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# On the concept of virtual constraints as a tool for walking robot control and balancing

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**Abstract**

In this paper we review a class of control methods for the walking control and the balancing problem. The methods under review are based on the notion of *virtual constraints* which are forced by feedback. The paper describes several examples where this notion has been used, in a variety of control problems, as a main tool for constructing orbitally stable feedback laws. We also underline the main stability mechanisms behind such approaches, and present the Rabbit 7-DOF walking robot which has been used as a testbed robot for studying controllers based on these concepts.


*Keywords:* Robot control; Control systems; Nonlinear analysis; Walking; Stabilizing controllers; Limit cycles; Virtual constraints

## 1. Introduction

La pensée ne doit jamais se soumettre, ni à un dogme, ni à un parti, ni à une passion, ni à un intérêt, ni à une idée préconçue, ni à quoi que ce soit, si ce n'est aux faits eux-mêmes, parce que, pour elle, se soumettre, ce serait cesser d'être. (Henri Poincaré)

A canonical problem in bipedal robots is how to design a controller that generates closed-loop motions—walking, running, or balancing (limit cycle generation without ground contact)—that are periodic and stable. Due to the inherent underactuation and the changing contact conditions with the ground, this task is far from being solved through existing control methods, and makes the planning of asymptotically stabilizable, dynamic motions extremely difficult, if not impossible.

Some existing concepts heavily relying on heuristics, such as the zero momentum point (ZMP) principle (Hirai, Hirose, Haikawa, & Takenake, 1998; Vukobratovic, Borovac, Surla, & Stokic, 1990), often suffer from a deficit of fundamental concepts, and relay on the limited view of a "frozen (or slow) dynamics." Stability is then constrained by sufficiently slow motions for which this static view is not overly violated. Truly dynamic motions, such as balancing, running or fast walking, are clearly excluded with these approaches (Goswami, 1999).

New paradigms, concepts and control analyses are thus needed to deal with the *truly dynamic* control of walking mechanisms.

Virtual constraints (VC) are relations among the links of the mechanism that are dynamically imposed through feedback control. Their function is to coordinate the evolution of the various links throughout a step—which is another way of saying that they reduce the degrees of freedom—with the goal of achieving a closed-loop mechanism that naturally gives rise to a desired periodic motion.

The concept of virtual constraints has recently been used in connection with the development of *stable walking* controllers for bipeds (Chevallereau et al., 2003; Grizzle, Abba, & Plestan, 2001; Westervelt, Grizzle, & Canudas-de-Wit, 2003; Westervelt, Grizzle, & Koditschek, 2002), and for the problem of *balancing* (keeping the mechanism oscillating about a pre-specified orbit) (Canudas-de-Wit, Espiau, & Urrea, 2002; Perram, Shiriaev, Canudas-de-Wit, & Grognard, 2003; Shiriaev & Canudas-de-Wit, 2003a, 2003b). In the former, the notion of VC is combined with the concept of hybrid zero dynamics (HZD). The result is that

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Fig. 1. Rabbit robot, located at the Laboratory of Automatic Control, at Grenoble France.

the stability of the full-order system can be studied on the basis of a two-state differential equation with jumps (the hybrid zero dynamics). Whereas in the latter (Shiriaev & Canudas-de-Wit, 2003a), the VC are used to construct the full integral form (orbit) of the resulting zero dynamics. This is an important ingredient for constructing nonstationary and nonlinear control law.

The paper is organized as follows, we first describe the notion of virtual constraint and provide some examples in how to use such a constraint for walking and balancing. This constraint leads to the virtually constrained system. In connection with these notions, we then describe how these ideas are used in connection with a recently developed construction procedure for designing stable controllers dealing with the balancing task. Finally, we describe similar issues relating to the walking control law.

## 2. Rabbit: a testbed for advanced control theory

The Rabbit testbed shown in Fig. 1 is the result of a joint effort by several French research laboratories, spanning Mechanical Engineering, Automatic Control and Robotics (Rabbit, 2002). The project is funded by the CNRS[1] and the National Research Council.[2] Initiated in 1997 in France, its central mission is to build a prototype for studying truly dynamic motion control. In particular, the mechanism was designed to allow for high speed walking and running. RABBIT's lateral stabilization is assured by a rotating bar, and thus only 2-D motion in the sagittal plane is considered. This apart, the prototype captures the main difficulties inherent in this type of nonlinear system: underactuation (no feet), variable structure (the state dimension varies as a function of the motion phase), and state jumps (sudden state variations resulting from impacts with the ground).

The prototype is located at the Laboratory of Automatic Control at Grenoble in France and is intented to be used as an open and a remote operated testbed, for researchers interested in the field.

## 3. Virtually constrained systems

In this paper we will be concerned with under-actuated Lagrangian systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = B(q)u \tag{1}$$

where $q, \dot{q}$ are vectors of generalized coordinates and velocities; $u$ is a vector of independent control inputs; $\mathcal{L}(q, \dot{q})$ is a Lagrangian of the system (1); $B(q)$ is a matrix function of an appropriate dimension, with rank equal to the number of inputs. The underactuation in (1) means that $\dim u < \dim q$, i.e., a number of actuators in (1) is less than a number of its degrees of freedom.

### 3.1. Virtual constraints and virtual limit system

Virtual constraints are relations among the links of the mechanism that are dynamically imposed through feedback control. Their function is to coordinate the evolution of the various links throughout a single variable—which is another way of saying that they reduce the degrees of freedom—with the goal of achieving a closed-loop mechanism whose dynamic behaviour is fully determined by the evolution of simplest lower-dimension system (*virtual limit system*).

