Data Structures Using C++ 2E

Chapter 12
Graphs

Objectives

- Learn about graphs
- Become familiar with the basic terminology of graph theory
- Discover how to represent graphs in computer memory
- Examine and implement various graph traversal algorithms

Objectives (cont'd.)

- Learn how to implement a shortest path algorithm
- Examine and implement the minimum spanning tree algorithm
- Explore topological sort
- Learn how to find Euler circuits in a graph

Introduction

- Königsberg bridge problem
  - Given: river has four land areas
    - *A*, *B*, *C*, *D*
  - Given: land areas connected using seven bridges
    - *a*, *b*, *c*, *d*, *e*, *f*, *g*
  - Starting at one land area
    - Is it possible to walk across all the bridges exactly once and return to the starting land area?
- Euler represented problem as a graph
  - Answered question in the negative
  - Marked birth of graph theory

Introduction (cont'd.)

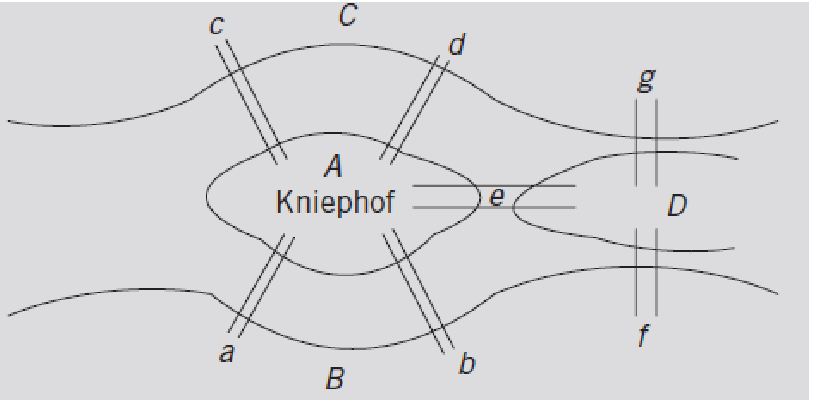


FIGURE 12-1 The Königsberg bridge problem

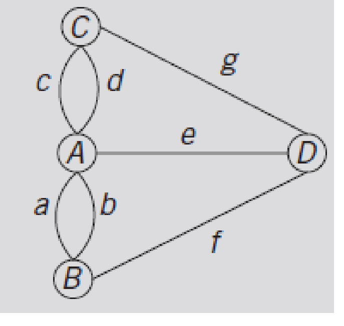


FIGURE 12-2 Graph representation of the Königsberg bridge problem

Graph Definitions and Notations

- Borrow definitions, terminology from set theory
- Subset
  - Set *Y* is a subset of *X*: $Y \subseteq X$
    - If every element of *Y* is also an element of *X*
- Intersection of sets *A* and *B*: $A \cap B$
  - Set of all elements that are in *A* and *B*
- Union of sets *A* and *B*: $A \cup B$
  - Set of all elements in *A* or in *B*
- Cartesian product: $A \times B$
  - Set of all ordered pairs of elements of *A* and *B*

Graph Definitions and Notations (cont'd.)

- Graph *G* *p*air
  - $G = (V, E)$, where V is a finite nonempty set
    - Called the set of vertices of *G*, and $E \subseteq V \times V$
  - Elements of *E*
    - Pairs of elements of *V*
- *E*: *s*et of edges of *G*
  - *G* called trivial if it has only one vertex
- Directed graph (digraph)
  - Elements in set of edges of graph *G*: ordered
- Undirected graph: not ordered

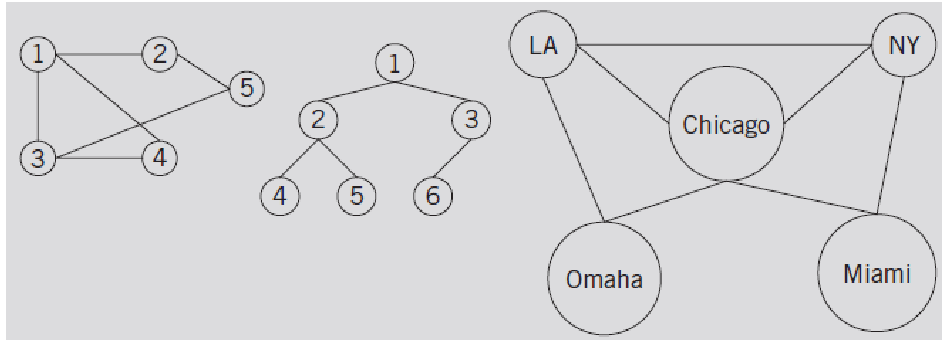


FIGURE 12-3 Various undirected graphs

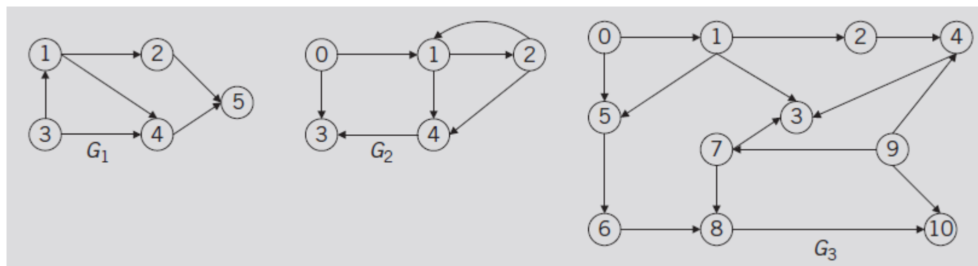


FIGURE 12-4 Various directed graphs

$V(G_1) = \{1, 2, 3, 4, 5\}$

$V(G_2) = \{0, 1, 2, 3, 4\}$

$V(G_3) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$E(G_1) = \{(1, 2), (1, 4), (2, 5), (3, 1), (3, 4), (4, 5)\}$

$E(G_2) = \{(0, 1), (0, 3), (1, 2), (1, 4), (2, 1), (2, 4), (4, 3)\}$

$E(G_3) = \{(0, 1), (0, 5), (1, 2), (1, 3), (1, 5), (2, 4), (4, 3), (5, 6), (6, 8), (7, 3), (7, 8), (8, 10), (9, 4), (9, 7), (9, 10)\}$

Graph Definitions and Notations (cont'd.)

