Convergence of sequences of random variables:

General setup: (Y_n) a sequence of random variables.

Want to give a meaning to $Y_n \longrightarrow Y$ as n tends to infinity.

random variable, or deterministic number.

Several modes of convergence:

Example: Weak law of large numbers.

(X_i) iid with $\mathbb{E} X_i = \mu$, $\operatorname{Var} X_i = \sigma^2 < \infty$

Consider the empirical mean

$$\overline{X}_n = \frac{1}{n} S_n := \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow[n \to \infty]{?} \mathbb{E}[X_i] = \mu$$

deterministic number

Thm 1: (X_i) iid as above. Then $S_n \xrightarrow{P} \mu$, i.e. $\forall \varepsilon > 0$

$$\mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) \longrightarrow 0 \text{ as } n \to \infty.$$

"convergence in probability"

Do we need independence?

Def: (X_i) with $\mathbb{E}X_i^2 < \infty$ are said to be uncorrelated if

$$\mathbb{E}[X_i \cdot X_j] = \mathbb{E}[X_i] \cdot \mathbb{E}[X_j] \text{ whenever } i \neq j$$

Remark: (X_i) independent $\Rightarrow$ (X_i) are uncorrelated.
$\cancel{\Leftarrow}$

Exercise: (X_i) uncorrelated $\Rightarrow$ $\mathrm{Cov}(X_i, X_j) := \mathbb{E}\left((X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)\right) = 0$ if $i \neq j$. //

Lemma: Let (X_i) uncorrelated, then

$$\mathrm{Var}(X_1 + X_2 + \dots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \dots + \mathrm{Var}(X_n)$$ //

Proof:

$$\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \mathbb{E}\left[\left(\sum_{i=1}^n (X_i - \mathbb{E}X_i)\right)^2\right]$$

"expand the square"

$$= \mathbb{E}\left[\sum_{i=1}^n (X_i - \mathbb{E}X_i) \cdot \sum_{j=1}^n (X_j - \mathbb{E}X_j)\right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \underbrace{\mathbb{E}\left((X_i - \mathbb{E}X_i) \cdot (X_j - \mathbb{E}X_j)\right)}_{= \mathrm{Cov}(X_i, X_j) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{Var}X_i & \text{if } i = j \end{cases}}$$

$$= \sum_{i=1}^n \mathrm{Var}X_i .$$ ∎

Thm 2: (weak law of large numbers, $\mathcal{L}^2$-convergence)

Let (X_i) be identically distributed and uncorrelated with $\mathbb{E}[X_i] = \mu$ and $\operatorname{Var}(X_i) = \sigma^2 < \infty$.

Then

$$\mathbb{E}\Big[\Big(\overbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}^{\bar{X}_n} - \mu\Big)^2\Big] \longrightarrow 0, \text{ as } n \to \infty$$

//

Def: (Y_n) a sequence of random variables, then Y_n is said to converge in mean-square (in the sense of $\mathcal{L}^2$) to Y, if

$$\mathbb{E}\left[(Y_n - Y)^2\right] \to 0, \text{ as } n \to \infty.$$

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Notation: $Y_n \xrightarrow{2} Y$ as $n \to \infty$.

Proof of weak law: $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\mathbb{E}\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}X_i}_{\mu} = \mu$

$$\mathbb{E}\left[(\bar{X}_n - \mu)^2\right] = \frac{1}{n^2}\underbrace{\mathbb{E}\Big[\Big(\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])\Big)^2\Big]}_{=\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)}$$

X_i are uncorrelated

$$= \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var} X_i = \frac{n}{n^2}\sigma^2 = \frac{\sigma^2}{n} \longrightarrow 0, \text{ as } n \to \infty. \quad \blacksquare$$

Recall: Markov / Tschebyschev inequality:

