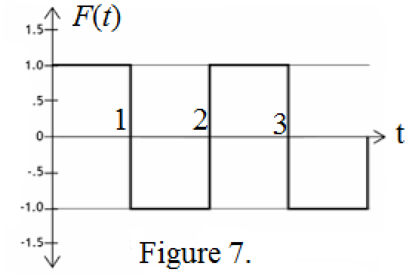


Figure 7.

35. Find $\mathcal{L}\{F(t)\}$ where $F(t)$ is the periodic function shown graphically in Fig. 8 above.

36. Let $F(t) = \begin{cases} 3t & 0 < t < 2 \\ 6 & 2 < t < 4 \end{cases}$

where $F(t)$ has period 4.

(a)Graph $F(t)$ $\quad (b)$ Find $\mathcal{L}\{F(t)\}$.

37. If $F(t) = t^2, 0 < t < 2$ and $F(t+2) = F(t)$, find $\mathcal{L}\{F(t)\}$

38. Find $\mathcal{L}\{F(t)\}$where $F(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$

and $F(t+2) = F(t)$for $t > 0$

39.(a) Show that the function $F(t)$ whose graph is the triangular wave shown in Fig. 9 has the Laplace transform $\frac{1}{s^2}\tanh\frac{s}{2}$

(b) How can the result in (a) be obtained from Problem 34? Explain.

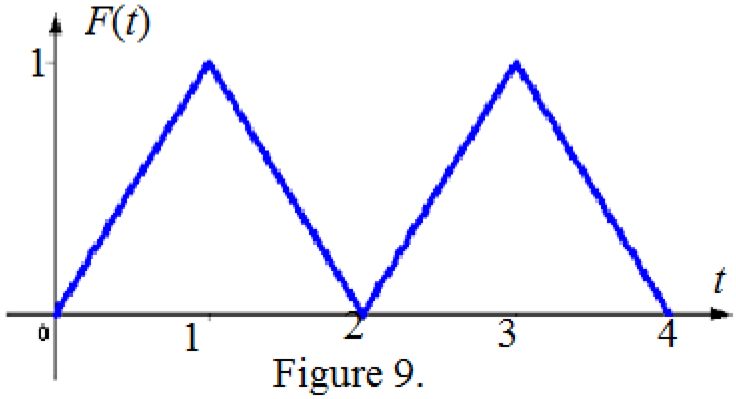


Figure 9.

40. Verify the initial – value theorem for:

$(a)\ 3-2\cos t, (b)(2t+3)^2, \quad (b)\ t+\sin 3t.$

41. Verify the final – value theorem for:

$(a)\ 1+e^{-t}(\sin t+\cos t), \quad (b)t^3e^{2t}$

42. Discuss the applicability of the final –value theorem for the function $\cos t$.

43. If $F(t) \sim c\, t^p$ as $t \to 0$ where $p > -1$, prove that: $f(s) \sim c\,\Gamma(p+1)/s^{p+1}$ as $s \to \infty$

44. If $F(t) \sim ct^p$ as $t \to \infty$ where $p > -1$, prove $f(s) \sim c\,\Gamma(p+1)/s^{p+1}$ as $s \to \infty$

45. Evaluate

$(a)\Gamma(5)$, $(b)\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}$, $(c)\Gamma(5/2)$, $(d)\frac{\Gamma(3/2)\Gamma(4)}{\Gamma(11/2)}$

46. Find

$(a)\ \mathcal{L}\left\{t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right\}, (b)\ \mathcal{L}\{t^{-1/3}\}, (c)\ \mathcal{L}\left\{\left(1+\sqrt{t}\right)^4\right\}$

47.Find$(a)\ \mathcal{L}\left\{\frac{e^{-st}}{\sqrt{t}}\right\}$, $(b)\ \mathcal{L}\left\{t^{7/2}e^{3t}\right\}$

48.Show that $\mathcal{L}\{e^{-at}J_0\} = \frac{1}{\sqrt{s^2-2as+a^2+b^2}}$

49.Show that $\mathcal{L}\{tJ_0(at)\} = \frac{s}{(s^2+a^2)^{2/2}}$

50. *Find* $(a)\ \mathcal{L}\{e^{-3t}J_0(4t)\}$, $(b)\ \mathcal{L}\{t\,J_0(2t\}$

51. Prove that$(a) {J'}_0(t) = -J_1(t)$,

$(b)\ \frac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t)$

52.If $I_0(t) = J_0(it)$, prove $\mathcal{L}\{I_0(at)\} = \frac{1}{\sqrt{s^2-a^2}}, a > 0$

53.Find $\mathcal{L}\{tJ_0(t)e^{-t}\}$ Ans. $(s-1)/(s^2-2s+2)^{3/2}$

54.Show that $(a)\ \int_0^\infty J_0(t)dt = 1$, $(b)\ \int_0^\infty e^{-t} J_0(t)dt = \frac{\sqrt{2}}{2}$

55. Find the Laplace transform of $\frac{d^2}{dt^2}\{e^{2t}J_0(2t)\}$

56.Show that $\mathcal{L}\{tJ_1(t)\} = \frac{1}{(s^2+1)^{3/2}}$

57.Prove that $\mathcal{L}\left\{J_0(a\sqrt{t})\right\} = \frac{e^{-a^2/4s}}{s}$

58. Evaluate $\int_0^\infty te^{-3t}J_0(4t)dt$ Ans. $3/125$

59.Prove that $\{J_n(t)\} = \frac{\left(\sqrt{s^2+1}-s\right)^n}{\sqrt{s^2+1}}$ and thus we obtain $\mathcal{L}\{J_n(at)\}$

60.Evaluate $(a)\ \mathcal{L}\{e^{2t}\ Si(t)\}$, $(b)\ \mathcal{L}\{t\ Si\ (t)\}$

61.Show that $\mathcal{L}\{t^2\ Ci\ (t)\} = \frac{In\ (s^2+1)}{s^3} - \frac{3s^2+1}{s(s^2+1)^2}$

62. Find $(a)\ \mathcal{L}\{e^{-3t}\ Ei(t)\}$, $(b)\ \mathcal{L}\{t\ Ei\ (t)\}$

63. Find(a) $\mathcal{L}\,\{e^{-t}\,Si\,(2t)\}$, $\quad (b)\,\mathcal{L}\,\{te\;Ei\,(3t)\}$

64.Find(a) $\mathcal{L}\left\{e^{3t}\operatorname{erf}\sqrt{t}\right\}$, $\quad (b)\mathcal{L}\left\{t\operatorname{erf}\left(2\sqrt{t}\right)\right\}$

65.Show that $\quad \mathcal{L}\left\{erfc\,\sqrt{t}\right\} = \frac{1}{\sqrt{s+1}\left\{\sqrt{s+1}+1\right\}}$

66. Find $\mathcal{L}\left\{\int_0^t \operatorname{erf}\sqrt{u}\,du\right\}$ Ans. $\;1/s^2\sqrt{s+1}$

67.(a) Show that in terms of Heaviside's unit step function, the function $F(t) = \begin{cases} e^{-t} & 0 < t < 3 \\ 0 & t > 3 \end{cases}$ can be written as $e^{-t}\{1 - u(t-3)\}$

(b)Use $\mathcal{L}\,\{u(t-a)\} = e^{-as}/s$ to find $\mathcal{L}\,(F(t)$

68. Show that $F(t) = \begin{cases} F_1(t) & 0 < t < a \\ F_2(t) & t > a \end{cases}$ can by wrtten as$F(t) = F_1(t) + \{F_2(t)\}\,u(t-a)$

69.If $F(t) = F_1(t)\;for\;\; 0 < t < a_1, F_2(t)$for

$a_1 < t < a_2, \ldots, F_{n-1}(t)fora_{n-2} < t < a_{n-1}$,

and$F_n(t)$for $t > a_{n-1}$show that.

$$F(t) = F_1(t) + \{F_2(t) - F_1(t)\}\, u(t - a_1)$$

$$+ \cdots + \{F_n(t) - F_{n-1}(t) u(t - a_{n-1})\}$$

70. Express in terms of Heaviside's unit step functions.

$$(a)\ F(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4t & t > 2 \end{cases}$$

$$(b)\ F(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$$

71. Show that

$$\mathcal{L}\,\{t^2 u(t-2)\} = \frac{2}{s^3} - \frac{2e^{-2s}}{s^3}(1 + 2s + 2s^2), s > 0.$$

72. Evaluate $(a) \int_{-\infty}^{\infty} cos\ 2t\ \delta\ (t - \pi/3)\ dt$ $(b) \int_{-\infty}^{\infty} e^{-t} u(t-2) dt$

73. (a) If $\delta'(t-a)$ denotes the formal derivative of the delta function, Show that:

$$\int_0^{\infty} F(t)\delta'(t-a)dt = -F'(a)$$

(b) Evaluate $\displaystyle\int_0^{\infty} e^{-4t}\delta'(t-2)dt$

74. Let $G_\epsilon(t) = \frac{1}{\epsilon}$ for $0 \leqq t < \epsilon,$ for

$\epsilon \leqq t < 2\epsilon, -\frac{1}{\epsilon}$ for $2\epsilon \leqq t < 3\epsilon$ and 0 for $t \geqq 3\epsilon$.

$(a)\ Find\ \ \mathcal{L}\{G_\epsilon\}.\ (b)\ Find\ \ \lim_{\epsilon\to 0} \mathcal{L}\{G_\epsilon(t)\}.$

$(c)\ is\ \lim_{\epsilon\to 0} \mathcal{L}\{G_\epsilon(t)\} = \mathcal{L}\left\{\lim_{\epsilon\to 0} G_\epsilon(t)\right\}$?

(d) Discuss geometrically the results of (a) and (b).

75. Generalize Problem 74 by defining a function $G_\epsilon(t)$ in terms of $\epsilon\ and\ n\ so\ that\ \lim_{\epsilon\to 0} G_\epsilon(t) = s^n$

where $n = 2,3,4, \ldots$.

Evaluation of integrals

76. Evaluate $\int_0^\infty t^3 e^{-t} \sin t\, dt$. Ans. 0

77. Show that $\int_0^\infty \frac{e^{-t} \sin t}{t}\, dt = \frac{\pi}{4}$.

78. Prove that $(a) \int_0^\infty J_n(t)dt = 1, (b) \int_0^\infty tJ_n(t)dt = 1$

79. Prove that $\int_0^\infty ue^{-st}J_0(au)du = \frac{1}{2}e^{-a^3/4}$

80. Show that $\int_0^\infty te^{-t}\ Ei\ (t) = In\ 2 - \frac{1}{2}$

81. Show that $\int_0^\infty ue^{-u^2} \operatorname{erf} u\, du = \frac{\sqrt{2}}{4}$.

CHAPTER V

The Inverse Laplace Transform

CHAPTER (II)

THE INVERSE LAPLACE TRANSFORM

1. Properties of Inverse Laplace Transform

Definition 1.

If the Laplace transform of a function$F(t)$ is $f(s)$, i.e. if$\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace transform of$f(s)$ and we write symbolically

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

where$\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

For example, since $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$we can write $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$.

● **Uniqueness of inverse Laplace Transforms**

Larch's Theorem

Since the Laplace transform of a null function$\mathcal{N}(t)$ is zero [see Chapter I], it is clear that if $\mathcal{L}\{F(t)\} = f(s)$ thatalso $\mathcal{L}\{F(t)\} + \mathcal{N}(t) = f(s)$. From this it follows that we canhave two different functions with the same Laplace transform. For example, the two different functions$F_1(t) = e^{-3t}$ and $F_2(t) = \begin{cases} 0, & t = 1 \\ e^{-3t}, & \text{otherwise} \end{cases}$

have the same Laplace transform, i.e.$1/(s+3)$.

If we allow null functions, we see that the inverse Laplace Transform is not unique. It is unique, however, if we disallow null functions [which do not in general arise in case of physical interest]. This result is indicated in:

Theorem 1. Larch's Theorem.

If we restrict ourselves to function $F(t)$ which are piecewise continuous in every finite interval $0 \leqq t \leqq N$ and of exponential order for $t > N$,then theinverse Laplace transform of $f(s)$, $i.e. \mathcal{L}^{-1}\{f(s)\} = F(t),$ isunique. We shall always assume such uniqueness otherwise stated.

►Some inverse Laplace Transforms

The following results follow at once from corresponding entries on Chapter I. The example follows the table proves some of these results.

Table of Inverse Laplace Transforms

	$f(s)$	$\mathcal{L}^{-1}\{f(s)\} = F(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^{n+1}}$ $n = 0,1,2,\ldots$	$\frac{t^n}{n!}$
4.	$\frac{1}{s-a}$	e^{at}
5.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
6.	$\frac{s}{s^2+a^2}$	$\cos at$
7.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
8.	$\frac{s}{s^2-a^2}$	$\cosh at$

Example 1.

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$,

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n = 0,1,2,3, \ldots\ 0! = 1$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$,

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$,

(e) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

(f) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$.

Proof.

(a) Since $\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$, $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$.

(b) $\mathcal{L}\left\{\frac{t^n}{n!}\right\} = \frac{1}{n!}\mathcal{L}\left\{t^n\right\} = \frac{1}{n!}\left(\frac{n!}{s^{n+1}}\right) = \frac{1}{s^{n+1}}$. Then

$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ for $n = 0,1,2,3, \ldots$

(c) $\mathcal{L}\left\{\frac{\sin at}{a}\right\} = \frac{1}{a}\mathcal{L}\left\{\sin at\right\} = \frac{1}{a} \cdot \frac{a}{s^2+a^2}$

$$= \frac{1}{s^2+a^2}.$$

Then$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$.

(d) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$.So $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$.

(e) $\mathcal{L}\left\{\frac{\sinh at}{a}\right\} = \frac{1}{a}\,\mathcal{L}\{\sinh at\}$

$= \frac{1}{a}\cdot\frac{a}{s^2-a^2} = \frac{1}{s^2-a^2}.$

Then $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}.$

(f) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

Then $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at.$◄

Example 2.

Prove that $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$ for $n > -1.$

Proof.

$$\mathcal{L}\left\{\frac{t^n}{\Gamma(n+1)}\right\} = \frac{1}{\Gamma(n+1)}.\mathcal{L}\{t^n\} = \frac{1}{\Gamma(n+1)}\cdot\frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{1}{s^{n+1}}\ .\ n > -1$$

Then $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}.\quad n > -1$

Note that if $n = 0,1,2,3,\ldots$, then $\Gamma(n+1) = n!$ and the result is equivalent to that of Example 1(b).

Example 2.

Find each of the following inverses Laplace transforms.

$(a)\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$ $(b)\mathcal{L}^{-1}\left\{\frac{4}{s-2}\right\}$

$(c)\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$ $(d)\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$

$(e)\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\}$ $(f)\mathcal{L}^{-1}\left\{\frac{1}{s^2-3}\right\}$

$(g)\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\}$.

Solution.

