

We follow the presentation in the Stacks Project and Raynaud's *Anneaux locaux henseliens*. Most proofs are different from the ones there however the differences usually turn out to be minimal and are mostly a matter of style. Contents of Ch.1 of Matsumura are assumed. **No claim of originality is made.**

1. QUASI-FINITENESS

The setting here is that of a ring morphism $\phi : R \rightarrow S$ of finite type.

Definition 1.1. ϕ is said to be quasi-finite at $\mathfrak{q} \in \text{spec}(S)$ with $\phi(\mathfrak{q}) = \mathfrak{p}$, if we have that \mathfrak{q} is isolated in $\phi^{-1}(\mathfrak{p})$. We say ϕ is quasi-finite if it is quasi-finite at all $\mathfrak{q} \in \text{spec}(S)$.

The following is an essential tool for giving other (equivalent) definitions of quasi-finiteness.

Lemma 1.2. *Given a field k and a finite type k -algebra S , for $\mathfrak{q} \in \text{spec}(S)$ TFAE:*

- (1) \mathfrak{q} is isolated in $\text{spec}(S)$.
- (2) $\{\mathfrak{q}\}$ is a clopen set in S .
- (3) $S_{\mathfrak{q}}$ is finite over k .
- (4) $\kappa(\mathfrak{q})/k$ is finite and $\dim S_{\mathfrak{q}} = 0$.

If any of these conditions is satisfied, then $S = S_{\mathfrak{q}} \times S'$ for some finite type k -algebra S' .

Proof. • (1) implies (2): \mathfrak{q} isolated in $\text{spec}(S)$ implies that $\exists f \in \text{spec}(S)$ such that $D(f) = \{\mathfrak{q}\}$. Moreover, this implies that $f \in \mathfrak{p}'$ whenever $\mathfrak{p} \subset \mathfrak{p}'$, which implies that $f \in \sqrt{\mathfrak{p}} = \mathfrak{p}$. So \mathfrak{p} is necessarily maximal in S .

- (2) implies (3): $\{\mathfrak{q}\}$ clopen in $\text{spec}(S) \Rightarrow S_{\mathfrak{q}}$ is an artinian k -algebra and $S = S_{\mathfrak{q}} \times S'$. Since a direct product is of finite type *iff* each component is of finite type, we also have $S_{\mathfrak{q}}$ is of finite type $/k$. Moreover, by Nakayama we get that $\mathfrak{q}^n = 0$ for some $n \in \mathbb{N}$. Now as a k -modules, $S_{\mathfrak{q}} \simeq \mathfrak{q} \times S_{\mathfrak{q}}/\mathfrak{q}$ and $\mathfrak{q}^r/\mathfrak{q}^{r+1}$ is isomorphic as a k -module to a finite direct sum of copies of $S_{\mathfrak{q}}/\mathfrak{q}$. Hence the proof would be complete if we can show that $S_{\mathfrak{q}}/\mathfrak{q}$ is finite dimensional over k . This is established by the following lemma whose proof which can be deduced from Noether Normalization and the fact that for x integral over a ring A , A is a field *iff* $A[x]$ is a field.

Lemma: *Let F be a finite type k -algebra which is also a field, then $\dim_k F < \infty$.*

- (3) implies (4): Cohen-Seidenberg gives $\dim S_{\mathfrak{q}} = \dim k$. Finiteness of $\kappa(\mathfrak{q})/k$ follows from it being a quotient of $S_{\mathfrak{q}}$.
- (4) implies (1): $\dim S_{\mathfrak{q}} = 0$ implies that \mathfrak{q} is minimal. Our goal is to show that \mathfrak{q} is maximal, which follows since $S/\mathfrak{q} \hookrightarrow \kappa(\mathfrak{q})$ the last being finite dimensional over k .

□

Remark 1.3. Observe that quasi-finiteness is a local property.

We now move on to some "permanence" type properties of quasi-finite maps.

