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Section 5

The inverse Laplace transform

$\mathcal{L}^{-1}(F(s)) = f(t)$

→ Table of inverse Laplace transform:

	$F(s)$	$f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^{n+1}}$, $n = 0, 1, 2, \ldots$	$\frac{t^n}{n!}$
4	$\frac{1}{s^{n+1}}$, n كسر	$\frac{t^n}{\Gamma(n+1)}$
5	$\frac{1}{s-a}$	e^{at}
6	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
7	$\frac{s}{s^2+a^2}$	$\cos at$
8	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
9	$\frac{s}{s^2-a^2}$	$\cosh at$

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Ex Find $\mathcal{L}^{-1}\left[\frac{4}{S-2} - \frac{3S}{S^2+16} + \frac{5}{S^2+4}\right]$ Linearity property:

Sol.

$$\mathcal{L}^{-1}\left\{\frac{4}{S-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{S}{S^2+16}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{S^2+4}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

* first translation or shifting property:-

$$\mathcal{L}^{-1}(F(S-a)) = e^{at} f(t).$$

Ex. find $\mathcal{L}^{-1}\left(\left[\frac{6S-4}{S^2-4S+20}\right] + \left[\frac{1}{\sqrt{2S+3}}\right]\right)$.

Sol.

$$* \quad \mathcal{L}^{-1}\left[\frac{6S-4}{S^2-4S+20}\right] = \mathcal{L}^{-1}\left[\frac{6(S-2)+8}{(S-2)^2-4+20}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{6(S-2)}{(S-2)^2+16} + \frac{8}{(S-2)^2+16}\right]$$

$$= 6\mathcal{L}^{-1}\frac{(S-2)}{(S-2)^2+16} + 8\mathcal{L}^{-1}\frac{1}{(S-2)^2+16}$$

$$= 6e^{2t}\cos 4t + \frac{8}{4}e^{2t}\sin 4t.$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t.$$

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$$* \; \mathcal{L}^{-1} \frac{1}{\sqrt{2S+3}} = \mathcal{L}^{-1} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{S+\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{2}} \mathcal{L}^{-1} \frac{1}{(S+\frac{3}{2})^{1/2}} \quad , \quad n+1 = \frac{1}{2}, \; n = -\frac{1}{2}$$

$$= \frac{1}{\sqrt{2}} \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} \cdot e^{-\frac{3}{2}t} \quad , \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \; t^{-1/2} e^{-\frac{3}{2}t}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{6S-4}{S^2-4S+20} + \frac{1}{\sqrt{2S+3}}\right)$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t + \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-3/2\, t}$$

* Second translation :-

$$\mathcal{L}^{-1}(e^{-as}F(S)) = \begin{cases} F(t-a) & , \; t > a \\ 0 & \; t < a \end{cases}$$

Ex:- Find $\mathcal{L}^{-1}\left[\frac{e^{4-3S}}{(S+4)^{5/2}}\right]$

Sol: $\mathcal{L}^{-1} \frac{1}{(S+4)^{5/2}} = e^{-4t} \frac{t^{3/2}}{\Gamma(\frac{5}{2})} = \frac{e^{-4t}\, t^{3/2}}{3/2 \cdot \frac{1}{2} \Gamma(\frac{1}{2})} = \frac{4}{3\sqrt{\pi}} e^{-4t} t^{3/2}$

$n+1 = \frac{5}{2}$

$n = \frac{5}{2} - 1 = \frac{3}{2}$

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$$e^{4-3s} = e^4 \cdot e^{-3s} \quad \therefore a=3$$

$$\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = \begin{cases} \frac{4e^4}{3\sqrt{\pi}} \cdot e^{-4(t-3)} \cdot (t-3)^{3/2}, & t>3 \\ 0 & t<3 \end{cases}$$

Ex:- find $\mathcal{L}^{-1}\left[\frac{3s+16}{s^2-s-6}\right]$

Sol.

$$\mathcal{L}^{-1}\left\{\frac{3\left(s+\frac{16}{3}\right)}{\left(s-\frac{1}{2}\right)^2-\frac{1}{4}-6}\right\} = \mathcal{L}\, 3\left\{\frac{s-\frac{1}{2}+\frac{1}{2}+\frac{16}{3}}{\left(s-\frac{1}{2}\right)^2-\frac{25}{4}}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{\left(s-\frac{1}{2}\right)+\frac{35}{6}}{\left(s-\frac{1}{2}\right)^2-\frac{25}{4}}\right\} = 3\mathcal{L}\left\{\frac{\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2-\frac{25}{4}} + \frac{35/6}{\left(s-\frac{1}{2}\right)^2-\frac{25}{4}}\right\}$$

$$= 3\left[\cos\frac{5}{2}t\, e^{\frac{1}{2}t} + \frac{35}{6}\cdot\frac{2}{5}\sin\frac{5}{2}t\, e^{1/2 t}\right]$$

$$= 3\left[e^{\frac{1}{2}t}\left(\cos\frac{5}{2}t + \frac{7}{3}\sin\frac{5}{2}t\right)\right]$$