Section 1,2

(Chapter I)

Laplace transform

يستخدم تحويل لابلاس للتحويل من t-domain إلى s-domain.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$f(t) \xrightarrow{\text{تحويل لابلاس}} F(s)$$

* جدول بعض الدوال المحولة.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
[1] C	$\frac{C}{s}$	[8] $\sinh at$	$\frac{a}{s^2-a^2}$
[2] t	$\frac{1}{s^2}$	[9] $\cosh at$	$\frac{s}{s^2-a^2}$
[3] عدد صحيح موجب t^n	$\frac{n!}{s^{n+1}}$		
[4] عدد كسر t^n	جاما $\frac{\Gamma(n+1)}{s^{n+1}}$		
[5] e^{at}	$\frac{1}{s-a}$		
[6] $\sin at$	$\frac{a}{s^2+a^2}$		
[7] $\cos at$	$\frac{s}{s^2+a^2}$		

← بعض القوانين المثلثية والأسية المستخدمة.

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

* Properties of Laplace transform:-

1 → Linearity property,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s).$$

Ex.- Find $\mathcal{L}\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$

Sol.

$$4\mathcal{L}\{e^{5t}\} + 6\mathcal{L}\{t^3\} - 3\mathcal{L}\{\sin 4t\} + 2\mathcal{L}\{\cos 2t\}$$

$$= \frac{4}{s-5} + \frac{6(3!)}{s^4} - \frac{3(4)}{s^2+16} + \frac{2(s)}{s^2+4}$$

$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}, \quad s > 5$$

2 → First translation or shifting property.

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at} F(t)\} = f(s-a)$.

Ex.- find $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

Sol.

أولاً: الدوال هي $\cos 6t$, $\sin 6t$

$$\mathcal{L}\{\cos 6t\} = \frac{s}{s^2+36}, \quad \mathcal{L}\{\sin 6t\} = \frac{6}{s^2+36}$$

ثانياً: نستخدم خاصية التوزيع

$$3\mathcal{L}\{e^{-2t}\cos 6t\} - 5\mathcal{L}\{e^{-2t}\sin 6t\}$$

نستخدم الإزاحة

$$= 3\frac{s+2}{(s+2)^2+36} - 5\frac{6}{(s+2)^2+36} = \frac{3s-24}{(s+2)^2+36}$$

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3 → Second translation or shifting property.

If $\mathcal{L}\{f(t)\} = f(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

Then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$.

Ex. Find $\mathcal{L}(f(t))$ if $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$

Sol. $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$

$$\mathcal{L}\{f(t)\} = e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1}$$

4 → Change of scale property.

If $\mathcal{L}\{f(t)\} = f(s)$, then $\mathcal{L}\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

Ex. Given $\mathcal{L}\left[\frac{\sin t}{t}\right] = \tan^{-1}(1/s)$. Find $\mathcal{L}\left[\frac{\sin 3t}{t}\right]$.

Sol.

$\mathcal{L}\left\{\frac{\sin 3t}{t}\right\}$

لكي تصبح الـ $f(at)$
للابلاس يوجد أمام $3t$
نضرب في 3 ونقسم على 3

$$3\mathcal{L}\left\{\frac{\sin 3t}{3t}\right\} = \cancel{3} \cdot \frac{1}{\cancel{3}} \tan^{-1} \frac{1}{(s/3)} = \tan^{-1}\left(\frac{3}{s}\right).$$

4/5

5 → Laplace transforms of derivatives.

*→ If $\mathcal{L}\{f(t)\} = F(s)$, then for $n = 1, 2, 3, \ldots$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

, $n=2$ لما يكون هناك الأس

EX: Find $\mathcal{L}\{t^2 \cos 3t\}$

Sol.

$$F(s) = \mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$$

, $n=2$

نفاضل مرتين

$$\frac{dF(s)}{ds} = \frac{(s^2+9) - 2s^2}{(s^2+9)^2} = \frac{9-s^2}{(s^2+9)^2}$$

$$\frac{d^2F(s)}{ds^2} = \frac{-2s(s^2+9)^2 - 4s(s^2+9)(9-s^2)}{(s^2+9)^4}$$

$$= \frac{\cancel{(s^2+9)}\left[-2s(s^2+9) - 4s(9-s^2)\right]}{(s^2+9)^{\cancel{4}}}$$

$$= \frac{2s^3 - 54s}{(s^2+9)^3}$$

$$\mathcal{L}\{t^2 \cos 3t\} = (-1)^2 \frac{d^2F(s)}{ds^2} = \frac{2s^3 - 54s}{(s^2+9)^3}.$$