$(a)\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3}$ [Example 1 (c)]

$(b)\mathcal{L}^{-1}\left\{\frac{4}{s-2}\right\} = 4e^{2t}$ [Example 1 (a)]

$(c)\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{3!} = \frac{t^3}{6}$[Example 1 (b) or 2]

$(d)\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} = \cos\sqrt{2}\,t$ [Example 1 (d)]

$(e)\mathcal{L}^{-1}\left\{\frac{6s}{s^2-16}\right\} = 6\cosh 4t$ [Example 1 (f)]

$(f)\mathcal{L}^{-1}\left\{\frac{1}{s^2-3}\right\} = \frac{\sinh\sqrt{3}\,t}{\sqrt{3}}$ [Example 1 (e)]

$(g)\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{2t^{1/2}}{\sqrt{\pi}} = 2\sqrt{\frac{t}{\pi}}.$

[Example 2]◄

►Various properties of inverse Laplace transforms

In the following list we have indicated various important properties of inverse Laplace transforms.

1. Linearity property

Theorem 2.

If $c_1 \; and \; c_2$ are any constants while $f_1(s)$and $f_2(s)$are the Laplace transforms of $F_1(t)$and $F_2(t)$respectively, then.

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\}$$
$$= c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$
$$= c_1 F_1(t) + c_2 F_2(t).$$

The result is easily extended to more than two functions.

Proof.

From the linearity of Laplace transform,

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\}$$
$$= c_1 \mathcal{L}\{F_1(t)\} + c_2 \mathcal{L}\{F_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s).$$

Then

$$\mathcal{L}^{-1}\{c_1f_1(s)+c_2f_2(s)\}=c_1F_1(t)+c_2F_2(t)$$

$$=c_1\mathcal{L}^{-1}\{f_1(s)\}+c_2\mathcal{L}^{-1}\{f_2(s)\}.◀$$

The result is easily generalized.

Example 3.

Find

$(a)\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right\};$

$(b)\mathcal{L}^{-1}\left\{\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right\}.$

Solution.

$$(a)\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{5}{s^2}+\frac{4}{s^3}-\frac{2s}{s^2+9}+\frac{18}{s^2+9}+\frac{24}{s^4}-\frac{30}{s^{7/2}}\right\}$$

$$=5t+4\left(\frac{t^2}{2!}\right)-2\cos 3t+18\left(\frac{1}{3}\sin 3t\right)$$

$$+24(t^3/3!)-30\left\{t^{5/2}\Gamma(7/2)\right\}$$

$$=5t+2t^2-2\cos 3t+6\sin 3t+4t^3-$$

$$16t^{5/2}/\sqrt{\pi}.$$

Since $\Gamma(7/2)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$

$$(b)\mathcal{L}^{-1}\left\{\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s-3/2} - \frac{1}{3}\left(\frac{1}{s^2-16/9}\right)\right.$$

$$\left. - \frac{4}{9}\left(\frac{s}{s^2-16/9}\right)\right\}$$

$$+\frac{1}{2}\left(\frac{1}{s^2+9/16}\right) - \frac{3}{8}\left(\frac{s}{s^2+9/16}\right)$$

$= 3e^{\frac{3}{2}t} - \frac{1}{4}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4t}{3} + \frac{2}{3}\sin\frac{3t}{4} -$

$\frac{3}{8}\cos\frac{3t}{4}$.◄

Exercise.

Find:$\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$.

2. First translation or shifting property

Theorem 3.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}\, F(t)$$

Proof.

By shifting property of Laplace transform, we have $\mathcal{L}\,\{e^{at}F(t)\} = f(s-a)$.then$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$.

Another method.

Since $f(s) = \int_0^\infty e^{-st}F(t)dt,$

$$f(s-a)=\int_0^\infty e^{-(s-a)}F(t)dt$$

$$=\int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt+0$$

$$=\int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt+\int_{-\infty}^0 e^{-st}\{e^{at}F(t)\}\,dt$$

$$=\int_{-\infty}^\infty e^{-st}\{e^{at}F(t)\}\,dt\ =\mathcal{L}\{e^{at}F(t)\}.$$

Then $\mathcal{L}^{-1}\{f(s-a)\}=e^{at}F(t)$.◄

Example 3. Find

$(a)\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$; $(c)\mathcal{L}^{-1}\left\{\frac{3s-7}{s^2-2s+3}\right\}$;

$(b)\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$; $(d)\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$.

Solution.

$$(a)\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}=\mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$=6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\}+2\,\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$$

$$=6\,e^{2t}\cos 4\mathrm{t}+2e^{2t}\sin 4t$$

$$=2e^{2t}(3\cos 4t+\sin 4t).$$

$$(b)\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= 4e^{-4t} - 4te^{-4t} = 4e^{-4t}(1-t).$$

$$(c)\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\} + 5\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2-4}\right\}$$

$$= 3\,e^{t}\cosh 2t + 5\,e^{t}\sinh 2t$$

$$= e^{t}(3\cosh 2t + 5\sinh 2t)$$

$$= 4\,e^{3t} - e^{-t}.$$

The student tries another method.

$$(d)\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\} = \frac{1}{\sqrt{2}}\ \mathcal{L}^{-1}\left\{\frac{1}{(s+3/2)^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{3t}{2}}\frac{t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2\pi}}t^{-1/2}\,e^{-3t/2}.\blacktriangleleft$$

3. Second translation or shifting property-

Theorem 4.

$$\text{If}\mathcal{L}^{-1}\{f(s)\} = F(t), \qquad \text{then}$$

$$\mathcal{L}^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

Proof.

Method 1. By Second translation or shifting property

$$\mathcal{L}\left\{G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}\right\} = e^{-as}f(s)$$

Then $\mathcal{L}^{-1}\{e^{-as}f(s)\} = G(t)$.

Method 2.

Since $f(s) = \int_0^\infty e^{-st}F(t)dt,$ we have

$$e^{-as}f(s) = \int_0^\infty e^{-as}e^{-st}F(t)dt$$

$$= \int_0^\infty e^{-s(t+a)}F(t)dt$$

$$= \int_a^\infty e^{-su}F(u-a)du\ ,\ t+a = u$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}F(t-a)dt$$

$$= \int_0^\infty e^{-st}G(t)dt.$$

From which the required result follows. ◄

Note.

It should be noted that we can write $G(t)$ in terms of the Heaviside null step function as

$$F(t-a)u(t-a).$$

Example 4. Find each of the following:

$(a)\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\}$; $\quad (b)\mathcal{L}^{-1}\left\{\frac{se^{-4\pi s/5}}{s^2+25}\right\}$;

$(c)\mathcal{L}^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+s+1}\right\}$, $\quad (d)\ \mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\}$.

Solution.

(a) Since$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\}$

$= \frac{t^3e^{2t}}{3!} = \frac{1}{6}t^3e^{2t}$,we have by Theorem 4,

$$\mathcal{L}^{-1}\left\{\frac{e^{5s}}{(s-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3e^{2(t-5)} & t>5 \\ 0 & t<5 \end{cases}$$

$= \frac{1}{6}(t-5)^3e^{2(t-5)}\ u(t-5)$.

(b)Since$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+25}\right\} = \cos 5t,$

$$\mathcal{L}^{-1}\left\{\frac{se^{-4\pi s/5}}{s^2+25}\right\} = \begin{cases} \cos 5\left(t-\frac{4\pi}{5}\right), & t>4\pi/5 \\ 0 \qquad , & t<4\pi/5 \end{cases}$$

$$= \begin{cases} \cos 5t\,, & t > 4\pi/5 \\ 0 \quad\quad , & t < 4\pi/5 \end{cases} = \cos 5t\, u(t - 4\pi/5).$$

(c) We have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} + \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}\,/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= e^{-1/2t}\cos\frac{\sqrt{3}\,t}{2} + \frac{1}{\sqrt{3}}e^{-1/2t}\sin\frac{\sqrt{3}\,t}{2}$$

$$= \frac{e^{-1/2t}}{\sqrt{3}}\left(\sqrt{3}\cos\frac{\sqrt{3}\,t}{2} + \sin\frac{\sqrt{3}\,t}{2}\right)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+s+1}\right\} =$$

$$\begin{cases} \dfrac{e^{-1/2(t-\pi)}}{\sqrt{3}}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}}{2}(t-\pi) + \sin\dfrac{\sqrt{3}}{2}(t-\pi)\right\}, & t > \pi \\ 0\ , & t < \pi \end{cases}$$

$$= \frac{e^{-1/2(t-\pi)}}{\sqrt{3}}\left\{\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi)+\sin\frac{\sqrt{3}}{2}(t-\pi\right\}\ u\,(t-\pi).$$

(d) We have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\} = e^{4t}\mathcal{L}^{-1}\left\{\frac{1}{s^{5/2}}\right\} = e^{-4t}\,\frac{t^{3/2}}{\Gamma(5/2)}$$

$$= \frac{4t^{3/2}e^{-4t}}{3\sqrt{\pi}}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^{4}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\}$$

$$= \begin{cases} \dfrac{4e^{4}(t-3)^{3/2}e^{-4(t-3)}}{3\sqrt{\pi}}, & t>3 \\ 0 & ,t<3 \end{cases}$$

$$= \begin{cases} \dfrac{4(t-3)^{3/2}e^{-4(t-4)}}{3\sqrt{\pi}}, & t>3 \\ 0, & t<3 \end{cases}$$

$$= \frac{4(t-3)^{3/2}e^{-4(t-4)}}{3\sqrt{\pi}}\ u(t-3).◄$$

4. Change of scale property

Theorem 5.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$,

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}\ F\left(\frac{t}{k}\right).$$

Proof.

Method 1. From the change of scale property for Laplace transforms, we have on replacing a by $1/k,\ \mathcal{L}\{F(t/k)\} = kf(ks)$.Then $\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}\,F(t/k)$.

Method 2. Since $f(s) = \int_0^\infty e^{-st}F(t)dt,$ we have

$$f(ks) = \int_0^\infty e^{-kst}F(t)dt$$

$$= \int_0^\infty e^{-su}F\left(\frac{u}{k}\right)d\left(\frac{u}{k}\right)$$

[letting $u = kt$]

$$= \frac{1}{k}\int_0^\infty e^{-su}F(u/k)\,du = \frac{1}{k}\,\mathcal{L}\{F(t/k)\}.$$

Then $\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}\,F(t/k)$.◄

Example 5.

If $\mathcal{L}^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find $\mathcal{L}^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}$, where $a > 0$.

Solution.

By Theorem 5, replacing s by ks, we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-1/ks}}{(ks)^{1/2}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi(t/k)}} = \frac{1}{\sqrt{k}}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}},$$

or

$$\mathcal{L}^{-1}\left\{\frac{e^{-1/ks}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}.$$

Then letting $k = 1/a$,

$$\mathcal{L}^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}. \blacktriangleleft$$

5. Inverse Laplace transform of derivatives

Theorem 6.

If $\mathcal{L}^{-1}\{f(s)\} = F(t),\ \ then$

$$\mathcal{L}^{-1}\{f^{(n)}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}\ f(s)\right\}$$

$$= (-1)^n t^n F(t)$$

$$n = 1, 2, 3, \ldots.$$

Proof.

By Laplace transform of derivatives, we have

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s),$$

and so $\mathcal{L}^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t).\blacktriangleleft$

Example 6.

Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

Solution.

We have $\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-2s}{(s^2+a^2)^2}$.

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{1}{2}\ t\ \left(\frac{\sin at}{a}\right) = \frac{t\sin at}{2a}.$$

Another method.

Differentiating with respect to the parameter a, we find,

$$\frac{d}{da}\left(\frac{s}{s^2+a^2}\right) = \frac{-2as}{(s^2+a^2)^2}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}.$$

Or

$$\frac{d}{da}\left\{\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)\right\} = -2a\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

i.e.,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2a}\frac{d}{da}(\cos at)$$

$$= -\frac{1}{2a}(-t\sin at) = \frac{t\sin at}{2a}.◀$$

Example 7.

Find $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$.

Solution.

Let $f(s) = \ln\left(1+\frac{1}{s^2}\right) = \mathcal{L}\{F(t)\}$.

Then$f'(s) = \frac{-2}{s(s^2+1)} = -2\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}$.

Thus

since $\mathcal{L}^{-1}\{f'(s)\} = -2(1-\cos t) = -t\,F(t)$, we

have$F(t) = \frac{2(1-\cos t)}{t}$.◀

6. Inverse Laplace transform of integrals

Theorem 7.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)du\right\} = \frac{F(t)}{t}.$$

Example 8.

Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$,

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\} = \mathcal{L}^{-1}\left\{In\left(1+\frac{1}{s}\right)\right\}$$

$$= \frac{1-e^{-t}}{t}$$

.

7. Multiplication by s^n.

Theorem 8.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$and $F(0) = 0$, then

$$\mathcal{L}^{-1}\{\, s\, f(s)\} = F'(t).$$

Thus, multiplication by *s* has the effect of differentiating$F(t)$. If $F(0) \neq 0$, then

$$\mathcal{L}^{-1}\{\, s\, f(s)\} - F(0) = F'(t)$$

or $\mathcal{L}^{-1}\{\, s\, f(s)\} = F'^{(t)} = F(0)\, \delta\,(t)$,where $\delta\,(t)$ is the Dirac delta function or unit impulse function.

Example 9.

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\sin 0 = 0$, we have

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{d}{dt}(\sin t) = \cos t$.

Generalizations to $\mathcal{L}^{-1}\{s^n f(s)\}$, $n = 2,3, ...$are possible.◄

8. Division by s.

Theorem 9. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)du$$

Thus, division by s (or multiplication by 1/s) has the effect ofIntegrating $F(t)$ from 0 to t.

Proof.

Let $G(t) = \int_0^t F(u)du$. Then $G'(t) = F(t), G(0) = 0$.

Thus

$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) = s\,\mathcal{L}\{G(t)\} = f(s)$.

So

$\mathcal{L}\{G(t)\} = \frac{f(s)}{s}$ or $\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(u)du$.◄

Example 10.

Prove that $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^v F(u)du\,dv$.

Solution. Let $G(t) = \int_0^t \int_0^v F(u)du\,dv$.

Then $G'(t) = \int_0^t F(u)du$ and$G''(t) = F(t)$.

Since $G(0) = G'(0) = 0$, We have

$\mathcal{L}\{G''(t)\} = s^2\,\mathcal{L}\{G(t)\} - s\,G(0) - G'(0)$

$= s^2\,\mathcal{L}\{G(t)\} = f(s)$.

Thus $\mathcal{L}\{G(t)\} = \frac{f(s)}{s^2}$

$$or\ \ \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = G(t) = \int_0^t \int_0^v F(u)du\,dv.$$

The result can be written

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^t F(u)dt^2$$

In general, $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \dots \int_0^t F(t)dt^n.$◄

Example 11. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Solution. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, we have by

repeated application of Example 10,

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = 1 - \cos t,$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - \cos u)\,du = t - \sin t,$

$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u - \sin u)\,du = \frac{t^2}{2} + cost - 1.$

Check:

$\mathcal{L}\left\{\frac{t^2}{2} + \cos t - 1\right\} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{1}{s} =$

$$= \frac{s^2+1+s^4-s^2(s^2+1)}{s^2(s^2+1)} = \frac{1}{s^2(s^2+1)}.$$◄

Example 12.

Given that $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}\ t\sin t$.

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Solution.

