Convergence in distribution:

Example 1: De Moivre-Laplace Thm. (X_i) i.i.d. with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2 \Rightarrow \mathbb{E} X_i = 0.$$

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} 0 \quad (LLN)$$

empirical mean

Law of large numbers.

Consider

$$Z_n := \sqrt{n}\,\overline{X}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i.$$

$$\Rightarrow \mathbb{E} Z_n = 0 \; ; \; \operatorname{Var} Z_n = \operatorname{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)$$

independence

$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{\operatorname{Var}(X_i)}_{=1} = \frac{n}{n} = 1.$$

Thm. (De Moivre-Laplace)

$$\mathbb{P}(a < Z_n \le b) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx, \quad a<b. \quad //$$

Rephrased

$$F_{Z_n}(x) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy = F_Z(x), \; Z \in \mathcal{N}(0,1)$$

$$F_{Z_n}(x) \longrightarrow F_Z(x), \; Z \in \mathcal{N}(0,1) \quad \text{"convergence in distribution"}$$

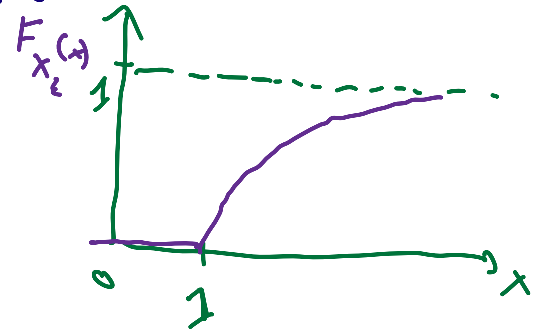
$//$

CDF of Z_n

CDF of a standard Normal r.v.

Example 2: $X_1, X_2, \dots$ iid with CDF

$$F_{X_k}(x) = \begin{cases} 1 - \frac{1}{x^\alpha} & \text{if } x > 1 \\ 0 & \text{else}. \end{cases}, \quad \alpha > 0$$



Set $Y_n := n^{-1/\alpha} \max_{1 \le k \le n} (X_k)$.

$$\Rightarrow F_{Y_n}(x) = \mathbb{P}\left(\max_{1\le k\le n} X_k \le n^{1/\alpha} x\right)$$

$$= \mathbb{P}(X_1 \le n^{1/\alpha}x, X_2 \le n^{1/\alpha}x, \dots, X_n \le n^{1/\alpha}x)$$

$$\overset{\text{independence}}{=} \prod_{k=1}^{n} \mathbb{P}(X_k \le n^{1/\alpha}x) \overset{\text{identically distributed}}{=} \begin{cases} \left(1 - \frac{1}{nx^\alpha}\right)^n, & \text{if } x > n^{-1/\alpha}, \\ 0 & \text{else}. \end{cases}$$

$$= \begin{cases} \left(1 - \frac{x^{-\alpha}}{n}\right)^n & \text{if } x > n^{-1/\alpha} \xrightarrow[\text{as } n\to\infty]{} 0 \\ 0 & \text{else}. \end{cases}$$

$$\boxed{e^x = \lim_{n\to\infty}\left(\frac{x}{n}+1\right)^n}$$

Euler identity

$$\xrightarrow{n\to\infty} \begin{cases} e^{-x^{-\alpha}} & \text{if } x > 0 \\ 0 & \text{else}. \end{cases}$$

Check that this is a CDF of a random variable: $F_{Y_n}(x) \to F_Y(x)$.

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Def: X_n converges in distribution to a random variable X, as $n \to \infty$, if

$$F_{X_n}(x) \longrightarrow F_X(x) \quad \text{as } n \to \infty \quad \text{for all } x \in C(F_X)$$

where $C(F_X) := \{ x \in \mathbb{R} : F_X \text{ is continuous at } x \}$. //

Notation: $X_n \xrightarrow{d} X$, $n \to \infty$. Convergence in distribution.

