1 NC-smooth affine schemes (following Kapranov)

We start by recalling the main definitions following Kapranov's approach to NC-geometry.

An NC-complete algebra R over a field k, is a k-algebra that is complete with respect to the decreasing *"commutator lie algebra filtration"* on R. The d^{th} filtration $\mathbf{F^d}R$ is generated by expressions of the form

$$a_1\mathbf{n}_{i_1}a_2\dots a_n\mathbf{n}_{i_n}a_{n+1}$$

where each $\mathbf{n}_{i_k}$ is formed of i_k nested commutators and such that $i_1+i_2+\cdots+i_n-n=d$. For example, $\mathbf{F^0}R=R$ and $\mathbf{F^1}R$ is generated (*additively*) by elements of the elements of the form $a[x_1,[x_2,x_3]]b$ and $a[x_1,x_2]b[x_3,x_4]c$.

Kapranov defines the notions of d-**smoothness** and NC-**smoothness** following the classical definitions of "*formal smoothness*" in algebraic geometry and commutative algebra. In the following, denote by $\mathcal{N}_d$ the the category of k-algebras S with $\mathbf{F}^{d+1}S=0$.

Definition 1.0.1. Let R be a k-algebra, then we say that

(a) $R\in\mathcal{N}_d$ is d-**smooth** if it is finitely generated and the representable functor $h^R_{\mathcal{N}_d}$ is formally smooth.

(b) R is NC-**complete** if the canonical map $R\to\varprojlim R/\mathbf{F}^kR$ is an isomorphism.

(c) R is said to be NC-**smooth** if it is NC-complete and if $R/\mathbf{F}^{d+1}R$ is d-smooth for all d.

Next, we would like to define the category of affine NC-**schemes** as certain ringed spaces, where the underlying topological space is the *spectrum* of the abelianization of a given NC-complete algebra. We start by defining the appropriate sheaf corresponding to an NC-nilpotent algebra R.

Lemma 1.0.2. Let R be an NC–nilpotent algebra and $\pi:R\to R_{ab}$ the quotient onto its abelianization. Then for any multiplicative subset $S\subset R_{ab}-\{0\}$, the set $\pi^{-1}(S)$ satisfies the **Ore** conditions for the calculus of fractions.

Using this fact, we define the stalks of the desired ringed space as follows:

Definition 1.0.3. Let R be an NC-**nilpontent** algebra, we define the accociated NC-**nilpotent** scheme $X:=\mathrm{spec}(R_{ab})$ with the sheaf rings $\widetilde{\mathcal{O}}$ defined as the sheaf associated to the presheaf defined on the basis of standard open subsets via

$$\Gamma(D(g),\widetilde{\mathcal{O}}):=R[g^{-1}]$$

Remark. (????? The presheaf on the basis defined above is actually a sheaf on the basis)

Proposition 1.0.4. Given an affine NC-**nilpotent** scheme X as above, the we have:

(a) The stalk $\widetilde{\mathcal{O}}_{\mathfrak{P}}$ is a local ring.

(b) $\Gamma(X,\widetilde{\mathcal{O}})=R$.

Now given an NC-**complete** algebra R, we define X_d to be the affine NC-nilpotent scheme associated to $R/\mathbf{F}^{d+1}R$. The NC–**complete** affine scheme associated to R is now defined as follows:

Definition 1.0.5. Given an NC–complete algebra R, the associated NC-complete affine scheme has underlying topological space $X:=\mathrm{Spec}\,(R_{ab})$ and sheaf of rings given by

$$\varprojlim\mathcal{O}_{X_d}.$$