Coordinates and Transformations

MIT ECCS 6.837
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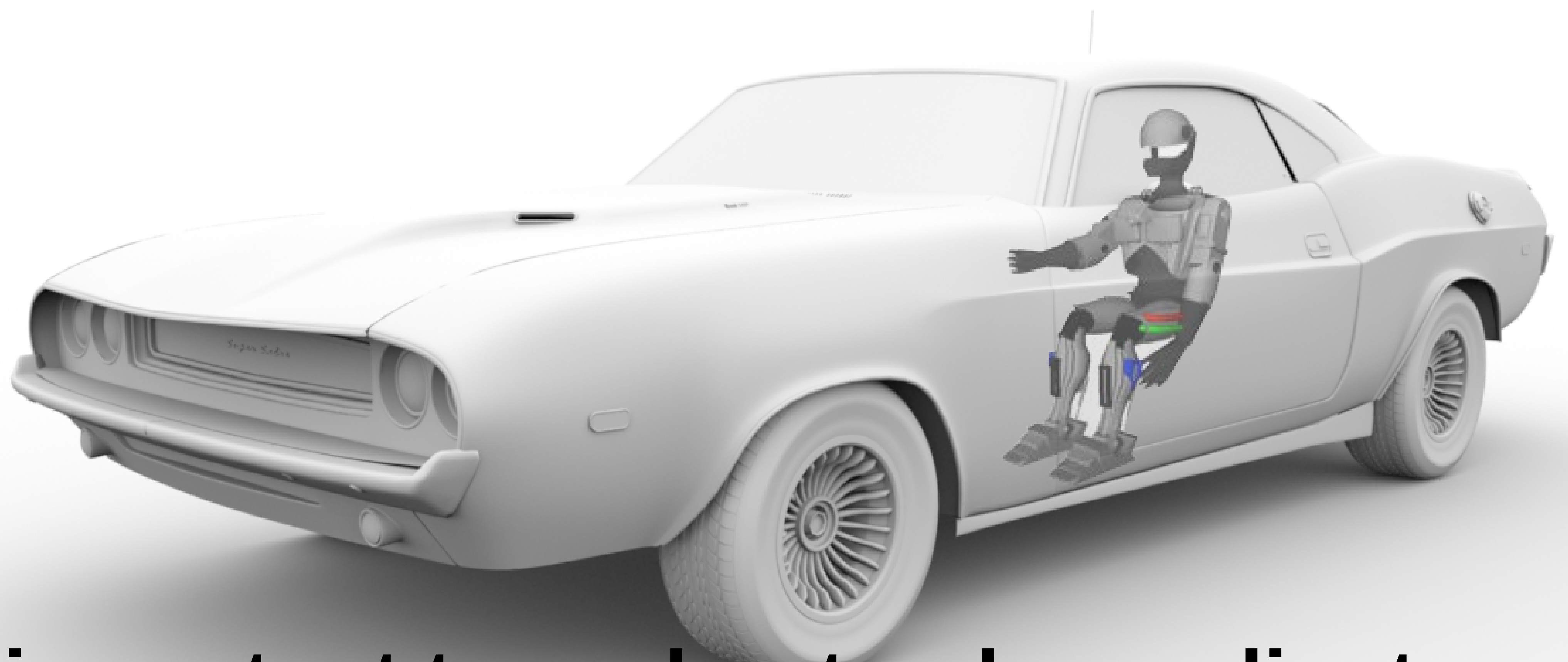
many slides follow Steven Gortler's book

Hierarchical modeling

- **Many coordinate systems:**
  - Camera
  - Static scene
  - car
  - driver
  - arm
  - hand
  - ...



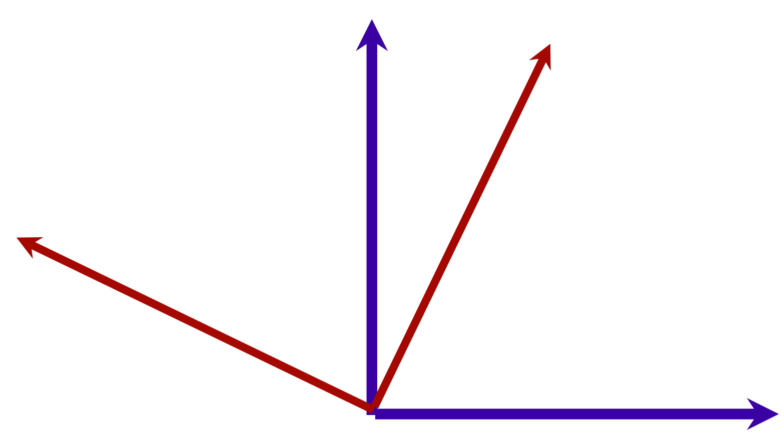
- **Makes it important to understand coordinate systems**

Coordinates

- We are used to represent points with tuples of coordinates such as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- But the tuples are meaningless without a clear coordinate system

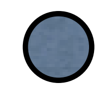
could be this point in the red coordinate system

could be this point in the blue coordinate system

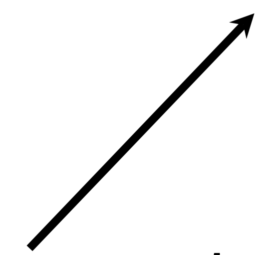


Different objects

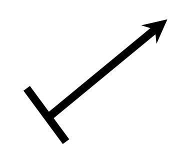
- **Points**
  - represent locations



- **Vectors**



  - represent movement, force, displacement from A to B
- **Normals**
  - represent orientation, unit length



- **Coordinates**
  - numerical representation of the above objects **in a given coordinate system**

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Points & vectors are different

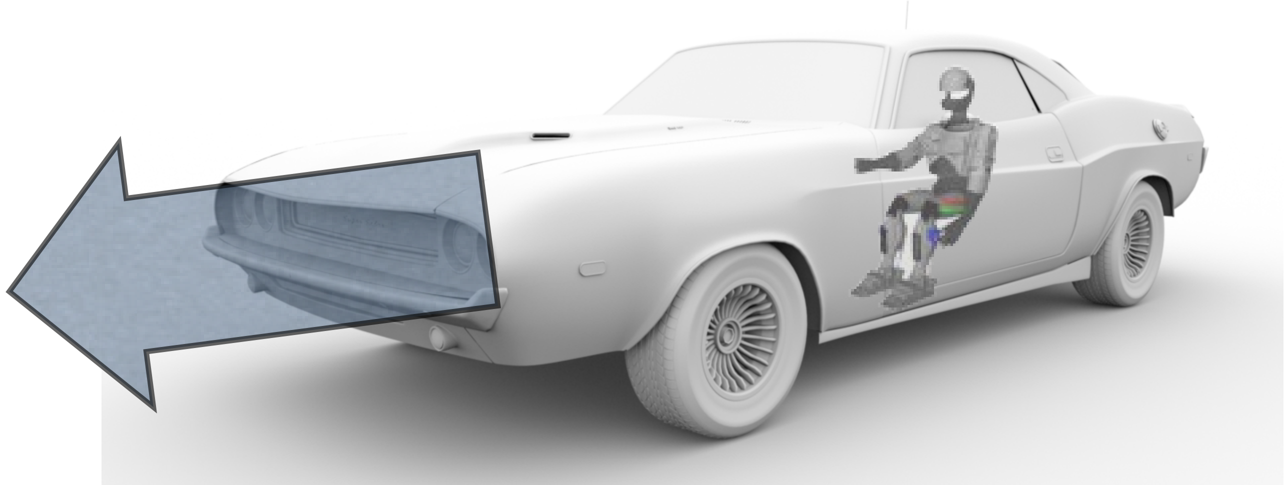
- The 0 vector has a fundamental meaning: no movement, no force
- Why would there be a special 0 point?
- It’s meaningful to add vectors, not points
  - Boston location + NYC location =?

+

=?

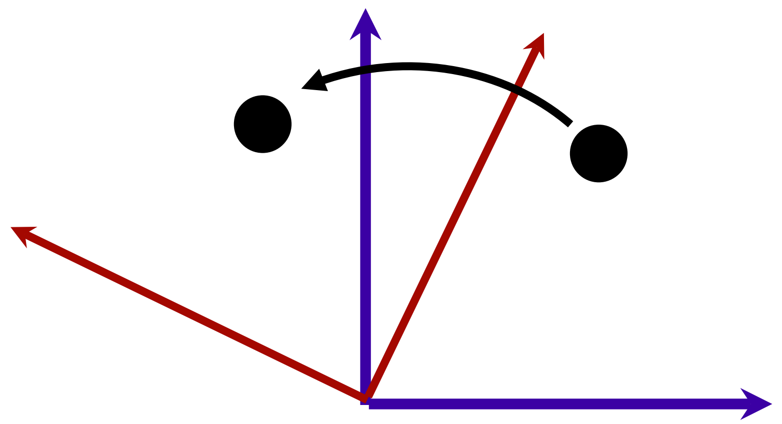
Points & vectors are different

- Moving car
  - points describe location of car elements
  - vectors describe velocity, distance between pairs of points
- If I **translate** the moving car to a different road
  - The points (location) change
  - The vectors (speed, distance between points) don’t

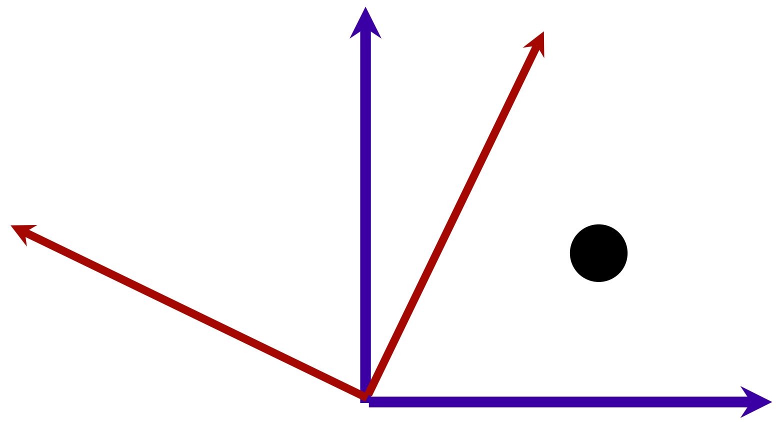


Matrices have two purposes

- (At least for geometry)
- Transform things
  - e.g. rotate the car from facing North to facing East



- Express coordinate system changes
  - e.g. given the driver's location in the coordinate system of the car, express it in the coordinate system of the world



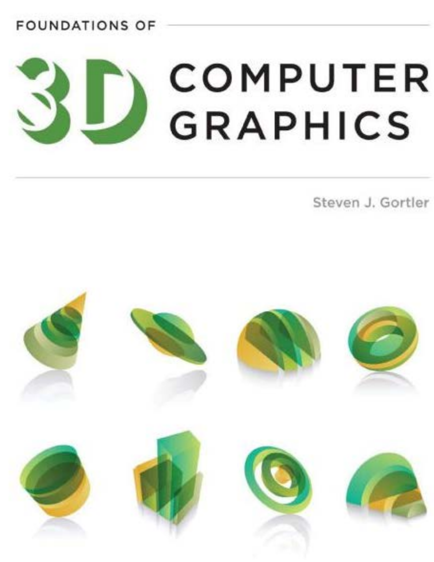
Goals for today

- Make it very explicit what coordinate system is used
- Understand how to change coordinate systems
- Understand how to transform objects
- Understand difference between points, vectors, normals and their coordinates

Questions?

