Parabolic PDEs, smearing property

$$\begin{cases} u_t - u_{xx} = 0 & x \in (0,\pi),\ t>0 \\ u(x,0) = g(x) & x \in (0,\pi) \\ u(0,t) = u(\pi,t) = 0 & t>0 \end{cases}$$

Exact solution by Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} \hat{u}_k(t) \sin(kx)$$

Boundary conditions

$u(0,t) = u(\pi,t) = 0$ are satisfied because

$$\sin(k\cdot 0) = \sin(k\cdot\pi) = 0,\ k = 1,2,\dots$$

$$u(x,0) = \sum_{k=1}^{\infty} \hat{u}_k(0)\sin(kx) = g(x)$$

$$= \sum_{k=1}^{\infty} \hat{g}_k \sin(kx),\ \text{so}\ \hat{u}_k(0) = \hat{g}_k$$

Compute u_t and u_{xx}:

$$u_t(x,t) = \sum_{k=1}^{\infty} \hat{u}_k'(t) \sin(kx)$$

$$u_{xx}(x,t) = \sum_{k=1}^{\infty} \hat{u}_k(t) \left(-k^2 \cdot \sin(kx)\right)$$

$$u_t - u_{xx} = \sum_{k=1}^{\infty} \left(\hat{u}_k'(t) - k^2 \hat{u}_k(t)\right) \sin(kx)$$

$$\Rightarrow \hat{u}_k'(t) - k^2 \hat{u}_k(t) = 0 \qquad y' - k^2 y = 0$$

ODE in fourier coefficients $\hat{u}_k(t)$!

Exact solution is

$$\hat{u}_k(t) = \hat{u}_k(0) e^{-k^2 t} = \hat{g}_k e^{-k^2 t}$$

and $u(x,t) = \sum_{k=1}^{\infty} \hat{g}_k e^{-k^2 t} \sin(kx)$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \cdots & \\ & & & \\ & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ \\ \\ \beta \end{bmatrix}$$

Robin BC:

$$\frac{u_1 - u_{-1}}{2\Delta x} = \alpha_0 u_0 + \alpha_1$$

Express u_{-1} (GP value) in terms of u_1, u_0:

$$u_1 - u_{-1} = 2\Delta x\, \alpha_0 u_0 + 2\Delta x\, \alpha_1$$

$$\Leftrightarrow u_{-1} = u_1 - \underbrace{2\Delta x\, \alpha_0}_{d_0} u_0 - \underbrace{2\Delta x\, \alpha_1}_{d_{-1}}$$

$$= u_1 - d_0 \cdot u_0 - d_{-1} \qquad (*)$$

Use (*) in equation for $j=0$:

$$\frac{du_0(t)}{dt} = \frac{u_{-1} - 2u_0 + u_1}{\Delta x^2} =$$

$$\frac{u_1 + d_0 u_0 + d_{-1} - 2u_0 + u_1}{\Delta x^2} =$$

$$\underbrace{\frac{2u_1 + (d_0 - 2)u_0}{\Delta x^2}}_{\text{change 1st row in A}} + \underbrace{\frac{d_{-1}}{\Delta x^2}}_{\text{add to } f}$$

Stability Euler forward:

Stability condition

$$-2 < \Delta t \lambda_k < 0$$

$$\lambda_k \in \left(-\frac{4}{\Delta x^2}, 0\right)$$

$$-2 < \frac{-4\Delta t}{\Delta x^2} < 0$$

always fulfilled

$\Delta t, \Delta x > 0$

$$-2 < \frac{-4\Delta t}{\Delta x^2} \quad \cdot \frac{-1}{4}\Delta x^2$$

$$\frac{\Delta x^2}{2} > \Delta t \quad \text{or} \quad \boxed{\Delta t < \frac{\Delta x^2}{2}}$$

Implicit Euler:

$$\bar{u}^{n+1} = \bar{u}^{n} + \Delta t \left(A \bar{u}^{n+1} + \bar{b}(t_{n+1}) \right)$$

$$\underbrace{\bar{u}^{n+1} - \Delta t A \bar{u}^{n+1}}_{\underbrace{(I - \Delta t A)}_{\text{tridiagonal matrix in 1D}} \bar{u}^{n+1}} = \underbrace{\bar{u}^{n} + \Delta t \cdot \bar{b}(t_{n+1})}_{\text{known vector}}$$

Crank-Nicolson

$$\bar{u}^{n+1} = \bar{u}^{n} + \frac{1}{2} \Delta t \Big(A \bar{u}^{n} + \bar{b}(t_n)$$

$$+ A \bar{u}^{n+1} + \bar{b}(t_{n+1}) \Big) :$$

$$\underbrace{\bar{u}^{n+1} - \frac{1}{2}\Delta t A \bar{u}^{n+1}}_{(I - \frac{1}{2}\Delta t A)\bar{u}^{n+1}} =$$

$$\underbrace{\bar{u}^{n} + \frac{1}{2}\Delta t A \bar{u}^{n}}_{(I + \frac{1}{2}\Delta t A)\bar{u}^{n}} + \frac{1}{2}\left(\bar{b}(t_n) + \bar{b}(t_{n+1})\right)$$

$$\left(I - \frac{1}{2}\Delta t A\right)\bar{u}^{n+1} = \underbrace{\left(I + \frac{1}{2}\Delta t A\right)\bar{u}^{n}}$$

$$\underbrace{+ \frac{1}{2}\left(\bar{b}(t_n) + \bar{b}(t_{n+1})\right)}_{\text{known vector}}$$

Stability for implicit methods

Test equation: $y'(t) = \lambda y(t)$

Euler backward: $\lambda < 0$

$$u^{n+1} = u^n + \Delta t \lambda u^{n+1}$$

$$(1 - \Delta t \lambda) u^{n+1} = u^n$$

$$u^{n+1} = \frac{1}{(1 - \Delta t \lambda)} u^n = \frac{1}{(1-\Delta t\lambda)^n} u^0$$

Amplification factor:

$$\frac{1}{|\underbrace{1 - \Delta t \lambda}_{>1}|} < 1$$

always fulfilled

$\Delta t > 0,\ \lambda < 0$

Crank-Nicolson:

$u^{n+1} = u^n + \frac{1}{2}\Delta t \lambda u^n +$

$\frac{1}{2}\Delta t \lambda u^{n+1}$

$$\left(1 - \frac{1}{2}\Delta t \lambda\right) u^{n+1} = \left(1 + \frac{1}{2}\Delta t \lambda\right) u^n$$

$$u^{n+1} = \frac{\left(1 + \frac{1}{2}\Delta t \lambda\right)}{\left(1 - \frac{1}{2}\Delta t \lambda\right)} u^n$$

for stability $\dfrac{\overbrace{\left|1 + \frac{1}{2}\Delta t \lambda\right|}^{<1}}{\left|1 - \frac{1}{2}\Delta t \lambda\right|} < 1$

$\Delta t > 0, \lambda < 0$

$\underbrace{\qquad\qquad}_{> 1}$

always fulfilled