6.837 Computer Graphics
Hierarchical Modeling

Wojciech Matusik, MIT EECS

Some slides from BarbCutler & Jaakko Lehtinen

russiantreasures.com

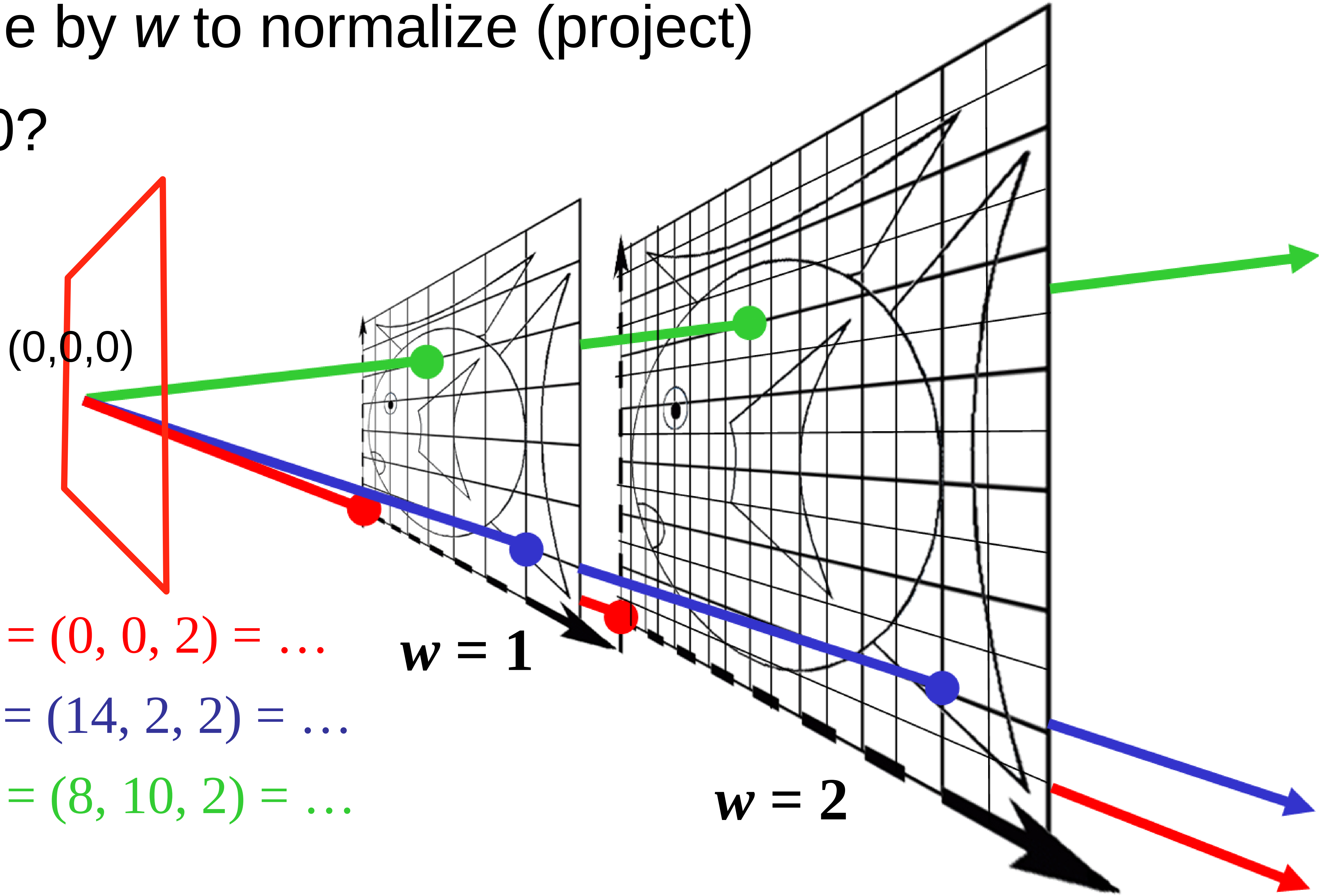
Recap

- Vectors can be expressed in a basis
  - Keep track of basis with left notation $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
  - Change basis $\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$
- Points can be expressed in a frame (origin+basis)
  - Keep track of frame with left notation
  - adds a dummy 4th coordinate always 1

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

Homogeneous Visualization

- Divide by w to normalize (project)
- $w = 0$?



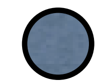
$(0, 0, 1) = (0, 0, 2) = \ldots$

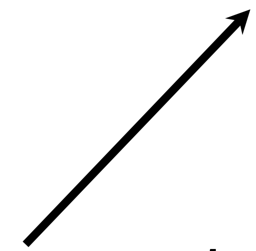
$(7, 1, 1) = (14, 2, 2) = \ldots$

$(4, 5, 1) = (8, 10, 2) = \ldots$

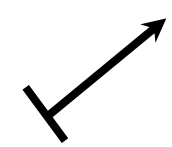
Different objects

- **Points**
  - represent locations





- **Vectors**
  - represent movement, force, displacement from A to B
- **Normals**
  - represent orientation, unit length

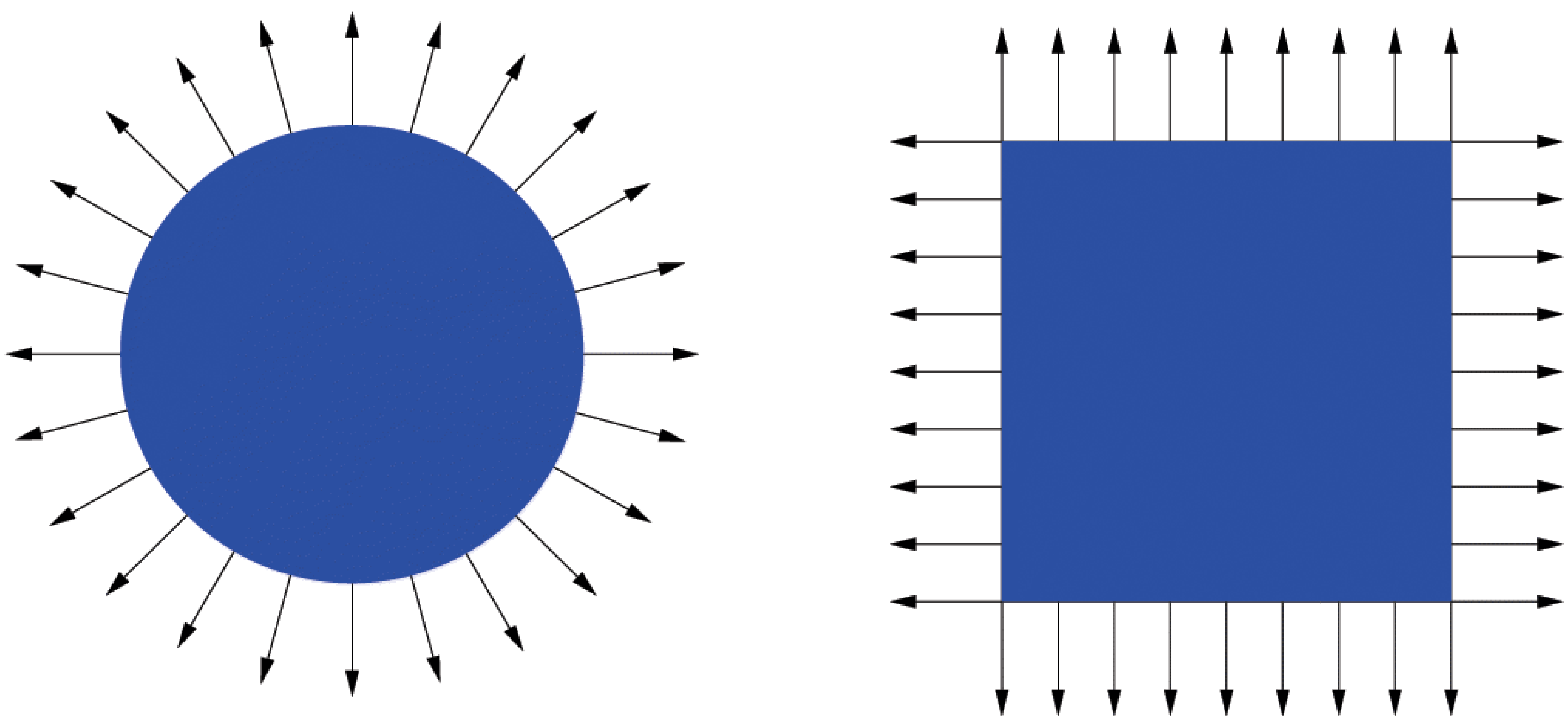


- **Coordinates**
  - numerical representation of the above objects **in a given coordinate system**

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

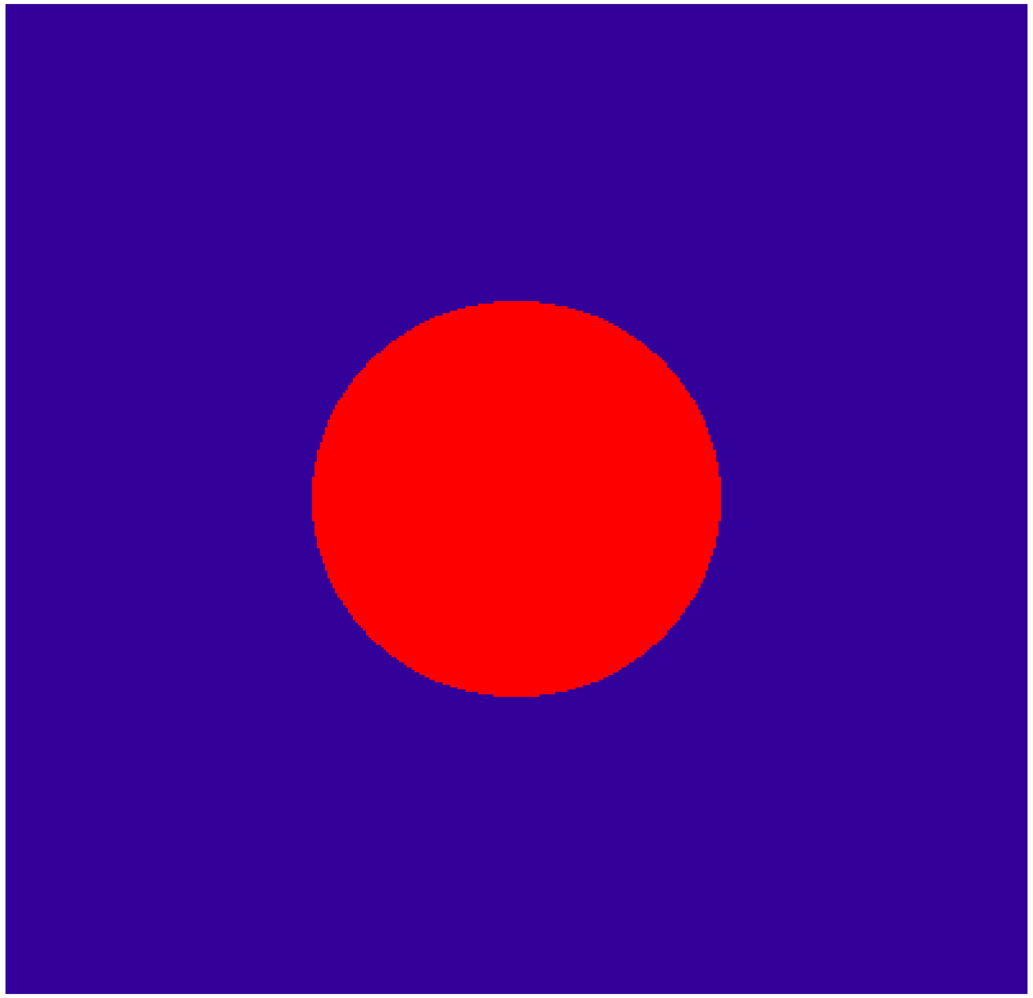
Normal

- Surface Normal: unit vector that is locally perpendicular to the surface

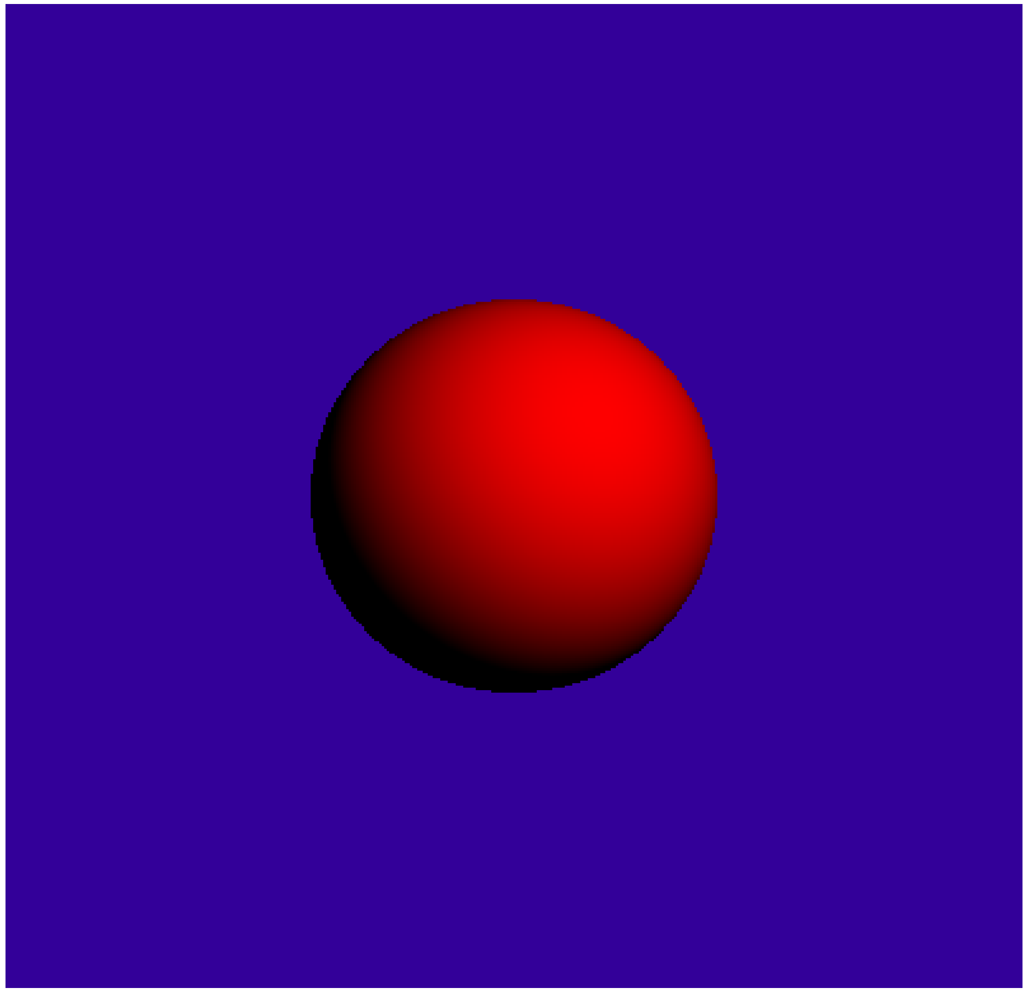


Why is the Normal important?

- It's used for shading — makes things look 3D!

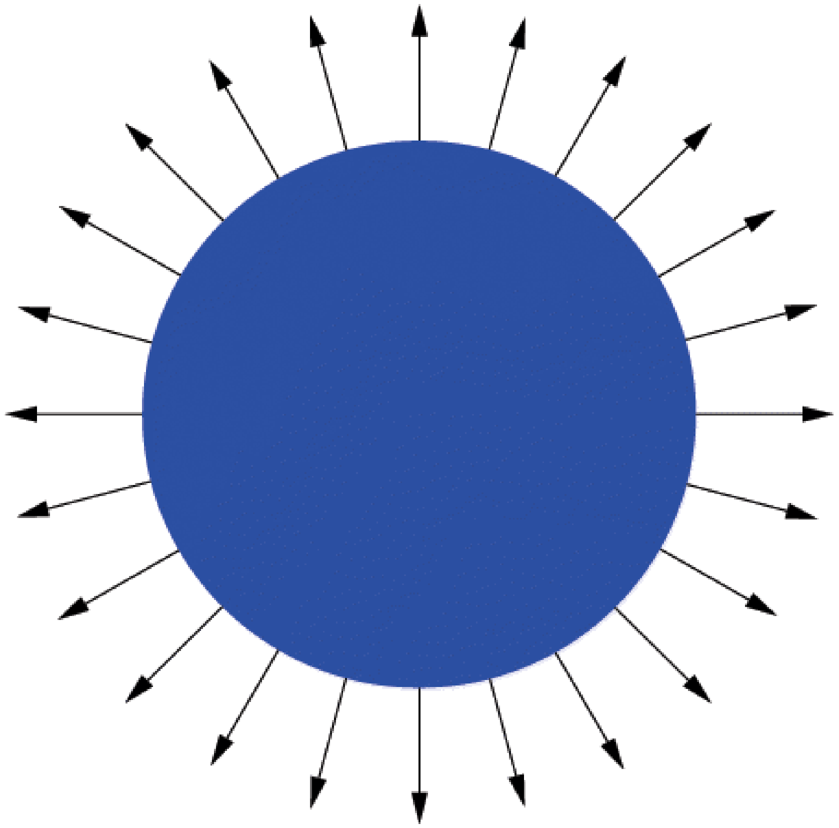


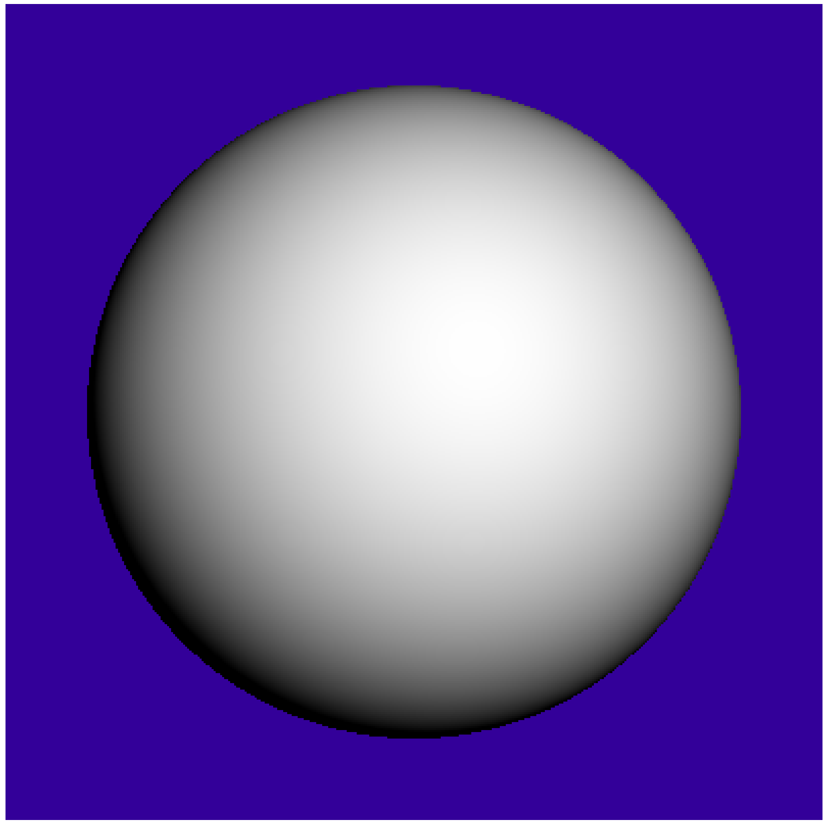
object color only

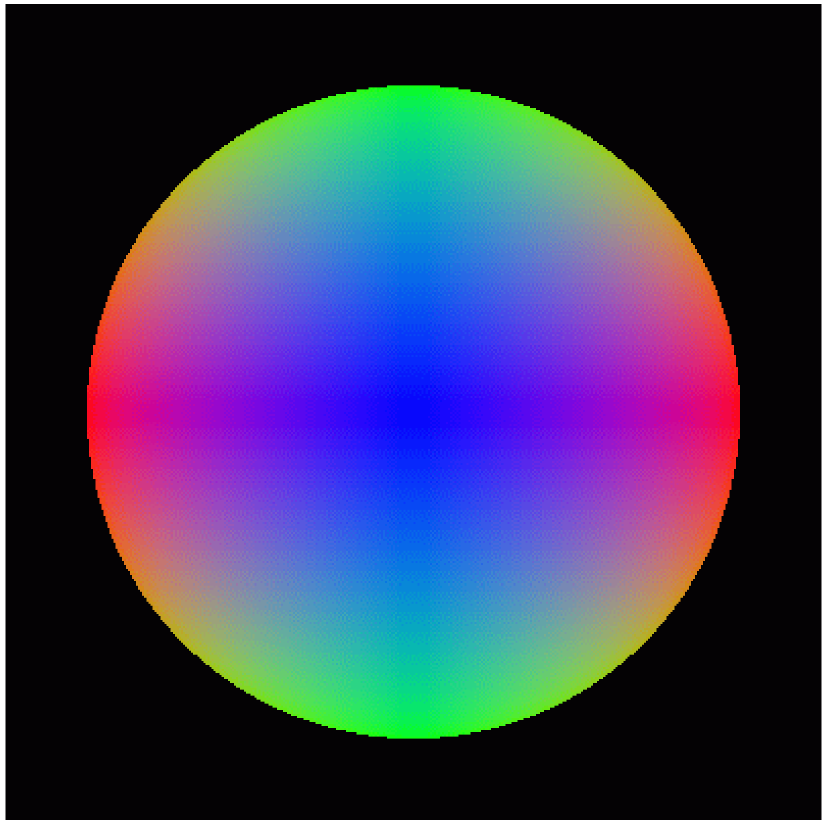


Diffuse Shading

Visualization of Surface Normal





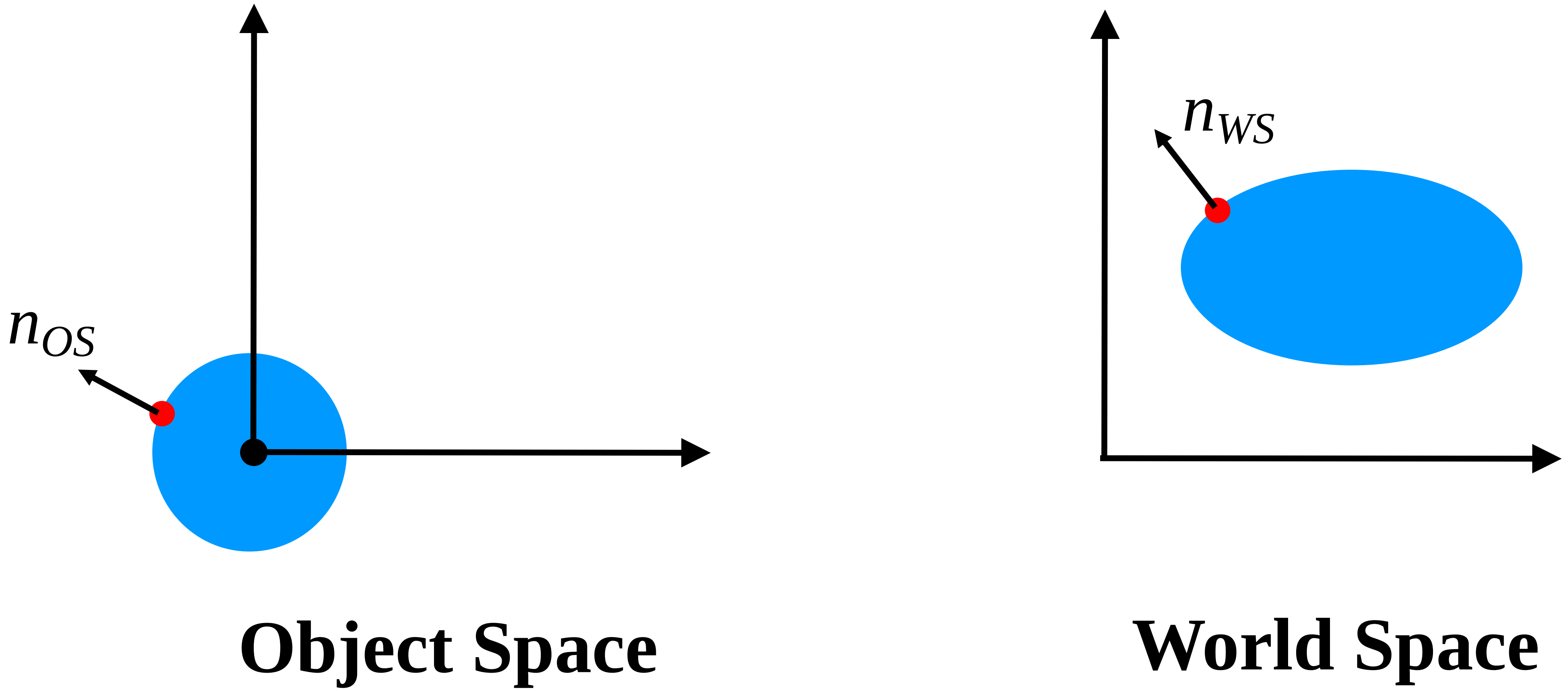


$\pm x$ = Red

$\pm y$ = Green

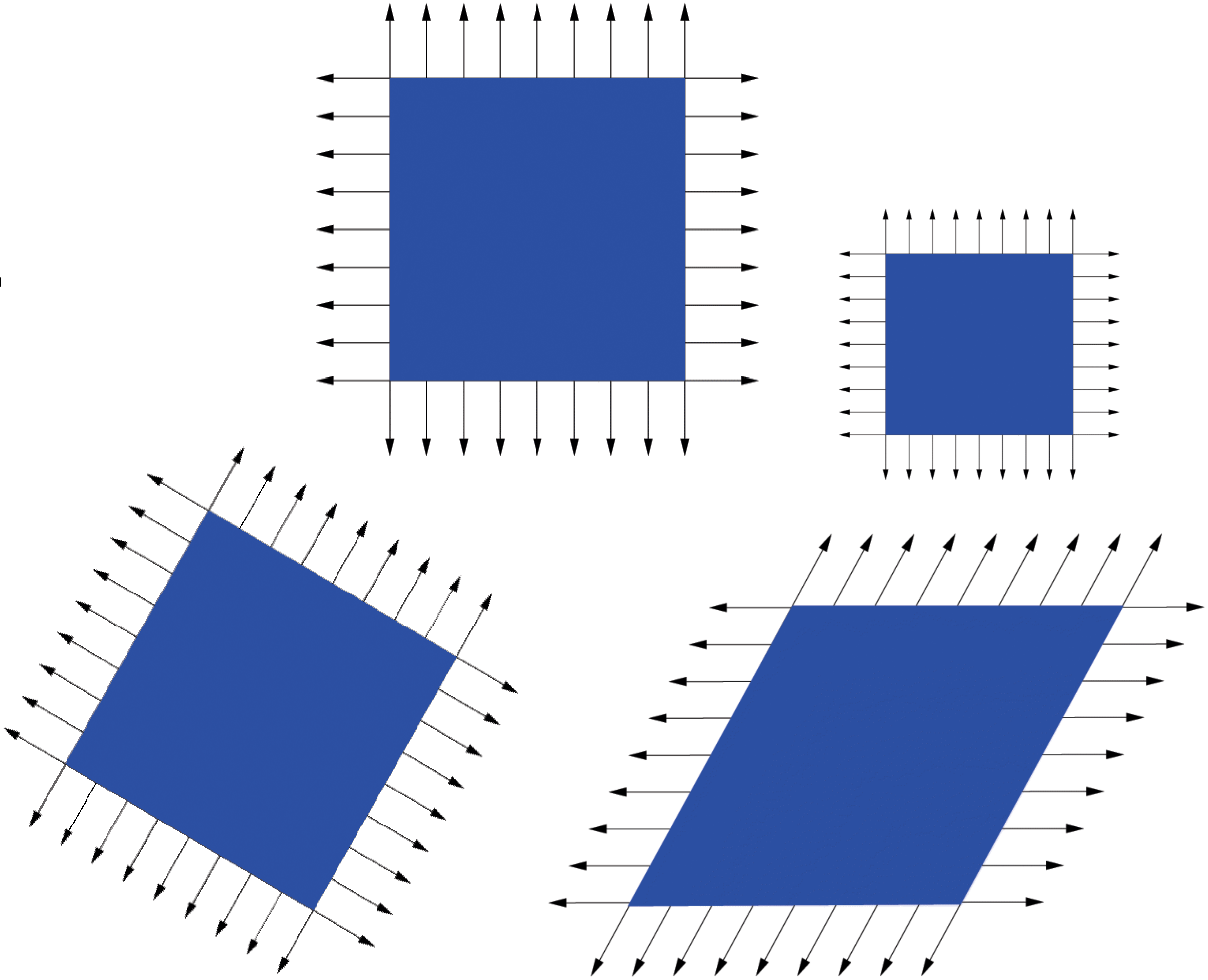
$\pm z$ = Blue

How do we transform normals?