Proposition 1.4. *The composition of quasi-finite maps is quasi-finite.*

Proof. This can be established via the following point set-topological argument. Consider the following commutative diagrams of topological spaces:

$$\begin{array}{ccc} Z_1 & \subseteq & X \\ \downarrow & & \downarrow \phi \\ Z_2 & \subseteq & Y \end{array}$$

such that Z_2 is clopen in Y and Z_1 is clopen in $\phi^{-1}(Z_2)$, then we necessarily have that Z_1 is clopen in X . Now suppose

$$\operatorname{spec}(C) \xrightarrow{f} \operatorname{spec}(B) \xrightarrow{g} \operatorname{spec}(A)$$

is a composition of quasi-finite maps, and suppose given prime ideals

$$\begin{array}{ccc} \mathfrak{r} & \in \operatorname{spec}(A) \\ \uparrow & \\ \mathfrak{q} & \in \operatorname{spec}(B) \\ \uparrow & \\ \mathfrak{p} & \in \operatorname{spec}(C) \end{array}$$

The proof of the proposition follows by taking $Y := g^{-1}(\mathfrak{r})$, $X := (g \circ f)^{-1}(\mathfrak{r})$, $Z_2 := \mathfrak{q}$ and $Z_1 := \mathfrak{p}$. □

Proposition 1.5. *The base change of quasi-finite maps is quasi-finite.*

Proof. Suppose $\text{spec}(B) \xrightarrow{f} \text{spec}(A)$ is quasi-finite and $A \rightarrow C$ is a ring map. Then $C \rightarrow B \otimes_A C$ is of finite type, and consider any diagram of the form

$$\begin{array}{ccc} \text{spec}(B \otimes_A C) \ni & \mathfrak{r} \longrightarrow \mathfrak{q} & \in \text{spec}(C) \\ & \downarrow & \downarrow \\ \text{spec}(B) \ni & \mathfrak{p} \longrightarrow \mathfrak{s} & \in \text{spec}(A) \end{array}$$

We have two push out diagrams D_1, D_2 :

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & \kappa(\mathfrak{q}) \\ & & \downarrow & & \downarrow \\ & D_1 & & D_2 & \\ & & B & \longrightarrow & B \otimes_A C \longrightarrow (B \otimes_A C) \otimes_C \kappa(\mathfrak{q}) \\ & & \downarrow & & \downarrow \\ & & B & \longrightarrow & B \otimes_A C \longrightarrow (B \otimes_A C) \otimes_C \kappa(\mathfrak{q}) \end{array}$$

and so their composition is also a pushout diagram. Hence we have an isomorphism $(B \otimes_A C) \otimes_C \kappa(\mathfrak{q}) \simeq B \otimes_A \kappa(\mathfrak{q})$.

Now since $B_{\mathfrak{p}} \otimes_A \kappa(\mathfrak{s}) \simeq (B \otimes_A \kappa(\mathfrak{s}))_{\mathfrak{p}} \simeq \kappa(\mathfrak{s})^n$, for some $n \in \mathbb{N}$ we have the following: $B \otimes_A \kappa(\mathfrak{s}) \otimes_{\kappa(\mathfrak{s})} \kappa(\mathfrak{q}) = B \otimes_A \kappa(\mathfrak{q})$ hence $(B \otimes_A \kappa(\mathfrak{q}))_{\mathfrak{r}} = (B \otimes_A \kappa(\mathfrak{q}) \otimes_B B_{\mathfrak{p}})_{\mathfrak{r}} = (\kappa(\mathfrak{q})^n)_{\mathfrak{r}}$.

Finally, the claim follows by the fact that the localization of an artinian ring is an artinian ring.

□

2. QUASI-FINITE VS. FINITE

It is clear from the definition that a finite map is automatically also quasi-finite. Although the converse is not true, there is a close relationship between the two notions. In this section we prove the Zariski Main Theorem.

We start however with some result, which in some sense relates finiteness to the finiteness of the cardinalities of fibers.

