In what follows, all varieties-schemes are defined over $\mathbb{C}$

1 Principal bundles

Let X be a algebraic variety and G an affine algebraic group.

Definition 1.0.1. A G-principal bundle is a variety is a morphism $\pi : P \to X$ and a right G-action on P preserving π. Moreover, we require that every $x \in X$ has an étale neighborhood U such that we have a G-equivariant isomorphism $P_{|U} \xrightarrow{\sim} G \times U$ making the following diagram commute:

$$\begin{tikzcd} P_{|U} \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & G \times U \arrow[dl, "p_2"] \\ & U & \end{tikzcd}$$

The following proposition allows us to work with zariski locally trivial bundles instead of étale, whenever the base scheme is a smooth scheme and G is a connected reductive group.

Proposition 1.0.2 (Borel-springer, Steinberg)**.** If X is a smooth curve and G is a connected reductive group, then any principal G-bundle on X is Zariski locally trivial.

Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-principal bundles on X.

Proposition 1.0.3. The pseudofunctor $\mathrm{Bun}_G(X)$ defined by

$$\mathbb{C} - alg \ni A \mapsto \mathrm{Bun}_G(X)(A) \in \mathbf{Grpd}$$

where $\mathrm{Bun}_G(X)(A)$ is the groupoid whose objects are $\Big\{\text{principal } G\text{-bundles } P \to X \times \mathrm{Spec}\,(A)\Big\}$ and whose morphisms are isomorphisms of G-bundles, is an algebraic stack.

We now restrict our attention to the $\mathrm{Bun}^\circ_G(X) \subset \mathrm{Bun}_G(X)$ defined as the substack of *stable* G-bundles. It will turn out to be represented by a nonsingular variety. We start by looking only at the case $G = GL_n$.

2 The case of $G = \mathbf{GL}_n$

Recall that a vector bundle E on a curve X is said to be stable if it is slope stable. This means:

Definition 2.0.1. A vector bundle E on a curve X is said to be **slope stable** if for every subbundle $F \subset E$ we have

$$\frac{deg(F)}{\text{rank } F} < \frac{deg(E)}{\text{rank } E}.$$

Lemma 2.0.2. Suppose E is stable, then $Hom_X(E, E) = \mathbb{C}$.

Proof. Suppose $0 \neq \phi \in Hom_X(E, E)$, then $\phi(E)$ is a sub-bundle of E and we get a short exact sequence

$$0 \to K \to E \to \phi(E) \to 0.$$

Now we claim that $deg(K) \geq 0$. Indeed, by assumption we have

$$0 \leq \mu(E) - \mu(\phi(E)) \leq \frac{deg(E)}{\text{rk } \phi(E)} - \frac{deg(\phi(E))}{\text{rk } \phi(E)} = \frac{deg(K)}{\text{rk } \phi(E)}.$$

Now since the *degree* function is additive on short exact sequences, we conclude that $deg(\phi(E)) \geq deg(E)$, giving

$$\mu(\phi(E)) \geq \mu(E).$$

From this we conclude that $\phi(E) = E$. It remains to show that ϕ is multiplication by a scalar. For this, let $x \in X$ be a point and $\lambda \in \mathbb{C}$ an eigenvalue of

$$\phi_{|_x} : E_{|_x} \to E_{|_x}$$

then by choice of λ we know that $\phi - \lambda$ is not an isomorphism on E. Therefore by the previous part of this proof, we conclude that $\phi - \lambda = 0$. $\square$

It is well know that the substack $\text{Bun}_G^\circ(X) \subset \text{Bun}_G(X)$ of stable bundles is representable by a smooth variety. Our next goal is to calculate the dimension of this variety. Since $\text{Bun}_G^\circ(X)$ is smooth, we only need to calculate the dimension of the tangent spaces at an arbitrary point $[E]$.

Proposition 2.0.3. $\dim_{\mathbb{C}} T_{[E]} \text{Bun}_G^\circ(X) = Ext_X^1(E, E) = (g-1)(\text{rk } E)^2 + 1$, where g is the genus of the curve X.

Proof. The first equality follows by the basic theory of infinitesimal deformations, the second equality follows from **Riemann-Roch** applied to $E \otimes E^*$, combined with *Lemma 2.0.2*. $\square$