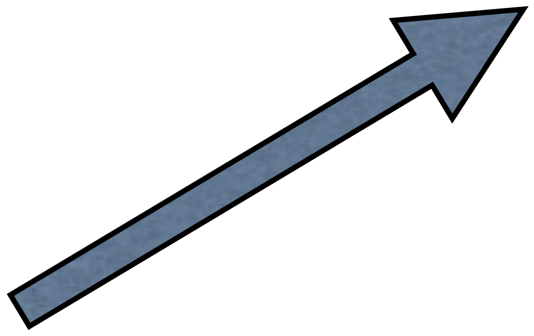
Reference

- This lecture follows the new book by Steven (Shlomo) Gortler from Harvard: Foundations of 3D Computer Graphics

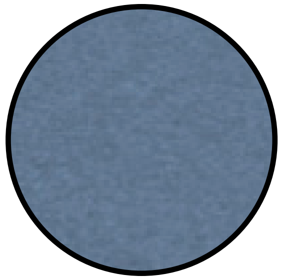


Plan

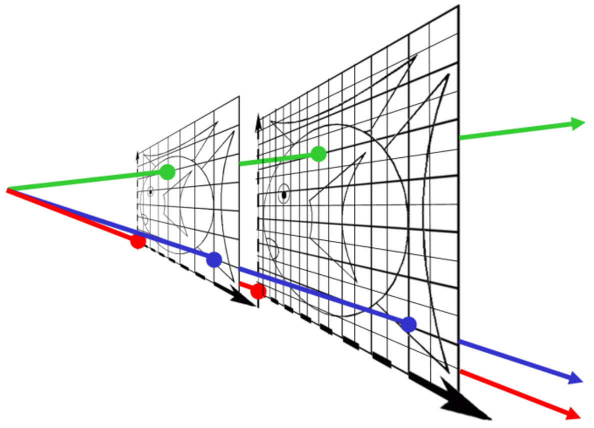
- Vectors



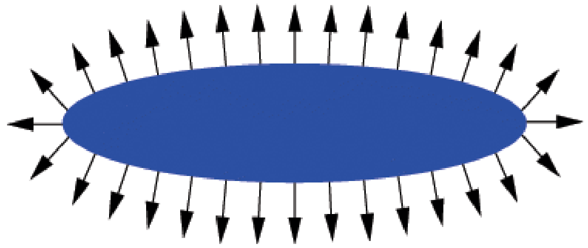
- Points



- Homogenous coordinates



- Normals (in the next lecture)

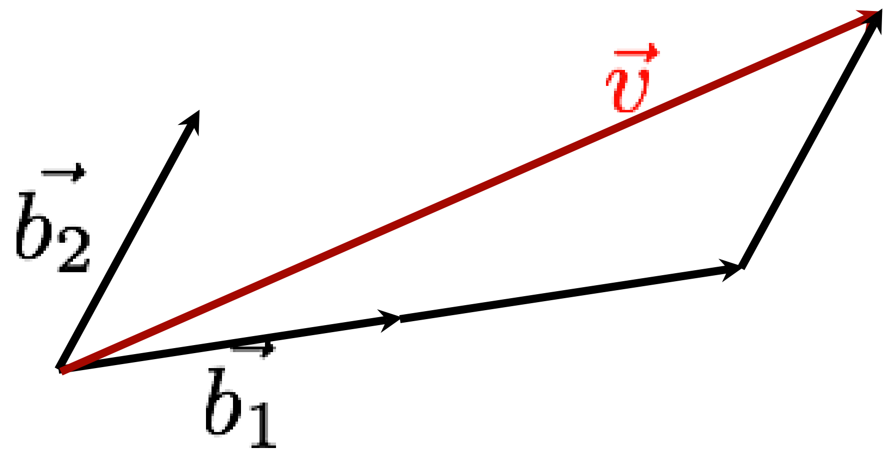


Vectors (linear space)

- Formally, a set of elements equipped with addition and scalar multiplication
  - plus other nice properties
- There is a special element, the zero vector
  - no displacement, no force

Vectors (linear space)

- We can use a ***basis*** to produce all the vectors in the space:
  - Given n basis vectors $\vec{b_i}$
    any vector $\vec{v}$ can be written as $\vec{v} = \sum_i c_i \vec{b_i}$



here:

$$\vec{v} = 2\vec{b_1} + \vec{b_2}$$

Linear algebra notation

$$\vec{v} = c_1 \vec{b_1} + c_2 \vec{b_2} + c_3 \vec{b_3}$$

- can be written as

$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

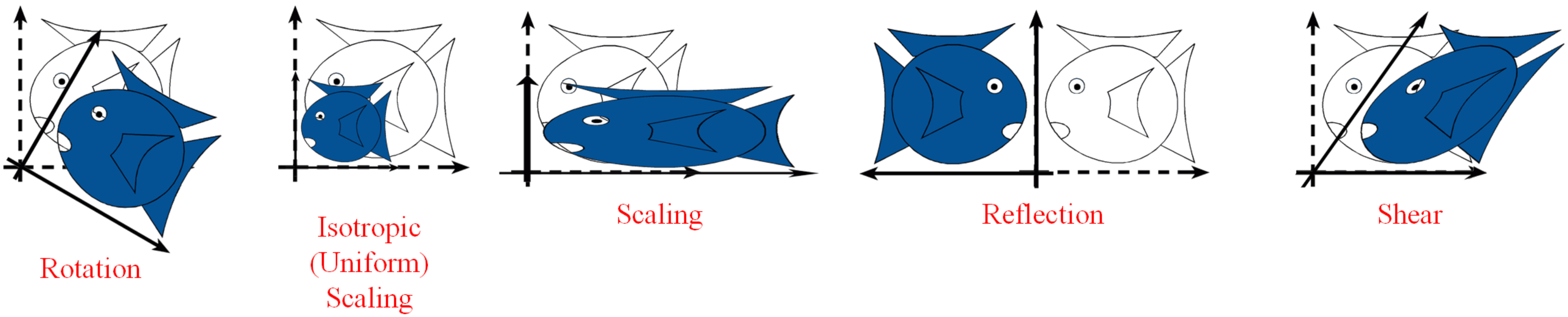
- Nice because it makes the basis (coordinate system) explicit
- Shorthand:

$$\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$$

- where bold means triplet, t is transpose

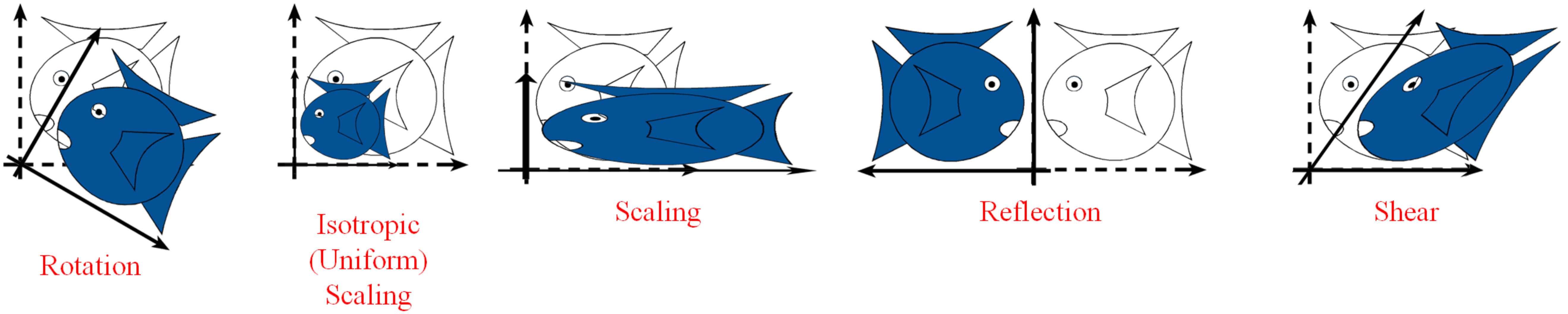
Questions?

Linear transformation



- Transformation $\mathcal{L}$ of the vector space

Linear transformation



- Transformation $\mathcal{L}$ of the vector space so that

$$\mathcal{L}(\vec{v}+\vec{u}) = \mathcal{L}(\vec{v}) + \mathcal{L}(\vec{u})$$

$$\mathcal{L}(\alpha\vec{v}) = \alpha\mathcal{L}(\vec{v})$$

- Note that it implies $\mathcal{L}(\vec{0}) = \vec{0}$
- Notation $\vec{v} \Rightarrow \mathcal{L}(\vec{v})$ for transformations

Matrix notation

- Linearity implies

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b_i}\right) = \qquad ?$$

Matrix notation

- Linearity implies

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b_i}\right) = \sum_i c_i \mathcal{L}(\vec{b_i})$$

- i.e. we only need to know the basis transformation
- or in algebra notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Algebra notation

- The $\mathcal{L}(\vec{b_i})$ are also vectors of the space
- They can be expressed in the basis

...

Algebra notation

- The $\mathcal{L}(\vec{b_i})$ are also vectors of the space
- They can be expressed in the basis
  for example:

$$\mathcal{L}(\vec{b}_1) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ M_{3,1} \end{bmatrix}$$

- which gives us

$$\begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

Recap, matrix notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Given the coordinates ***c*** in basis $\vec{\mathrm{b}}$
  the transformed vector has coordinates Mc in $\vec{\mathrm{b}}$

Why do we care

- We like linear algebra
- It's always good to get back to an abstraction that we know and for which smarter people have developed a lot of tools
- But we also need to keep track of what basis/coordinate system we use

Questions?

Change of basis

- Critical in computer graphics
  - From world to car to arm to hand coordinate system
  - From Bezier splines to B splines and back

- problem with basis change:
  you never remember which is M or M^{-1}
  it’s hard to keep track of where you are

Change of basis

- Assume we have two bases $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$
- And we have the coordinates of $\vec{\mathbf{a}}$ in $\vec{\mathbf{b}}$
- e.g.

$$\vec{a_1} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix}$$

- i.e.

$$\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$$

- which implies $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$

Change of basis

- We have $\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$ & $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$
- Given the coordinate of $\vec{v}$ in $\vec{\mathbf{b}}$: $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
- What are the coordinates in $\vec{\mathbf{a}}$?

