Some exam problems

Omtenta 2013: $X \in Exp(1)$, $Y \in Exp(1)$, X and Y are independent.

Find $\mathbb{P}(X<Y) = \mathbb{P}(X-Y<0)$

Use characteristic function: $\varphi_{X-Y}(t) \overset{\text{indep}}{=} \varphi_X(t) \cdot \varphi_{-Y}(t) = \varphi_X(t) \cdot \varphi_Y(-t)$

$$\overset{Exp(1)}{=} \frac{1}{1-it} \cdot \frac{1}{1-i(-t)} \quad \text{cf from table}$$

$$= \frac{1}{1-it} \cdot \frac{1}{1+it}$$

$$= \frac{1}{1+t^2}.$$

$\Rightarrow X-Y \in La(1)$. $\varphi_{X-Y}(t)$ real valued for all t $\Leftrightarrow$ $X-Y \overset{d}{=} Y-X \overset{d}{=} -(X-Y)$

$\mathbb{P}(X-Y<0) = \mathbb{P}(Y-X<0) = 1/2.$

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2.

Tenta 2020 $X = (X_1, X_2)$ bivariate Gaussian $\mathcal{N}(0, \Lambda)$

mean $= 0$

covariance matrix.

$$\Rightarrow \Lambda = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

a.) Find the joint probability density $f_{X_1,X_2}(x_1,x_2)$.

b.) $\mathbb{E}[X_1 | X_2] = ?$

Solution a.)

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} \frac{1}{\sqrt{\det \Lambda}} e^{-\frac{1}{2}(x_1,x_2)\Lambda^{-1}(x_1,x_2)'}$$

Good to remember!

$\det \Lambda = 4 - 1 = 3$

$$\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

[Cramer's rule $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det M \neq 0$

$$\Rightarrow M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$]

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} \frac{1}{\sqrt{3}} e^{-\frac{1}{6}(x_1,x_2)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}(x_1,x_2)'}$$

[also good to remember]

$$= \frac{1}{2\pi\sqrt{3}} e^{-\frac{1}{3}(x_1^2 - x_1 \cdot x_2 + x_2^2)}$$

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b.) $\mathbb{E}[X_1 | X_2]$

definition

$$f_{X_1|X_2=x_2}(x_1) \overset{\downarrow}{=} \frac{f_{X_1 X_2}(x_1, x_2)}{f_{X_2}(x_2)} = \frac{f_{X_1,X_2}(x_1,x_2)}{\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1}$$

Marginal density of X_2

Already know $X_2 \in \mathcal{N}(0,2)$

$$= \frac{\frac{1}{2\pi\sqrt{3}} e^{-\frac{(x_1^2 - x_1 \cdot x_2 + x_2^2)}{3}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{x_2^2}{4}}}$$

$$= \frac{1}{\sqrt{3\pi}} e^{-\frac{1}{3}(x_1^2 - x_1 \cdot x_2 + x_2^2) + \frac{x_2^2}{4}},$$

computation

$$= \frac{1}{\sqrt{3\pi}} e^{-\frac{1}{3}(x_1 - \frac{x_2}{2})^2}$$

$$\mathbb{E}[X_1 | X_2 = x_2] = \frac{1}{\sqrt{3\pi}} \int_{-\infty}^{\infty} x_1 f_{X_1|X_2=x_2}(x_2)\,dx_1 = \frac{x_2}{2}$$

Answer: $\mathbb{E}[\underline{X}_1 | \underline{X}_2] = \frac{\underline{X}_2}{2}$

(4.)

(Omtenta 2018) $X_1, X_2, \dots,$ iid rv

with $\mathbb{E} X_1 = -2$ and $\operatorname{Var} X_1 = \mathbb{E} X_1^2 - (\mathbb{E} X_1)^2 = 4$.

$$S_n := \sqrt{n} \; \frac{X_1 + X_2 + \dots + X_n + 2n}{X_1^2 + X_2^2 + \dots + X_n^2}$$

Find the limiting distribution of S_n.

CLT

Rewrite:
$$\frac{\frac{\sqrt{n}}{n} \sum_{i=1}^{n} (X_i + 2)}{\frac{1}{n} \sum_{i=1}^{n} X_i^2} = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mathbb{E} X_i)}{\frac{1}{n} \sum_{i=1}^{n} X_i^2}$$

LLN

By CLT: $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mathbb{E} X_i) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,4)$.

By LLN: $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \xrightarrow[n\to\infty]{P} \mathbb{E}[X_i^2] = \underbrace{\operatorname{Var} X_1}_{4} + \underbrace{(\mathbb{E}(X_1))^2}_{4} = 8$.

deterministic number.

By Cramér-Slutsky $S_n \xrightarrow[n\to\infty]{d} \frac{\mathcal{N}(0,4)}{8} = \mathcal{N}(0, \frac{4}{64}) = \mathcal{N}(0, \frac{1}{16})$

Answer: $S_n \xrightarrow{d} \mathcal{N}(0, \frac{1}{16})$.