The way to perform this is first to define a set of $n-1$ outputs (or constraints), i.e.

$$y = \bar{q} - h(\theta, p) = \varphi(q, p)$$

where $\bar{q} \in R^{n-1}$ describes the actuated coordinates, $\theta \in R$ is the unactuated variable, and $p$ is the set of design parameters. Then an inner-feedback loop of the form

$$\psi(q)u = \kappa(q, \dot{q}) + v \tag{2}$$

is used to perform output feedback linearization in a local domain where the $(n-1) \times (n-1)$ matrix $\psi(q)$ is invertible. As a result, we get a partially linearized system of the

[1] Centre National de la Recherche Scientifique Française.

[2] Le ministre de la recherche et de la technologie Française.

form,

$$\ddot{y} = v$$

$$\underbrace{\alpha(\theta,p)\ddot{\theta} + \beta(\theta,p)\dot{\theta}^2 + \gamma(\theta,p)}_{\text{zero dynamics}} = g(p,\theta,\dot{\theta},y,\dot{y},v)$$

where $\theta(q)$, and $\alpha(\theta)$, $\beta(\theta)$, $\gamma(\theta)$ are some scalar functions depending on the structure of (2). Other property is that $g(p,\theta,\dot{\theta},0,0,0) = 0$.

The second equation describes the internal dynamics of the system. The homogeneous part of this equation represents the zero dynamics. If an outer feedback loop $v$ is designed to zeroing the output $y$, then the *full* system dynamic is captured by the solutions of

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0 \tag{3}$$

together with the imposed constraint,

$$\bar{q} = h(\theta,p)$$

The resulting system is here named *virtually constrained mechanical system* or *virtual limit system*. This process allows to deal with high-dimensional systems with under-actuated degree one, by only analyzing this second-order nonlinear equation.

Some examples of virtually constrained systems are shown in Fig. 2. In the camless engine example, the linear motion of each piston, $q_i$s, can be shifted and synchronized to the rotation of the cramshaft angle $\theta$ by imposing the following set of constraints,

$$\varphi_1 = q_1 - p_1\sin(p_2\theta) = 0,$$
$$\varphi_2 = q_2 - p_1\sin(p_2\theta) + \pi = 0,$$
$$\varphi_3 = q_3 - p_1\sin(p_2\theta) + 2\pi = 0,$$
$$\varphi_4 = q_2 - p_1\sin(p_2\theta) + 3\pi = 0$$

where $p_i$ are arbitrary design constants.

In the constrained acrobot shown in Fig. 2(b), the robot tip can be constrained to move along the vertical axis, at $x = 0$, by imposing the constraint

$$\varphi_1 = q_1 - \arcsin\left(\frac{l_2}{l_1}\sin(\theta)\right) = 0$$

where $q_2 = \theta$.

Another example is the biped mecanisms shown in Fig. 2(d). If the contact foot is assumed not to slide, the kinematic of such a system is equivalent to the one of a multi-link open-chain robot. When the degree of actuation is one, the $n-1$ control inputs can be used to set $n-1$ constraints, and the *virtual limit system* results in a second-oder nonlinear equation, which is easier to analyze. In this example the function of the virtual constraints is to coordinate the evolution of the various links throughout a step (or a balancing cycle) with the goal of achieving a closed-loop mechanism that naturally gives rise to a desired periodic motion.

The notion of virtual constraints can also be used for applications other than the generation of periodic motions. An example is shown in Fig. 2(c). This example shows a possible control strategy to perform simultaneous lateral and longitudinal control between two vehicles (leader-follower), by imposing a *virtual toolbar* between the two vehicles.

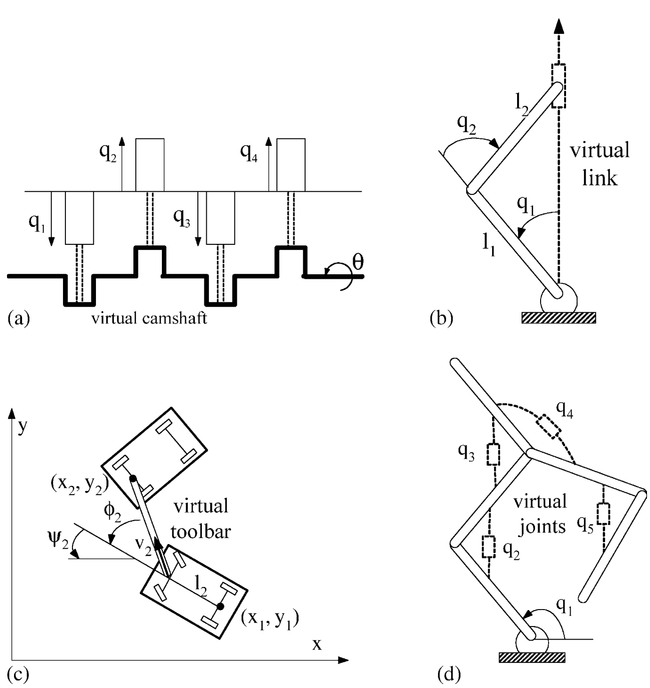


Fig. 2. Examples of virtually constrained systems: (a) the constrained camshaft engine (top left), (b) the constrained acrobot (top right), (c) the vehicle with virtual tool bar (bottom left) and (d) the multi-link open-chain robot (bottom right).

