1 Exponents

$$X^A X^B = X^{A+B}$$
$$\frac{X^A}{X^B} = X^{A-B}$$
$$(X^A)^B = X^{AB}$$
$$X^N + X^N = 2X^N \neq X^{2N}$$
$$2^N + 2^N = 2^{N+1}$$

2 Logarithm

$$X^A = B \Leftrightarrow \log_X B = A$$
$$\log_A B = \frac{\log_C B}{\log_C A}$$
$$\log AB = \log A + \log B$$
$$\frac{logA}{\log B} = \log A - \log B$$
$$\log(A^B) = B \log A$$
$$logX < X \quad for \quad all \quad X > 0$$

3 Series

$$\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$
$$\sum_{i=0}^{N} A^i = \frac{A^{N+1} - 1}{A - 1}$$
$$\sum_{i=0}^{N} A^i \leq \frac{1}{1 - A} \quad if \quad 0 < A < 1, \quad N \to \infty$$

3.1 Proof for $\sum_{i=0}^{infty}$

$$S = \sum_{i=0}^{N} = 1 + A + A^2 + A^3 + \cdots + A^{N-1} + A^{N+1}$$

then

$$AS = A + A^2 + A^3 + A^4 + \cdots + A^N + A^{N+1}$$

then

$$AS = S - 1 + A^{N+1}(1)$$

then

$$S = \frac{A^{N+1} - 1}{A - 1}$$

if $0 < A < 1$ then $A^{N+1} \to 0, N \to \infty$ then

$$(1) \Leftrightarrow AS = S - 1 \Leftrightarrow S = \frac{1}{1 - A}$$

3.2 Proof for $\sum_{i=1}^{\infty} i/2^i$

Suppose

$$S = \sum_{i=1}^{N} i/2^i = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots + \frac{N-1}{2^{N-1}} + \frac{N}{2^N}$$

then

$$2S = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \cdots + \frac{N}{2^{N-1}}$$

then

$$2S - S = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{N-1}} + \frac{N}{2^N}$$

Since $\lim \frac{N}{2^N} = \lim \frac{1}{2^N \ln N} = 0$, then

$$S = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{N-1}}$$

Or

$$S = \sum_{i=0}^{N-1} (1/2)^i = \sum_{i=0}^{\infty} (1/2)^i = \frac{1}{1 - 1/2} = 2$$

3.3 Arithmetic series

$$\sum_{i=1}^{N} = \frac{N(N+1)}{2} \approx \frac{N^2}{2}$$

$$2 + 5 + 8 + \cdots + (3k - 1) = 3(1 + 2 + 3 + \cdots + k) - (1 + 1 + \cdots + 1) = \frac{3k(k+1)}{2} - k$$

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} \approx \frac{N^3}{3}$$

$$\sum_{i=1}^{N} i^k \approx \frac{N^{k+1}}{|k+1|}, k \neq -1$$

If, $k = -1$, the sum is then the harmonic number H_N

$$H_N = \sum_{i=1}^{N} \frac{1}{i} \approx \log_\epsilon N$$

with error tends to $\gamma = 0.57721566$, also known as **Euler's constant**

$$\sum_{i=1}^{N} f(N) = Nf(N)$$

$$\sum_{i=n_o}^{N} f(i) = \sum_{i=1}^{N} f(i) - \sum_{i=1}^{n_o-1} f(i)$$

4 Modular Arithmetic

- A is congruent to B modulo N, written $A \equiv B \pmod{N}$, if N divides $A - B$.
- Intuitively, the remainder is the same when either A or B is divided by N.
If $A \equiv B \pmod{N}$

$$A + C \equiv B + C \pmod{N}$$

$$AD \equiv BD \pmod{N}$$

If N is a prime number:
- $ab \equiv 0 \pmod{N} \Leftrightarrow a \equiv 0 \pmod{N} \quad or \quad b \equiv 0 \pmod{N}$. In other words, if a prime number N divides a product of two numbers, it divides at least one of the two numbers.
- $ax \equiv 1 \pmod{N}$ has a unique solution $\mod N$ for all $0 < a < N$. This solution , $0 < x < N$, is the multiplicative inverse.
- The equation $x^2 \equiv a(\mod N)$ has either two solutions for all $0 < a < N$, or it has no solution.

5 The P Word

5.1 Proof by induction

- Proving a base case that is establishing that a theorem is true.
- Assuming an inductive hypothesis that is true for all case. Showing that the theorem is true for the next value, k + 1.

5.2 Proof by Counterexample

Find something that counter the statements

5.3 Proof by contradiction