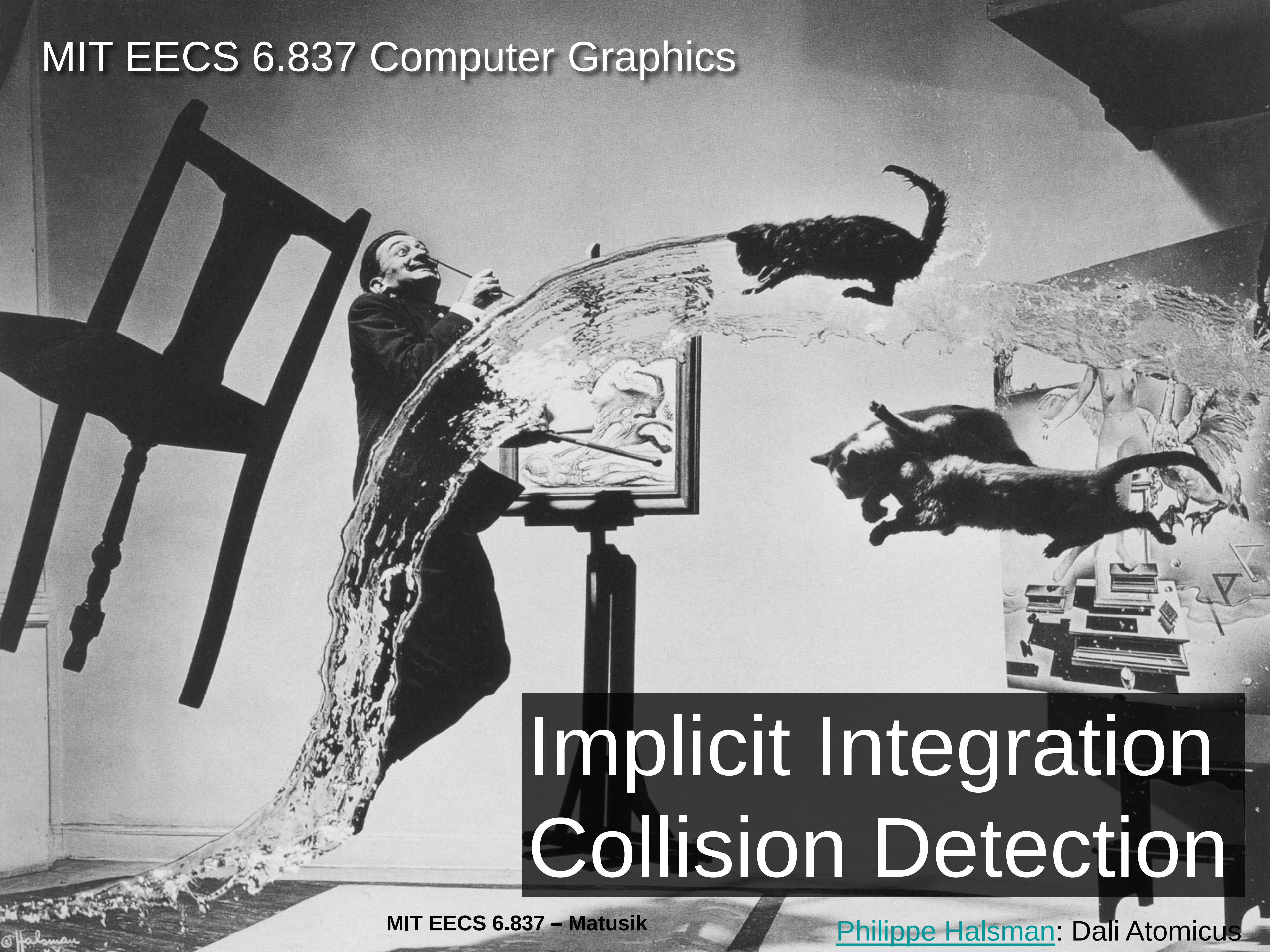


Implicit Integration
Collision Detection

[Philippe Halsman](): Dali Atomicus

Midterm

- Tuesday, October 16th 2:30pm – 4:00pm
- In class
- Two-pages of notes (double sided) allowed

Plan

- Implementing Particle Systems
- Implicit Integration
- Collision detection and response
  - Point-object and object-object detection
  - Only point-object response

ODEs and Numerical Integration

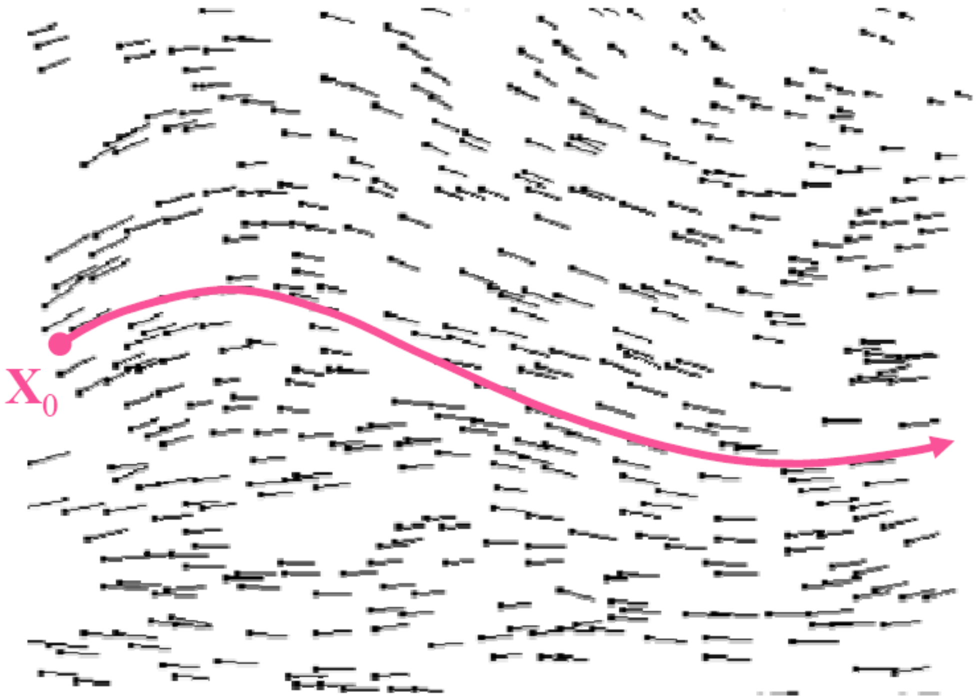
$$\frac{d\,\mathbf{X}(t)}{dt} = f(\mathbf{X}(t), t)$$

- Given a function $f(\mathbf{X},t)$ compute $\mathbf{X}(t)$
- Typically, *initial value problems*:
  - Given values $\mathbf{X}(t_0)=\mathbf{X}_0$
  - Find values $\mathbf{X}(t)$ for $t > t_0$

- We can use lots of standard tools

ODE: Path Through a Vector Field

- ***X****(t):* path in multidimensional <u>phase space</u>



$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X} = f(\boldsymbol{X}, t)$$

"When we are at state **X** at time *t*, where will **X** be after an infinitely small time interval d*t* ?"

- f=d/dt ***X*** is a vector that sits at each point in phase space, pointing the direction.

Many Particles

- We have N point masses
  - Let's just stack all $\boldsymbol{x}$s and $\boldsymbol{v}$s in a big vector of length $6N$
  - $\boldsymbol{F}^i$ denotes the force on particle i
    - When particles do not interact, $\boldsymbol{F}^i$ only depends on $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$.

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{x}_N \\ \boldsymbol{v}_N \end{pmatrix} \qquad f(\boldsymbol{X}, t) = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{F}^1(\boldsymbol{X}, t) \\ \vdots \\ \boldsymbol{v}_N \\ \boldsymbol{F}^N(\boldsymbol{X}, t) \end{pmatrix}$$

***f* gives d/dt X, remember!**

Implementation Notes

- It pays off to abstract (as usual)
  - It's easy to design your "Particle System" and "Time Stepper" to be unaware of each other

- Basic idea
  - "Particle system" and "Time Stepper" communicate via floating-point vectors **X** and a function that computes f(**X**,t)
    - "Time Stepper" does not need to know anything else!

Implementation Notes

- Basic idea
  - “Particle System” tells “Time Stepper” how many dimensions (N) the phase space has
  - “Particle System” has a function to write its state to an N-vector of floating point numbers (and read state from it)
  - “Particle System” has a function that evaluates f(**X**,t), given a state vector **X** and time t
  - “Time Stepper” takes a “Particle System” as input and advances its state

Particle System Class

```
class ParticleSystem
{
    virtual int getDimension()
    virtual setDimension(int n)
    virtual float* getStatePositions()
    virtual setStatePositions(float* positions)
    virtual float* getStateVelocities()
    virtual setStateVelocities(float* velocities)
    virtual float* getForces(float* positions, float* velocities)
    virtual setMasses(float* masses)
    virtual float* getMasses()

    float* m_currentState
}
```

Time Stepper Class

```
class TimeStepper
{
    virtual takeStep(ParticleSystem* ps, float h)
}
```

Forward Euler Implementation

```
class ForwardEuler : TimeStepper
{
    void takeStep(ParticleSystem* ps, float h)
    {
        velocities = ps->getStateVelocities()
        positions = ps->getStatePositions()
        forces = ps->getForces(positions, velocities)
        masses = ps->getMasses()
        accelerations = forces / masses
        newPositions = positions + h*velocities
        newVelocities = velocities  + h*accelerations
        ps->setStatePositions(newPositions)
        ps->setStateVelocities(newVelocities)
    }
}
```

