Maxwells equations

In 3D we have $\bar{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$, $\bar{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$

$$\nabla \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} B_3 - \frac{\partial}{\partial z} B_2 \\ \frac{\partial}{\partial z} B_1 - \frac{\partial}{\partial x} B_3 \\ \frac{\partial}{\partial x} B_2 - \frac{\partial}{\partial y} B_1 \end{pmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{\partial}{\partial x} B_3 \\ \frac{\partial}{\partial x} B_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial y} B_3 \\ 0 \\ -\frac{\partial}{\partial y} B_1 \end{bmatrix} + \begin{bmatrix} -\frac{\partial}{\partial z} B_2 \\ \frac{\partial}{\partial z} B_1 \\ 0 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{R_1} \frac{\partial}{\partial x} \underbrace{\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}}_{\bar{B}} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{R_2} \frac{\partial}{\partial y} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{R_3} \frac{\partial}{\partial z} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Since $\nabla \times \bar{B} = R_1 \bar{B}_x + R_2 \bar{B}_y + R_3 \bar{B}_z$

we have

$$\nabla \times \bar{E} = R_1 \bar{E}_x + R_2 \bar{E}_y + R_3 \bar{E}_z$$

Let $\bar{u} = \begin{bmatrix} \bar{E} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$

Then

$$\bar{u}_t = \begin{bmatrix} \bar{E}_t \\ \bar{B}_t \end{bmatrix} = \begin{bmatrix} c^2 \nabla \times \bar{B} \\ -\nabla \times \bar{E} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & c^2 R_1 \\ -R_1 & 0 \end{bmatrix}}_{A_1 \text{ (6x6-matrix)}} \underbrace{\begin{bmatrix} \bar{E}_x \\ \bar{B}_x \end{bmatrix}}_{\bar{u}_x}$$

$$+ \underbrace{\begin{pmatrix} 0 & c^2 R_2 \\ -R_2 & 0 \end{pmatrix}}_{A_2} \underbrace{\begin{pmatrix} \bar{E}_y \\ \bar{B}_y \end{pmatrix}}_{\bar{u}_y} + \underbrace{\begin{pmatrix} 0 & c^2 R_3 \\ -R_3 & 0 \end{pmatrix}}_{A_3} \underbrace{\begin{pmatrix} \bar{E}_z \\ \bar{B}_z \end{pmatrix}}_{\bar{u}_z}$$

$$= A_1 \bar{u}_x + A_2 \bar{u}_y + A_3 \bar{u}_z$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c^2 \\ 0 & 0 & 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_2 R_1$

$-R_1$

Symmetric if $c = 1$

Isentropic Euler equations in 1D

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + \varkappa \rho^\gamma)_x = 0 \end{cases}$$

$m = \rho u$, then

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \left(\dfrac{m^2}{\rho} + \varkappa \rho^\gamma\right)_x = 0 \end{cases}$$

With $\bar{u} = \begin{bmatrix} \rho \\ m \end{bmatrix}$ the system is

$$\begin{bmatrix} \rho \\ m \end{bmatrix}_t + \begin{bmatrix} m \\ \frac{m^2}{\rho} + \chi \rho^\gamma \end{bmatrix}_x := \bar{u}_t + \bar{F}(\bar{u})_x = 0$$

Systems of hyperbolic PDEs

Diagonalize $A = S \Lambda S^{-1}$

$$\begin{cases} \bar{u}_t + A \bar{u}_x = 0 \\ \bar{u}(x,0) = \bar{g}(x) \end{cases}$$

Define $\bar{v}(x,t) = S^{-1} u(x,t)$

$$\bar{v}_t = S^{-1} \bar{u}_t = -S^{-1} A \bar{u}_x =$$

$$-S^{-1} \underbrace{S \Lambda S^{-1}}_{A} \bar{u}_x = - \underbrace{S^{-1} S}_{I} \Lambda \underbrace{S^{-1} \bar{u}_x}_{\bar{v}_x}$$

$$= - \Lambda \bar{v}_x$$

Our new system of equations is

$$\begin{cases} \bar{v}_t + \Lambda \bar{v}_x = 0 \\ \bar{v}(x,0) = S^{-1} g(x) \end{cases}$$