Symmetric simple random walks

(X_i) iid with $\mathbb{P}(X_i=1)=\mathbb{P}(X_i=-1)=1/2$.

$$\begin{cases} S_n := X_1+X_2+\dots+X_n \\ S_0 := 0. \end{cases}$$

Lemma: Let $u_n := \mathbb{P}(S_n=0)$

Then $u_{2m+1}=0$

$$u_{2m}=\binom{2m}{m}\left(\frac{1}{2}\right)^{2m}, \quad m\geq 0$$

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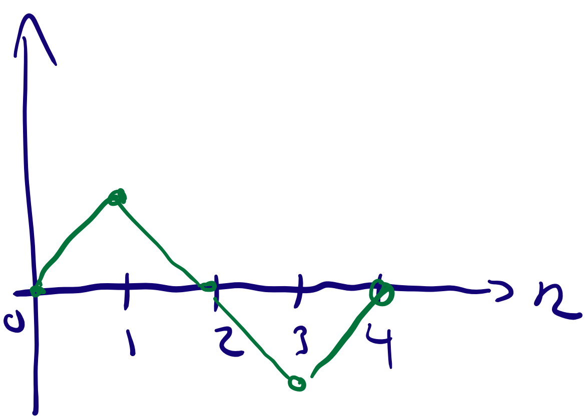
Proof:

$S_n(\omega)$



Need to count the number of paths starting at $(0,0)$ and ending at $(2m,0)$.

Notation: $N_n(a,b)$, $a,b\in\mathbb{Z}$

$:=$ # of possible paths from a to b of length n.

②

Exercise:

$$N_n(a,b) = \begin{cases} \binom{n}{\frac{n+b-a}{2}} & \text{if } \frac{n+b-a}{2} \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

$$\Longrightarrow \quad N_{2m}(0,0) = \binom{2m}{m}.$$

"m steps to the left, m steps to the right."

Thus $u_{2m} = \binom{2m}{m} \cdot \left(\frac{1}{2}\right)^{2m}.$

Remark: $u_{2m} \sim \frac{1}{\sqrt{\pi m}}$ ($a_m \sim b_m \iff \frac{a_m}{b_m} \to 1$ as $m \to \infty$)

Sterling approximation for $n!$

⌈ Sterling: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ ⌋

Probability of first return:

$$\begin{cases} f_n := \mathbb{P}(S_n = 0,\ S_k \neq 0 \text{ for all } 0<k<n), & n>0. \\ f_0 := 0. & \end{cases}$$

Convention.

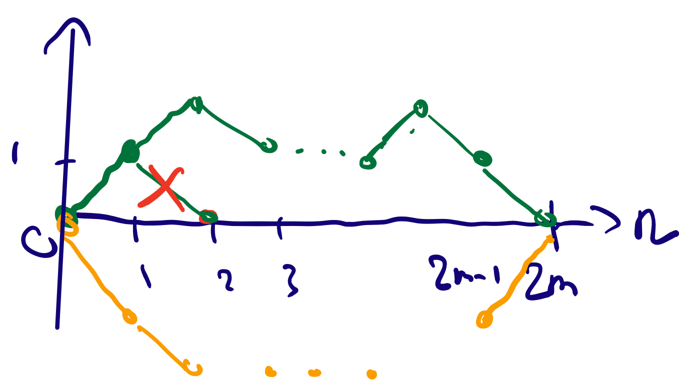
Clearly $f_{2m+1} = 0$, $m \geq 0$.

Notation: $N_n(a,b) :=$ # of paths from a to b of length n.

$N_n^{\neq 0}(a,b) :=$ # of paths from a to b of length n, that do not visit 0.

$N_n^{0}(a,b) :=$ # of paths from a to b of length n that visit 0.

Then $f_{2m} = N_{2m}^{\neq 0}(0,0)\left(\frac{1}{2}\right)^{2m}$.



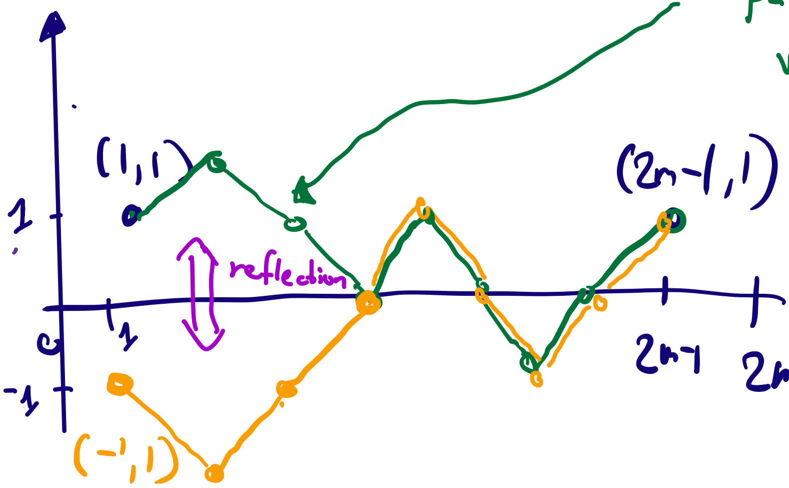
$$N_{2m}^{\neq 0}(0,0) = N_{2m-1}^{\neq 0}(1,0) + N_{2m-1}^{\neq 0}(-1,0)$$

Symmetry

$$= 2N_{2m-1}^{\neq 0}(1,0)$$

$$= 2N_{2m-2}^{\neq 0}(1,1). \qquad (*)$$

Reflection principle:



path from (1,1) to (2m-1,1) visiting 0.

a path connecting -1 to 1 has to go through 0.

