6.837 Introduction to Computer Graphics
Quiz 1

Tuesday, October 16, 2012 2:40-4pm
One sheet of notes (2 pages) allowed

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1 Transformations[/16]

1.1 Frames

In this question, we consider homogeneous coordinates in 2D affine space. Let $\{\vec{x}, \vec{y}\}$ be an orthonormal basis in 2D ($\mathcal{R}^2$) of a frame $\vec{\mathbf{f}}$. Let $\tilde{o}$ be the origin of the frame $\vec{\mathbf{f}}$.

Provide a matrix M that transforms coordinates from frame $\vec{\mathbf{f}}$ to the world coordinate frame. [**/2**]

$$M = \begin{pmatrix} x_1 & y_1 & o_1 \\ x_2 & y_2 & o_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Point P has coordinates $\tilde{v}$ in the world coordinate frame. In the local frame $\vec{\mathbf{f}}$, P is transformed by a matrix U. What are the new coordinates of P in the world coordinate frame after the transformation? [**/4**]

1. Expressing $\tilde{v}$ in frame $\vec{\mathbf{f}}$ gives $M^{-1}\tilde{v}$.
2. Apply transformation U.
3. Express new position back into the world frame using M.

$$\tilde{v}' = MUM^{-1}\tilde{v}$$

Provide a matrix S that transforms normals from frame $\vec{\mathbf{f}}$ to the world coordinate frame. [**/3**]

$$M, (M^{-1})^T$$

or

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

A line in the world coordinate system is defined as $\tilde{x}(t) = \tilde{e} + t\vec{d}$, where $t \in \mathcal{R}$, $\tilde{e}$ is a point on the line, and $\vec{d}$ is the line direction vector. In frame $\vec{\mathbf{f}}$ this line is expressed as $\tilde{x}_f(t) = \tilde{e}_f + t\vec{d}_f$. What are $\tilde{e}_f$ and $\vec{d}_f$?
[**/4**]

$$\tilde{e}_f = M^{-1}\tilde{e}, \vec{d}_f = M^{-1}\vec{d}$$

1.2 Linear Transformations

Describe the effects of the following three 2D transformation matrices:

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$T_3 = T_1 T_2 T_1.$$

Recall that matrices can represent scaling, rotation, and shear. (Hint: what happens to a unit square?)
[**/3**]

T_1 shears in x direction by 1.
T_2 shears in y direction by -1.
T_3 is rotation 90^o clockwise.

2 Splines [/16]

We have defined a new type of cubic spline. The spline has the following four basis functions: $\frac{1}{4}t^3 + \frac{1}{4}t^2$, $\frac{1}{2} - \frac{1}{4}t^3 - \frac{1}{4}t^2$, $\frac{1}{2}t$, $\frac{1}{2} - \frac{1}{2}t$.

Prove or disprove that this new type of cubic spline lies in the convex hull of its control points. [**/6**]

1. The sum of basis/weights are 1, $\forall t$.

2. Show that the weights are always non-negative. You can be sloppy. We accept answers even if you just drew diagrams. To be more rigorous, each basis is monotonic for $t \in [0, 1]$, and both end points are non-negative.

Write down the spline basis matrix $\mathbf{B}$ for this new spline type. [**/4**]

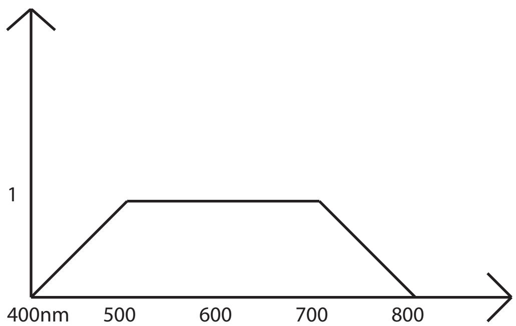
$$\frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{pmatrix}$$

We want to chain two splines of this type. We have 4 control points for the first segment: P_1, P_2, P_3, P_4 and 4 control points for the second segment: Q_1, Q_2, Q_3, Q_4. What conditions must be satisfied to ensure C^0 continuity? [**/6**]

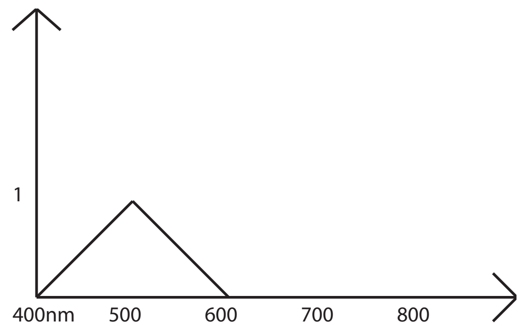
C^0 continuity is just making sure the end points match.
The end point $t = 1$ of first spline is $\frac{1}{2}(P_1 + P_3)$.
The starting point $t = 0$ of the second spline is $\frac{1}{2}(Q_2 + Q_4)$.
Therefore $P_1 + P_3 = Q_2 + Q_4$ must be satisfied.

3 Color[/16]

We have an object with the reflectance spectrum (shown on the left) illuminated by a light sources with the spectral power distribution (shown on the right).



