1. Not doing question 1.

2. (a) The formula for a general linear multistep method of order k is:

$$\sum_{i=0}^{l} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}$$

The method is explicit if $\beta_k = 0$, and implicit if $\beta_k \neq 0$. If the method is implicit, then the value $y_n + k$ must be computed using:

$$y_{n+k} = h\beta_k f(x_{n+k}, y_{n+k}) + h \sum_{i=0}^{k-1} \beta_i f_{n+i} - \sum_{i=0}^{k-1} \alpha_i y_{n+i}$$

(b) In order to determine the order of convergence of a linear multistep method, we must determine the constants $C_0, \ldots, C_j$:

$$C_0 = \sum_{i=0}^{k} \alpha_i$$
$$C_1 = \sum_{i=1}^{k} i\alpha_i - \sum_{i=0}^{k} \beta_i$$
$$C_j = \frac{1}{j!} \sum_{i=1}^{k} i^j \alpha_i - \frac{1}{(j-1)!} \sum_{i=1}^{k} j^{i-1} \beta_i$$

The order of the method p is given by $C_0 = ... = C_p = 0, C_{p+1} \neq 0$.
For consistency, p must be greater than or equal to 1.
For stability, the roots of the first characteristic polynomial must have modulus less than or equal to one and any root that does have a modulus equal to one must be simple. The first characteristic polynomial is given by:

$$p(\zeta) = \sum_{i=0}^{k} \alpha_i \zeta^i$$

(c) i. In order to determine the order of method two, we must find the values of α and β and the constants:

$$\alpha_0 = 0$$
$$\alpha_1 = \frac{1}{6}$$
$$\alpha_2 = -1$$
$$\alpha_3 = \frac{1}{2}$$
$$\alpha_4 = \frac{1}{3}$$
$$\beta_0 = 0$$
$$\beta_1 = 0$$
$$\beta_2 = 1$$

Now we can find the constants:

$$
\begin{aligned}
C_0 &= \sum_{i=0}^{k} \alpha_i = \frac{1}{6} - 1 + \frac{1}{2} + \frac{1}{3} = 0 \\
C_1 &= \sum_{i=1}^{k} i\alpha_i - \sum_{i=0}^{k} \beta_i = \frac{1}{6} - 2(-1) + \frac{3}{2} + \frac{3}{3} - 1 = -\frac{1}{3}
\end{aligned}
$$

Therefore we can see that the method is not consistent as it is of order 0.
The first characteristic polynomial is:

$$
p(\zeta) = \frac{1}{3}\zeta^3 + \frac{1}{2}\zeta^2 - \zeta + \frac{1}{6}
$$

Since we can (relatively easily) see that $\zeta = 1$ is a root, we can use polynomial division to get the rest:

$$
\begin{array}{rl}
 & \frac{1}{3}x^2 + \frac{5}{6}x - \frac{1}{6} \\ \cline{2-2}
x-1 \big) & \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + \frac{1}{6} \\
 & -\frac{1}{3}x^3 + \frac{1}{3}x^2 \\ \cline{2-2}
 & \frac{5}{6}x^2 - x \\
 & -\frac{5}{6}x^2 + \frac{5}{6}x \\ \cline{2-2}
 & -\frac{1}{6}x + \frac{1}{6} \\
 & \frac{1}{6}x - \frac{1}{6} \\ \cline{2-2}
 & 0
\end{array}
$$

Now we have $(\zeta - 1)\left(\frac{\zeta^2}{3} + \frac{5\zeta}{6} - \frac{1}{6}\right)$, we can use the quadratic formula to find the rest of the roots:

$$
\begin{aligned}
& a = \frac{1}{3}, b = \frac{5}{6}, c = \frac{-1}{6} \\
& \zeta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{15 \pm \sqrt{2 - 0}}{12} \\
& \zeta = 2.556, \zeta = 0.0453
\end{aligned}
$$

Since 2.556 is greater than 1, the method is not stable.

3.