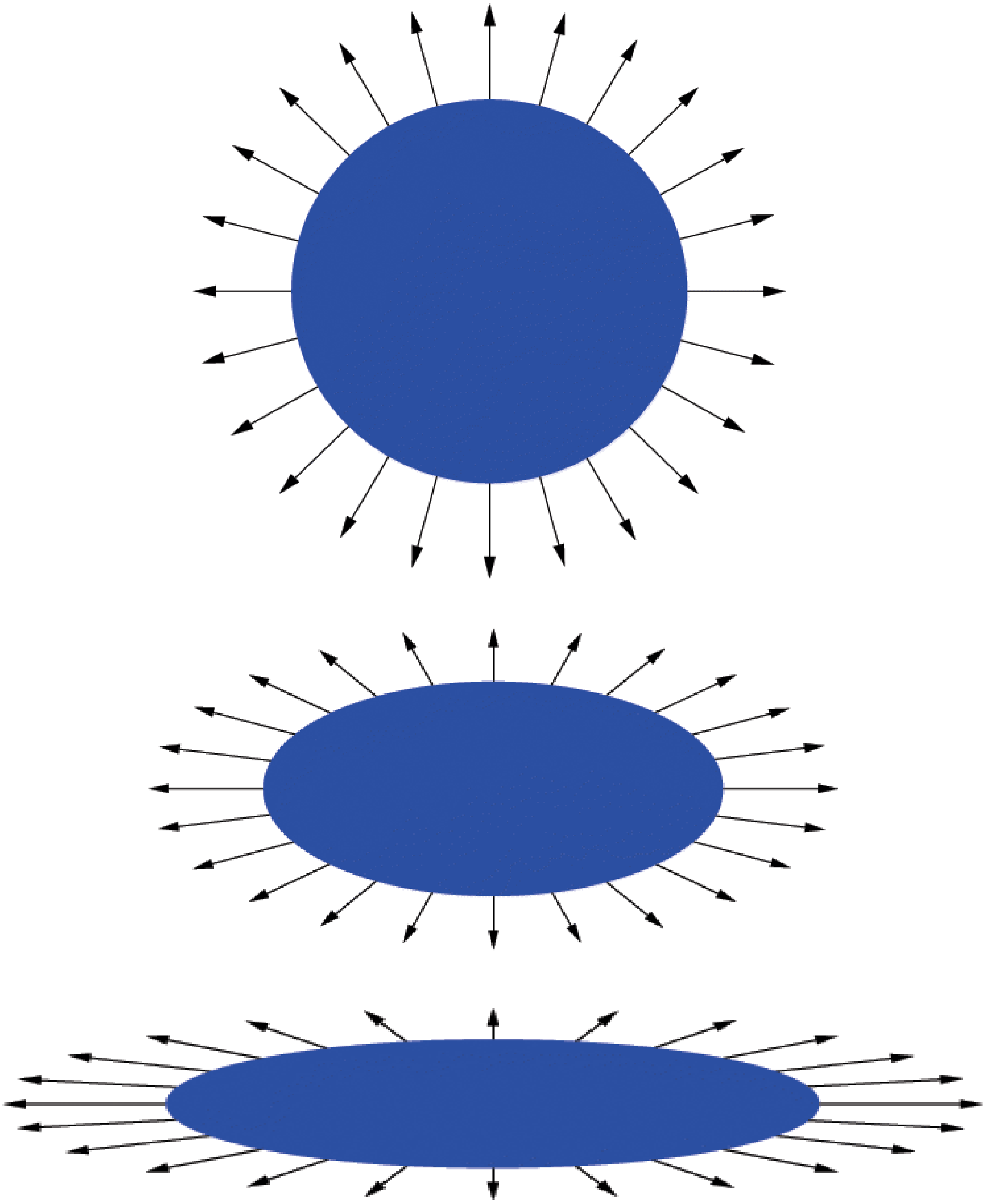
Transform Normal like Object?

- translation?
- rotation?
- isotropic scale?
- scale?
- reflection?
- shear?
- perspective?



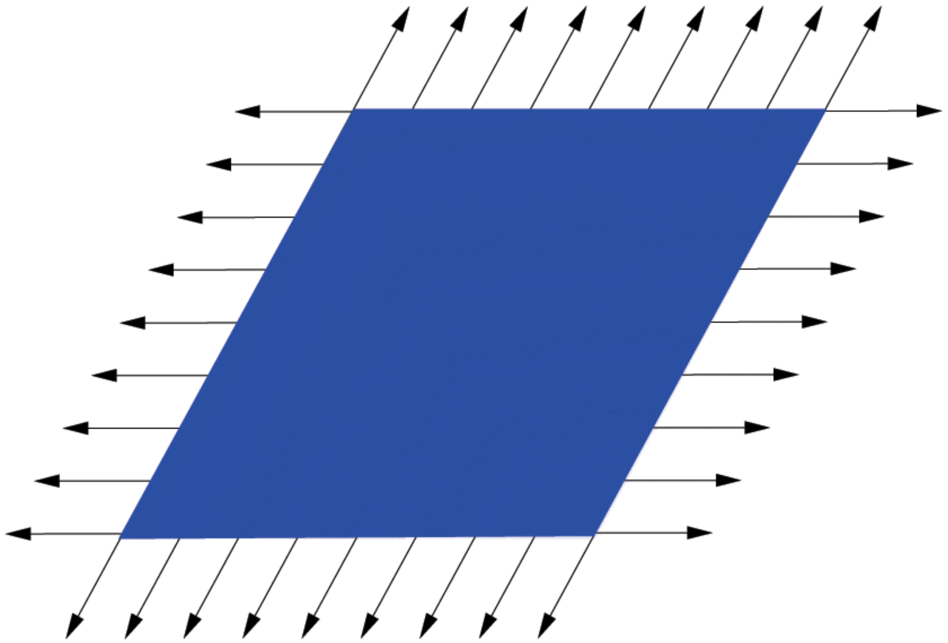
Transform Normal like Object?

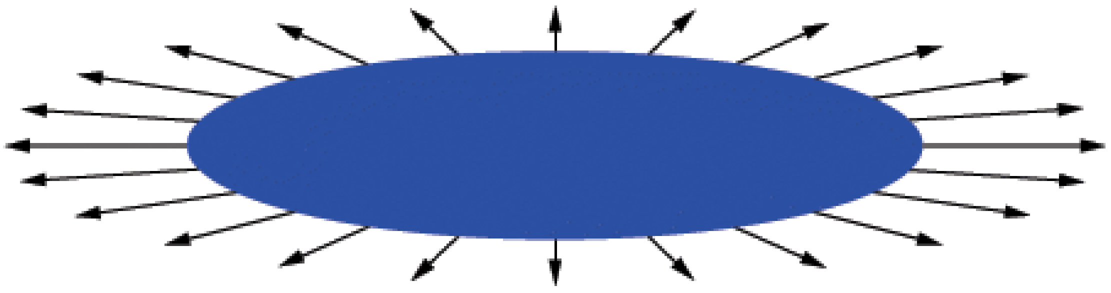
- translation?
- rotation?
- isotropic scale?
- scale?
- reflection?
- shear?
- perspective?



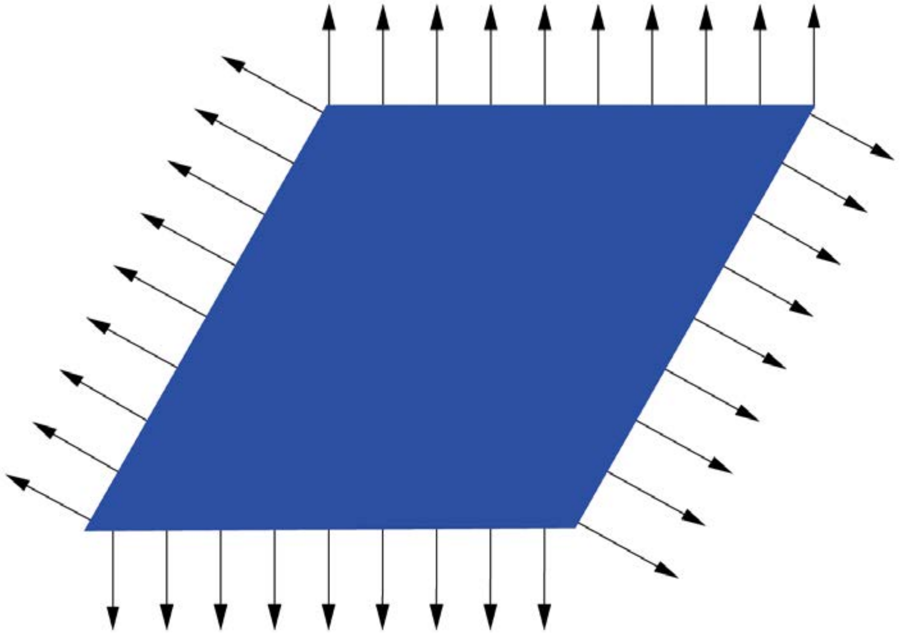
Transformation for shear and scale

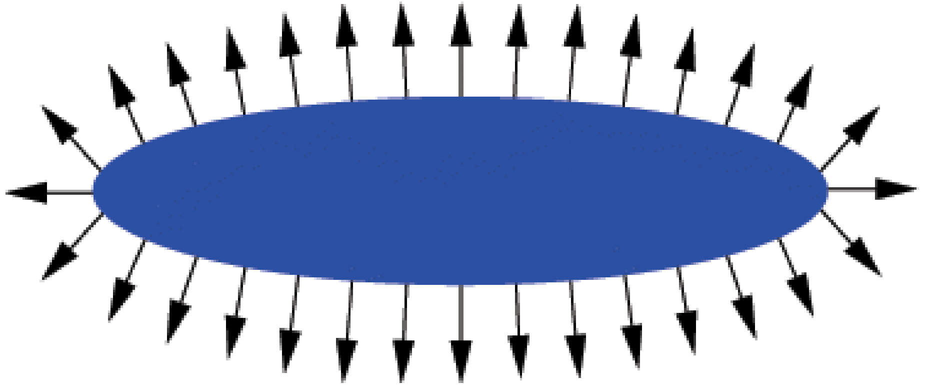
Incorrect
Normal
Transformation



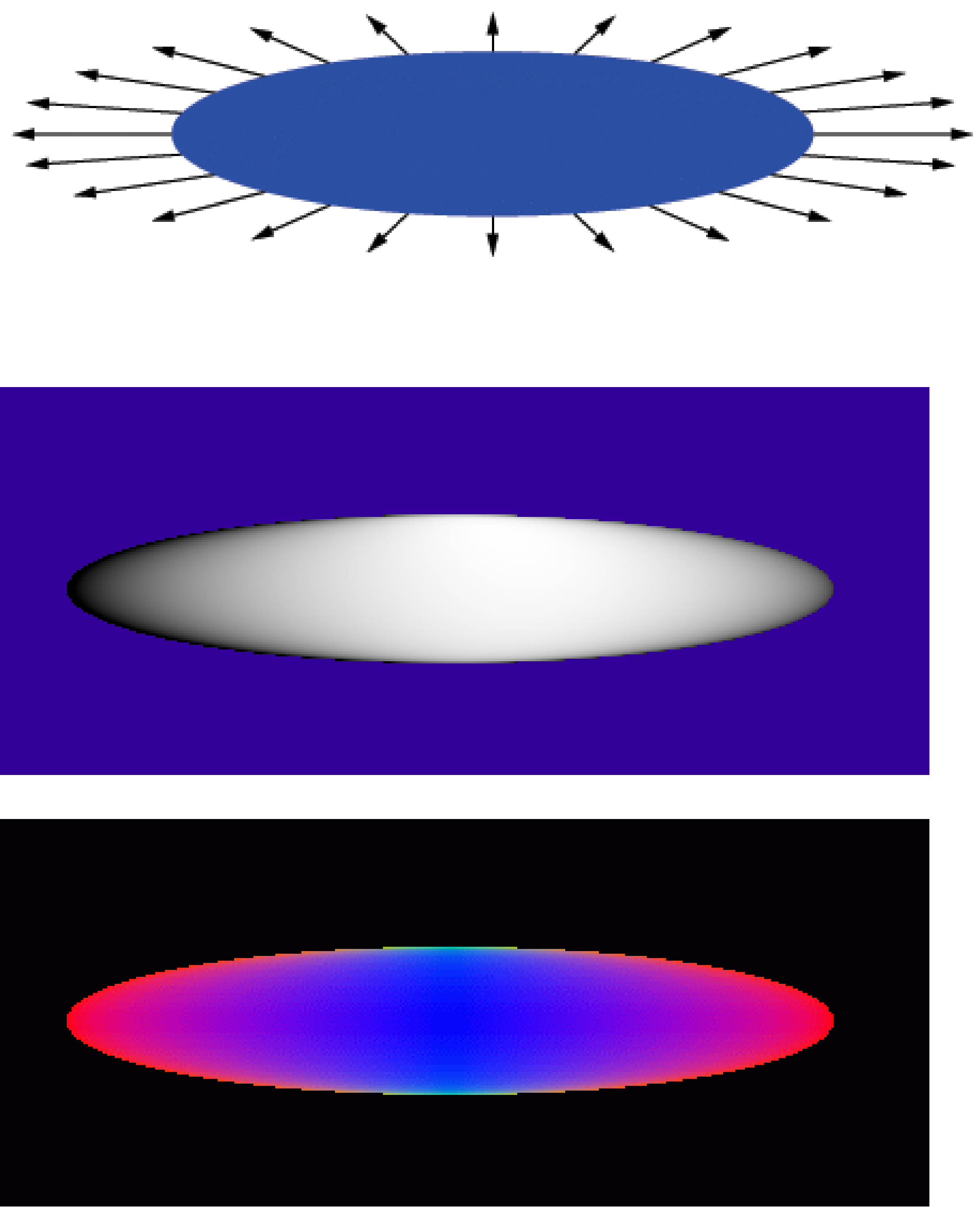


Correct
Normal
Transformation

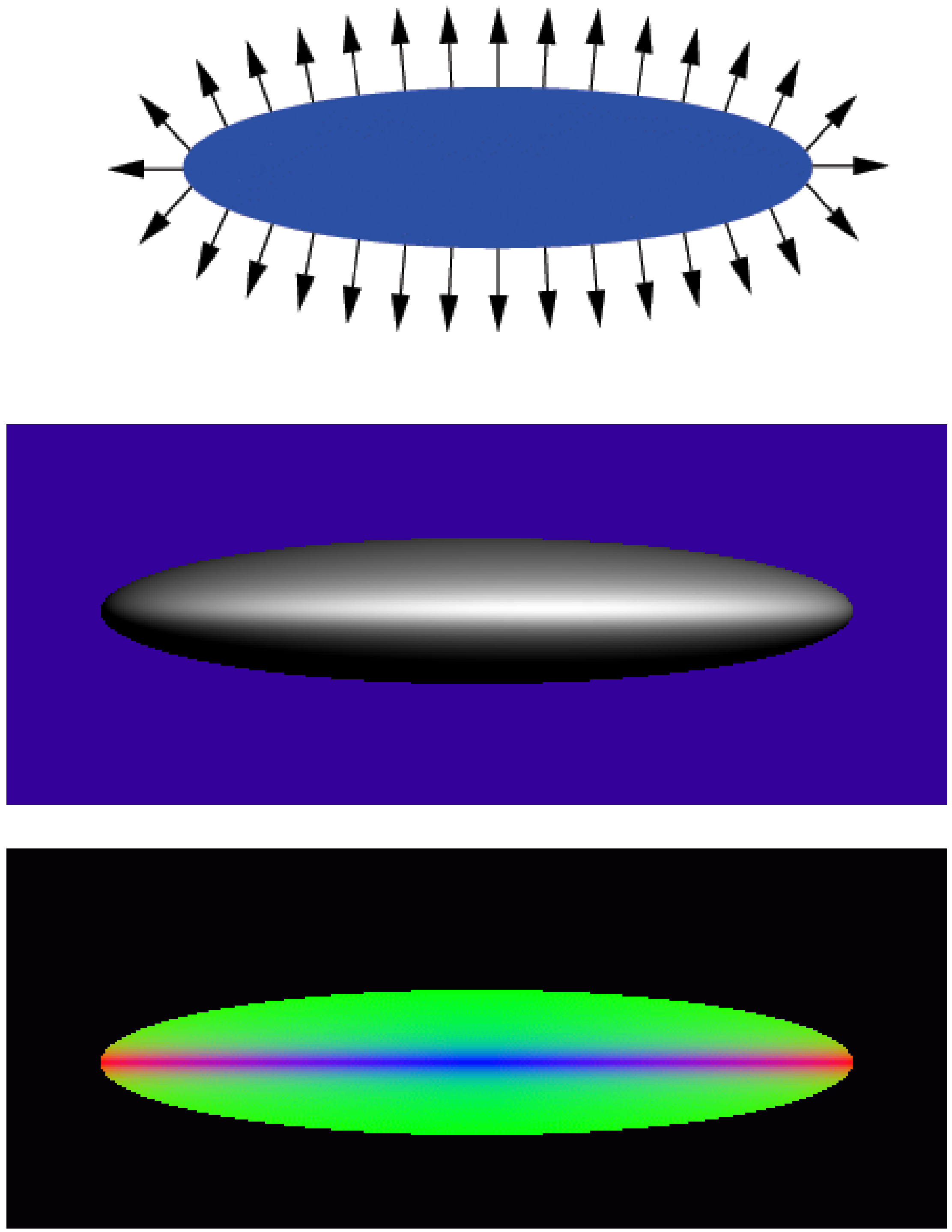




More Normal Visualizations



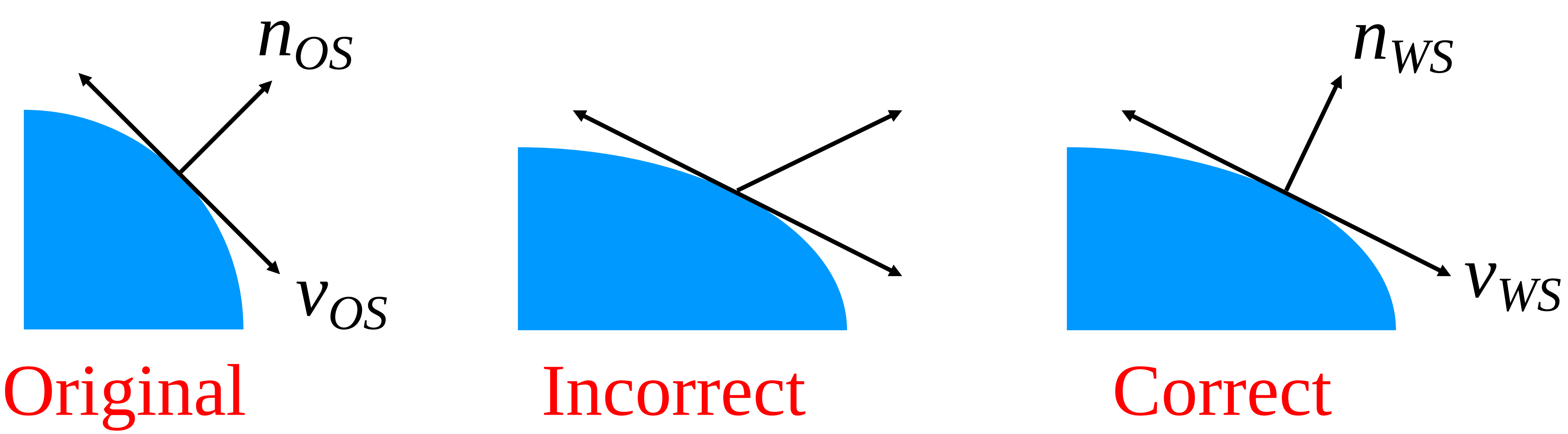
Incorrect Normal Transformation



Correct Normal Transformation

So how do we do it right?

- Think about transforming the *tangent plane* to the normal, not the normal *vector*



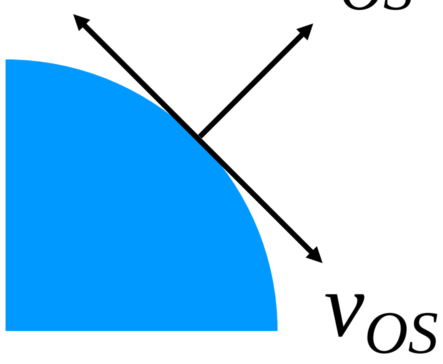
Pick any vector v_{OS} in the tangent plane, how is it transformed by matrix **M**?

$$v_{WS} = \mathbf{M} \; v_{OS}$$

Transform tangent vector *v*

v is perpendicular to normal *n*:

Dot product $$n_{OS}^{\mathbf{T}}\, v_{OS} = 0$$

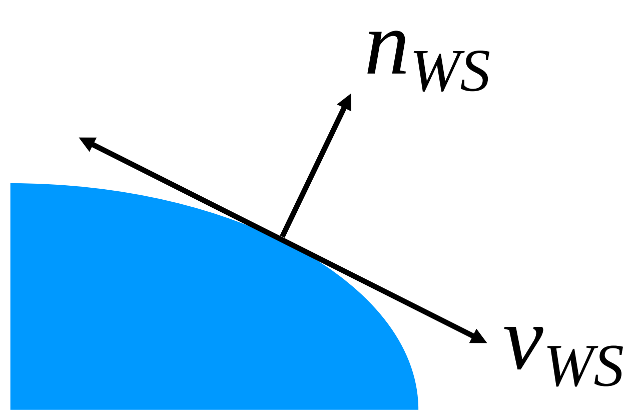


$$n_{OS}^{\mathbf{T}}\ (\mathbf{M}^{-1}\ \mathbf{M})\ v_{OS} = 0$$

$$(n_{OS}^{\mathbf{T}}\ \mathbf{M}^{-1})\ (\mathbf{M}\ \ v_{OS}) = 0$$

$$(n_{OS}^{\mathbf{T}}\ \mathbf{M}^{-1})\ v_{WS} = 0$$

v_{WS} is perpendicular to normal n_{WS}:



$$n_{WS}^{\mathbf{T}} = n_{OS}^{\mathbf{T}}\, (\mathbf{M}^{-1})$$

$$\boxed{n_{WS} = (\mathbf{M}^{-1})^{\mathbf{T}}\, n_{OS}}$$

$$n_{WS}^{\mathbf{T}}\, v_{WS} = 0$$

Connections

- Not part of class, but cool
  - “Covariant”: transformed by the matrix
    - e.g., tangent
  - “Contravariant”: transformed by the inverse transpose
    - e.g., the normal
    - a normal is a “co-vector”

- Google “differential geometry” to find out more

- Further Reading
  - Buss, Chapter 2

- Other Cool Stuff
  - Algebraic Groups
  - http://phototour.cs.washington.edu/
  - http://phototour.cs.washington.edu/findingpaths/
  - Free-form deformation of solid objects
  - Harmonic coordinates for character articulation

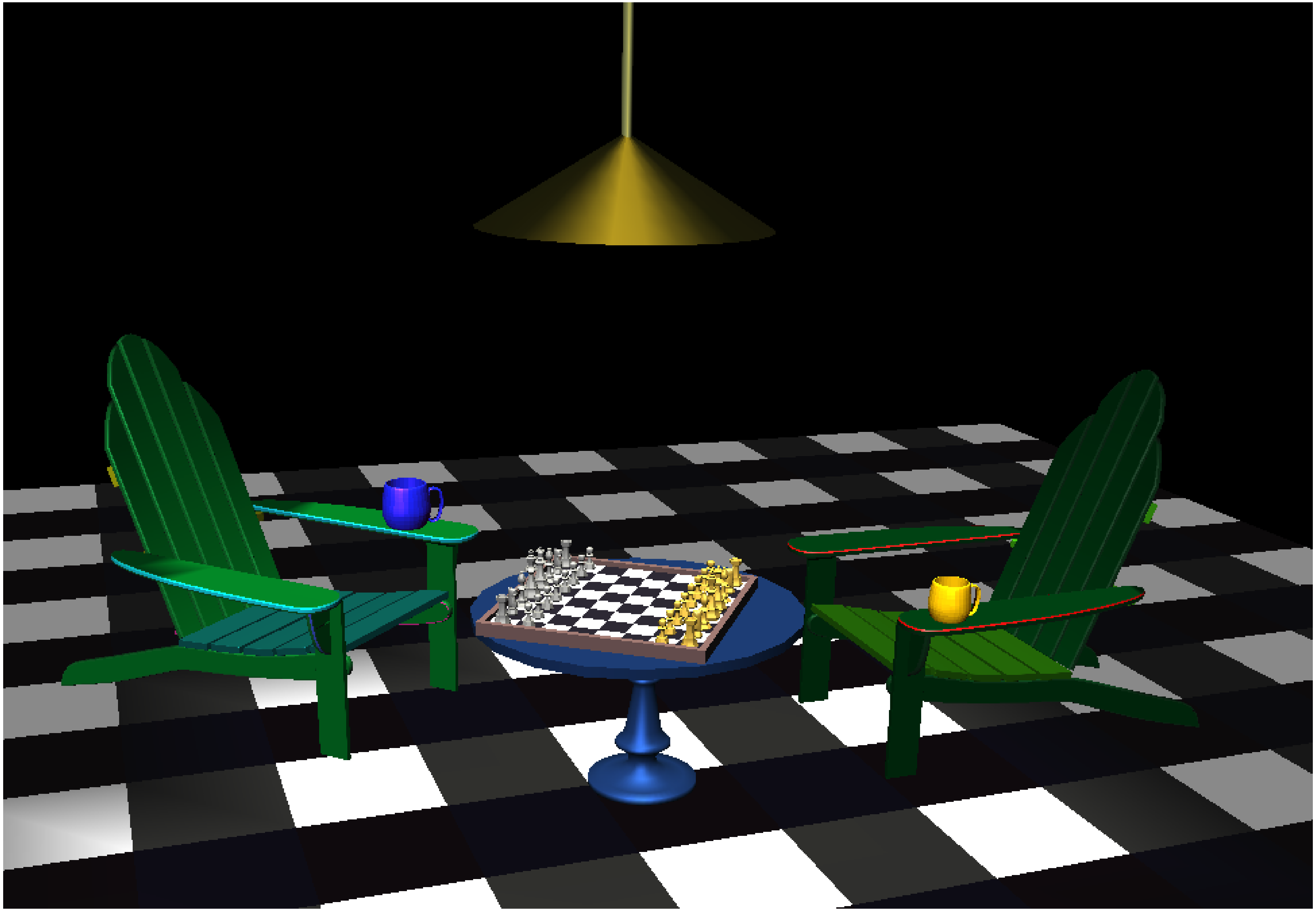
Question?

Hierarchical Modeling

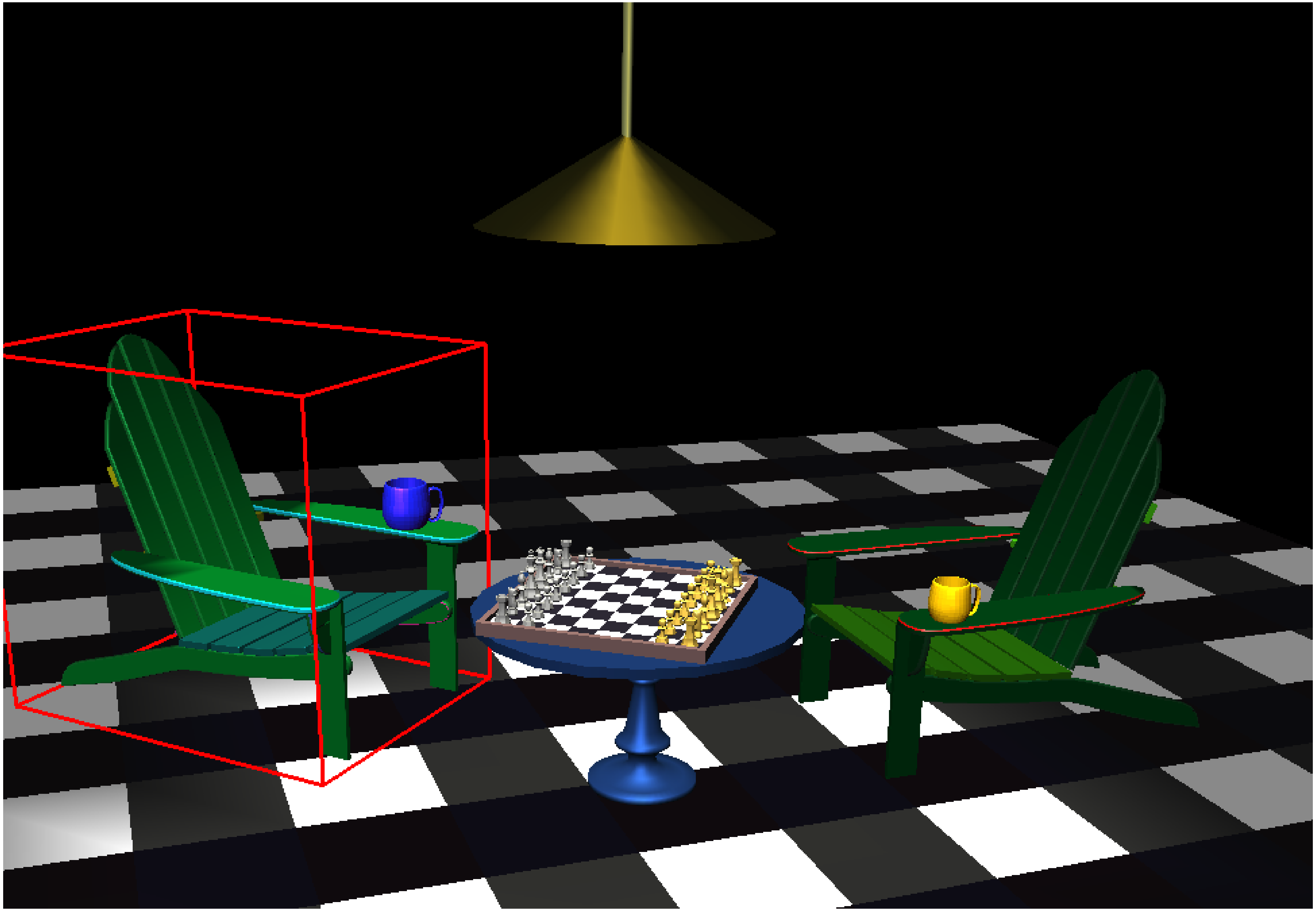
- Triangles, parametric curves and surfaces are the building blocks from which more complex real-world objects are modeled.