Method 1. By Theorem 9, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \ \mathcal{L}^{-1}\left\{\frac{1}{s}\ .\frac{s}{(s^2+1)^2}\right\}$$

$$= \int_0^t \frac{1}{2}\ u\sin u\, du = \frac{1}{2}(\sin t - t\cos t).$$

Method 2. By Theorem 8, we have

$$\mathcal{L}^{-1}\left\{\ s.\frac{s}{(s^2+1)^2}\right\} = \ \mathcal{L}^{-1}\left\{\frac{s^2+1-1}{(s^2+1)^2}\right\}$$

$$= \ \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} -$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{d}{dt}\left(\frac{1}{2}\ t\sin t\right) = \frac{1}{2}(t\cos t + \sin t).$$

Then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$= \ \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{2}(t\cos t + \sin t)$$

$$= \frac{1}{2}(\sin t - t\cos t).◄$$

Example 13.

Find $\mathcal{L}^{-1}\left\{\frac{1}{s}\ln\left(1+\frac{1}{s^2}\right)\right\}$.

Solution.

Using Example 7, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\ln\left(1+\frac{1}{s^2}\right)\right\} = \int_0^t \frac{2(1-\cos u)}{u}\,du$$

$= 2\int_0^t \frac{1-\cos u}{u}\,du.$◀

9. The Convolution property.

Theorem 10.

If $\mathcal{L}^{-1}\{f(s)\} = F(t)and\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = F * G.$$

We call $F * G$ the convolution or fatling of $F$and G, and the theorem is called the convolution theorem or property.

Proof.

Method 1. The required result follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)G(t-u)du\right\} = f(s)g(s) \quad (*)$$

where $f(s) = \mathcal{L}\{F(t)\}, g(s) = \mathcal{L}\{G(t)\}$.

To show this we note

That the left side of (*) is

$$\int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} F(u)G(t-u)du\right\} dt$$

$= \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}F(u)G(t-u)du\,dt = \lim_{M\to\infty} s_M$, where

$$s_M = \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}\,F(u)\,G(t-u)du\,dt \quad (**)$$

The region in the tu plane over which the integration (**) is performed is shown shaded in Fig. 1.

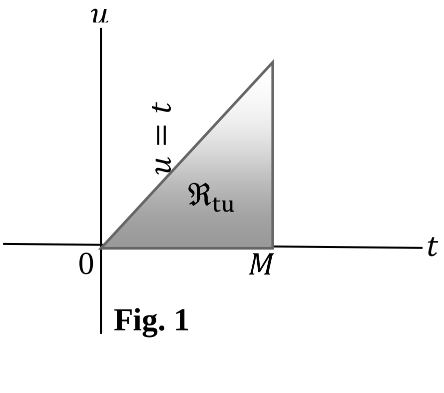


Fig. 1

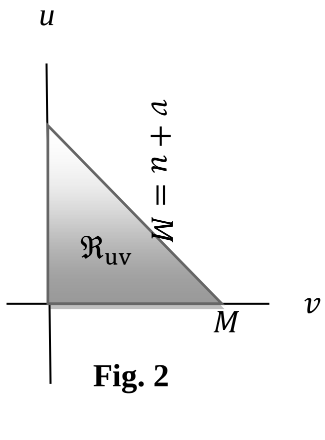


Fig. 2

Letting $t - u = v$ or $t = u + v$, the shaded region $\Re_{tu}$ of the tu plane is transformed into the shaded region $\Re_{uv}$ of the uv plane shown in Fig. 2. Then by a theorem on transformation of multiple integrals, we have

$$s_M = \iint\limits_{\Re_{tu}} e^{-st} F(u)G(t-u)du\,dt$$

$$= \iint\limits_{\Re_{uv}} e^{-s(u+v)}\, F(u)G(v) \left|\frac{\partial(u,t)}{\partial(u,v)}\right| du\,dv\,,$$

$$(***)$$

where the Jacobean of the transformation is.

$$J = \left|\frac{\partial(u,t)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus, the right side of (***) is

$$s_M = \int_{v=0}^{M} \int_{u=0}^{M-v} e^{-s(u+v)}\, F(u)G(v)du\,dv \qquad (\$)$$

Let us define a new function

$$K(u,v) = \begin{cases} e^{-s(u+v)}\, F(u)G(v), & \text{if } u+v \le M \\ 0 & \text{, if } u+v > M \end{cases} \quad (\$\$)$$

This function is defined over the square of Fig. 3 but, as indicated in ($$), is zero over the unshaded portion of the square. In terms of this new function we can write ($) as $s_M = \int_{v=0}^{M} \int_{u=0}^{M} K(u,v)du\, dv$. Then

$$\lim_{M\to\infty} s_M = \int_{v=0}^{\infty} \int_{u=0}^{\infty} K(u,v)du\, dv$$

$$= \int_0^{\infty} \int_0^{\infty} e^{-s(u+v)}\, F(u)G(v)du\, dv$$

$$= \left\{\int_0^{\infty} e^{-su}\, F(u)du\right\}\left\{\int_0^{\infty} e^{-sv}\, G(v)dv\right\}$$

$$= f(s)g(s).$$

Which establishes the theorem.

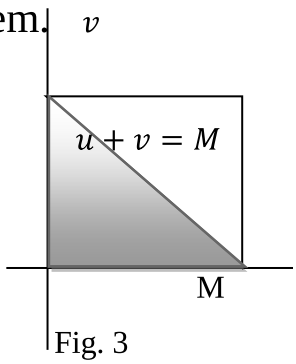


Fig. 3

We call $\int_0^t F(u)G(t-u)du = F * G$ the convolution integral or, briefly, convolution of *F* and *G*.◀

Example 14. Prove that $F * G = G * F$.

Solution.

Letting $t - u = v$ or $u = t - v$, we have

$$F * G = \int_0^t F(u)\, G(t-u)du$$

$$= \int_0^t F(t-v)\, G\,(v)dv$$

$$= \int_0^t G(v)\, F(t-v)dv = G * F$$

This shows that the convolution of *F* and *G* obeys the commutative law of algebra. It also obeys the associative law and distributive law [left as exercise].◀

Example 15.

Evaluate each of the following by use of the convolution Theorem:

$(a)\ \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$, $\qquad (b)\ \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

Solution.

(a) We can write $\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$.

Then since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{\sin at}{a}$,we have by the convolution theorem,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \,.\frac{\sin a\ (t-u)}{a}\ du$$

$$= \frac{1}{a}\int_0^t (\cos au)(\sin at \cos au - \cos at \sin au)\ du$$

$$= \frac{1}{a}\sin at \int_0^t \cos^2 au\ du$$

$$-\frac{1}{a}\cos at \int_0^t \sin au \cos au\ du$$

$$= \frac{1}{a}\sin at \int_0^t \left(\frac{1+\cos 2au}{2}\right)\ du$$

$$-\frac{1}{a}\cos at \int_0^t \frac{\sin 2au}{2}\ du$$

$$= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at\ \left(\frac{1-\cos 2at}{4a}\right)$$

$= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right)-\frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right)$

$= \frac{t\ \sin at}{2a}.$

(b) We have $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t,\ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} =$

te^{-t}Then by the convolution theorem,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t (ue^{-u})(t-u)\,du$$

$$= \int_0^t (ut-u^2)e^{-u}\,du$$

$= (ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) +$

$(-2)(-e^{-u})|_0^t\ = te^{-t} + 2e^{-t} + t - 2.$

Check:

$\mathcal{L}\left\{te^{-t} + 2e^{-t} + t - 2\right\} = \frac{1}{(s+1)^2} + \frac{2}{s+1} + \frac{1}{s^2} - \frac{2}{s}$

$= \frac{s^2+2s^2(s+1)+(s+1)^2-2s(s+1)^2}{s^2(s+1)^2} = \frac{1}{s^2(s+1)^2}.$ ◀

Example 16.

Show that $\int_0^t \int_0^v F(u)du\,dv = \int_0^t (t-u)\,F(u)du.$

Solution.

By the convolution theorem, if $f(s) = \mathcal{L}\left\{F(t)\right\},$

we have

$$\mathcal{L}\left\{\int_0^t (t-u)\, F(u)du\right\} = \mathcal{L}\,\{t\}\,\mathcal{L}\,\{F(t)\} = \frac{f(s)}{s^2}$$

Then by Example 10,

$\int_0^t (t-u)\, F(u)du = \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} =$

$\int_0^t \int_0^v F(u)du\, dv.$

The result can be written

$\int_0^t \int_0^t F(t)dt^2 = \int_0^t (t-u)F(u)du.$

In general, we can prove that [Left as exercise],

$\int_0^t \int_0^t \dots \int_0^t F(t)dt^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!}\, F(u)du.$◄

2. Methods of finding inverse Laplace transforms

1. Partial fractions method.

Any rational function $P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials, with the degree of $P(s)$ less than that of$Q(s)$, can be written as the sum of rational functions

having the form $\frac{A}{(as+b)^r}, \frac{As+B}{(as^2+bs+c)^r}$, where $r =$ 1,2, ... By finding the inverse Laplace transform of each of the partial fractions, we can find $\mathcal{L}^{-1}\{P(s)/Q(s)\}$.

Example 1.

$$\frac{2s-5}{(3s-4)(2s+1)^3} = \frac{A}{3s-4} + \frac{B}{(2s+1)^3} + \frac{C}{(2s+1)^2} + \frac{D}{2s+1}$$

Example 2.

$$\frac{3s^2-4s+2}{(s^2+2s+4)^2(s-5)} = \frac{As+B}{(s^2+2s+4)^2} + \frac{Cs+D}{s^2+2s+4} + \frac{E}{s-5}$$

The constants A, B, C,etccan be found by clearing of fractions and equating of like powers of s on both sides of the resulting equation or by using special methods [see Examples 3-7].

Example 3. Find $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$.

Method 1. $\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

Multiplying by $(s-3)(s+1)$, we obtain.

$3s + 7 = A\,(s+1) + B(s-3) = (A+B)s + A - 3B.$

Equating coefficients, $A + B = 3$ and $A - 3B = 7$; then $A = 4$, $B = -1$,

$$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$$

and $\mathcal{L}^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$$= 4e^{3t} - e^{-t}$$

Method 2. Multiply both sides of (1) by $s-3$ and let $s \to 3$. Then

$\lim_{s\to 3}\frac{3s+7}{s+1} = A + \lim_{s\to 3}\frac{B(s-3)}{s+1}$ or $A = 4$

Similarly multiplying both sides of (1) by $s+1$ and letting $s \to -1$, we have

$\lim_{s\to -1}\frac{3s+7}{s-3} = \lim_{s\to -1}\frac{A(s+1)}{s-3} + B$ or $B = -1$

Using these values we obtain the result in Method 1◄

Example 4.

Find $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

We have $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3} \quad (*)$.

Let us use the procedure of Method 2, Problem 3.

Multiply both sides of (*) by $s+1$ and let $s \to -1$;then $A = \lim_{s\to -1} \frac{2s^2-4}{(s-2)(s-3)} = -\frac{1}{6}$.

Multiply both sides of (*) by $s+2$ and let $s \to -2$; then $B = \lim_{s\to -2} \frac{2s^2-4}{(s+1)(s-3)} = -\frac{4}{3}$.

Multiply both sides of (*) by $s+3$ and let $s \to -3$; then $C = \lim_{s\to -3} \frac{2s^2-4}{(s+1)(s-2)} = -\frac{7}{2}$.

Thus

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{6}}{s+1} + \frac{-\frac{4}{3}}{s-2} + \frac{\frac{7}{2}}{s-3}\right\}$$

$$= \frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}.$$

The procedure of Method 1, Example 3, can also be used. However, it will be noted that the present method is less tedious. It can be used whenever the denominator has distinct linear factors. ◄

Example 5.

Find $\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}$$
$$=\frac{A}{s+1}+\frac{B}{(s-2)^3}+\frac{C}{(s-2)^2}$$
$$+\frac{D}{s-2} \qquad (1)$$

Multiply both sides of (1) by $s+1$ and let $s\to -1$;then$A=\lim_{s\to-1}\frac{5s^2-15s-11}{(s-2)^3}=-\frac{-1}{3}$.

Multiply both sides of (1) by $(s-2)^3$ and let $s\to 2$;then$B=\lim_{s\to2}\frac{5s^2-15s-11}{s+1}=-7$.

The method fails to determine C and D. However, since we know A and B, we have from (1),

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}$$
$$=\frac{-1/3}{s+1}+\frac{-7}{(s-2)^3}+\frac{C}{(s-2)^2}$$
$$+\frac{D}{s-2} \qquad (2)$$

To determine C and D we can substitute two values for s, says$s=0$ and $s=1$,we find respectively,

$$\frac{11}{8}=-\frac{1}{3}+\frac{7}{8}+\frac{C}{4}-\frac{D}{2}, \quad \frac{21}{2}=-\frac{1}{6}+7+C-D$$

i.e.$3C-6D=10$ and $3C-3D=11$, from which $C=4, D=1/3$. Thus

$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{-1/3}{s+1}+\frac{-7}{(s-2)^3}+\frac{4}{(s-2)^2}+\frac{1/3}{s-2}\right\}$$
$$=-\frac{1}{3}e^{-t}-\frac{7}{2}t^2e^{2t}+4t\,e^{2t}+\frac{1}{3}e^{2t}$$

Another method.

On multiplying both sides of (2) by s and letting $s\to\infty$, we find $0=-\frac{1}{3}+D$ which gives $D=\frac{1}{3}$

Then C can be found as above by letting $s=0$.

This method can be used when we have some repeated linear factors.◄

Example 6.

Find $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$.

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

Multiply both sides by $s-1$ and let $s \to 1$; then

$A = \lim_{s\to 1}\frac{3s+1}{s^2+1} = 2and\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{Bs+C}{s^2+1}$

To determine *B* and *C*, let $s = 0$ and 2 (for example); then $-1 = -2 + C,\ \ \frac{7}{5} = 2 + \frac{2B+C}{5}$from which $C = 1$ and $B = -2$. Thus we have

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \left\{\frac{2}{s-1} + \frac{2s+1}{s^2+1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= 2e^t - 2\ \cos t + \sin t$$

Another method.

Multiplying both sides of (2) by s and lettings$s \to \infty$, we find at once that $B = -2$.

Example 7.

Find $\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

Method 1.

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$
$$=\frac{As+B}{s^2+2s+2}+\frac{Cs+D}{s^2+2s+5}$$

Multiplying by $(s^2+2s+2)(s^2+2s+5)$.

$$s^2+2s+3$$
$$=(As+B)(s^2+2s+5)$$
$$+(Cs+D)(s^2+2s+2)$$
$$=(A+C)s^3+(2A+B+2C+D)s^2$$
$$+(5A+2B+2C+2D)+5B$$
$$+2D$$

Then $A+C=0,\ 2A+B+2C+D=1,$

$5A+2B+2C+2D=2, 5B+2D=3.$

Solving, $A=0, B=\frac{1}{3}\ ,\ C=0,\ D=\frac{2}{3}.$ Thus

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}=\mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2}+\frac{2/3}{s^2+2s+5}\right\}$$
$$=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$= \frac{1}{3} e^{-1} \sin t + \frac{2}{3} \cdot \frac{1}{2} e^{-1} \sin 2t$

$= \frac{1}{3} e^{-1} (\sin t + \sin 2t).$

Method 2. Let $s = 0$ in (1): $\frac{3}{10} = \frac{B}{2} + \frac{D}{5}$

Multiply (1) by s and let $s \to \infty$: $0 = A + C$

Let $s = 1$: $\frac{3}{20} = \frac{A+B}{5} + \frac{C+D}{8}$

Let $s = 1$: $\frac{1}{2} = -A + B + \frac{C+D}{8}$

Solving $A = 0,\ B = \frac{1}{3}\ ,\ C = 0,\ D = \frac{2}{3}$ as in Method 1. This illustrates the case of non-repeated quadratic factors.