- Graph H called subgraph of G
  - If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
  - Every vertex of H: vertex of G
  - Every edge in H: edge in G
- Graph shown pictorially
  - Vertices drawn as circles
    - Label inside circle represents vertex
- Undirected graph: edges drawn using lines
- Directed graph: edges drawn using arrows

Graph Definitions and Notations (cont'd.)

- Let u and v be two vertices in G
  - u and v adjacent
    - If edge from one to the other exists: $(u, v) \in E$
- Loop
  - Edge incident on a single vertex
- e_1 and e_2 called parallel edges
  - If two edges e_1 and e_2 associate with same pair of vertices $\{u, v\}$
- Simple graph
  - No loops, no parallel edges

Graph Definitions and Notations (cont'd.)

- Let $e = (u, v)$ be an edge in G
  - Edge e is incident on the vertices u and v
  - Degree of u written deg(u) or d(u)
    - Number of edges incident with u
- Each loop on vertex u
  - Contributes two to the degree of u
- u is called an even (odd) degree vertex
  - If the degree of u is even (odd)

Graph Definitions and Notations (cont'd.)

- Path from *u* to *v*
  - If sequence of vertices $u_1, u_2, \ldots, u_n$ exists
    - Such that $u = u_1$, $u_\mathrm{n} = v$ and $(u_i, u_i + 1)$ is an edge for all $i = 1, 2, \ldots, n - 1$
- Vertices *u* and *v* called connected
  - If path from *u* to *v* exists
- Simple path
  - All vertices distinct (except possibly first, last)
- Cycle in *G*
  - Simple path in which first and last vertices are the same

Graph Definitions and Notations (cont'd.)

- *G* is connected
  - If path from any vertex to any other vertex exists
- Component of *G*
  - Maximal subset of connected vertices
- Let *G* be a directed graph and let *u* and *v* be two vertices in *G*
  - If edge from *u* to *v* exists: $(u, v) \in E$
    - *u* is adjacent to *v*
    - *v* is adjacent from *u*

Graph Definitions and Notations (cont'd.)

- Definitions of paths and cycles in *G*
  - Similar to those for undirected graphs
- *G* is strongly connected
  - If any two vertices in *G* are connected

Graph Representation

- Graphs represented in computer memory
  - Two common ways
    - Adjacency matrices
    - Adjacency lists

Adjacency Matrices

- Let G be a graph with n vertices where n > zero
- Let $V(G) = \{v_1, v_2, \ldots, v_n\}$
  - Adjacency matrix

$$A_G(i,j) = \begin{cases} 1 & \text{if}(v_i, v_j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

$$A_{G_1} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A_{G_2} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Adjacency Lists

- Given:
  - Graph G with n vertices, where n > *zero*
  - $V(G) = \{v_1, v_2, \ldots, v_n\}$
- For each vertex v: linked list exists
  - Linked list node contains vertex u: $(v, u) \in E(G)$
- Use array A, of size n, such that $A[i]$
  - Reference variable pointing to first linked list node containing vertices to which v_i adjacent
- Each node has two components: vertex, link
  - Component vertex
    - Contains index of vertex adjacent to vertex i

Adjacency Lists (cont'd.)

- Example 12-4

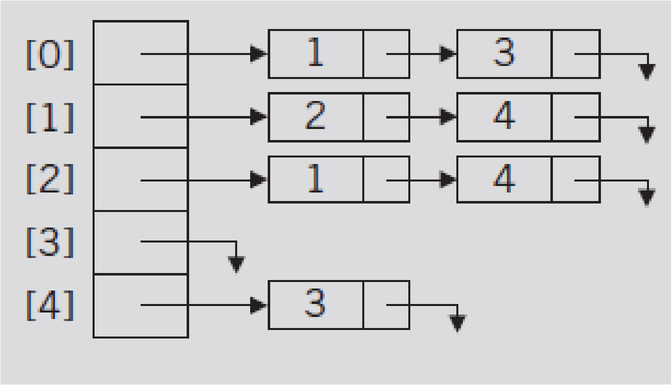


FIGURE 12-5 Adjacency list of graph *G2* of Figure 12-4

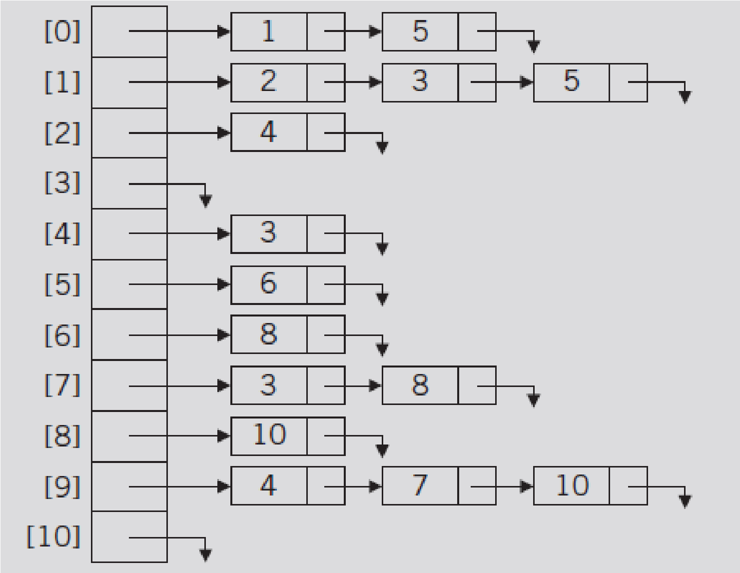


FIGURE 12-6 Adjacency list of graph *G3* of Figure 12-4

Operations on Graphs

- Commonly performed operations
  - Create graph
    - Store graph in computer memory using a particular graph representation
  - Clear graph
    - Makes graph empty
  - Determine if graph is empty
  - Traverse graph
  - Print graph

Operations on Graphs (cont'd.)