Mid-Point Implementation

```
class MidPoint : TimeStepper
{
    void takeStep(ParticleSystem* ps, float h)
    {
        velocities = ps->getStateVelocities()
        positions = ps->getStatePositions()
        forces = ps->getForces(positions, velocities)
        masses = ps->getMasses()
        accelerations = forces / masses
        midPositions = positions + 0.5*h*velocities
        midVelocities = velocities  + 0.5*h*accelerations
        midForces = ps->getForces(midPositions, midVelocities)
        midAccelerations = midForces / masses
        newPositions = positions + 0.5*h*midVelocities
        newVelocities = velocities  + 0.5*h*midAccelerations
        ps->setStatePositions(newPositions)
        ps->setStateVelocities(newVelocities)
    }
}
```

Particle System Simulation

```
ps = new MassSpringSystem(particleCount, masses, springs, externalForces)
stepper = new ForwardEuler()
time = 0
while time < 1000
    stepper->takeStep(ps, 0.0001)
    time = time + 0.0001
    // render
```

Particle System Simulation

```
ps = new MassSpringSystem(particleCount, masses, springs, externalForces)
stepper = new MidPoint()
time = 0
while time < 1000
    stepper->takeStep(ps, 0.0001)
    time = time + 0.0001
    // render
```

Computing Forces

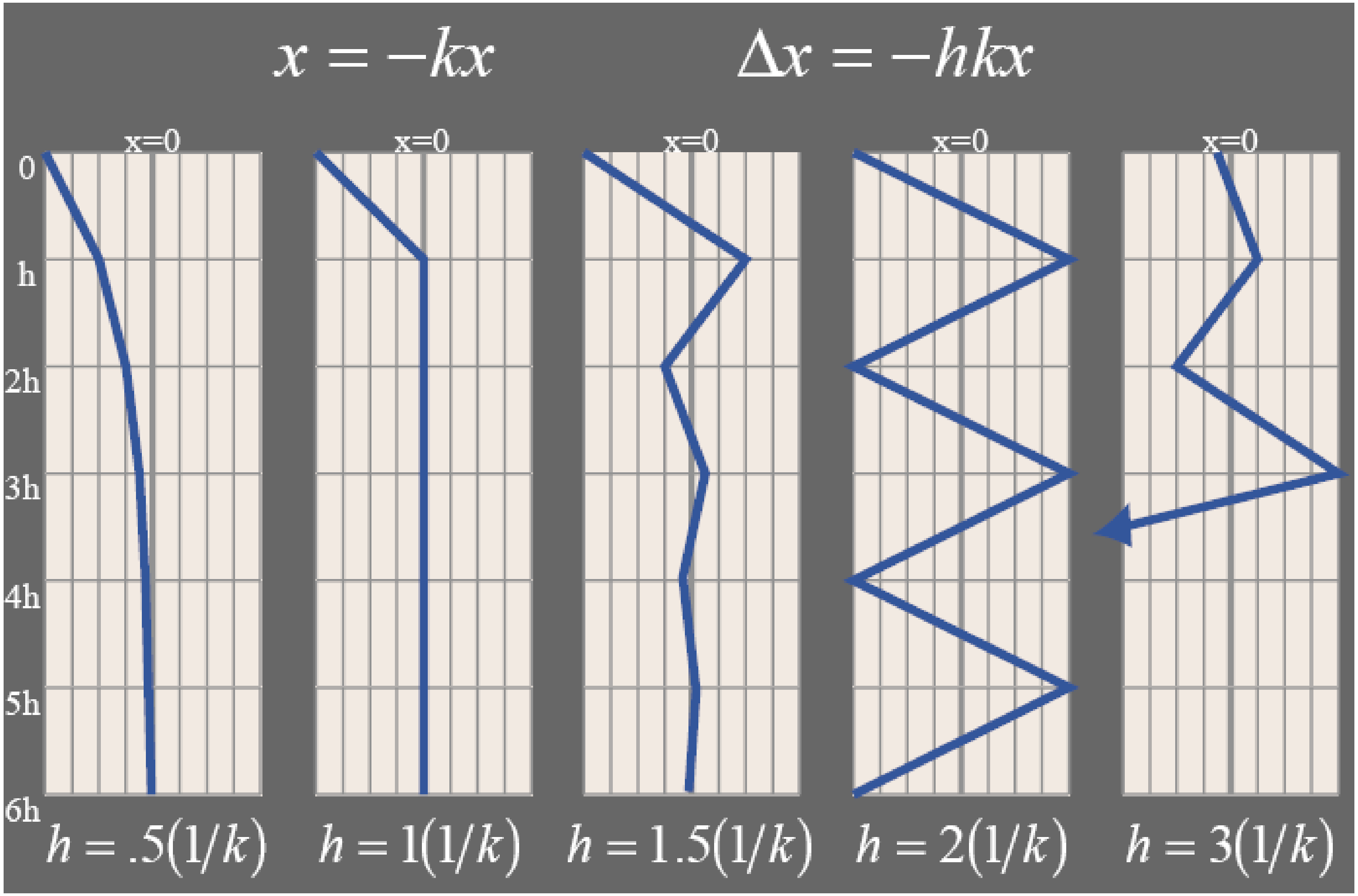
- When computing the forces, initialize the force vector to zero, then sum over all forces for each particle
  - Gravity is a constant acceleration
  - Springs connect two particles, affects both
  - $d\boldsymbol{v}_i/dt = \boldsymbol{F}^i(\boldsymbol{X}, t)$ is the vector sum of all forces on particle i
  - For 2nd order $\boldsymbol{F}^i = m_i\boldsymbol{a}_i$ system, $d\boldsymbol{x}_i/dt$ is just the current $\boldsymbol{v}_i$

$$f(\boldsymbol{X}, t) = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{F}^1(\boldsymbol{X}, t) \\ \vdots \\ \boldsymbol{v}_N \\ \boldsymbol{F}^N(\boldsymbol{X}, t) \end{pmatrix}$$

Questions?

Euler Has a Speed Limit!

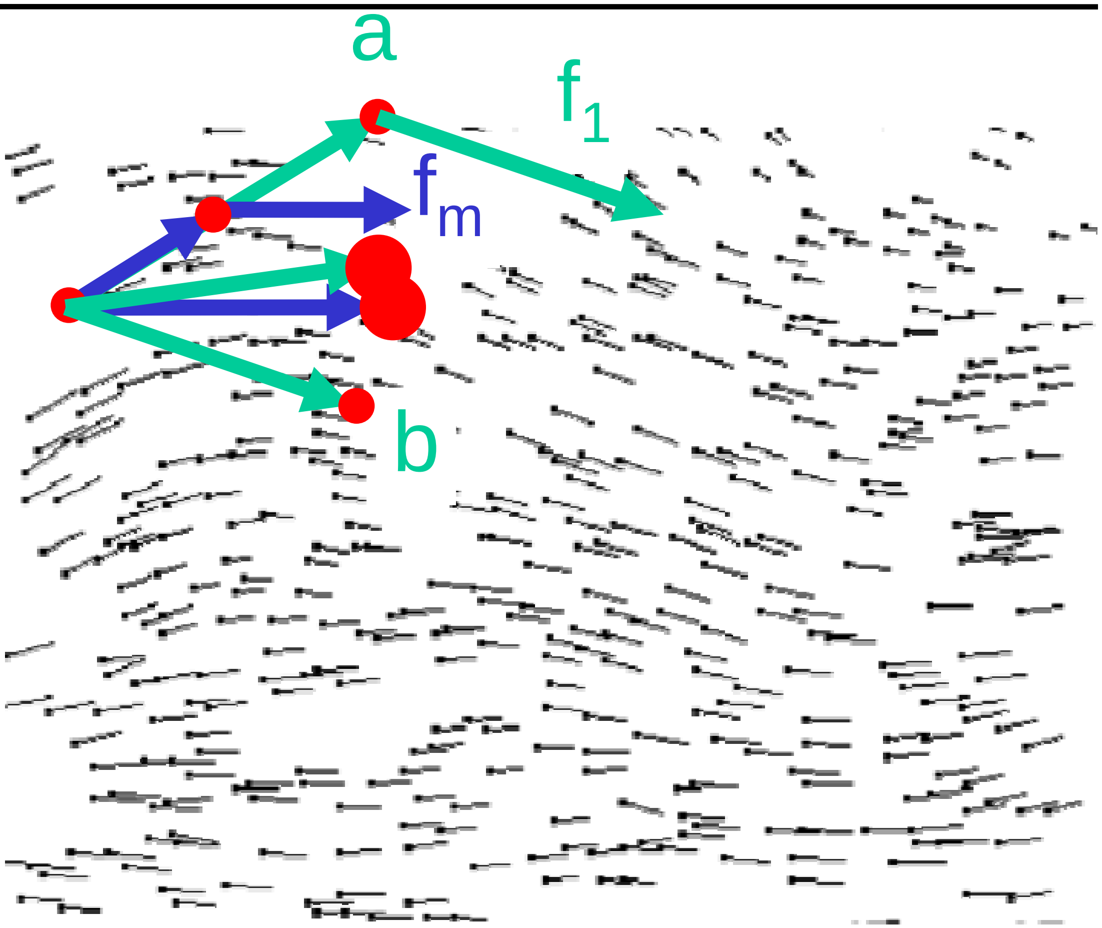
- h > 1/k: oscillate. h > 2/k: explode!



From the SIGGRAPH PBM notes

Integrator Comparison

- Midpoint:
  - ½ Euler step
  - evaluate f_m
  - full step using f_m
- Trapezoid:
  - Euler step (a)
  - evaluate f_1
  - full step using f_1 (b)
  - average (a) and (b)
- Better than Euler but still a speed limit



Midpoint Speed Limit

- $x' = -kx$
- First half Euler step: $x_m = x - 0.5\ hkx = x(1 - 0.5\ hk)$
- Read derivative at x_m: $f_m = -kx_m = -k(1 - 0.5\ hk)x$
- Apply derivative at origin:
  $x(t+h) = x + hf_m = x - hk(1 - 0.5hk)x = x(1 - hk + 0.5\ h^2k^2)$
- Looks a lot like Taylor...
- We want $0 < x(t+h)/x(t) < 1$

  $-hk + 0.5\ h^2k^2 < 0$

  $hk(-1 + 0.5\ hk) < 0$

  For positive values of h & k => $h < 2/k$
- Twice the speed limit of Euler

Stiffness

- In more complex systems, step size is limited by the largest k.
  - One stiff spring can ruin things for everyone else!

