1// **Section 3**

2) *→ Laplace transforms of derivatives.

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}[F'(t)] = sf(s) - F(0)$.

3) If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}[F''(t)] = s^2 f(s) - sF(0) - F'(0)$.

6] *→ Division by t.

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(u)\,du$.

where $f(u) = \mathcal{L}[F(t)]$.

Ex:- Find $\mathcal{L}\left[\frac{\sin t}{t}\right]$

$$\mathcal{L}\{\sin t\} = \frac{1}{u^2+1} = f(u)$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty f(u)\,du = \int_s^\infty \frac{1}{u^2+1}\,du = \tan^{-1}u\Big|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$= \tan^{-1}\frac{1}{s}.$$

7] *→ Laplace transforms of integrals.

If $\mathcal{L}\{F(t)\} = f(u)$, then $\mathcal{L}\left[\int_0^t F(u)\,du\right] = \frac{f(s)}{s}$.

Ex:- Find $\mathcal{L}\left[\int_0^t \frac{\sin u}{u}\,du\right]$

Sol. $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}s$, $\mathcal{L}\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s} f(s)$

$= f(s)$

$$= \frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}s\right).$$

8] → Periodic functions.

If $F(t)$ have period $T>0$. So, $F(t+T)=F(t)$. Then

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$$

Ex:- $F(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$

$T = 2\pi$

$$\mathcal{L}\{F(t)\} = \frac{\int_0^{2\pi} e^{-st}\sin t\,dt}{1-e^{-2\pi s}}$$, integeration by part.

$I = \int_0^{2\pi} e^{-st}\sin t\,dt$, Let $u = e^{-st} \longrightarrow du = -se^{-st}\,dt$

$dv = \sin t\,dt \longrightarrow v = -\cos t$

$$I = -e^{-st}\cos t\Big|_0^{\pi} - s\int_0^{\pi} e^{-st}\cos t\,dt = -(-e^{-s\pi} - 1) - s\int_0^{\pi} e^{-st}\cos t\,d$$

$= (1+e^{-s\pi}) - \ldots$

Let $u = e^{-st}$, $du = -se^{-st}\,dt$

$dv = \cos t\,dt$, $v = \sin t$

$$I = (1+e^{-s\pi}) - s\Big[e^{-st}\sin t\Big|_0^{\pi} + s\underbrace{\int_0^{\pi} e^{-st}\sin t\,dt}_{I}\Big]$$

$$(1+s^2)\,I = (1+e^{-s\pi}) \underbrace{\cancel{-s[0]}}_{\text{zero}}$$

$$I = \frac{1+e^{-s\pi}}{1+s^2}$$

$$\boxed{\mathcal{L}\{F(t)\} = \frac{1+e^{-s\pi}}{(1+s^2)(1-e^{-2\pi s})}}$$

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9] → Evaluation of integrals.

If $\mathcal{L}[F(t)] = f(s)$, then $\int_0^{\infty} e^{-st} F(t)\,dt = f(s)$

$s \to 0$, $\int_0^{\infty} F(t)\,dt = f(0)$.

Ex.:- Evaluate.

1] $\int_0^{\infty} t e^{-2t} \cos t \, dt$

Sol. $\int_0^{\infty} e^{-2t} (t \cos t)\, dt$

$\mathcal{L}[t \cos t]$, laplace transform of derivatives.

$\mathcal{L}[\cos t] = \frac{s}{s^2+1}$, $\frac{d}{ds} f(s) = \frac{(s^2+1) - 2s^2}{(s^2+1)^2} = \frac{-s^2+1}{(s^2+1)^2}$

$f(s) = \mathcal{L}[t \cos t] = (-1)^1 \frac{d f(s)}{ds} = \frac{s^2-1}{(s^2+1)^2}$

$\int_0^{\infty} e^{-2t} (t \cos t)\, dt = f(2)$, $s = 2$

$= \frac{2^2-1}{(2^2+1)^2} = \frac{3}{5^2} = \frac{3}{25}$

2] $\int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t}\, dt$.

Sol. $\mathcal{L}\left[\frac{e^{-t} - e^{-3t}}{t}\right]$, Division by t.

$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$, $\mathcal{L}[e^{-3t}] = \frac{1}{s+3}$

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$$\mathcal{L}\left\{\frac{e^{-t}-e^{-3t}}{t}\right\}=\int_S^\infty \frac{1}{u+1}-\frac{1}{u+3}\,du$$

$$=\ln|u+1|-\ln|u+3|\Big|_S^\infty$$

$$=\ln\left|\frac{u+1}{u+3}\right|\Bigg|_S^\infty=-\ln\left|\frac{S+1}{S+3}\right|$$

$$=\ln\left(\frac{S+3}{S+1}\right)=F(S)$$

$$\int_0^\infty\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt=F(0)\ ,\ S=0$$

$$=\ln\left(\frac{3}{1}\right)=\ln 3 .$$

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$\mathcal{L}\left\{\frac{\sinh t}{t}\right\}=$ division by t.

$$\mathcal{L}\{\sinh t\}=\frac{1}{S^2-1}$$

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\}=\int_S^\infty \frac{1}{u^2-1}\,du=-\tanh^{-1}u\Big|_S^\infty=\tanh^{-1}S=F(S)$$

$$2\int_0^\infty\left(\frac{e^{-t}-e^{-3t}}{2\,t}\right)dt=2\int_0^\infty e^{-2t}\left(\frac{e^{t}-e^{-t}}{2}\cdot\frac{1}{t}\right)dt$$

$$=2\int_0^\infty e^{-2t}\left(\frac{\sinh t}{t}\right)dt\ ,\ S=2$$

$$=2\,F(2)=2\tanh^{-1}2 .$$

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