Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

Sample space Ω: Random experiment cannot predict the outcome with certainty, e.g. a coin toss. Yet we know all the possible outcomes.

The set of all possible outcomes is denoted by Ω and called Sample space (or state space). A element $\omega \in \Omega$ is called a sample point.

Examples:

1.) Toss a coin once: Outcomes heads H or tails T

$$\Omega = \{H, T\}.$$

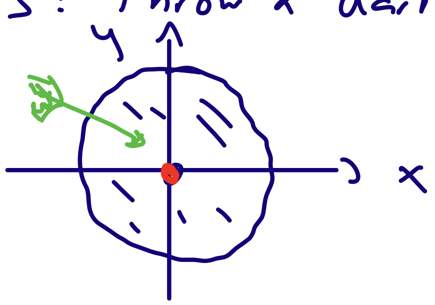
2.) Toss a coin twice

$$\Omega = \{HH, HT, TH, TT\}.$$

3.) Toss a die once $\Omega = \{1, 2, 3, 4, 5, 6\}$.

4.) Toss a die n-times $\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_n) : \omega_i \in \{1, 2, \dots 6\}\}$

5.) Darts: Throw a dart on a circular board:



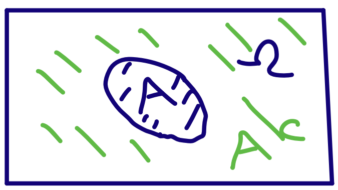
$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

<u>Events</u>: An <u>event</u> is a property that can be observed to hold or not.

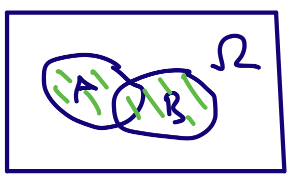
In example 2, $\Omega = \{HH, HT, TH, TT\}$. The event that the first toss is heads is $A = \{HH, HT\}$. Note $A \subset \Omega$.

Mathematically, events are subsets of Ω. If A and B are events, then

- $A^c := \{\omega \in \Omega : \omega \notin A\}$, <u>complement</u> / complementary event.

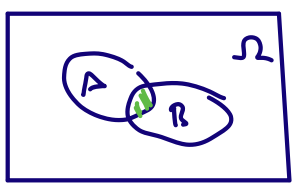


- $A \cup B := \{\omega \in \Omega : \omega \in A$ <u>or</u> $\omega \in B\}$ <u>union</u>



$A \cup B$

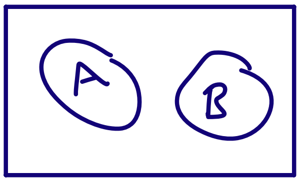
- $A \cap B := \{\omega \in \Omega : \omega \in A$ <u>and</u> $\omega \in B\}$ <u>intersection</u>



$A \cap B$

- $A \subset B$ if $\omega \in A$ implies $\omega \in B$, <u>subset</u>

- Sure event Ω, impossible event ϕ the empty set.
- A and B are <u>disjoint</u> if $A \cap B = \phi$



<u>Lemma</u> (de Morgan formulas), $A_1, A_2, A_3, \ldots$ collection of sets.

$$\left(\bigcup_{k=1}^{n} A_k \right)^c = \bigcap_{k=1}^{n} A_k^c ,$$

$$\left(\bigcap_{k=1}^{n} A_k \right)^c = \bigcup_{k=1}^{n} A_k^c .$$

<u>Remark</u>: We can choose "$n = \infty$":

$$\bigcup_{k=1}^{\infty} A_k = \{ \omega \in \Omega : \omega \in A_k \text{ for at least one } k \}$$

$$\bigcap_{k=1}^{\infty} A_k = \{ \omega \in \Omega : \omega \in A_k \text{ for all } k \}$$

<u>Exercise</u>: Check de Morgan for $n = 2$.

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Let now $\mathcal{A}$ be a collection of events (collection of subsets of Ω). Then $\mathcal{A}$ should be stable under the logical operations above.

