SF2520 — Applied numerical methods

Lecture 13

Numerical methods for hyperbolic equations

Anna Nissen
Numerical analysis
Department of Mathematics, KTH

2023-10-31

Today’s lecture

- Finite difference methods for hyperbolic equations
- Numerical analysis, smooth solutions
- Numerical analysis, discontinuous solutions

Hyperbolic PDEs

We want to solve the advection equation

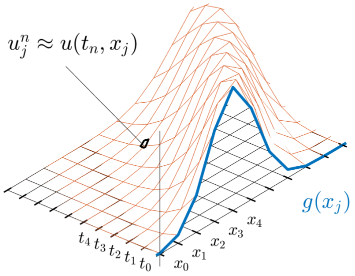
$$
\begin{aligned}
u_t + a u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0,\\
u(x,0) &= g(x),
\end{aligned}
$$

numerically. (No boundary for now.)

- Semi-discretizations (MoL) path not so practical:
    - Most common methods not on this form.
    - Natural discretizations lead to unstable methods.
    - Absolute stability of MoL often not sufficient for stability of full discretization.
- Instead do full discretization directly,

$$x_j = j\Delta x, \qquad t_n = n\Delta t,$$

$$u_j^n \approx u(x_j, t_n).$$



Hyperbolic PDEs

We want to solve the advection equation

$$
\begin{aligned}
u_t + a u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= g(x).
\end{aligned}
$$

- Introduce the CFL number λ as the relation between time and spatial step,
$$\lambda = \frac{\Delta t}{\Delta x} \qquad (\sigma = \frac{a\Delta t}{\Delta x} \text{ in the book.})$$
- In hyperbolic problems:
  - λ is held constant (typically) as Δt and Δx are refined.
    (I.e. not $\Delta t/\Delta x^2$ as in parabolic case with explicit methods.)
  - Discretizations are not stiff so explicit methods are most often used.
    (I.e. not implicit methods as in parabolic case.)

Examples of methods

1. A cautionary example

   Consider the natural discretization

$$\underbrace{\frac{u_j^{n+1}-u_j^n}{\Delta t}}_{\approx u_t}+a\underbrace{\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}}_{\approx u_x}=0,$$

   or

$$u_j^{n+1}=u_j^n-a\frac{\lambda}{2}(u_{j+1}^n-u_{j-1}^n).$$

   - Based on forward differences in time and central differences in space ("FTCS")
   - This method is unstable for all fixed $\lambda=\Delta t/\Delta x$. It cannot be used to solve the advection equation.

2. **Lax–Friedrichs**

   Replace u_j^n in the previous method by the average of u_{j+1}^n and u_{j-1}^n,

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - a\frac{\lambda}{2}(u_{j+1}^n - u_{j-1}^n).$$

   - This is stable for $|a|\lambda \leq 1$.

3. **Upwind**

Replace central differences in unstable method by backward/forward differences ("FTBS"/"FTFS")

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \begin{cases} a\frac{u_j^n - u_{j-1}^n}{\Delta x}, & a > 0, \\ a\frac{u_{j+1}^n - u_j^n}{\Delta x}, & a < 0, \end{cases} = 0$$

or

$$u_j^{n+1} = \begin{cases} u_j^n - a\lambda(u_j^n - u_{j-1}^n), & a > 0, \\ u_j^n - a\lambda(u_{j+1}^n - u_j^n), & a < 0. \end{cases}$$

- This is stable for $|a|\lambda \leq 1$.
- Change makes method first order in space. Not big problem since it is only first order in time anyway.
- When $a > 0$ the approximation u_j^{n+1} only depends on the values in the points to the left: u_j^n and u_{j-1}^n. This mimics the exact solution where $u(x,t) = g(x - at)$ only depends on values to the left.
- This property is important when discontinuous solutions are computed. (It is e.g. not true for Lax–Friedrichs.)

Numerical methods

4. **Lax–Wendroff**

   Subtract a small difference approximating u_{xx}

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}-\underbrace{\Delta t\frac{a^2}{2}\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}}_{\approx \Delta t\ \frac{a^2}{2}u_{xx}}=0,$$

   or

$$u_j^{n+1}=u_j^n-a\frac{\lambda}{2}(u_{j+1}^n-u_{j-1}^n)+a^2\frac{\lambda^2}{2}(u_{j+1}^n-2u_j^n+u_{j-1}^n).$$

   - This is stable for $|a|\lambda\leq 1$.
   - Added term vanishes as $\Delta t\to 0$ but precisely cancels the leading part of error. (Note: $u_t=-au_x\Rightarrow u_{tt}=-au_{xt}=a^2u_{xx}$.)
   - Makes the method second order in time and space!

