In what follows, all varieties-schemes are defined over $\mathbb{C}$

1 Principal bundles

Let X be a algebraic variety and G an affine algebraic group.

Definition 1.0.1. A G-principal bundle is a variety is a morphism $\pi : P \to X$ and a right G-action on P preserving π. Moreover, we require that every $x \in X$ has an étale neighborhood U such that we have a G-equivariant isomorphism $P_{|U} \xrightarrow{\sim} G \times U$ making the following diagram commute:

$$\begin{tikzcd} P_{|U} \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & G \times U \arrow[dl, "p_2"] \\ & U & \end{tikzcd}$$

The following proposition allows us to work with zariski locally trivial bundles instead of étale, whenever the base scheme is a smooth scheme and G is a connected reductive group.

Proposition 1.0.2 (Borel-springer, Steinberg)**.** If X is a smooth curve and G is a connected reductive group, then any principal G-bundle on X is Zariski locally trivial.

Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-principal bundles on X.

Proposition 1.0.3. The pseudofunctor $\mathrm{Bun}_G(X)$ defined by

$$\mathbb{C} - alg \ni A \mapsto \mathrm{Bun}_G(X)(A) \in \mathbf{Grpd}$$

where $\mathrm{Bun}_G(X)(A)$ is the groupoid whose objects are $\Big\{ G\text{-bundles } P \to X \times \mathrm{Spec}\,(A) \Big\}$ and whose morphisms are isomorphisms of G-bundles, is an algebraic stack.

We now restrict our attention to the $\mathrm{Bun}^{\circ}_G(X) \subset \mathrm{Bun}_G(X)$ defined as the substack of *stable* G-bundles. It will turn out to be represented by a nonsingular variety. We start by looking only at the case $G = GL_n$.

2 The case of $G = \mathbf{GL}_n$

Recall that a vector bundle E on a curve X is said to be stable if it is slope stable. This means:

Definition 2.0.1. A vector bundle E on a curve X is said to be **slope stable** if for every subbundle $F \subset E$ we have

$$\frac{deg(F)}{\mathrm{rank}\ F} < \frac{deg(E)}{\mathrm{rank}\ E}.$$

Remark. Suppose E is stable, then $Hom_X(E, E) = 0$. Indeed, suppose $0 \neq \phi \in Hom_X(E, E)$ then $\phi(E)$ is a subbundle of E.