*→ ~~If $\mathcal{L}\{$~~ * **Exercise**

1) Evaluate:

1) $\mathcal{L}\{4\cos^2 2t + \cosh^2 4t + 2t^2 - e^{-t}\}$.

$$4\mathcal{L}\{\cos^2 2t\} + \mathcal{L}\{\cosh^2 4t\} + 2\mathcal{L}\{t^2\} - \mathcal{L}\{e^{-t}\}.$$

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$$\mathcal{L}\{\cos^2 2t\} = \mathcal{L}\left[\frac{1}{2}(1+\cos 4t)\right] = \frac{1}{2}\left[\mathcal{L}\{1\} + \mathcal{L}\{\cos 4t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right].$$

$$\mathcal{L}\{\cosh^2 4t\} = \mathcal{L}\left\{\left(\frac{e^{4t}+e^{-4t}}{2}\right)^2\right\} = \frac{1}{4}\left[\mathcal{L}\{e^{8t}+2+e^{-8t}\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-8} + \frac{2}{s} + \frac{1}{s+8}\right]$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

$$\therefore \mathcal{L}\{4\cos^2 2t + \cosh^2 4t + 2t^2 - e^{-t}\}$$

$$= \frac{4}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right] + \frac{1}{4}\left[\frac{1}{s-8} + \frac{2}{s} + \frac{1}{s+8}\right] + \frac{4}{s^3} + \frac{1}{s+1},$$

$s > 8$.

2] If $f(t) = \begin{cases} 0 & 0<t<2 \\ 4 & t>0 \end{cases}$

Sol. Second translation, $f(t)=4$, $\mathcal{L}\{4\} = \frac{4}{s} = f(s)$

$$\mathcal{L}\{f(t)\} = e^{-0s} f(s) = \frac{4}{s}.$$

3] $\mathcal{L}\{(t+2)^2 e^t\}$, First translation

$$\mathcal{L}\{(t+2)^2 e^t\} = \mathcal{L}\{(t^2+4t+4)e^t\} = \mathcal{L}\{t^2 e^t\} + 4\mathcal{L}\{te^t\} + 4\mathcal{L}\{e^t\}$$

$$= \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}, \quad s>1.$$

4] If $\mathcal{L}\{F(t)\} = \dfrac{s^2 - s + 1}{(2s+1)^2(s-1)}$

Find $\mathcal{L}\{e^{-t}\sin^2 t + F(2t)\}$

Sol.

$$\mathcal{L}\{e^{-t}\sin^2 t + F(2t)\} = \mathcal{L}\{e^{-t}\sin^2 t\} + \mathcal{L}\{F(2t)\}$$

$$\mathcal{L}\{e^{-t}\sin^2 t\} = \mathcal{L}\{e^{-t}\left(\tfrac{1}{2}(1 - \cos 2t)\right)\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{-t}\} - \mathcal{L}\{(\cos 2t)e^{-t}\}\right], \text{ First transl.}$$

$$= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right]$$

$$\mathcal{L}\{F(2t)\} = \frac{1}{2} f\left(\frac{s}{2}\right), \text{ change of scale.}$$

$$f(s) = \mathcal{L}\{F(t)\} = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$$

$$\mathcal{L}\{F(2t)\} = \frac{1}{2}\left[\frac{(s/2)^2 - s/2 + 1}{(2\,s/2 + 1)^2(s/2 - 1)}\right]$$

$$= \frac{1}{4}\left[\frac{s^2 - 2s + 4}{(s+1)^2(s-2)}\right]$$

$$\therefore \mathcal{L}\{e^{-t}\sin^2 t + F(2t)\} = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right] + \frac{1}{4}\left[\frac{s^2 - 2s + 4}{(s+1)^2(s-2)}\right]$$