Method 3. Since $s^2 + 2s + 2 = 0$ for $s = -1 \pm i$, we can write $s^2 + 2s + 2 = (s + 1 - i)(s + 1 + i)$. Similarly,

$$s^2 + 2s + 5 = (s + 1 - 2i)(s + 1 + 2i)$$

Then

$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

$= \frac{s^2+2s+3}{(s+1-i)(s+1+i)(s+1-2i)(s+1+2i)}$

$= \frac{A}{s+1-i} + \frac{B}{s+1+i} + \frac{C}{s+1-2i} + \frac{D}{s+1+2i}$

Solving for A, B, C, D, we find $A = \frac{1}{6i}$, $B = -\frac{1}{6i}$, $C = \frac{1}{6i}, D = -1/6i$.

Thus, the required inverse Laplace transform is

$$\frac{e^{-(1-i)t}}{6i} - \frac{e^{-(1+i)t}}{6i} + \frac{e^{-(1-2i)t}}{6i} - \frac{e^{-(1+2i)t}}{6i}$$

$$= \frac{1}{3}e^{-t}\left(\frac{e^{it}-e^{-it}}{2i}\right) + \frac{1}{3}e^{-t}\left(\frac{e^{2it}-e^{-2it}}{2i}\right)$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$

$$= \frac{1}{3}e^{-t}(\sin t + \sin 2t).$$

This shows that the case of non-repeated quadratic factors can be reduced to non-repeated linear factors using complex numbers.◄

2. Series methods.

If $f(s)$ has a series expansion in inverse powers of s given by

$$f(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \cdots, \quad (1)$$

then under suitable conditions we can invert term by term to obtain.

$$F(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \cdots \quad (2)$$

Series expansions other than those of the form (1) can sometimes be used.

3. Method of differential equations.

4.Differentiation with respect to a parameter.

5. Miscellaneous methods using the above theorems.

6. Use of Tables.

7.The Complex inversion formula.

This formula, which supplies a powerful direct method for finding inverse Laplace transforms, uses complex variable theory and is out of our consideration.

►The Heaviside expansion formula

Let $P(s)$ and $Q(s)$ be polynomials where $P(s)$ has degreeLess than that of $Q(s)$.Suppose that $Q(s)$ has n distinct zeros$a_k, k = 1,2,3, \dots, n$.Then

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(a_k)}{Q\prime(a_k)} e^{a_k t}.$$

This is often called Heaviside's expansion theorem or formula

Proof.

Since $Q(s)$ is a polynomial with n distinct zeros $a_1, a_2, \dots, a_n$,we can write according to the method of partial fractions,

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_k}{s-a_k} + \cdots + \frac{A_n}{s-a_n}$$

Multiplying both sides by $s - a_k$ and letting $s \to a_k$, we find using L' Hospital's rule,

$$A_k = \lim_{s \to a_k} \frac{P(s)}{Q(s)}(s-a_k) = \lim_{s \to a_k} P(s)\left\{\frac{s-a_k}{Q(s)}\right\}$$

$$= \lim_{s\to a_k} P(s) \lim_{s\to a_k} \left(\frac{s-a_k}{Q(s)}\right) = P(a_k) \lim_{s\to a_k} \frac{1}{Q\prime(s)} = \frac{P(a_k)}{Q(a_k)}.$$

Thus $\frac{P(s)}{Q(s)}$ can be written

$$\frac{P(s)}{Q(s)} = \frac{P(a_1)}{Q'(a_1)}\frac{1}{s-a_1} + \cdots + \frac{P(a_k)}{Q'(a_k)}\frac{1}{s-a_k} + \cdots + \frac{P(a_n)}{Q'(a_n)}\frac{1}{s-a_n}$$

Then taking the inverse Laplace transform, we have as required

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(a_1)}{Q\prime(a_1)} e^{a_1 t} + \cdots + \frac{P(a_k)}{Q\prime(a_k)} e^{a_k t} + \cdots + \frac{P(a_n)}{Q\prime(a_n)} e^{a_n t} = \sum_{k=1}^{n} \frac{P(a_k)}{Q\prime(a_k)} e^{a_k t}. \blacktriangleleft$$

Example 8.

Find $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

Solution.

We have

$$P(s) = 2s^2 - 4, \qquad Q(s) = (s+1)(s-2)(s-3)$$

$$= s^2 - 4s^2 + s + 6,$$

$$Q'(s) = 3s^2 - 8s + 1$$

And$a_1 = -1, a_2 = 2, a_3 = 3$.

Then the required inverse is by Heaviside's expansion theorem,

$$\frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{2t}$$

$$= \frac{-2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t}$$

$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$.◀

Example 9.

Find $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$.

Solution.

We have $P(s) = 3s + 1, Q(s) = (s-1)(s^2+1)$
$= s^3 - s^2 + s - 1, Q'(s) = 3s^2 - 2s + 1$,and
$a_1 = 1, a_2 = i, a_3 = -i$, since $s^2 + 1 =$
$(s-i)(s+i)$. Then by the Heaviside expansion formula the required inverse is

$$\frac{P(1)}{Q'(1)}e^{t} + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{it}$$

$$= \frac{4}{2}e^{t} + \frac{3i+1}{-2-2i}e^{it} + \frac{3i+1}{-2+2i}e^{-it}$$

$$= 2e^t + \left(-1 - \frac{1}{2}\ i\right)(\cos t + i\ \sin t) +$$

$$(-1 + \frac{1}{2}\ i)(\cos t + i\ \sin t)$$

$$= 2e^t - \cos t + \frac{1}{2}\sin t - \cos t + \frac{1}{2}\sin t$$

$= 2e^t - 2\ \cos t + \sin t.$◄

Note that some labor can be saved by observing that the last two Terms in (1) are complex conjugates of each other.

►The Beta Function

If $m > 0,\ n > 0,$ we define the beta function as

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\, du$$

Example 10.

Prove that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$,where $m > 0,$ $n > 0.$

Proof.

Consider $G(t) = \int_0^t x^{m-1}(1-x)^{n-1}\, dx$

Then by the convolution theorem, we have

$\mathcal{L}\,\{G(t)\} = \mathcal{L}\,\{t^{m-1}\}\,\mathcal{L}\,\{t^{n-1}\}$

$$= \frac{\Gamma(m)}{s^m} \cdot \frac{\Gamma(n)}{s^n} = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

Thus $G(t) = \mathcal{L}^{-1}\left\{\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\right\} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} t^{m+n-1}$

or $\int_0^t x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}} t^{m+n-1}$

Letting $t = 1$, we obtain the required result. ◄

Example 11.

Prove that:

$\int_0^{\pi/2} \sin^{2\mathrm{m}-1}\theta \cos\theta \ d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$.

Proof.

From Example 10, we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Letting $x = \sin^2\theta,$ this becomes.

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2\mathrm{m}-1}\theta \cos^{2\mathrm{n}-1}\theta \ d\theta$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

from which the required result follows.◄

Example 12.

Evaluate

$(a)\ \int_0^{\pi/2} \sin^4\theta \cos^6\theta\ d\theta,\qquad (b)\ \int_0^{\pi} \cos^4\theta\ d\theta$

$(c)\ \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}}.$

Solution.

(a) Let $2m-1=4,\ 2n-1=6$ in Example 11.

Then $m=5/2,\ n=7/2,$ and we have

$$\int_0^{\pi/2} \sin^4\theta \cos^6\theta\ d\theta = \frac{\Gamma(5/2)\Gamma(7/2)}{2\Gamma(6)}$$

$$= \frac{(3/2)(1/2)\sqrt{\pi}\,.(5/2)(3/2)(1/2)\sqrt{\pi}}{2\,.5\,.4\,.3\,.2\,.1} = \frac{3\pi}{512}.$$

(b) Since $\cos\theta$ is symmetric about $\theta=\pi/2$, we have $\int_0^{\pi} \cos^4\theta\ d\theta = 2\int_0^{\pi/2} \cos^4\theta\ d\theta$. Then letting $2m-1=0$ and $2n-1=4$, i.e $m=1/2$

and$n=5/2$ in Example 11, we find.

$$\int_0^{\pi/2} \cos^4\theta\ d\theta = 2\left[\frac{\Gamma(1/2)\Gamma(5/2)}{2\Gamma(3)}\right]$$

$$= 2\left[\frac{\sqrt{\pi}\,.(3/2)(1/2)\sqrt{\pi}}{2\,.2\,.1}\right] = \frac{3\pi}{8}$$

$(c)\ \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\ d\theta.$

Letting $2m-1=-1/2$ and $2n-1=1/2$, or $m=1/4$and $n=3/4$in Example 11, we find

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} = \frac{\Gamma(1/4)\Gamma(3/4)}{2\Gamma(1)} = \frac{1}{2}\frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{2}.$$

Using the result $\Gamma(p)\Gamma(1-p)=\pi/(\sin p\pi)$, $0<p<1$.

►**Evaluation of integrals**

The Laplace transformation is often useful in evaluating definite integrals. See the following two examples.

Example 13.

Evaluate $\int_0^t J_0(u)J_0(t-u)du.$

Solution.

Let $G(t)=\int_0^t J_0(u)J_0(t-u)du$.Then by the convolution theorem

$\mathcal{L}\{G(t)\} = \mathcal{L}\{J_0(t)\}\mathcal{L}\{J_0(t)\}$

$$= \left(\frac{1}{\sqrt{s^2+1}}\right)\left(\frac{1}{\sqrt{s^2+1}}\right) = \frac{1}{s^2+1}$$

Hence $\quad G(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

and so $\quad G(t) = \int_0^t J_0(u)J_0(t-u)du = \sin t.$ ◀

Example 14.

Show that $\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$.

Solution.

Let $\quad G(t) = \int_0^\infty \cos tx^2\, dx.$ Then taking the Laplace transform, we find.

$$\mathcal{L}\,\{G(t)\} = \int_0^\infty e^{-st}dt \int_0^\infty \cos tx^2\, dx$$

$$= \int_0^\infty dx \int_0^\infty e^{-st} \cos tx^2\, dt$$

$$= \int_0^\infty \mathcal{L}\,\{\cos tx^2\}\, dx = \int_0^\infty \frac{s}{s^2+x^4}\, dx$$

Letting $x^2 = s\tan\theta$ or $x = \sqrt{s}\sqrt{\tan\theta},$ this integral becomes onusing Example 12 (c),

$$\frac{1}{2\sqrt{s}}\int_0^{\pi/2} (\tan\theta)^{-1/2}\, d\theta = \frac{1}{2\sqrt{s}}\left(\frac{\pi\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

Inverting, we find

$$G(t) = \int_0^\infty \cos tx^2\, dx = \frac{\pi\sqrt{2}}{4}\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s}}\right)$$

$$= \left(\frac{\pi\sqrt{2}}{4}\right)\left(\frac{t^{-1/2}}{\sqrt{\pi}}\right) = \frac{\sqrt{2\pi}}{4}t^{-1/2}$$

Letting $t = 1$ we have, as required,

$\int_0^\infty \cos tx^2\, dx = \frac{\sqrt{2\pi}}{4} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.◄

MISCELLANEOUS PROBLEMS

1. Determine each of the following:

$(a)\mathcal{L}^{-1}\left\{\frac{3}{s+4}\right\}$ $(c)\mathcal{L}^{-1}\left\{\frac{8s}{s^2+16}\right\}$

$(e)\mathcal{L}^{-1}\left\{\frac{3s-12}{s^2+8}\right\}$ (g) $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$(i)\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$ $(b)\mathcal{L}^{-1}\left\{\frac{1}{2s-5}\right\}$

$(d)\mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\}$ $(f)\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-9}\right\}$

$(h)\mathcal{L}^{-1}\left\{\frac{1}{s^{7/2}}\right\}$ $(i)\mathcal{L}^{-1}\left\{\frac{s+1}{s^{4/3}}\right\}$

2. Find $(a)\mathcal{L}^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\}$, $(b)\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

3. Find$(a)\mathcal{L}^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$, $(b)\mathcal{L}^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}$

4. (a) Shown that the functions $F(t) = \begin{cases} t & t \neq 3 \\ 5 & t = 3 \end{cases}$

and $G(t) = t$have the same Laplace transforms.

(b) Discuss the significance of the result in (a) as far as uniqueness of inverse Laplace transforms is concerned.

5. Find $(a)\mathcal{L}^{-1}\left\{\frac{3s-8}{s^2+4}-\frac{4s-24}{s^2-16}\right\}$, $(b)\left\{\frac{3s-2}{s^{5/2}}-\frac{7}{3s+2}\right\}$

6. $(a)\ If\ \ F_1(t)=\mathcal{L}^{-1}\{f_1(s)\},\ F_2(t)=\mathcal{L}^{-1}\{f_2(s)\}, F_3(t)=\mathcal{L}^{-1}\{f_3(s)\},$

and c_1, c_2, c_3 are any constants, prove that:

$$\mathcal{L}^{-1}\{\, c_1f_1(s)+c_2f_2(s)+c_3f_3(s)\}$$
$$= c_1F_1(t)+c_2F_2(t)+c_3F_3(t)$$

Stating any restrictions.

(b) Generalize the result of part (a) to*n* functions.

7. Find $\mathcal{L}^{-1}\left\{\frac{3(s^2-1)}{5s^5}+\frac{4s-18}{9-s^2}+\frac{(s+1)(2-s^{1/2})}{s^{5/2}}\right\}$

8. Find $(a)\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^5}\right\}$. $(b)\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\}$

9. Find $(a)\mathcal{L}^{-1}\left\{\frac{3s-14}{s^2-4s+8}\right\}$, $(b)\mathcal{L}^{-1}\left\{\frac{8s+20}{s^2-12s+32}\right\}$.

10. Find $(a)\mathcal{L}^{-1}\left\{\frac{3s-2}{4s^2+12s+9}\right\}$, $(b)\mathcal{L}^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}$

11. Find $(a)\mathcal{L}^{-1}\left\{\frac{1}{\sqrt[3]{8s-27}}\right\}$, $(b)\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2-4s+20}}\right\}$

12. Find

$(a)\ \mathcal{L}^{-1}\left\{\frac{e^{2s}}{s^2}\right\},\ \ (b)\ \mathcal{L}^{-1}\left\{\frac{8e^{-3s}}{s^2+4}\right\},\ \ (c)\ \mathcal{L}^{-1}\left\{\frac{e^{-s}}{\sqrt{s+1}}\right\}$

13. Find $\mathcal{L}^{-1}\left\{\frac{8e^{-3s}}{s^2+3s+2}\right\},\ \ (b)\ \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+2s+5}\right\}$

14. Use Theorem 6, to find

$$(a)\mathcal{L}^{-1}\{1/(s-a)^2\}$$

When $\mathcal{L}^{-1}\{1/(s-a)^2\} = e^{at}$

$$(b)\mathcal{L}^{-1}\{s/(s^2-a^2)^2\}$$

When $\mathcal{L}^{-1}\{1/(s^2-a^2)^2\} = (\sinh \text{at})/\text{a}$

15. Use the fact that $\mathcal{L}^{-1}\{1/s\} = 1$ to find

$\mathcal{L}^{-1}\{1/s^n\}$ where$n = 2,3,4, ...$

Thus find $\mathcal{L}^{-1}\{1/(s-a)^n\}$.