Numerical methods

Linear systems of equations: $\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$

2. **Lax–Friedrichs** (as scalar case, just replace $a \to A$ and $u_j^n \to \boldsymbol{u}_j^n$)

$$\boldsymbol{u}_j^{n+1} = \frac{1}{2}(\boldsymbol{u}_{j+1}^n + \boldsymbol{u}_{j-1}^n) - \frac{\lambda}{2}A(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_{j-1}^n).$$

3. **Upwind**
   Split A-matrix into positive and negative parts

$$A = S\Lambda S^{-1}, \qquad A^+ = S\Lambda^+ S^{-1}, \qquad A^- = S\Lambda^- S^{-1},$$

   where $\Lambda^\pm$ is diagonal with negative/positive eigenvalues replaced by zero. (Note: $\Lambda = \Lambda^+ + \Lambda^-$ and $A = A^+ + A^-$.) Then do backward difference with A^+ and forward difference with A^-:

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \lambda A^+(\boldsymbol{u}_j^n - \boldsymbol{u}_{j-1}^n) - \lambda A^-(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_j^n).$$

4. **Lax–Wendroff** (as scalar case, just replace $a \to A$ and $u_j^n \to \boldsymbol{u}_j^n$)

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\lambda}{2}A(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_{j-1}^n) + \frac{\lambda^2}{2}A^2(\boldsymbol{u}_{j+1}^n - 2\boldsymbol{u}_j + \boldsymbol{u}_{j-1}^n).$$

- These methods are stable if $\max|\mu_k|\lambda \le 1$, where μ_k = eigenvalues of A.

Numerical methods

Linear systems with source functions: $\boldsymbol{u}_t + A\boldsymbol{u}_x = \boldsymbol{f}$

Let $\boldsymbol{f}_j^n := \boldsymbol{f}(\boldsymbol{x}_j, t_n)$. Then

2. **Lax–Friedrichs** (just add a term at the end)

$$\boldsymbol{u}_j^{n+1} = \frac{1}{2}(\boldsymbol{u}_{j+1}^n + \boldsymbol{u}_{j-1}^n) - \frac{\lambda}{2}A(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_{j-1}^n) + \Delta t\, \boldsymbol{f}_j^n$$

3. **Upwind** (just add a term at the end)

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \lambda A^+(\boldsymbol{u}_j^n - \boldsymbol{u}_{j-1}^n) - \lambda A^-(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_j^n) + \Delta t\, \boldsymbol{f}_j^n$$

4. **Lax–Wendroff** (more complicated)

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\lambda}{2}A(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_{j-1}^n) + \frac{\lambda^2}{2}A^2(\boldsymbol{u}_{j+1}^n - 2\boldsymbol{u}_j + \boldsymbol{u}_{j-1}^n) + \Delta t\, \tilde{\boldsymbol{f}}_j^n$$

$$\tilde{\boldsymbol{f}}_j^n = \boldsymbol{f}_j^n - \frac{\lambda}{4}A(\boldsymbol{f}_{j+1}^n - \boldsymbol{f}_{j-1}^n) + \frac{1}{2}(\boldsymbol{f}_j^{n+1} - \boldsymbol{f}_j^n)$$

Nonlinear systems of equations: $\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0$

2. **Lax–Friedrichs** (as scalar case, just replace $au \to \boldsymbol{F}(\boldsymbol{u})$ and $u_j^n \to \boldsymbol{u}_j^n$)

$$\boldsymbol{u}_j^{n+1} = \frac{1}{2}(\boldsymbol{u}_{j+1}^n + \boldsymbol{u}_{j-1}^n) - \frac{\lambda}{2}\Big(\boldsymbol{F}(\boldsymbol{u}_{j+1}^n) - \boldsymbol{F}(\boldsymbol{u}_{j-1}^n)\Big).$$

3. **Upwind**

   Much more complicated. In the nonlinear case the upwind scheme is referred to as the Godunov scheme.

4. **Lax–Wendroff**

   More complicated. A two-step approach typically used.

Numerical analysis of hyperbolic PDEs

For hyperbolic equations we are concerned with convergence in two different settings:

- Convergence for smooth solutions.
  This is governed by the usual order of accuracy and local truncation errors.
- Convergence for non-smooth, discontinuous, solutions.
  Important since discontinuous solutions are common in applications. Governed by "modified equations".