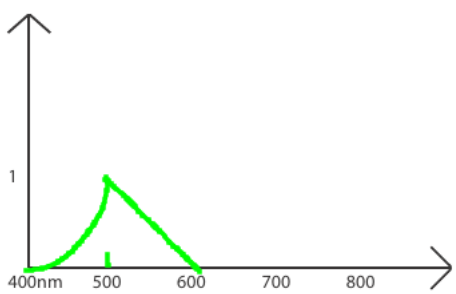
Reflectance



Light

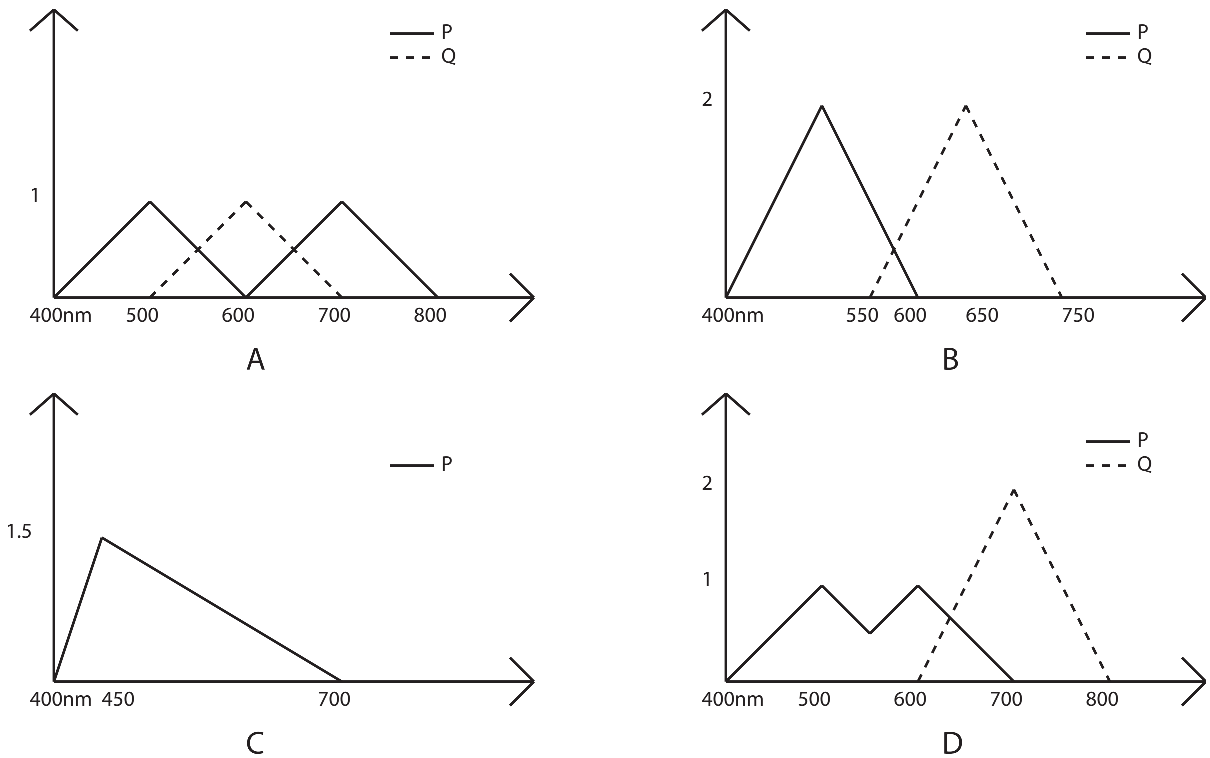
Draw the spectrum of the reflected light. [**/4**]

A quadratic curve going up followed by a line going down.

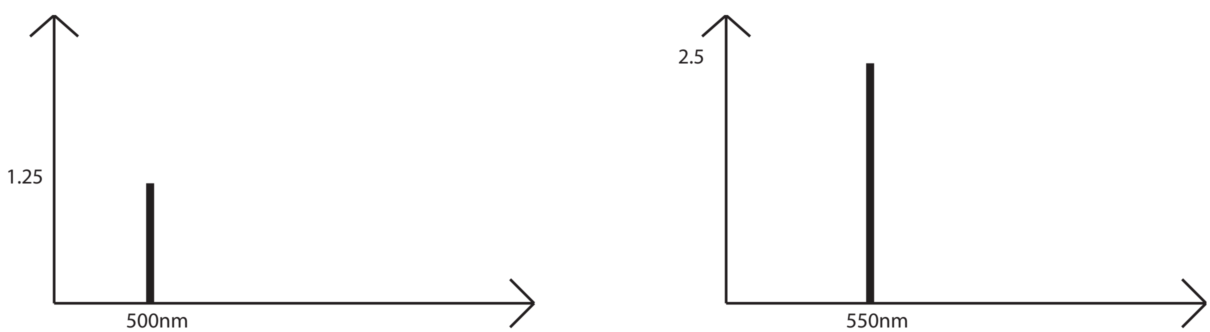


Draw a different reflectance spectrum that would result in the same reflected light spectrum given the same lighting condition. [**/4**]
Any reflectance that's the same as the original reflectance between $[400nm, 600nm]$.