16. Find $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$.

17. Prove $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^v \int_0^w F(u)du\, dv\, dw$.

Can the integral be written as $\int_0^t \int_0^t \int_0^t F(t)dt^3$?

Explain.

18. Evaluate $(a)\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\},\ (b)\mathcal{L}^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\},$

$(c)\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^3}\right\}$

19. Find $(a)\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+4}}\right\}$, $(b)\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s^2+a^2}}\right\}$

20. Use the convolution theorem to find

$$(a)\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\},\quad (b)\ \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$$

21. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$.

22. Find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$.

23. Use partial fractions to find

$$(a)\ \mathcal{L}^{-1}\left\{\frac{3s+16}{s^2-s-6}\right\},\qquad (b)\ \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-s}\right\}$$

24. Find

$(a)\ \mathcal{L}^{-1}\left\{\frac{s+1}{6s^2+7s+2}\right\}$, $(b)\ \mathcal{L}^{-1}\left\{\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}\right\}$.

25. Find $(a)\ \mathcal{L}^{-1}\left\{\frac{27-12s}{(s+4)(s^2+9)}\right\}$,

$(b)\ \mathcal{L}^{-1}\left\{\frac{s^3+16s-24}{s^4+20s^2+64}\right\}$.

26. Using Heaviside's expansion formula find

$$(a)\ \mathcal{L}^{-1}\left\{\frac{2s-11}{(s+2)(s-3)}\right\},$$

$$(b)\ \mathcal{L}^{-1}\left\{\frac{19s+37}{(s-2)(s+1)(s+3)}\right\}$$

27. Find $\mathcal{L}^{-1}\left\{\frac{2s^3-6s+5}{s^3-6s^2+11s-6}\right\}$.

28. Find $\mathcal{L}^{-1}\left\{\frac{s+5}{(s+1)(s^2+1)}\right\}$.

29. Evaluate each of the following:

$$(a)\int_0^1 x^{3/2}(1-x)^2 dx, (b)\int_0^1 x^3(4-x)^{-\frac{1}{2}} dx,$$

$(c)\int_0^2 y^4\sqrt{4-y^2}\, dy$.

30. Show that $\int_0^1 \sqrt{1-x^2}\, dx = \pi/4$.

31. Evaluate each of the following;

(a) $\int_0^{\pi/2} \cos^6 d\theta$,

$$(b)\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta,$$

$$(c)\int_0^{\pi} \sin^4\theta\cos^4\theta\, d\theta$$

32. Show that $\int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$

33. Evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$.

34. Show that $\int_0^\infty x\cos x^3\, dx = \frac{\pi}{3\sqrt{3}\Gamma(\frac{1}{3})}$.

Chapter VI

The z-Transform and the Difference Equations

CHAPTER VI

THE z–TRANSFORM AND THE DIFFERENCE EQUATIONS

z-Transform is one of the mathematical tools used for the analysis and design of discrete-time control systems. The role of the z-transform in digital control systems is analogous to that of the Laplace transform in the continuous-time control systems. Just like a linear ordinary differential equation characterizes the dynamics of a linear time invariant system, a constant coefficient difference equation characterizes the dynamics of a linear discrete-time system. In order to determine the response of a discrete-time system to a given input, a difference equation must be solved. Just as the Laplace transformation transforms linear time invariant differential equations into algebraic equations of the Laplace variable s, the z-transformation transforms the constant coefficient difference equations into algebraic equations of the z-transform variable z. By using the z-transformation, a linear discrete-time system may be represented by a transfer function called the pulse transfer function. The z-transform of the output signal can then be expressed as the product between the system's pulse transfer function and the z-transform of the input signal. The main objective of this chapter is to introduce the z-transform and the associated properties that are often used in analyzing and synthesizing discrete-time controllers. The pulse transfer function will also be discussed.

For simplicity, for the remainder of this chapter we will use $x(k)$or $x_k$to represent $x(kT)$ and will assume that the signal (sequence) has zero values for $k < 0$, i.e.

$x(k) = 0$ for $k < 0$.

1. Difference Equations

In digital control, we are concerned with the generation of a sequence of control inputs $u(k)$ given a sampled sequence of the system output $y(k)$. In general, we would like to determine the control effort at sample instance k base on the sampled system output at sample instance k and a finite number of previous sampled outputs and control efforts. Mathematically this can be written as

$$u(k) = f[y(k), y(k-1), y(k-2), \dots, y(k-m), u(k-1), u(k-2), \dots, u(k-n)].$$

There are an infinite number of ways the $n + m - 1$ values on the right-hand side of the above equation can be combined to form $u(k)$. In this course, we shall only focus on the cases where the right-hand side of the above equation involves a linear combination of the past samples of the measurements and controls, i.e.

$$u(k) = a_m . y(k) + a_{m-1} . y(k-1) + a_{m-2} . y(k-2) + \cdots$$
$$+ a_0 . y(k-m)$$
$$+ b_{n-1} . u(k-1) + b_{n-2} . u(k-2) + \cdots$$
$$+ b_0 . u(k-n) \qquad (1)$$

Equation (1) is a linear difference equation. If the coefficients a_i and b_j are constants (independent of sample instance k), it is a constant coefficient difference equation.

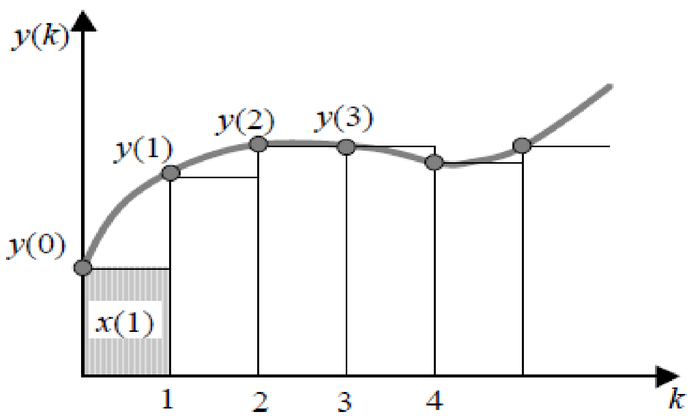
Example 1

(Numerical Analysis and Difference Equation)

In numerical analysis, there are many methods to approximate the integral of a function.

$$x(t) = \int_0^t y(\tau) d\tau$$

The backward rectangular rule is one of the simplest. It uses a rectangular approximation to approximate the area under the function $y(t)$ as shown in the following figure.



Let $x(k)$ be the value of the integral at sample instance k, while $x(k-1)$ is the value of the integral at the previous sample instance. Then, $x(k)$ can be calculated using

$$x(k) = x(k-1) + T\,.\,y(k-1)(2)$$

The above equation provided an algorithm to successively approximate the integral of the function $y(t)$. This algorithm takes the form of a constant coefficient difference equation.

Example 2.

(Solving Difference Equation)

Solving a constant coefficient difference equation often involve tedious algebraic manipulations. To find the solution for Eq (2), i.e. find an expression for solving $x(k)$given current and previous values of $y(k)$, first write out Eq. (2) for $k = 1, 2, ...$

$$x(k) = x(k-1) + T\,.\,y(k-1)$$

$$x(k-1) = x(k-2) + T\,.\,y(k-2)$$

$$x(k-2) = x(k-3) + T\,.\,y(k-3)$$

$$\vdots$$

$$x(2) = x(1) + T \, . \, y(1)$$

$$x(1) = x(0) + T \, . \, y(0)$$

Adding the above k equations and noting that many $x(k)$s can be cancelled, we can obtain.

$$x(k) = x(0) + T \, . \sum_{j=0}^{k-1} y(j) \qquad (3)$$

Equation (3) represented the solution to the difference equation (2). The solution for a constant coefficient difference equation requires the initial value of the solution as well as all the past values of the input, in this case,

$$y(k).$$

Not all constant coefficient difference equations can be solved using the manipulation illustrated in Example 2. A more general, systematic approach is needed.

2. The z- Transform

The Laplace transform is used as an analysis tools for continuous-time linear time invariant systems. The reasons are two folds,

(1) The input/output relationship for continuous-time systems, a convolution operation in the time domain, is simplified to an algebraic relationship in the s- domain.

(2) The Laplace transform relates the time domain and frequency domain characteristic of the system.

The discrete-time analogy of the Laplace transform is the z-transform, which transforms a discrete sequence of time domain signal into a function of the z-transform variable z.

Definition1.(z-Transform)

The z-transform of a sampled sequence $x(kT)$ or $x(k)$, where k represents non-negative integers and T is the sampling period, is defined by

$$x(z) = Z[x^*(t)] = Z[x(kT)] = Z[x(k)]$$

$$= \sum_{k=0}^{\infty} x(kT)Z^{-k} = \sum_{k=0}^{\infty} x(k)Z^{-k} \quad (4)$$

where the complex variable z must be selected so that the summation converges. This z-transform is called the one-sided z-transform. The symbol Z[•] denotes "the z- transform, we assume $x^*(t) = 0$ for $t < 0$ or $x(kT) = x(k) = 0$ for $k < 0$.
For most engineering applications, the one-sided z-transform will have a convenient closed form representation in its region of convergence. From Eq. (3), $X(z)$ is an infinite series of z^{-1}that converges outside the circle $|z| = R$, where R is called the radius of convergence.
How to calculate the z-transform of several common functions directly from the definition?

Example 3. (Unit Step Function(Sequence))
Consider the unit step function defined by

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

The z-transform of $u(k)$is

$$U(z) = Z[u(k)]$$

$$= \sum_{k=0}^{\infty} 1 \,.\, z^{-k} = 1 + z^{-1} + z^{-2} + \cdots$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

for $|z^{-1}| < 1(\text{or}|z| > 1)$.◄

Example 4. (Polynomial Function)

Consider a polynomial function defined by

$$x(k) = \begin{cases} a^k, & k \geq 0 \\ 0\,, & k < 0 \end{cases}$$

The z-transform of $x(k)$is

$$x(z) = Z[x(k)] = \sum_{k=0}^{\infty} a^k \,.\, z^{-k} = \sum_{k=0}^{\infty} (a^{-1}z)^{-k}$$

$$= \frac{1}{1 - (a^{-1}z)^{-1}} = \frac{z}{z - a}$$

for $|z| > a$

Example 5. (Exponential Function)

Consider the exponential function defined by

$$x(k) = \begin{cases} e^{-akT}\,, & k \geq 0 \\ 0 \quad\,, & k < 0 \end{cases}$$

The z-transform of $x(k)$is

$$X(z) = Z[x(k)] = \sum_{k=0}^{\infty} a^{-akT} \, . z^{-k} = \sum_{k=0}^{\infty} (a^{aT} z)^{-k}$$

$$= \frac{1}{1-(a^{aT}z)^{-1}} = \frac{z}{z-e^{-aT}}$$

for $\quad |z| > e^{-aT}$.

Example 6. (Sinusoidal Function)

Consider the sinusoidal function defined by

$$x(k) = \begin{cases} \sin(\omega kT), & k \geq 0 \\ 0 \qquad\qquad , & k < 0 \end{cases}$$

The z-transform of $x(k)$is

$X(z) = Z[x(k)]$

$$= \sum_{k=0}^{\infty} \sin(\omega kT) \, . z^{-k} = \sum_{k=0}^{\infty} \frac{e^{j\omega kT} - e^{-j\omega kT}}{2j} \, . z^{-k}$$

$$= \frac{1}{2j} \left(\frac{1}{1 - e^{j\omega T} z^{-1}} - \frac{1}{1 - e^{-j\omega T} z^{-1}} \right)$$

$$= \frac{1}{2j} \frac{\left(e^{j\omega kT} - e^{-j\omega kT}\right) z^{-1}}{1 - (e^{j\omega kT} + e^{-j\omega kT}) z^{-1} + z^{-2}}$$

$$= \frac{z^{-1} \sin(\omega T)}{1 - 2z^{-1} \cos(\omega T) + z^{-2}}$$

$$= \frac{z \, . \sin(\omega T)}{z^2 - 2z \, . \cos(\omega T) + 1}$$

for $|z| > 1$.Notice that the expansion of the right-hand side of Eq. (4) gives

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots$$

The above equation implies that the z-transform of any discrete sequence can be written in the infinite series form by inspection. The z^{-k}in this series indicates the position in time at which the amplitude $x(kT)$occurs.Conversely, if $X(z)$is given in the series form as above, the inverse z- transform can be obtainedby inspection as a sequence of values $x(kT)$ that correspond tothe values of $x(t)$at the sampling instances. Just as in workingwith Laplace transformation, we will make extensive use of a table of z- transforms for commonly encountered function, rather than calculating the z-transform from definition.

$f(t),\ t \geq 0$	$F(s)$	$f(kT),\ k \geq 0$	$F(z)$
–	–	$\begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$	1
–	–	$\begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$	z^{-n}
1	$\frac{1}{s}$	1	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{2}t^2$	$\frac{1}{s^3}$	$\frac{1}{2}(kT)^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$(kT)e^{-akT}$	$\frac{T\,e^{-aT}z}{(z-e^{-aT})^2}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-at}-e^{-bt})z}{(z-e^{-aT})(z-e^{-bt})}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega kT)$	$\frac{\sin(\omega t)}{z^2-2\cos(\omega T)z+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega kT)$	$\frac{z^2-\cos(\omega T).z}{z^2-2\cos(\omega T)z+1}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-akT}\sin(\omega kT)$	$\frac{e^{-aT}\sin(\omega T).z}{x^2-2e^{-aT}\cos(\omega T)z+e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-akT}\cos(\omega kT)$	$\frac{z^2-e^{-aT}\cos(\omega T).z}{z^2-2e^{-aT}\cos(\omega T)z+e^{-2aT}}$
—	—	a^k	$\frac{z}{z-a}$
—	—	$k.a^{k-1}$	$\frac{z}{(z-a)^2}$

Table 1 The z-Transform Table

Poles and Zeros

As shown from Example 3 to Example 6, the *z*-transform of a signal often takes the form of a rational function of *z*

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

$$= \frac{b_0(z - z_1)(z - z_2) \ldots (z - z_m)}{(z - p_1)(z - p_2) \ldots (z - p_n)}$$

where the p_i's are the *poles* of $X(z)$and thez_i's are the *zeros* of $X(z)$.The location of the poles and zeros of $X(z)$determines thecharacteristics of $x(k)$, a sequence of values or numbers. As in the case of the *s* plane analysis of continuous-time signals, we often use a graphical representation on the *z* plane of the location of the poles and zeros of $X(z)$.

Note in signal processing and control engineering, $X(z)$is represented as a ratio of polynomials in z^{-1}.

$X(z) = X(z^{-1})$

$= \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_n z^{-2} + \cdots + a_n z^{-n}}.$

Both Eqs. (5) and (6) are valid representations of the same function in the *z* domain. However, in determining the poles and zeros of$X(z)$, it is more convenient to express$X(z)$ as a ratio ofpolynomials in *z*. For example

$$X(z) = \frac{z^2 + 0.5z}{z^2 + 3z + 2} = \frac{z(z + 0.5)}{(z + 1)(z + 2)}$$

Clearly, $X(z)$has poles at $z = -1$ and $z = -2$ and zeros at $z = 0$ and$z = -0.5$.