<u>Definition</u> (σ-algebra, σ-field). A collection $\mathcal{A}$ of subset of Ω is called a σ-algebra if

1.) $\Omega \in \mathcal{A}$

2.) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

3.) If $A_1, A_2, \ldots, A_n, \ldots$ is a countable sequence of events in $\mathcal{A}$, i.e. $A_k \in \mathcal{A}$ for every k, then

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}.$$

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<u>Remark</u>: $\Omega \in \mathcal{A}$ by 1.) $\Rightarrow \emptyset \in \Omega$, because $\emptyset = (\Omega)^c$, then 2.)

Example: $\Omega = \{HH, TH, HT, TT\}$

- $\mathcal{A}_1 = \{\phi, \Omega\}$ is a σ-algebra "trivial σ-algebra".
- $\mathcal{A}_2 = \{\phi, \{HH\}, \{TH, HT, TT\}, \Omega\}$ is a σ-algebra
- $\mathcal{B} = \{\phi, \{HH\}, \{TT\}, \Omega\}$ is not a σ-algebra.

  e.g. $\{HH\}^c = \{HT, TH, TT\}$ missing.
- $\mathcal{A}_3 = 2^{\Omega}$ = collection of all subsets of Ω is a σ-algebra: "power set"

$$\mathcal{A}_3 = \{\phi, \Omega, \{HH\}, \{HT\}, \ldots, \{HH, HT\}, \ldots, \{HH, HT, TH\} \ldots\}$$

16 elements, $2^4 = 16$.

Note that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3$. //

Terminology: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be σ-algebras over Ω.
If $\forall A \in \mathcal{A}_1$ we have $A \in \mathcal{A}_2$, we write $\mathcal{A}_1 \subset \mathcal{A}_2$.
$\mathcal{A}_1$ is called a σ-subalgebra of $\mathcal{A}_2$.

"$\mathcal{A}_1$ is smaller than $\mathcal{A}_2$"

<u>Example 2:</u> $\Omega = \mathbb{R}$. Borel σ-algebra $\mathcal{B}$ or $\mathcal{B}(\mathbb{R})$ is "the smallest" σ-algebra containing all the open intervals of $\mathbb{R}$.

<u>Facts:</u> • $\mathcal{B}$ contains all $x \in \mathbb{R}$ (exercise)

• $\mathcal{B}$ contains all closed intervals $[a,b] = (a,b) \cup \{a\} \cup \{b\}$. //

<u>Lemma:</u> $\mathcal{A}$ a σ-algebra. Let $A_n \in \mathcal{A}$, $\forall n$. Then

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

meaning: $\mathcal{A}$ is also stable under countable intersections.

<u>Proof:</u> $A_n^c \in \mathcal{A}$ because 2.)

$\bigcup_{n=1}^{\infty} A_n^c \in \mathcal{A}$ because 3.)

$\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{A}$ because 2.)

$$\mathcal{A} \ni \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \underset{\text{de Morgan}}{=} \bigcap_{n=1}^{\infty} (A_n^c)^c = \bigcap_{n=1}^{\infty} A_n$$

used that $(A_n^c)^c = A_n$.

$$\Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$$

$\blacksquare$

Final remarks:

- σ stands for countable unions.
- The pair $(\Omega, \mathcal{A})$ is called a <u>measurable space</u>.

Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

sample space "elementary events"

event space

probability measure.

Idea: Assign to every event in $\mathcal{A}$ a probability, a number in $[0,1]$:

Def: ($\mathbb{P}$ - probability measure) $\mathbb{P} : \mathcal{A} \longrightarrow [0,1]$

$$A \longmapsto \mathbb{P}(A)$$

such that

1.) $\mathbb{P}(A) \geq 0 \quad \forall A \in \mathcal{A}$

2.) $\mathbb{P}(\Omega) = 1$

3.) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ for a collection $(A_n)_{n=1}^{\infty}$ of pairwise disjoint sets.

$(A_n \cap A_m = \phi \text{ if } n \neq m)$.

"Kolmogorov axioms".

Lemma: $A \in \mathcal{A}$

- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(\emptyset) = 0$
- If $A \subset B$, then $\mathbb{P}(A) \le \mathbb{P}(B)$. //
- $0 \le \mathbb{P}(A) \le 1$.