$U \geq 0$ a non-negative random variable, then $\forall a > 0$ and $\forall r \geq 1$, we have

$$\mathbb{P}(U > a) \leq \frac{1}{a^r} \mathbb{E}[U^r].$$

(see (8.2)–(8.1) Introduction [PG])

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Apply this to the weak law of large numbers: $a = \varepsilon > 0$ and $r = 2$

$$\mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) \overset{\text{Markov}}{\leq} \frac{1}{\varepsilon^2} \mathbb{E}[(\bar{X}_n - \mu)^2] \overset{\text{Thm 2.}}{=} \frac{\sigma^2}{\varepsilon^2 \cdot n} \longrightarrow 0, \text{ as } n \to \infty.$$

for any $\varepsilon > 0$.

$\Longrightarrow S_n \xrightarrow{P} \mu$

Upshot: Thm 2 $\Rightarrow$ Thm 1.

Moreover, convergence in mean-square implies convergence in probability

$$Y_n \xrightarrow{2} Y \underset{\cancel{\Leftarrow}}{\Longrightarrow} Y_n \xrightarrow{P} Y$$

"Meaning: Mean-square convergence is a stronger form of convergence than convergence in probability."

<u>Remark:</u> $\mathbb{E}|Y_n - Y|^r \xrightarrow{r} 0$ convergence in r-th mean.
$r > 0$.

Focus on $r=2$: mean-square convergence.

<u>Facts of life:</u>

- $\left. \begin{array}{l} Y_n \xrightarrow{2} Y \\ Y_n \xrightarrow{2} X \end{array} \right\} \Longrightarrow \mathbb{P}(X = Y) = 1$

  "limit is unique"

- $\left. \begin{array}{l} Y_n \xrightarrow{P} Y \\ Y_n \xrightarrow{P} X \end{array} \right\} \Longrightarrow \mathbb{P}(X = Y) = 1$ //

<u>Exercise:</u> (Monte-Carlo Integration) Let $f:[0,1] \to \mathbb{R}$ be a (measurable) function on $[0,1]$ with $\int_0^1 |f(x)|\,dx < \infty$.

a.) Let $U_1, U_2, \dots$ be independent and uniformly distributed on $[0,1]$. Set

$$I_n := \frac{1}{n}\big(f(U_1) + f(U_2) + \dots + f(U_n)\big).$$

Show that
$$I_n \xrightarrow{P} I := \int_0^1 f(x)\,dx.$$
as $n \to \infty$.

b.) Suppose in addition that $\int_0^1 |f(x)|^2\,dx < \infty$. Use Markov's inequality to estimate

$$\mathbb{P}\left(|I_n - I| > \frac{a}{\sqrt{n}}\right)$$ "speed of convergence"

for once we start enumerating at 2.

Example: $X_2, X_3, \ldots$ such that

$$\mathbb{P}(X_n = 1) = 1 - \frac{1}{n^2}, \quad \mathbb{P}(X_n = n) = \frac{1}{n^2}, \quad n \geq 2.$$

So X_n takes the values 1 or n.

- $$\mathbb{P}(|X_n - 1| > \varepsilon) = \mathbb{P}(X_n = n) = \frac{1}{n^2} \longrightarrow 0, \text{ as } n \to \infty \quad \forall \varepsilon > 0.$$

So $X_n \xrightarrow{P} 1$, as $n \to \infty$.

- However, $$\mathbb{E}[(X_n - 1)^2] = \underbrace{0^2 \cdot \mathbb{P}(X_n = 1)}_{=0} + (n-1)^2 \underbrace{\mathbb{P}(X_n = n)}_{= \frac{1}{n^2}}$$

$$= \frac{(n-1)^2}{n^2} \xrightarrow[n \to \infty]{} 1 \neq 0.$$

Hence, $X_n \not\xrightarrow{2} 1$ as $n \to \infty$.

but $X_n \xrightarrow{P} 1$.

In words, X_n does converge in probability to 1, but the sequence of r.v. does <u>not</u> converge in mean-square to 1.