If$X(z)$ is written as a ration of polynomials inz^{-1},the proceeding $X(z)$can be written as

$X(z) = X(z^{-1})$

$= \frac{1 + 0.5\, z^{-1}}{1 + 3z^{-1} + 2z^{-2}} = \frac{(1 + 0.5\, z^{-1})}{(1 + z^{-1})(1 + 2z^{-1})}.$

From the above representation, although the poles at $z = -1$ and $z = -2$ and the zero $z = -0.5$ are clearly seen, the zeroat$z = 0$ is not explicitly shown, and will often be overlooked. Therefore, when determining the poles and zeros of *z*-

transforms it is preferable to express the z-transform as ratios of polynomials in z, rather than in z^{-1}.

Frequency Response

Recall that the Laplace transform of a sample sequence $x^*(t) = x(t).\delta_T$

Is$X^*(s) = \sum_{k=0}^{\infty} x(kT).e^{-kTs}$

If we introduce a change of variable$z = e^{sT}$, the above equation can be written as

$$X(z) = \sum_{k=0}^{\infty} x(kT).z^{-k}$$

In other words, the z-transform of a time sequence and the Laplace transform of the sampled signal are the same. This is not surprising since both transformations concerns with the signal values at the sample instances. Another implication of the similarity is that for continuous-time systems the frequency response of a system can be analyzed by substituting s with $j\omega$in the Laplace transform. Similarly, to study the frequencyresponse of a discrete-time system, we can substitute $z = e^{j\omega T}$inthe *z*-transform,

i.e.

$$X\left(e^{j\omega T}\right) = \sum_{k=0}^{\infty} x(kT).\left(e^{j\omega T}\right)^{-k}$$

A more formal derivation of the frequency response of discrete-time systems will be discussed in later sections.

Important Properties of the *z*-Transform

a) Linearity

Theorem 1.

z-Transform is a linear transformation which implies that

$$Z[a.x(k)] = a.Z[x(k)z] = a.X(z)$$

and that

$$Z[a.x(k) + b.y(k] = a.Z[x(k)] + b.Z[y(k)]$$
$$= a.X(z) + b.Y(z)$$

where$X(z)$ and $Y(z)$are the *z*-transform of $x(k)$and$y(k)$, respectively.

Proof.

The proof of the property can be easily done by applying the definition of the *z*-transform.◄

b) Time Shift

Theorem 2.

If $x(k) = 0$ for $k < 0$ and $x(k)$ has the *z*-transform $X(z)$, then

$$Z[x(k-d)] = z^{-d}.X(z)$$

and

$$Z[x(k+d)] = z^{d}\left[X(z) - \sum_{j=0}^{d-1} x(j).z^{-j}\right]$$

$$= z^{d}.X(z) - \sum_{i=1}^{d} x(d-i).z^{i}$$

$$= z^{d}X(z) - z^{d}x(0) - z^{d-1}x(1) -$$

$z^{d-2}x(2) - \cdots - z.x(d-1),$

where *d* is a zero or a positive integer.

Proof.

$Z[x(k-d)] = \sum_{k=0}^{\infty} x(k-d).z^{-k}$, let $k - d = j$,

then

$$Z[x(k-d)] = \sum_{k=0}^{\infty} x(k-d).z^{-k}$$

$$= \sum_{j=-d}^{\infty} x(j).z^{-d-j}$$

$$= z^{-d} \sum_{j=-d}^{\infty} x(j).z^{-j},$$

$$= z^{-d} \sum_{j=0}^{\infty} x(j).z^{-j} = z^{-d}.Z[x(j)]$$

$$= z^{-d}.X(z)$$

$Z[x(k+d)] = \sum_{k=0}^{\infty} x(k+d).z^{-k},$ let $k+d = j,$ then

$$Z[x(k+d)] = \sum_{k=0}^{\infty} x(k+d).z^{-k}$$

$$= \sum_{j=d}^{\infty} x(j).z^{d-j} = \sum_{j=d}^{\infty} x(j).z^{-j}$$

$$= z^{d}\left[\sum_{j=0}^{\infty} x(j).z^{-j}, - \sum_{j=0}^{d-1} x(j).z^{-j}\right]$$

$$= z^d Z[x(j)] - \sum_{j=0}^{d-1} x(j).\, z^{-j}$$

$$= z^d \left[X(z) - \sum_{j=0}^{d-1} x(j).\, z^{-j} \right]$$

$= z^d.\, X(z) - \sum_{i=1}^{d} x(d-i).\, z^j$,

where $i = d - j$.

The time shift theorem implies that a delay of one sample period in the time domain (a one-step delay operator q^{-1}operating on the sample sequence $x(k)$, $q^{-1}\, x(k) = x(k-1)$

is equivalent to multiplying z^{-1}in the z domain, i.e.

$$Z[x(k-1)] = z^{-1}.\, X(z)$$

Similarly,

$Z[x(k+1)] = z.\, X(z) - z.\, x(0) = z.\, [X(z) - x(0)]$.◄

c) Initial Value Theorem (IVT)

Theorem 3.

If the *z*-transform of $x(k)$ is $X(z)$and if $\lim_{z\to\infty} X(z)$ exists, then the initial value of $x(k)(i.e., x(0))$ is

$$x(0) = \lim_{z\to\infty} X(z)$$

Proof.

The proof is obvious by examine the definition of the *z*-transform.◄

d) Final Value Theorem (FVT)

Theorem 4.

If the *z*-transform of $x(k)$ is $(z\ )$ and if $\lim_{k\to\infty} x(k)$ exists, then the value of $x(k)$ as $k \to \infty$ is given by

$$x(\infty) = \lim_{k\to\infty} x(k) = \lim_{z\to 1}[(z-1).X(z)]$$

Proof.

$$Z[x(k+1) - x(k)] = z.X(z) - z.x(0) - X(z)$$

$$= (z-1)X(z) - z.x(0)$$

$$= \sum_{j=0}^{\infty} x(k+1).z^{-k} - \sum_{k=0}^{\infty} x(k).z^{-k}$$

$$= \sum_{k=0}^{\infty}[x(k+1) - x(k)].z^{-k}.$$

In the above equation let $z \to 1$,

i.e.,

$$\lim_{z\to 1}[(z-1)X(z) - z.x(0)] = \lim_{z\to 1}[(z-1)X(z)] - x(0)$$

$$= \lim_{z\to 1}\sum_{k=0}^{\infty}[x(k+1) - x(k)]z^{-k}$$

$$= \sum_{k=0}^{\infty}[x(k+1) - x(k)] = \lim_{k\to\infty} x(k) - x(0).$$

So, $x(\infty) = \lim_{k\to\infty} x(k) = \lim_{z\to 1}[(z-1).X(z)].$◄

e) Convolution

Discrete convolution in the time domain is equivalent to multiplication in the z domain. Let the operand $\otimes$be the convolution operator, i.e.,

$$g(k)\otimes u(k) = \sum_{j=0}^{k} g(k-j).u(j)$$

$$= \sum_{j=0}^{\infty} g(j).u(k-j)$$

Then $Z[g(k)\otimes u(k)] = G(z).U(z)$

where $G(z)$and $U(z)$are the z-transform of $g(k)$ and $u(k)$ respectively.

Proof.

$$Z[g(k) \otimes u(k)] = Z\left[\sum_{j=0}^{k} g(k-j)..u(j)\right]$$

$$= \sum_{k=0}^{\infty}\left\{\left[\sum_{j=0}^{k} g(k-j)..u(j)\right] z^{-k}\right\}$$

$$= \sum_{j=0}^{\infty}\left[\sum_{k=j}^{\infty} g(k-j).z^{-k}\right].u(j)$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} g(i).z^{-i-j}u(j)$$

$$= \left[\sum_{i=0}^{\infty} g(i).z^{-i}\right].\left[\sum_{j=0}^{\infty} u(j).z^{-j}\right].$$

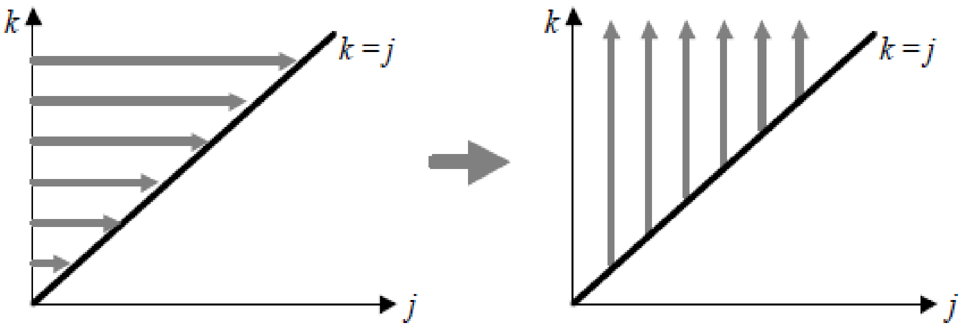


Figure 3.2 Chanege of Summation in Covolutions

Figure 2 shows the graphical interpretation of the interchanging of summation in the proof.◄

3. Inverse z-Transform

Given a *z*-transform function $X(z)$the corresponding time domain sequence $x(k)$can be obtained using the inversez-transform. The inverse *z*-transform is defined to be.

$$x(k) = Z^{-1}[X(z)]$$

$$= \frac{1}{2\pi j}\oint_c X(z).z^{k-1}dz \quad (8)$$

where the contour integration can be evaluated using the Cauchy Residue Theorem. The integration contour in Eq. (8) should enclose all the singularities of (z) .

Example 7.

Inverse z-Transform Using the Cauchy Residue Theorem.

Find the inverse *z*-transform of the following function:

$$X(z) = \frac{z}{(z-1)(z-2)}$$

Solution.

The function has poles at $z = 1$ and $z = 2$. The inverse z-transform, using Eq. (8), is

$$x(k) = \frac{1}{2\pi j}\oint_{c} \frac{z}{(z-1)(z-2)} . z^{k-1} dz$$

where C is a constant that contains all finite poles of function the $X(z). z^{k-1}$. The Cauchy Residue Theorem from complex

variable theory states that

$$x(k) = \frac{1}{2\pi j} .2\pi j$$

(sum of the residue of the integral)

$$= \frac{1}{2\pi j} .2\pi j . \left(\sum (z-p_i)X(z)z^{k-1}\Big|_{z=p_i}\right)$$

$$= \left[\frac{z}{z-1} . z^{k-1}\Big|_{z=2} + \frac{z}{z-1} . z^{k-1}\Big|_{z=1}\right]$$

$= 2^k - 1.$

Similar to the inverse Laplace transformation, there are several other methods to calculate the inverse z-transform. Generally, the z-transform of interest in linear discrete-time control applications can be represented as ratios of polynomials in the complex variable z with the numerator polynomial being of no higher order than the denominator polynomial. The long division and the partial fraction expansion are two methods that are especially suited in calculating the inverse z-transform of rational functions of z.

Direct Long Division Method

From the definition of the z-transform, Eq. (4), if we can represent the z-transform function $X(z)$ as an infinite series

in z^{-1} that fits Eq. (4), then the coefficients of the series will give the time domain sequence of $x(k)$. Given a rational functionof the complex variable z, long division can be used to decompose the rational function into infinite series of z^{-1}

Example 8.

(Inverse z-Transform Using Long Division)
Find the inverse z-transform sequence of the following function$X(z) = \frac{z^2+z}{z^2-3z+4}$.

Solution:

Represent the z-transform function $X(z)$in terms of $z^{-1}$by dividing $z^2$from both the numerator and the denominator.

$$X(z) = \frac{z^2 + z}{z^2 - 3z + 4} = \frac{1 + z^{-1}}{1 - 3z^{-1} + 4z^{-2}}$$

Carrying out the long division:

$$1 - 3z^{-1} + 4z^{-2} | \frac{1+4z^{-1}+8z^{-2}+8z^{-3}}{1+z^{-1}}$$

$$\frac{1 - 3z^{-1} + 4z^{-2}}{4z^{-1} - 4z^{-2}}$$

$$\frac{4z^{-1} - 12z^{-2} + 16z^{-3}}{8z^{-2} - 16z^{-3}}$$

$$\frac{8z^{-2} - 24z^{-3} + 32z^{-4}}{8z^{-3} - 32z^{-4}}$$

By examination, the sequence $x(k)$is

$x(0) = 1, x(1) = 4, x(2) = 8, x(3) = 8, \ldots$◄

The long division procedure used in the previous example can be carried out to any desired number of steps. The disadvantage of this technique is that it does not give a closed form representation of the resulting sequence. In many applications, we need to obtain a closed form result to infer general qualitative insights into the sequence $x(k)$.

For most engineering investigation, the method of partial fraction expansion and a good z-transform table is often sufficient to generate the desired closed form solution.

Partial Fraction Expansion

The procedure of using partial fraction expansion to calculate the inverse *z*-transform is identical to the one used in solving the inverse Laplace transform.

Distinct Real Poles

Assuming the *z* domain rational function can be written as

$$X(z) = \frac{N(z)}{(z-p_1)(z-p_2)(z-p_n)} \quad (9)$$

where the p_i's are distinct real numbers.

Examining the *z*-transform table (Table1), we see that the form of the inverse *z*-transform should be

$$X(z) = \frac{N(z)}{(z-p_1)(z-p_2)(z-p_n)}$$

$$= A_0 + A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + \cdots + A_n \frac{z}{z-p_n} \quad (10)$$

where the constant term A_0 is added to ensure equality in the case where there is a constant term in the numerator of Eq. (9). Thecoefficients of Eq. (10) can be calculated by the following formula.

$$A_1 = \frac{z - p_i}{z} \; X(z)\Big|_{z=p_i}, i = 1,2,3, \ldots \qquad (11)$$

The coefficient A_0 can be found by evaluating $X(z)$at $z = 0$. Consequently, the inverse *z*-transform of $X(z)$ is

$$x(k) = Z^{-1}[X(z)]$$

$$= A_0 \,.\, \delta_0(k) + A_1(p_1)^k + A_2(p_2)^k + \cdots + A_n \,.\, (p_n)^k.$$

Example 9.

(Solving inverse z-Transform Using Partial Fraction Expansion)

Find the time sequence that corresponds to the following *z*-transform:

$$X(z) = \frac{0.5(1 - e^{-T})^2(z^2 + e^{-T}z)}{(z-1)(z - e^{-T})(z - e^{-2T})}$$

Solution.

