Presentation Flute Recorder

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Background

Figure: Recorder flute taken from project paper

In order to simulate the wave propagation inside a flute, we implemented this simple mathematical model for P, which is a variant of the the wave equation:

$$\frac{\partial^2 P}{\partial t^2} - c_0^2 \frac{\partial^2 P}{\partial x^2} + D(x)P = G(x), \quad 0 \leq x \leq L, \quad t > 0,$$

$$\text{BC: } P(t,0) = P(t,L) = 0$$

$$\text{IC: } P(0,x) = 0, \quad 0 \leq x \leq L$$

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Rescaling in order to obtain the dimensionless form

Consider P_0 and T as reference values for the pressure P and time t. Introducing the rescaled variables u, τ, y as:

$P = uP_0$, $t = \tau T$, $x = yL$

Rescale the equations to dimensionless form and obtain:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial y^2} + d(y)u = g(y), \quad 0 \leq y \leq 1, \quad \tau > 0,$$

$$\text{BC: } u(\tau, 0) = u(\tau, 1) = 0$$

$$\text{IC: } u(0, y) = 0, \quad 0 \leq y \leq 1$$

Rescaling in order to obtain the dimensionless form

Insert the rescaled variables into the wave equation:

$$\frac{\partial^2(uP_0)}{\partial(\tau T)^2} - c_0^2\frac{\partial^2(uP_0)}{\partial(yL)^2} + D(yL)uP_0 = G(yL)$$

Rewriting the system and we will obtain:

$$\frac{P_0}{T^2}\frac{\partial^2 u}{\partial\tau^2} - c_0^2\frac{P_0}{L^2}\frac{\partial^2 u}{\partial y^2} + D(yL)uP_0 = G(yL)$$

Rewrite the coefficient D(yL) and forcing function G(yL)

$$D(yL) = \ldots = d(y)\cdot\frac{S\alpha c_0^2}{LR^2 s}$$

$$G(yL) = \ldots = g(y)\cdot\frac{G_0 L}{w}, \quad \beta = \frac{4L}{w}$$

Insert the rewriting of the coefficient and forcing function into the system, afterwards divide by $\frac{P_0}{T^2}$:

$$\frac{\partial^2 u}{\partial \tau^2} - c_0^2 \frac{T^2}{L^2} \frac{\partial^2 u}{\partial y^2} + \frac{T^2 S \alpha c_0^2}{L R^2 s} d(y) u = \frac{G_0 L T^2}{w P_0} g(y)$$

In order to obtain the rescaled equations, consider the coefficients:

$$\frac{c_0^2 T^2}{L^2} = 1 \rightarrow T = \frac{L}{c_0}$$
$$\frac{T^2 S \alpha c_0^2}{L R^2 s} = 1 \rightarrow s = \frac{S \alpha L}{R^2}$$
$$\frac{G_0 L T^2}{w P_0} = 1 \rightarrow P_0 = \frac{G_0 L^3}{w c_0^2}$$

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Diameter of the hole (S): The hole diameter influences the coefficient D(x) in the wave equation. The hole diameter (S [m]) was chosen so the dimensionless hole size (s) is between 0 (no holes) and 100 (very large holes).

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A uniform discretization in time and space was made as,

Uniform Discretization

$$y_j = j\Delta y, \quad \tau_n = n\Delta\tau, \quad u_{n,j} \approx u(\tau_n, y_j)$$

$\Delta y = 1/N_y$ being the step size in space and $\Delta\tau = 1/N_\tau$ being the step size in time.

With central difference approximation of the second order derivatives the dimensionless DE discretized can be written on the form:

Dimensionless DE

$$\frac{\boldsymbol{u}^{n+1} - 2\boldsymbol{u}^{n} + \boldsymbol{u}^{n-1}}{\Delta\tau^2} = A\boldsymbol{u}^n - D\boldsymbol{u}^n + \boldsymbol{g}, \quad \boldsymbol{u}^n = (u_{n,1}, u_{n,2}, \ldots, u_{n,N_y-1})^T \Re^{(N_y-}$$

$$A = \frac{1}{\Delta y^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & 1 & -2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix} \in \Re^{(N_y-1)x(N_y-1)},$$

$$D = \mathrm{diag}(d(y_j)) \in \Re^{(N_y-1)x(N_y-1)},$$

$$\boldsymbol{g} = \left[g(y_1), g(y_2), \ldots, g(y_N)\right]^T \in \Re^{(N_y-1)x1}.$$

This was the rewritten as

$$\boldsymbol{u}^{n+1} = 2\boldsymbol{u}^n - \boldsymbol{u}^{n-1} + \Delta\tau^2(A\boldsymbol{u}^n - D\boldsymbol{u}^n + \boldsymbol{g})$$

and this is our discretization scheme.
With the given initial condition $\mathbf{u}(0, y) = 0$ and boundary conditions $u(\tau, 0) = u(\tau, 1) = 0$ and $\mathbf{u}^{-1} = \mathbf{u}^0 \equiv 0$, implemented there was no change to the A matrix.