- Systems that have some big k values are called *stiff systems*.

- In the general case, k values are eigenvalues of the local Jacobian!

From the siggraph PBM notes

Stiffness

Questions?

- In more complex systems,
  step size is limited by the largest *k*.
  - One stiff spring can ruin things for everyone else!

- Systems that have some big *k* values
  are called *stiff systems*.

- In the general case, *k* values are eigenvalues of the local Jacobian!

From the siggraph PBM notes

Explicit Integration

- So far, we have seen **explicit** Euler
  - $\boldsymbol{X}(t+h) = \boldsymbol{X}(t) + h\,\boldsymbol{X}'(\boldsymbol{t})$

- We also saw midpoint and trapezoid methods
  - They took small Euler steps, re-evaluated $\boldsymbol{X}'$ there, and used some combination of these to step away from the original $\boldsymbol{X}(t)$.
  - Yields higher accuracy, but not impervious to stiffness (twice the speed limit of Euler)

Implicit Integration

- So far, we have seen **explicit** Euler
  - $\boldsymbol{X}(t+h) = \boldsymbol{X}(t) + h\,\boldsymbol{X}'(\boldsymbol{t})$
- Implicit Euler uses the derivative at the destination!
  - $\boldsymbol{X}(t+h) = \boldsymbol{X}(t) + h\,\boldsymbol{X}'(\boldsymbol{t+h})$
  - It is implicit because we do not have **X**'(t+h), it depends on where we go (HUH?)
  - aka backward Euler

Difference with Trapezoid

- Trapezoid
  - take “fake” Euler step
  - read derivative at “fake” destination
- Implicit Euler
  - take derivative at the real destination
  - harder because the derivative depends on the destination and the destination depends on the derivative

Implicit Integration

- Implicit Euler uses the derivative at the destination!
  - $\boldsymbol{X}(t+h) = \boldsymbol{X}(t) + h\,\boldsymbol{X}'(\boldsymbol{t+h})$
  - It is implicit because we do not have $\boldsymbol{X}'(t+h)$, it depends on where we go (HUH?)
  - Two situations
    - $\boldsymbol{X}'$ is known analytically and everything is closed form (*doesn't happen in practice*)
    - **We need some form of iterative non-linear solver.**

Simple Closed Form Case

- Remember our model problem: $x' = -kx$
  - Exact solution was a decaying exponential $x_0 e^{-kt}$
- Explicit Euler: $x(t+h) = (1-hk)\,x(t)$
  - Here we got the bounds on h to avoid oscillation/explosion

Simple Closed Form Case

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  - Exact solution was a decaying exponential $x_0\, e^{-kt}$

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- Implicit Euler: $x(t+h) = x(\mathrm{t}) + h\, x'(t+h)$

Simple Closed Form Case

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- Implicit Euler: $x(t+h) = x(\mathrm{t}) + h\, x'(t+h)$

$$x(t+h) = x(t) - hk\, x(\mathrm{t+h})$$

$$x(t+h) + hkx(t+h) = x(t)$$

$$x(t+h) = x(t) / (1+hk)$$

  - It is a hyperbola!

Simple Closed Form Case

Implicit Euler is unconditionally stable!

- Explicit Euler: $x(t+h) = (1\text{-}hk)\, x(t)$

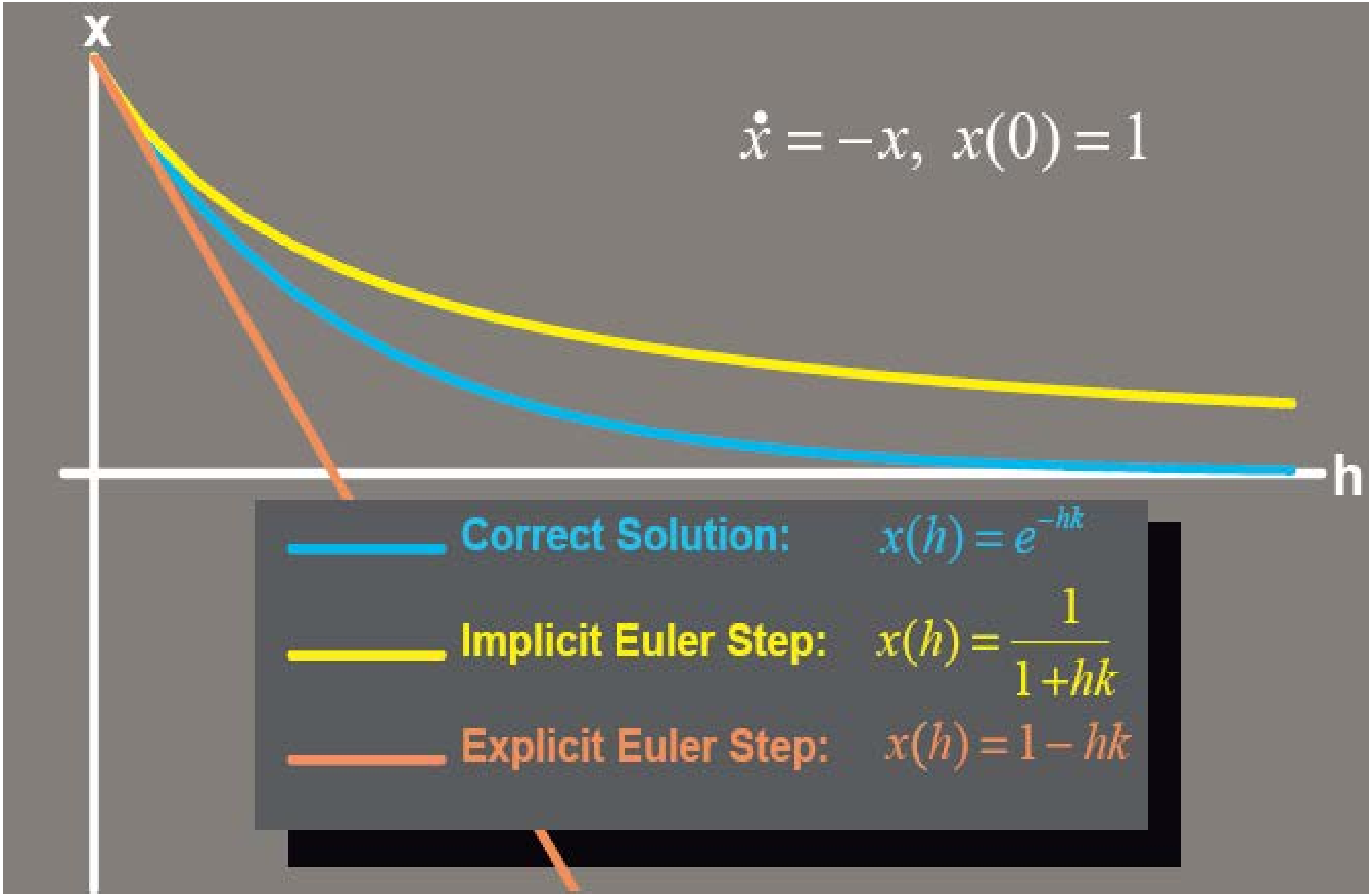
- Implicit Euler: $x(t+h) = x(\mathrm{t}) + h\, x'(t+h)$

$$x(t+h) = x(t) - h\,k\,x(\mathrm{t+h})$$
$$= x(t) \,/\, \boxed{(1+hk)}$$

  – It is a hyperbola!

$1/(1+hk) < 1$,
when h,k > 0

Implicit vs. Explicit



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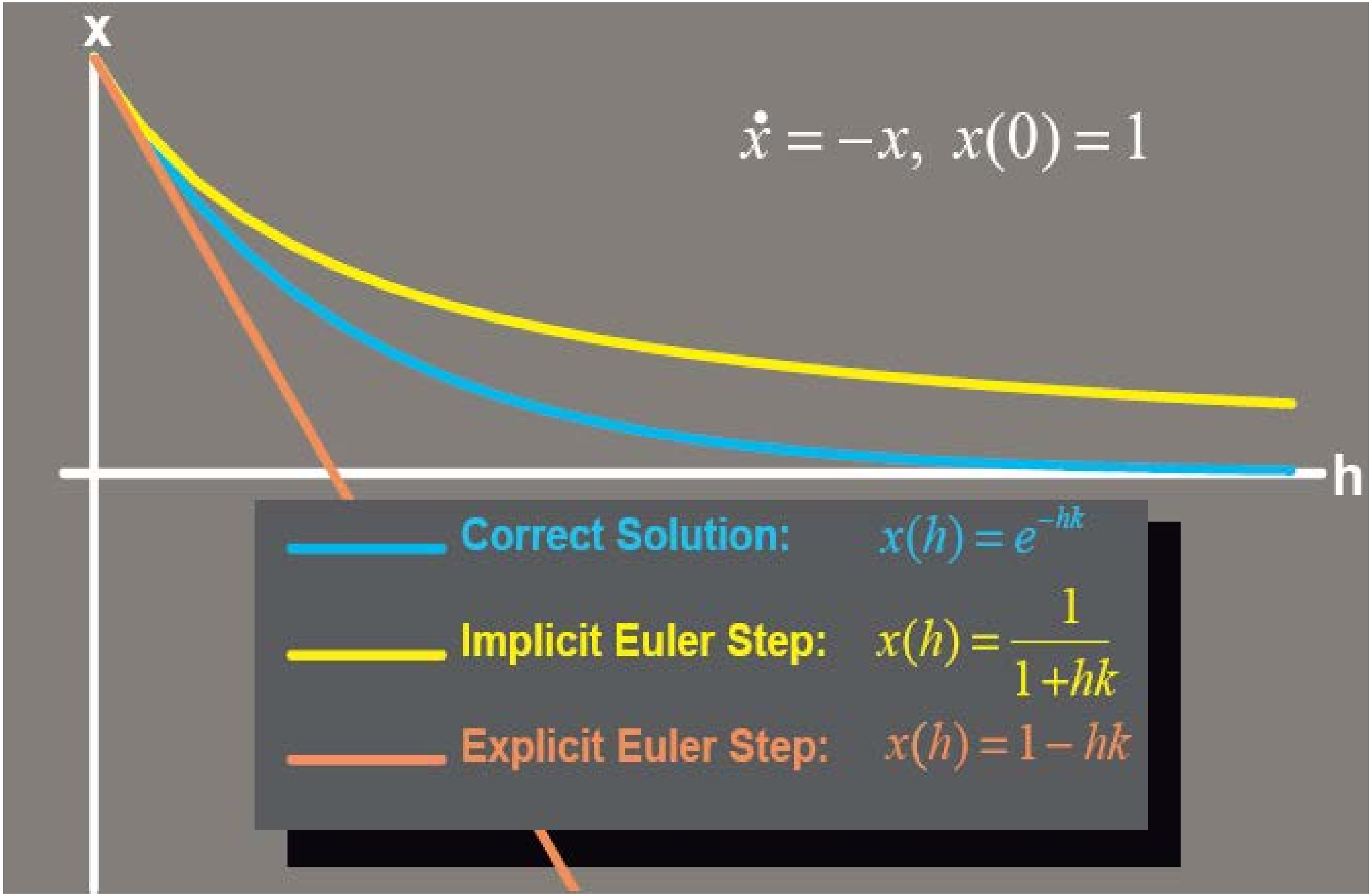
Implicit vs. Explicit