**Remark.** The constraints can be defined in a more general manner. For instance, instead of always use the unactuated angle as the unique argument of the function $h(\theta,p)$ we can let $\theta(q)$ be a function of the generalized coordinates $q$. In this case, the constraints will then write as

$$\varphi(q,p) = \bar{q} - h(\theta(q),p), \tag{4}$$

We next discuss some important properties associated to the Eq. (3)

### 3.2. Virtual energy and associated Lagrangian

For the case of the example shown in Fig. 2(d) where the contact leg is not actuated, the virtual limit system (3) can, in some special cases, be rewritten in terms of the angular momentum $\sigma$ and the orientation $\delta$ of the instantaneous center of mass $M(\delta)$ as

$$\dot{\delta} = \frac{1}{M(\delta)}\sigma, \quad \dot{\sigma} = f(\delta) \tag{5}$$

This expression may be integrated by eliminating time,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\delta}\sigma = f(\delta)M(\delta)$$

leading to

$$\frac{\sigma^2}{2} - \frac{\sigma_0^2}{2} = \int_{\delta_0}^{\delta} f(s)M(s)\,\mathrm{d}s \tag{6}$$

or equivalent to

$$I(\delta,\sigma,\delta_0,\sigma_0) = K(\sigma) - P(\delta) - K_0 + P_0 = 0 \tag{7}$$

where $P(\delta) = \int f(\delta)M(\delta)\,\mathrm{d}\delta$, may have a close form, since it describes the potential energy of the equivalent virtual mechanisms, and $K = \sigma^2/2$ describes the kinetic energy. Expression (7) reflects the energy balance of the virtual constraint system in the transformed coordinates $(\delta, \sigma)$. This relation can also be used to define the *virtual Lagrangian*[3]

$$\mathcal{L}_V(\delta,\sigma) = I(\delta,\sigma,\delta_0,\sigma_0) + K_0 - P_0 = K(\sigma) - P(\delta)$$

which is invariant along *any* solution of (5). This special closed-form for $I$ correspond to the *first integral* of (5). There are cases when it is not possible to obtain this explicitly form, i.e. when $f(\delta,\sigma)$ depends on $\sigma$ as well. An example is the pendubot which is pivoting about its actuated joint. The angular momentum rate of change depends then on momentum as well.

In Perram et al. (2003), it has been shown that the function

$$I(\theta,\dot\theta,\theta_0,\dot\theta_0) = \dot\theta^2 - \psi(\theta_0,\theta)\dot\theta_0^2 + \psi(\theta_0,\theta)\int_{\theta_0}^{\theta}\psi(s,\theta_0)\frac{2\gamma(s)}{\alpha(s)}\,\mathrm{d}s \tag{8}$$

with

$$\psi(\theta_0,\theta_1) = \exp\left\{-2\int_{\theta_0}^{\theta_1}\frac{\beta(\tau)}{\alpha(\tau)}\,\mathrm{d}\tau\right\}$$

preserves its value along the solutions of the Eq. (3), which is a generalization of the previous expression for $I$, since represents the *full integral* of the virtual limit system.

Function $I$ is an important ingredient to be used in the process of designing the control law. The way that this function is used during the control synthesis depend upon the final control task specifications, and problem in hand.

In the *balancing control problem* the constraint set $\varphi = 0$, is first designed such that the resulting function $I = 0$ exhibits closed curves in the $(\theta - \dot\theta)$ plane. No impacts occurs during this task. Since function $I = 0$ does not define an attractive manifold, but just a family of invariant sets (depending on the initial conditions), it is then necessary to design an additional control loop that renders the target manifold (i.e. the desired orbit) attractive. In that sense, the control for balance task is considered to be an *active* one.

In the *walking control problem*, the design goal is slightly different. In this case this equation needs to be completed with the impact map, leading to a hybrid model. The control inputs are only used here to set the constraints to zero. The virtual constraints are designed so that, together with the impact map, the Eq. (7) leads to a stable limit cycle. The closed-loop system can be compared to a *passive* mechanism designed to produce stable walking when subject to ground impacts. The mechanism can be render active if the constraint parameters $p$ are used as an additional degree of freedom to adapt the biped walking gaits as to obtain a particular velocity profile.

We now elaborate further these two different control perspectives.

## 4. Balancing control

Balancing consists, to some extent, in viewing the robot in single support as a multi-link inverted pendulum. The goal is to find a feedback control law that induces a nontrivial, limit cycle in the high-dimensional inverted pendulum. We may also wish to move from one limit cycle to another, possibly by just changing the period and/or the amplitude of the motion.

The balancing problem has been studied by several authors and using a diversity of approaches. The Hamiltonian formalism has been used by Aracil, Gordillo, and Acosta (2002), Aracil, Gomez-Estern, and Gordillo (2003) to stabilize a class of orbits in underactuated systems. In Vivas-Venegas and Rubio (2003), the authors propose to compute sub-optimal controllers for orbital stabilization. Hauser and Chung (1994) present an analysis framework the computation of Lyapunov functions. The idea was applied to the cart and pendulum system.