There are four types of organisms: A, B, C, and D. Organisms A, B, and D have two types of cones: P and Q. Organism C has only P-type cone. The cone responses for each organism type are shown below:



There are two light sources with spectral characteristics shown below: one emits light at 500nm, one at 550nm but with the intensity twice as the first.

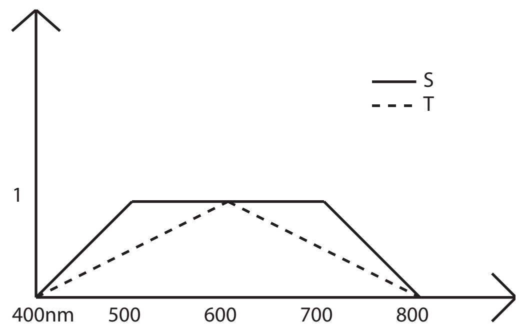


Which organism types can distinguish these two light sources? [**/4**]

A,C

The response of a cone is computed as multiplying the intensity by the cone spectrum at that wave length. As long as one cone has a different response to the two input signals, the organism can distinguish the two light sources.

We have created a color system with two primaries S and T as shown below.



Is there a linear color matching from organism A's cone spectral response PQ to ST color system? Explain and write out the transformation matrix if there is one. [**/4**]

Notice that for any wavelength λ,

$$S(\lambda) = P(\lambda) + Q(\lambda)$$

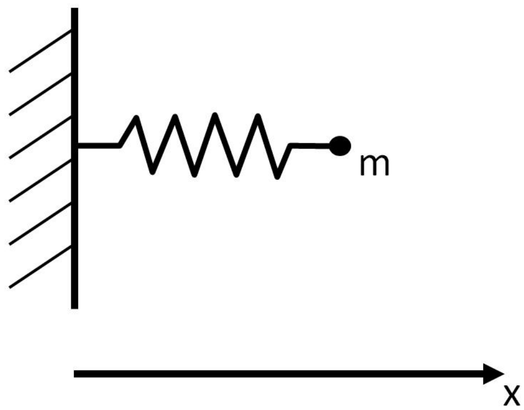
$$T(\lambda) = \frac{1}{2}(P(\lambda) + 2Q(\lambda))$$

Therefore the matrix that converts from PQ's response to ST's response is

$$\begin{pmatrix} 1 & 1 \\ 0.5 & 1 \end{pmatrix}$$

4 Physically-based animation [/24]

We are given a system in which a single particle is connected to a wall using a spring as shown in the figure below. The horizontal position of the spring is described by the scalar function $x(t)$. The spring constant is equal to k and the rest length of the spring is L. Initially, the particle is at x_0 and travels with velocity v_0. The particle has mass m. Assume that the particle is **not** affected by any other forces (e.g., gravity).



Write the second-order ODE that describes this particle system. [**/4**]

$$\frac{d^2x}{dt^2} = \frac{k(L-x)}{m}$$

$$x(0) = x_0, \frac{dx}{dt}(x_0) = v_0$$

Transform this second-order ODE into a first-order ODE system. [**/4**]

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{k(L-x)}{m} \end{pmatrix}$$

Use the trapezoid method to estimate the state of the system after time step h. [**/4**]
The trapzoidal rule is $x(t+h) = x(t) + \frac{h}{2}(f(x,t) + f(x,t+h))$.

$$f_0 = \begin{pmatrix} v_0 \\ \frac{k(L-x_0)}{m} \end{pmatrix}, \begin{pmatrix} v_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0 + hv_0 \\ v_0 + h\frac{k(L-x_0)}{m} \end{pmatrix}$$

$$f_1 = \begin{pmatrix} v_0 + h\frac{k(L-x_0)}{m} \\ \frac{k(L-x_1)}{m} \end{pmatrix}$$

Let x_h, v_h be the new position and velocity after the trapzoidal step.

$$\begin{pmatrix} x_h \\ v_h \end{pmatrix} = \begin{pmatrix} x_0 + hv_0 + \frac{h^2}{2}\frac{k(L-x_0)}{m} \\ v_0 + \frac{h}{2}(\frac{k(L-x_0)}{m} + \frac{k(L-x_1)}{m}) \end{pmatrix}$$

We didn't expect an answer for updated velocity, but a few people got it right.

Sketch $x(t)$ that the trapezoid method would produce (sketch sufficient number of steps to show the long-term behavior of the system). [**/4**]

Any curve that oscilates and diverges to infinity.

Sketch $x(t)$ that the implicit Euler would produce (sketch sufficient number of steps to show the long-term behavior of the system). [**/4**]

Any curve that oscilates and converges to the middle.

Describe advantages and disadvantages of both methods (trapezoid and implicit Euler). [**/4**]
Implicit Euler: Disadvantages: Hard to compute. Damps energy. Advantage: Stable. Trapzoidal: Disadvantages: Unstable. Advantage: Easy to compute. Does not damp away energy.