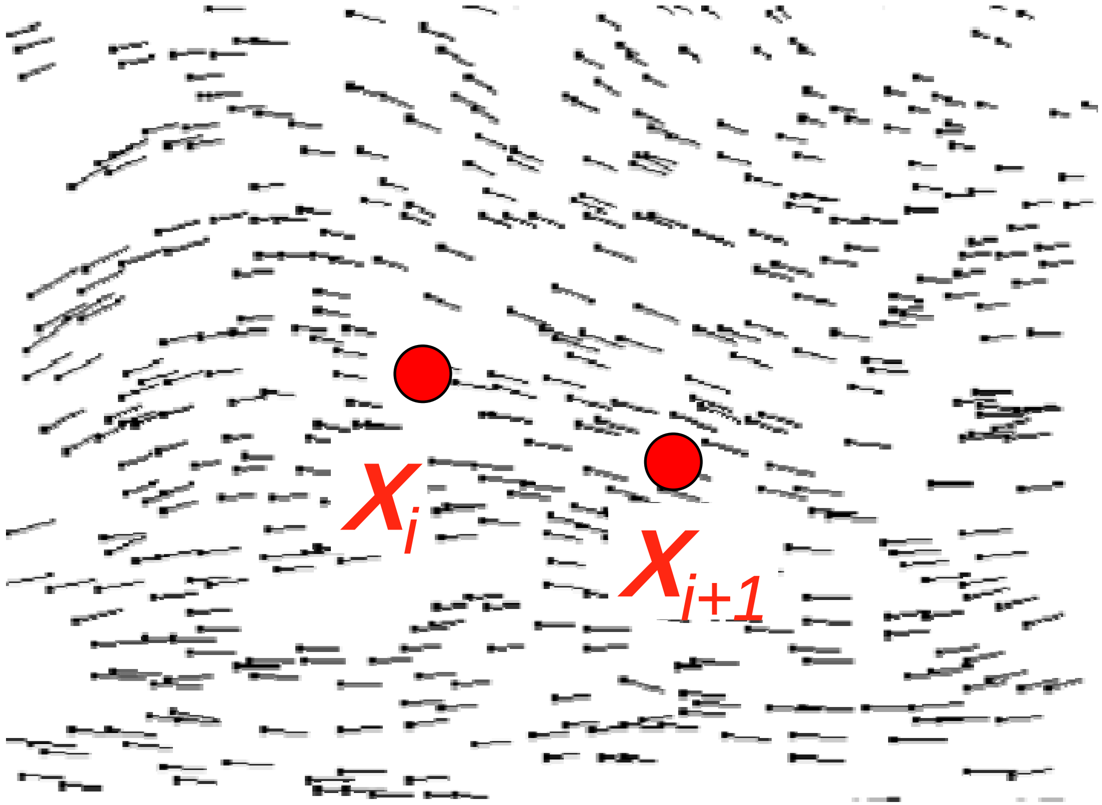
Questions?



From the Siggraph PBM notes

Implicit Euler, Visually

$$\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + h\, f(\, \boldsymbol{X}_{i+1},\, \boldsymbol{t+h}\,)$$
$$\boldsymbol{X}_{i+1} - h\, f(\, \boldsymbol{X}_{i+1},\, \boldsymbol{t+h}\,) = \boldsymbol{X}_i$$

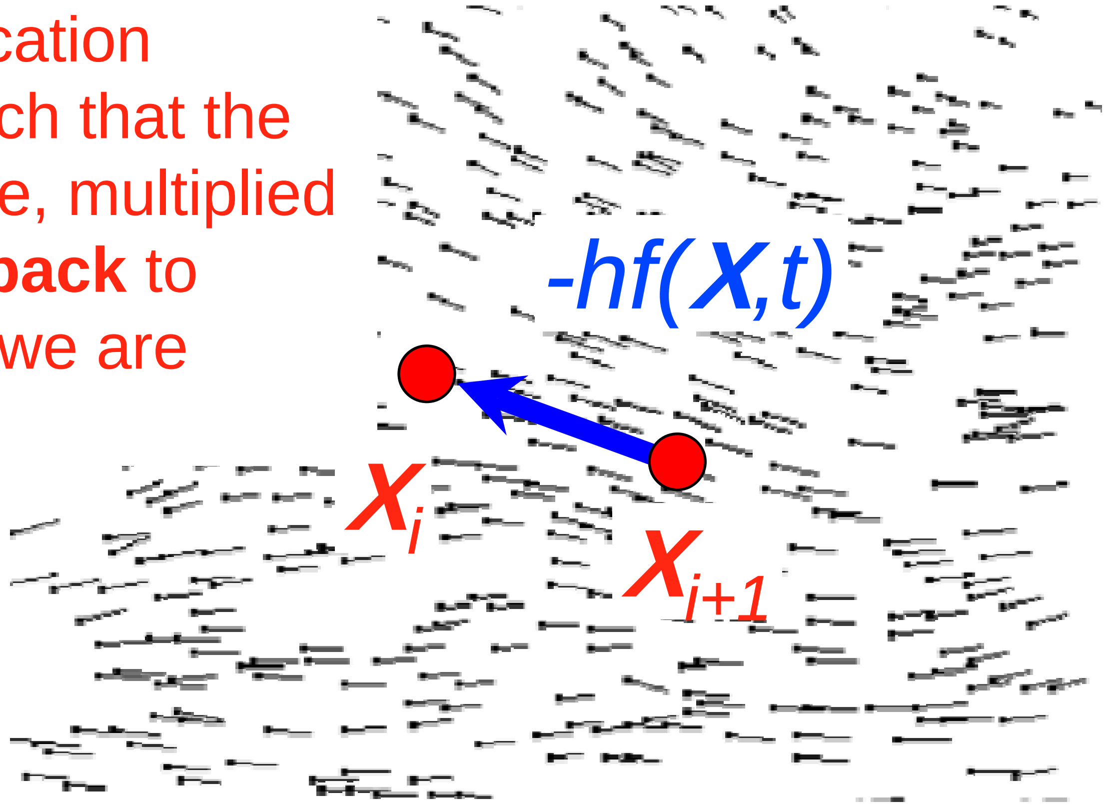


Implicit Euler, Visually

$$\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + h\, f(\, \boldsymbol{X}_{i+1},\ \boldsymbol{t+h}\,)$$

$$\boldsymbol{X}_{i+1} - h\, f(\, \boldsymbol{X}_{i+1},\ \boldsymbol{t+h}\,) = \boldsymbol{X}_i$$

What is the location $X_{i+1}=X(t+h)$ such that the derivative there, multiplied by *-h*, **points back** to $X_i=X(t)$ where we are starting from?