- Graph representation in computer memory
  - Depends on specific application
- Use linked list representation of graphs
  - For each vertex *v*
    - Vertices adjacent to *v* (directed graph: called immediate successors)
    - Stored in the linked list associated with *v*
- Managing data in a linked list
  - Use `class unorderedLinkedList`
- Labeling graph vertices
  - Depends on specific application

Graphs as ADTs

- See code on pages 692-693
  - Defines a graph as an ADT
  - Class specifying basic operations to implement a graph
- Definitions of the functions of the `class graphType`

```
bool graphType::isEmpty() const
{
    return (gSize == 0);
}
```

Graphs as ADTs (cont'd.)

- Function `createGraph`
  - Implementation
    - Depends on how data input into the program
  - See code on page 694
- Function `clearGraph`
  - Empties the graph
    - Deallocates storage occupied by each linked list
    - Sets number of vertices to zero
  - See code on page 695

Graph Traversals

- Processing a graph
  - Requires ability to traverse the graph
- Traversing a graph
  - Similar to traversing a binary tree
    - A bit more complicated
- Two most common graph traversal algorithms
  - Depth first traversal
  - Breadth first traversal

Depth First Traversal

- Similar to binary tree preorder traversal
- General algorithm

```
for each vertex, v, in the graph
   if v is not visited
      start the depth first traversal at v
```

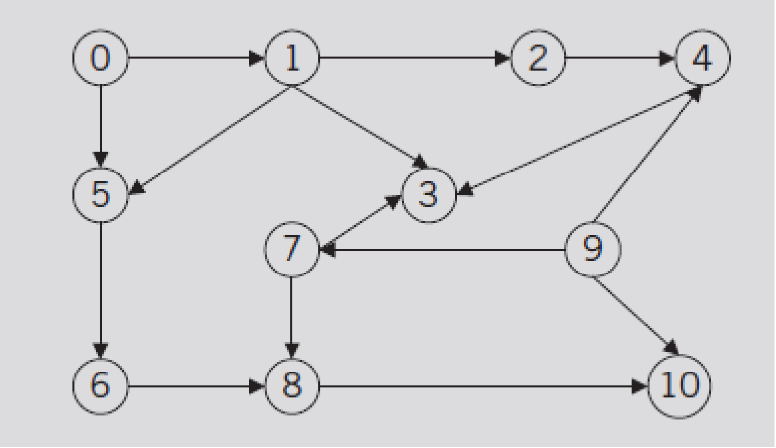


FIGURE 12-7 Directed graph G_3

Depth First Traversal (cont'd.)

- General algorithm for depth first traversal at a given node *v*
  - Recursive algorithm

```
1.  mark node v as visited
2.  visit the node
3.  for each vertex u adjacent to v
          if u is not visited
             start the depth first traversal at u
```

Depth First Traversal (cont'd.)

- Function `dft` implements algorithm

```
void graphType::dft(int v, bool visited[])
{
    visited[v] = true;
    cout << " " << v << " ";  //visit the vertex

    linkedListIterator<int> graphIt;

        //for each vertex adjacent to v
    for (graphIt = graph[v].begin(); graphIt != graph[v].end();
                                      ++graphIt)
    {
        int w = *graphIt;
        if (!visited[w])
            dft(w, visited);
    } //end while
} //end dft
```

Depth First Traversal (cont'd.)

- Function `depthFirstTraversal`
  - Implements depth first traversal of the graph

```
void graphType::depthFirstTraversal()
{
    bool *visited; //pointer to create the array to keep
                   //track of the visited vertices
    visited = new bool[gSize];

    for (int index = 0; index < gSize; index++)
        visited[index] = false;

        //For each vertex that is not visited, do a depth
        //first traverssal
    for (int index = 0; index < gSize; index++)
        if (!visited[index])
            dft(index,visited);
    delete [] visited;
} //end depthFirstTraversal
```

Depth First Traversal (cont'd.)

- Function `depthFirstTraversal`
  - Performs a depth first traversal of entire graph
- Function `dftAtVertex`
  - Performs a depth first traversal at a given vertex

```
void graphType::dftAtVertex(int vertex)
{
    bool *visited;

    visited = new bool[gSize];

    for (int index = 0; index < gSize; index++)
        visited[index] = false;

    dft(vertex, visited);

    delete [] visited;
} // end dftAtVertex
```

Breadth First Traversal

- Similar to traversing binary tree level-by-level
  - Nodes at each level
    - Visited from left to right
  - All nodes at any level *i*
    - Visited before visiting nodes at level *i* + one

Breadth First Traversal (cont'd.)

- General search algorithm
  - Breadth first search algorithm with a queue

```
1. for each vertex v in the graph
       if v is not visited
           add v to the queue //start the breadth first search at v
2. Mark v as visited
3. while the queue is not empty
   3.1. Remove vertex u from the queue
   3.2. Retrieve the vertices adjacent to u
   3.3. for each vertex w that is adjacent to u
            if w is not visited
              3.3.1. Add w to the queue
              3.3.2. Mark w as visited
```

```
void graphType::breadthFirstTraversal()
{
    linkedQueueType<int> queue;

    bool *visited;
    visited = new bool[gSize];

    for (int ind = 0; ind < gSize; ind++)
        visited[ind] = false;    //initialize the array
                                //visited to false

    linkedListIterator<int> graphIt;

    for (int index = 0; index < gSize; index++)
        if (!visited[index])
        {
            queue.addQueue(index);
            visited[index] = true;
            cout << " " << index << " ";

            while (!queue.isEmptyQueue())
            {
                int u = queue.front();
                queue.deleteQueue();

                for (graphIt = graph[u].begin();
                     graphIt != graph[u].end(); ++graphIt)
                {
                    int w = *graphIt;
                    if (!visited[w])
                    {
                        queue.addQueue(w);
                        visited[w] = true;
                        cout << " " << w << " ";
                    }
                }
            } //end while
        }

    delete [] visited;
} //end breadthFirstTraversal
```

Shortest Path Algorithm

- Weight of the graph
  - Nonnegative real number assigned to the edges connecting to vertices
- Weighted graphs
  - When a graph uses the weight to represent the distance between two places
- Weight of the path *P*
  - Given *G* as a weighted graph with vertices *u* and *v* in *G* and *P* as a path in *G* from *u* to *v*
    - Sum of the weights of all the edges on the path
- Shortest path: path with the smallest weight

Shortest Path Algorithm (cont'd.)