- Hierarchical modeling creates complex real-world objects by combining simple primitive shapes into more complex aggregate objects.

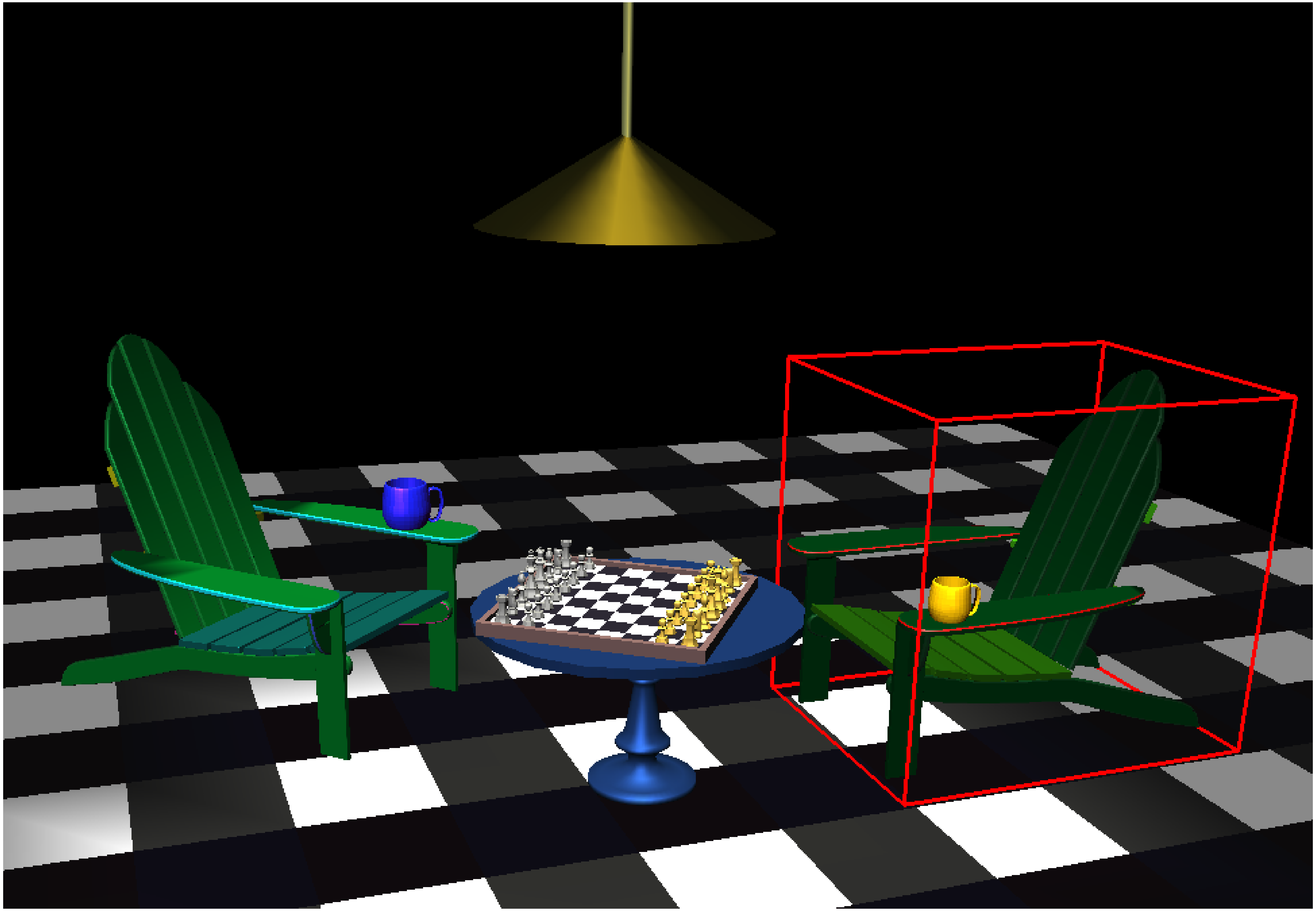
Hierarchical models



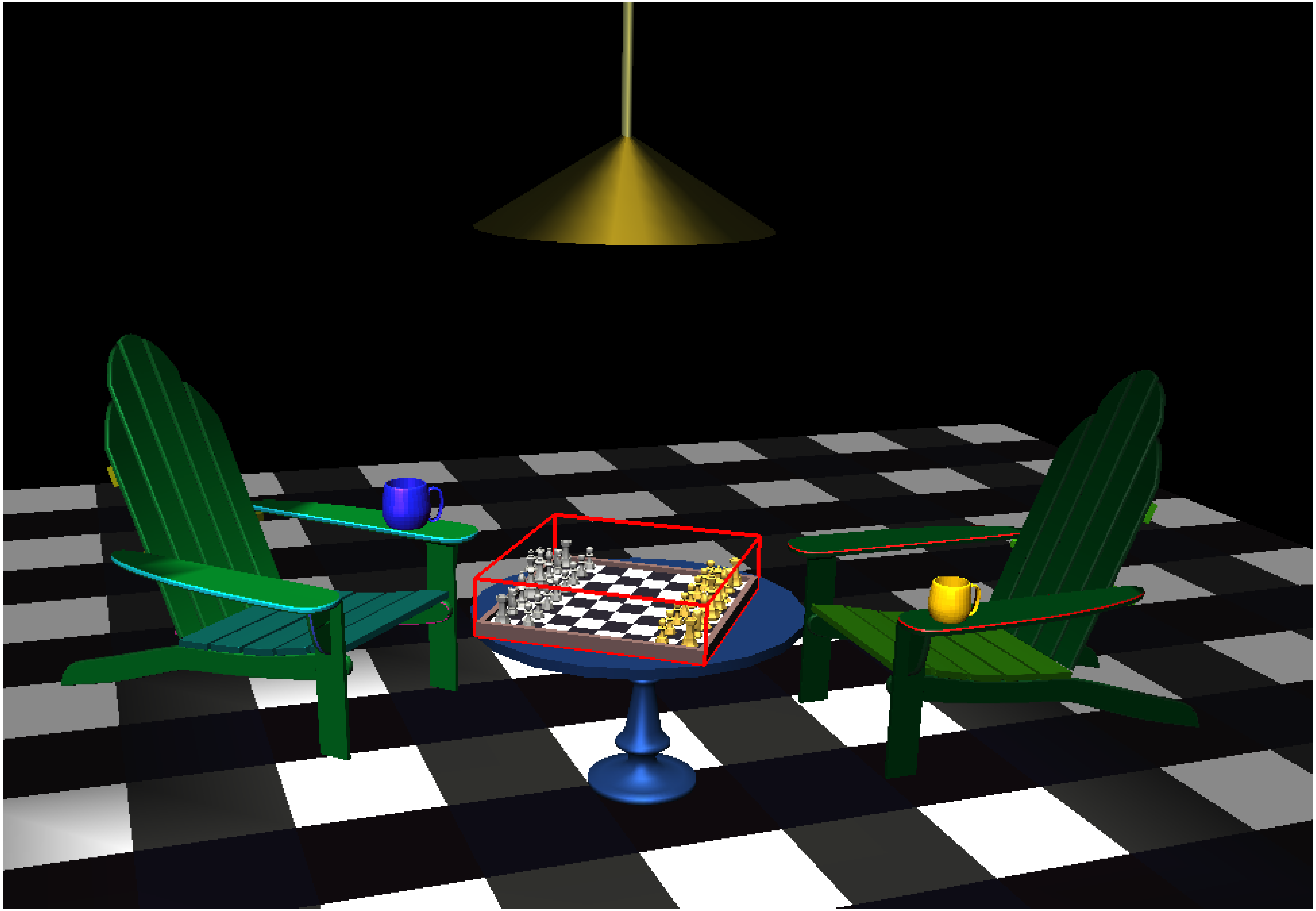
Hierarchical models



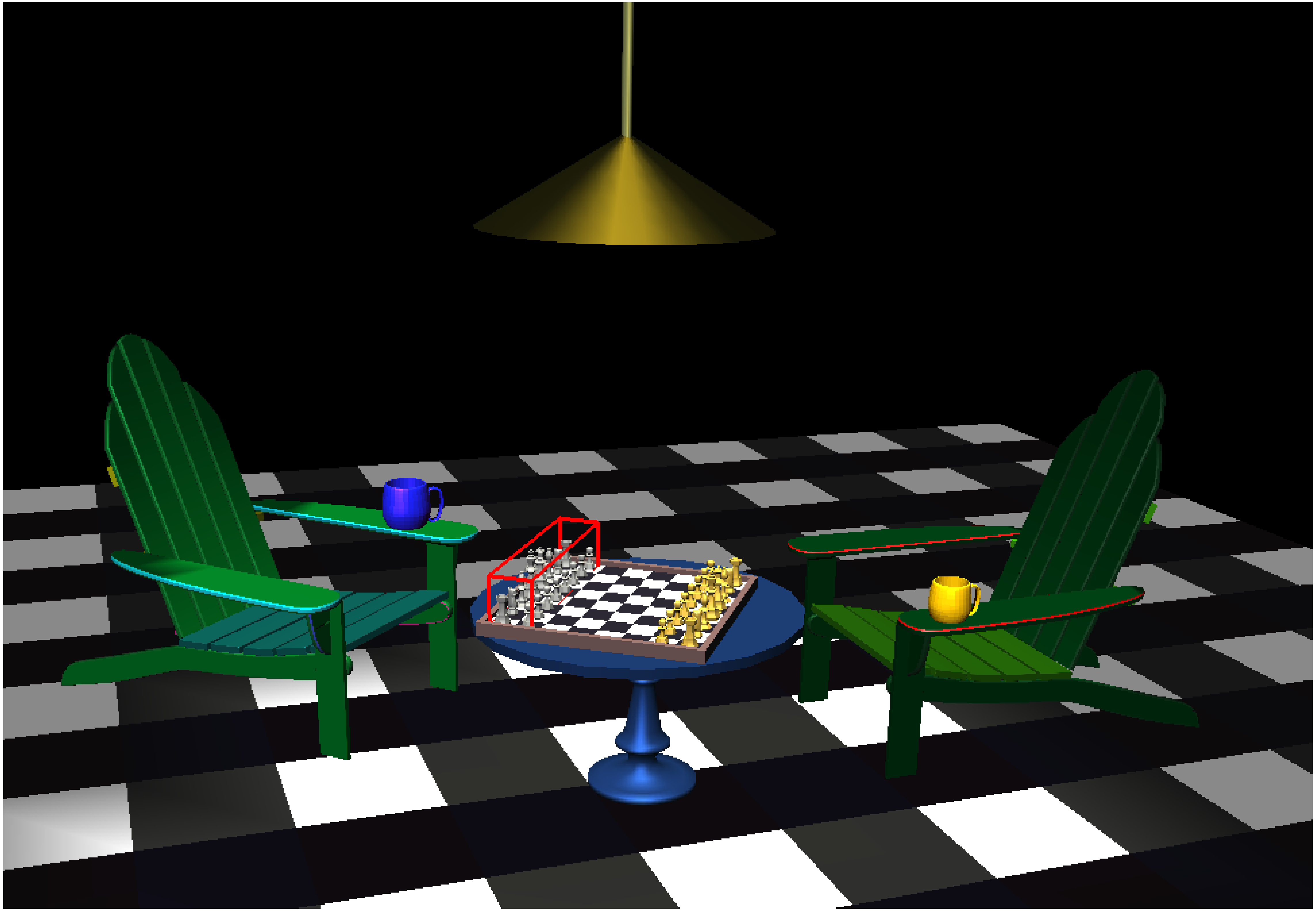
Hierarchical models



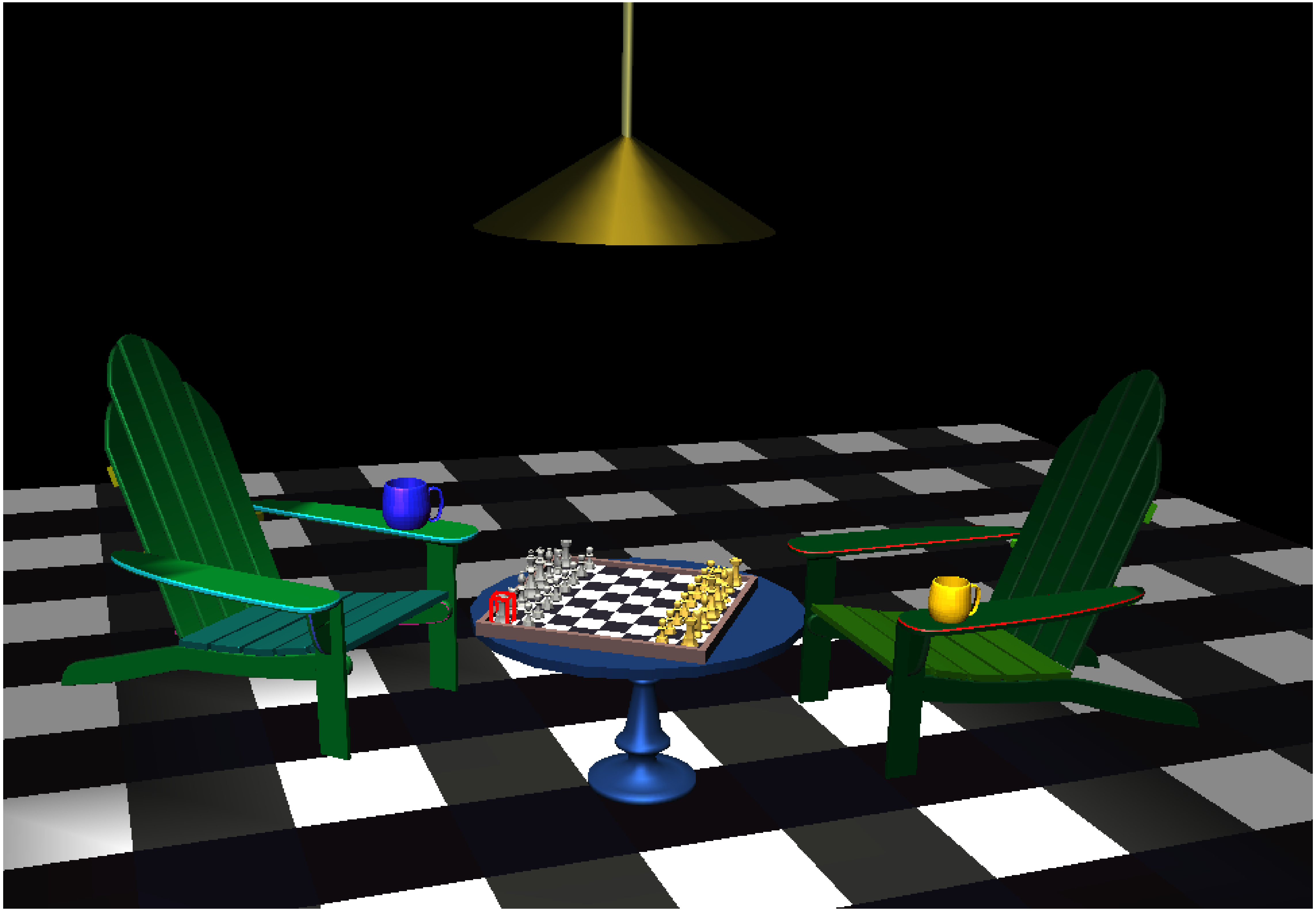
Hierarchical models



Hierarchical models

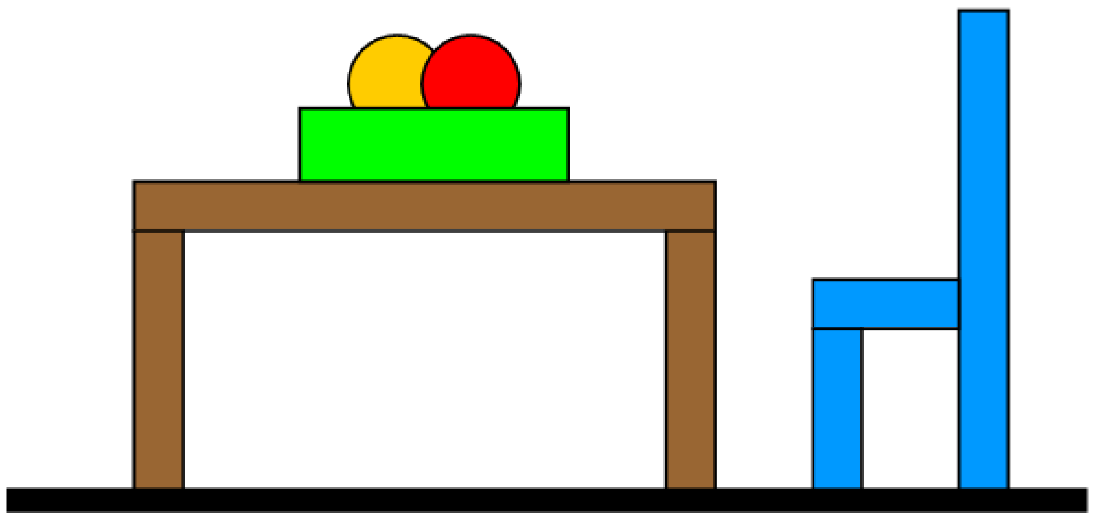


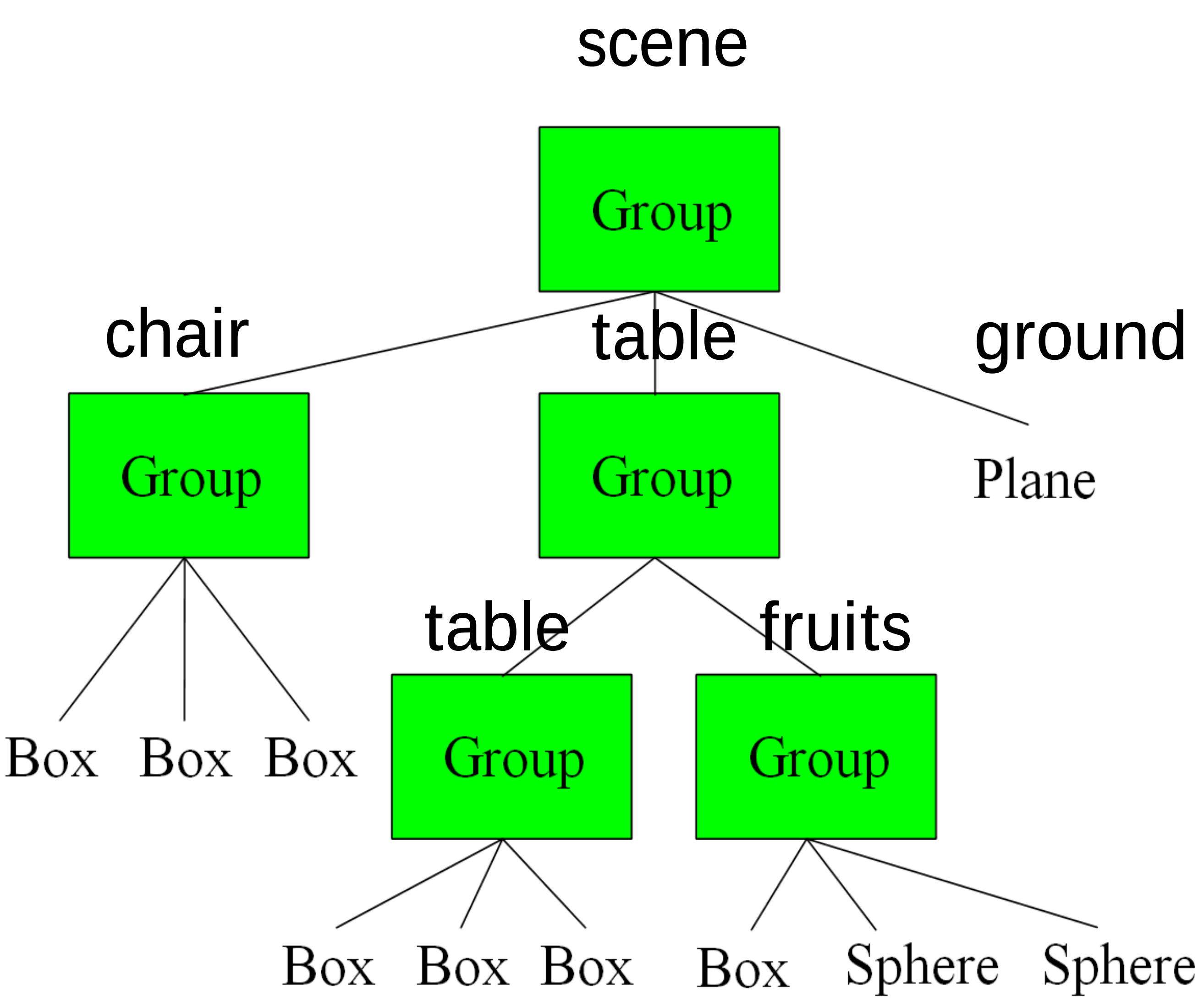
Hierarchical models



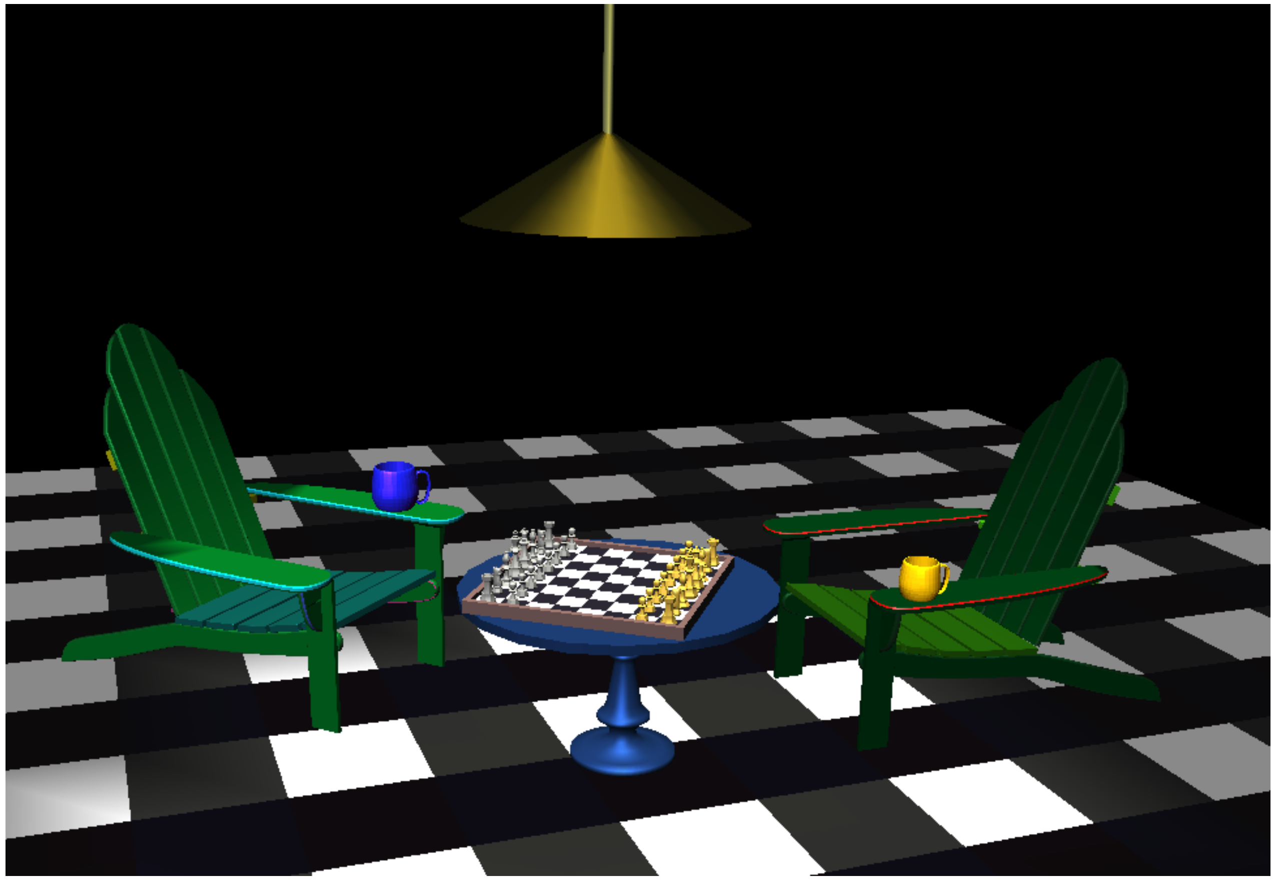
Hierarchical Grouping of Objects

- The "scene graph" represents the logical organization of scene

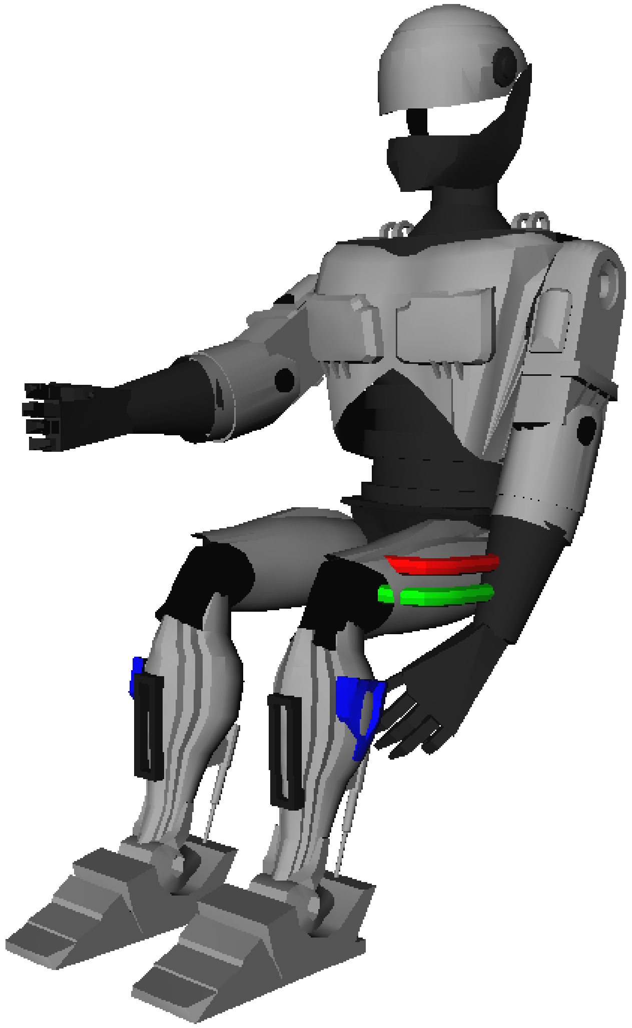




Scene Graph

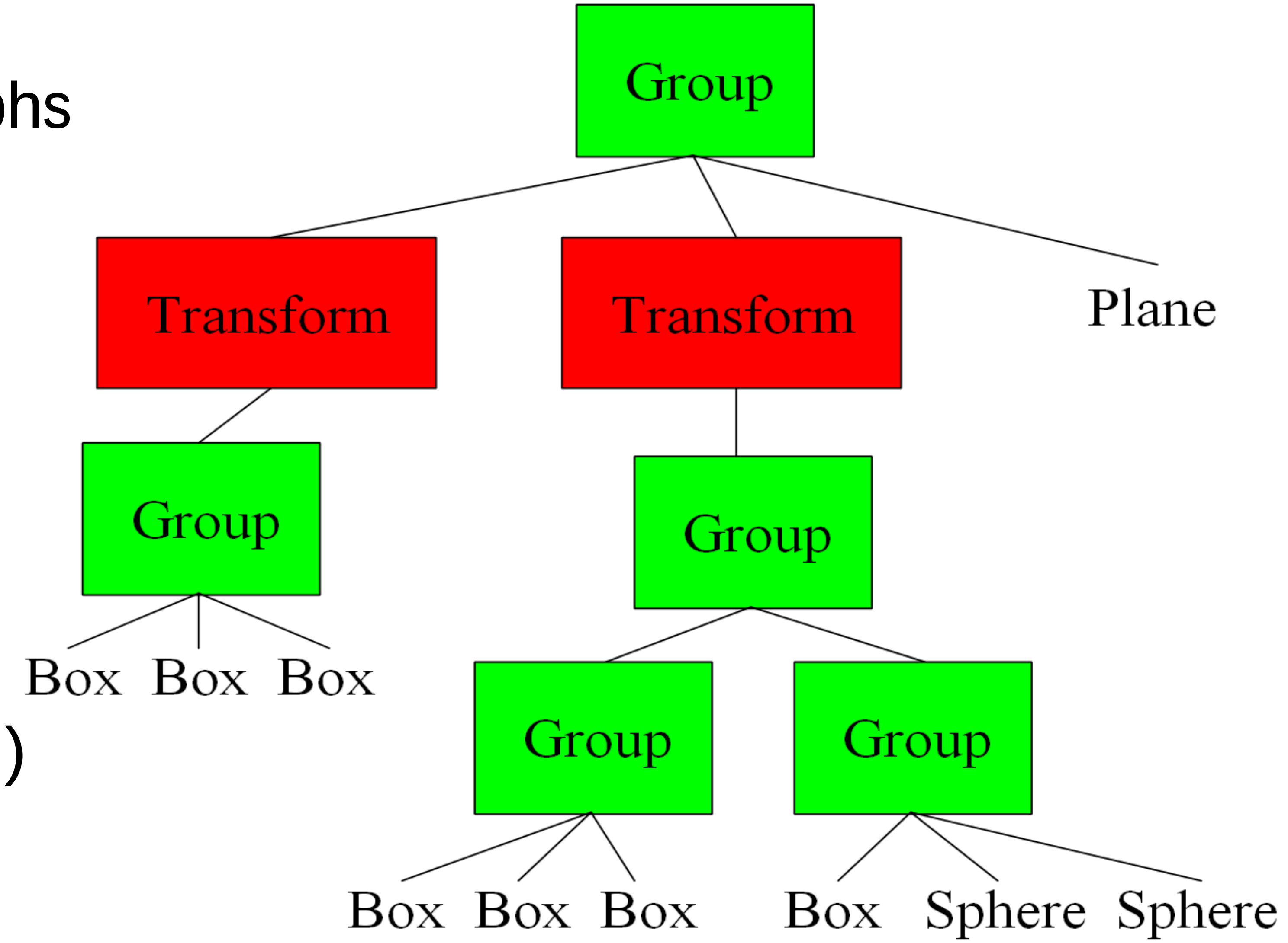


- Convenient Data structure for scene representation
  - Geometry (meshes, etc.)
  - Transformations
  - Materials, color
  - Multiple instances
- Basic idea: Hierarchical Tree
- Useful for manipulation/animation
  - Also for articulated figures
- Useful for rendering, too
  - Ray tracing acceleration, occlusion culling
  - But note that two things that are close to each other in the tree are NOT necessarily spatially near each other