Change of basis

- We have $\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$ & $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$
- Given the coordinate of $\vec{v}$ in $\vec{\mathrm{b}}$: $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
- Replace $\vec{\mathrm{b}}$ by its expression in $\vec{\mathbf{a}}$

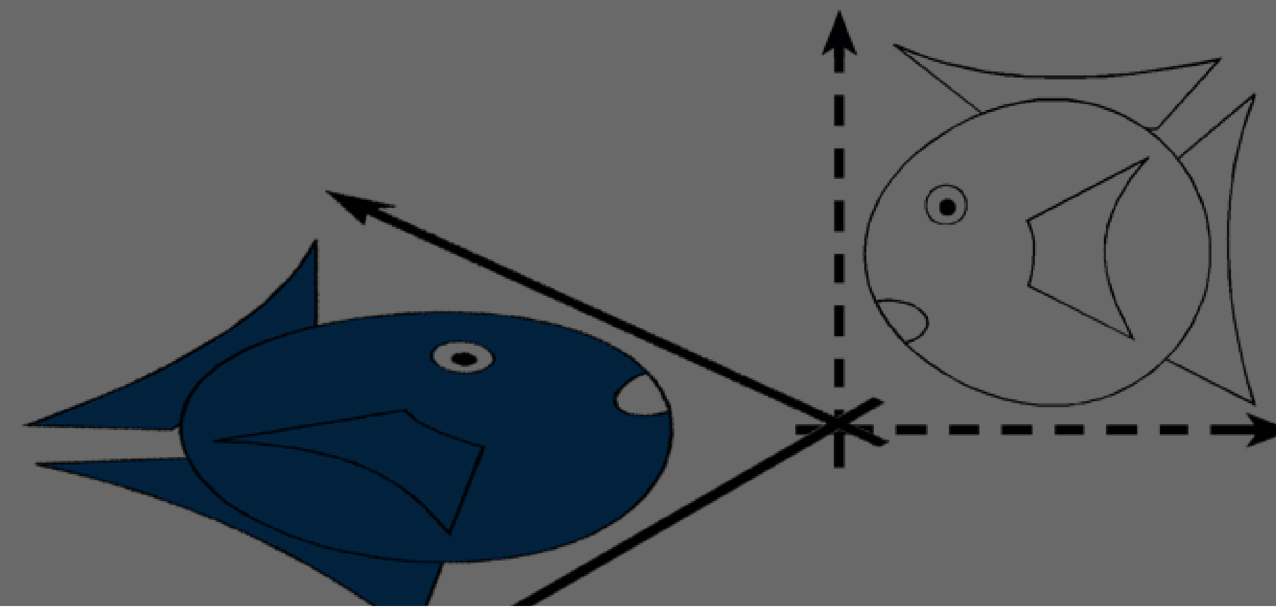
$$\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$$

- $\vec{v}$ has coordinates $M^{-1}\mathbf{c}$ in $\vec{\mathbf{a}}$
- Note how we keep track of the coordinate system by having the basis on the left

Questions?

Linear Transformations

- $L(p + q) = L(p) + L(q)$
- $L(ap) = a\,L(p)$



Translation is not linear:

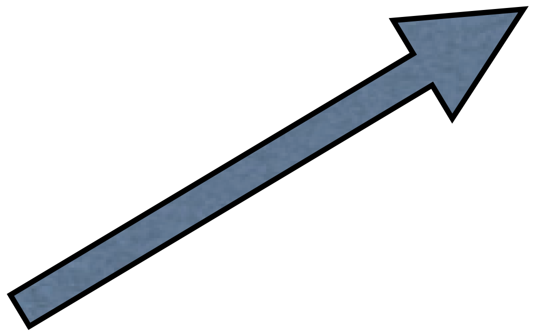
$$f(p) = p+t$$

$$f(ap) = ap+t \neq a(p+t) = a\,f(p)$$

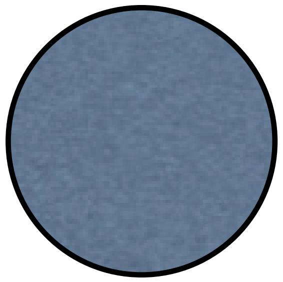
$$f(p+q) = p+q+t \neq (p+t)+(q+t) = f(p) + f(q)$$

Plan

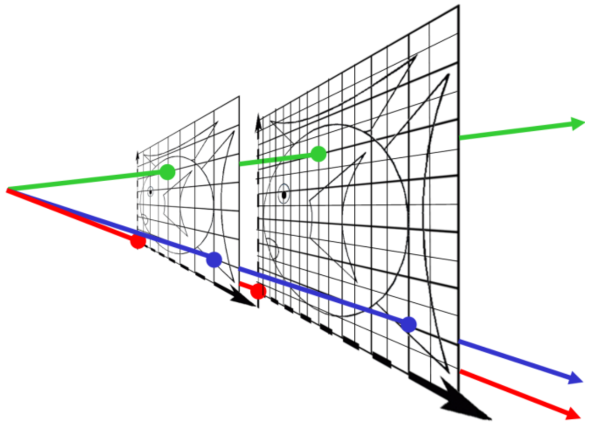
- Vectors



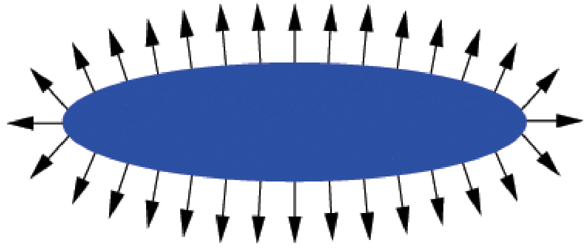
- **Points**



- Homogenous coordinates



- Normals



Points vs. Vectors

- A point is a location
- A vector is a motion between two points
- Adding vectors is meaningful
  - going 3km North + 4km East = going 5km North-East
- Adding points is not meaningful
  - Boston location + New York location = ?
- Multiplying a point by a scalar?
- The zero vector is meaningful (no movement)
- Zero point ?

Affine space

- Points are elements of an affine space
- We denote them with a tilde $\tilde{p}$

- Affine spaces are an extension of vector spaces

Point-vector operations

- Subtracting points gives a vector

$$\tilde{p} - \tilde{q} = \vec{v}$$

- Adding a vector to a point gives a point

$$\tilde{q} + \vec{v} = \tilde{p}$$

Frames

- A frame is an origin $\tilde{o}$ plus a basis $\vec{\mathrm{b}}$
- We can obtain any point in the space by adding a vector to the origin

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i$$

- using the coordinates ***c*** of the vector in $\vec{\mathrm{b}}$

Algebra notation

- We like matrix-vector expressions
- We want to keep track of the frame
- We're going to cheat a little for elegance and decide that 1 times a point is the point

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

- $\tilde{p}$ is represented in $\vec{\mathbf{f}}$ by 4 coordinate, where the extra dummy coordinate is always 1 (for now)

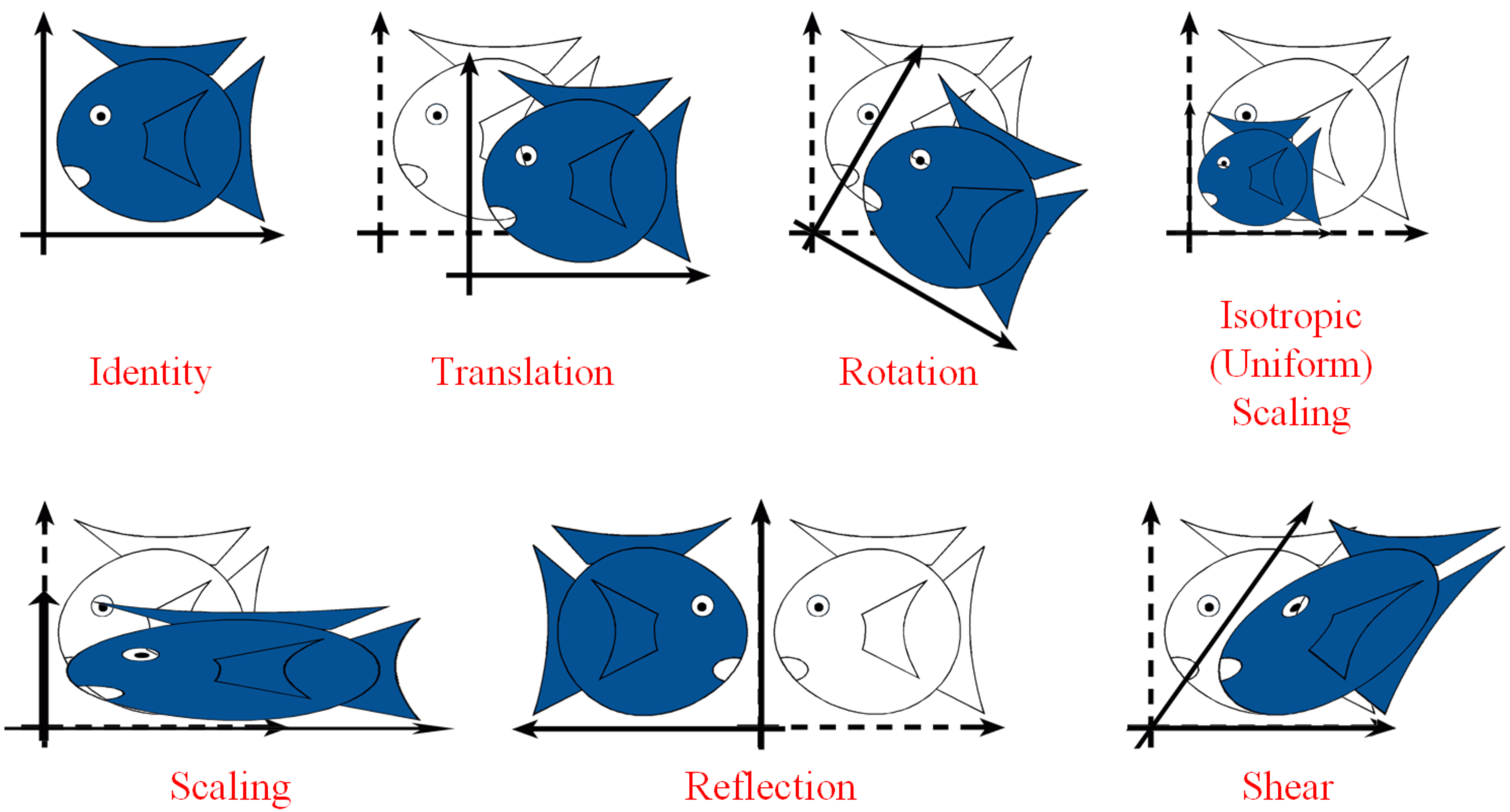
Recap

- Vectors can be expressed in a basis
  - Keep track of basis with left notation $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
  - Change basis $\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$
- Points can be expressed in a frame (origin+basis)
  - Keep track of frame with left notation
  - adds a dummy 4th coordinate always 1

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

Affine transformations

- Include all linear transformations
  - Applied to the vector basis
- Plus translation



Matrix notation

- We know how to transform the vector basis

$$\begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

- We will soon add translation by a vector $\vec{t}$

$$\tilde{p} \Rightarrow \tilde{p} + \vec{t}$$

Linear component

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\tilde{o} + \sum_i c_i \mathcal{L}(\vec{b_i}) = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

- Note how we leave the fourth component alone

Translation component

$$\tilde{p} \Rightarrow \tilde{p} + \vec{t}$$

- Express translation vector t in the basis

$$\vec{t} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} M_{14} \\ M_{24} \\ M_{34} \end{bmatrix}$$

Translation

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\tilde{o} + \vec{t} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & M_{14} \\ 0 & 1 & 0 & M_{24} \\ 0 & 0 & 1 & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Full affine expression

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\tilde{o} + \vec{t} + \sum_i c_i \mathcal{L}(\vec{b_i}) = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Which tells us both how to get a new frame $f^t M$
or how to get the coordinates Mc after transformation

Questions?