Proposition 2.1. *Let $\phi : R \rightarrow S$ be a map of finite type. Suppose that for some minimal prime $\mathfrak{p} \in \text{spec}(R)$, the fiber $\text{spec}(S \otimes_R \kappa(\mathfrak{p}))$ is finite. Then $\exists g \in R - \mathfrak{p}$ such that $\phi|_{D(g)}$ is finite.*

Proof. The set theoretic finiteness on the fiber implies that it has the discrete topology, which implies that the fiber is the spectrum of a finite direct product of artinian local rings, hence the spectrum of an artinian local ring and $S \otimes_R \kappa(\mathfrak{p})$ is a finite $\kappa(\mathfrak{p})$ -algebra.

Since \mathfrak{p} is minimal, every element in \mathfrak{p} and hence $\mathfrak{p}S_{\mathfrak{p}}$ is nilpotent. Moreover by the previous paragraph, for any presentation $\psi : R[X_1, \dots, X_n] \rightarrow S$, the image of $\psi(X_i)$ in $S \otimes_R \kappa(\mathfrak{p})$ is a root of some monic polynomial $\overline{P}_i \in \kappa(\mathfrak{p})[T]$ for which we can choose a representative in $P_i \in R_{\mathfrak{p}}[T]$. Finally, combining the two facts: Since $P_i(X_i) = 0 \in S \otimes_R \kappa(\mathfrak{p}) = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$, we have $P_i(X_i) \in \mathfrak{p}S_{\mathfrak{p}}$ and so $\exists e_i \in \mathbb{N}$ such that $P_i(X_i)^{e_i} = 0 \in S_{\mathfrak{p}}$. We conclude that $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{p}}$ is finite.

Claim: For any ring map $R \rightarrow S$ of finite type, if $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{p}}$ is finite then $\exists g \in R$ such that $R_g \rightarrow S_g$ is also finite.

proof of claim: Suppose $\psi : R[X_1, \dots, X_n] \rightarrow S$ is a presentation for S . Then each X_i is integral over $R_{\mathfrak{p}}$. Hence there exists some monic $P_i \in R_{\mathfrak{p}}[T]$ such that $P_i(X_i) = 0 \in S_{\mathfrak{p}}$ and moreover $\exists s_i \in R - \mathfrak{p}$ such that $s_i P_i(X_i) = 0 \in S$. Finally, chose g to be the product of all the s_i 's multiplied by the product of all the denominators of coefficients of the P_i 's. \square

We shall also need some lemmata on integral elements. The following is an essential tool in the proof of ZMT.

Proposition 2.2. *For a map $\phi : R \rightarrow S$ of finite type and ϕ is quasi-finite at $\mathfrak{q} \in \text{spec}(S)$ and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Then TFAE:*

- (ZMT) $\exists f \in R'$, where R' is the integral closure of R in S , such that $f \notin \mathfrak{q}$ and $R'_f \simeq S_f$.
- (ZMT)' if \exists an algebra $R \subseteq T \subseteq S$ such that T is of finite type over R and S is finite over T and R is integrally closed in S , then $R_{\mathfrak{p}} \simeq S_{\mathfrak{p}}$.

Proof. • (ZMT) \Rightarrow (ZMT)': If R is integrally closed in S , then by (ZMT) we must have $R_f \simeq S_f$ for some $f \notin \mathfrak{q}$. Now since $f \notin \mathfrak{q}$, we necessarily have $R_{\mathfrak{p}} \simeq R_f \otimes_R R_{\mathfrak{p}} \simeq S_f \otimes_R R_{\mathfrak{p}} \simeq S_{\mathfrak{p}}$.