Implicit Euler in 1D

- To simplify, consider only 1D time-invariant systems
  - This means $\boldsymbol{x}' = f(\boldsymbol{x},t) = f(\boldsymbol{x})$ is independent of t
  - Our spring equations satisfy this already

- $x(t+h)=x(t)+dx = x(t)+h\, f(x(t+h))$
- f can be approximated it by 1st order Taylor: $f(x+dx)=f(x)+dxf'(x)+O(dx^2)$
- $x(t+h)=x(t)+h\,[f(x) + dx\, f'(x)]$
- $dx=h\,[f(x) +dx\, f'(x)]$
- $\boxed{dx=hf(x)/[1-hf'(x)]}$
- Pretty much Newton solution

Newton's Method (1D)

- Iterative method for solving non-linear equations

$$f(x) = 0$$

- Start from initial guess x_0, then iterate

Newton’s Method (1D)

- Iterative method for solving non-linear equations

$$f(x) = 0$$

- Start from initial guess x_0, then iterate

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Also called *Newton-Raphson iteration*

Newton's Method (1D)

- Iterative method for solving non-linear equations

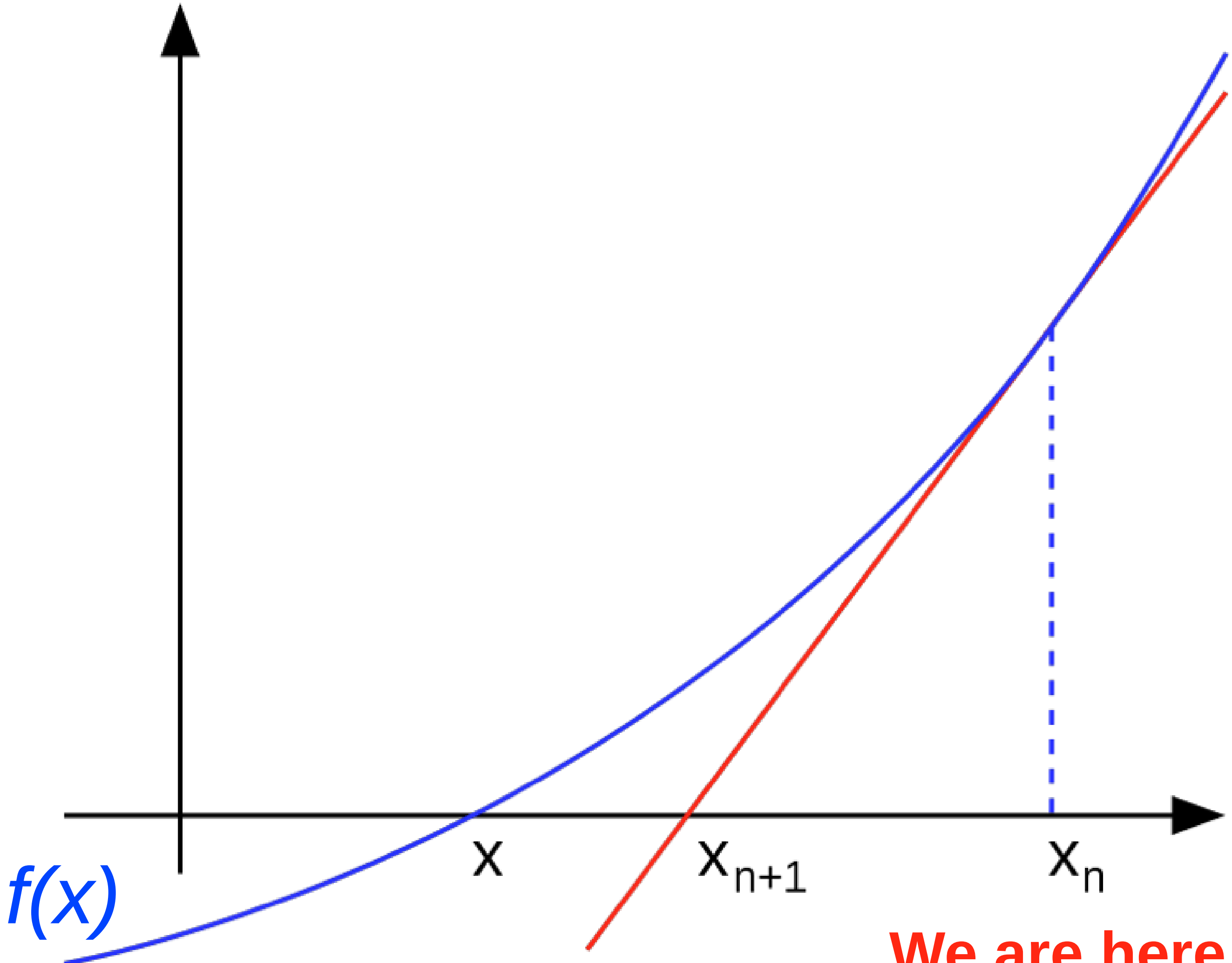
$$f(x) = 0$$

- Start from initial guess x_0, then iterate

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Leftrightarrow \; f'(x_i)\underbrace{(x_{i+1} - x_i)}_{\text{one step}} = -f(x_i)$$

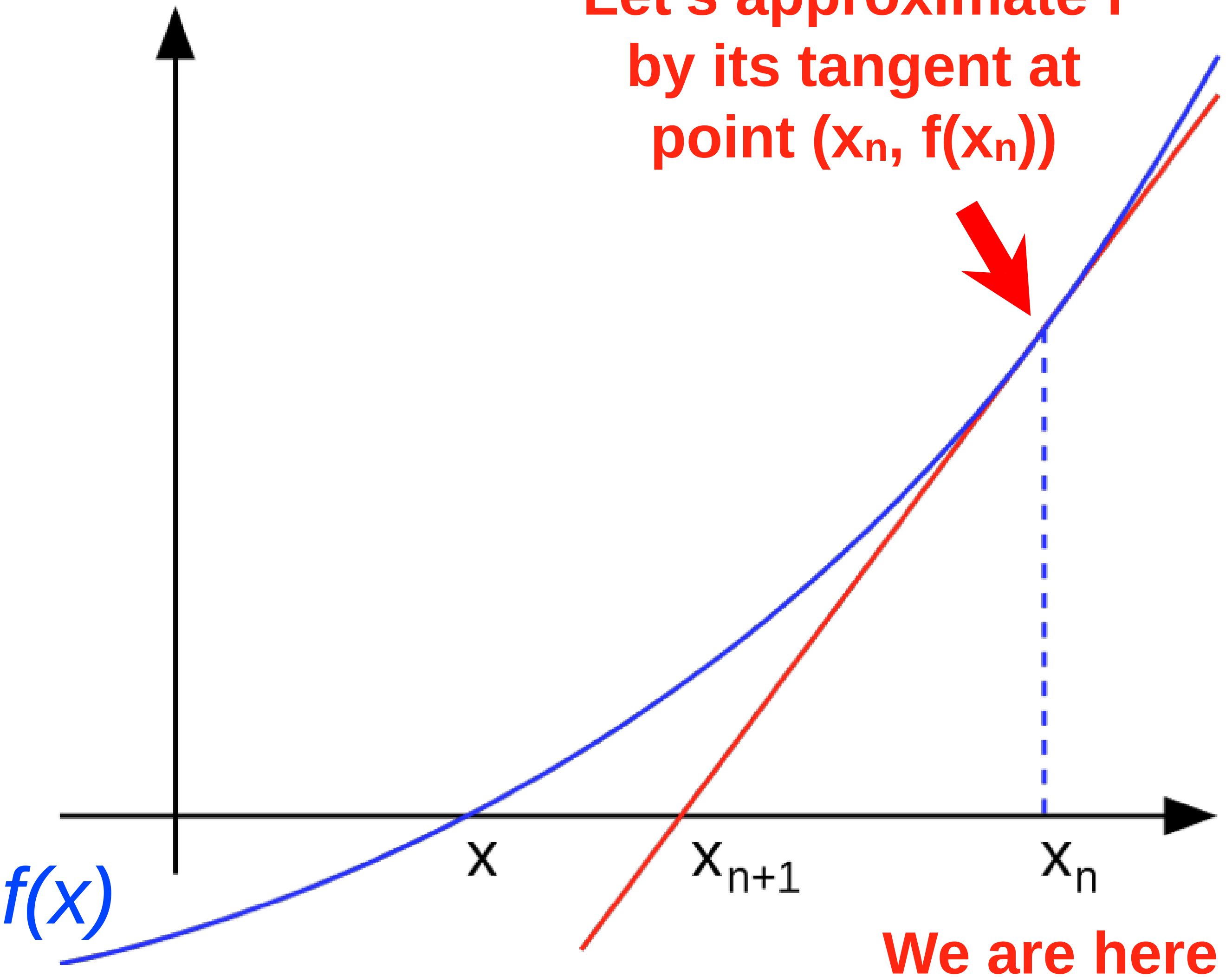
Newton, Visually



Wikipedia users Olegalexandrov, Pbroks13

Newton, Visually

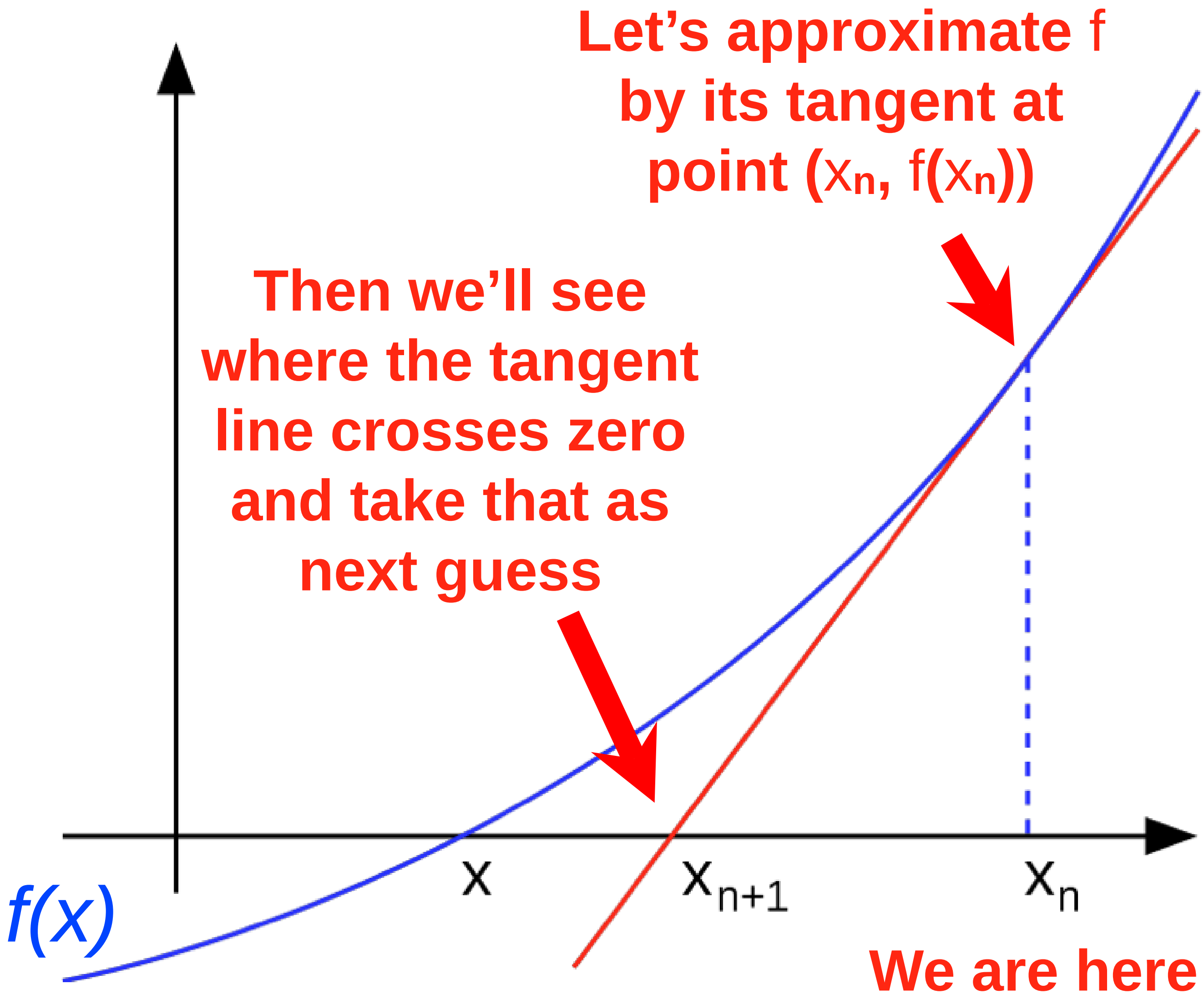
Let's approximate f by its tangent at point $(x_n, f(x_n))$