The balancing problem has been studied within the context of virtual constraints in Canudas-de-Wit et al. (2002), Grognard and Canudas-De-Wit (2003), and Shiriaev and Canudas-de-Wit (2003a, 2003b). Employing the concept of virtual constraints developed earlier (Canudas-de-Wit et al., 2002), a constraint of the form

$$y = \bar q - h(\theta, p(t))$$

is defying and zeroed by a feedback controller. In this case, the resulting zero dynamics become time-varying. One part of the balancing problem may be restated as finding the generator of $p(t)$ that induce a suitable orbit. In Canudas-de-Wit et al. (2002)$p(t)$ is viewed as an additional control variable (dynamic state feedback) that can be adjusted on line so as to match an arbitrarily exo-system describing the desired orbit. However, the stability of the resulting internal dynamic of the controller (dynamics of $p(t)$), needs to be studied case-by-case.

[3] This function can be understood as the Lagrange function for the limit system (3), which is different from the Lagrange function of the full mechanism projected into the constraints $\varphi = 0$.

To cope with this problem, a different construction procedure proposed by Shiriaev and Canudas-de-Wit (2003a, 2003b) has been recently developed. This procedure has as a particularity the design of a feedback loop that *simultaneously* sets the constraints to zero while rendering the desired orbit locally exponentially stable. The procedure is summarized in the following section.

### 4.1. Existence of the orbits

To provide a constructive method for controller design, the original nonlinear system (1) is first transformed, via *partial feedback linearization*, to a form

$$\underbrace{\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta)}_{\text{virtual system}} = \underbrace{g_1 + g_2 v}_{=0,\text{ if } y=0} \tag{9}$$

$$\ddot{y} = v \tag{10}$$

where $g_1 = g_1(\theta, \dot{\theta}, y, \dot{y})$, and $g_2 = g_2(\theta, \dot{\theta}, y, \dot{y})$. The explicit dependency on the parameters $p$ is drop in this notation for seek of simplicity. Note that the homogeneous part of the internal dynamics, matches the structure of the *virtual limit system* (3), and hence has the general full integral form $I$ given before. Function $I$ preserves its value along the solution of the homegenous part of the above equation. However, $I$ depends on the particularly chosen initial conditions $(\theta_0, \dot{\theta}_0)$.

As mentioned previously, the set of constraints should first be designed so that $I = 0$ defines a set of closed curves (parametrized as a function of the initial conditions $(\theta_0, \dot{\theta}_0)$) in the $\theta\dot{\theta}$–plane, as shown in the Fig. 3. The associated time-trajectories are shown in Fig. 4. Once this is done, we can select one particular target orbit by using the corresponding values for $(\theta_0, \dot{\theta}_0) = (a, b)$ (see the red orbit in Fig. 3). The target trajectory is noted $I = I(\theta, \dot{\theta}, a, b) = 0$, where the specifically chosen constants $(a, b)$ replace the initial conditions values.

By construction, the target orbit $I = 0$ does not describe an attractive set, and hence we need to use the control variable $v \in R^{n-1}$ to simultaneously bring $y$ and the relevant $I(\theta, \dot{\theta}, a, b)$ to zero. This is done throughout the control design procedure described next.

### 4.2. Orbit stabilization

To study simultaneous stabilization of the output and the orbit, it is useful to write the ‘‘error’’ equations in the coordinates $(I, y, \dot{y})$. This leads to the auxiliary system

$$\dot{I} = \dot{\theta}\left\{\frac{2}{\alpha(\theta)}\left[g_y y + g_{\dot{y}}\dot{y} + g_2 v\right] - \frac{2\beta(\theta)}{\alpha(\theta)} I\right\}, \quad \ddot{y} = v$$

where $g_y = g_y(\theta, \dot{\theta}, y, \dot{y})$, $g_{\dot{y}} = g_{\dot{y}}(\theta, \dot{\theta}, y, \dot{y})$, and $g_2 = g_2(\theta, \dot{\theta}, y, \dot{y})$. By construction these functions have the property of vanishing when $y$ and $\dot{y}$ tend to zero.

The local controllability properties of this system around the target orbit can be studied by linearizing the above error system around its target time-periodic solution (see Fig. 4) $[\theta_\gamma(t), \dot{\theta}_\gamma(t)] = [\theta_\gamma(t+T), \dot{\theta}_\gamma(t+T)]$, $\forall\, t$, of a period $T$.

The resulting auxiliary linearized system is time-variant, and has the form

$$\dot{I} = \kappa_1(t)y + \kappa_2(t)\dot{y} + \kappa_3(t)I + b_1(t)v, \quad \ddot{y} = v$$

where $\kappa_i(t)$ and $b_1(t)$ are time-periodic functions. This system belongs to the class of systems,

$$\dot{x}(t) = A(t)x(t) + b(t)v$$

where $A(t) = A(t+T)$, and $b(t) = b(t+T)$. The linear time-varying controllability test for periodic systems can thus be checked for each of the selected solutions. An easy to analyse case is when $\kappa_2(t) = \kappa_3(t) = 0$. Then the system is controllable if the following inequality holds

$$\int_0^T \frac{b_1^2(t)}{\tilde{\kappa}^2(t)}\,\mathrm{d}t > \frac{12}{T^3}\left|\int_0^T \frac{t}{\tilde{\kappa}(t)}\, b_1(t)\,\mathrm{d}t\right|^2 \tag{11}$$

where $\tilde{\kappa}(t) = \exp\{\int_0^t \kappa_1(\tau)\,\mathrm{d}\tau\}$.