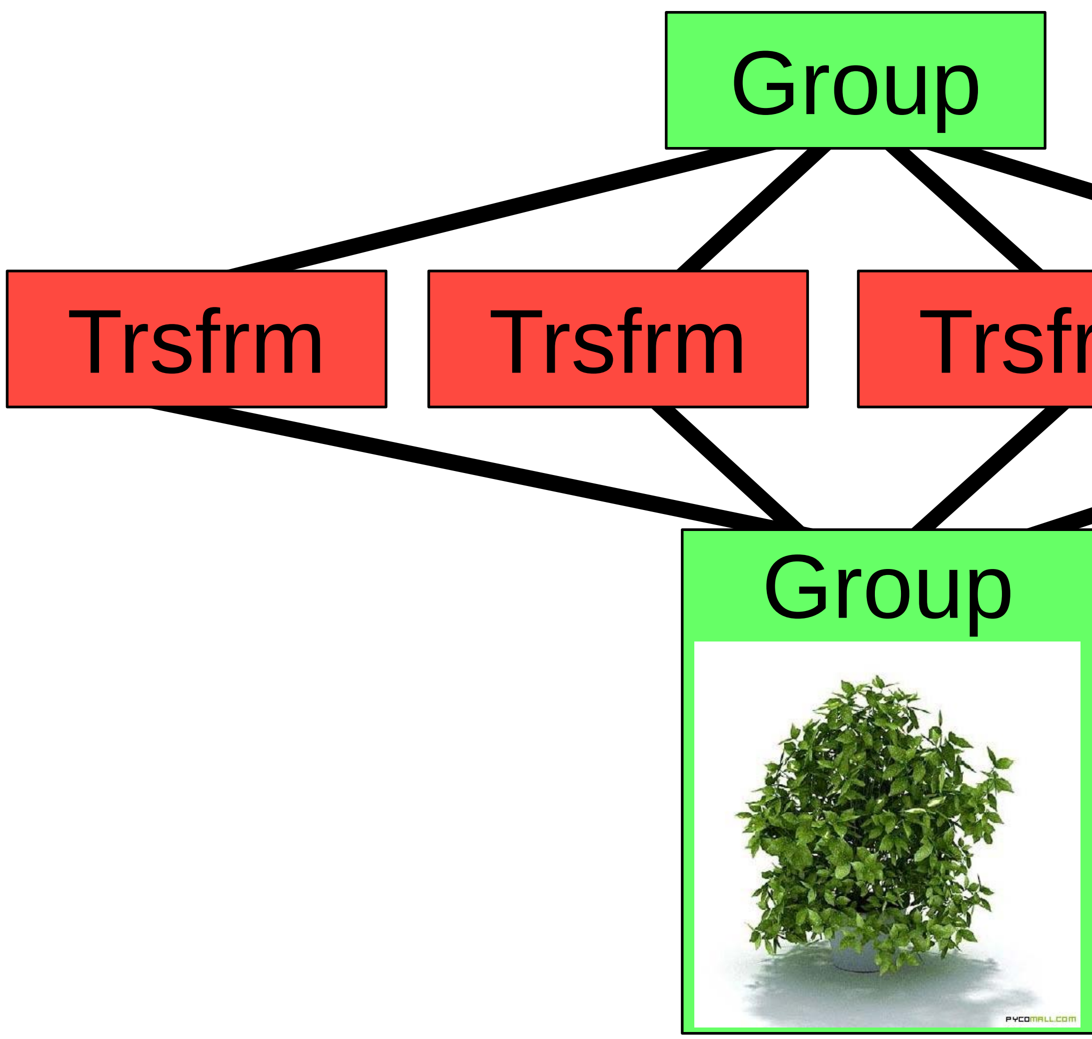
Scene Graph Representation

- Basic idea: Tree
- Comprised of several node types
  - Shape: 3D geometric objects
  - Transform: Affect current transformation
  - Property: Color, texture
  - Group: Collection of subgraphs
- C++ implementation
  - base class Object
    - children, parent
  - derived classes for each node type (group, transform)



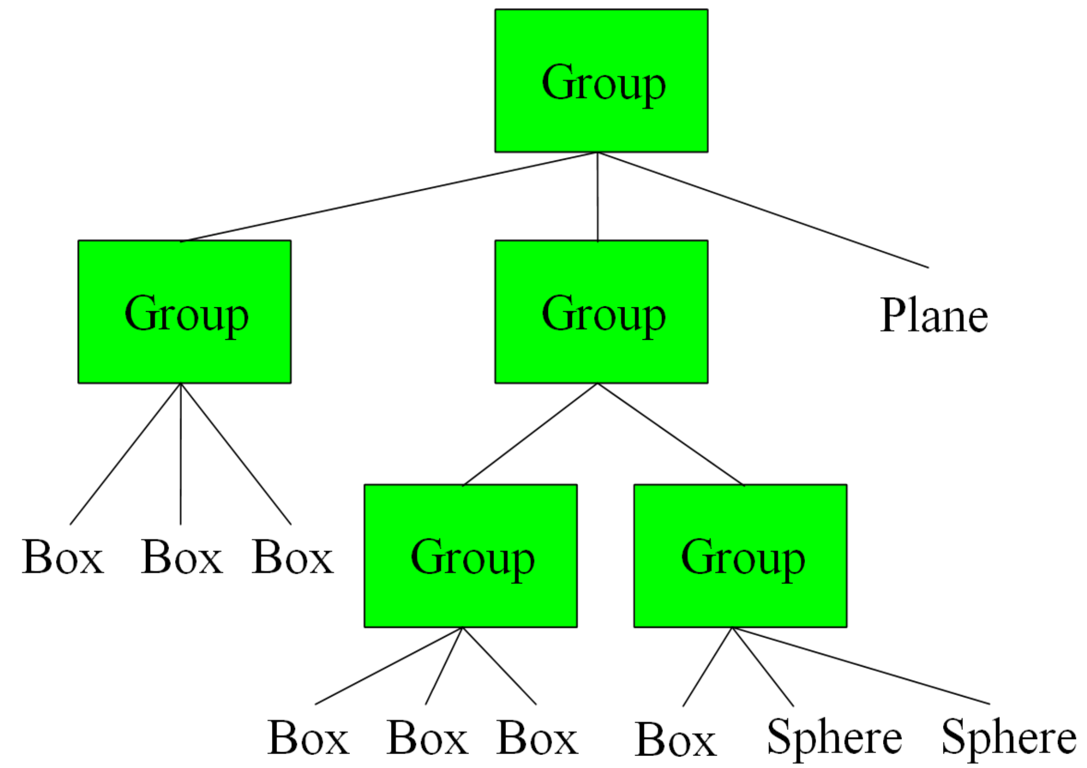
Scene Graph Representation

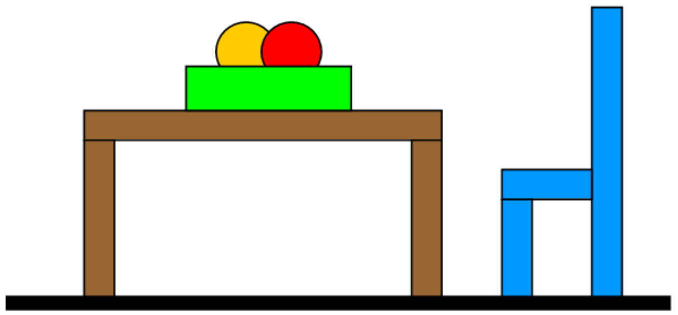
- In fact, generalization of a tree: Directed Acyclic Graph (DAG)
  - Means a node can have multiple parents, but cycles are not allowed
- Why? Allows multiple instantiations
  - Reuse complex hierarchies many times in the scene using different transformations (example: a tree)
    - Of course, if you only want to reuse meshes, just load the mesh once and make several geometry nodes point to the same data



Simple Example with Groups

```
Group {
    numObjects 3
    Group {
        numObjects 3
        Box { <BOX PARAMS> }
        Box { <BOX PARAMS> }
        Box { <BOX PARAMS> } }
    Group {
        numObjects 2
        Group {
            Box { <BOX PARAMS> }
            Box { <BOX PARAMS> }
            Box { <BOX PARAMS> } }
        Group {
            Box { <BOX PARAMS> }
            Sphere { <SPHERE PARAMS> }
            Sphere { <SPHERE PARAMS> } } }
    Plane { <PLANE PARAMS> } }
```

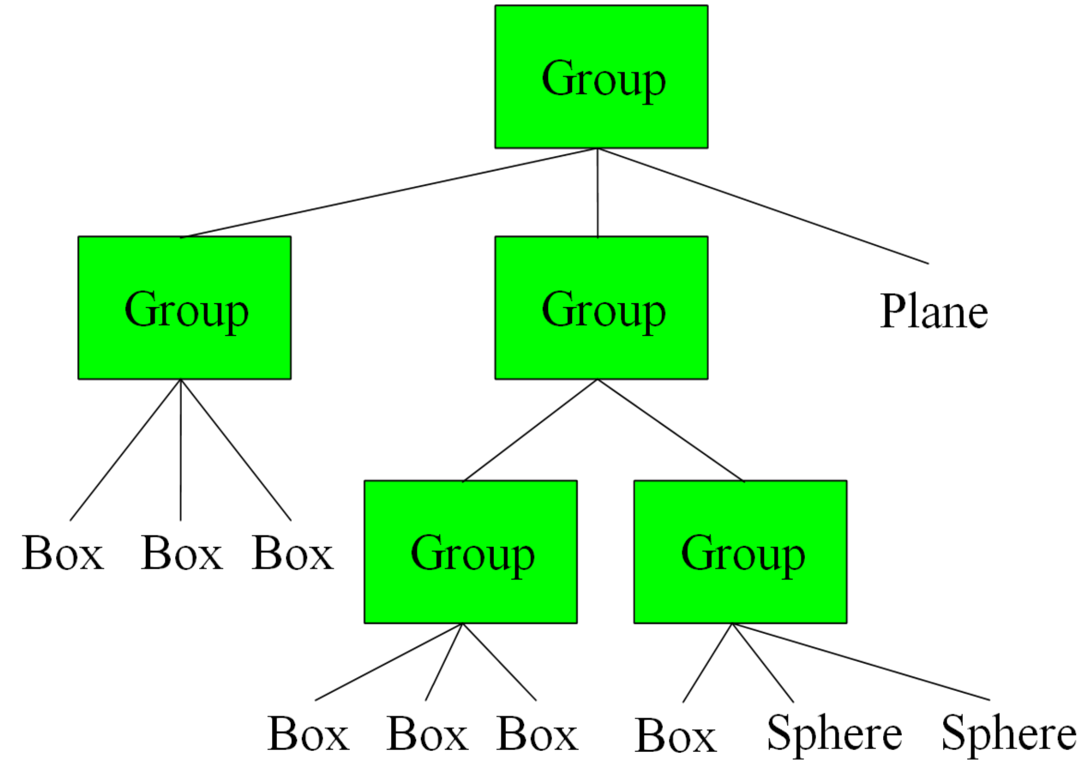


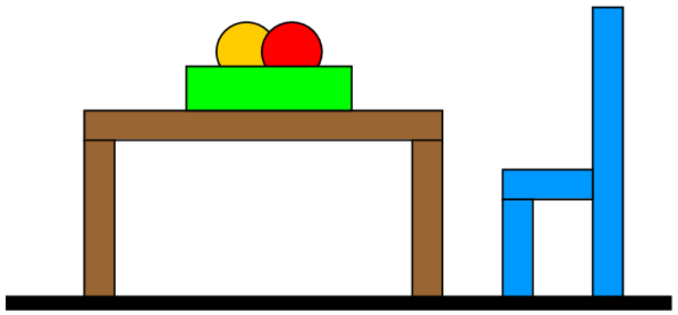


Text format is fictitious, better to use XML in real applications

Simple Example with Groups

```
Group {
    numObjects 3
    Group {
        numObjects 3
        Box { <BOX PARAMS> }
        Box { <BOX PARAMS> }
        Box { <BOX PARAMS> } }
    Group {
        numObjects 2
        Group {
            Box { <BOX PARAMS> }
            Box { <BOX PARAMS> }
            Box { <BOX PARAMS> } }
        Group {
            Box { <BOX PARAMS> }
            Sphere { <SPHERE PARAMS> }
            Sphere { <SPHERE PARAMS> } } }
    Plane { <PLANE PARAMS> } }
```



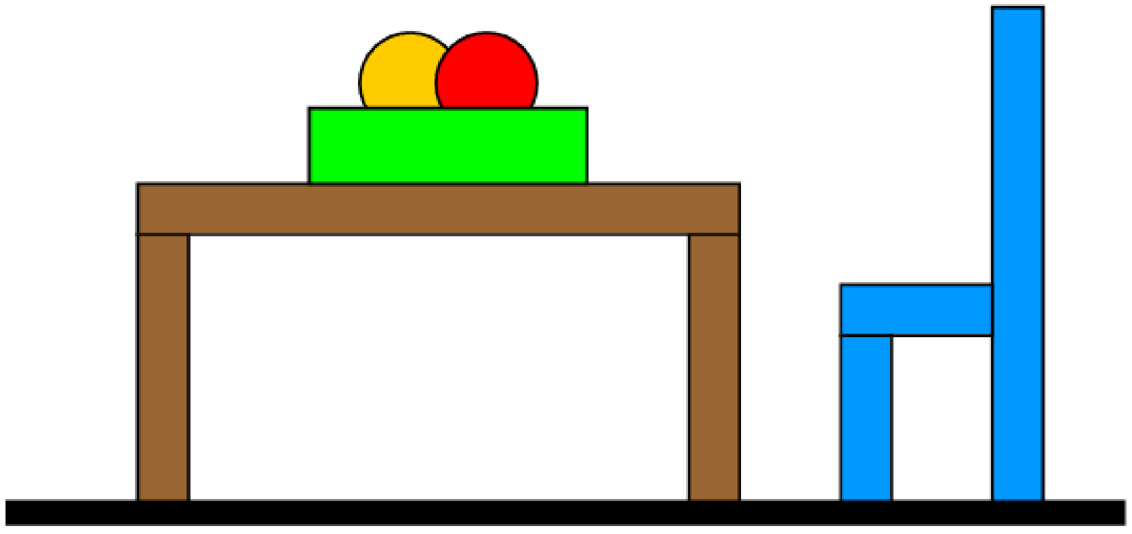


Here we have only simple shapes, but easy to add a “Mesh” node whose parameters specify an .OBJ to load (say)

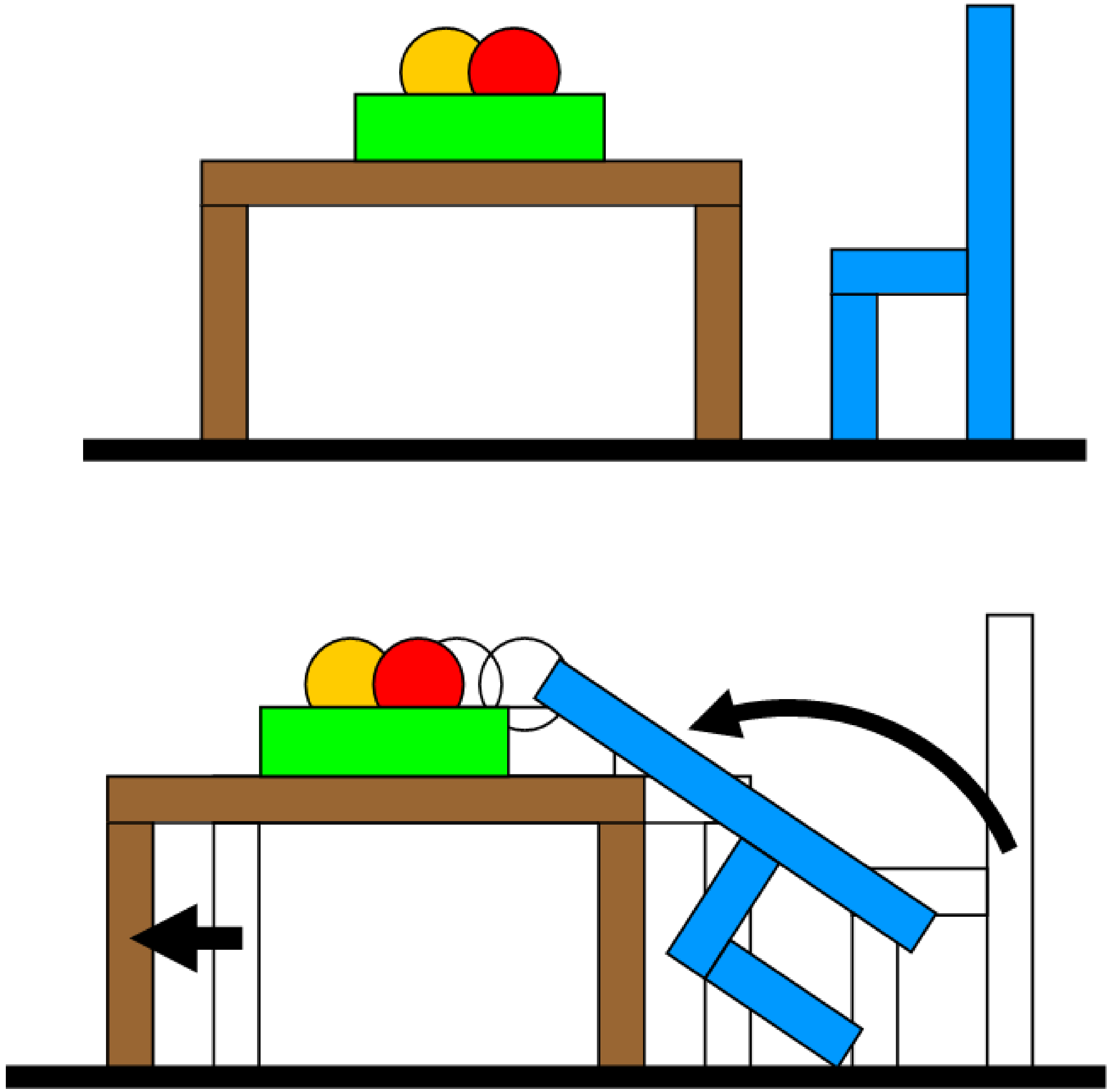
Adding Attributes (Material, etc.)

```
Group {
    numObjects 3
    Material { <BLUE> }
    Group {
        numObjects 3
        Box { <BOX PARAMS> }
        Box { <BOX PARAMS> }
        Box { <BOX PARAMS> } }
    Group {
        numObjects 2
        Material { <BROWN> }
        Group {
            Box { <BOX PARAMS> }
            Box { <BOX PARAMS> }
            Box { <BOX PARAMS> } }
        Group {
            Material { <GREEN> }
            Box { <BOX PARAMS> }
            Material { <RED> }
            Sphere { <SPHERE PARAMS> }
            Material { <ORANGE> }
            Sphere { <SPHERE PARAMS> } } }

    Plane { <PLANE PARAMS> } }
```



Adding Transformations

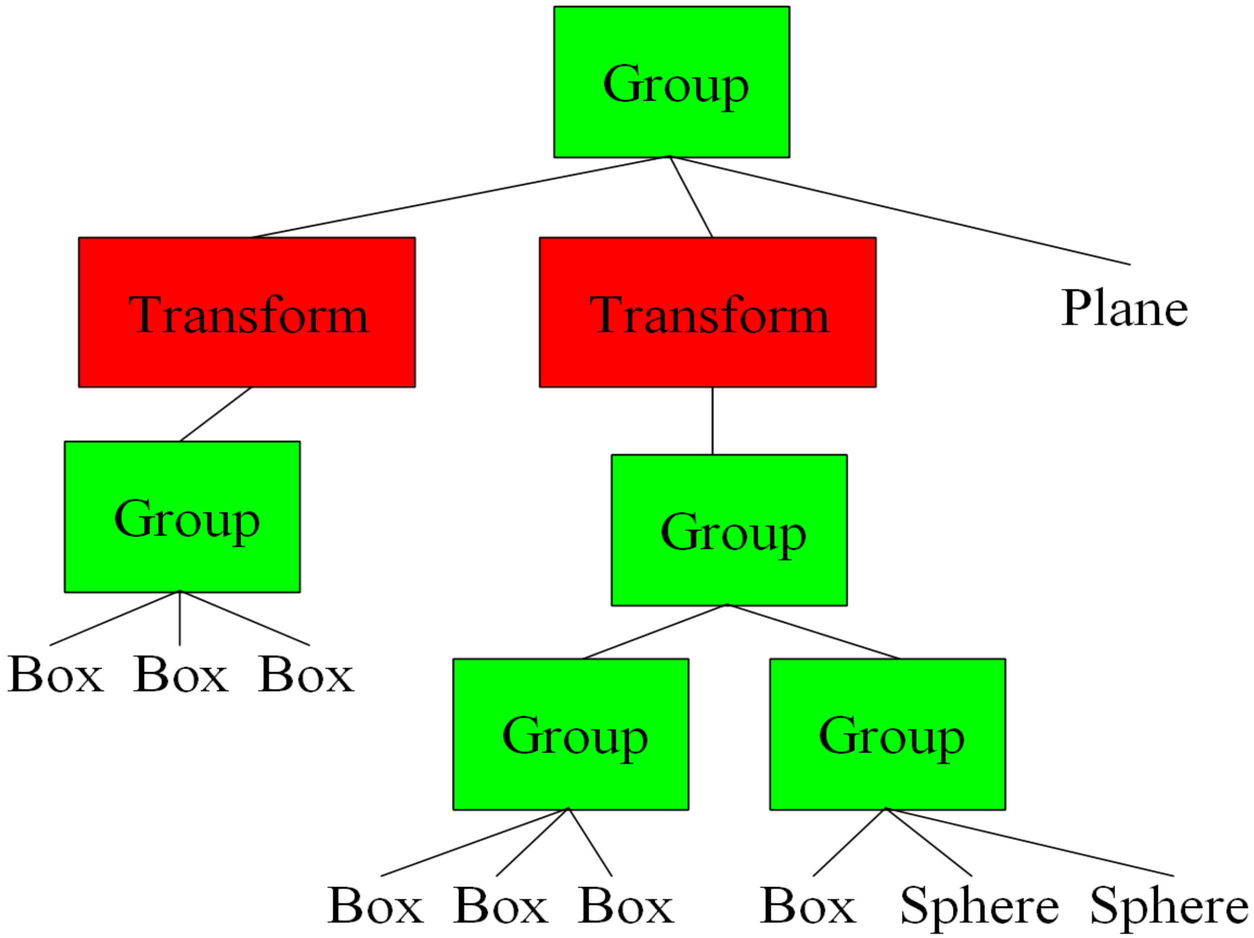


Questions?

Scene Graph Traversal

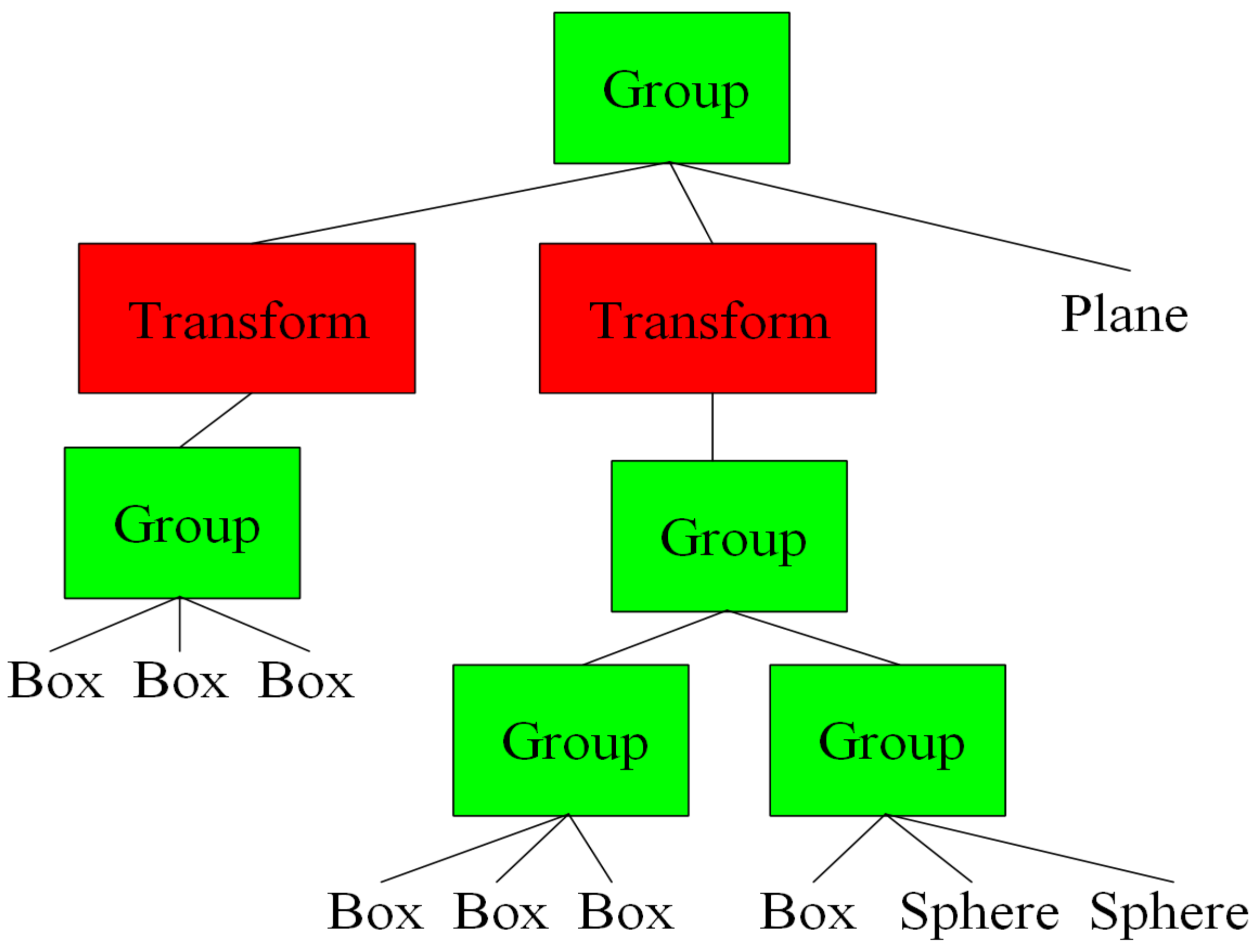
- Depth first recursion
  - Visit node, then visit subtrees (top to bottom, left to right)
  - When visiting a geometry node: Draw it!