We are here

Wikipedia users Olegalexandrov, Pbroks13

Newton, Visually



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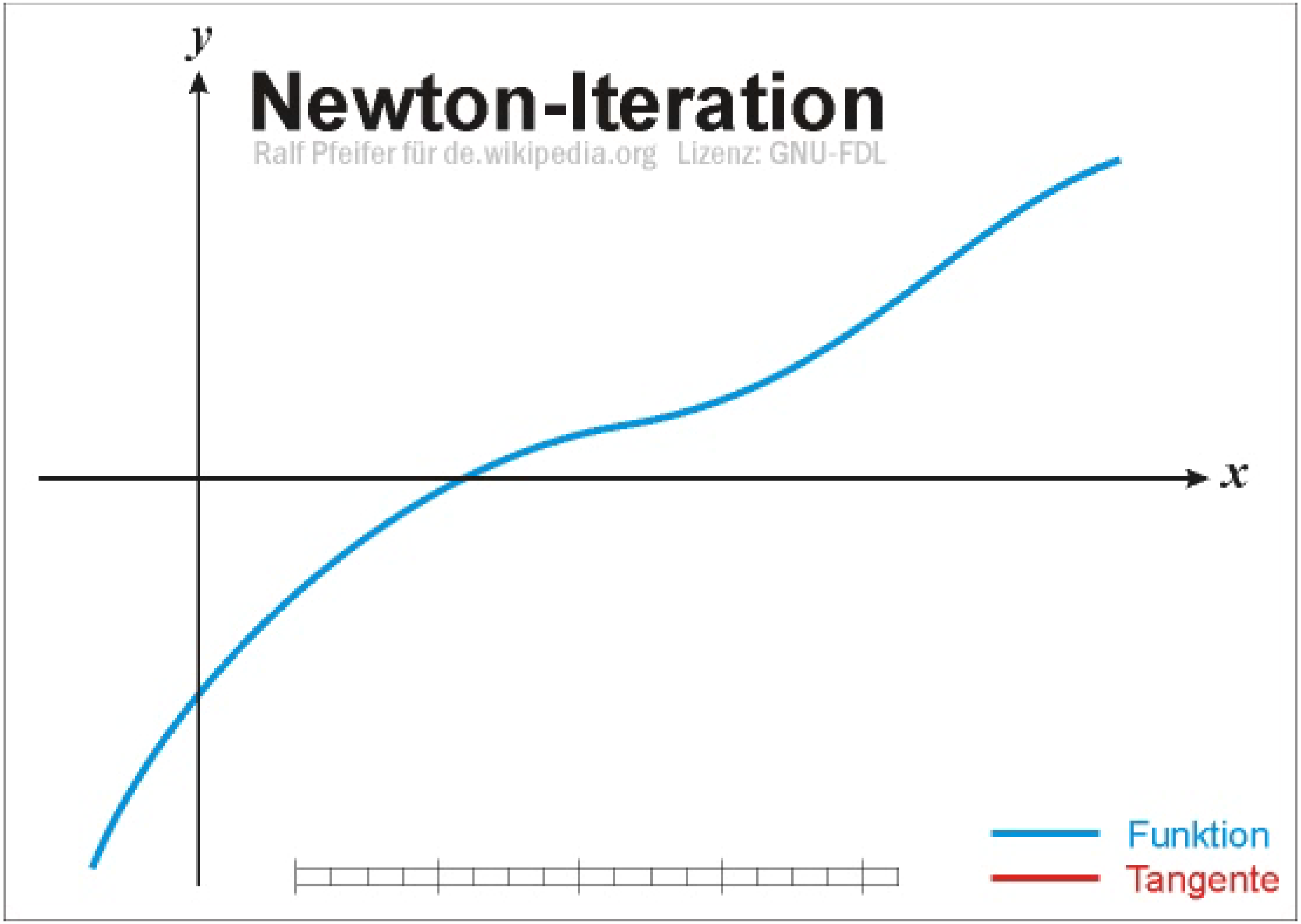
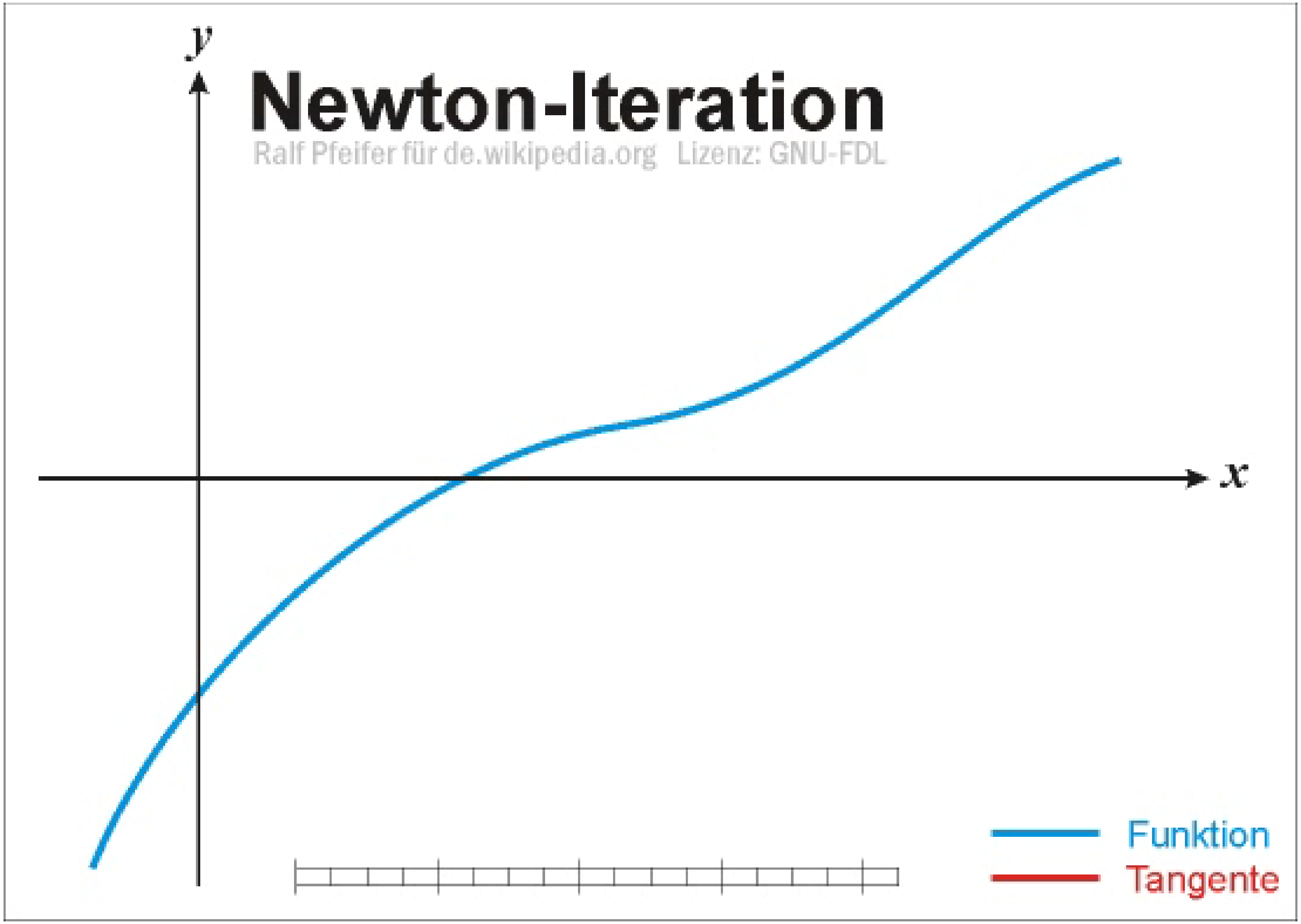
Newton, Visually



Newton, Visually Questions?