If this condition is fulfilled then the control can be of the form,

$$v = -\Gamma^{-1}\, b(\theta, \dot{\theta}, y, \dot{y})^T R(t) \begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix}$$

with $R(t)$ given by

$$\dot{R}(t) + A(t)^T R(t) + R(t)A(t) + G = R(t)\, b(t)\Gamma^{-1} b(t)^T R(t)$$

where $G = G^T > 0$, and $\Gamma = \Gamma^T > 0$. The resulting control $v$ is thus nonlinear and periodic on time, and yield local exponentially stability about the desired orbit. More details in the stability issues can be found in Shiriaev and Canudas-de-Wit (2003a), and the treatment of many other studied cases can be found in Shiriaev and Canudas-de-Wit (2003b). One example treated in this reference is the Rabbit robot with two fixed joints (Equivalent to a 3-DOF lilnk robot). The motion of the target orbit is shown in Fig. 5, and convergence of the closed-loop system solution to the desired orbit is shown in Fig. 6.

The control block scheme is shown by Fig. 7.

## 5. Walking control

Walking control using the virtual control principle has been studied in the references given in the introductory chapter. A quite general and broader review with a complete list of references can be found in (Chevallereau et al., 2003). A very complete treatment can be found in the recent Ph.D dissertation (Westervelt, 2003), and one important milestone reference is (Grizzle et al., 2001). In this section we summarize some of these major achievements.

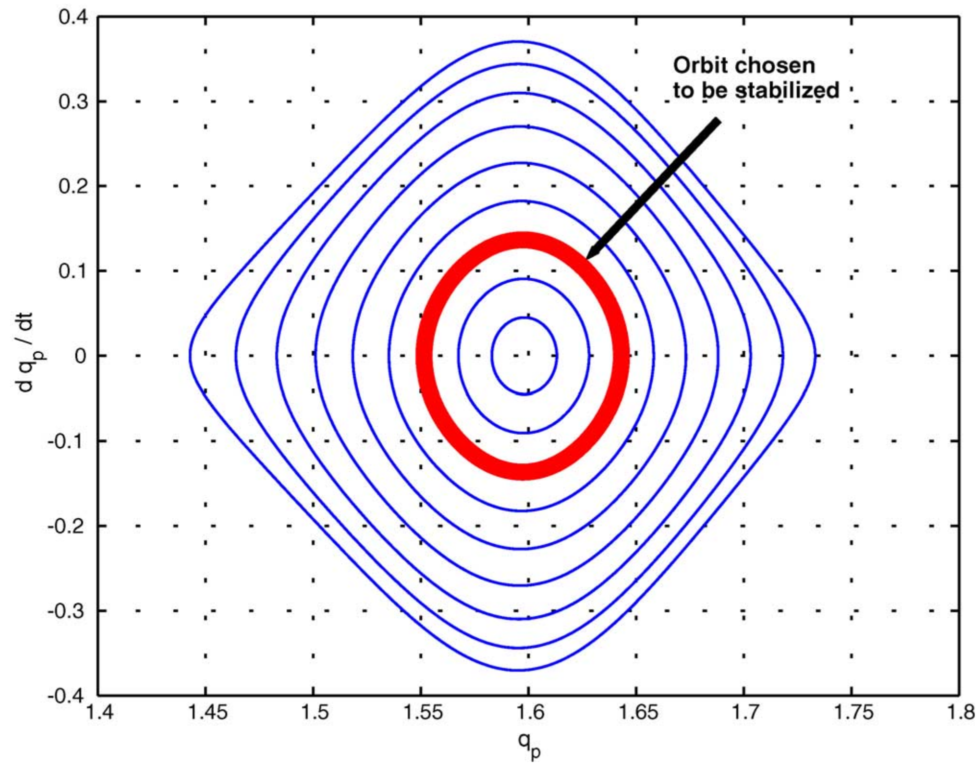


Fig. 3. Example of sets of orbits for the limit system. The desired one is indicated in red.

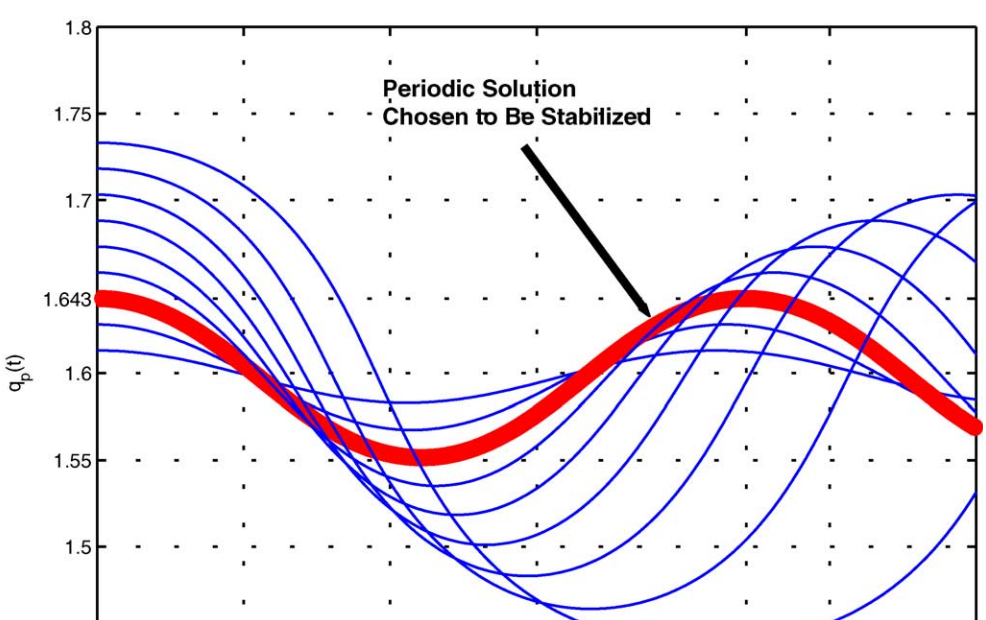


Fig. 4. Example of time solutions corresponding to the desired orbit sets. The desired solution $\theta_\gamma(t)$ is indicated in red.

