10 Some Special Functions — Section 4

I The Gamma Function:-

If $n>0$ $\quad \Gamma(n)=\int_0^{\infty} u^{n-1} e^{-u}\, du$

→ Some properties:-

1. $\Gamma(n+1)=n\Gamma(n), \ n>0$
2. $\Gamma(\frac{1}{2})=\sqrt{\pi}$.
3. $\Gamma(p)\Gamma(1-p)=\dfrac{\pi}{\sin p\pi}, \quad 0<p<1$
4. For large n. $\Gamma(n+1)\sim\sqrt{2\pi n}\, n^{n} e^{-n}$
5. $\Gamma(n)=\dfrac{\Gamma(n+1)}{n}$
6. $\Gamma(n+1)=n!\ , \quad n=1,2,3,\dots$

Ex:- Find $\mathcal{L}\{t^{-1/2}\}$

Sol.

$$\mathcal{L}\{t^{-1/2}\}=\frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}=\frac{\Gamma(\frac{1}{2})}{s^{1/2}}=\frac{\sqrt{\pi}}{s^{1/2}}=\sqrt{\frac{\pi}{s}}, \quad s>0$$

* II → The Sine and Cosine integrals:-

$$* \ S_i(t)=\int_0^{t}\frac{\sin u}{u}\, du \quad , \quad * \ C_i(t)=\int_t^{\infty}\frac{\cos u}{u}\, du$$

*** Ex:-**

Find $\mathcal{L}[S_i(t)]$

Sol.

$$\mathcal{L}\{S_i(t)\}=\mathcal{L}\left\{\int_0^{t}\frac{\sin u}{u}\, du\right\}$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\}=\tan^{-1}\frac{1}{s}=f(s) \quad , \quad \mathcal{L}\left\{\int_0^{t}\frac{\sin u}{u}\, du\right\}=\frac{f(s)}{s}=\frac{\tan^{-1}\frac{1}{s}}{s}$$

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Ex.- Find $\mathcal{L}\{Ci(t)\}$

Sol.- $\mathcal{L}\{Ci(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\, du\right\}$.

$\mathcal{L}\left\{\frac{\cos t}{t}\right\} = \frac{1}{2}\ln(s^2+1) = f(s)$, t is a time domain, $t>0$

$\therefore \mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\, du\right\} = \frac{f(s)}{s} = \frac{\ln(s^2+1)}{2s}$.

* 12 → The Exponential integral:

$*\; E_i(t) = \int_t^\infty \frac{e^{-u}}{u}\, du$

Ex.-

Find $\mathcal{L}\{E_i(t)\}$.

Sol.

$\mathcal{L}\{E_i(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u}\, du\right\}$, integral laplace

$\mathcal{L}\left\{\frac{e^{-t}}{t}\right\} = \ln(s+1) = f(s)$, division by t

, $t \to \infty$: $\mathcal{L}\{E_i(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u}\, du\right\} = \frac{f(s)}{s}$

$= \frac{\ln(s+1)}{s}$, $s>0$

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* جدول تحويل الأشكال المميزة :-

$f(t)$	$f(s)$
1] $\mathcal{L}\{f(t)\}=f(s)$, $\mathcal{L}\{e^{at}f(t)\}$	$f(s-a)$
2] $\mathcal{L}\{f(t)\}=f(s)$ $F(t)=\begin{cases} f(t-a), & t>a \\ 0 & t<a \end{cases}$	$e^{-as}f(s)$
3] $\mathcal{L}\{f(t)\}=f(s)$ $\mathcal{L}\{t^n f(t)\}$	$(-1)^n \dfrac{d^n f(s)}{ds^n}$
4] $f'(t)$ المشتقة الأولى	$s f(s) - f(0)$ $t=0$
5] $f''(t)$ الثانية "	$s^2 f(s) - s f(0) - f'(0)$ تفاضل الدالة الأصلية
6] $f'''(t)$ الثالثة "	$s^3 f(s) - s^2 f(0) - s f'(0) - f''(0)$
7] $\mathcal{L}\{f(t)\}=f(s)$ $\mathcal{L}\{\int_0^t f(u)\,du\}$	$\dfrac{f(s)}{s}$
8] $\mathcal{L}\{f(t)\}=f(s)$ $\int_0^\infty f(t)\,dt$	$s=0$, $f(0)$ $s=0$
9] $F(t+T)=F(t)$ دالة دورية $T>0$, periodic function	$\mathcal{L}\{F(t)\}=\dfrac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$

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77 Find $\int_0^\infty e^{-t} \frac{\sin t}{t} dt$

Sol. $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1} s$, division by t.

$$\int_0^\infty e^{-t} \frac{\sin t}{t} dt = \mathcal{L}\left\{\frac{\sin t}{t}\right\}_{at\ s=1} = \frac{\pi}{2} - \tan^{-1} 1 =$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