- (ZMT)' \Rightarrow (ZMT): Let R' be the integral closure of $\phi(R) \subseteq S$, then $R' \subset S \subset S$ satisfies the conditions of (ZMT)'. Hence $R'_{\mathfrak{p}} \simeq S_{\mathfrak{p}}$. Now by the claim in the proof of Proposition 2.1 we get that $\exists f \in R'$ such that $R'_f \rightarrow S_f$ is finite. But the integral closure of

R' implies (by a quick computation) that R'_f is also integrally closed in S_f . We conclude that $R'_f \simeq S_f$. \square

Our main goal now is to prove (ZMT)', which we will do via a series of lemmas.

Lemma 2.3. *Suppose S is a domain containing $R[X]$, such that S is integral over $R[X]$. Then S is not quasi-finite over R at any $\mathfrak{q} \in \text{spec}(S)$.*

Proof. First, we reduce to the case where R is integrally closed in its fraction field. Denote the integral closure of R by \overline{R} . Then consider the following diagram:

$$\begin{array}{ccccc} R & \hookrightarrow & R[X] & \hookrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ \overline{R} & \hookrightarrow & \overline{R}[X] & \longrightarrow & S \otimes_R \overline{R} \end{array}$$

Notice that the map $R \rightarrow \overline{R}$ is the localization by some multiplicative set. Hence $\overline{R}[X] \rightarrow S \otimes_R \overline{R}[X]$ is injective and $S \otimes_R \overline{R}$ is a domain. Moreover, from the (proof of the) base change property of quasi-finite maps we know that $\overline{R} \rightarrow S \otimes_R \overline{R}$ is quasi finite at any prime lying over \mathfrak{q} . So assume without loss of generality that R is integrally closed in its fraction field.

Next the following claim allows us to use the full power of Cohen-Seidenberg:

Claim: If a domain A is integrally closed, then so is $A[X]$.

proof of claim: suppose $0 = a_0 + \dots + (\frac{P(X)}{Q(X)})^n$ for some $Q(X), P(X) \in A[X]$ and $a_j \in \text{Frac}(A), 0 \leq j \leq n-1$. Then multiplying by $Q(X)^n$ shows that $Q(X) | P(X)$ in $\text{Frac}(A)[X]$, so $\frac{P(X)}{Q(X)} = F(X)$ for some $F(X) = f_0 + \dots + f_k X^k \in \text{Frac}(A)[X]$. Moreover, observe that evaluating at $X = 0$ immediately gives $f_0 \in A$ by the fact that A is integrally closed.

Now since both f_0 & $F(X)$ are integral over $A[X]$ we must have $F(X) - f_0$ also integral over A . Hence \exists an equation $0 = a'_0 + \dots + (f_1 X + \dots + f_k X^k)^n$ where $a'_j \in A[X]$ for $0 \leq j \leq n-1$.

Observe that $a'_0(0) = 0$ and so $f_1 + \dots + f_k X^{k-1}$ is also integral over

$A[X]$. And so the proof follows by iterating the previous argument k more times, giving $F(X) \in A[X]$. ■

Now let $\mathfrak{q} \cap R[X] =: \mathfrak{r}$ and $\mathfrak{q} \cap R =: \mathfrak{p}$. Then if $\exists \mathfrak{r}' \supset \mathfrak{r}$ such that $\mathfrak{r}' \cap R = \mathfrak{p}$, then by Cohen-Seidenberg $\exists \mathfrak{q}' \supset \mathfrak{q}$ such that $\mathfrak{q}' \cap R[X] = \mathfrak{r}'$. But then $\mathfrak{q}' \cap R = \mathfrak{p}$ too. Hence if \mathfrak{q} is minimal in the fiber over \mathfrak{p} then so is \mathfrak{r} . Similarly, Cohen-Seidenberg gives us that if \mathfrak{q} is minimal in the fiber over \mathfrak{p} then so is \mathfrak{r} .

Hence we conclude that if S is quasi-finite over R at \mathfrak{q} then $R[X]$ is quasi-finite over R at \mathfrak{r} . But this is impossible since no prime ideal of $\kappa(\mathfrak{p})[X] = R[X] \otimes_R \kappa(\mathfrak{p})$ is both minimal and maximal at the same time. □

We now move on to establish the first case of (ZMT)'. This will allow us to use an induction argument for the general case. We start with the case where S is generated over R by one element.