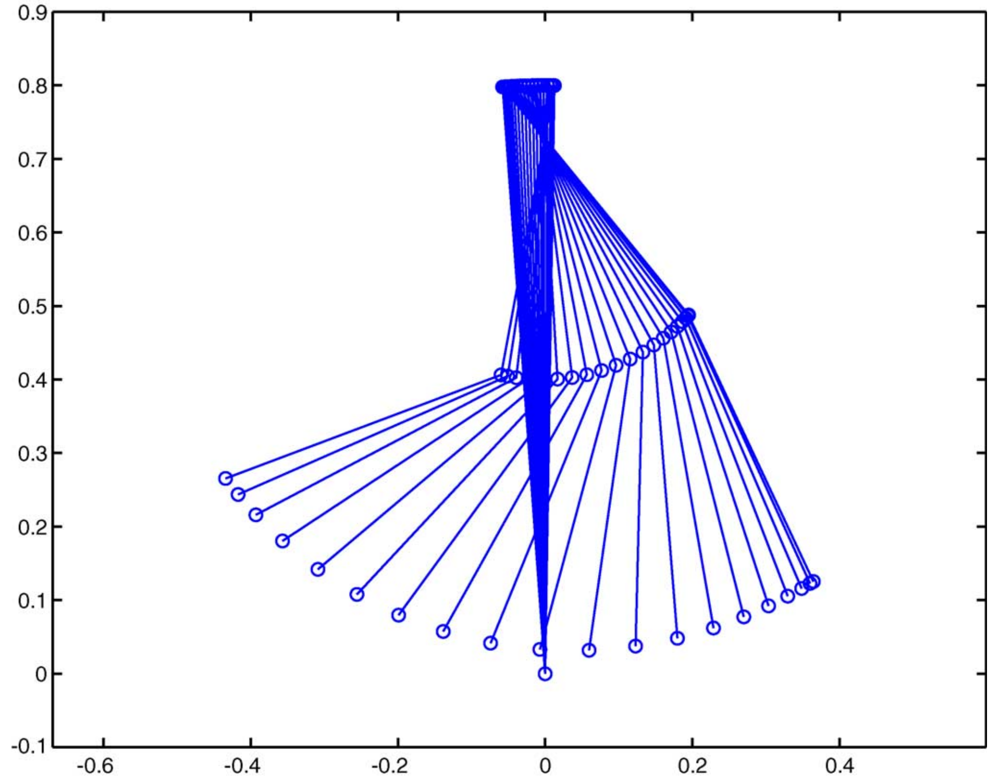


Fig. 5. The motion of the 3-link robot corresponding to the chosen orbit over period.

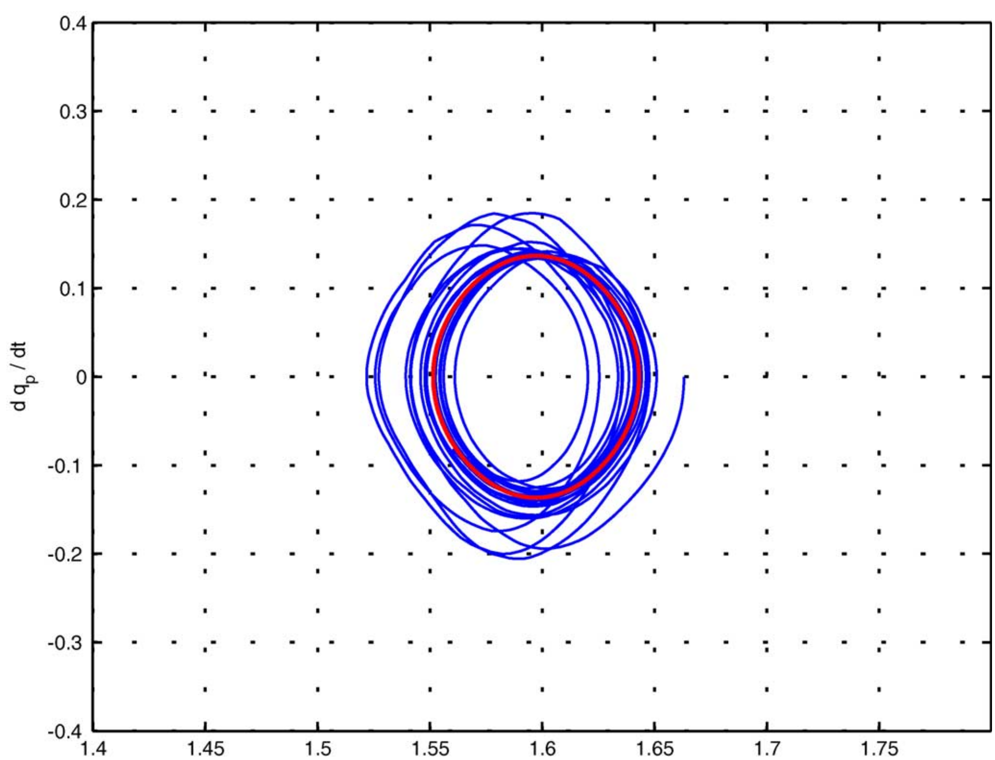


Fig. 6. Convergence of the closed-loop system solutions to the desired orbit. Here the desired orbit chosen to be stabilized is shown in red.

