

In what follows, all varieties-schemes are defined over \mathbb{C}

1 Principal bundles

Let X be a algebraic variety and G an affine algebraic group.

Definition 1.0.1. A G -principal bundle is a variety is a morphism $\pi : P \rightarrow X$ and a right G -action on P preserving π . Moreover, we require that every $x \in X$ has an étale neighborhood U such that we have a G -equivariant isomorphism $P|_U \xrightarrow{\sim} G \times U$ making the following diagram commute:

$$\begin{array}{ccc} P|_U & \xrightarrow{\sim} & G \times U \\ \pi \searrow & & \swarrow p_2 \\ & U & \end{array}$$

The following proposition allows us to work with zariski locally trivial bundles instead of étale, whenever the base scheme is a smooth scheme and G is a connected reductive group.

Proposition 1.0.2 (Borel-springer, Steinberg). If X is a smooth curve and G is a connected reductive group, then any principal G -bundle on X is Zariski locally trivial.

Let $\text{Bun}_G(X)$ denote the moduli stack of G -principal bundles on X .

Proposition 1.0.3. The pseudofunctor $\text{Bun}_G(X)$ defined by

$$\mathbb{C} - \text{alg} \ni A \mapsto \text{Bun}_G(X)(A) \in \mathbf{Grpd}$$

where $\text{Bun}_G(X)(A)$ is the groupoid whose objects are $\{\text{principal } G\text{-bundles } P \rightarrow X \times \text{Spec}(A)\}$ and whose morphisms are isomorphisms of G -bundles, is an algebraic stack.

We now restrict our attention to the $\text{Bun}_G^\circ(X) \subset \text{Bun}_G(X)$ defined as the substack of *stable* G -bundles. It will turn out to be represented by a nonsingular variety. We start by looking only at the case $G = \text{GL}_n$.

2 The case of $G = \text{GL}_n$

Recall that a vector bundle E on a curve X is said to be stable if it is slope stable. This means:

Definition 2.0.1. A vector bundle E on a curve X is said to be **slope stable** if for every subbundle $F \subset E$ we have

$$\frac{\deg(F)}{\text{rank } F} < \frac{\deg(E)}{\text{rank } E}.$$

Lemma 2.0.2. Suppose E is stable, then $\text{Hom}_X(E, E) = \mathbb{C}$.

Proof. Suppose $0 \neq \phi \in \text{Hom}_X(E, E)$, then $\phi(E)$ is a sub-bundle of E and we get a short exact sequence

$$0 \rightarrow K \rightarrow E \rightarrow \phi(E) \rightarrow 0.$$

Now we claim that $\deg(K) \geq 0$. Indeed, by assumption we have

$$\begin{aligned} \frac{\deg(E)}{\operatorname{rk} E} &\geq \frac{\deg(K)}{\operatorname{rk} K} = \frac{\deg(E) - \deg(\phi(E))}{\operatorname{rk} K} \\ \Rightarrow \frac{\deg(\phi(E))}{\operatorname{rk} K} &\geq \left(\frac{1}{\operatorname{rk} K} - \frac{1}{\operatorname{rk} E} \right) \deg(E) \\ \Rightarrow \mu(\phi(E)) &\geq \mu(E). \end{aligned}$$

From this we conclude that $\phi(E) = E$. It remains to show that ϕ is multiplication by a scalar. For this, let $x \in X$ be a point and $\lambda \in \mathbb{C}$ an eigenvalue of

$$\phi|_x : E|_x \rightarrow E|_x$$

then by choice of λ we know that $\phi - \lambda$ is not an isomorphism on E . Therefore by the previous part of this proof, we conclude that $\phi - \lambda = 0$. \square

It is well known that the substack $\operatorname{Bun}_G^\circ(X) \subset \operatorname{Bun}_G(X)$ of stable bundles is representable by a smooth variety. Our next goal is to calculate the dimension of this variety. Since $\operatorname{Bun}_G^\circ(X)$ is smooth, we only need to calculate the dimension of the tangent spaces at an arbitrary point $[E]$.

Proposition 2.0.3. $\dim_{\mathbb{C}} T_{[E]} \operatorname{Bun}_G^\circ(X) = \operatorname{Ext}_X^1(E, E) = (g - 1)(\operatorname{rk} E)^2 + 1$, where g is the genus of the curve X .

Proof. The first equality follows by the basic theory of infinitesimal deformations, the second equality follows from **Riemann-Roch** applied to $E \otimes E^*$, combined with *Lemma 2.0.2*. \square