Some exam problems

Omtenta 2015 $\quad N \in Po(\lambda),\ \lambda > 0$

$X \mid N = n \in Bin(n, p), \quad p \in (0,1)$

a.) Show that $\mathbb{E}[X] = \lambda \cdot p$

b.) Determine the distribution of X.

Solution a.)

$$\mathbb{E}[X] \overset{\text{law of total expectation}}{=} \mathbb{E}\big[\mathbb{E}[X \mid N]\big] \overset{\text{Claim}}{=} \mathbb{E}[N \cdot p] = p\,\mathbb{E}[N] \overset{N \in Po(\lambda)}{=} p \cdot \lambda.$$

Claim: $\mathbb{E}[X \mid N] = N \cdot p$ i.e. $\mathbb{E}[X \mid N = n] = n \cdot p$

$Y \in Bin(n,p)$ $\quad$ Table: $\boxed{\varphi_Y(t) = (q + pe^{it})^n}$, $\quad p + q = 1.$

$\mathbb{E}[Y] = n \cdot p$

$$\mathbb{E}Y = \frac{1}{i} \frac{d}{dt}\Big|_{t=0} \varphi_Y(t) \overset{\text{above}}{=} \frac{1}{i} n (q + p\underbrace{e^{it}}_{=1})^{n-1} p\, i\, \underbrace{e^{it}}_{=1}\Big|_{t=0}$$

$$= n(\underbrace{q + p \cdot 1}_{=1}) p \cdot 1$$

$$= n \cdot p.$$

Bonus problem: $\operatorname{Var} X = \lambda \cdot p$.

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6.

b) Find distribution of X

$$\varphi_X(t) = \mathbb{E}[e^{itX}]$$

$$= \mathbb{E}[\mathbb{E}[e^{itX} | N]]$$

$$= \mathbb{E}[(q+pe^{it})^N]$$

used that

$\varphi_Y(t) = (q+pe^{it})^n$

use $\mathbb{E}[e^{itX}|N=n]$

$= (q+pe^{it})^n$

because $X|N=n \in Bin(n,p)$

$N \in Po(\lambda)$, table

$$= e^{-\lambda} \sum_{n=0}^{\infty} (q+pe^{it})^n \frac{\lambda^n}{n!}$$

use $p+q=1$

$$= e^{-\lambda} \underbrace{\sum_{n=0}^{\infty} \frac{(1-p+pe^{it})^n \lambda^n}{n!}}_{\text{Taylor series of exponential}}$$

$$= e^{-\lambda} e^{(1-p+pe^{it})\lambda}$$

$$= e^{\lambda p (e^{it}-1)}$$

Table: characteristic function $Po(\lambda \cdot p)$.

$\Rightarrow X \in Po(\lambda p)$.

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$\mathbb{W} = \{W(t) : 0 \le t < \infty\}$ a Brownian motion, Wiener process.

Let $h > 0$. Define

$$X_n := \sum_{k=1}^{n} \frac{1}{k} \big(\underbrace{W(h \cdot k) - W(h(k-1))}_{=: Y_k} \big).$$

a.) Determine the distribution of X_n.

b.) $X_n \xrightarrow[n\to\infty]{d} X$

Solution a) $Y_k := W(h \cdot k) - W(h(k-1)) \in \mathcal{N}(0, h)$

increment of Brownian motion for time h.

Y_k's are independent: $X_n = \sum_{k=1}^{n} \frac{1}{k} Y_k$ is Gaussian by Cramér-Wold device.

1.) $\mathbb{E} X_n = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E} Y_k = 0$

2.) $\operatorname{Var} X_n = \mathbb{E}\left(\sum_{k=1}^{n} \frac{1}{k} Y_k \right)^2 = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \frac{1}{k_1} \frac{1}{k_2} \mathbb{E}\big[\underbrace{Y_{k_1} \cdot Y_{k_2}}_{0 \text{ unless } k_1 = k_2}\big]$

$= \sum_{k_1=1}^{n} \frac{1}{k_1^2} \cdot \underbrace{\mathbb{E}\left[Y_{k_1}^2\right]}_{h}$

$= h \sum_{k=1}^{n} \frac{1}{k^2}$

Answer:

$X_n \in \mathcal{N}\left(0, h \sum_{k=1}^{n} \frac{1}{k^2}\right)$

b.) $X_n \xrightarrow[n\to\infty]{d} X$

$$\varphi_{X_n}(t) = e^{-\frac{t^2}{2} n \sum_{k=1}^{n} \frac{1}{k^2}} \xrightarrow[n\to\infty]{} e^{-\frac{t^2}{2} n \cdot \frac{\pi^2}{6}}$$

From hint:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{(Euler)}$$

$$X_n \xrightarrow{d} \mathcal{N}\left(0, n\frac{\pi^2}{6}\right).$$

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