- Shortest path algorithm space (greedy algorithm)
- See code on page 700
  - `class weightedGraphType`
    - Extend definition of `class graphType`
    - Adds function `createWeightedGraph` to create graph and weight matrix associated with the graph

Let G be a graph with n vertices, where $n \geq 0$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let W be a two-dimensional $n \times n$ matrix such that

$$W(i,j) = \begin{cases} w_{ij} & \text{if } (v_i, v_j) \text{ is an edge in } G \text{ and } w_{ij} \text{ is the weight of the edge } (v_i, v_j) \\ \infty & \text{if there is no edge from } v_i \text{ to } v_j \end{cases}$$

Shortest Path

- General algorithm
  - Initialize array `smallestWeight`

    `smallestWeight[u] = weights[vertex, u]`
  - Set `smallestWeight[vertex] = zero`
  - Find vertex *v* closest to vertex where shortest path is not determined
  - Mark *v* as the (next) vertex for which the smallest weight is found

Shortest Path (cont'd.)

- General algorithm (cont'd.)
  - For each vertex *w* in *G*, such that the shortest path from vertex to w has not been determined and an edge (*v*, *w*) exists
    - If weight of the path to *w* via *v* smaller than its current weight
    - Update weight of *w* to the weight of *v* + weight of edge (*v*, *w*)

Shortest Path (cont'd.)

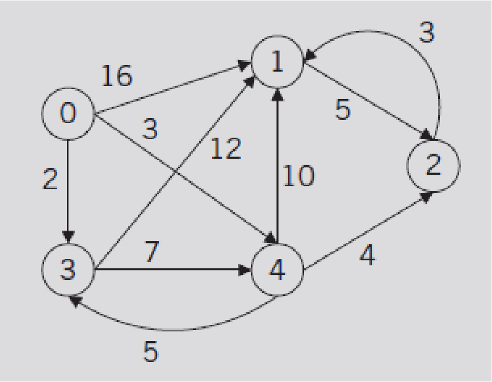
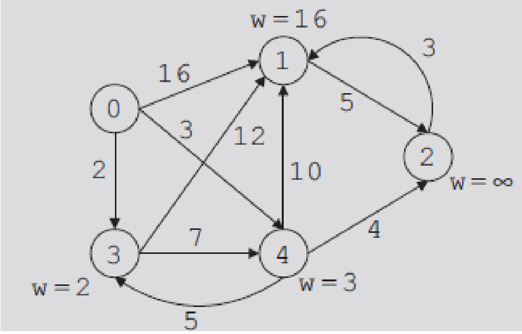


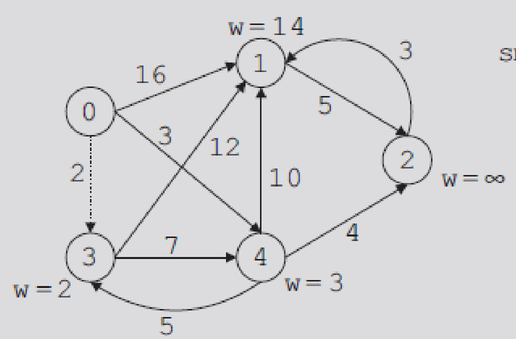
FIGURE 12-8 Weighted graph *G*



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	16	∞	2	3
weightFound	T	F	F	F	F

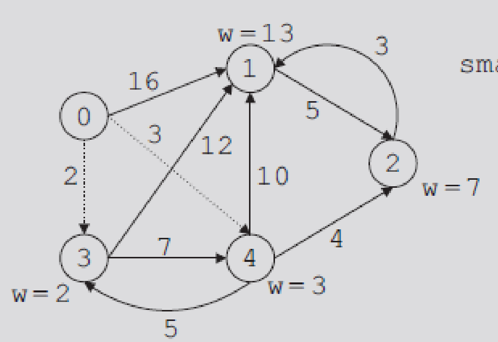
FIGURE 12-9 Graph after Steps 1 and 2 execute

Shortest Path (cont'd.)



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	14	∞	2	3
weightFound	T	F	F	T	F

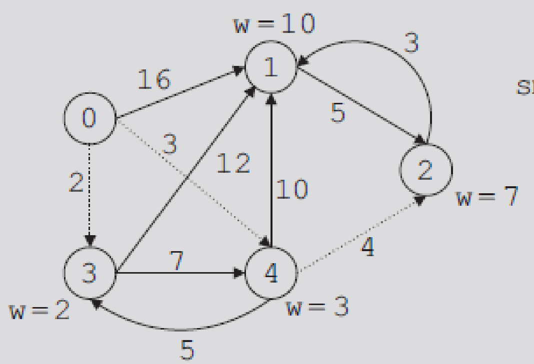
FIGURE 12-10 Graph after the first iteration of Steps 3 to 5



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	13	7	2	3
weightFound	T	F	F	T	T

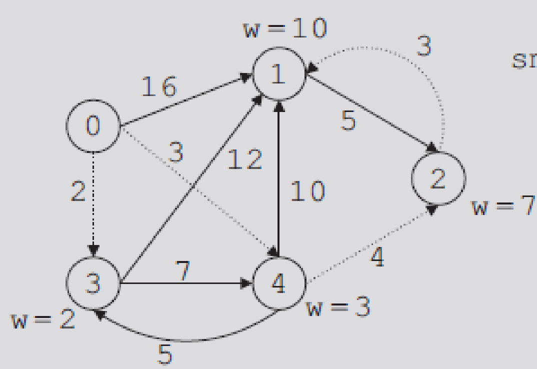
FIGURE 12-11 Graph after the second iteration of Steps 3 to 5

Shortest Path (cont'd.)



