Time-limited assignment 1, SF2940HT23

Date: September 22, 2023. Regular writing time 08:15-09:15. Canvas upload closes 15 minutes after the end of the writing time sharp. No late submission will be accepted.

Examiner: Kevin Schnelli.

Aids: The following aids are allowed: Book: *An Intermediate Course in Probability*, by Allan Gut, Lecture Notes: *Probability and Random Processes* by Timo Koski. Notes from online lectures and exercise sessions available in Canvas. Digital versions of the books etc. are also allowed. In addition, all personal notes for the course. A basic pocket calculator is allowed. No other aids are allowed. Collaboration with others is forbidden.

Examiners may request that individual students explain and present their solutions.

Instructions:

- On the first page write 'Assignment SF2940, 22nd September 2023', your personal number, your full name (as found in Ladok), total number of pages submitted. On top of each page write down the problem number solved on that page, number your pages. Write only one exercise per sheet.

- **Upload your solutions to Canvas as a single pdf file. Do not forget to sign the code of conduct in Canvas.**

- Your solutions and answers should be legible, easy to follow and not lack in explanations. In particular, all notation used should be explained.

- Design: There are two independent problems, each giving maximal 5 points.

Problem 1. Let the pair of random variables (X, Y) have joint density

$$f_{X,Y}(x,y) = \begin{cases} 21y^2\,, & \text{if} \quad 0 < y < x^2 < 1\,, \\ 0\,, & \text{else}\,. \end{cases}$$

a.) Determine the marginal density $f_X(x)$ of the random variable X........ (2points)
b.) Determine the conditional density $f_{Y|X=x}(y)$. .... (1point)
c.) Determine the conditional expectation $\mathbb{E}[Y|X]$. .... (2points)

Problem 2. Let X and Y be two random variables. Assume that X has probability density function

$$f_X(x) = \begin{cases} 2x\,, & \text{if} \quad 0 < x < 1\,, \\ 0\,, & \text{else}\,. \end{cases}$$

Further let $Y|X = x \in U(0, x)$, that is $Y|X = x$ is uniformly distributed on the interval $(0, x)$.

a.) Determine the conditional expectation $\mathbb{E}[Y|X]$ and the expectation $\mathbb{E}[Y]$.(3points)
b.) Compute the moment generating function $\psi_Y(t)$ of Y.................. (2points)

SF2940: Distributions and their characteristic functions

Distribution	Notation	Probability function	$\varphi_X(t)$
One point	$\delta(a)$, $a \in \mathbb{R}$	$p(a) = 1$	e^{ita}
Bernoulli	$\mathrm{Be}(p)$; $0 \leq p \leq 1$	$p(0) = q$, $p(1) = p$; $q = 1 - p$	$q + p\mathrm{e}^{\mathrm{i}t}$
Binomial	$\mathrm{Bin}(n, p)$, $n = 1, 2, \ldots$; $0 \leq p \leq 1$	$p(k) = \binom{n}{k} p^k q^{n-k}$, $k = 0, 1, 2, \ldots, n$; $q = 1 - p$	$(q + p\mathrm{e}^{\mathrm{i}t})^n$
Geometric	$\mathrm{Ge}(p)$, $0 \leq p \leq 1$	$p(k) = pq^k$, $k = 0, 1, 2, \ldots$; $q = 1 - p$	$\frac{p}{1 - q\mathrm{e}^{\mathrm{i}t}}$
Poisson	$\mathrm{Po}(\lambda)$, $\lambda > 0$	$p(k) = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$	$\mathrm{e}^{\lambda(\mathrm{e}^{\mathrm{i}t} - 1)}$

Table 1. Discrete distributions

Distribution	Notation	Density	$\varphi_X(t)$
Uniform	$\mathrm{U}(a, b)$, $a < b$	$f(x) = \frac{1}{b-a}$, $a < x < b$	$\frac{\mathrm{e}^{itb} - \mathrm{e}^{ita}}{\mathrm{i}t(b-a)}$
Exponential	$\mathrm{Exp}(\lambda)$, $\lambda > 0$	$f(x) = \frac{1}{\lambda} \mathrm{e}^{-x/\lambda}$, $x > 0$	$\frac{1}{1 - \lambda \mathrm{i}t}$
Laplace	$\mathrm{L}(\lambda)$, $\lambda > 0$	$f(x) = \frac{1}{2\lambda} \mathrm{e}^{-\lvert x \rvert/\lambda}$, $x \in \mathbb{R}$	$\frac{1}{1 + \lambda^2 t^2}$
Beta	$\beta(r, s)$, $r, s > 0$	$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1 - x)^{s-1}$, $0 < x < 1$	$*$
Normal	$\mathrm{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$, $x \in \mathbb{R}$	$\mathrm{e}^{\mathrm{i}\mu t - \frac{1}{2}\sigma^2 t^2}$
Standard normal	$\mathrm{N}(0, 1)$	$f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}$, $x \in \mathbb{R}$	$\mathrm{e}^{-\frac{1}{2}t^2}$
Log-normal	$\mathrm{LN}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{1}{2}(\log x - \mu)^2/\sigma^2}$, $x > 0$	$*$
Cauchy	$\mathrm{C}(m, \eta)$, $\eta > 0$, $m \in \mathbb{R}$	$f(x) = \frac{1}{\pi} \frac{\eta}{(x-m)^2 + \eta^2}$, $x \in \mathbb{R}$	$\mathrm{e}^{imt - \eta\lvert t \rvert}$

Table 2. Continuous distributions