Proof: $\Omega = A \cup A^c$, $A \cap A^c = \emptyset$ disjoint.

$$1 \overset{2.)}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) \underset{\text{disjoint}}{\overset{3.)}{=}} \mathbb{P}(A) + \mathbb{P}(A^c)$$

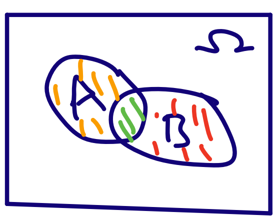
$$\Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$ ∎

Lemma: $A, B \in \mathcal{A}$ (not necessarily disjoint). Then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$ //

Proof: Note that $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$

pairwise disjoint!



$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) \quad \text{by 3.)}$$

$$+ \mathbb{P}(A \cap B) - \mathbb{P}(A \cap B) \quad \text{disjoint.}$$

$= \text{exercise}$ //

<u>Example</u>: Toss a coin twice. Let T_1 be the event that tails occurs on the first toss. Let T_2 be event that tails occurs on second toss. If the coin is fair

$$\mathbb{P}(T_1 \cup T_2) = \mathbb{P}(T_1) + \mathbb{P}(T_2) - \mathbb{P}(T_1 \cap T_2)$$
$$= 1/2 + 1/2 - 1/4 = \frac{3}{4}.$$

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Independence: $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space.

Two events $A, B \in \mathcal{A}$ are called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

- A collection of events $\{A_i : i \in I\}$ is independent if

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i)$$

for any finite subset J of I.

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Example 1: Toss a fair die once. Let

$$A = \{2,4,6\}, \quad B = \{1,2,3,4\}.$$

Then $A \cap B = \{2,4\}$.

$$\mathbb{P}(A \cap B) = \frac{2}{6} = \frac{1}{3}, \quad \mathbb{P}(A) = \frac{3}{6}, \quad \mathbb{P}(B) = \frac{4}{6}.$$

$$\Rightarrow \quad \mathbb{P}(A \cap B) = \frac{1}{3} = \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{3 \cdot 4}{6 \cdot 6} = \frac{1}{3}.$$

$\Rightarrow$ A and B are independent.

//.

Exercise: Assume $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$ and $A \cap B = \emptyset$.

Are A and B independent?

<u>Example 2:</u> $\Omega = \{1,2,3,4\}$, $\mathcal{A} = 2^{\Omega}$.

$\mathbb{P}(\{i\}) = \frac{1}{4}$, $i = 1,2,3,4$.

Let $A = \{1,2\}$, $B = \{1,3\}$ and $C = \{2,3\}$.

Then A and B are independent: $\mathbb{P}(A \cap B) = \mathbb{P}(\{1\}) = \frac{1}{4}$

$\mathbb{P}(A) = \mathbb{P}(B) = \frac{2}{4} = \frac{1}{2}$

so $\mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A \cap B)$.

But A, B and C are not independent: $A \cap B \cap C = \emptyset$

$$0 = \mathbb{P}(\underbrace{A \cap B \cap C}_{= \emptyset}) \neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

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<u>Conditional probability</u>: Two events $A, B \in \mathcal{A}$ with $\mathbb{P}(B) > 0$. <u>Conditional probability</u> of A given B denoted $\mathbb{P}(A|B)$, is defined as

$$\Rightarrow \mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

<u>Note</u>: If A and B are independent then $\mathbb{P}(A|B) = \mathbb{P}(A)$.

<u>Exercise</u>: Show that $\mathbb{P}(\cdot\,|B)$ satisfies the Kolmogorov axioms.

- $\mathbb{P}(A|B) \geq 0 \quad \forall A \in \mathcal{A}$
- $\mathbb{P}(\Omega|B) = 1$
- $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n \Big| B\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n|B)$, A_n disjoint. //

<u>Example</u>: Toss a fair die. $A = \{1, 3, 5\}$ odd number

$B = \{1, 2, 3\}$ outcome is less equal to three.

$A \cap B = \{1, 3\}$.

$$\mathbb{P}(B) = \frac{1}{2}.$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{2}{6}}{\frac{1}{2}} = \frac{2}{3}.$$

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$\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$ in general.