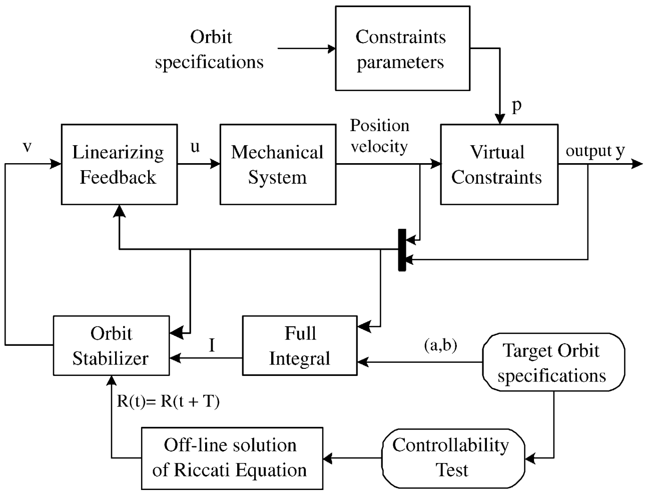


Fig. 7. Block diagram of the orbit stabilizer.

Assume the following

- ground contact generates jumps in the velocity vector, while position remains continuous,
- no rebound and no slipping of the swinging leg,
- the double support phase is instantaneous[4]
- the $n-1$ control inputs are used to zeroed the $n-1$ output vector $y$ with

  $y = \varphi_i(q,p)$
- the choice of the outputs are such that the virtual limit system (3) takes the form (5).

Then the complete mathematical model describing the reduced order mechanism is called the *hybrid zero dynamics*, and can be expressed in terms of the angular momentum by Eq. (5) during the single phase, together with the memoryless map $\Delta$ resulting from the ground contact, i.e.

$$\sigma^+ = \Delta(\sigma^-) \tag{12}$$

where $\sigma^-$, and $\sigma^+$ describe the angular momentum before and after the impact. It is important to notice that $\Delta$ depends on the impact model and on the selected constraints $\varphi$. Therefore, it can be modified by changing the design parameters $p$.

In a more general form the model can be expressed as

$$\dot{z} = f(z) \qquad z^- \notin S \cap Z$$
$$z^+ = \begin{bmatrix} \delta^+ \\ \Delta(\sigma^-) \end{bmatrix} \qquad z^- \in S \cap Z,$$

where $z = [\delta, \sigma]^T$. $S$ defines the contact surface, and $Z$ the zero-dynamic (free motion) manifold. This equation is of hybrid nature due to the same impacts experienced by the full order mechanism during locomotion. For a biped robot such as RABBIT that has one degree of underactuation, the hybrid zero dynamics evolve on a two-dimensional invariant surface. The result is that the stability of the full-order closed-loop system can be studied on the basis of a nonlinear, two-state differential equation with jumps.

### 5.1. Existence of a periodic orbit

The first question here is to find a condition that ensures the existence of a periodic orbit. Concatenating the solution of the zero dynamics in the manifold[5] $Z$ to those in $S$, we can write the variation of the angular momentum along a cycle as a function of the variation in the potential energy along the same cycle. This leads to the discrete-event expression

$$\frac{1}{2}(\sigma_{k+1}^-)^2 - \frac{1}{2}\Delta(\sigma_k^-)^2 = P(\delta_{k+1}) - P(\delta_k) = \Delta P$$

where the sub-script $(\cdot)_k$ describes the impact event number, i.e. the number of times that the Poincaré's map is crossed. Since the positions are invariant with respect to ground contact, potential energy is not affected by the jumps of the angular momentum. The right-hand side of the above equation expresses the variation of potential energy only during the motion on the manifold $Z$. In opposition to this, there is a loss of kinetic energy due to the ground contact. The left-hand side of this relation describes thus the variation of the kinetic energy including energy losses during contact. A fixed point $\sigma^* = \sigma_{k+1}^- = \sigma_k^-$ exists, if and only if there exist $\Delta$ and $\Delta P$ such that the following equation holds,

$$\frac{1}{2}(\sigma^*)^2 - \frac{1}{2}\Delta(\sigma^*)^2 = \Delta P$$

which indicate that the energy is preserved along a cycle. For the case when map $\Delta$ turns out to be linear in $\delta$, i.e. $\Delta(\sigma) = \eta(\cdot)\sigma$, then a sufficient condition so that $\sigma^*$ exists, is

$$\frac{\Delta P(\theta)}{1-\eta^2} > 0.$$

### 5.2. Stability of a cycle

Assume that there exists some $\sigma^*$ satisfying the equation above. Assume also that the map $\Delta$ is linear in $\sigma$. Then stability of the cycle can be studied by defining $V_k = 1/2(\sigma_{k+1}^-)^2$, $V^* = 1/2(\sigma^*)^2$, and $\tilde{V}_k = V_k - V^*$, then we have

$$\tilde{V}_{k+1} - \eta^2 \tilde{V}_k = 0$$

[4] Walking is defined as a succession of: single support, impact. The double support phase is not discussed here.