Implicit Euler and Large Systems

- To simplify, consider only time-invariant systems
  - This means $\boldsymbol{X}' = f(\boldsymbol{X},t) = f(\boldsymbol{X})$ is independent of t
  - Our spring equations satisfy this already

- Implicit Euler with *N*-*D* phase space:
  $\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + h\, f(\boldsymbol{X}_{i+1})$

Implicit Euler and Large Systems

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- Implicit Euler with N-D phase space:
  $\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + h\, f(\boldsymbol{X}_{i+1})$

- Non-linear equation,
  unknown $\boldsymbol{X}_{i+1}$ on both the LHS and the RHS

Newton's Method – N Dimensions

- 1D: $f'(x_i)(x_{i+1} - x_i) = -f(x_i)$

- Now locations $\boldsymbol{X}_i$, $\boldsymbol{X}_{i+1}$ and F are N-D
- N-D Newton step is just like 1D:

$$J_F(\boldsymbol{X}_i)(\boldsymbol{X}_{i+1} - \boldsymbol{X}_i) = -F(\boldsymbol{X}_i)$$

NxN Jacobian matrix replaces f'

unknown N-D step from current to next guess

Newton's Method – N Dimensions

- Now locations $\boldsymbol{X}_i$, $\boldsymbol{X}_{i+1}$ and F are N-D
- Newton solution of $F(\boldsymbol{X}_{i+1}) = 0$ is just like 1D:

$$J_F(\boldsymbol{X}_i)(\boldsymbol{X}_{i+1} - \boldsymbol{X}_i) = -F(\boldsymbol{X}_i)$$

NxN Jacobian matrix

unknown N-D step from current to next guess

$$J_F(\boldsymbol{X}_i) = \left[\frac{\partial F}{\partial X}\right]_{\boldsymbol{X}_i}$$

- Must solve a linear system at each step of Newton iteration
  - Note that also Jacobian changes for each step

Newton's Method – N Dimensions

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Questions?

Implicit Euler – N Dimensions

- Implicit Euler with *N*-*D* phase space:
  $\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + h\, f(\boldsymbol{X}_{i+1})$

- Let's rewrite this as $F(\boldsymbol{Y}) = 0,$ with

$$F(\boldsymbol{Y}) = \boldsymbol{Y} - \boldsymbol{X}_i - hf(\boldsymbol{Y})$$

Implicit Euler – N Dimensions

- Implicit Euler with *N*-*D* phase space:
  $\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + h\, f(\, \boldsymbol{X}_{i+1}\,)$

- Let's rewrite this as $F(\boldsymbol{Y}) = 0,$ with

$$F(\boldsymbol{Y}) = \boldsymbol{Y} - \boldsymbol{X}_i - hf(\boldsymbol{Y})$$

- Then the $\boldsymbol{Y}$ that solves $F(\boldsymbol{Y})=0$ *is* $\boldsymbol{X}_{i+1}$

Implicit Euler – N Dimensions

$$F(\boldsymbol{Y}) = \boldsymbol{Y} - \boldsymbol{X}_i - hf(\boldsymbol{Y})$$

Y is variable **X**$_i$ is fixed

- Then iterate
  - Initial guess $\boldsymbol{Y}_0 = \boldsymbol{X}_i$ (or result of explicit method)
  - For each step, solve $J_F(\boldsymbol{Y}_i)\Delta\boldsymbol{Y} = -F(\boldsymbol{Y}_i)$
  - Then set $\boldsymbol{Y}_{i+1} = \boldsymbol{Y}_i + \Delta\boldsymbol{Y}$

What is the Jacobian?

$$F(\boldsymbol{Y}) = \boldsymbol{Y} - \boldsymbol{X}_i - hf(\boldsymbol{Y})$$

- Simple partial differentiation...

$$J_F(\boldsymbol{Y}) = \left[\frac{\partial F}{\partial \boldsymbol{Y}}\right] = \boldsymbol{I} - hJ_f(\boldsymbol{Y})$$

- Where $J_f(\boldsymbol{Y}) = \left[\frac{\partial f}{\partial \boldsymbol{Y}}\right]$

The Jacobian of the Force function f

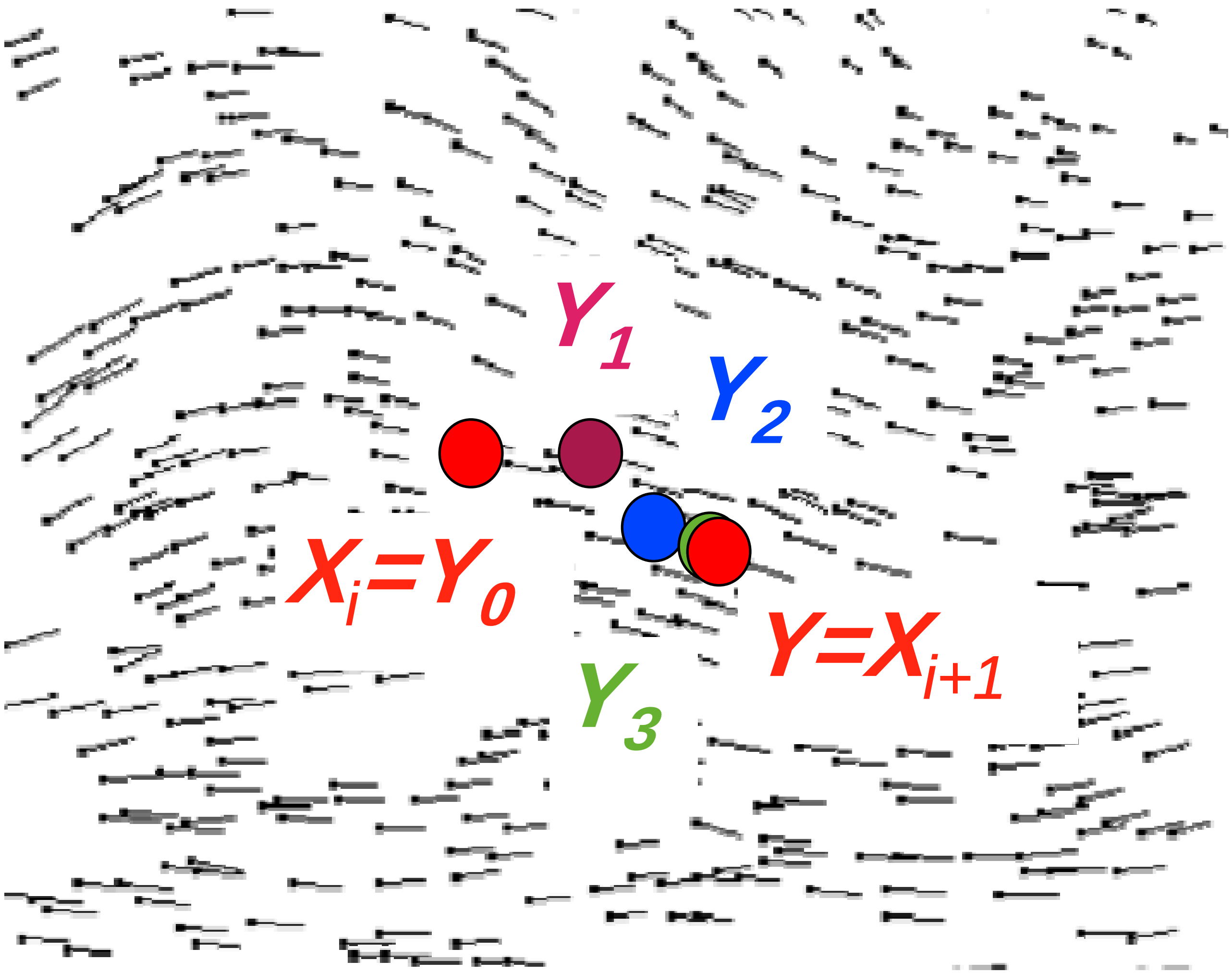
Putting It All Together

- Iterate until convergence
  - Initial guess $\boldsymbol{Y}_0 = \boldsymbol{X}_i$ (or result of explicit method)
  - For each step, solve

$$\Big(\boldsymbol{I} - hJ_f(\boldsymbol{Y}_i)\Big)\Delta\boldsymbol{Y} = -F(\boldsymbol{Y}_i)$$

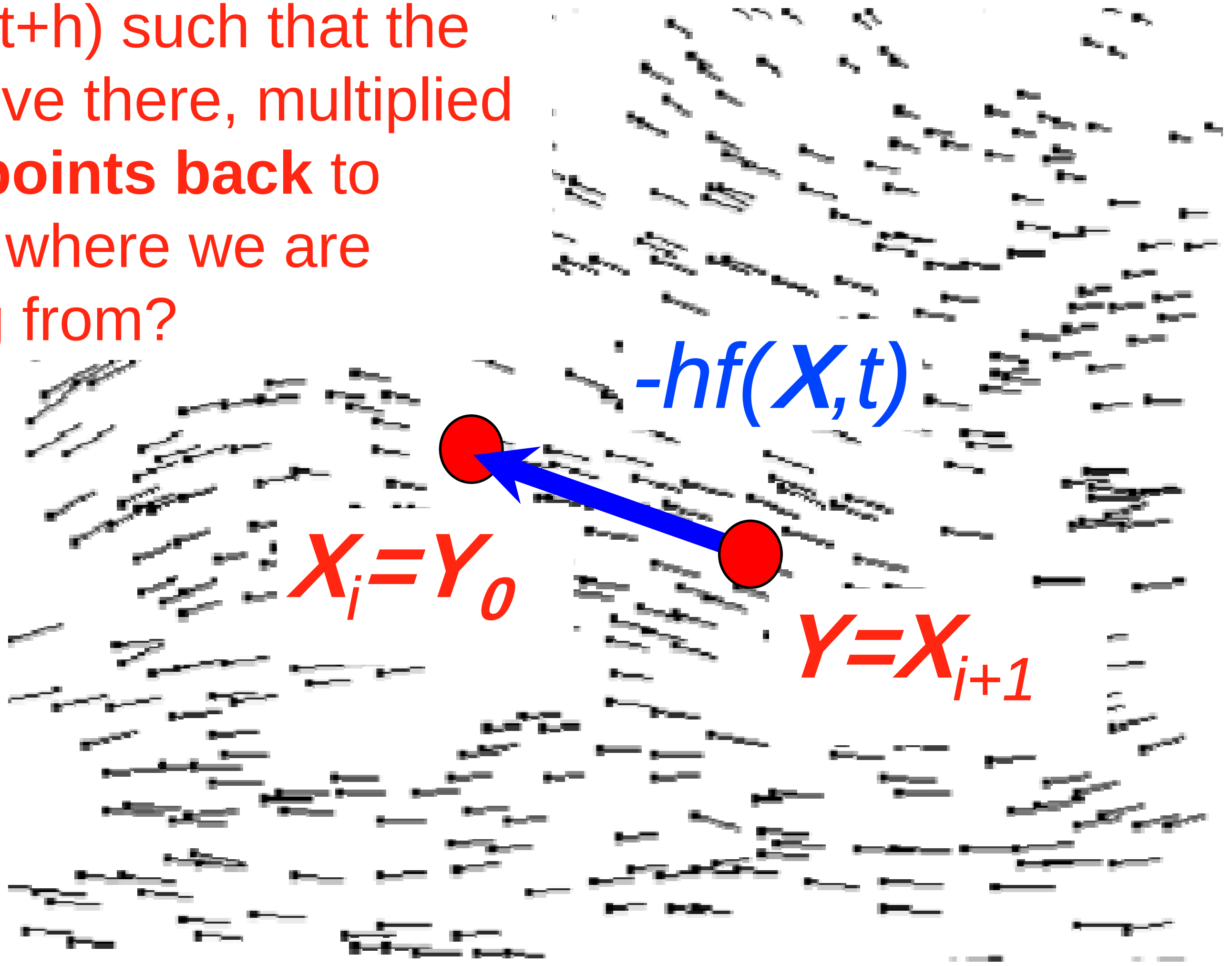
  - Then set $\boldsymbol{Y}_{i+1} = \boldsymbol{Y}_i + \Delta\boldsymbol{Y}$

