In what follows, all varieties-schemes are defined over $\mathbb{C}$

# 1 Principal bundles

Let $X$ be a algebraic variety and $G$ an affine algebraic group.

**Definition 1.0.1.** A $G$-principal bundle is a variety is a morphism $\pi : P \to X$ and a right $G$-action on $P$ preserving $\pi$. Moreover, we require that every $x \in X$ has an étale neighborhood $U$ such that we have a $G$-equivariant isomorphism $P_{|U} \xrightarrow{\sim} G \times U$ making the following diagram commute:

$$\begin{tikzcd} P_{|U} \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & G \times U \arrow[dl, "p_2"] \\ & U & \end{tikzcd}$$

The following proposition allows us to work with zariski locally trivial bundles instead of étale, whenever the base scheme is a smooth scheme and $G$ is a connected reductive group.

**Proposition 1.0.2** (Borel-springer, Steinberg)**.** If $X$ is a smooth curve and $G$ is a connected reductive group, then any principal $G$-bundle on $X$ is Zariski locally trivial.

Let $\mathrm{Bun}_G(X)$ denote the moduli stack of $G$-principal bundles on $X$.

**Proposition 1.0.3.** The pseudofunctor $\mathrm{Bun}_G(X)$ defined by

$$\mathbb{C} - alg \ni A \mapsto \mathrm{Bun}_G(X)(A) \in \mathbf{Grpd}$$

where $\mathrm{Bun}_G(X)(A)$ is the groupoid whose objects are $\left\{ G\text{-bundles } P \to X \times \mathrm{Spec}\,(A) \right\}$ and whose morphisms are isomorphisms of $G$-bundles, is an algebraic stack.

We now restrict our attention to the $\mathrm{Bun}_G^\circ(X) \subset \mathrm{Bun}_G(X)$ defined as the substack of *stable* $G$-bundles. It will turn out to be represented by a nonsingular variety. We start by looking only at the case $G = GL_n$.

# 2 The case of $G = \mathbf{GL}_n$

Recall that a vector bundle $E$ on a curve $X$ is said to be stable if it is slope stable. This means:

**Definition 2.0.1.** A vector bundle $E$ on a curve $X$ is said to be **slope stable** if for every subbundle $F \subset E$ we have

$$\frac{deg(F)}{\mathrm{rank}\ F} < \frac{deg(E)}{\mathrm{rank}\ E}.$$

*Remark.* Suppose $E$ is stable, then $Hom_X(E, E) = 0$. Indeed, suppose $0 \neq \phi \in Hom_X(E, E)$ then $\phi(E)$ is a subbundle of $E$, which has slope $\leq$ that of $E$. To see this last, use the basic fact that the existence of a nonzero map of line bundles $L_1 \to L_2$ implies that $deg(L_1) < deg(L_2)$.

**Lemma 2.0.2.** Suppose $E$ is stable, then $Hom_X(E, E) = 0$.

*Proof.* Suppose $0 \neq \phi \in Hom_X(E, E)$, then $\phi(E)$ is a sub-bundle of $E$. Moreover, we have a map

$$\wedge^{\text{rk } E} E \to \wedge^{\text{rk } \phi(E)} \phi(E).$$

But then $\wedge^{\text{rk } E} E^* \otimes \wedge^{\text{rk } \phi(E)} \phi(E)$ corresponds to an effective divisor. This gives

$$0 \leq deg(\wedge^{\text{rk } E} E^* \otimes \wedge^{\text{rk } \phi(E)} \phi(E)) = deg(\phi(E)) - deg(E).$$

We conclude that $\frac{deg(\phi(E))}{\text{rk } \phi(E)} \geq \frac{deg(E)}{\text{rk } E}$ which contradicts the assumption that $E$ is stable. $\square$