More notation properties

- If the fourth coordinate is zero, we get a vector
- Subtracting two points:

$$\tilde{p} = \vec{f}^t \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \qquad \tilde{p'} = \vec{f}^t \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ 1 \end{bmatrix}$$

- Gives us $$\tilde{p} - \tilde{p'} = \vec{f}^t \begin{bmatrix} c_1 - c'_1 \\ c_2 - c'_2 \\ c_3 - c'_3 \\ 0 \end{bmatrix}$$

  a vector (last coordinate = 0)

More notation properties

- Adding a point

$$\tilde{p} = \vec{f}^t \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

to a vector

$$\vec{v} = \vec{f}^t \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ 0 \end{bmatrix}$$

- Gives us

$$\tilde{p} + \vec{v} = \vec{f}^t \begin{bmatrix} c_1 + c'_1 \\ c_2 + c'_2 \\ c_3 + c'_3 \\ 1 \end{bmatrix}$$

a point (4th coordinate=1)

More notation properties

- vectors are not affected by the translation part

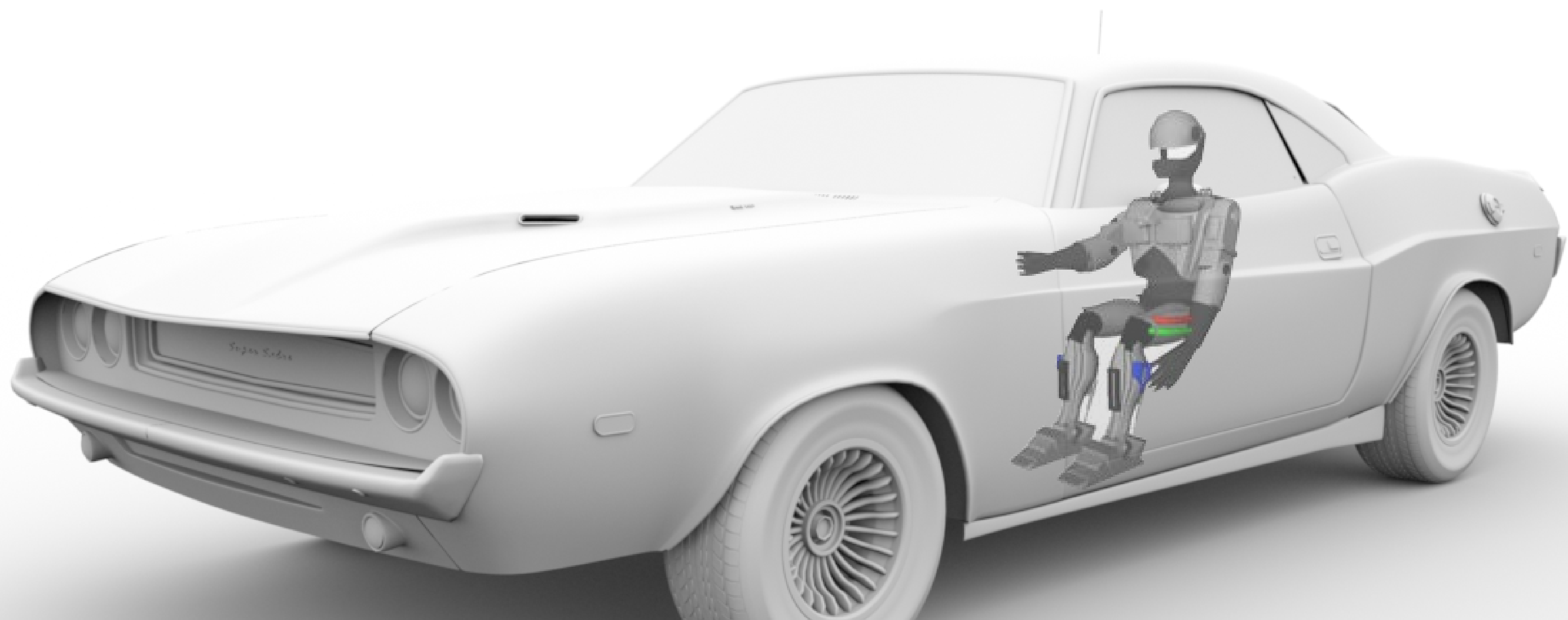
$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$$

- because their 4th coordinate is 0
- If I rotate my moving car in the world, I want its motion to rotate
- If I translate it, motion should be unaffected

Questions?

Frames & hierarchical modeling

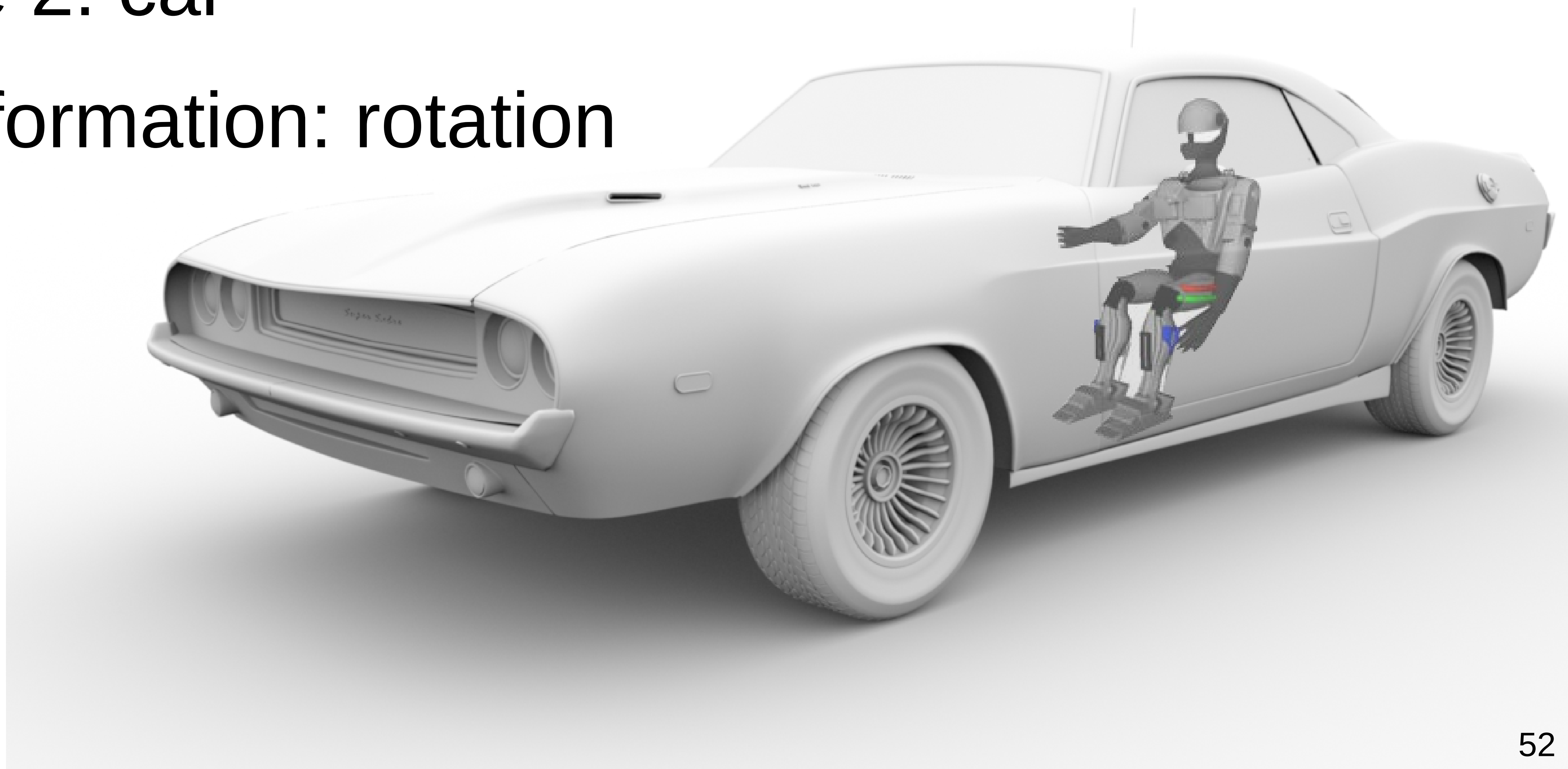
- Many coordinate systems (frames):
  - Camera
  - Static scene
  - car
  - driver
  - arm
  - hand
  - ...



- Need to understand nested transformations

Frames & hierarchical modeling

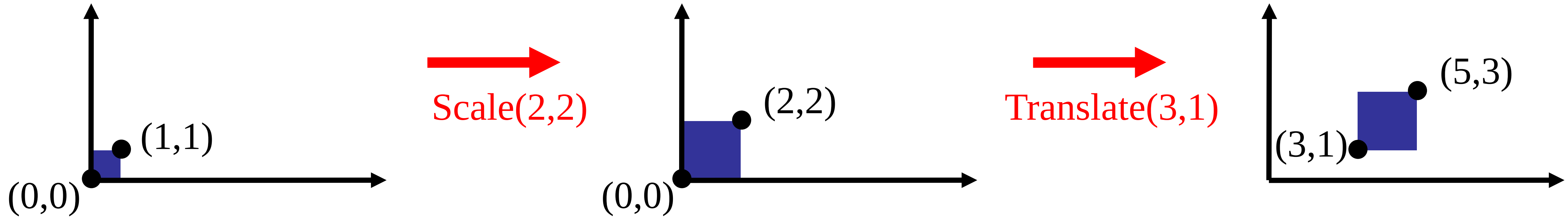
- Example: what if I rotate the wheel of the moving car:
- frame 1: world
- frame 2: car
- transformation: rotation



Questions?

How are transforms combined?