- How to handle transformations?
  - Remember, transformations are always specified in coordinate system of the parent



Scene Graph Traversal

- How to handle transformations?
  - Traversal algorithm keeps a **transformation state S** (a 4x4 matrix)
    - from world coordinates
    - Initialized to identity in the beginning
  - Geometry nodes always drawn using current **S**
  - When visiting a transformation node **T**: multiply current state **S** with **T**, then visit child nodes
    - Has the effect that nodes below will have new transformation
  - When all children have been visited, **undo the effect of T!**



Recall frames

- An object frame has coordinates O in the world (of course O is also our 4x4 matrix)

$$\vec{\mathbf{o}}^t = \vec{\mathbf{w}}^t O$$

- Then we are given coordinates c in the object frame

$$\vec{\mathbf{o}}^t \mathbf{c} = \vec{\mathbf{w}}^t O \mathbf{c}$$

- Indeed we need to apply matrix O to all objects

Frames and hierarchy

- Matrix M_1 to go from world to torso $\vec{\mathbf{t}}^t = \vec{\mathbf{w}}^t M_1$
- Matrix M_2 to go from torso to arm $\vec{\mathbf{a}}^t = \vec{\mathbf{t}}^t M_2$

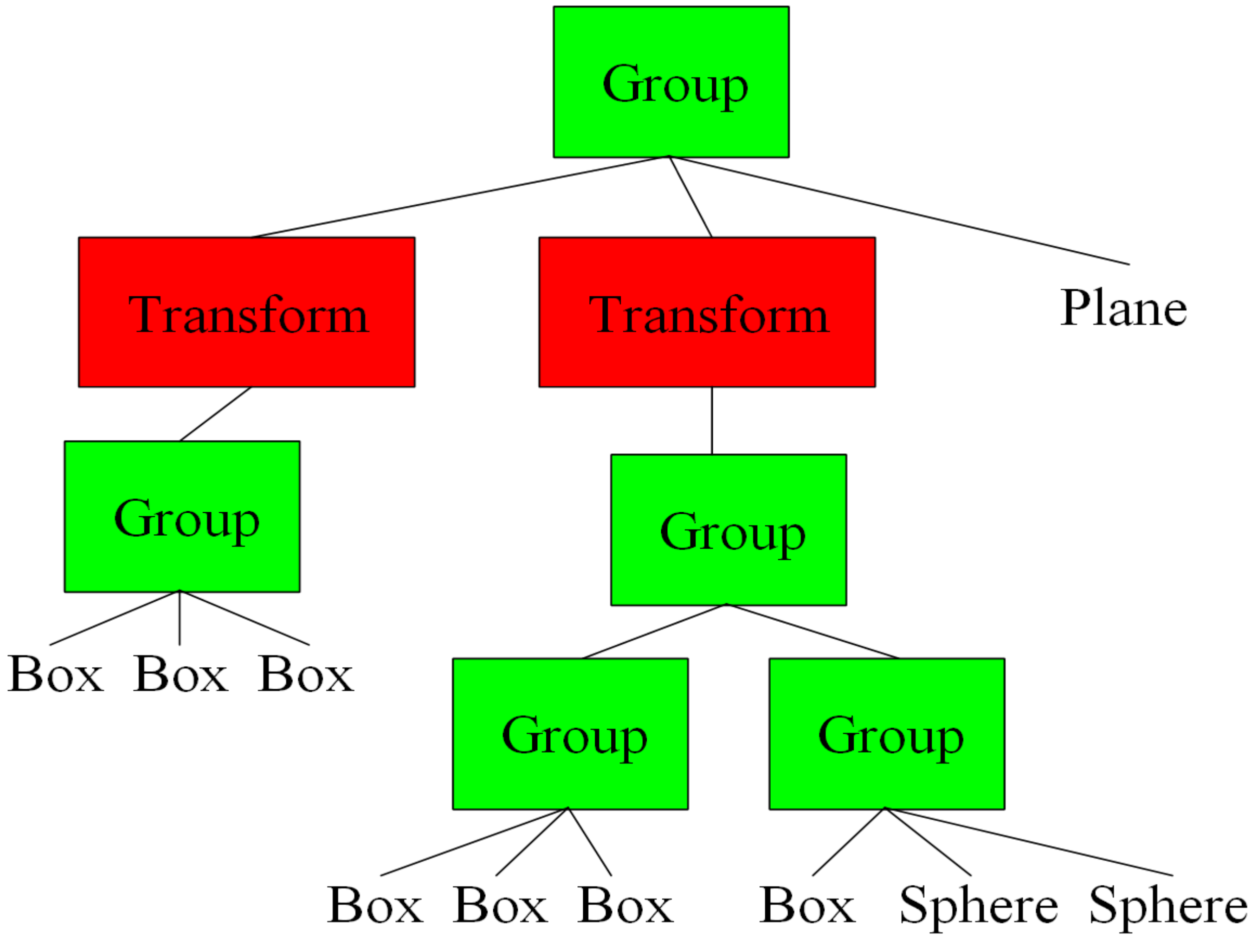
- How do you go from arm coordinates to world?

$$\vec{\mathbf{a}}^t \mathbf{c} = \vec{\mathbf{t}}^t M_2 \mathbf{c} = \vec{\mathbf{w}}^t M_1 M_2 \mathbf{c}$$

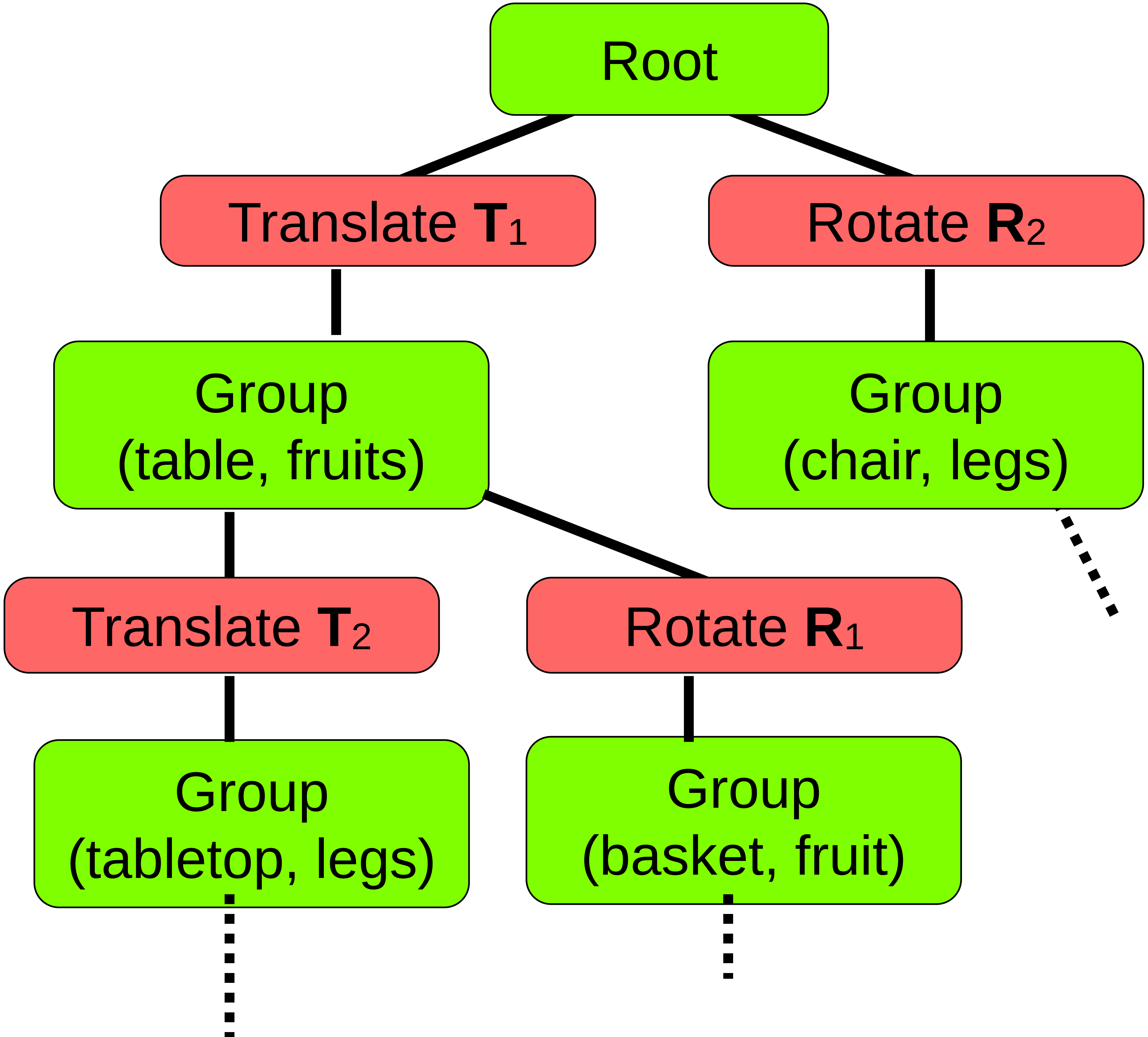
- We can concatenate the matrices
- Matrices for the lower hierarchy nodes go to the right

Recap: Scene Graph Traversal

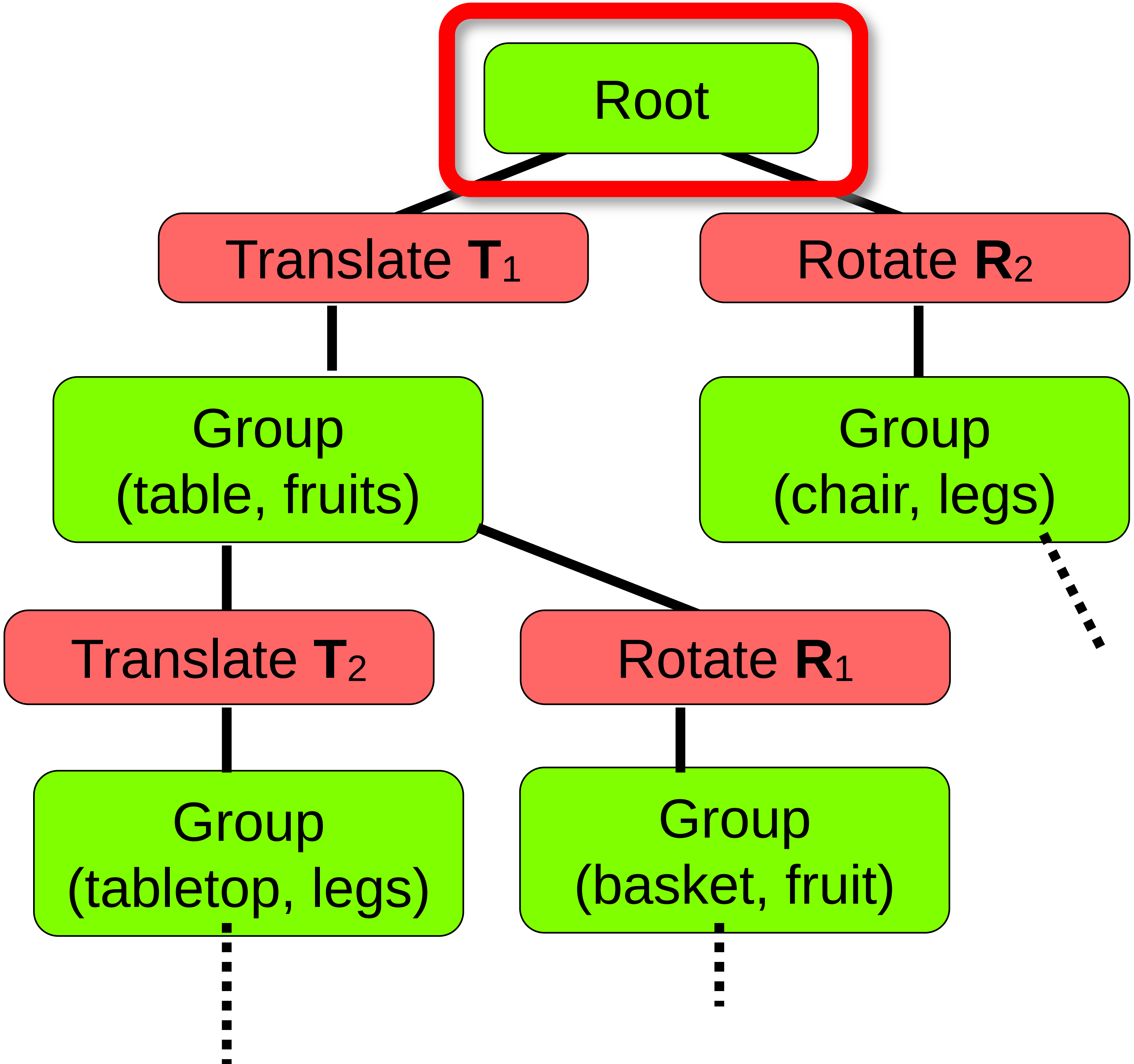
- How to handle transformations?
  - Traversal algorithm keeps a **transformation state S** (a 4x4 matrix)
    - from world coordinates
    - Initialized to identity in the beginning
  - Geometry nodes always drawn using current **S**
  - When visiting a transformation node **T**: multiply current state **S** with **T**, then visit child nodes
    - Has the effect that nodes below will have new transformation
  - When all children have been visited, **undo the effect of T!**



Traversal Example

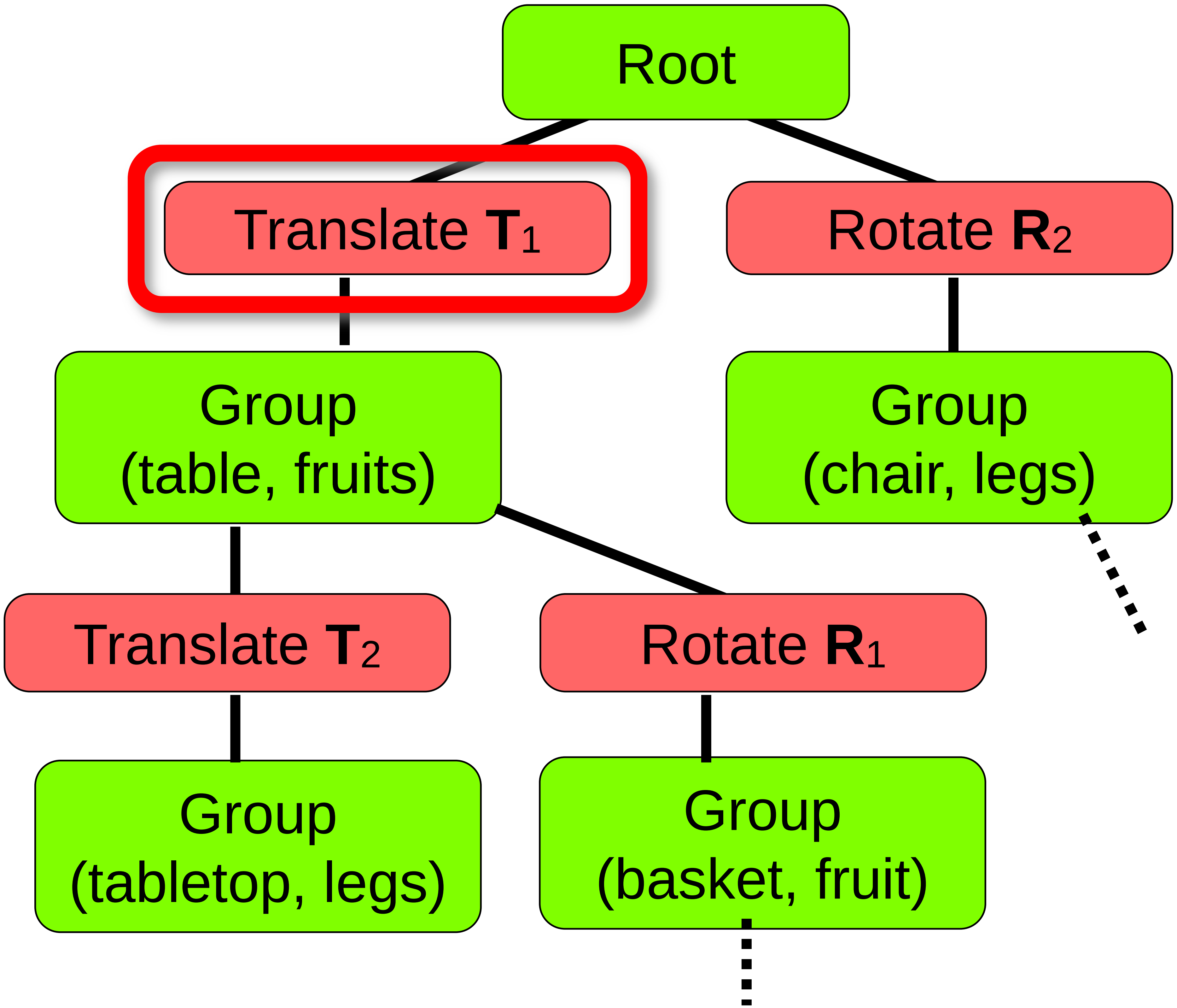


Traversal Example



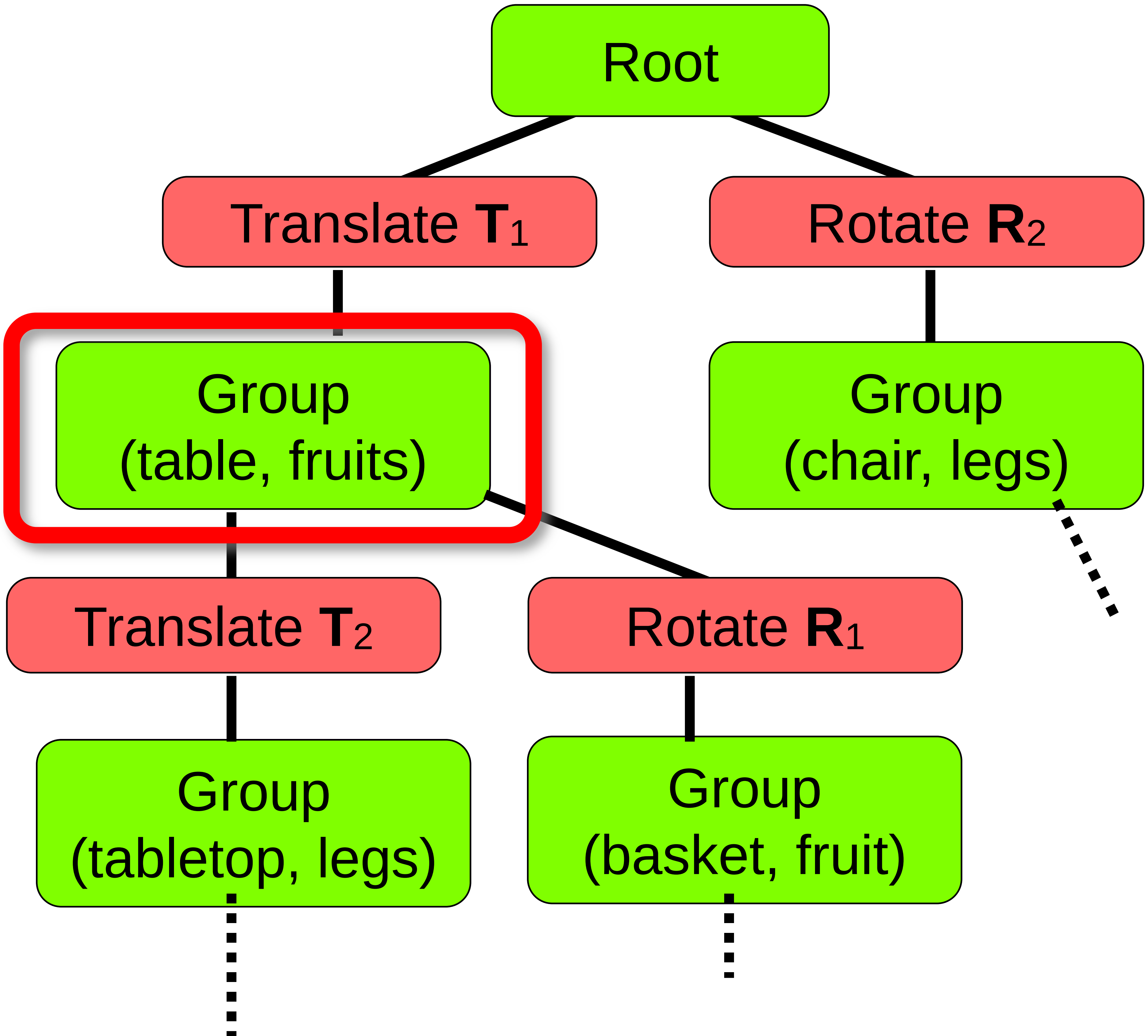
S = **I**

Traversal Example



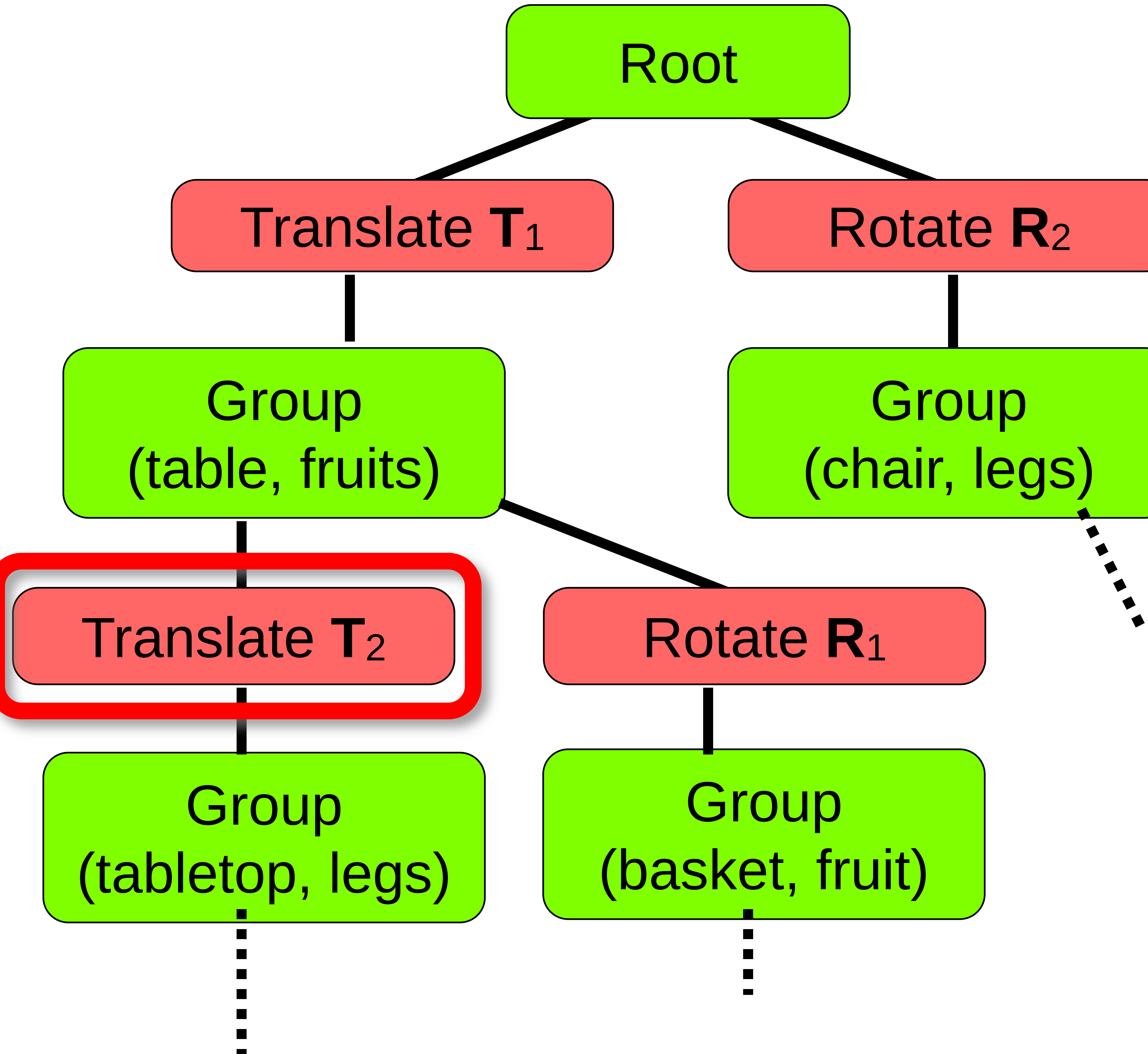
$$\mathbf{S} = \mathbf{T}_1$$

Traversal Example



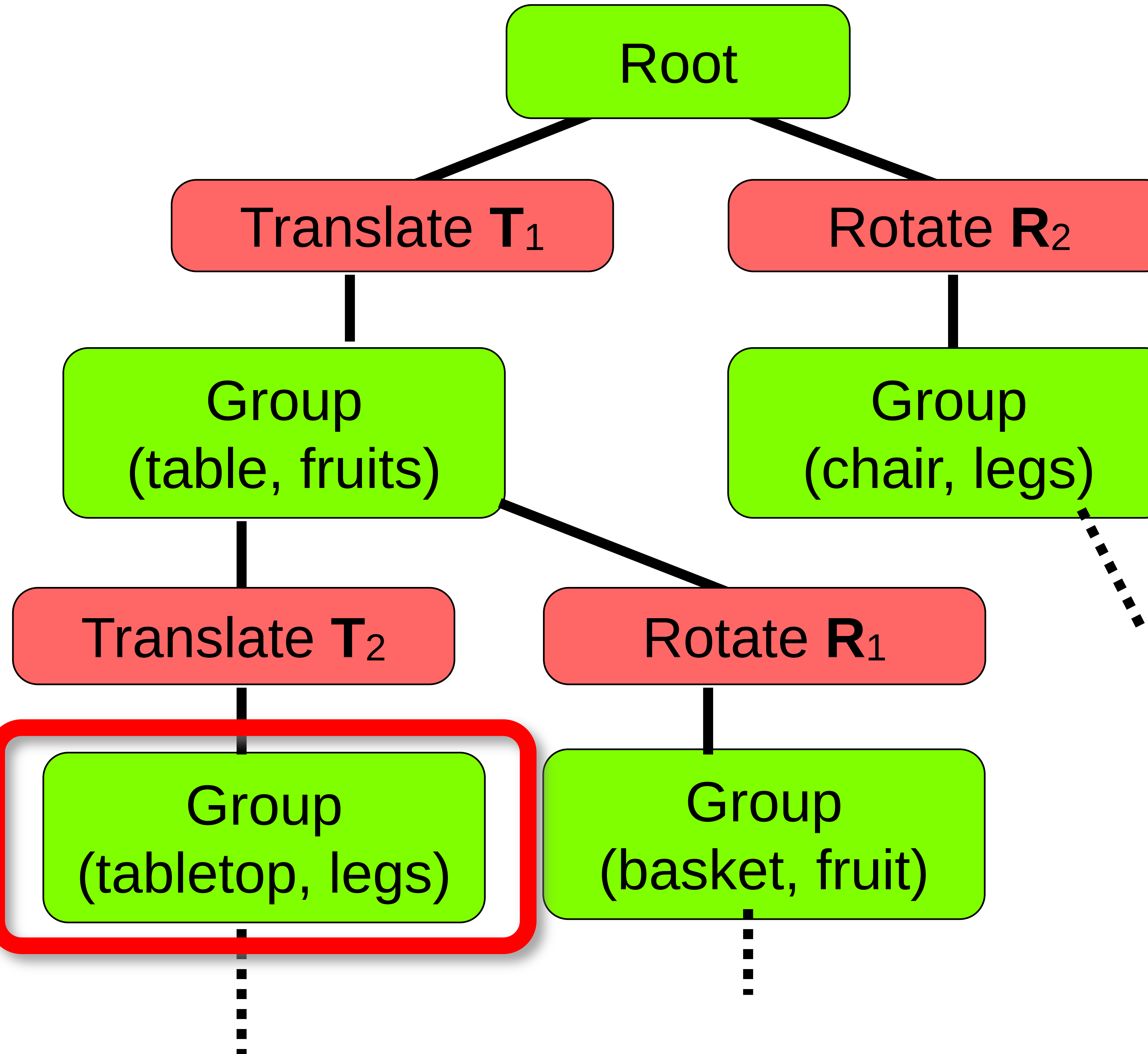
$$\mathbf{S} = \mathbf{T}_1$$

Traversal Example



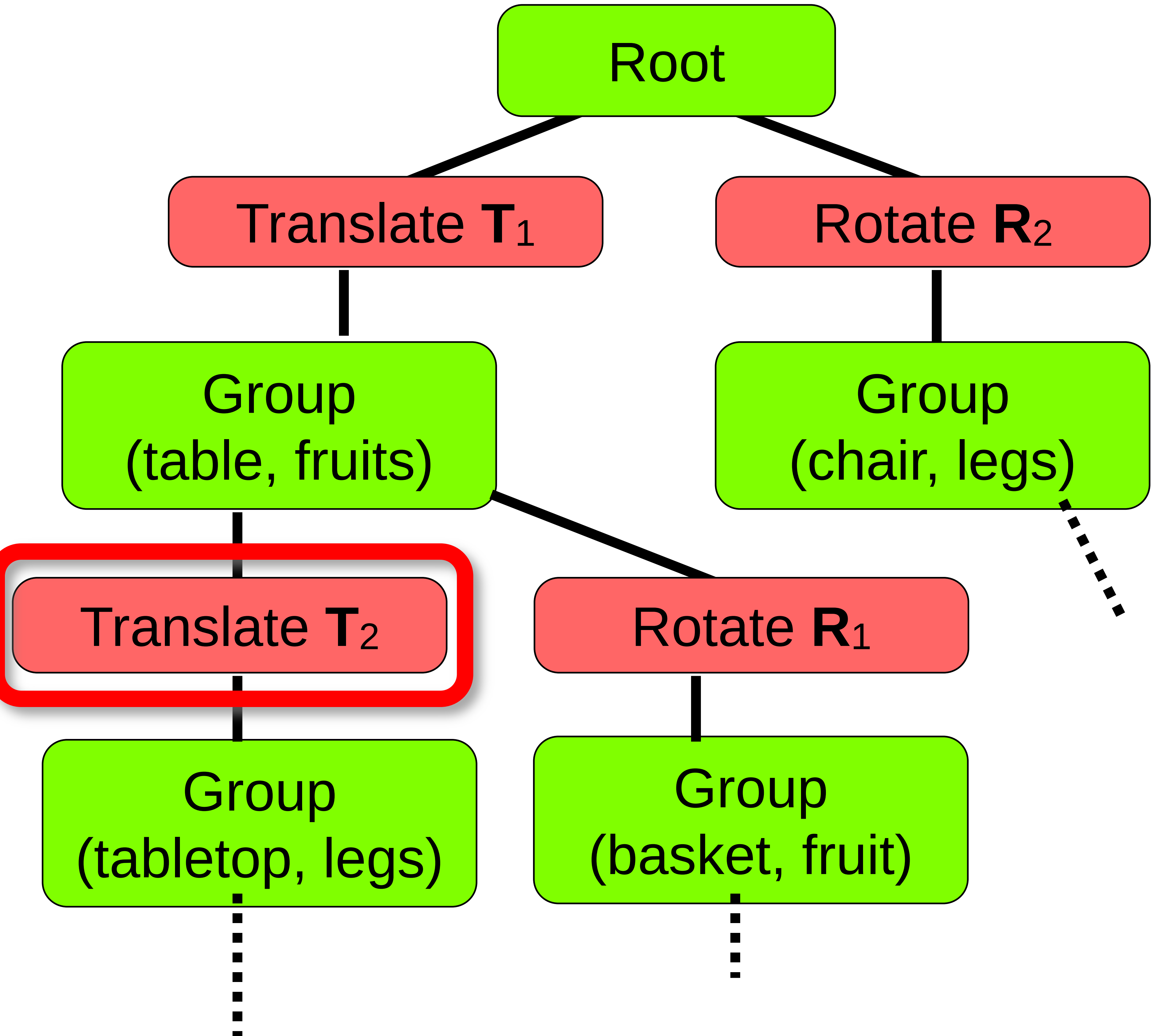
$$\mathbf{S} = \mathbf{T}_1\ \mathbf{T}_2$$