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	10	7	2	3
weightFound	T	F	T	T	T

FIGURE 12-12 Graph after the third iteration of Steps 3 to 5



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	10	7	2	3
weightFound	T	T	T	T	T

FIGURE 12-13 Graph after the fourth iteration of Steps 3 through 5

Shortest Path (cont'd.)

- See code on pages 704-705
  - C++ function `shortestPath` implements previous algorithm
    - Records only the weight of the shortest path from the source to a vertex
- Review the definitions of the function `printShortestDistance` and the constructor and destructor on pages 705-706

Minimum Spanning Tree

- Airline connections of a company
  - Between seven cities

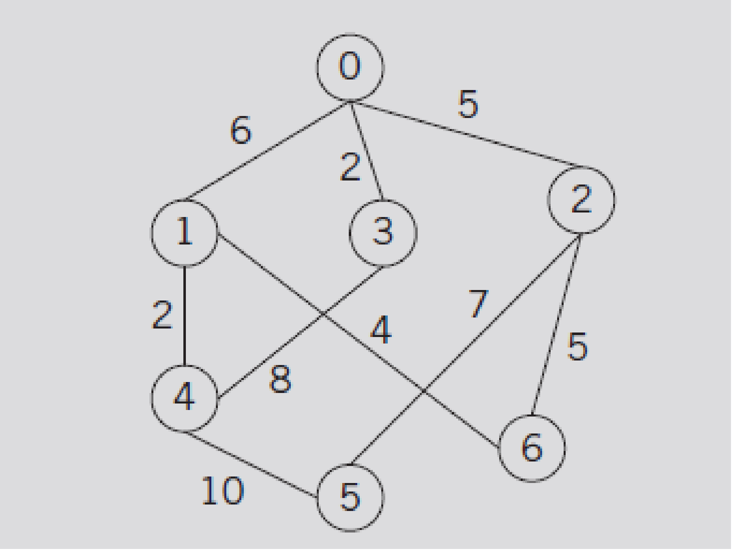


FIGURE 12-14 Airline connections between cities and the cost factor of maintaining the connections

Minimum Spanning Tree (cont'd.)

- Due to financial hardship
  - Company must shut down maximum number of connections
    - Still be able to fly (maybe not directly) from one city to another

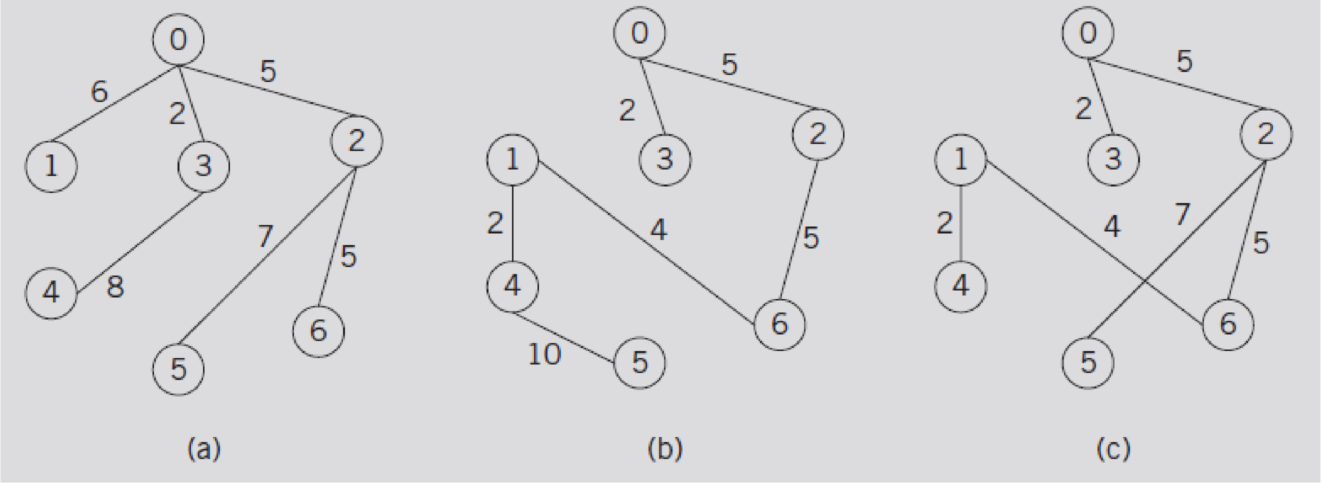


FIGURE 12-15 Possible solutions to the graph of Figure 12-14

Minimum Spanning Tree (cont'd.)

- Free tree *T*
  - Simple graph
  - If *u* and *v* are two vertices in *T*
    - Unique path from *u* to *v* exists
- Rooted tree
  - Tree with particular vertex designated as a root

Minimum Spanning Tree (cont'd.)

- Weighted tree T
  - Weight assigned to edges in T
  - Weight denoted by $W(T)$: sum of weights of all the edges in T
- Spanning tree T of graph G
  - T is a subgraph of G such that $V(T) = V(G)$

Minimum Spanning Tree (cont'd.)

- Theorem 12-1
  - A graph *G* has a spanning tree if and only if *G* is connected
  - From this theorem, it follows that to determine a spanning tree of a graph
    - Graph must be connected
- Minimum (minimal) spanning tree of *G*
  - Spanning tree with the minimum weight

Minimum Spanning Tree (cont'd.)

- Two well-known algorithms for finding a minimum spanning tree of a graph
  - Prim's algorithm
    - Builds the tree iteratively by adding edges until a minimum spanning tree obtained
  - Kruskal's algorithm

Minimum Spanning Tree (cont'd.)