Lemma 2.4. *(ZMT)' holds in the case $S = T = R[x]$.*

Proof. Let $\mathfrak{p} := \mathfrak{q} \cap R$ and suppose $S = R[x] = R[X]/I$ for some ideal I . We begin by showing that $\exists r_0, \dots, r_n \in R$ such that $r_0 + \dots + r_n X^n \in I$ and $r_j \notin \mathfrak{p}$ for some $0 \leq j \leq n$. For this, we show that $I \not\subseteq \mathfrak{p}R$:

Suppose $I \subseteq \mathfrak{p}R[X]$, then we get

$$\begin{aligned} (R[x] \otimes_R \kappa(\mathfrak{p}))_{\mathfrak{q}} &= ((R[X]/I) \otimes_R \kappa(\mathfrak{p}))_{\mathfrak{q}} \\ &= ((R[X] \otimes_R \kappa(\mathfrak{p})/I \otimes_R \kappa(\mathfrak{p}))_{\mathfrak{q}} \\ &= (\kappa(\mathfrak{p})[X]/I \otimes_R \kappa(\mathfrak{p}))_{\mathfrak{q}} \end{aligned}$$

but the quasi-finiteness of S at \mathfrak{q} implies that the map $I \otimes_R \kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{p})[X]$ is not the zero map. Which is the desired result.

Now, we move on to show that given a polynomial $r_0 + \dots + r_n X^n \in I$ such that $r_j \notin I$ for some $0 \leq j \leq n$ then \exists another polynomial in I of degree ≥ 1 and whose leading coefficient is not in \mathfrak{p} . We argue by induction:

- $n=1$: If $r_1 \in \mathfrak{p}$, then $r_1 X \in \mathfrak{q}$ and so $(r_0 + r_1 X) - r_1 X \in \mathfrak{q} \cap R = \mathfrak{p}$. Since this contradicts what we proved earlier, we must have $r_1 \notin \mathfrak{p}$.
- Suppose the claim holds for $n = N-1$: Then if $r_0 + \dots + r_N X^N \in I$ and $r_N \in \mathfrak{p}$, then $r_0 + \dots + r_{N-2} X^{N-2} + (r_{N-1} + r_N) X^{N-1} \in I$ with $r \in R$ such that $r - r_N X \in I$, which can be done since

$r_N x$ is integral over R and the latter is integrally closed. Then the claim follows by the induction hypothesis.

Finally, now we know that \exists an equation $0 = r_0 + \cdots + r_n x^n$, with $r_n \notin \mathfrak{p}$ and $n \geq 1$. This implies that x is integral over $R_{\mathfrak{p}}$. But since R is integrally closed in S , $R_{\mathfrak{p}}$ is integrally closed in $S_{\mathfrak{p}}$. Hence $R_{\mathfrak{p}} = S_{\mathfrak{p}}$. \square

We now attempt to weaken the assumptions in the previous lemma a little bit.

Lemma 2.5. *(ZMT)' holds in the case $T = R[x]$ and S is finite over T .*

Proof. Let $I := \{P(x) \in R[x] \mid P(x)S \subset R[x]\}$. If $I \not\subseteq \mathfrak{r}$, where $\mathfrak{r} := \mathfrak{q} \cap R[X]$, then $R[x]_{\mathfrak{r}} = S_{\mathfrak{r}}$ and therefore also equal to $S_{\mathfrak{q}}$.