Traversal Example



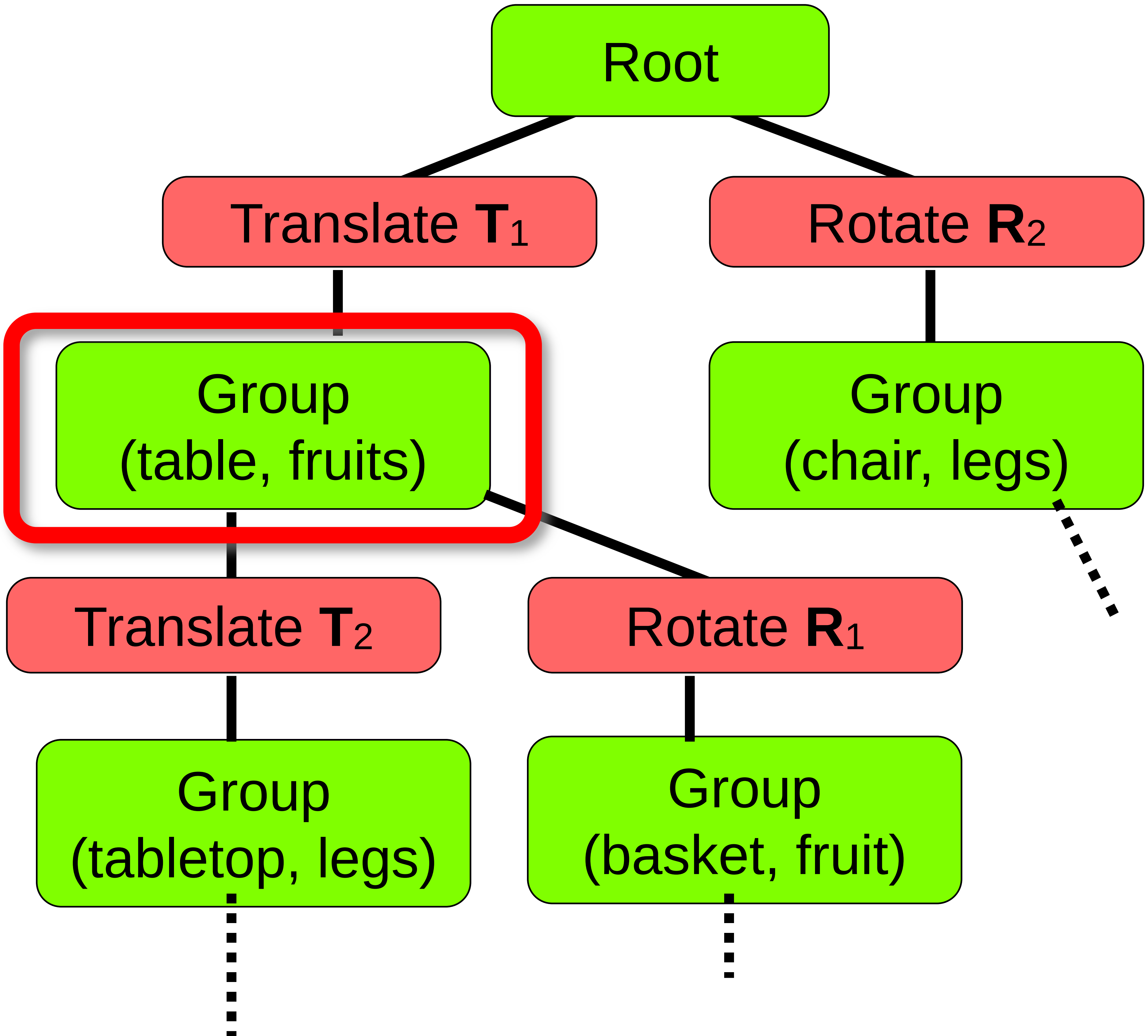
$$\mathbf{S} = \mathbf{T}_1 \, \mathbf{T}_2$$

Traversal Example



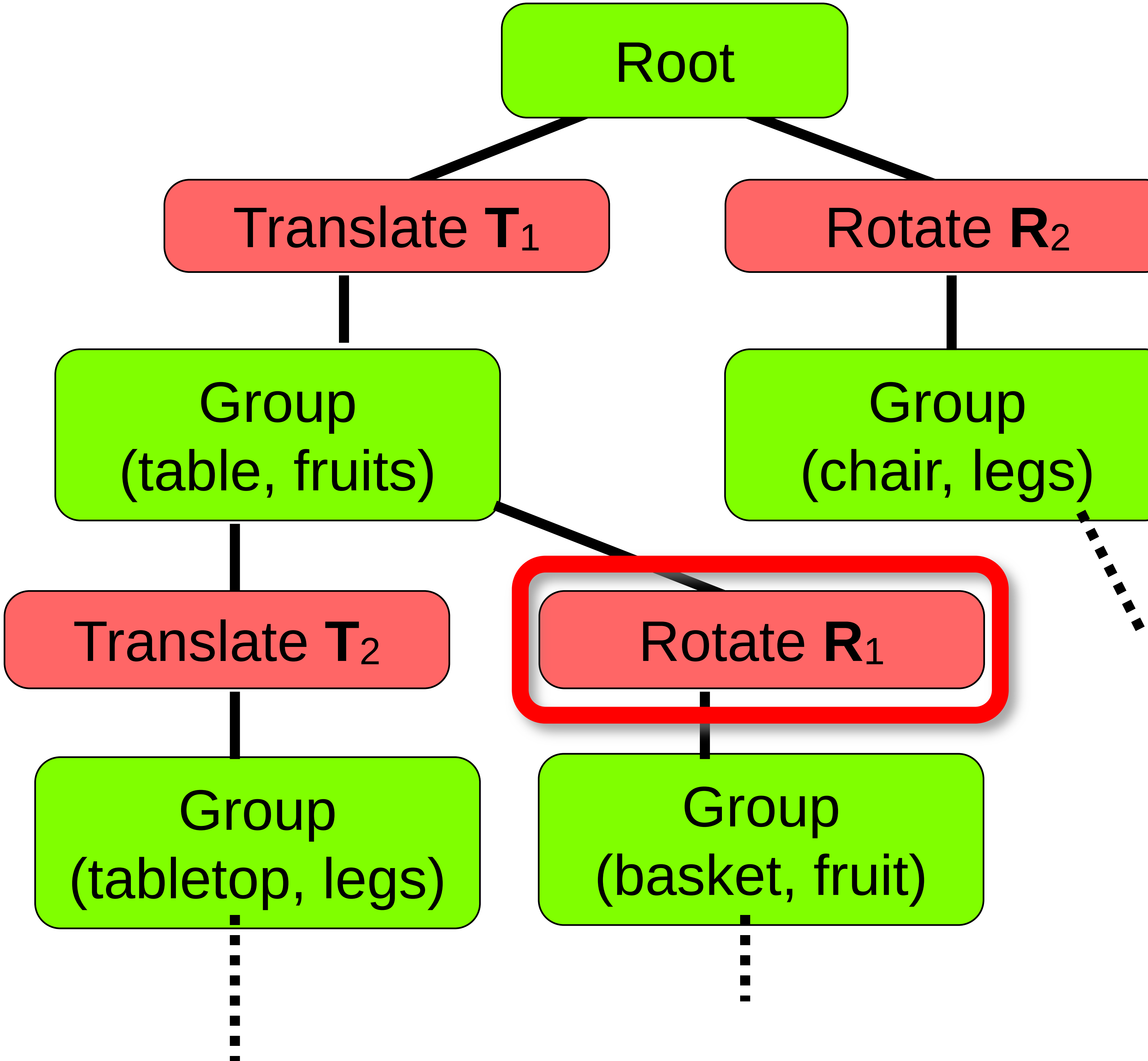
$$\mathbf{S} = \mathbf{T}_1\,\mathbf{T}_2$$

Traversal Example



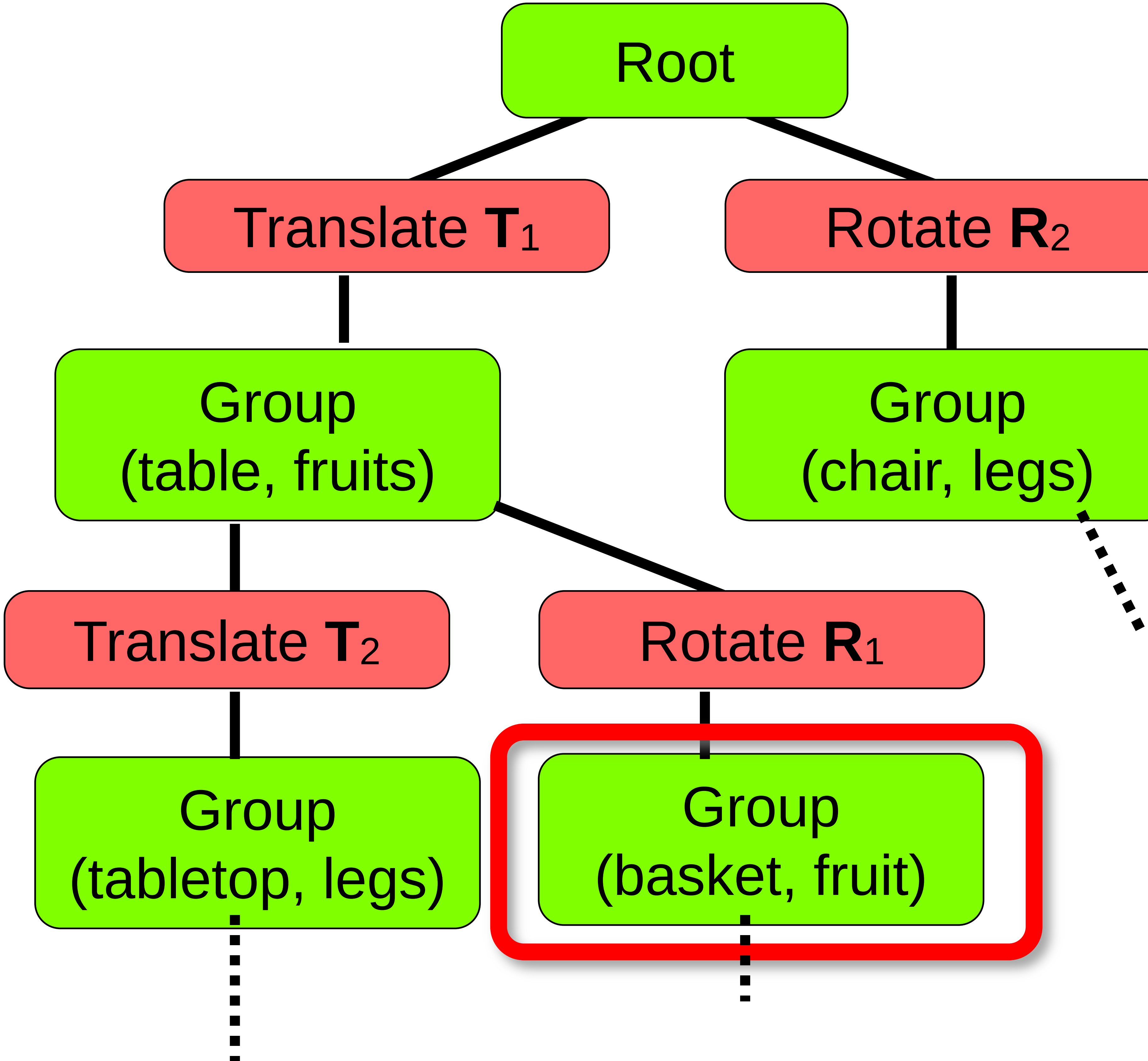
$\mathbf{S} = \mathbf{T}_1$

Traversal Example



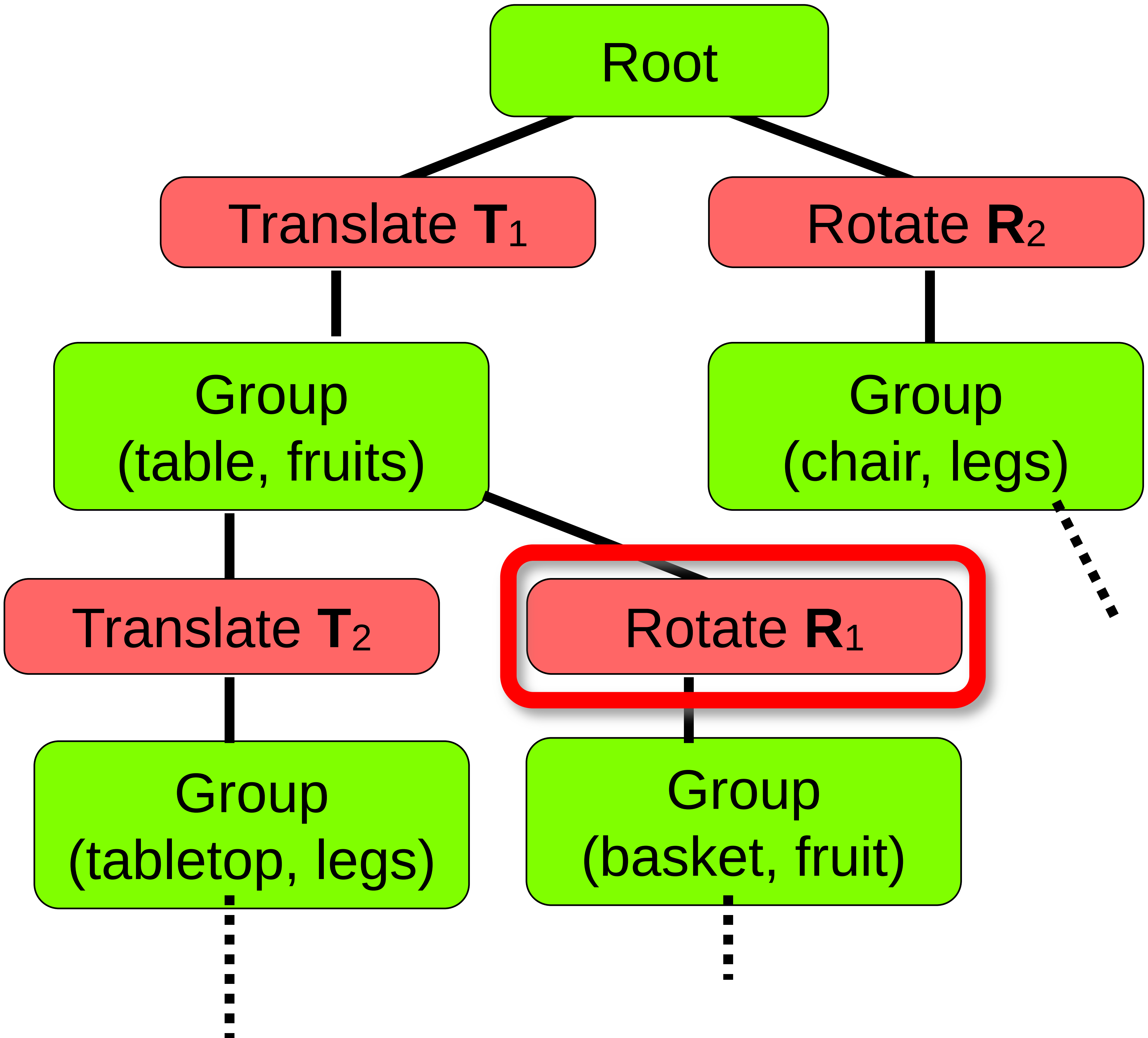
$$\mathbf{S} = \mathbf{T}_1\ \mathbf{R}_1$$

Traversal Example



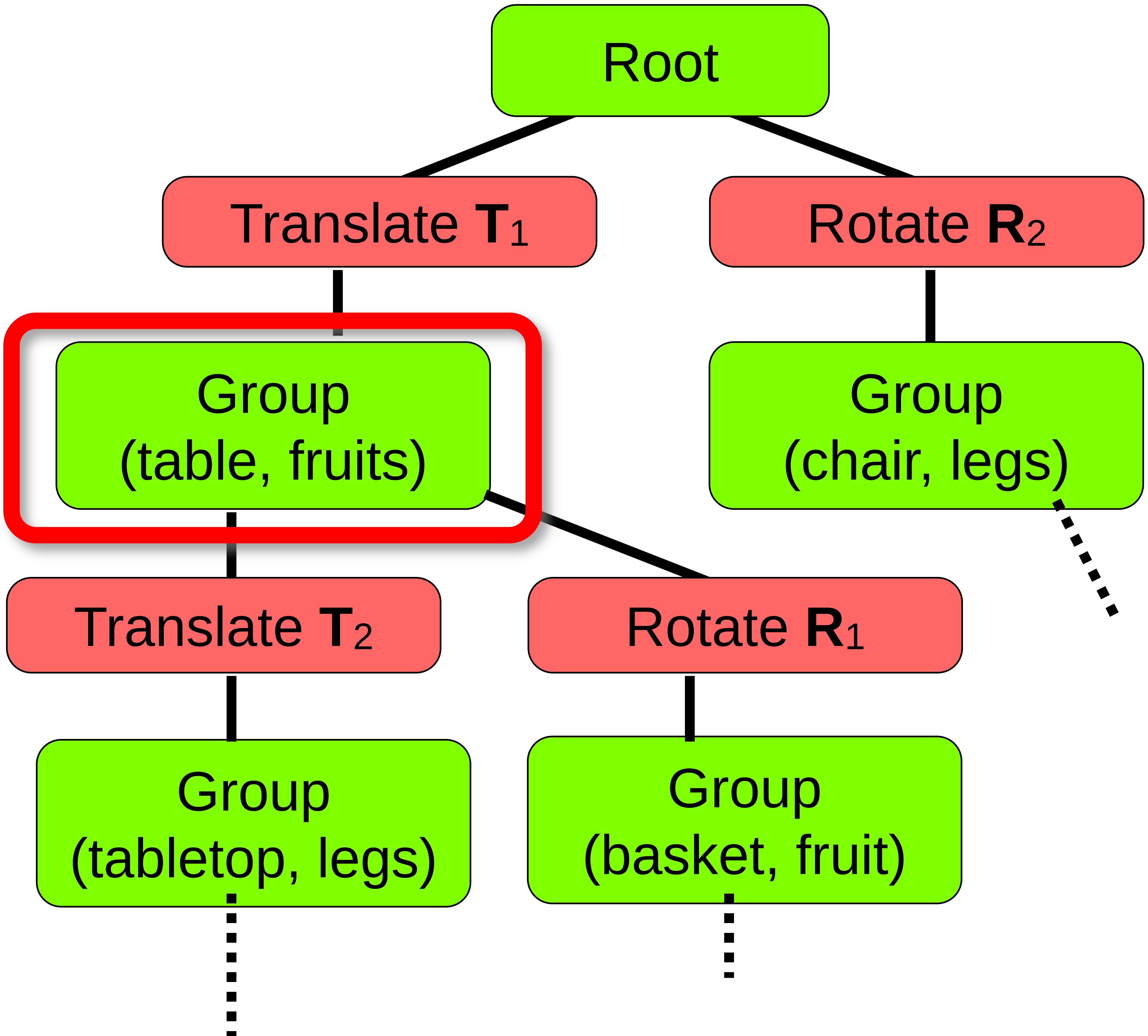
$$\mathbf{S} = \mathbf{T_1}\,\mathbf{R_1}$$

Traversal Example



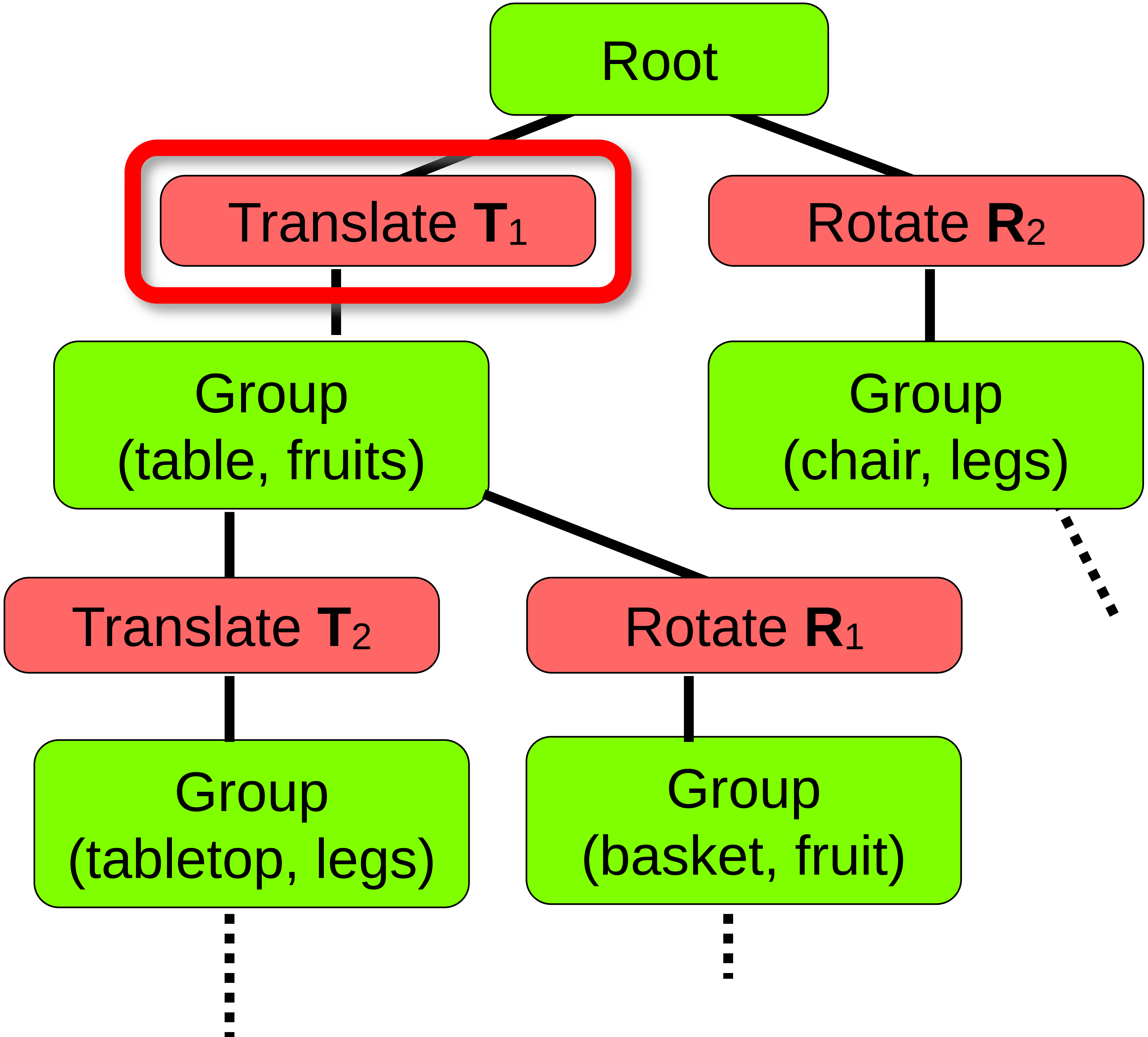
$$\mathbf{S} = \mathbf{T}_1\,\mathbf{R}_1$$

