KTH Matematik

Homework 1
Mathematical Systems Theory, SF2832
2023 Fall
Passing grade: 12p

1. (a) Solve $\dot{x}(t) = \begin{bmatrix} -1 & t & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .... (2p)

(b) Solve $\dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$, where ω is a positive constant, and show that $x_1(t) = a\sin(\omega t + b)$ where $a,\ b$ are functions of $x(0)$. .... (2p)

2. (a) Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.$$

Show that $det\ \Phi(t, t_0) = e^{\int_{t_0}^{t}(a_{11}(s)+a_{22}(s))ds}$. .... (2p)

(b) Verify that $\Psi(t) = \Phi(t, t_0)\Psi_0\Phi^T(t, t_0)$ is the solution to

$$\dot{\Psi}(t) = A(t)\Psi(t) + \Psi(t)A^T(t),\ \ \Psi(t_0) = \Psi_0,$$

where $\Phi(t, t_0)$ is the state transition matrix of $\dot{x} = A(t)x$. .... (2p)

3. Consider

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x. \end{aligned}$$

Show that controllability and observability of linear time-varying systems are invariant under linear transformation $\bar{x} = P(t)x$, where $P(t)$ is nonsingular and continuously differentiable for all $t \in (-\infty, \infty)$. .... (3p)

4. Consider the inverted pendulum as we did in the lecture (see Figure 1). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0,\ \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

(a) We consider $\ddot{x}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e. let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$), and show that the model you derive is both controllable and observable. .... (2p) (*b*) Setting $u(t) = 0$ and using the linearized model, can we find an initial state $x_1(0) \neq 0$ and $x_2(0)$ such that $x_1(t) = 0$ for all $t \geq T$ where $T > 0$ is some finite time? .... (2p)

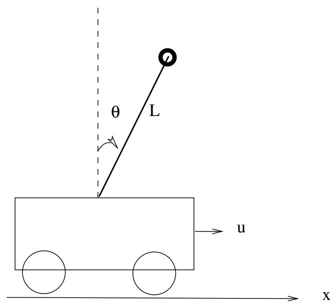
5. Now consider an inverted pendulum with oscillatory base (see Figure 2). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0,\ \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

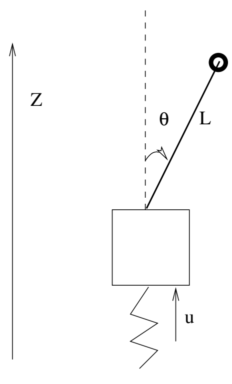
(*a*) We consider $\ddot{z}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the *linearized* system. .... (1p)

(*b*) Is the model you derive in (a) controllable? .... (1p)

(*c*) Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab (or any other software of your choice) simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is "slow" and when the oscillation is "fast". Take $L = 1,\ \theta(0) = 0.02$ and $\dot{\theta}(0) = 0$. .... (3p)



Figur 1: Inverted pendulum on a cart.



Figur 2: Pendulum with oscillatory base.