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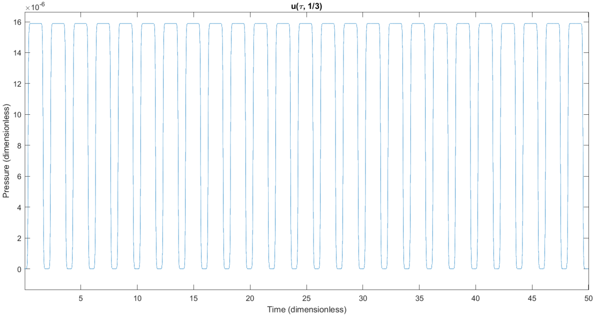

Figure: Pressure change at $y = \frac{1}{3}\Delta y = \Delta t = 0.002$

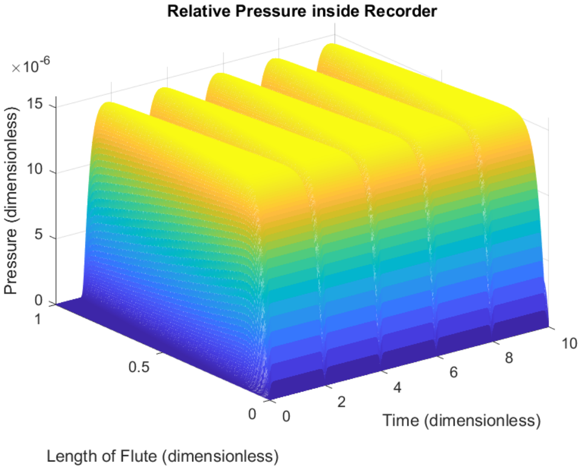

Figure: Pressure over time with no holes

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To analyze the frequencies of the recorder we used Fast Fourier Transform.

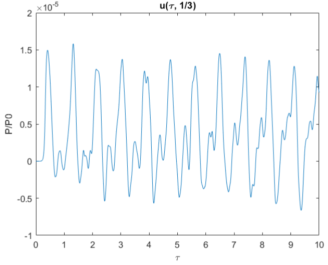

Figure: Pressure inside the flute with 7 holes

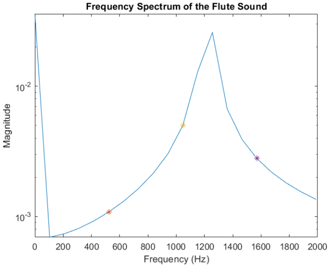

Figure: Frequency for $\tau = 10$

The low resultion is due to low N, which is the length of the time vector, which creates large steps in frequencies, $\Delta f = f_s/N$, where f_s is the Nyquist frequency $f_s = 1/2dt$

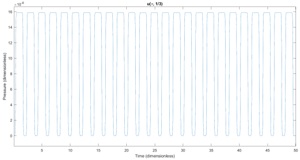

Figure: Pressure inside the flute with 0 holes

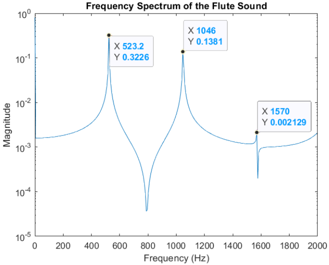

Figure: Frequency for $\tau = 150$

The first spike is a $f_0 = 523.25Hz$, the resonance frequency. The following spikes are the overtones. We can thus verify the theory.

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Pressure mesh plot with holes