[5] The manifold is render inviariant with respect to impacts by a suitable choice of the constraint functions $\varphi$.

The orbit is *exponentially stable* if and only if

$$\eta^2 < 1.$$

### 5.3. Designing good walking

It is worth noting that the degree of freedom for fulfilling the previous existence and stability condition, are the structure and the parameters of the functions used to define the virtual constraints. Assuming that the structure is fixed, the final degree of freedom are the parameter set $p$. The parameters will also influence the shape of the walking gait. Therefore a *good walking* will be one that respects a certain number of physical constrains, the stability restrictions, and some walking specifications. One way to do that is by optimization. Here below we summarize the main steps to do that.

Let the constraints $\varphi$ be of the form

$$\varphi = \bar{q} - h(\theta, p)$$

with $h(\theta, p)$ expressed in some polynomial form having $p$ as a set of free matrix parameters.

- Specifications: Initial and final values of the swing phase, i.e. $q(s_0)$, $q(s_T)$, step length $s_T$, Time of a step: $T$.
- Torque Constraints $u = u^*(\theta(q), \dot{\theta}(q), p) \in \mathcal{U} = \bigcap \mathcal{U}_i$
  - no sliding $u \in \mathcal{U}_1$,
  - no ground penetration $u \in \mathcal{U}_2$
- Stability Constraints
  - $\frac{\Delta P(p)}{1-\eta^2(p)} > 0$
  - $\eta^2(p) < 1$

Find the set of parameters $p$ such that injected energy is minimized, i.e.

$$p^* = \min\arg\{J(p)\}, \quad J(p) := \frac{1}{s_T}\int_0^T \|u^*(t,p)\|^2\,\mathrm{d}t,$$

subject to the above set of nonlinear design constraints.

An video file of experimental results applied to our Rabbit robot can be found in the web page of the rabbit project (Rabbit, 2002)

The above method may be extended to cover other interesting cases such as:

- walking with variable speed,
- walking with double support phases,
- running.

An interesting case is the ability to perform motions with different walking velocities. This can be obtained by making the set of parameters $p$ change as a function of desired velocity. One possibility is to pre-compute two sets of parameters $p$ of interest, and then to move from one to another in one step. This can also be donne progressively, by computing an on-line variation of such a parameter, i.e. $p(t) = p^* + \Delta p(t)$. In Westervelt et al. (2003), walking at a continuum of speeds was obtained via an event-based PI controller. The novelty here is that the controller uses integral action to adjust parameters in an exponentially stabilizing hybrid zero dynamic controller. Parameter adjustment takes place just after impact (swing leg touching the ground). The control analysis is based on the restricted Poincaré map of the hybrid zero dynamics. The resulting control block scheme is shown by Fig. 8. Finally, some preliminary ideas on how to use this technique for running can be found in (Westervelt, 2003), and further work is in progress along this problem.

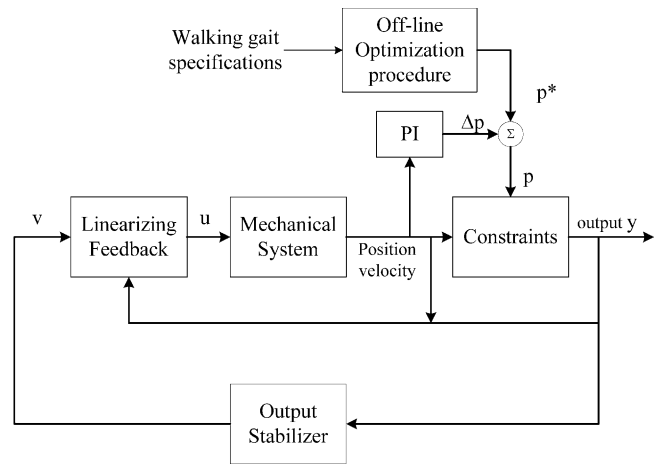


Fig. 8. Block diagram of the controller used to perform walking. When $p$ is changing on-line by the event-based PI controller the mechanisms becomes an active one.

## 6. Conclusions

In this paper we have presented an overview of different control problems using the notion of virtual constraints. This notion has been very useful to design controllers for bipeds in task such as walking, running, and balancing. The notion of virtual constraints can also be used in tasks that are not necessarily periodic. The paper has presented some examples of such systems.

This notion of Virtual constraint rolls out the time-dependency of the reference trajectories, making them space dependent. This property enhances the closed-loop robustness properties because delays in time are no longer important. This notion leads to virtual constraint systems (or zero dynamics) that basically contain all useful information for studying the stability of the full system. In other words, this process leads to a reduction procedure to study cycles in high-dimensional systems.

We have summarized some of the main control achievements in walking and balancing. It is worth underlining that these two approaches lead to two different control solutions. The control law for walking is nonlinear and time-invariant. Its stability relaies on the impact map and to some extent the resulting closed-loop system can be viewed as a passive mechanical mechanism shaped to produce stable cycles when subject to ground contact.

The balancing task leads to a control law that is composed of a first nonlinear time invariant loop (partial linearization) plus a linear time-varying periodic feedback. Thus, this control structure results in a nonlinear time-varying periodic feedback. The control has been proved to be locally orbitally exponentially stable.

Finally, it is important to note that this method has been recently used to perform real-time experiments in our Rabbit walking mecanisms that are worth observing in (Rabbit, 2002).

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