Implicit Euler with Newton, Visually



Implicit Euler with Newton, Visually

What is the location $\mathbf{X}_{i+1}$=$\mathbf{X}$(t+h) such that the derivative there, multiplied by *-h*, **points back** to $\mathbf{X}_i$=$\mathbf{X}$(t) where we are starting from?



One-Step Cheat

- Often, the 1st Newton step may suffice
  - People often implement Implicit Euler using only one step.
  - This amounts to solving the system

$$\left(I - h\frac{\partial f}{\partial X}\right)\Delta X = hf(X)$$

where the Jacobian and *f* are evaluated at $\boldsymbol{X}_i$, and we are using $\boldsymbol{X}_i$ as an initial guess.

One-Step Cheat Questions?

- Often, the 1st Newton step may suffice
  - People often implement Implicit Euler using only one step.
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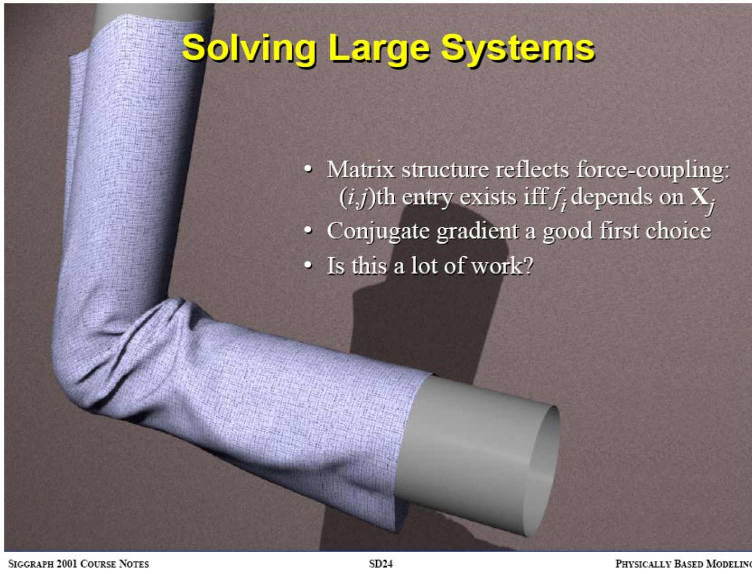
$$\left(I - h\frac{\partial f}{\partial X}\right)\Delta X = hf(X)$$

where the Jacobian and f are evaluated at $\boldsymbol{X}_i$, and we are using $\boldsymbol{X}_i$ as an initial guess.

Good News

- The Jacobian matrix J_f is usually sparse
  - Only few non-zero entries per row
  - E.g. the derivative of a spring force only depends on the adjacent masses' positions
- Makes the system cheaper to solve
  - Don't invert the Jacobian!
  - Use iterative matrix solvers like conjugate gradient, perhaps with preconditioning, etc.

$$\Big(\boldsymbol{I} - J_f(\boldsymbol{Y}_i)\Big)\Delta\boldsymbol{Y} = -F(\boldsymbol{Y}_i)$$



Implicit Euler Pros & Cons

- Pro: Stability!

- Cons:
  - Need to solve a linear system at each step
  - Stability comes at the cost of “numerical viscosity”, but then again, you do not have to worry about explosions.
    - Recall exp vs. hyperbola

- Note that accuracy is not improved
  - error still $O(h)$
  - There are lots and lots of implicit methods out there!

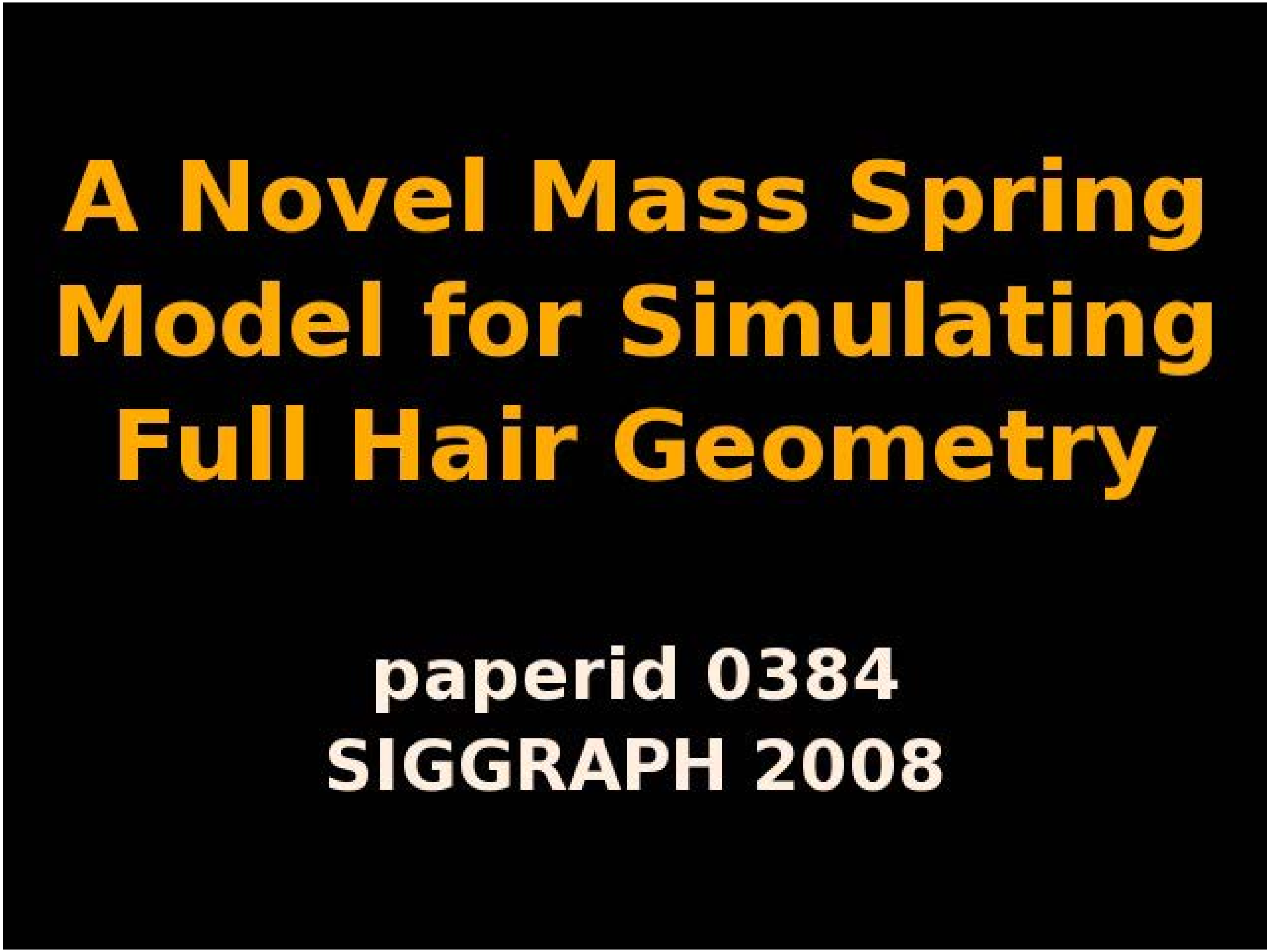
Reference

- **Large steps in cloth simulation**
- David Baraff   Andrew Witkin
- http://portal.acm.org/citation.cfm?id=280821

Figure 5 (top row): Dancer with short skirt; frames 110, 136 and 155. Figure 6 (middle row): Dancer with long skirt; frames 185, 215 and 236. Figure 7 (bottom row): Closeups from figures 4 and 6.

A Mass Spring Model for Hair Simulation

Selle, A., Lentine, M., G., and Fedkiw



Simulating Knitted Cloth at the Yarn Level

Jonathan Kaldor, Doug L. James, and Steve Marschner

Efficient Simulation of Inextensible Cloth

Rony Goldenthal, David Harmon, Raanan Fattal, Michel Bercovier, Eitan Grinspun

Questions?