Traversal Example



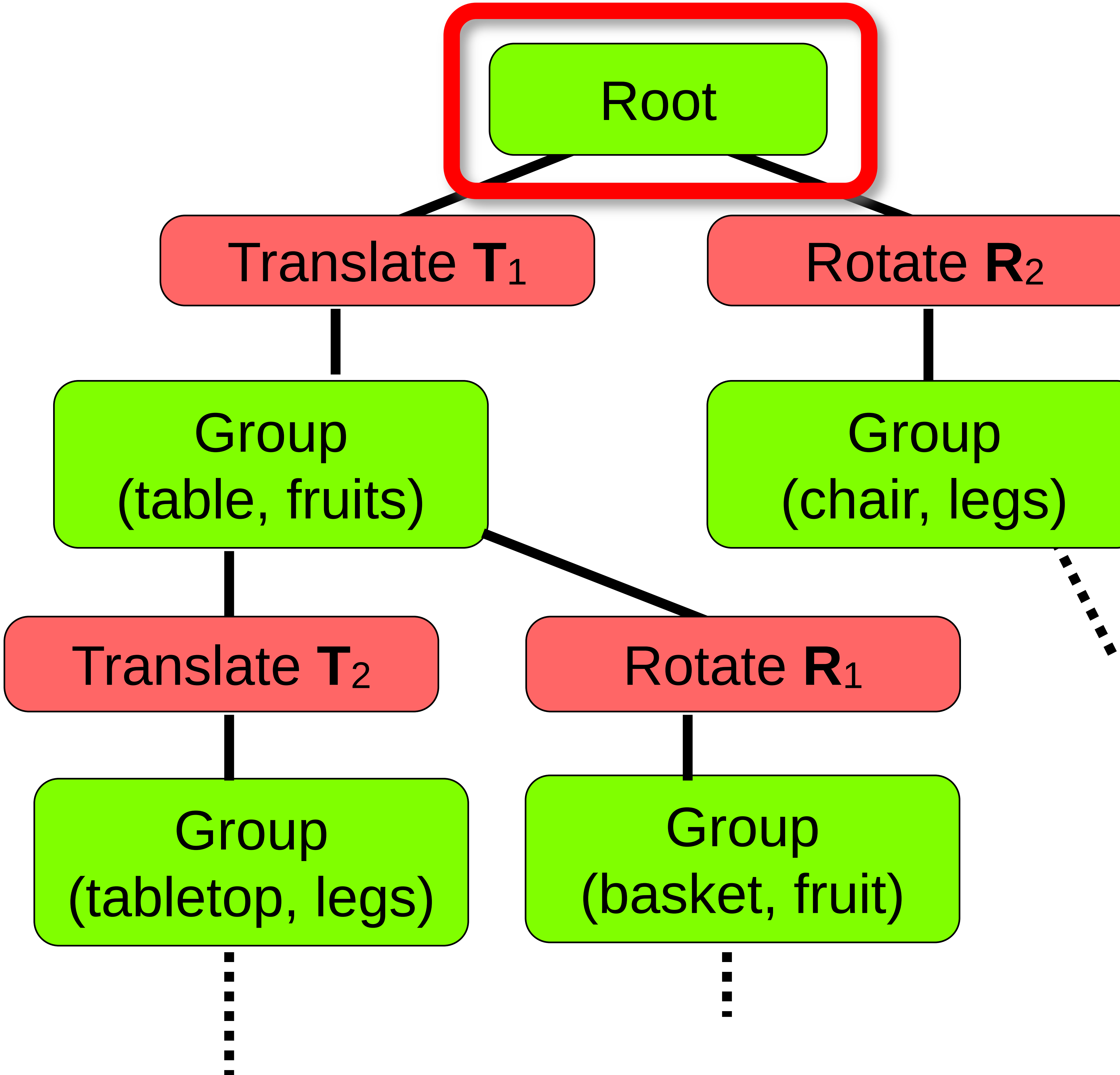
$\mathbf{S} = \mathbf{T}_1$

Traversal Example



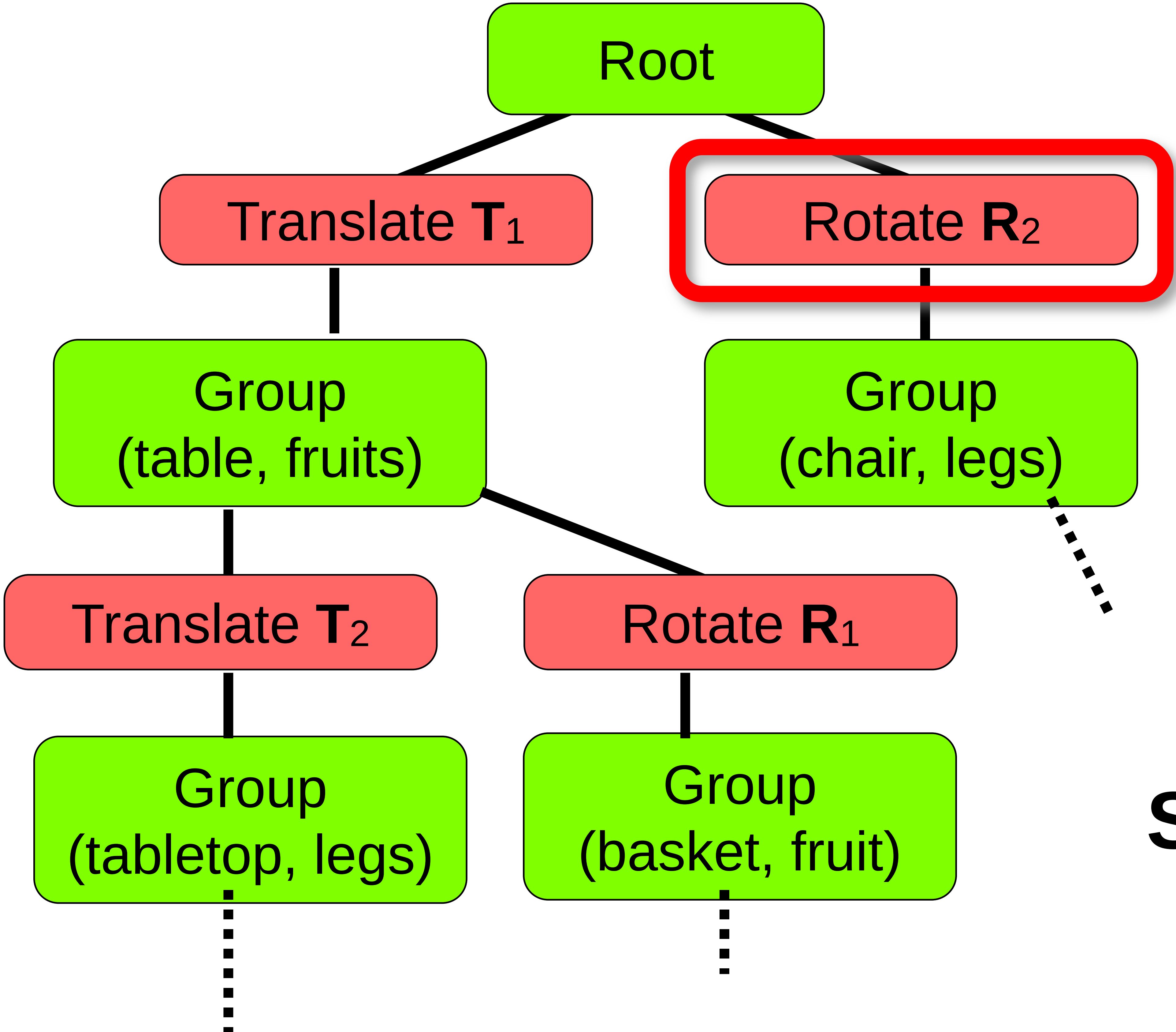
$$\mathbf{S} = \mathbf{T}_1$$

Traversal Example



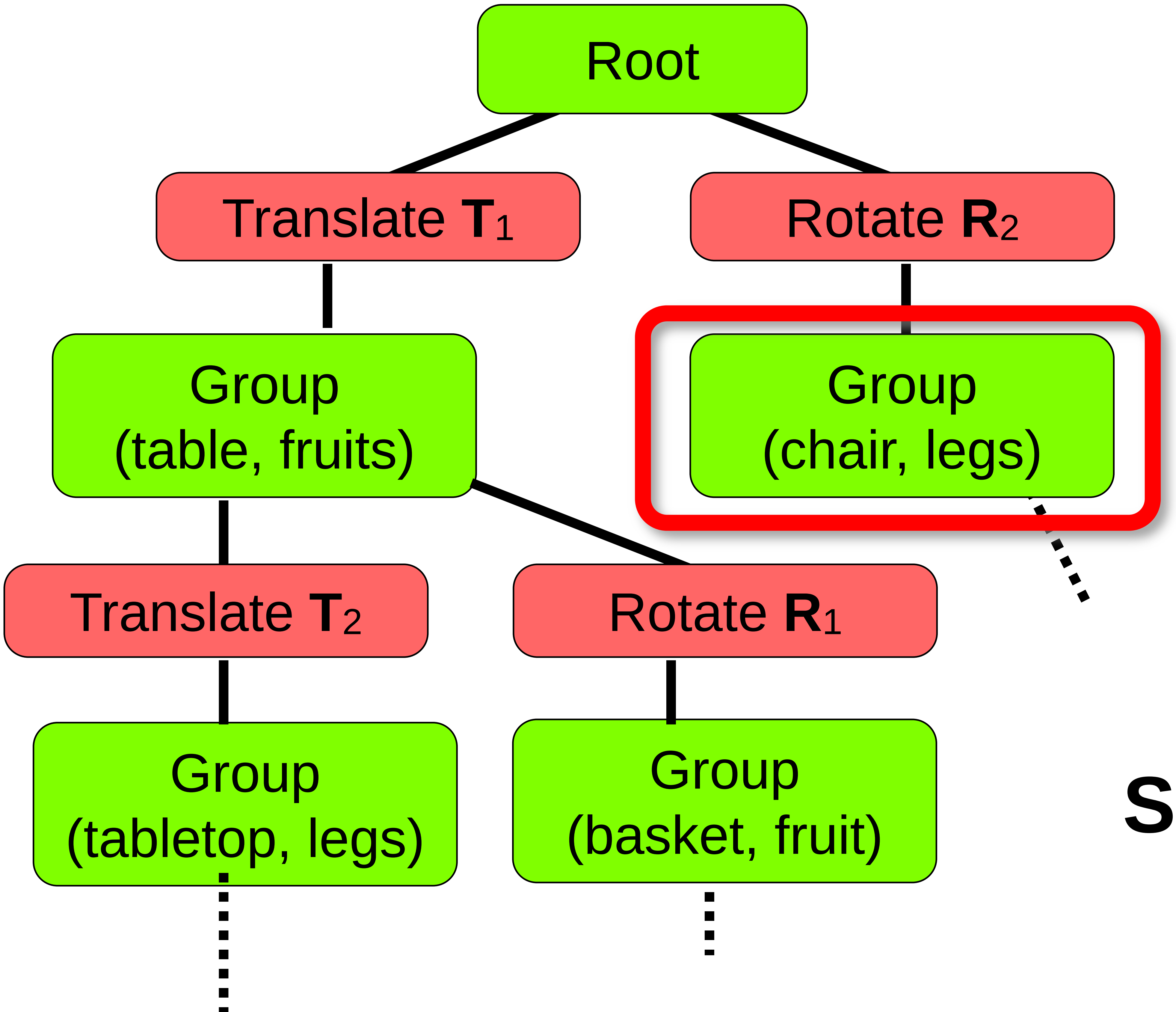
$$\mathbf{S} = \mathbf{I}$$

Traversal Example



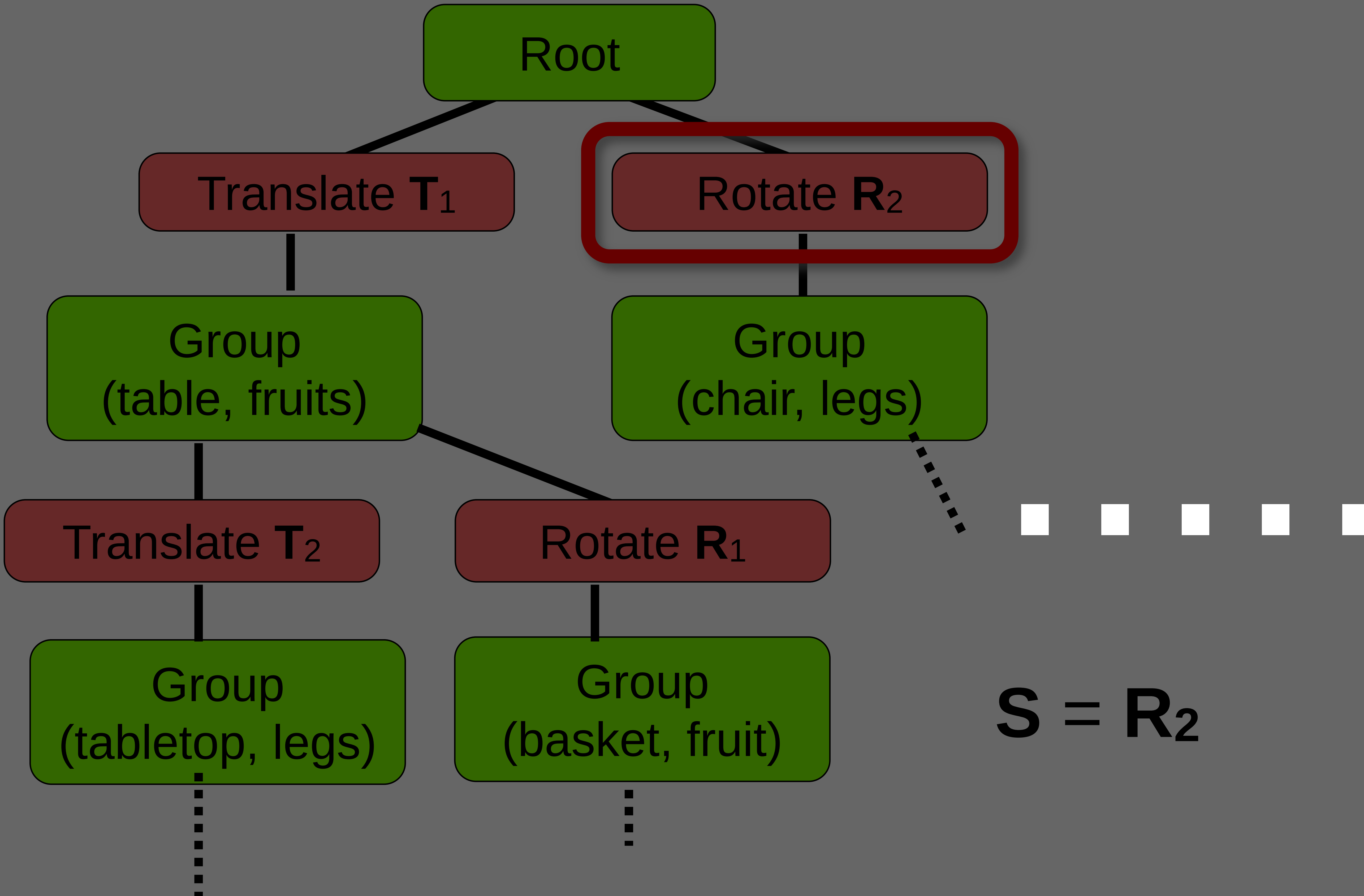
$\mathbf{S} = \mathbf{R}_2$

Traversal Example



$\mathbf{S} = \mathbf{R}_2$

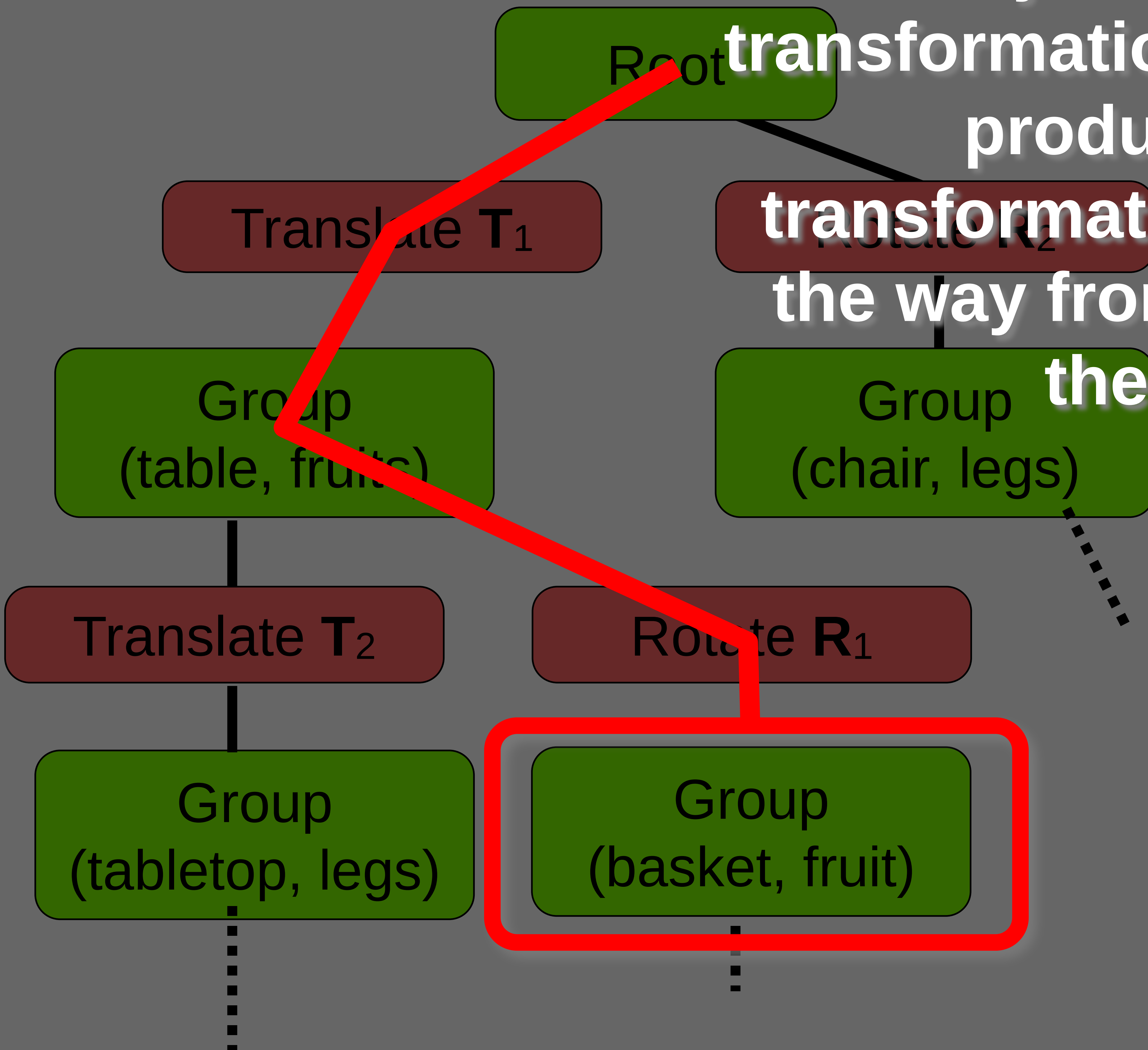
Traversal Example



$$\mathbf{S} = \mathbf{R}_2$$

Traversal Example

At each node, the current object-to-world transformation is the matrix product of all transformations found on the way from the node to the root.



$$\mathbf{S} = \mathbf{T}_1\mathbf{R}_1$$

Traversal State

- The state is updated during traversal
  - Transformations
  - But also other properties (color, etc.)
  - **Apply when entering node, “undo” when leaving

- How to implement?
  - Bad idea to undo transformation by inverse matrix **(Why?)**

Traversal State

- The state is updated during traversal
  - Transformations
  - But also other properties (color, etc.)
  - **Apply when entering node, "undo" when leaving

- How to implement?
  - Bad idea to undo transformation by inverse matrix
  - Why I? $\mathbf{T}*\mathbf{T}^{-1} = \mathbf{I}$ does not necessarily hold in floating point even when T is an invertible matrix – you accumulate error
  - Why II? **T** might be singular, e.g., could flatten a 3D object onto a plane – no way to undo, inverse doesn't exist!

Traversal State

- The state is updated during traversal
  - Transformations
  - But also other properties (color, etc.)
  - **Apply when entering node, “undo” when leaving

- How to implement?
  - Bad idea to undo transformation by inverse matrix
  - Why I? $\mathbf{T}*\mathbf{T}^{-1} = \mathbf{I}$ does not necessarily hold in floating point even when T is an invertible matrix – you accumulate error
  - Why II? **T** might be singular, e.g., could flatten a 3D object onto a plane – no way to undo, inverse doesn’t exist!

Can you think of a data structure suited for this?

Traversal State – Stack

- The state is updated during traversal
  - Transformations
  - But also other properties (color, etc.)
  - **Apply when entering node, “undo” when leaving

- How to implement?
  - Bad idea to undo transformation by inverse matrix
  - Why I? $\mathbf{T}*\mathbf{T}^{-1} = \mathbf{I}$ does not necessarily hold in floating point even when T is an invertible matrix – you accumulate error
  - Why II? **T** might be singular, e.g., could flatten a 3D object onto a plane – no way to undo, inverse doesn’t exist!

- **Solution:** Keep state variables in a **stack**
  - Push current state when entering node, update current state
  - Pop stack when leaving state-changing node
  - See what the stack looks like in the previous example!

Questions?

Plan

- Hierarchical Modeling, Scene Graph
- OpenGL matrix stack
- Hierarchical modeling and animation of characters
  - Forward and inverse kinematics

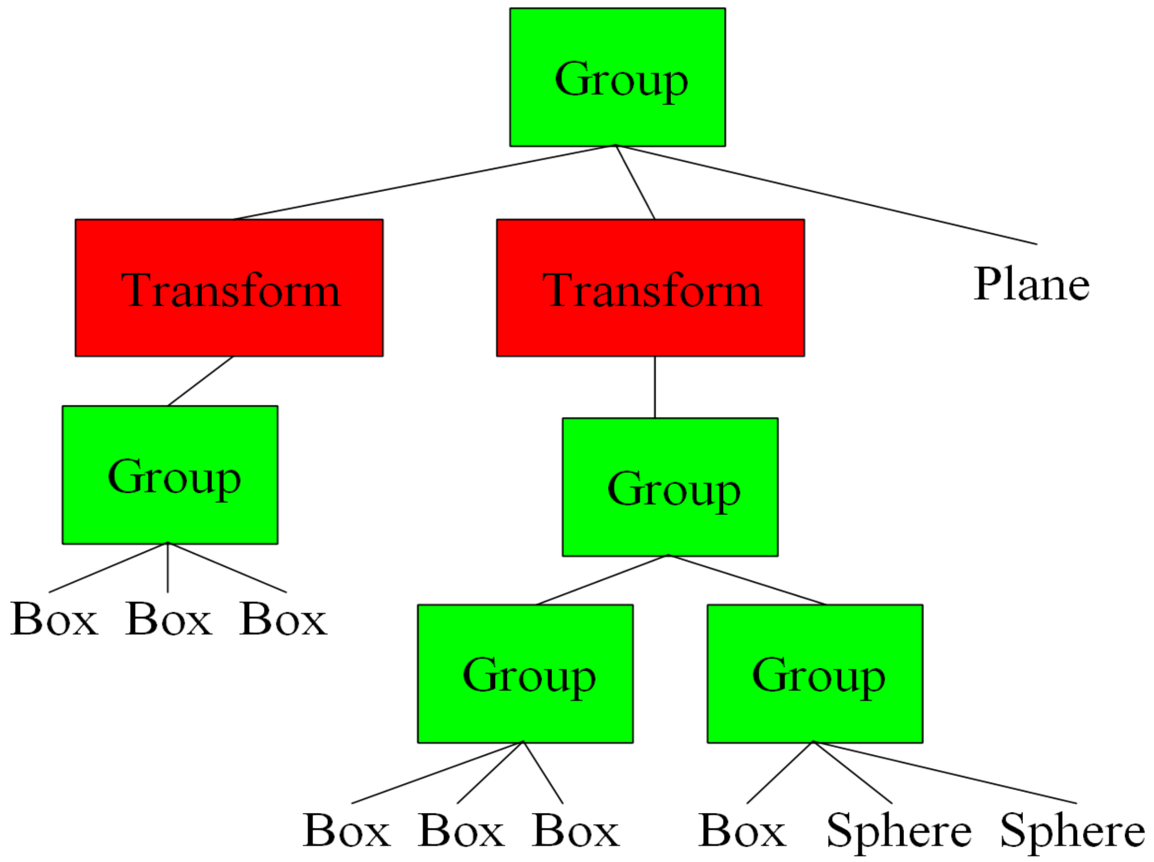
Hierarchical Modeling in OpenGL

- The OpenGL Matrix Stack implements what we just did!

- Commands to change current transformation
  - glTranslate, glScale, etc.
- Current transformation is part of the OpenGL state, i.e., all following draw calls will undergo the new transformation
  - Remember, a transform affects the whole subtree
- Functions to maintain a matrix stack
  - glPushMatrix, glPopMatrix
- Separate stacks for modelview (object-to-view) and projection matrices

When You Encounter a Transform Node

- Push the current transform using glPushMatrix()
- Multiply current transform by node’s transformation
  - Use glMultMatrix(), glTranslate(), glRotate(), glScale(), etc.
- Traverse the subtree
  - Issue draw calls for geometry nodes
- Use glPopMatrix() when done.
- Simple as that!



Questions?

- Further reading on OpenGL
  Matrix Stack and hierarchical model/view transforms
  - http://www.glprogramming.com/red/chapter03.html

- It can be a little confusing if you don't think the previous through, but it's really quite simple in the end.
  - I know very capable people who after 15 years of experience still resort to brute force (trying all the combinations) for getting their transformations right, but it's such a waste :)

Plan

- Hierarchical Modeling, Scene Graph
- OpenGL matrix stack
- Hierarchical modeling and animation of characters
  - Forward and inverse kinematics

Animation

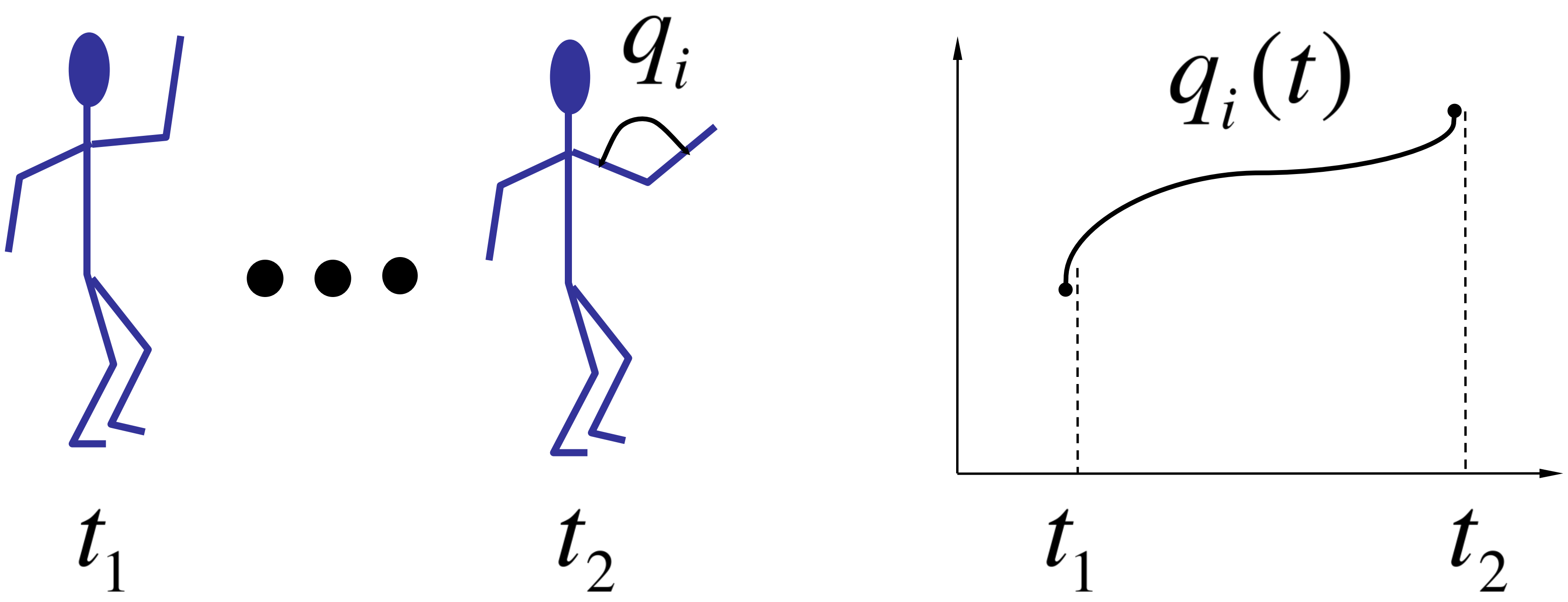
- Hierarchical structure is essential for animation
- Eyes move with head
- Hands move with arms
- Feet move with legs
- ...

- Without such structure the model falls apart.

Articulated Models

- **Articulated models** are rigid parts connected by joints
  - each joint has some angular degrees of freedom

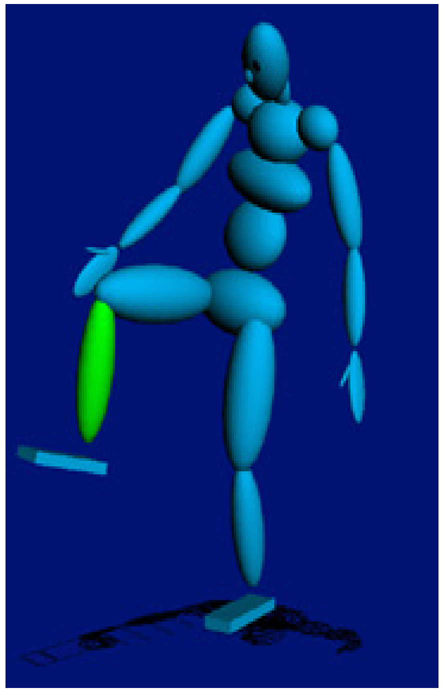
- Articulated models can be animated by specifying the joint angles as functions of time.