Scale then Translate



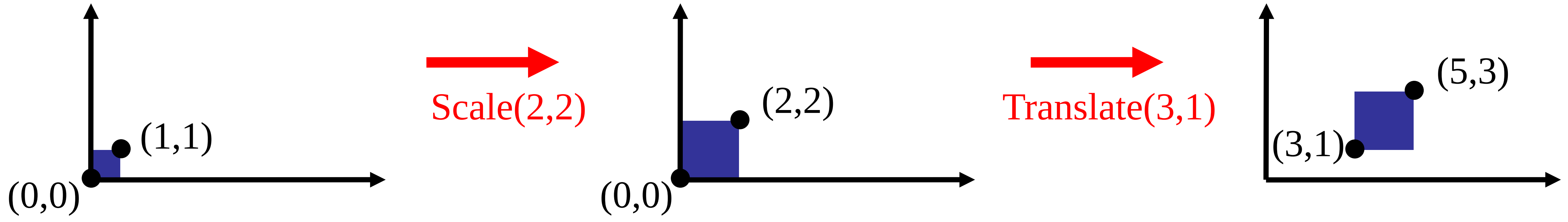
Use matrix multiplication: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

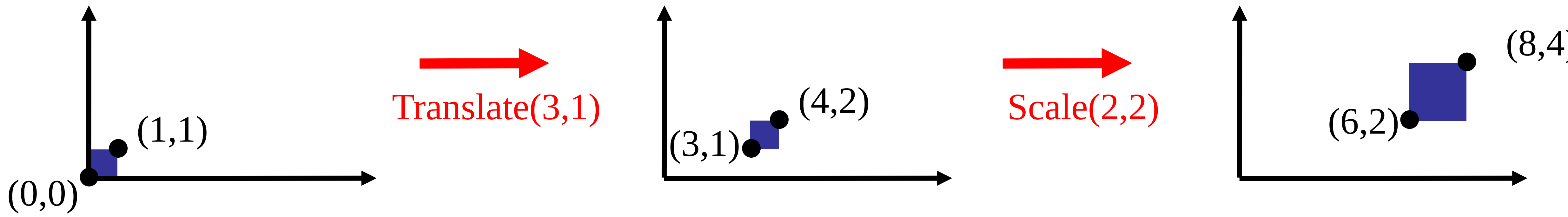
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: $p' = T(Sp) = TSp$



Translate then Scale: $p' = S(Tp) = STp$



Non-commutative Composition

Scale then Translate: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

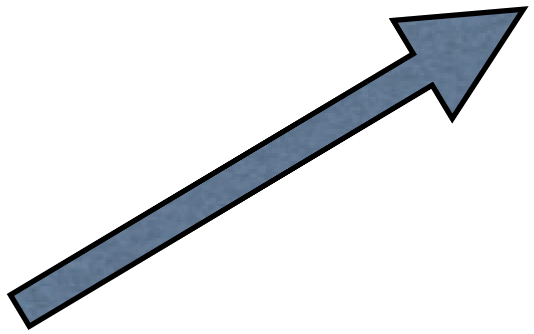
Translate then Scale: p' = S (T p) = ST p

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

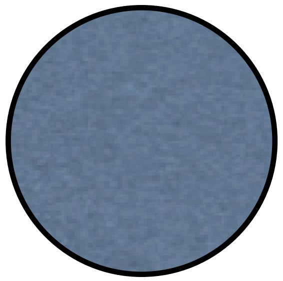
Questions?

Plan

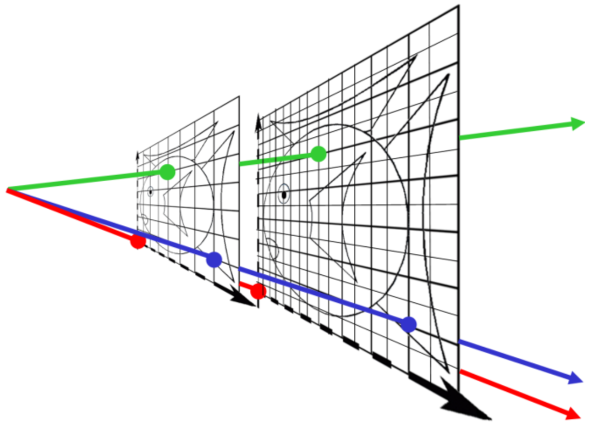
- Vectors



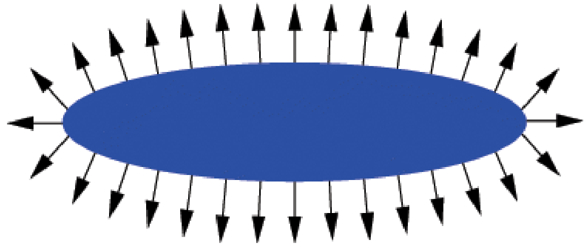
- Points



- **Homogenous coordinates**



- Normals



Forward reference and eye

- The fourth coordinate is useful for perspective projection
- Called homogenous coordinates

Homogeneous Coordinates

- Add an extra dimension (same as frames)
  - in 2D, we use 3-vectors and 3 x 3 matrices
  - In 3D, we use 4-vectors and 4 x 4 matrices
- The extra coordinate is now an **arbitrary** value, w
  - You can think of it as “scale,” or “weight”
  - For all transformations except perspective, you can just set *w*=1 and not worry about it

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Projective Equivalence

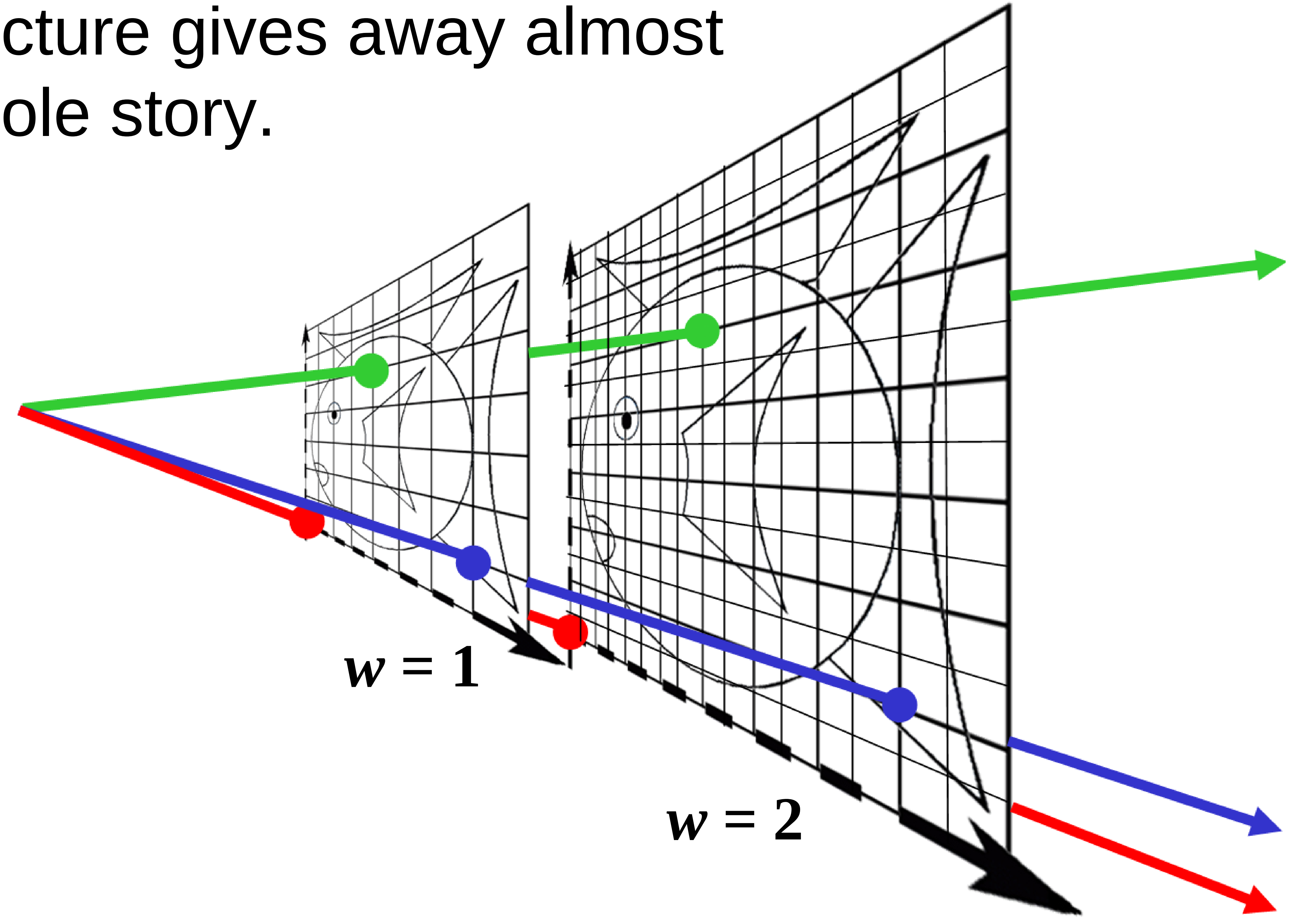
- All non-zero scalar multiples of a point are considered identical
- to get the equivalent Euclidean point, divide by *w*

