Martingales: Invitation to SF2971.

Bonus notes

"Fair game": Gambler betting on a "fair" game.

Let X_n denote the outcome of the n'th game.

$X_1, X_2, \ldots,$ random variables on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

After to n'th game, we know the values of $X_1, X_2, \ldots, X_n$.
Let $\mathcal{A}_n := \sigma(X_1, X_2, \ldots, X_n)$, the σ-algebra generated by $(X_1, X_2, \ldots, X_n)$.
The fortune, S_n, of the gambler after the n-th game is a function of $(X_1, X_2, \ldots, X_n)$, i.e. S_n is $\mathcal{A}_n$-measurable.

"Fair game": $\mathbb{E}[S_{n+1} \mid \mathcal{A}_n] = S_n$

"Best guess for the fortune after the $n+1^{st}$ game."

Note: $\mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \mathcal{A}_{n+2} \ldots \subset \mathcal{A}$.

Def: • A sequence of σ-algebras

$$\underline{\mathcal{A}} := (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n, \dots)$$

such that $\mathcal{A}_i \subset \mathcal{A}$ and $\mathcal{A}_n \subset \mathcal{A}_m$ if $m > n$

is called a <u>filtration</u>.

• $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, \mathbb{P})$ is called a <u>filtered probability space</u>. //

Ex: $\mathcal{A}_n = \sigma(X_1, \dots, X_n)$, (X_n) sequence of random variables.

"natural filtration"

Def: A sequence of random variables, (S_n), $n \geq 1$, is called a <u>martingale</u> with respect to the filtration $\underline{\mathcal{A}}$, if, for any n,

i.) S_n is $\mathcal{A}_n$-measurable. (S_n is <u>adapted</u> to $\underline{\mathcal{A}}$)

ii) $\mathbb{E}|S_n| < \infty$

iii) $\mathbb{E}[S_{n+1} | \mathcal{A}_n] = S_n$ (martingale property). //

<u>Example I:</u> (X_i) iid with $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_i| < \infty$.

$$S_n := \sum_{i=1}^{n} X_i .$$

with respect to the filtration $\mathcal{A}_n = \sigma(X_1, \dots, X_n)$

is a martingale.

i.) S_n is adapted by definition of $\mathcal{A}_n$.

ii) $\mathbb{E}|S_n| \leq n\,\mathbb{E}|X_1| < \infty$

iii) $\mathbb{E}[S_{n+1} | \mathcal{A}_n] = \mathbb{E}[X_{n+1} + S_n | \mathcal{A}_n]$

S_n is $\mathcal{A}_n$-measurable

X_{n+1} is independent from $\mathcal{A}_n$

$$= \underbrace{\mathbb{E}[X_{n+1} | \mathcal{A}_n]}_{= \mathbb{E}[X_{n+1}] = 0} + S_n$$

$$= S_n .$$

//

Example II: Let (X_i) be iid. r.v. with moment generating function $\varphi_{X_1}(t) = E[e^{tX_1}] =: \varphi(t)$, $|t|<1$.

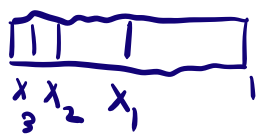
Define:
$$S_n := \frac{e^{tY_n}}{(\varphi(t))^n} \quad \text{with } Y_n = \sum_{i=1}^{n} X_i .$$

Claim: S_n is a martingale with respect $\mathcal{A}_n = \sigma(X_1, \dots, X_n)$ //

Example III: S_n as in example I, but with $E X_i^2 = \sigma^2$

Show that
$$M_n := S_n^2 - n\sigma^2$$
is a martingale.

Example IV: Broken stick:



Claim: $S_n := 2^n X_n$, $\mathcal{A}_n := \sigma(X_1, \dots, X_n)$.

is a martingale. We know $E[X_{n+1} \mid \mathcal{A}_n] = \frac{1}{2} X_n$ ✓

$$E[S_{n+1} \mid \mathcal{A}_n] = E[2^{n+1} X_{n+1} \mid \mathcal{A}_n] = \frac{2^{n+1}}{2} X_n = 2^n X_n$$
$$= S_n \quad //$$

Example V: Brownian motion is a (continuous-time) martingale. //

Lemma: S_n a martingale.

$$\mathbb{E}[S_n] = \mathbb{E}[S_1] = \text{const} \quad \forall n. \ //$$

Proof:

$$\mathbb{E}[S_{n+1}] \overset{\text{law of total expectation}}{=} \mathbb{E}\big[\underbrace{\mathbb{E}[S_{n+1} | \mathcal{A}_n]}_{= S_n \text{ martingale property}}\big] = \mathbb{E}[S_n] \overset{\text{iterate}}{=} \mathbb{E}[S_1] \quad \blacksquare$$

In fact, it can be shown that

$$\mathbb{E}[S_{n+m} | \mathcal{A}_n] = S_n \text{ for all } n, m \geq 1.$$

Thm: (A martingale convergence thm): S_n be a martingale,

$$\mathbb{E}[S_n^2] < M < \infty, \text{ for some } M \text{ and all } n.$$

Then there exists a random variable S such that

$$S_n \xrightarrow[n\to\infty]{a.s.} S. \qquad //$$

Example: Harmonic series

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \text{ diverges as } n \to \infty$$

$\sim \log n$

but the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ converges to } \ln(2) \text{ as } n \to \infty.$$

What happens when we choose the signs at random? (X_i) i.i.d. with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$$

$$S_n := \sum_{i=1}^{n} \frac{X_i}{i}, \quad n \geq 1$$

Claim: S_n is a martingale (exercise). Moreover $\mathbb{E}[S_n^2] = \sum_{i=1}^{n} \frac{1}{i^2} \leq \sum_{i\geq 1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} < \infty$

Apply the theorem: $S_n \xrightarrow[n\to\infty]{a.s.} S$. (However, little is known about the distribution of S)