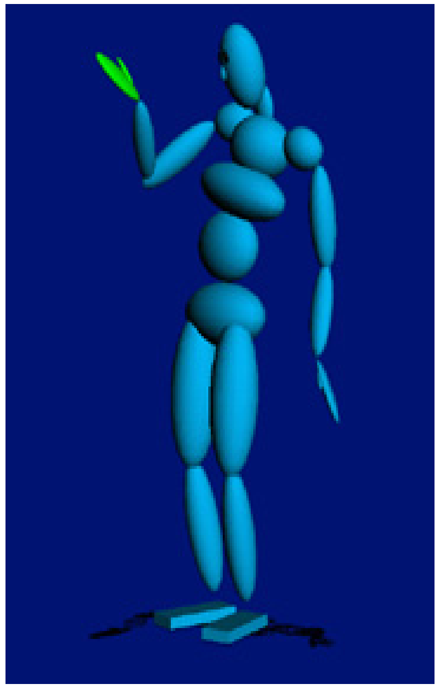
Joints and bones

- Describes the positions of the body parts as a function of joint angles.
  - Body parts are usually called “bones”
- Each joint is characterized by its degrees of freedom (dof)
  - Usually rotation for articulated bodies

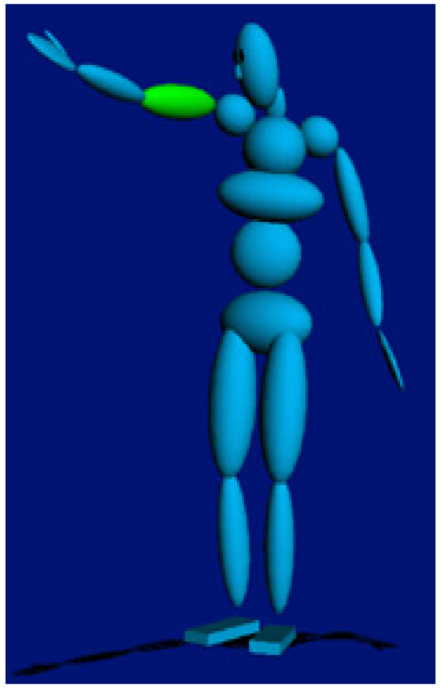
1 DOF: knee



2 DOF: wrist

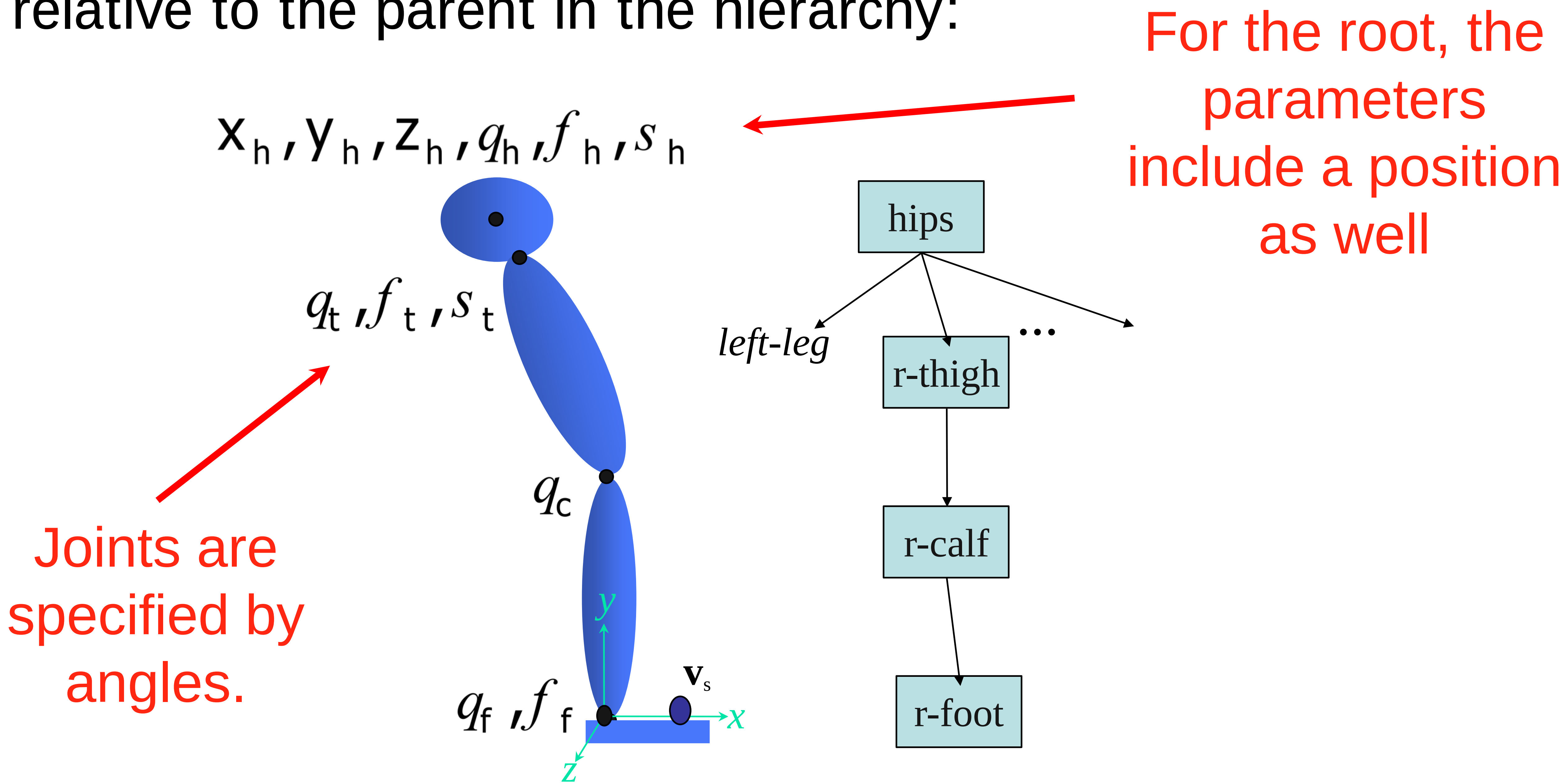


3 DOF: arm

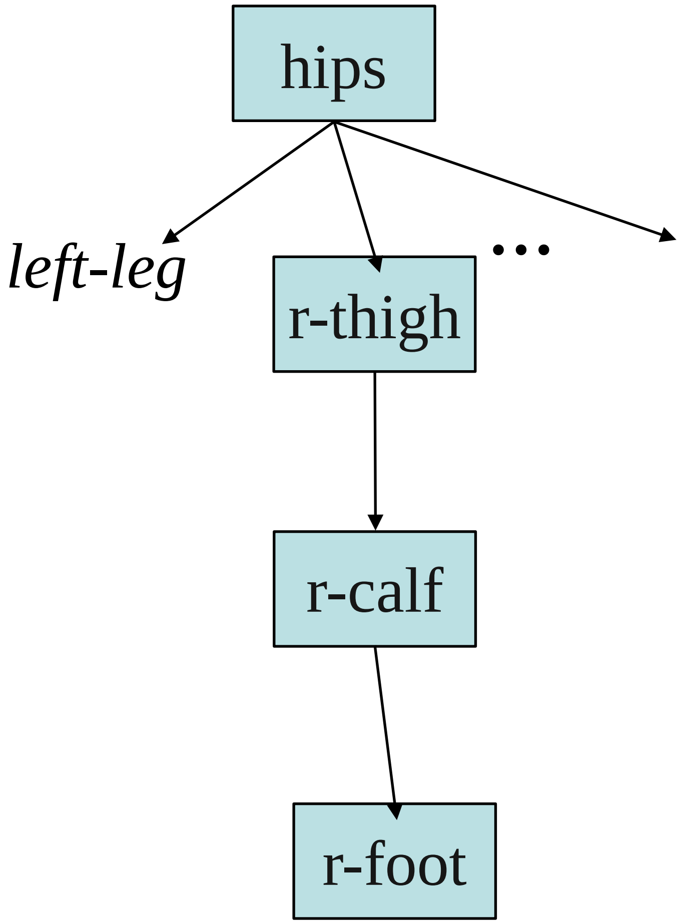


Skeleton Hierarchy

- Each bone position/orientation described relative to the parent in the hierarchy:



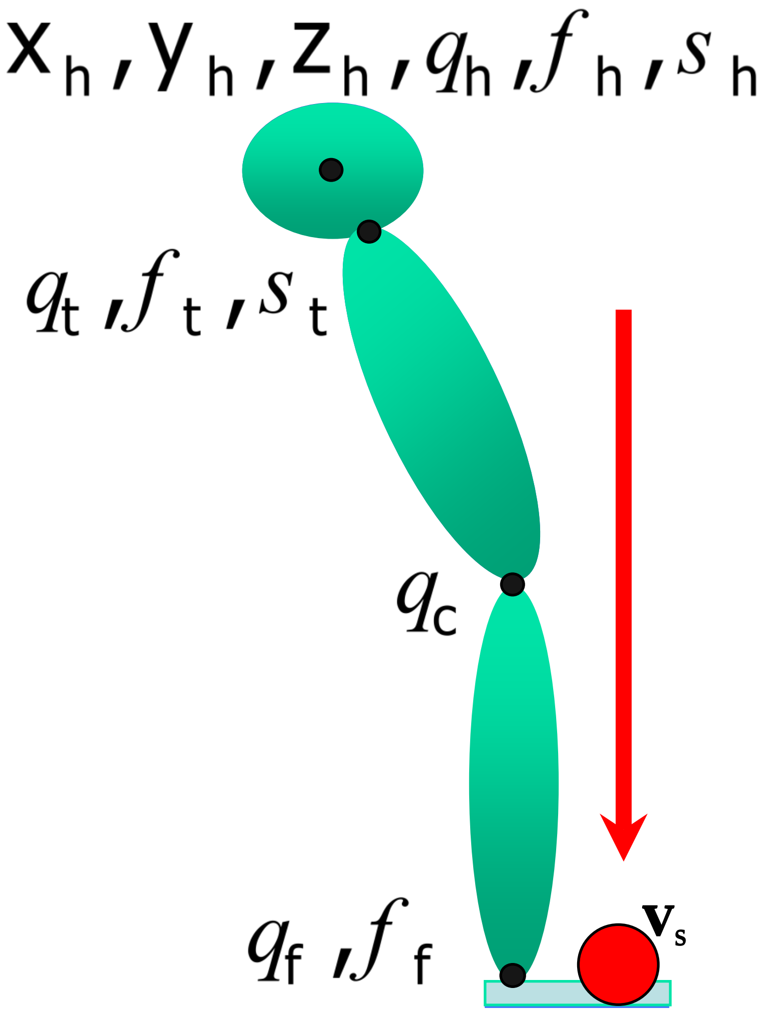
Draw by Traversing a Tree




```
glLoadIdentity();
glPushMatrix();
  glTranslatef(…);
  glRotate(…);
  drawHips();
  glPushMatrix();
    glTranslate(…);
    glRotate(…);
    drawThigh();
    glTranslate(…);
    glRotate(…);
    drawCalf();
    glTranslate(…);
    glRotate(…);
    drawFoot();
  glPopMatrix();
  left-leg
```

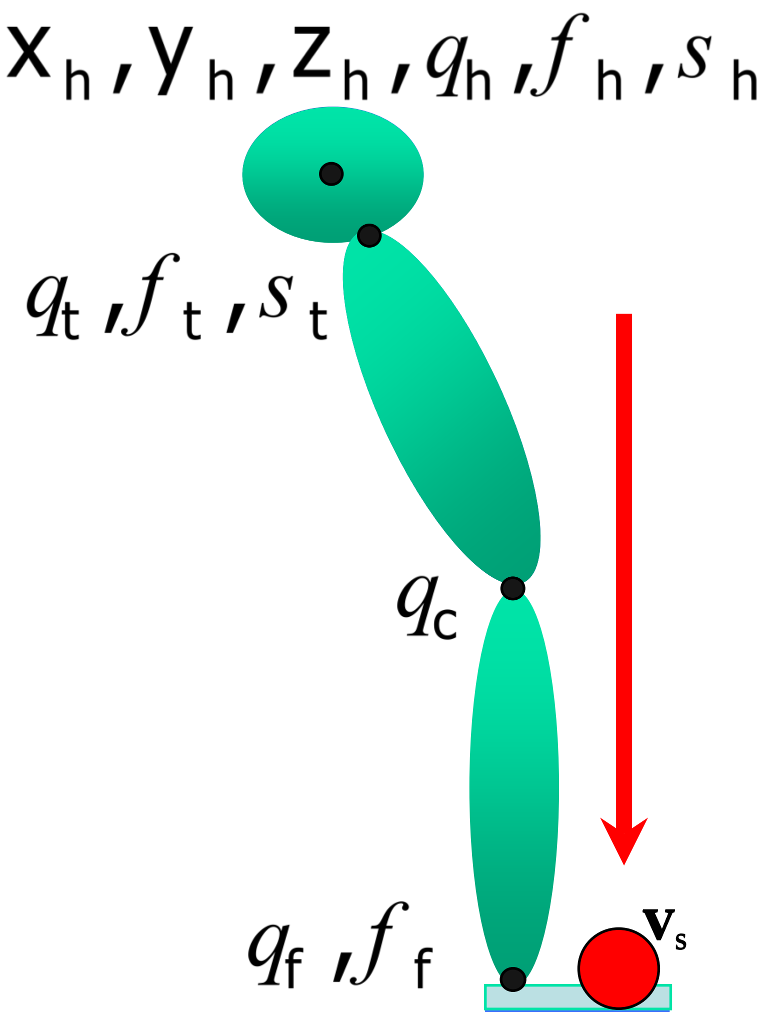
- Assumes drawing procedures for thigh, calf, and foot use joint positions as the origin for a drawing coordinate frame

Forward Kinematics



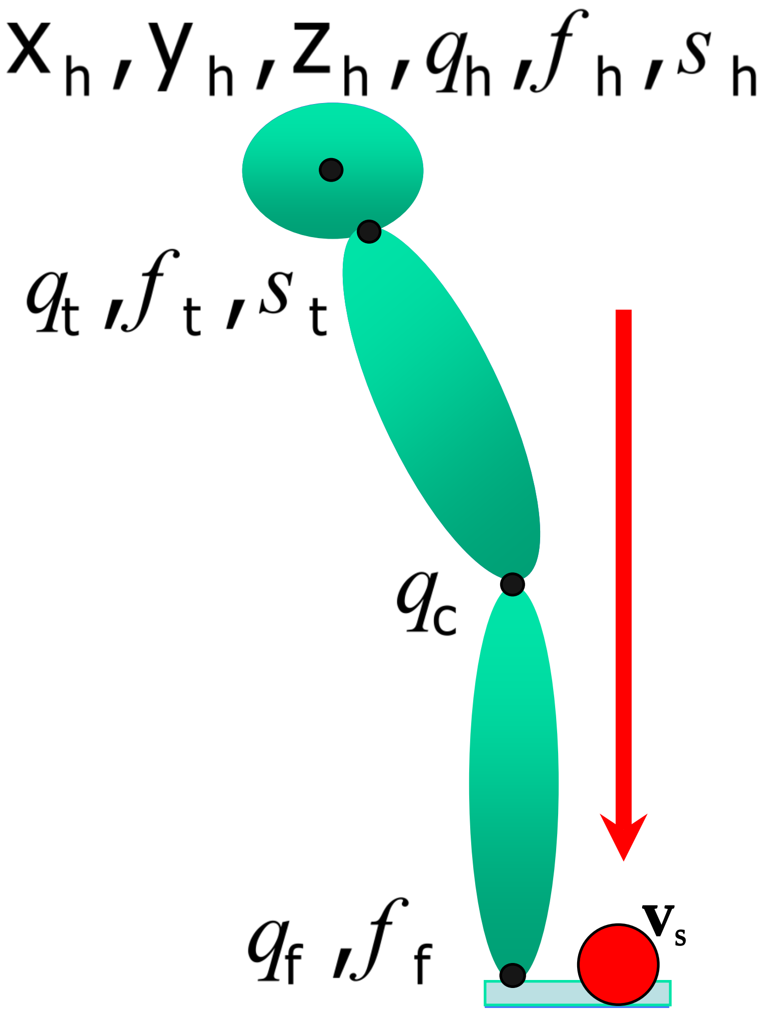
How to determine the world-space position for point $\mathbf{v}_s$?

Forward Kinematics



Transformation matrix **S** for a point $\mathbf{v}_s$ is a matrix composition of all joint transformations between the point and the root of the hierarchy. **S** is a function of all the joint angles between here and root.

Forward Kinematics



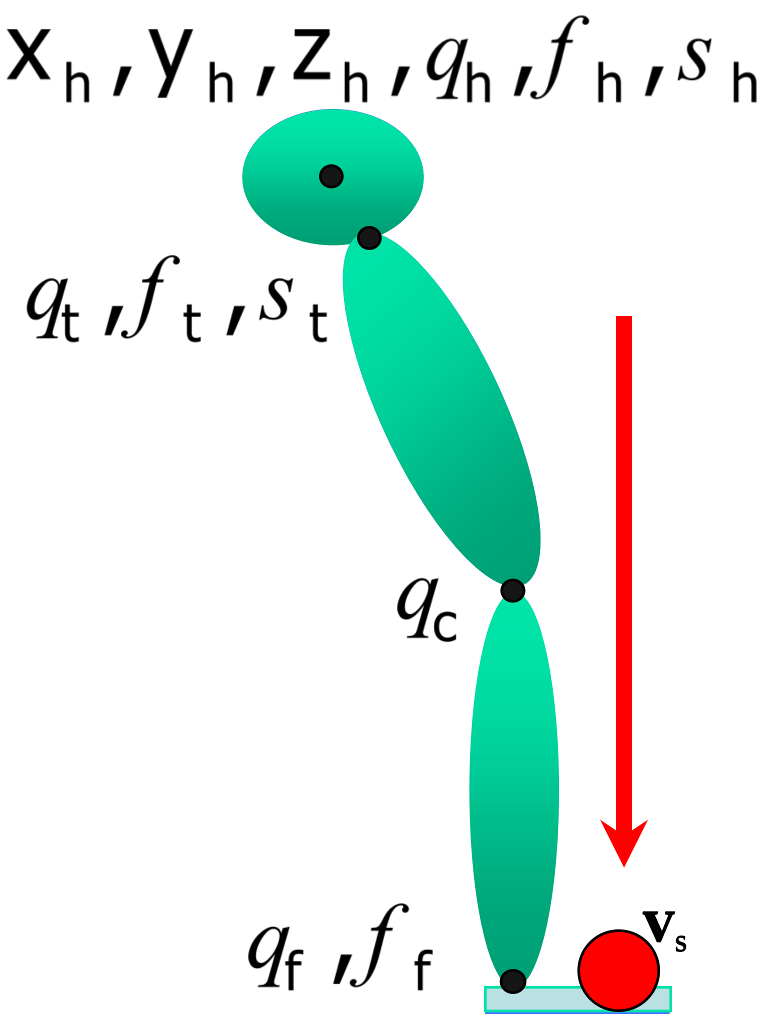
Transformation matrix **S** for a point **v**$_s$ is a matrix composition of all joint transformations between the point and the root of the hierarchy. **S** is a function of all the joint angles between here and root.

Note that the angles have a non-linear effect.

This product is **S**

$$\mathbf{v}_w = \boxed{\mathbf{T}(x_h, y_h, z_h)\mathbf{R}(q_h, f_h, s_h)\ \mathbf{TR}(q_t, f_t, s_t)\ \mathbf{TR}(q_c)\ \mathbf{TR}(q_f, f_f)}\ \mathbf{v}_s$$

Forward Kinematics



Transformation matrix **S** for a point $\mathbf{v}_s$ is a matrix composition of all joint transformations between the point and the root of the hierarchy. **S** is a function of all the joint angles between here and root.

Note that the angles have a non-linear effect.

This product is **S**

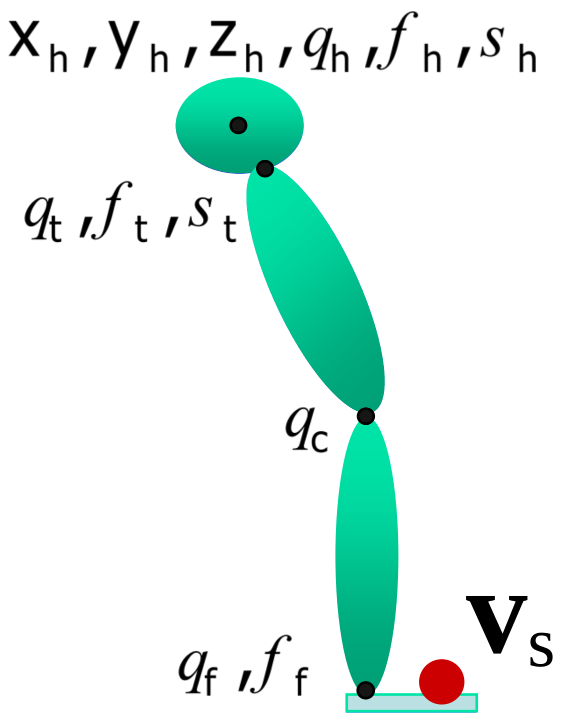
$$\mathbf{v}_w = \boxed{\mathbf{T}(x_h,y_h,z_h)\mathbf{R}(q_h,f_h,s_h)\ \mathbf{TR}(q_t,f_t,s_t)\ \mathbf{TR}(q_c)\ \mathbf{TR}(q_f,f_f)}\ \mathbf{v}_s$$

$$v_w = S\left(\underbrace{x_h, y_h, z_h, \theta_h, \phi_h, \sigma_h, \theta_t, \phi_t, \sigma_t, \theta_c, \theta_f, \phi_f}_{\text{parameter vector } \mathbf{p}}\right) v_s = S(p)\, v_s$$

Questions?

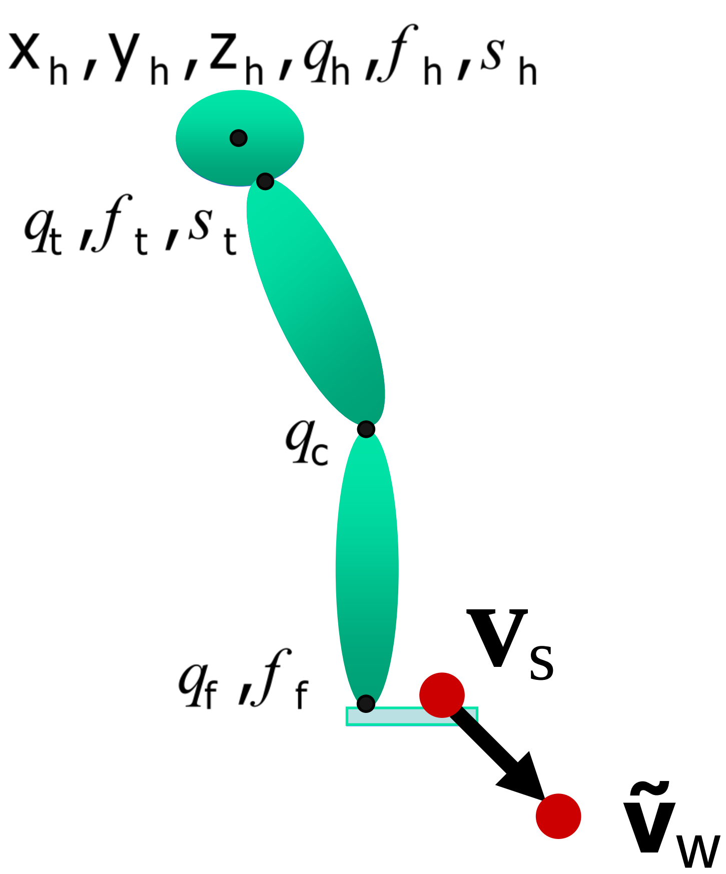
Inverse Kinematics

- Forward Kinematics
  - Given the skeleton parameters **p** (position of the root and the joint angles) and the position of the point in local coordinates $\mathbf{v}_s$, what is the position of the point in the world coordinates $\mathbf{v}_w$?
  - Not too hard, just apply transform accumulated from the root.



Inverse Kinematics

- Forward Kinematics
  - Given the skeleton parameters **p** (position of the root and the joint angles) and the position of the point in local coordinates $\mathbf{v}_s$, what is the position of the point in the world coordinates $\mathbf{v}_w$?
  - Not too hard, just apply transform accumulated from the root.



- Inverse Kinematics
  - Given the current position of the point and the desired new position $\tilde{\mathbf{v}}_w$ in world coordinates, what are the skeleton parameters **p** that take the point to the desired position?

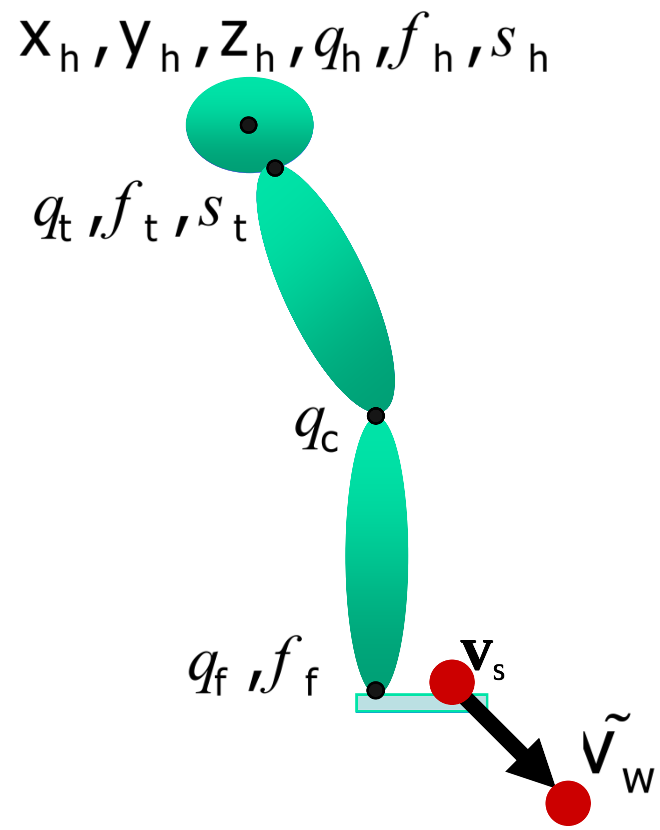
Inverse Kinematics

- Given the position of the point in local coordinates $\mathbf{v}_s$ and the desired position $\tilde{\mathbf{v}}_w$ in world coordinates, what are the skeleton parameters $\mathbf{p}$?

$$\tilde{v}_w = S\left(\underbrace{x_h, y_h, z_h, \theta_h, \phi_h, \sigma_h, \theta_t, \phi_t, \sigma_t, \theta_c, \theta_f, \phi_f}_{\text{skeleton parameter vector } \mathbf{p}}\right) v_s = S(p)v_s$$

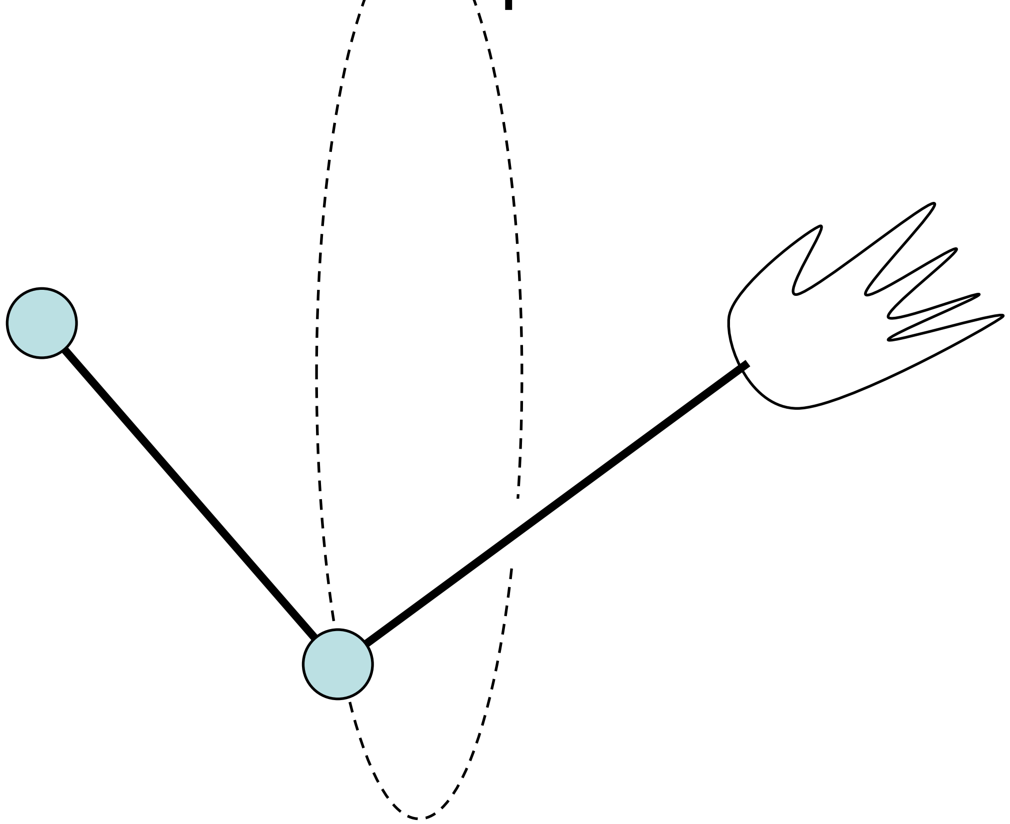
- Requires solving for $\mathbf{p}$, given $\mathbf{v}_s$ and $\tilde{\mathbf{v}}_w$
  - Non-linear and ...

It’s Underconstrained



- Count degrees of freedom:
  - We specify one 3D point (3 equations)
  - We usually need more than 3 angles
  - **p** usually has tens of dimensions

- Simple geometric example (in 3D):
  specify hand position, need elbow & shoulder
  - The set of possible elbow location is a circle in 3D



How to tackle these problems?

$$\boldsymbol{v}_{\mathrm{WS}} = \boldsymbol{S}(\boldsymbol{p})\,\boldsymbol{v}_{\mathrm{S}}$$

- Deal with non-linearity:
  Iterative solution (steepest descent)
  - Compute Jacobian matrix of world position w.r.t. angles
    - Jacobian: "If the parameters **p** change by tiny amounts, what is the resulting change in the world position $\mathbf{v}_{WS}$?"
  - Then invert Jacobian.
    - This says "if $\mathbf{v}_{WS}$ changes by a tiny amount, what is the change in the parameters **p**?"
  - But wait! The Jacobian is non-invertible (3xN)
  - Deal with ill-posedness: Pseudo-inverse
    - Solution that displaces things the least
    - See http://en.wikipedia.org/wiki/Moore-Penrose_pseudoinverse
  - Deal with ill-posedness: Prior on "good pose" (more advanced)

$$\left[\frac{\partial(\boldsymbol{v}_{\mathrm{WS}})_i}{\partial p_j}\right]$$

- Additional potential issues: bounds on joint angles, etc.
  - Do not want elbows to bend past 90 degrees, etc.

Example: Style-Based IK

- Video

- Prior on “good pose”

- Link to paper: Grochow, Martin, Hertzmann, Popovic: Style-Based Inverse Kinematics, ACM SIGGRAPH 2004

Mesh-Based Inverse Kinematics

- Video
- Doesn’t even need a hierarchy or skeleton: Figure proper transformations out based on a few example deformations!
- Link to paper:
  Sumner, Zwicker, Gotsman, Popovic: Mesh-Based Inverse Kinematics, ACM SIGGRAPH 2005

That’s All for Today!

Further reading

- OpenGL Matrix Stack and hierarchical model/view transforms
  - http://www.glprogramming.com/red/chapter03.html