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ az \\ aw \end{pmatrix} \overset{w\,!=0}{=} \begin{pmatrix} x/w \\ y/w \\ z/w \\ 1 \end{pmatrix}$$

$$a\ != 0$$

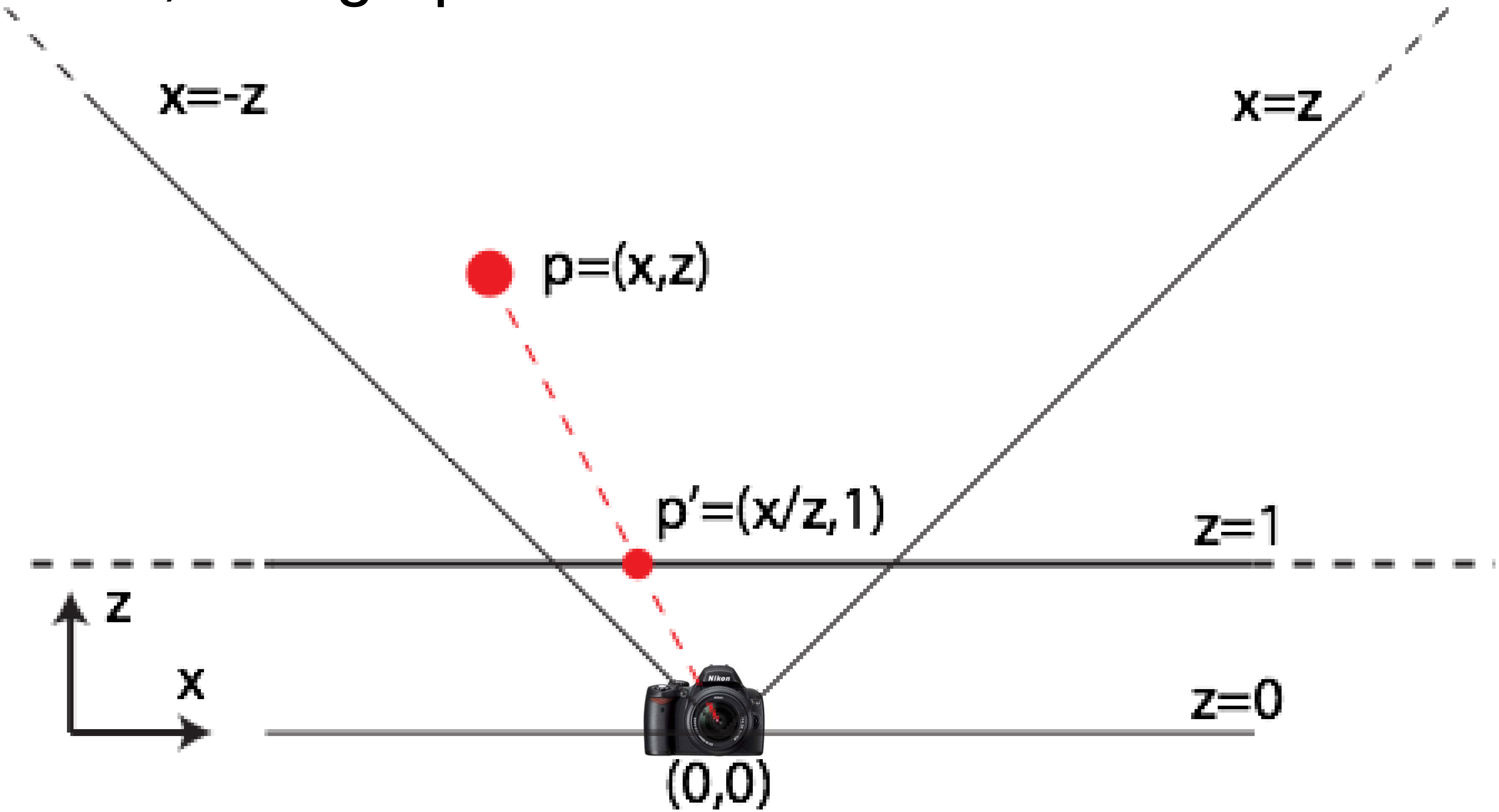
Why bother with extra coord?

- This picture gives away almost the whole story.



Perspective in 2D

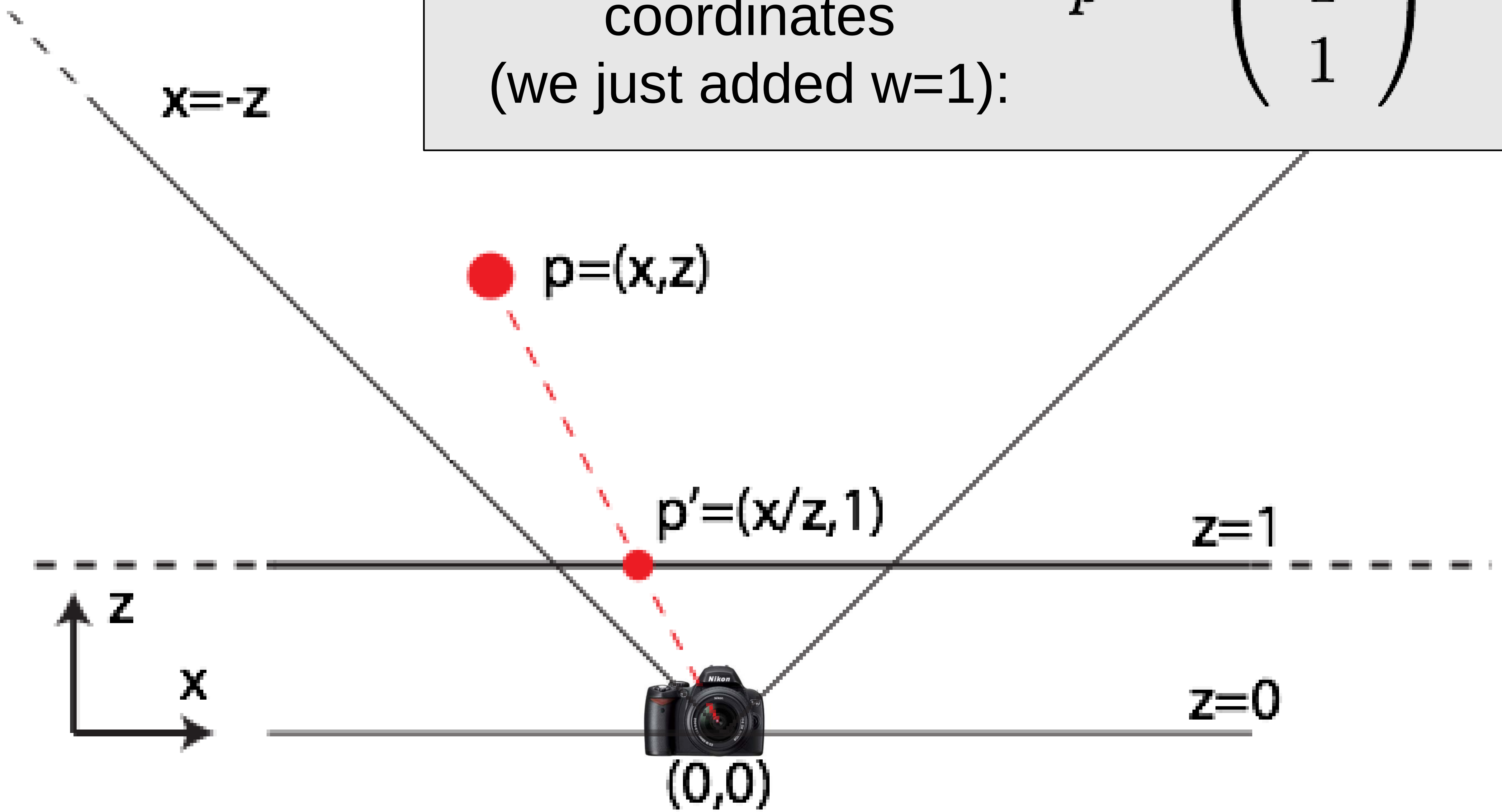
- Camera at origin, looking along z, 90 degree f.o.v., “image plane” at z=1



Perspective in 2D

The projected point in homogeneous coordinates (we just added w=1):

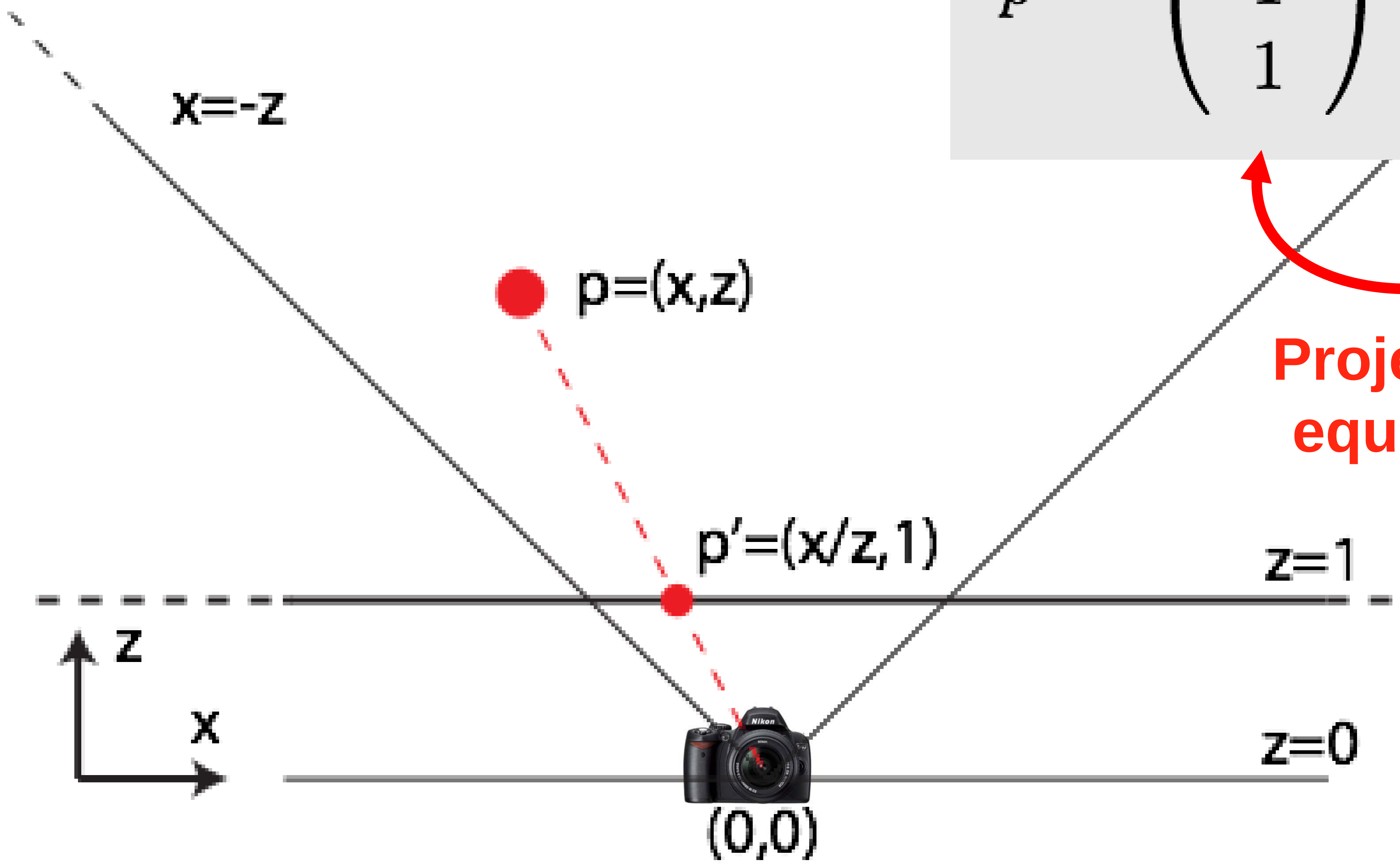
$$p' = \begin{pmatrix} x/z \\ 1 \\ 1 \end{pmatrix}$$



Perspective in 2D

$$p' = \begin{pmatrix} x/z \\ 1 \\ 1 \end{pmatrix} \propto \begin{pmatrix} x \\ z \\ z \end{pmatrix}$$