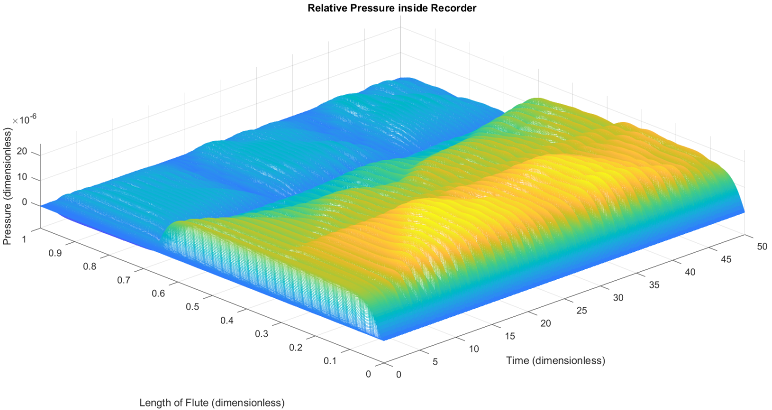

Figure: Pressure with one large hole s=100

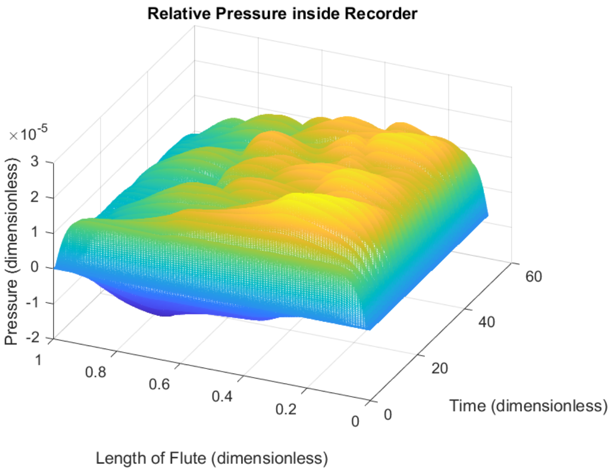

Figure: Pressure with one small hole, s = 1

Holes with small s

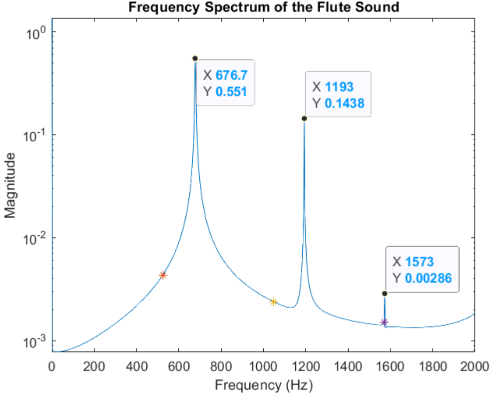

Figure: Frequency spectrum with one hole at $y = 0.381$, $dy = dt = 0.002$. $s = 7.866$

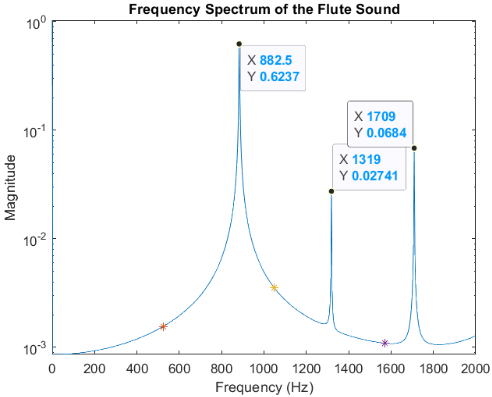

Figure: Frequency spectrum with two holes at y = 0.381 and 0.4286, dy = dt = 0.002. s = 7.866, 15.732

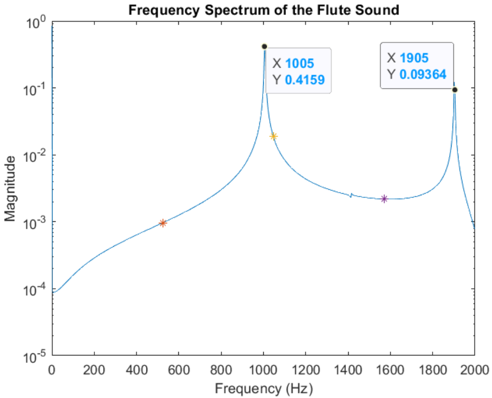

Figure: Frequency spectrum with three holes at $y = 0.381, 0.4286, 0.4762$, $dy = dt = 0.002$. $s = 7.866, 15.732, 23.599$

The peaks start to appear for higher frequencies up on the frequency axis with more number of holes. This tells us that the sound wave inside the flute oscilates faster when small holes are added.

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Direct way of computing the fundamental frequency

Standing wave U fullfills the homegenous rescaled PDE

$$\begin{cases} U_{\tau\tau} - U_{yy} + d(y)u = 0 \\ U(0, y) = 0, \qquad U(\tau, 0) = U(\tau, 1) = 0 \end{cases} \tag{1}$$

$$U(\tau, y) = a(y)e^{2\pi i f \tau}. \tag{2}$$

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The amplitude $a(y)$

By (1) and (2):

$$-a(y)(2\pi f)^2 e^{2\pi i f \tau} - a_{yy} e^{2\pi i f \tau} + d(y)a(y)e^{2\pi i f \tau} = 0$$

$$\Leftrightarrow -a_{yy} + d(y)a(y) = a(y)(2\pi f)^2.$$

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Thus discretising using central difference we get:

$$-A\mathbf{a} + D\mathbf{a} = (2\pi f)^2\mathbf{a},$$

$$\text{where } A = \frac{1}{\Delta y^2}\begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix},$$

$D = diag(d(y_i))$ and $\mathbf{a} = (a(y_1), \dots, a(y_{N-1}))^T$, $i = 1, \dots, N-1$.

Eigenfunctions corresponding to fund. f and nearest overtones

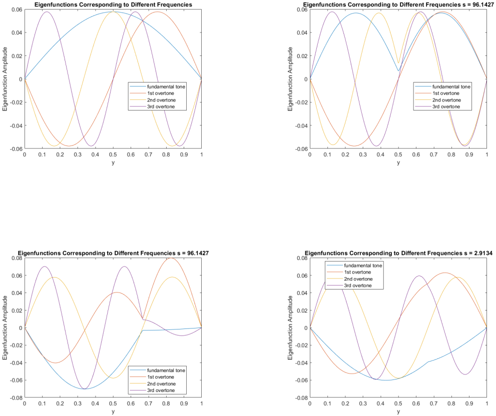

Figure: Eigenfunction/amplitude with no holes, one large hole in y=1/2 and y=2/3, and 1 small hole at y = 2/3, respectively.