Computer experiments with Lax-Friedrichs (LxF), Upwind and Lax-Wendroff (LW).

Conclusions:

- Smooth solutions: LW much more accurate than LxF and Upwind.
- Non-smooth solutions: Upwind best. Smears discontinuities less than LxF. LW introduces a lot of spurious oscillations.
- All three methods stable iff $|a|\lambda \leq 1$.

Theory for smooth solutions

Define the local truncation error (LTE), denoted ℓ_j^n ,as the residual when the exact solution is entered into the scheme.

Example: Lax–Friedrichs

$$u(x_j, t_{n+1}) = \frac{u(x_{j+1}, t_n) + u(x_{j-1}, t_n)}{2} - a\Delta t \frac{u(x_{j+1}, t_n) - u(x_{j-1}, t_n)}{2\Delta x} + \underbrace{\ell_j^n}_{LTE}.$$

Convergence conditions are given by the Lax Equivalence Theorem, which in this case implies

Theorem (Lax)

Suppose $\lambda = \Delta t/\Delta x$ is fixed as $\Delta t \to 0$. Then

$$|u_j^n - u(x_j, t_n)| = O(\Delta t^p),$$

if the method is

- *Consistent with order p: $|\ell_j^n| = O(\Delta t^{p+1})$ with $p \geq 1$,*
- *Stable: $\sum_j (u_j^n)^2 \leq C \sum_j (u_j^0)^2$, with C independent of Δt and $n \leq \frac{T}{\Delta t}$.*

Theory for smooth solutions

Need to check:

- Consistency/accuracy
  - Done by Taylor expansions
  - Derivations give:

Lax–Friedrichs	$\ell_j^n = O\left(\Delta t(\Delta t + \Delta x^2 + \frac{\Delta x^2}{\Delta t})\right)$
Upwind	$\ell_j^n = O(\Delta t(\Delta t + \Delta x))$
Lax–Wendroff	$\ell_j^n = O(\Delta t(\Delta t^2 + \Delta x^2))$

  - Hence, if $\Delta t/\Delta x =$ fixed:

Lax–Friedrichs	$\lvert u_j^n - u(x_j, t_n)\rvert = O(\Delta x)$,	order of accuracy 1
Upwind	$\lvert u_j^n - u(x_j, t_n)\rvert = O(\Delta x)$,	order of accuracy 1
Lax–Wendroff	$\lvert u_j^n - u(x_j, t_n)\rvert = O(\Delta x^2)$,	order of accuracy 2

- Stability

  Checked by
  - CFL condition $\Rightarrow$ necessary conditions
  - von Neumann analysis $\Rightarrow$ sufficient conditions (next course)

Stability – the CFL condition

CFL condition

For any consistent three-point scheme of the form

$$u_j^{n+1} = c_0 u_{j-1}^n + c_1 u_j^n + c_2 u_{j+1}^n, \qquad u_j^0 = g(x_j),$$

the condition

$$|a|\lambda \leq 1$$

is necessary for stability.

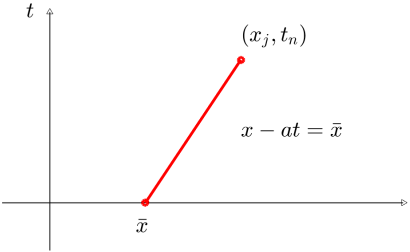
- This condition is also sufficient for Lax–Friedrichs, Upwind and Lax–Wendroff. (Although not for the unstable FTCS scheme.)
- For systems of d equations it is necessary that

$$|\mu_k|\lambda \leq 1, \qquad k = 1, \ldots, d,$$

  where μ_k are the eigenvalues of the system matrix A. (Again sufficient for the methods mentioned above.)

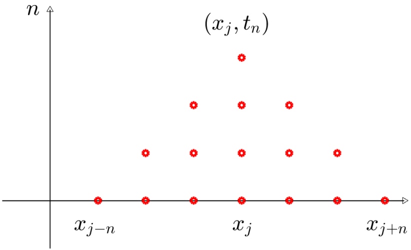
Stability – the CFL condition

Continuous case



Exact solution $u(x_j, t_n)$ depends on value of g in the point $\bar{x}$, only.
(= domain of dependence)

Numerical case



Numerical solution u_j^n depends on value of g in the points $x_{j-n}, \ldots, x_{j+n}$.
(= numerical domain of dependence)

- The method is unstable if $\bar{x} \notin [x_{j-n}, x_{j+n}]$ because then it has no way of "knowing" the correct solution value $g(\bar{x})$ $\Rightarrow$ need

$$x_{j-n} \le \bar{x} \le x_{j+n} \quad \Rightarrow \quad x_j - n\Delta x \le x_j - an\Delta t \le x_j + n\Delta x$$

$$\Rightarrow \quad -\Delta x \le -a\Delta t \le \Delta x \quad \Rightarrow \quad |a|\frac{\Delta t}{\Delta x} \le 1.$$

- Require: domain of dependence $\subset$ numerical domain of dependence.

