Conditional distributions

$(\Omega, \mathcal{A}, \mathbb{P})$, $A, B \in \mathcal{A}$, $\mathbb{P}(B) > 0$.

Conditional probability of A given B was defined as

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Note: If A and B are independent, then

$$P(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

Example: Throw a symmetric die twice.

X_1: # of eyes in first toss. $1 \leq X_i \leq 6$,

X_2: # of eyes in second toss.

$$Y = X_1 + X_2 \qquad 2 \leq Y \leq 12$$

Consider the events: $A = \{Y = 8\}$, $B = \{X_1 \leq 4\}$

$$\mathbb{P}(B) = \mathbb{P}(\{X_1 \leq 4\}) = \frac{4}{6} = \frac{2}{3}$$

$$\mathbb{P}(A) = 2\,\frac{1}{6}\cdot\frac{1}{6} + 2\cdot\frac{1}{6}\cdot\frac{1}{6} + 1\,\frac{1}{6}\cdot\frac{1}{6} = \frac{5}{36}.$$

$8 = 2+6 = 6+2$ ✓ ✗ $\quad 8 = 3+5 = 5+3$ ✓ ✗ $\quad 8 = 4+4$ ✓

$$\mathbb{P}(A \cap B) = \frac{3}{36} = \frac{1}{12} \Rightarrow \mathbb{P}(A|B) = \frac{\frac{1}{12}}{\frac{2}{3}} = \frac{1}{8}$$

//

Can compute $\mathbb{P}(Y=y \mid X_1=x_1)$ for any given y and x_1.

Def: (X,Y) be discrete, jointly distributed random variables. For $\mathbb{P}(X=x)>0$ the conditional probability function of Y given that $X=x$ is defined as

joint p.m.f.

$$p_{Y|X=x}(y) := \mathbb{P}(Y=y \mid X=x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

⚠ function of y

its marginal

Conditional distribution function of Y given that $X=x$ is

$$F_{Y|X=x}(y) = \sum_{z \leq y} p_{Y|X=x}(z) .$$

//

Continuous setup: $\mathbb{P}(Y=y \mid X=x)$ has no meaning.

Way out: Consider densities.

Def: Let X and Y have a joint continuous distribution with density $f_{X,Y}(x,y)$.

For $f_X(x)>0$, the conditional density function of Y given that $X=x$ is defined as

$$f_{Y|X=x}(y) := \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,v)\,dv}.$$

Conditional distribution function:

$$F_{Y|X=x}(y) = \int_{-\infty}^{y} f_{Y|X=x}(v)\,dv.$$

Exercise: Show that $f_{Y|X=x}(y)$ is a density function, e.g.

$$\int_{-\infty}^{\infty} f_{Y|X=x}(y)\,dy = 1, \quad f_{Y|X=x}(y) \geq 0 \quad //$$

4.

Example: (Problem 2.6.1 in AG)

X, Y independent $\operatorname{Exp}(1)$-distributed random variables. Find:

$$f_{X \mid X+Y=c}(x) \text{ for some given } c>0.$$

Solution:

indicator function

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = e^{-x} \mathbb{1}_{\{x \geq 0\}}$$

$$f_{X \mid X+Y=c}(x) = \frac{f_{X, X+Y}(x,c)}{f_{X+Y}(c)} \underset{x+y=c}{=} \frac{f_{X,Y}(x, \overset{y}{\overset{\shortparallel}{c-x}})}{f_{X+Y}(c)}$$

independence of X and Y.

$$= \frac{f_X(x) \cdot f_Y(c-x)}{f_{X+Y}(c)} = \frac{e^{-x} e^{-(c-x)}}{f_{X+Y}(c)} \mathbb{1}_{\{x \geq 0\}} \mathbb{1}_{\{c-x \geq 0\}}$$

$$= \frac{e^{-c} \mathbb{1}_{\{0 \leq x \leq c\}}}{f_{X+Y}(c)}.$$

Remains to find $f_{X+Y}(c)$:

X, Y are independent: Convolution formula

$$f_{X+Y}(c) = \int_{-\infty}^{\infty} f_X(c-y)\, f_Y(y)\,dy$$

$$= \int_0^c e^{-(c-\cancel{y})}\, e^{\cancel{-y}}\,dy$$

$$= e^{-c}\int_0^c dy = c e^{-c}.$$

$$f_{X|X+Y=c}(x) = \frac{\cancel{e^{-c}}\,\mathbb{1}_{\{0\le x\le c\}}}{c\,\cancel{e^{-c}}} = \begin{cases}\frac{1}{c} & \text{if } 0\le x\le c\\ 0 & \text{else.}\end{cases}$$

<u>Answer:</u> Uniform$(0,c)$.

//