OR $\mathcal{L}\left\{\frac{\sin t}{t}\right\}_{at\ s=1} = \tan^{-1}\frac{1}{s}$, $s = 1$,

$$= \tan^{-1} 1 = \boxed{\frac{\pi}{4}}$$

63 Find $\mathcal{L}\{e^{-t} Si(2t)\}$

Sol. $\mathcal{L}\{e^{-t} Si(2t)\}$, First Translation, $F(t) = Si(2t)$

$\mathcal{L}\{Si(2t)\} = \frac{\tan^{-1} 2/s}{s}$, $\frac{1}{a} f\left(\frac{s}{a}\right)$, $\mathcal{L}\{Si(t)\} = \frac{\tan^{-1} 1/s}{s}$

$\downarrow$ Change of scale.

$$\mathcal{L}\{e^{-t} Si(2t)\} = \frac{\tan^{-1}\frac{2}{s+1}}{s+1}$$

62 Find $\mathcal{L}\{e^{-3t} Ei(t)\}$, $\mathcal{L}\{t\, Ei(t)\}$

Sol.

* $\mathcal{L}\{e^{-3t} Ei(t)\}$, First translation, $F(t) = Ei(t)$,

$\mathcal{L}\{Ei(t)\} = \ln(s+1)/s$

$\mathcal{L}\{e^{-3t} Ei(t)\} = \ln(s+4)/s+3$

5) * $\mathcal{L}\{t\,E_i(t)\}$ Laplace of derivatives

$f(t) = E_i(t)$, $\mathcal{L}\{E_i(t)\} = \dfrac{\ln(S+1)}{S}$

$\mathcal{L}\{t\,E_i(t)\} = (-1)^1 \dfrac{dF(S)}{dS}$, $\dfrac{dF(S)}{dS} = \dfrac{\frac{1}{S+1}\cdot S - \ln(S+1)}{S^2}$

$= \dfrac{S - (S+1)\ln(S+1)}{(S+1)\,S^2}$

$\mathcal{L}\{t\,E_i(t)\} = -\left[\dfrac{S-(S+1)\ln(S+1)}{S^2(S+1)}\right]$

$= \dfrac{(S+1)\ln(S+1) - S}{S^2(S+1)}$

6) $\mathcal{L}\{t\,S_i(t)\}$, $\mathcal{L}\{e^{2t}\,S_i(t)\}$

Sol.

$\mathcal{L}\{t\,S_i(t)\}$, Laplace of derivatives , $f(t) = S_i(t)$

$\mathcal{L}\{S_i(t)\} = \tan^{-1}\left(\frac{1}{S}\right) = F(S)$

$\dfrac{dF(S)}{dS} = \dfrac{\frac{1}{1+\left(\frac{1}{S}\right)^2}\left(\frac{S}{S^2}\right)\cdot S - \tan^{-1}\left(\frac{1}{S}\right)}{S^2}$

$= \dfrac{\frac{-1}{S+\frac{1}{S}} - \tan^{-1}\left(\frac{1}{S}\right)}{S^2}$

$= \dfrac{\frac{-S}{S^2+1} - \tan^{-1}\left(\frac{1}{S}\right)}{S^2}$

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* $\mathcal{L}\{e^{2t} S_i(t)\}$, First translation $f(t) = S_i(t)$

$$\mathcal{L}\{f(t)\} = \frac{\tan^{-1} 1/s}{s},$$

$$\mathcal{L}\{e^{2t} S_i(t)\} = \frac{\tan^{-1} \frac{1}{s-2}}{s-2}$$

47 * Find $\mathcal{L}\{\frac{e^{-st}}{\sqrt{t}}\}$, First translation $f(t) = \frac{1}{\sqrt{t}} = t^{-1/2}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{(-\frac{1}{2}+1)}} = \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$= \sqrt{\frac{\pi}{s}} = f(s)$$

$$\mathcal{L}\{\frac{e^{-st}}{\sqrt{t}}\} = \sqrt{\frac{\pi}{s+s}} = \sqrt{\frac{\pi}{2s}}.$$

* $\mathcal{L}\{t^{7/2} e^{3t}\}$, $\mathcal{L}\{t^{7/2}\} = \frac{\Gamma(\frac{7}{2}+1)}{s^{7/2+1}} = \frac{\Gamma(\frac{9}{2})}{s^{9/2}}$

$$\Gamma(\tfrac{9}{2}) = \tfrac{7}{2}\Gamma(\tfrac{7}{2}) = \tfrac{7}{2}\cdot\tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$= \frac{105}{8}\sqrt{\pi}$$

$\mathcal{L}\{t^{7/2}\} = \frac{105\sqrt{\pi}}{8 s^{9/2}}$, $\mathcal{L}\{t^{7/2} e^{3t}\}$, First trans.

$$\mathcal{L}\{t^{7/2} e^{3t}\} = \frac{105\sqrt{\pi}}{8(s-3)^{9/2}}.$$