Theory for non-smooth solutions

- For each method one can introduce a modified equation for which the method is more accurate, and the solution is smooth.
- The idea is to derive a precise expression for the local truncation error, and add this (scaled by Δt) to the original equation. Then the new local truncation error is smaller.
- **Example:** Lax–Friedrichs has local truncation error

$$\ell_j^n = \Delta t \, \frac{a}{2} \Delta x \left(\frac{1}{a\lambda} - a\lambda \right) u_{xx}(x_j, t_n) + O(\Delta t^3).$$

- Therefore, let v solve the modified PDE

$$v_t + a v_x = \frac{a}{2} \Delta x \left(\frac{1}{a\lambda} - a\lambda \right) v_{xx}.$$

- Lax–Friedrichs is now a second order accurate approximation of v,

$$|u_j^n - u(x_j, t_n)| = O(\Delta t), \qquad |u_j^n - v(x_j, t_n)| = O(\Delta t^2)$$

and the difference $u - v$ describes the leading order error term in u_j^n. (Also note that the v_{xx} term makes solution smooth.)

Modified equations

Modified equations for the three schemes above are:

$$v_t + av_x = \frac{a}{2}\Delta x \left(\frac{1}{a\lambda} - a\lambda \right) v_{xx}, \qquad \text{Lax–Friedrichs,}$$

$$v_t + av_x = \frac{|a|}{2}\Delta x \left(1 - |a|\lambda \right) v_{xx}, \qquad \text{Upwind,}$$

$$v_t + av_x = \frac{a}{2}\Delta x^2 \left((a\lambda)^2 - 1 \right) v_{xxx}, \qquad \text{Lax–Wendroff.}$$

Modified equations

Modified equations for Lax–Friedrichs and Upwind are advection-diffusion equations of the form

$$v_t + av_x = \varepsilon v_{xx},$$

with $\varepsilon = \varepsilon_{\mathrm{L}}$ and $\varepsilon = \varepsilon_{\mathrm{U}}$ respectively

$$\varepsilon_{\mathrm{L}} = \frac{a}{2}\Delta x \left(\frac{1}{a\lambda} - a\lambda \right), \qquad \varepsilon_{\mathrm{U}} = \frac{|a|}{2}\Delta x \,(1 - |a|\lambda).$$

- Schemes are unstable if $\varepsilon < 0$ = "backward heat equation". Stability condition $\varepsilon \geq 0$ is again $|a|\lambda \leq 1$.
- Solution to the advection-diffusion equation given by

$$v(x,t) = w(x - at, t), \qquad w_t = \varepsilon w_{xx}, \qquad w(x,0) = g(x).$$

- Diffusion coefficient ε determines amount of smearing of discontinuities. (Exact u has $\varepsilon = 0$.) Upwind smears less than Lax–Friedrichs since

$$1 - |a|\lambda < \frac{1}{a\lambda} - a\lambda$$

- Larger $|a|\lambda$ gives less smearing.
- $a\lambda = 1$ special "magic time step" $\Delta t = \Delta x / a$. Then $\varepsilon = 0$. No smearing occurs, but only works for simple scalar constant coefficient problems.

Modified equations

Modified equation for Lax–Wendroff is a dispersive advection equation (linear KdV) of the form

$$v_t + av_x = \varepsilon v_{xxx},$$

with

$$\varepsilon = \frac{a}{2}\Delta x^2 \left((a\lambda)^2 - 1\right).$$

- Here $\varepsilon = O(\Delta x^2)$, much smaller than for Lax–Friedrichs and Upwind.
- Stability condition not seen directly. (Need even more precise modified equation which also includes v_{xxxx}.)
- Solution to the dispersive equation given by

$$v(x,t) = w(x - at, t), \qquad w_t = \varepsilon w_{xxx}, \qquad w(x,0) = g(x).$$

- Coefficient ε determines amount of dispersion seen at discontinuities.
- Larger $|a|\lambda$ gives less dispersion. Magic time step also here.