In what follows, all varieties-schemes are defined over $\mathbb{C}$

1 Principal bundles

Let X be a algebraic variety and G an affine algebraic group.

Definition 1.0.1. A G-principal bundle is a variety is a morphism $\pi : P \to X$ and a right G-action on P preserving π. Moreover, we require that every $x \in X$ has an étale neighborhood U such that we have a G-equivariant isomorphism $P_{|U} \xrightarrow{\sim} G \times U$ making the following diagram commute:

$$
\begin{array}{ccc}
P_{|U} & \xrightarrow{\ \sim\ } & G \times U \\
& {\scriptstyle \pi}\searrow \quad \swarrow {\scriptstyle p_2} & \\
& U &
\end{array}
$$

The following proposition allows us to work with zariski locally trivial bundles instead of étale, whenever the base scheme is a smooth scheme and G is a connected reductive group.

Proposition 1.0.2 (Borel-springer, Steinberg)**.** If X is a smooth curve and G is a connected reductive group, then any principal G-bundle on X is Zariski locally trivial.

Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-principal bundles on X.

Proposition 1.0.3. The pseudofunctor $\mathrm{Bun}_G(X)$ defined by

$$\mathbb{C} - alg \ni A \mapsto \mathrm{Bun}_G(X)(A) \in \mathbf{Grpd}$$

where $\mathrm{Bun}_G(X)(A)$ is the groupoid whose objects are $\big\{ \text{principal } G\text{-bundles } P \to X \times \mathrm{Spec}\,(A) \big\}$ and whose morphisms are isomorphisms of G-bundles, is an algebraic stack.

We now restrict our attention to the $\mathrm{Bun}_G^{\circ}(X) \subset \mathrm{Bun}_G(X)$ defined as the substack of *stable* G-bundles. It will turn out to be represented by a nonsingular variety. We start by looking only at the case $G = GL_n$.

2 The case of $G = \mathbf{GL}_n$

Recall that a vector bundle E on a curve X is said to be stable if it is slope stable. This means:

Definition 2.0.1. A vector bundle E on a curve X is said to be **slope stable** if for every subbundle $F \subset E$ we have

$$\frac{deg(F)}{\mathrm{rank}\ F} < \frac{deg(E)}{\mathrm{rank}\ E}.$$

Remark. Suppose E is stable, then $Hom_X(E, E) = 0$. Indeed, suppose $0 \neq \phi \in Hom_X(E, E)$ then $\phi(E)$ is a subbundle of E, which has slope $\leq$ that of E. To see this last, use the basic fact that the existence of a nonzero map of line bundles $L_1 \to L_2$ implies that $deg(L_1) < deg(L_2)$.

Lemma 2.0.2. Suppose E is stable, then $Hom_X(E, E) = 0$.

Proof. Suppose $0 \neq \phi \in Hom_X(E, E)$, then $\phi(E)$ is a sub-bundle of E. Moreover, we have a map

$$\wedge^{\mathrm{rk}\ E} E \to \wedge^{\mathrm{rk}\ \phi(E)} \phi(E).$$

But then $\wedge^{\mathrm{rk}\ E} E^* \otimes \wedge^{\mathrm{rk}\ \phi(E)} \phi(E)$ corresponds to an effective divisor. This gives

$$0 \leq deg(\wedge^{\mathrm{rk}\ E} E^* \otimes \wedge^{\mathrm{rk}\ \phi(E)} \phi(E)) = deg(\phi(E)) - deg(E).$$

We conclude that $\frac{deg(\phi(E))}{\mathrm{rk}\ \phi(E)} \geq \frac{deg(E)}{\mathrm{rk}\ E}$ which contradicts the assumption that E is stable. $\square$

It is well know that the substack $\mathrm{Bun}_G^{\circ}(X) \subset \mathrm{Bun}_G(X)$ of stable bundles is representable by a smooth variety. Our next goal is to calculate the dimension of this variety. Since $\mathrm{Bun}_G^{\circ}(X)$ is smooth, we only need to calculate the dimension of the tangent spaces at an arbitrary point $[E]$.

Proposition 2.0.3. $\dim_{\mathbb{C}} T_{[E]} \mathrm{Bun}_G^{\circ}(X) = Ext_X^1(E, E) = (g - 1)(\mathrm{rk}\ E)^2$, where g is the genus of the curve X.

Proof. The first equality follows by the basic theory of infinitesimal doformations, the second equality follows from **Riemann-Roch** combined with *Lemma 2.0.2*. $\square$