From the picture: $N^{0}_{2m-2}(1,1) = N^{0}_{2m-2}(-1,1) = N_{2m-2}(-1,1)$

Hence, $N^{\neq 0}_{2m-2}(1,1) = N_{2m-2}(1,1) - N^{0}_{2m-2}(1,1)$

$$\overset{\text{reflection principle}}{=} N_{2m-2}(1,1) - N_{2m-2}(-1,1)$$

$$\overset{\text{previous exercise}}{=} \binom{2m-2}{m-1} - \binom{2m-2}{m}$$

$$\overset{\text{computation}}{\underset{(*)}{=}} \frac{1}{m}\binom{2m-2}{m-1}.$$

Thus $N^{\neq 0}_{2m}(0,0) \overset{(*)}{=} 2N^{\neq 0}_{2m-2}(1,1) \overset{(*)}{=} \frac{2}{m}\binom{2m-2}{m-1} \overset{\text{computation}}{=} \frac{1}{2m-1}\binom{2m}{m}$.

Finally, $f_{2m} = N^{\neq 0}_{2m}(0,0)\left(\frac{1}{2}\right)^{2m} = \frac{1}{2m-1}\underbrace{\binom{2m}{m}\left(\frac{1}{2}\right)^{2m}}_{=u_{2m}\text{, see first Lemma on page 1.}} = \frac{1}{2m-1}u_{2m}$.

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Exercise: For $b \geq 1$,

$$N_n^{\neq 0}(0,b) = \frac{b}{n} \cdot N_n(0,b)$$

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Solution: To reach $b \geq 1$ from 0 in n steps, I have to go $\frac{n+b}{2}$ steps to the right and $\frac{n-b}{2}$ steps to the left.

Hence $N_n(0,b) = \binom{n}{\frac{n+b}{2}}$.

But also

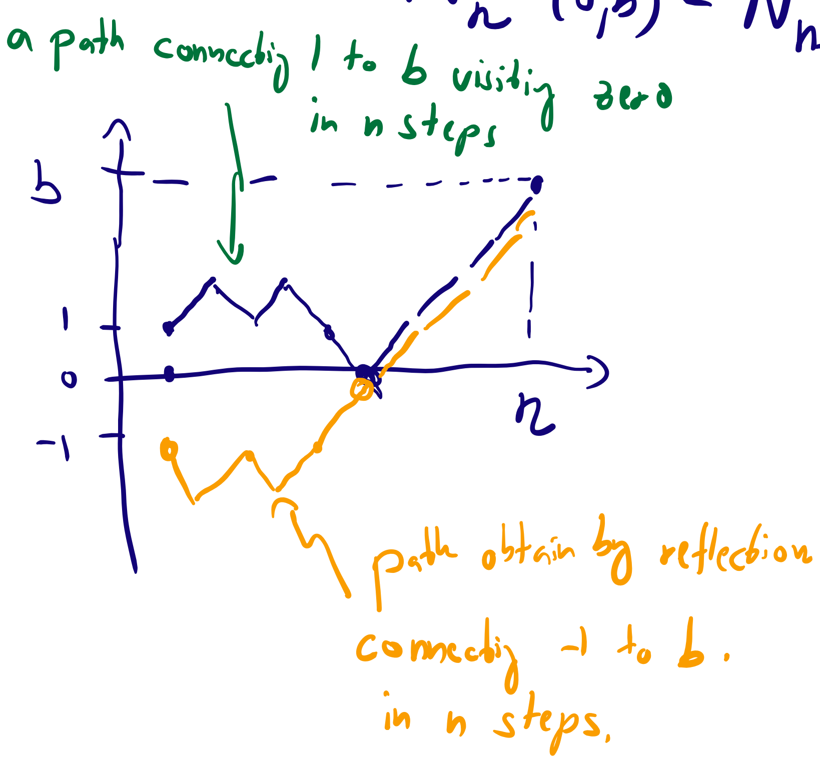
$$N_n^{\neq 0}(0,b) = N_{n-1}^{\neq 0}(1,b) = N_{n-1}(1,b) - \underbrace{N_{n-1}(-1,b)}_{N^{0}_{n-1}(1,b)}$$

reflection principle.

$$= \binom{n-1}{\frac{n+b}{2}-1} - \binom{n-1}{\frac{n+b}{2}}$$

$$= \dots = \binom{n}{\frac{n+b}{2}} \frac{b}{n}$$

$$= N_n(0,b) \cdot \frac{b}{n}.$$



Remark: $u_{2n} = \mathbb{P}(S_{2m}=0)$

$f_{2m} = \mathbb{P}(S_{2n}=0,\ S_k \neq 0 \text{ for all } 0<k<n)$

By Sterling approximation $u_{2m} \sim \dfrac{1}{\sqrt{\pi m}}$. $\left(u_{2m+1} = \mathbb{P}(S_{2m+1}=0) = 0\right)$

Thus $f_{2m} \sim \dfrac{1}{(2n-1)\sqrt{\pi m}} \sim \dfrac{1}{2\sqrt{\pi}\, m^{3/2}}$.