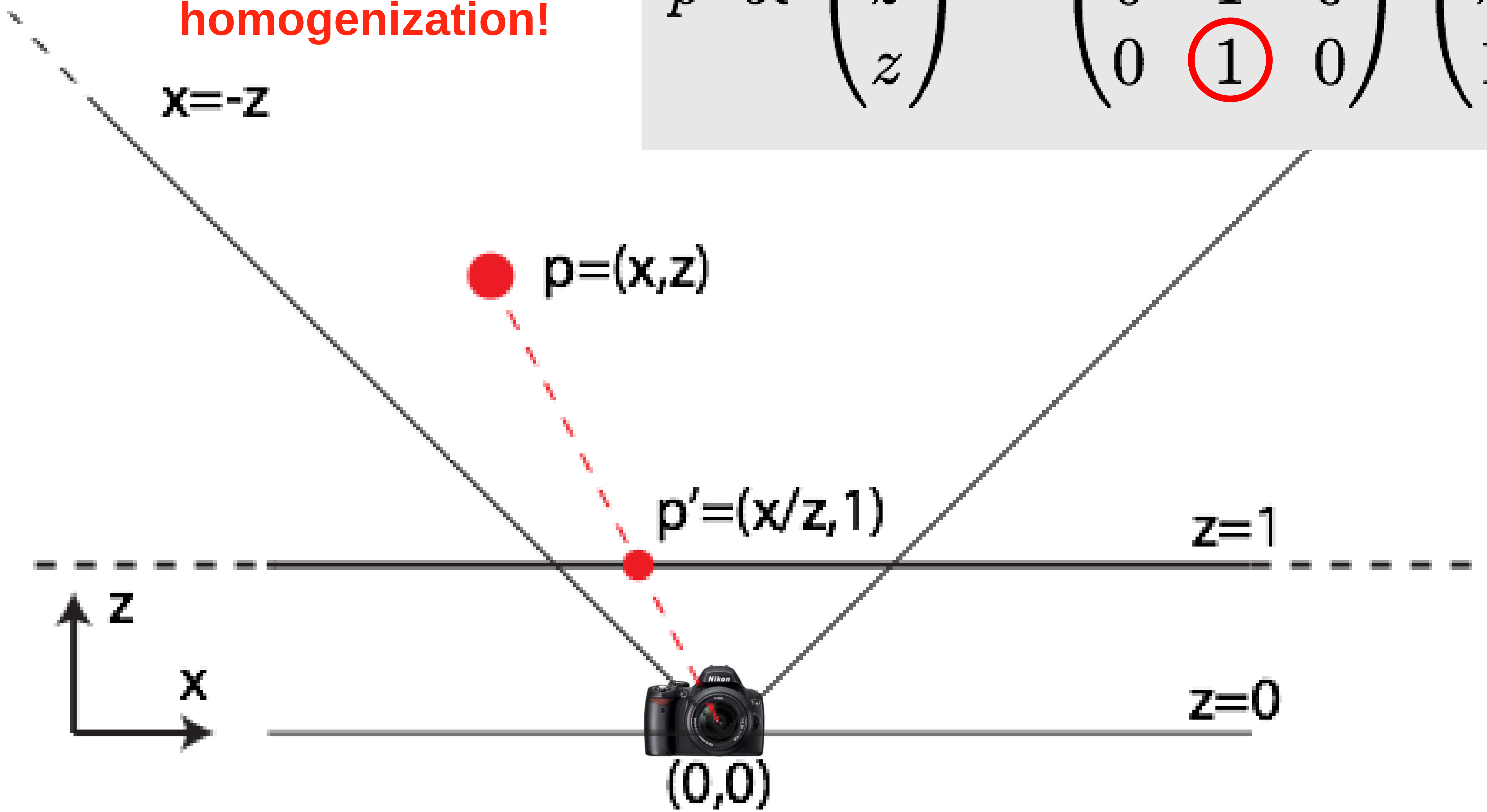
Projectively equivalent



Perspective in 2D

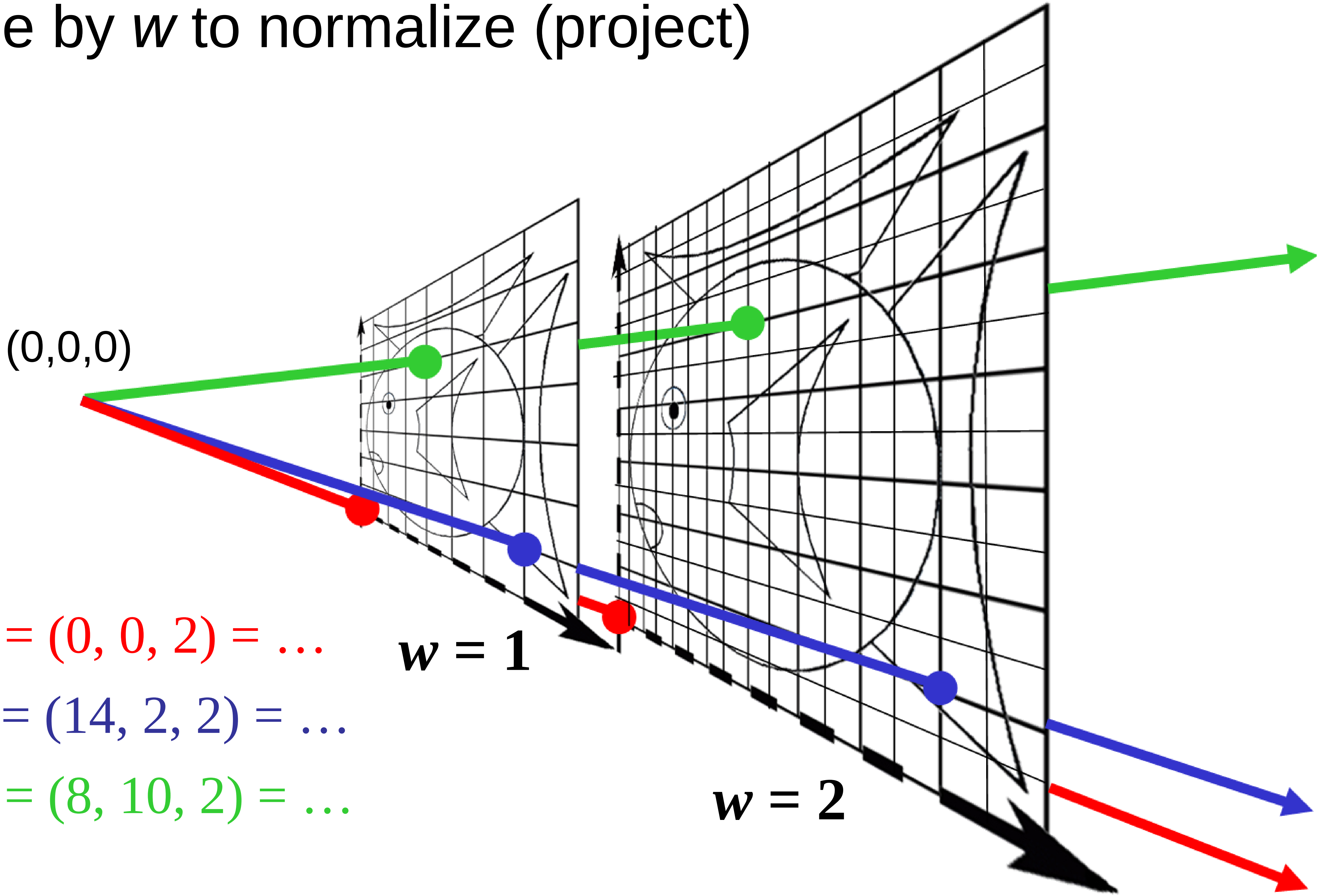
We'll just copy z to w, and get the projected point after homogenization!

$$p' \propto \begin{pmatrix} x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \\ 1 \end{pmatrix}$$



Homogeneous Visualization

- Divide by w to normalize (project)



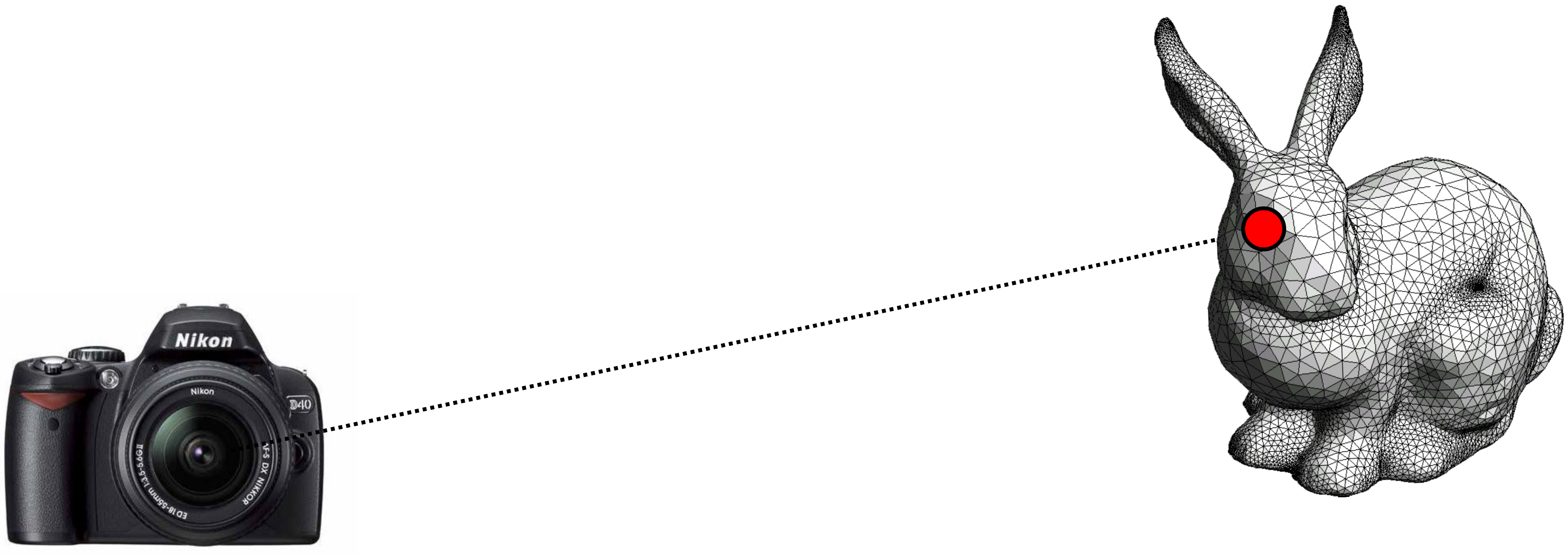
$(0, 0, 1) = (0, 0, 2) = \ldots$

$(7, 1, 1) = (14, 2, 2) = \ldots$

$(4, 5, 1) = (8, 10, 2) = \ldots$

Projective Equivalence – Why?