The denominator of the *z*-transform is already in the factored form. Following Eq. (10), we can write

$$X(z) = \frac{0.5(1 - e^{-T})^2(z^2 + e^{-T}z)}{(z-1)(z - e^{-T})(z - e^{-2T})}$$

$$= A_1 \frac{z}{z-1} + A_2 \frac{z}{z-e^{-T}} + \cdots + A_3 \frac{z}{z-e^{-2T}}.$$

In the above equation, since there is no constant term in the numerator, the constant coefficient A_0 is not needed. Using

Eq. (11), we can solve for the coefficients

$$A_1 = \frac{z-1}{z} X(z)\Big|_{z=1}$$

$$= \frac{0.5(1-e^{-T})^2(1+e^{-T})}{(z-e^{-T})(z-e^{-2T})}$$

$$= 0.5 \ \frac{(1-e^{-T})^2(1+e^{-T})}{(1-e^{-2T})} = 0.5$$

$$A_2 = \frac{z-e^{-T}}{z} X(z)\Big|_{z=e^{-T}}$$

$$= \frac{0.5(1-e^{-T})^2(e^{-2T}+e^{-T})}{e^{-T}(e^{-T}-e^{-2T})} = -1$$

$$A_3 = \frac{z-e^{-2T}}{z} X(z)\Big|_{z=e^{-2T}}$$

$$= \frac{0.5(1-e^{-T})^2(e^{-4T}+e^{-3T})}{e^{-2T}(e^{-2T}-1)(e^{-2T}+e^{-T})} = 0.5.$$

Hence the inverse *z*-transform of $X(z)$is

$$x(k) = Z^{-1}[X(z)] = 0.5 - (e^{-T})^k + 0.5(e^{-2T})^k$$

$$= 0.5 - e^{-kT} + 0.5e^{-2T}$$

for $k = 0,1,2, \ldots$◄

Distinct Complex Poles

For complex poles, one approach is to factor them as complex numbers and use the previous steps to obtain the complex coefficients in the partial fraction expansion, then combine the terms using Euler's identity to yield exponentially multiplied sinusoidal sequences. However, we can simplify the procedure by doing more work ahead of time. From the *z*-transform table, Table1, we see that

$$Z[e^{-akT}\cos(\omega kT)] = \frac{z^2 - e^{-aT}\cos(\omega kT).z}{z^2 - 2e^{-aT}\cos(\omega kT).z + e^{-2aT}},$$

and

$$Z[e^{-akT}\sin(\omega kT)] = \frac{z^2 - e^{-aT}\sin(\omega kT)\ .z}{z^2 - 2e^{-aT}\cos(\omega T).z + e^{-2aT}}$$

The poles of the both *z*-transforms can be represented by the following polar form.

$$p_{1,2} = e^{-aT}[\cos(\omega T) \pm \mathrm{j}\sin(\omega kT)]$$
$$= e^{-aT}\,.\,e^{\pm j\omega T} = R\,.\,e^{\pm j\theta} \quad (12)$$

where $R = e^{-aT}$ and $\theta = \omega T$. Consider a *z*-transform of the form

$$X(z) = \frac{N(z)}{D(z)(z^2 - 2R\beta z + R^2)}\ , 0 < \beta < 1 \quad (13)$$

where $R = e^{-at}$ and $\beta = \cos(\omega T)$. By examining the *z*-transform table and using the variables defined in Eq. (12), the partial fraction expansion of the *z*-transform represented in Eq.(13) should be

$$X(z) = \frac{N(z)}{D(z)(z^2 - 2R\beta z + R^2)}$$

$$= A\frac{(z^2 - R\beta z)}{z^2 - 2R\beta z + R^2} + B\frac{R\sin(\omega kT)\,z}{z^2 - 2R\beta z + R^2}$$

$$+ Q(z) \quad (14)$$

The corresponding inverse *z*-transform of Eq, (14) is

$$x(k) = Z^{-1}[X(z)]$$

$$= AZ^{-1}\left[\frac{(z^2 - R\beta z)}{z^2 - 2R\beta z + R^2}\right]$$

$$+ BZ^{-1}\left[\frac{R\sin(\omega kT)\,z}{z^2 - 2R\beta z + R^2}\right] +$$

$$= A.R^k\,.\cos(k\omega T) + B.R^k\sin(k\omega T) + Z^{-1}[Q(z)]$$

A good rule of thumb of finding the coefficients of the partial fraction expansion of Eq. (14) is to first find the coefficients related to $Q(z)$using similar approach as in the case of the distinct real poles and then evaluate$A$ and B by brute force (equate numerators and compare coefficients).

Example 10

(Solving Inverse z-Transform Using Partial Fraction Expansion)

Find the inverse z-transform of the following function

$$X(z) = \frac{z^2 + z}{(z^2 - 1.13z + 0.64)(z - 0.5)}$$

Solution.

The partial fraction of $X(z)$can be written as

$$X(z) = \frac{z^2 + z}{(z^2 - 1.13z + 0.64)(z - 0.5)}$$
$$= A\frac{(z^2 - R\beta z)}{z^2 - 1.13z + 0.64} + B\frac{R\sin(\omega T)z}{z^2 - 1.13z + 0.64} + C\frac{z}{z - 0.5} \quad (15)$$

First, we need to identify the parameters R, β , and ωT.By inspection$R = \sqrt{0.64} = 0.8$ and $\beta = \frac{1.13}{2R} = \frac{1.13}{1.6} = 0.7063$. Since $\beta = \cos(\omega T)$, $\omega T =$

$\cos^{-1}(0.7063) = 0.7865$ rad. Hence $\sin(\omega T) = 0.7079$. Substitute the numbers into Eq. (15), we get

$$X(z) = A\frac{(z^2 - 0.565z)}{z^2 - 1.13z + 0.64} + B\frac{0.5663z}{z^2 - 1.13z + 0.64} + C\frac{z}{z - 0.5} \quad (16)$$

The coefficient *C* can be calculated by

$$C = \frac{(z - 0.5)}{z} X(z)\bigg|_{z=0.5}$$

$$= \frac{0.5 + 1}{(0.5)^2 - 1.13.(0.5) + 0.64} = \frac{1.5}{0.325}$$

$$= 4.6154$$

Now if we combine the terms on the left-hand side of Eq. (16) and compare the numerator polynomial with that of Eq. (15) we see that

$$z^2 + z = A.(z^2 - 0.565z)(z - 0.5) + B.05663z(z - 0.5) + 4.6154z(z^2 - 1.13z + 0.64)$$

Equating the coefficients corresponding to the powers of *z*, and solving for *A* and *B*, we get$A = -4.6154$ and

$B = 2.2956$. Hence the time domain sequence corresponding to the given z-transform is

$$x(k) = -4.6154.(0.8)^k \cos(0.7865k) + 2.2956.(0.8)^k \sin(0.7865k) + 4.6154.(0.5)^k. \blacktriangleleft$$

4. Solving Linear Difference Equation Using z-Transform

If the numerical values of the coefficients and parameters are given, difference equations can be easily solved using a computer. However, closed-form expressions of the solution cannot be obtained from the computer (numerical) solution, except for very special cases. The z-transform provided an effective procedure to obtain the closed-form expression for the solution of a difference equation. The time shift property of the z-transform

$$Z[x(k-d] = z^{-d}.X(z)$$

$$Z[x(k-d] = z^{d}.X(z) - z^{d}x(0) - z^{d-1}x(1) - z^{d-2}x(2) - \cdots - z.x(d-1)$$

will be used extensively in solving linear difference equations. The process of solving difference equation using z-transform is very similar to that of using the Laplace transform to solve linear ODEs

Example 11

(Solving Difference Equation Using the z-Transform)

Find the solution to the following difference equation by using the *z*-transform and by using a program written in MATLAB.

$$x(k+2) + 3x(k+1) + 2x(k) = 0,$$

$$x(0) = 0, \qquad x(1) = 1$$

Solution.

a) z-transform approach

Take the *z*-transform of both side of the equation, we get

$$z^2X(z) - z^2x(0) - z.x(1) + 3z.X(z) - 3z.x(0) + 2X(z) = 0$$

Substituting in the initial conditions and simplifying gives

$$X(z) = \frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)}$$

$$= \frac{z}{z+1} - \frac{z}{z+2}$$

Take the inverse z-transform of the above equation we get

$$x(k) = Z^{-1}[X(z)]$$

$$= Z^{-1}\left[\frac{z}{z-(-1)}\right] - Z^{-1}\left[\frac{z}{z-(-2)}\right]$$

$$= (-1)^k - (-2)^k, k = 0,1,2, \ldots$$

***b)* MATLAB program**

The following MATLAB program will calculate the solution $x(k)$:

% Set up output vectors:

$\boldsymbol{x = zeros(1:10,1)};$

% Assign initial values:

$x\ (1,1) = 0, \quad \%\ x(0) = 0$

$x\ (2,1) = 1, \quad \%\ x(1) = 1$

% Calculate the output values:

for $\ k = 1:8,$

$x(k+2,1) = -3^{*}x\ (k+1,1) - 2^{*}x(k,1);$

and

% Calculated the closed-form solution:

$k = (0:9)';$

$x = (-1) . \text{^k} - (-2) . \text{^k} ;$

Plot $(k, x, 'x' , k, x, ' h');$

By the way, the solution $x(k)$oscillates and is unbounded.◄

Example 12.

(Solving Difference Equation Using the z-Transform)

Using the *z*-transform to solve the following difference equation

$$x(k+2) + 0.4. x(k+1) - 0.32. x(k) = u(k)$$

where $x(0) = 0$ and $x(1) = 1$. The input $u(k)$is a unit step input, i.e. $u(k) = 1$, for $k \geq 0$.

Solution.

Take the *z*-transform of the difference equation we get

$$z^2 X(z) - z^2 x(0) - z. x(1) + 04. z. X(z) - 0.4. zx(0) \\ - 0.32. X(z) = \frac{z}{z-1}$$

Substituting the initial conditions and simplifying, we obtain

$$X(z) = \frac{z^2}{(z-1)(z^2 + 0.4z - 0.32)}$$

$$= \frac{z^2}{(z-1)(z+0.8)(z-04)}$$

The partial fraction expansion of the solution $X(z)$is

$$X(z) = 0.926\ \frac{z}{z-1} - 0.3704\ \frac{z}{z+0.8} - 0.5556\frac{z}{z-0.4}$$

The corresponding time sequence can be obtained by taking the inverse *z*-transform of the above equation:

$x(k) = 0.926 - 0.3704.(-0.8)^k - 0.5556(0.4)^k,$

For $k = 0,1,2, \dots$

Note $x(k)$ will exhibit oscillatory response caused by the $(-0.8)^k$component.

5. Pulse Transfer Function and Impulse Response Sequence

The transfer function for the continuous-time system relates the Laplace transform of the continuous time output to that of the continuous-time input. For discrete-time systems, the pulse transfer function relates the *z*-transform of the output at the sample instance to that of the sampled input. Consider a linear time-invariant discrete-time system characterized by the following linear difference equation:

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) + \cdots + b_n u(k-n) \qquad (17)$$

where $u(k)$and $y(k)$are the system input and output, respectively, at the *k*th sample instances. If we take the
z-transform of the Eq. (17), by using the time shift property of the *z*-transform, we obtain.

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_n z^{-n} Y(z) = b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \cdots + b_n z^{-n} U(z)$$

or

$$(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}).Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}).U(z)$$

which can be written as

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\, U(z) = G(z).U(z) \qquad (18)$$

where

$$G(z) = \frac{Y(z)}{U(z)}$$
$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \qquad (19)$$

Consider the response of the linear discrete-time system described by Eq. (19), initially at rest $(y(k) = 0,\ k < 0)$,when the input $u(k)$is the Kronecker delta function$\delta_0(k), i.e.$

$$u(k) = \delta_0(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

Since

$$U(z) = Z[u(k)] = Z[\delta_0(k)] = 1$$

Then

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$
$$= G(z) \qquad (20)$$

Thus, $G(z)$ is the *z*-transform of the response of the system to the Kronecker delta function input. The function $G(z)$is called the pulse transfer function of the discrete-time system In the abovederivation, the role of the Kronecker delta function in discrete-time system is similar to that of the unit impulse function(the Dirac delta function) in continuous-time systems. The inverse transform of $G(z)$as given by Eq. (21)

$$g(k) = Z^{-1}[G(z)]$$

$$= Z^{-1}\left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\right] \qquad (21)$$

is called the impulse response function (sequence).

Remark: The system described by the difference equation

$$\begin{aligned} y(k+n) + a_1 y(k+n-1) &\\ + a_2 y(k+n-2) + \cdots + a_n y(k) &\\ = b_0 u(k+n) + b_1 u(k+n-1) &\\ + b_2 u(k+n-2) + \cdots + b_n u(k) & \end{aligned}$$

where the system is initially at rest $y(k) = 0, k < 0$) and the input $u(k) = 0$, for $k < 0$, can be represented by the same pulse transfer function$G(z)$, Eq. (19), as the system described by Eq. (17).

Example 13.

(Pulse Transfer Function)

Consider the difference equation

$$y(k+2) + a_1 y(k+1) + a_2 y(k) = b_0 u(k+2) + b_1 u(k+1) + b_2 u(k)$$

Assuming that the system is initially at rest and $u(k) = 0$ for $k < 0$, find the pulse transfer function.

Solution.

The z-transform of the difference equation is

$$[z^2 Y(z) - z^2 y(0) - z.y(1)] + a_1[zY(z) - z.y(0)] + a_2 Y(z) = b_0[z^2 U(z) - z^2 u(0) - z.u(1)] + b_1[zU(z) - zu(0)] + b_2 U(z)$$

Collect common terms

$$(z^2 + a_1 z + a_2).Y(z) = (b_0 z^2 + b_1 z + b_2).U(z) + z^2[y(0) - b_0 u(0)] + z[y(1) + a_1 y(0) - b_0 u(1) + b_1 u(0)]$$

Hence

$$Y(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \,.U(z) +$$

$$\frac{[y(0) - b_0 u(0)]z^2 + [y(1) + a_1 y(0) - b_0 u(1) + b_1 u(0)]z}{z^2 + a_1 z + a_2} \quad (22)$$

To determine the initial conditions $y(0)$and $y(1)$we substitute $k = -2$ into the original difference equation and obtain

$$y(0) + a_1 y(-1) + a_2 y(-2)$$
$$= b_0 u(0) + b_1 u(-1) + b_2 u(-2)$$

which implies$y(0) = b_0 . u(0) \quad (23)$

By substitute $k = -1$ into the original difference equation and obtain

$$y(1) + a_1 y(0) + a_2 y(-1)$$
$$= b_0 u(1) + b_1 u(0) + b_2 u(-1),$$

which implies

$y(1) = -a_1 y(0) + b_0 u(1) + b_1 u(0) \quad (24)$

By substituting Eqs. (23) and (24) into Eq. (22), we get

$$Y(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \,.U(z) \qquad (25)$$

Hence, if both $y(k)$and $u(k)$ are zero for$k < 0$, then the system's input and output are related by Eq. (25). The pulsetransfer function $G(z) = Y(z)/U(z)$can be written as

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \qquad (26)$$

Note that Eq. (26) is the same transfer function for the system described by the difference equation.

$$y(k) + a_1 y(k-1) + a_2 y(k-2)$$
$$= b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

Example 14.

(Impulse Response Function)

Assuming that the system is initially at rest, find the impulse response of the following discrete time system.

$$y(k+3) = 2u(k+3) - u(k+2) + 4u(k+1)$$
$$+ u(k)$$

Solution:

From the previous example, we see that the pulse transfer function of the system can be written as

$$G(z) = \frac{2z^3 - z^2 + 4z + 1}{z^3} = 2 - z^{-1} + 4z^{-2} + z^{-3}$$

The impulse response of the system with zero initial condition is then the inverse *z*-transform of the pulse transfer function,

$$g(k) = Z^{-1}[G(z)] = Z^{-1}[2 - z^{-1} + 4z^{-2} + z^{-3}]$$
$$= 2.\delta_0(k) - \delta_0(k-1) + 4.\delta_0(k-2) + \delta_0(k-3)$$

Hence

$g(0) = 2\,, g(1) = -, g(2) = 4, g(3) = 1, g(k) = 0,$ for $k > 3.$

Example 14 illustrated that the impulse response of a system with all of its poles at the origin will have finite non-zero terms. Impulse response of this type is often called finite impulse response and the system (digital filter) that has all its poles at the origin is called a finite impulse response (FIR) filter.