- General form of Prim's algorithm

1. Set V(T) = {source}
2. Set E(T) = empty
3. for i = 1 to n

    3.1. minWeight = infinity;

    3.2.
```
for j = 1 to n
      if v_j is in V(T)
        for k = 1 to n
          if v_k is not in T and weight[v_j, v_k] < minWeight
          {
              endVertex = v_k;
              edge = (v_j, v_k);
              minWeight = weight[v_j, v_k];
          }
```

    3.3. V(T) = V(T) ∪ {endVertex};

    3.4. E(T) = E(T) ∪ {edge};

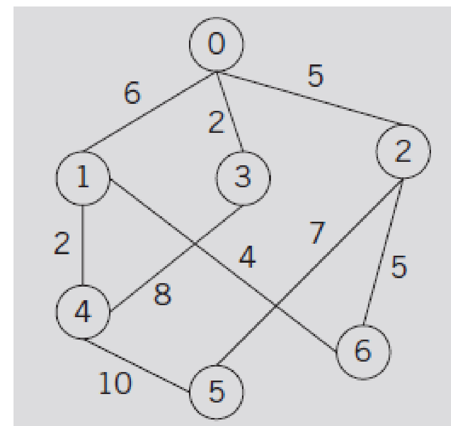


FIGURE 12-16 Weighted graph G

Minimum Spanning Tree (cont'd.)

- See code on page 710
  - `class msTreeType` defines spanning tree as an ADT
- See code on page 712
  - C++ function `minimumSpanning` implementing Prim's algorithm
  - Prim's algorithm given in this section: $O(n^3)$
    - Possible to design Prim's algorithm order $O(n^2)$
- See function `printTreeAndWeight` code
- See constructor and destructor code

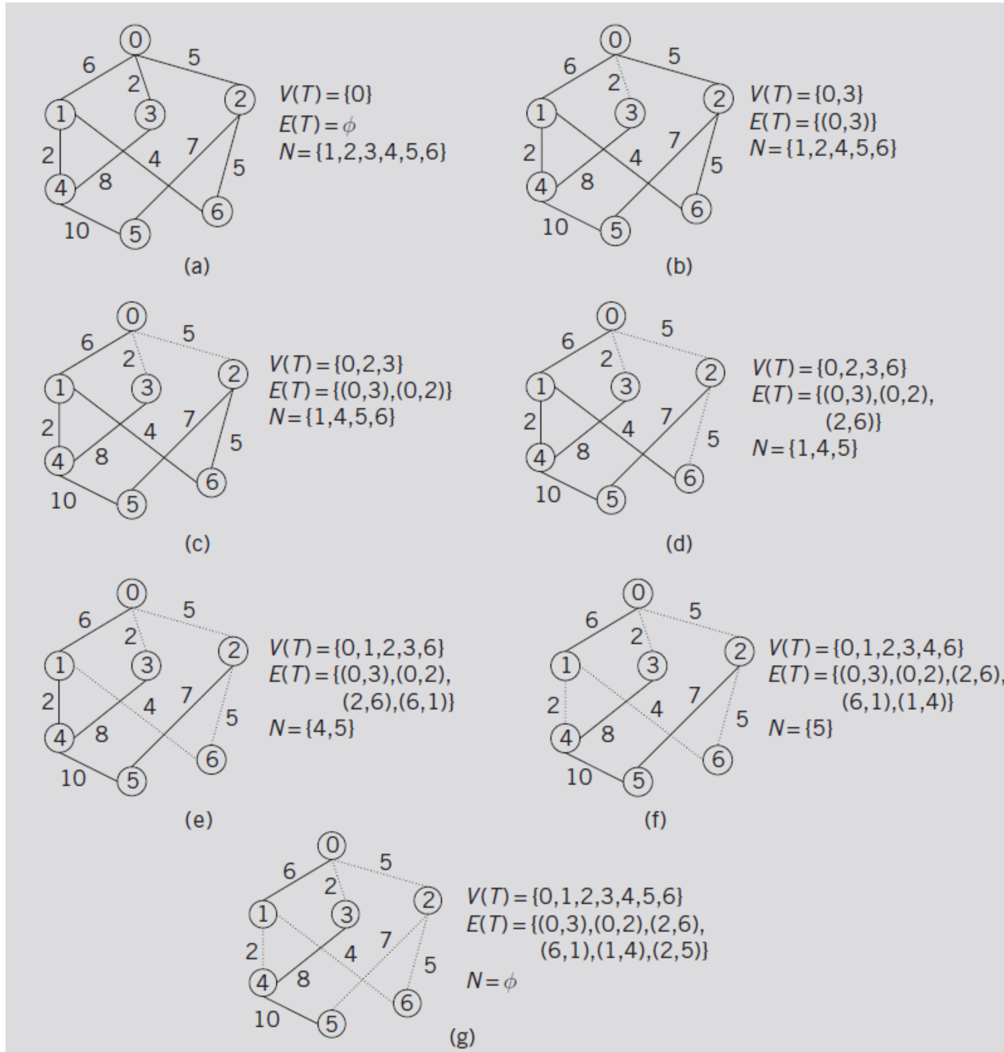


FIGURE 12-17 Graph *G*, *V*(*T*), *E*(*T*), and *N* after Steps 1 and 2 execute

Topological Order

- Topological ordering of $V(G)$
  - Linear ordering $v_{i1}, v_{i2}, \ldots, v_{in}$ of the vertices such that
    - If v_{ij} is a predecessor of v_{ik}, $j \neq k$, $1 \leq j \leq n$, $1 \leq k \leq n$
    - Then v_{ij} precedes v_{ik}, that is, $j < k$ in this linear ordering
- Algorithm topological order
  - Outputs directed graph vertices in topological order
  - Assume graph has no cycles
    - There exists a vertex v in G such that v has no successor
    - There exists a vertex u in G such that u has no predecessor

Topological Order (cont'd.)