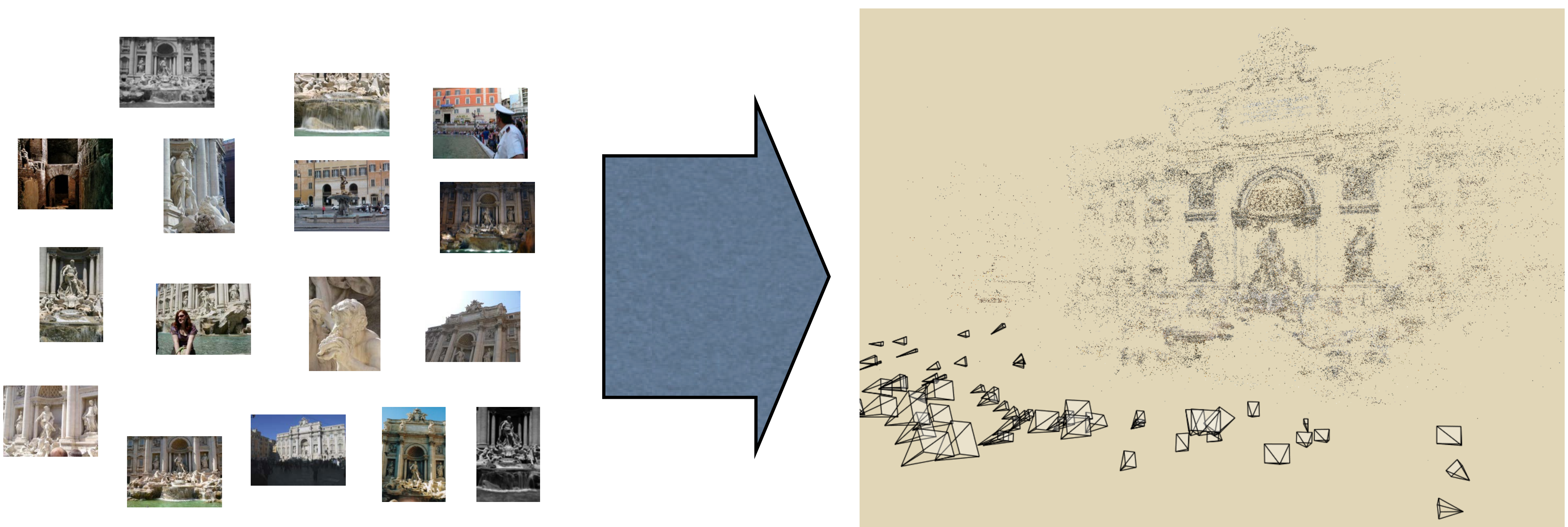
- For affine transformations,
  adding w=1 in the end proved to be convenient.
- The real showpiece is **perspective.**



Questions?

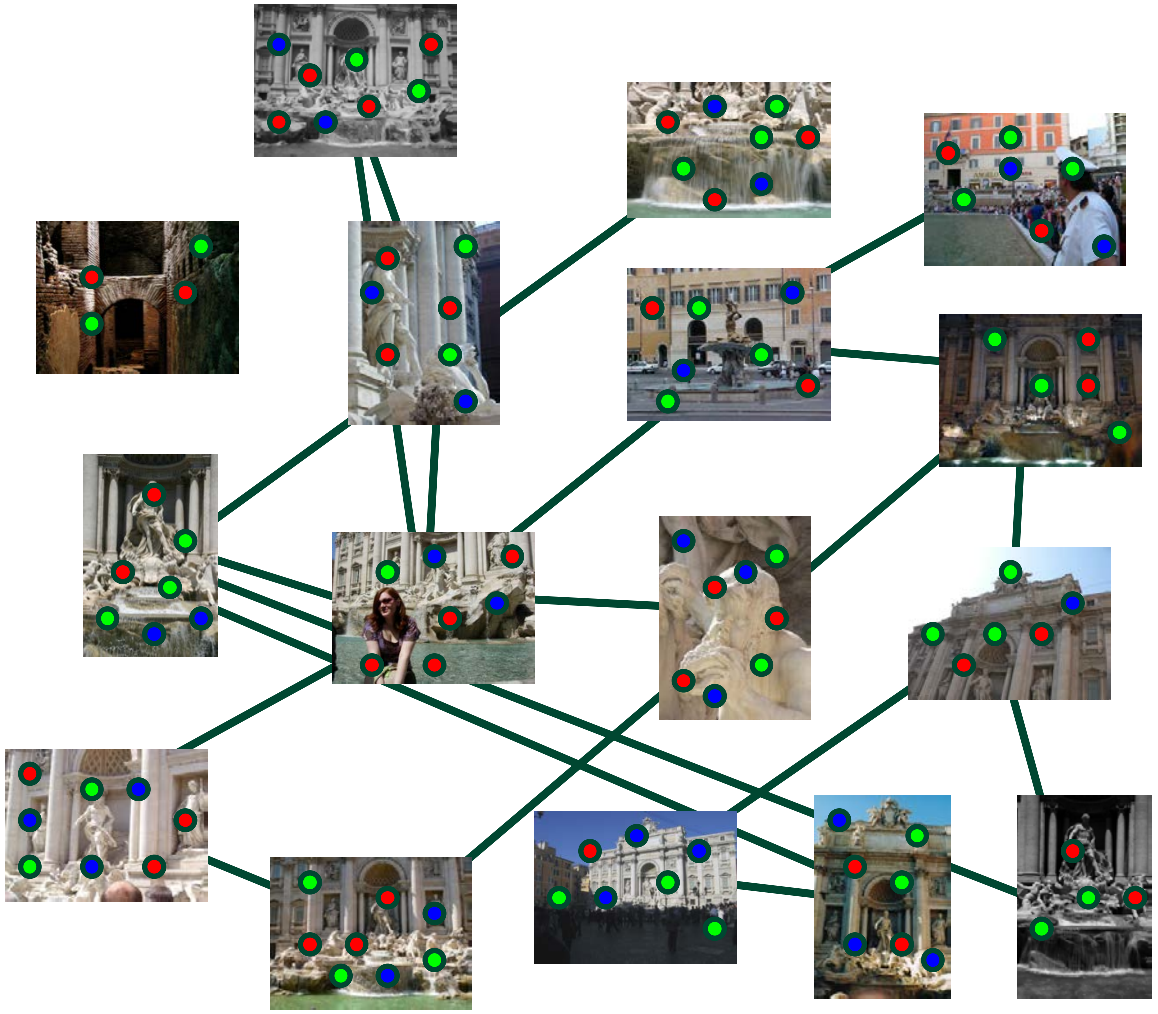
Eye candy: photo tourism

- Application of homogenous coordinates
- Goal: given N photos of a scene
  - find where they were taken
  - get 3D geometry for points in the scene



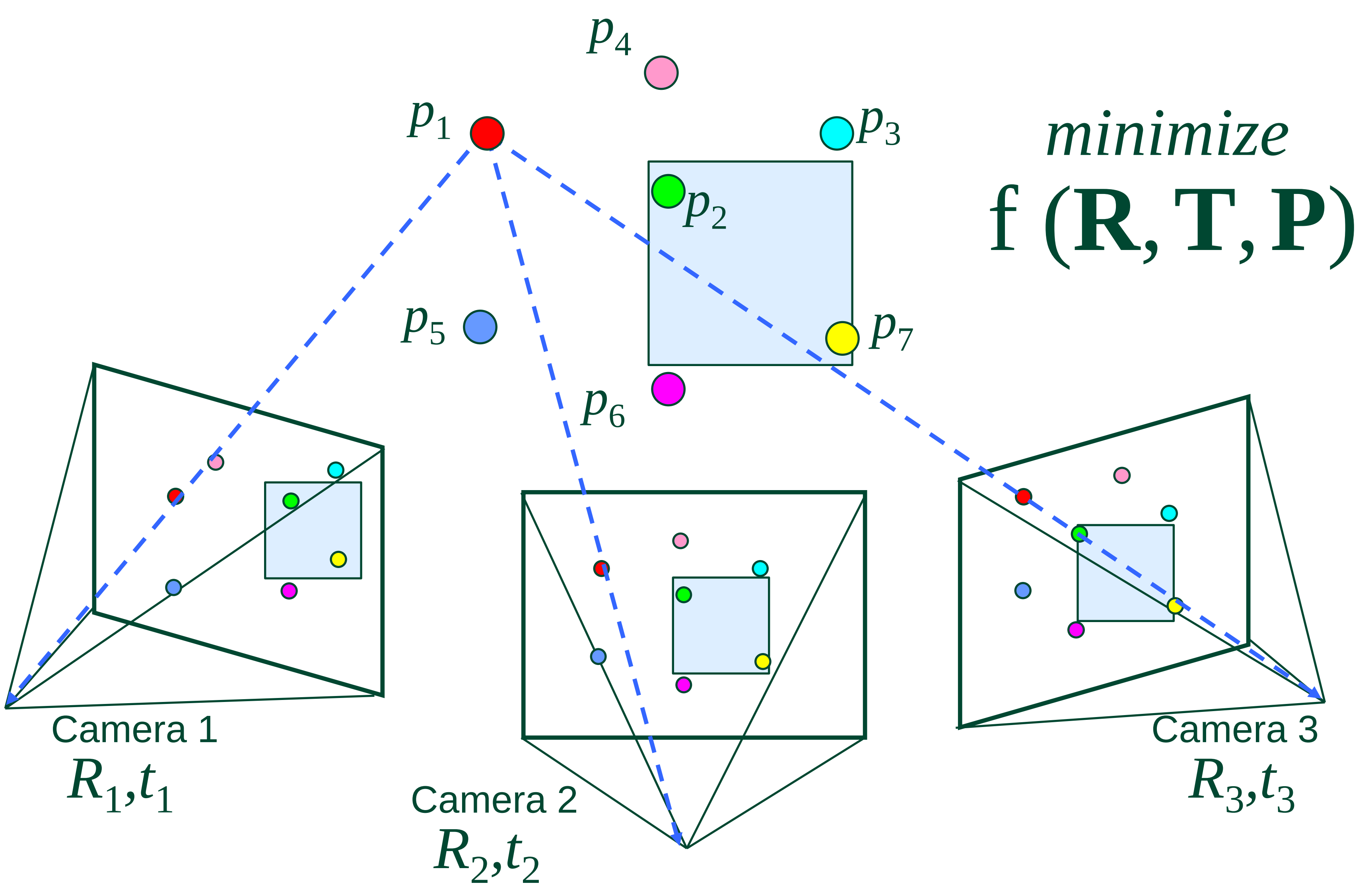
Step 1: point correspondences

- Extract salient points (corners) from images
- Find the same scene point in other images
- To learn how it's done, take 6.815



Structure from motion

- Given point correspondences
- Unknowns: 3D point location, camera poses
- For each point in each image, write perspective equations



Eye candy: photo tourism

And that’s it for today

- The rest on Thursday