Now we also have $R[x]_{\mathfrak{r}} \otimes_R \kappa(\mathfrak{p}) = S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$ is finite $/\kappa(\mathfrak{p})$. So by the previous lemma, we have $R_{\mathfrak{p}} = R[x]_{\mathfrak{p}}$. Finally, since R is integrally closed in S , $R_{\mathfrak{p}}$ is integrally closed in $S_{\mathfrak{p}}$ and since the latter is integral over $R[x]_{\mathfrak{p}}$, we must have

$$R_{\mathfrak{p}} = S_{\mathfrak{p}}.$$

Now it remains to study what happens if $I \subseteq \mathfrak{r}$. Suppose $\mathfrak{n} \subseteq \mathfrak{q}$ is a minimal prime containing IS and let $\mathfrak{m} := \mathfrak{n} \cap R$.

Claim: In this case, $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{n})$ is a transcendental extension.

Assuming the claim is true, let $\overline{R} := R/\mathfrak{m}$ and $\overline{S} := S/\mathfrak{n}$. One observes that from the claim, we necessarily have $R[x]$ transcendental over R hence we have inclusions $\overline{R} \subset \overline{R}[X] \subset \overline{S}$ where the last one is an integral extension. By base change, \overline{S} is quasi-finite over \overline{R} , which we know is impossible by *Lemma 2.3*. Therefore $I \not\subseteq \mathfrak{r}$ for which case we have already shown that (ZMT)' holds.

So it remains to prove the claim:

proof of claim: (to be added) \blacksquare \square

We now have all the ingredients to prove (ZMT)'.

Theorem 2.6. *(ZMT)' holds in general.*

Proof. We argue by induction on the number n of generators of T over R , the case $n = 0$ being immediate.

- Suppose (ZMT)' holds whenever T is generated by $< n$ elements over R .

Now if $T = R[x_1, \dots, x_n]$, let \tilde{R} be the integral closure (in S) of $R[x_1, \dots, x_{n-1}]$ and let $\tilde{T} := \tilde{R}[x_n]$. Then by *Lemma 2.5*, we necessarily have $\tilde{R}_{\tilde{\mathfrak{p}}} = S_{\tilde{\mathfrak{p}}}$ where $\tilde{\mathfrak{p}} := \mathfrak{q} \cap \tilde{R}$.

Let $\{T^\alpha\}_\alpha$ be the family of finite $R[x_1, \dots, x_{n-1}]$ -algebras that are contained in \tilde{T} . Then this family is directed and

$$\tilde{T} = \varinjlim_\alpha T^\alpha.$$

Moreover, letting $\mathfrak{r}_\alpha := T^\alpha \cap \mathfrak{q}$ and $\tilde{\mathfrak{r}} := \tilde{T} \cap \mathfrak{q}$, we have

$$\tilde{T}_{\tilde{\mathfrak{r}}} = \varinjlim_\alpha T^{\alpha}_{\mathfrak{r}_\alpha}$$

Now $\tilde{R}_{\tilde{\mathfrak{p}}} = S_{\tilde{\mathfrak{p}}} \Rightarrow \tilde{R}_{\tilde{\mathfrak{p}}} = \tilde{T}_{\tilde{\mathfrak{p}}} = \tilde{T}_{\tilde{\mathfrak{r}}} = S_{\tilde{\mathfrak{r}}} = S_{\mathfrak{q}}$. But since \tilde{T} is of finite type over \tilde{R} , the generators must lie in $T^{\alpha_0}_{\mathfrak{r}_{\alpha_0}}$ for some α_0 . In other words, we get

$$S_{\mathfrak{q}} = \tilde{T}_{\tilde{\mathfrak{r}}} = T^{\alpha_0}_{\mathfrak{r}_{\alpha_0}}$$

Finally, from the induction hypothesis applied to $R \subseteq R[x_1, \dots, x_{n-1}] \subseteq T^{\alpha_0}$, we get:

$$R_{\mathfrak{p}} = T^{\alpha_0}_{\mathfrak{p}} = T^{\alpha_0}_{\mathfrak{r}_{\alpha_0}} = S_{\mathfrak{q}}$$

but then we necessarily have $R_{\mathfrak{p}} = S_{\mathfrak{p}}$. □