- Topological sort algorithm
  - Implemented with the depth first traversal or the breadth first traversal
- Extend `class graphType` definition (using inheritance)
  - Implement breadth first topological ordering algorithm
    - Called `class topologicalOrderType`
  - See code on pages 714-715
    - Illustrating class including functions to implement the topological ordering algorithm

Breadth First Topological Ordering

- General algorithm

1. Create the array `predCount` and initialize it so that `predCount[i]` is the number of predecessors of the vertex v_i.
2. Initialize the queue, say `queue`, to all those vertices v_k so that `predCount[k]` is 0. (Clearly, `queue` is not empty because the graph has no cycles.)
3. `while` the queue is not empty

   3.1. Remove the front element, `u`, of the queue.

   3.2. Put `u` in the next available position, say `topologicalOrder[topIndex]`, and increment `topIndex`.

   3.3. For all the immediate successors `w` of `u`,

   3.3.1. Decrement the predecessor count of `w` by `1`.

   3.3.2. `if` the predecessor count of `w` is 0, add `w` to `queue`.

Breadth First Topological Ordering (cont'd.)

- Breadth First Topological order
  - 0 9 1 7 2 5 4 6 3 8 10

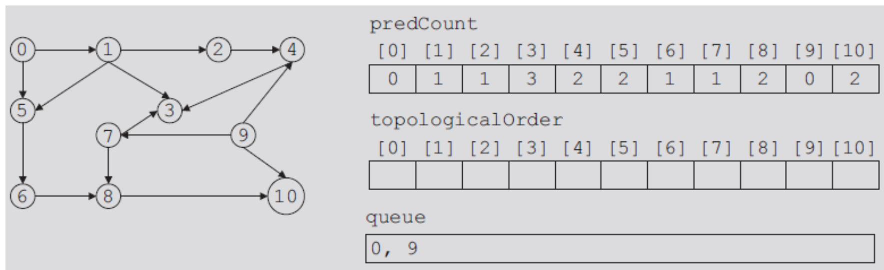


FIGURE 12-18 Arrays `predCount`, `topologicalOrder`, and `queue` after Steps 1 and 2 execute

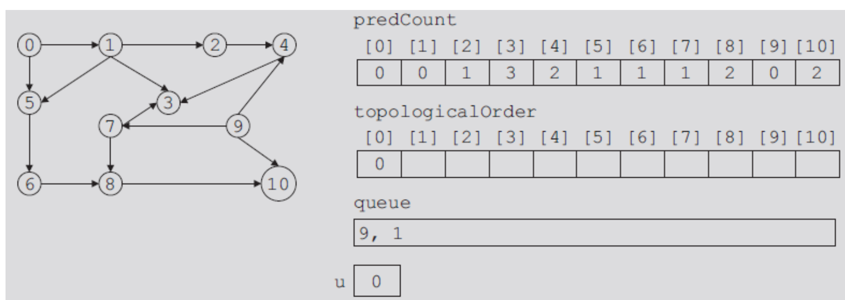


FIGURE 12-19 Arrays `predCount`, `topologicalOrder`, and `queue` after the first iteration of Step 3

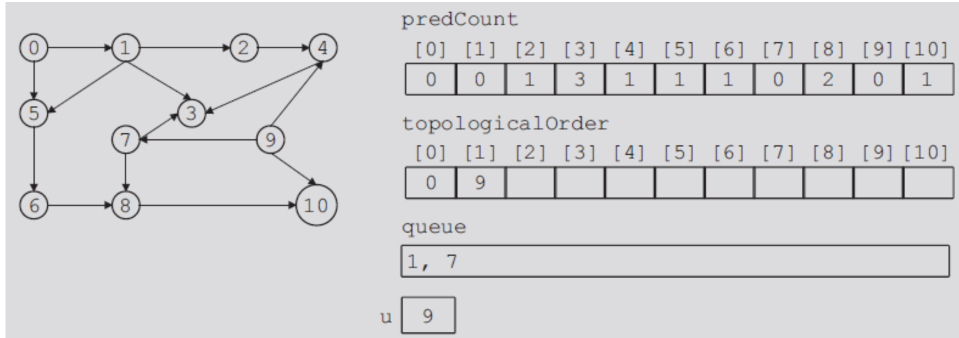


FIGURE 12-20 Arrays `predCount`, `topologicalOrder`, and `queue` after the second iteration of Step 3

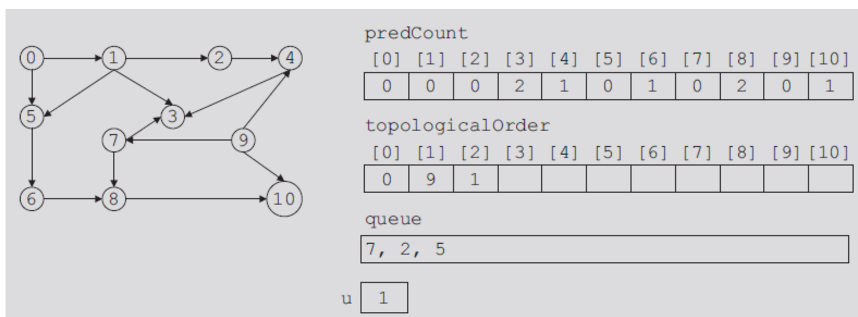


FIGURE 12-21 Arrays `predCount`, `topologicalOrder`, and `queue` after the third iteration of Step 3

Breadth First Topological Ordering (cont'd.)