Discrete-Time Convolution and Impulse Response Function

Given a linear discrete-time system described by its impulse transfer function.

$$G(z) = \frac{Y(z)}{U(z)} = Z[g(k)]$$

where $g(k)$is the impulse response of the system. We would like to find the system response to some arbitrary input sequence$u(k)$The arbitrary input $u(k)$can be representedby a sequence of weighted discrete impulses:

$$u(k) = u(0).\delta_0(k) + u(1).\delta_0(k-1) + u(2).\delta_0(k-2) + \cdots$$

$$= \sum_{j=0}^{\infty} u(j).\delta_0(k-j)$$

The system's response due to the first impulse is $u(0).g(k)$ and due to the second impulse to be $u(1).g(k-1)$. Hencethe system response at some time *k* can be represented as

$$y(k) = u(0).\, g(0) + u(1).\, g(k-1)$$
$$+ u(2).\, g(k-2) + \cdots + u(k).\, g(0)$$

This can be written as

$y(k) = \sum_{j=0}^{k} g(k-j).\, u(j) = g(k) \otimes u(k)$ for $k = 0,1,2,3, \dots$ (27)

or if we change the subscript by letting $k - j = i$, we get

$y(k) = \sum_{i=0}^{k} u(k-i).\, g(i) = u(k) \otimes g(k)$

for $k = 0,1,2,3, \dots$ (28)

Eqs. (27) and (28) represent discrete-time convolutions. The response of the system $G(z)$to the arbitrary input $u(k)$can be calculated in the z domain as

$$y(k) = Z^{-1}[Y(z)] = Z^{-1}[G(z).\, U(z)]$$

or equivalently by either expressions of Eqs. (27) and (4.28). In summary, the response of a discrete-time system to any arbitrary input can be calculated by the convolution of the system's impulse response function and the input sequence.

Frequency Response of Discrete-Time Systems

In order for systems to possess a steady-state response to a sinusoidal input, in must be stable (all the poles of the transfer function must lie within the unit circle of the complex *z* plane).Let the system of interest is

$$G(z) = \frac{Y(z)}{U(z)} = \frac{N(z)}{(z-p_1)(z-p_2)\dots(z-p_n)} \quad (29)$$

where $p_i$are the complex poles of the system. We further assume that the system is stable, i.e.$|p_i| < 1$ for all *i*. Let the input to the system be a cosine sequence of radian frequency ω, i.e.

$$u(k) = A\cos(\omega kT) = \frac{A}{2}\left(e^{j\omega kT} + e^{-j\omega kT}\right)$$

The corresponding z-transform of the input sequence is

$$U(z) = \frac{A}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{j\omega T}}\right) \qquad (30)$$

Substituting the input, Eq. (30), into Eq. (29), the output $Y(z)$is given by

$$Y(z) = G(z).U(z)$$

$$= \frac{N(z)}{(z-p_1)(z-p_2)\ldots(z-p_n)} \cdot \frac{A}{2}\left(\frac{z}{z-e^{j\omega T}} + \frac{z}{z-e^{j\omega T}}\right) \quad (31)$$

A partial fraction expansion of the above equation is

$$Y(z) = B\frac{z}{z-e^{j\omega T}} + C\frac{z}{z-e^{-j\omega T}} + \sum_{i=1}^{n} D_i \frac{z}{z-p_i} \quad (32)$$

Each term in the summation on the right hand side of Eq. (32) yields a time domain sequence of the form $D_i(p_i)^k$, which if $|p_i| < 1$ will vanish when k gets larger and hence does not contribute to the steady-state response. The coefficients *B* and *C* in Eq. (32) can be evaluated by the following formula

$$B = \frac{z-e^{j\omega T}}{z}\, Y(z)\Big|_{z=e^{j\omega T}} \text{and}$$

$$C = \frac{z-e^{-j\omega T}}{z}\, Y(z)\Big|_{z=e^{-j\omega T}}$$

Substituting $Y(z)$expressed by Eq. (31) into the above formula, we get

$$B = \frac{z - e^{j\omega T}}{z}\ Y(z)\Big|_{z=e^{j\omega T}}$$

$$= \frac{A}{2}\left[1 + \frac{z - e^{j\omega T}}{z - e^{-j\omega T}}\right] G(z)\Bigg|_{z=e^{j\omega T}}$$

$$= \frac{A}{2}\ G\left(e^{j\omega T}\right)$$

$$C = \frac{z - e^{-j\omega T}}{z}\ Y(z)\Big|_{z=e^{-j\omega T}}$$

$$= \frac{A}{2}\left[1 + \frac{z - e^{-j\omega T}}{z - e^{j\omega T}}\right] G(z)\Bigg|_{z=e^{j\omega T}}$$

$$= \frac{A}{2}\ G\left(e^{-j\omega T}\right)$$

Thus, the steady-state response of the system $Y_{ss}(z)$ under a cosine sequence input may be written as

$$Y_{ss}(z) = \frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z - e^{j\omega T}} + G\left(e^{-j\omega T}\right)\frac{z}{z - e^{-j\omega T}}\right] \qquad (33)$$

Since $G(z)$is a rational function of the complex variable z, $G\left(e^{j\omega T}\right)$ is a complex number that can be written in polarforma as

$$G\left(e^{j\omega T}\right) = \left|G\left(e^{j\omega T}\right)\right|.\, e^{j<G\left(e^{j\omega T}\right)}$$
$$= \left|G\left(e^{j\omega T}\right)\right|.\, e^{j\phi} \qquad (34)$$

where $\phi$is the phase angle of the complex number $G\left(e^{j\omega T}\right)$ With similar reasoning, $G\left(e^{-j\omega T}\right)$ will have the same magnitude and conjugate phase angle as $G\left(e^{j\omega T}\right)$

$$G\left(e^{-j\omega T}\right) = \left|G\left(e^{-j\omega T}\right)\right|.\, e^{j<G\left(e^{-j\omega T}\right)}$$
$$= \left|G\left(e^{j\omega T}\right)\right|.\, e^{-j\phi} \qquad (35)$$

Substituting Eqs. (34) and (35) into Eq. (33), the steady-state response can be written as

$$Y_{ss}(z) = \frac{A}{2}\left|G\left(e^{j\omega T}\right)\right|.\left[e^{j\phi}\frac{z}{z - e^{j\omega T}} + e^{-j\phi}\frac{z}{z - e^{-j\omega T}}\right]$$

Taking inverse z-transform of the above equation, we can obtain the time sequence of the steady-state sinusoidal response to be

$$y(k) = \frac{A}{2}\left|G\left(e^{j\omega T}\right)\right|.\left[e^{j\phi}\left(e^{j\omega T}\right)^k + e^{-j\phi}\left(e^{-j\omega T}\right)^k\right]$$
$$= A\left|G\left(e^{j\omega T}\right)\right|.\frac{1}{2}\left(e^{j(\omega kT+\phi)} + e^{-j(\omega kT+\phi)}\right)$$

Using the Euler identity, the above equation can be further simplified and the steady-state sinusoidal response is$y(k) = A.\left|G\left(e^{j\omega T}\right)\right|.\cos(\omega kT + \phi)$, where $\phi =< G\left(e^{j\omega T}\right)$ (36)

From Eq. (36) we see that, similar to the continuous-time case, the steady-state response of the system $G(z)$to a sinusoidalinput is still sinusoidal with the same frequency but scaled in amplitude and shifted in phase. The amplitude of the steady-state response is scaled by a factor of $\left|G\left(e^{j\omega T}\right)\right|$ which will

be referred to as the system gain associated with $G(z)$at frequency ω .The complex function of ω, $G\left(e^{j\omega T}\right)$is called the frequency response function of the system The frequency response function of a system $G(z)$can be obtained by replacingthe z-transform complex variable z with $e^{j\omega T}, i.e$

$$G\left(e^{j\omega T}\right) = G(z)|_{z=e^{j\omega T}}$$

$$= G(\cos(\omega T) + j\sin(\omega T))$$

As in the continuous-time case, we are usually interested in the magnitude and phase characteristics of this function as a function of frequency. It is interesting to note that the DC gain of the system corresponds to the magnitude of the frequency response function at $\omega = 0$,

$$DC\ Gain = G(e^{j\omega T})\big|_{\omega=0} = G(z)|_{z=1} = G(1) \quad (37)$$

This is slightly different from the continuous-time case where the $DC\ \ gain$ is evaluated by substituting the Laplace variable s by 0.

Periodicity of Discrete-Time Frequency Response Function

Since both $\cos(\omega T)$ and $\sin(\omega T)$are periodic (for fixed sample period T), the frequency response function $G\left(e^{j\omega T}\right)$ is also periodic in frequency ω and will repeat itself every sample frequency $\omega_s = 2\pi/T$ rad/sec. Since $e^{-j\omega T}$ is the complex conjugate of $e^{j\omega T}$, we can write for negativefrequencies.

$\left|G\left(e^{-j\omega T}\right)\right| = \left|G\left(e^{j\omega T}\right)\right|$and

$$\angle G\left(e^{-j\omega T}\right) = -\angle G\left(e^{j\omega T}\right)$$

The above equation along with the periodic condition for $G\left(e^{j\omega T}\right)$indicates that the magnitude part of the frequency response function will be "folded" about the Nyquist frequency$\omega_N = \omega_s/2$,

$$\left|G\left(e^{-j\omega T}\right)\right| = \left|G\left(e^{j(\omega_s-\omega)}\right)\right| = \left|G\left(e^{j(\omega-\omega_s)T}\right)\right|$$

and the phase shift is

$$\angle G\left(e^{j\omega T}\right) = -\angle G\left(e^{j(\omega_s-\omega)^T}\right) =< G\left(e^{j(\omega-\omega_s)T}\right)$$

Example 15:

Frequency Response of Discrete-Time Systems.
Find the frequency response for the discrete-time system described by the following difference equation:

$y(k) = e^{-2T} y(k-1) + u(k),$ where $T = \frac{\pi}{5}$

Solution:

The impulse transfer function of the system can be found by taking the z-transform of the difference equation and assuming zero initial conditions.

$Y(z) = e^{-2T} z^{-1} Y(z) + U(z)$which implies

$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{1-e^{-2T}z^{-1}} = \frac{z}{z-e^{-2T}}$

The frequency response of the system is.

$G\left(e^{j\omega T}\right) = \frac{e^{j\omega T}}{e^{j\omega T}-e^{-2T}}$

MATLAB can be used to calculate the frequency response:

```
T = pi/5;
% Define discrete-time transfer function
G = tf([1 0],[1 –exp(-2*T)],T);
% Set up frequency vector:
w = lenspiece(0,50,200);
% Calculate frequency response:
```

```
out = freqresp(G,w);
for i = 1:length(w)
fr(i,1) = out(:,:,i);
end
% Plot frequency response
subplot(211);plot(w,abs(fr));
subplot(212);plot(w,180/pi*angle(fr));
```

Figure 3 shows the MATLAB calculated frequency response:

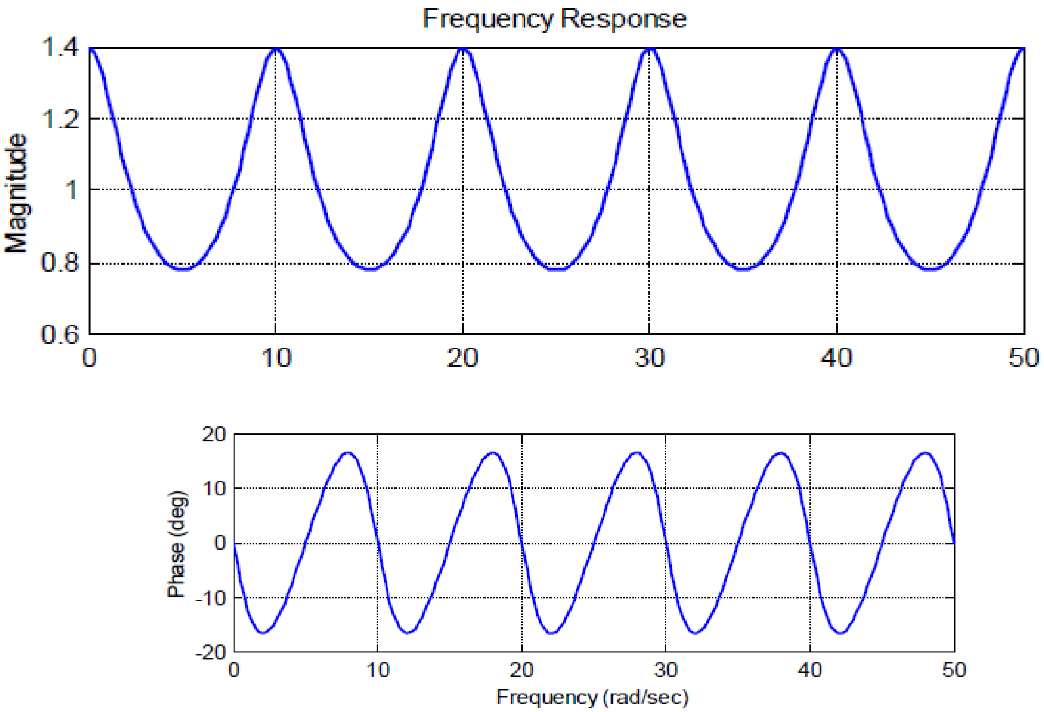


Figure 3.3 Frequency Response of Discrete-Time System for Example 3.15

Problems

1-Let $F(z) = \frac{2z}{(z-2)(z-1)^2}$, determine time series f_n by:

a) By Infinite Series

b) By Partial Fraction Expansion

c) Using the Inversion Integral

2- Find the inverse z-Transform of

$$F(z) = \frac{z(z^2 - 2z - 1)}{(z^2 + 1)^2}$$

3- Prove that the z-transform of the exponential $x_n = e^{2n}$ sequence is $X(z) = \frac{z}{z-e^2}$.

4- The z-transform of the sequence $x_n = \cos an$ is $X(z) = \frac{z(z-\cos a)}{z^2-2z\cos a+1}$.

REFERENCES

● Beyer, W. H. (Ed.). *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, 1987.

● Bracewell, R. *The Fourier Transform and Its Applications, 3rd ed.* New York: McGraw-Hill, 1999.

● Brown, J. W. and Churchill, R. V. *Fourier Series and Boundary Value Problems, 5th ed.* New York: McGraw-Hill, 1993.

● Carslaw, H. S. *Introduction to the Theory of Fourier's Series and Integrals, 3rd ed., rev. and enl.* New York: Dover, 1950.

● Davis, H. F. *Fourier Series and Orthogonal Functions.* New York: Dover, 1963.

● Dym, H. and McKean, H. P. *Fourier Series and Integrals.* New York: Academic Press, 1972.

● Folland, G. B. *Fourier Analysis and Its Applications.* Pacific Grove, CA: Brooks/Cole, 1992.

● James, J. F. *A Student's Guide to Fourier Transforms with Applications in Physics and Engineering.* New York: Cambridge University Press, 1995.

● Jury, E. I. *Theory and Application of the Z-Transform Method*. Krieger Pub Co. ISBN 0-88275-122-0.1973.

●Jury, E. I. *Theory and Application of the Z-Transform Method*. John Wiley & Sons. p. 1. 1964.