- See code on pages 718-719
  - Function implementing breadth first topological ordering algorithm

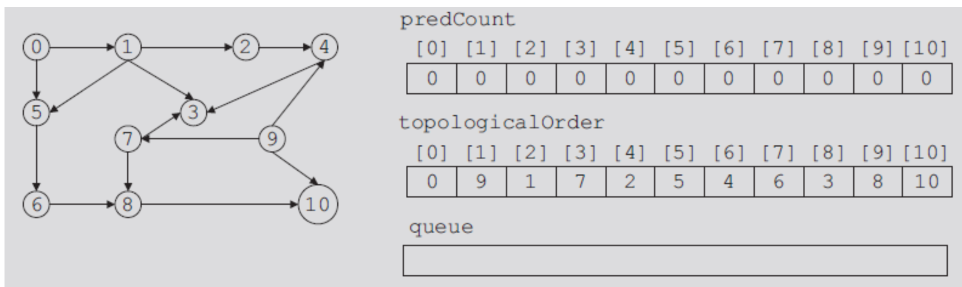


FIGURE 12-22 Arrays `predCount`, `topologicalOrder`, and `queue` after Step 3 executes

Euler Circuits

- Euler’s solution to Königsberg bridge problem
  - Reduces problem to finding circuit in the graph
- Circuit
  - Path of nonzero length
    - From a vertex *u* to *u* with no repeated edges
- Euler circuit
  - Circuit in a graph including all the edges of the graph
- Eulerian graph *G*
  - If either *G* is a trivial graph or *G* has an Euler circuit

Euler Circuits (cont'd.)

- Graph of Figure 12-24: Euler circuit

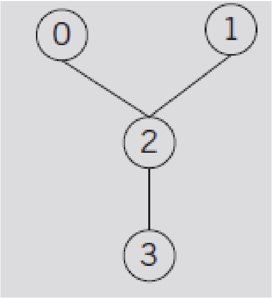


FIGURE 12-23 A graph with all vertices of odd degree

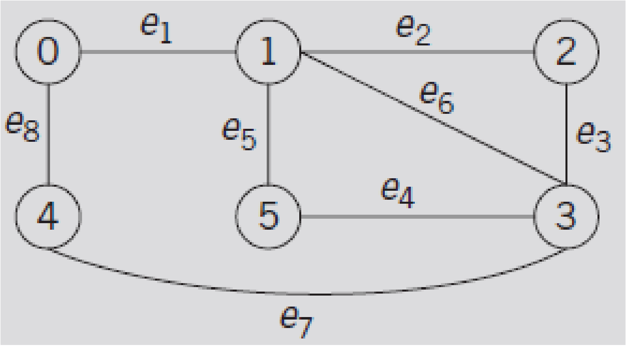


FIGURE 12-24 A graph with all vertices of even degree

Euler Circuits (cont'd.)

- Theorem 12-2
  - If a connected graph *G* is Eulerian, then every vertex of *G* has even degree
- Theorem 12-3
  - Let *G* be a connected graph such that every vertex of *G* is of even degree; then, *G* has an Euler circuit

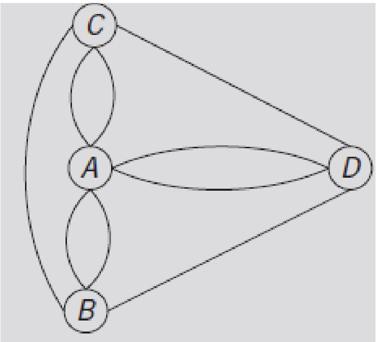


FIGURE 12-25 Graph of the Königsberg bridge problem with two additional bridges

Euler Circuits (cont'd.)

- Fleury's Algorithm

Step 1. Choose a vertex v as the starting vertex for the circuit and choose an edge e with v as one of the end vertices.

Step 2. If the other end vertex u of the edge e is also v, go to Step 3. Otherwise, choose an edge e_1 different from e with u as one of the end vertices. If the other vertex u_1 of e_1 is v, go to Step 3; otherwise, choose an edge e_2 different from e and e_1 with u_1 as one of the end vertices and repeat Step 2.

Step 3. If the circuit T_1 obtained in Step 2 contains all the edges, then stop. Otherwise, choose an edge e_j different from the edges of T_1 such that one of the end vertices of e_j, say, w is a member of the circuit T_1.

Step 4. Construct a circuit T_2 with starting vertex w, as in Steps 1 and 2, such that all the edges of T_2 are different from the edges in the circuit T_1.

Step 5. Construct the circuit T_3 by inserting the circuit T_2 at w of the circuit T_1. Now go to Step 3 and repeat Step 3 with the circuit T_3.

Euler Circuits (cont'd.)

- Fleury's Algorithm (cont'd.)

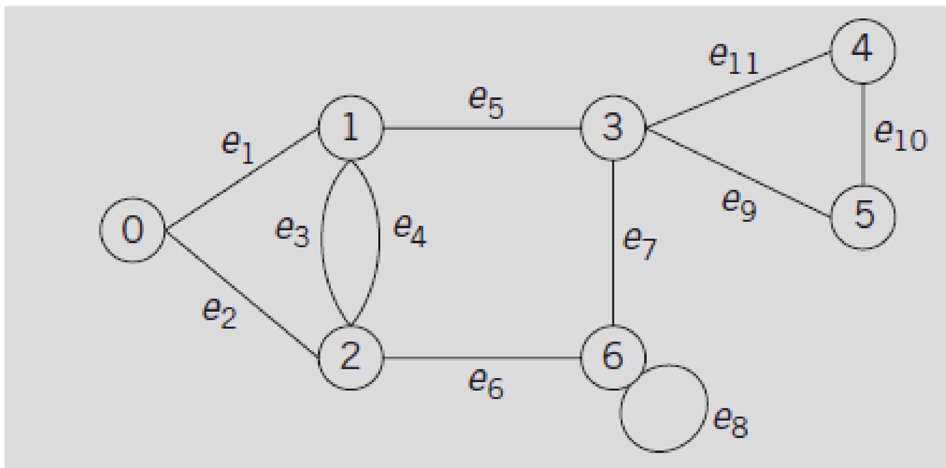


FIGURE 12-26 A graph with all vertices of even degree

Summary

- Many types of graphs
  - Directed, undirected, subgraph, weighted
- Graph theory borrows set theory notation
- Graph representation in memory
  - Adjacency matrices, adjacency lists
- Graph traversal
  - Depth first, breadth first
- Shortest path algorithm
- Prim’s algorithm
- Euler circuit