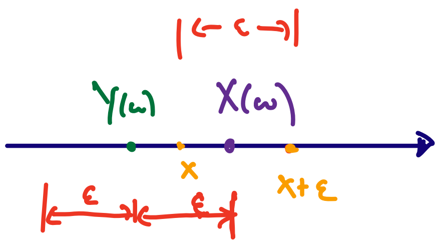
Lemma: $X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X$ //

Auxiliary claim: X, Y random variables, $x \in \mathbb{R}$, $\varepsilon > 0$. Then

$$\mathbb{P}(Y \le x) \le \mathbb{P}(X \le x + \varepsilon) + \mathbb{P}(|Y - X| > \varepsilon). //$$

Proof of claim:

$$\mathbb{P}(Y \le x) = \underbrace{\mathbb{P}(Y \le x, |X - Y| \le \varepsilon)}_{\le \mathbb{P}(X \le x + \varepsilon)} + \underbrace{\mathbb{P}(Y \le x, |X - Y| > \varepsilon)}_{\le \mathbb{P}(|X - Y| > \varepsilon)}$$



If $Y \le x$ and $|X - Y| \le \varepsilon$, then $X \le x + \varepsilon$.

Thus $\{Y \le x\} \cap \{|X - Y| \le \varepsilon\} \subset \{X \le x + \varepsilon\}$.

use $\mathbb{P}(A \cap B) \le \mathbb{P}(B)$

$$\le \mathbb{P}(X \le x + \varepsilon) + \mathbb{P}(|X - Y| > \varepsilon). \quad \blacksquare$$

Upshot: $F_Y(x) \le F_X(x + \varepsilon) + \mathbb{P}(|X - Y| > \varepsilon)$

4.

Back to the proof: Let $x \in \mathbb{R}$ be a continuity point of F_X. Let $\varepsilon > 0$.

$$(*)\quad F_{X_n}(x) = \mathbb{P}(X_n \le x) \underbrace{\le}_{\text{Claim with } Y = X_n} \mathbb{P}(X \le x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$$
$$= F_X(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon).$$

Repeat the argument by switching X and X_n, and changing x to $x-\varepsilon$ and $x+\varepsilon$ to x.

$$(**)\quad F_X(x-\varepsilon) \le F_{X_n}(x) + \mathbb{P}(|X_n - X| > \varepsilon).$$

$(*)$ & $(**)$

$$\Longrightarrow \underbrace{F_X(x-\varepsilon)}_{\text{no } n!} \overset{(**)}{\le} F_{X_n}(x) + \underbrace{\mathbb{P}(|X_n - X| > \varepsilon)}_{\to 0,\ n\to\infty.\ \text{as } X_n \xrightarrow{P} X} \overset{(*)}{\le} \underbrace{F_X(x+\varepsilon)}_{\text{no } n!} + 2\underbrace{\mathbb{P}(|X_n - X| > \varepsilon)}_{\to 0,\ n\to\infty}$$

We do not know at this point whether $F_{X_n}(x)$ has a limit as $n \to \infty$. But we must have from the above that

$$(\dagger)\quad F_X(x-\varepsilon) \le \liminf_{n\to\infty} F_{X_n}(x) \le \limsup_{n\to\infty} F_{X_n}(x) \le F_X(x+\varepsilon).$$

no ε!

By assumption x is a continuity point of F_X: $\lim_{\varepsilon\to 0} F_X(x-\varepsilon) = \lim_{\varepsilon \to 0} F_X(x+\varepsilon) = F_X(x)$.

Take now $\varepsilon \to 0$ in $(\dagger)$:

$$F_X(x) = \lim_{n\to\infty} F_{X_n}(x). \qquad \blacksquare$$

⚠ Used that x is a continuity point of F_X.

Summary:

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X \quad \text{as } n \to \infty$$

$$\Uparrow$$

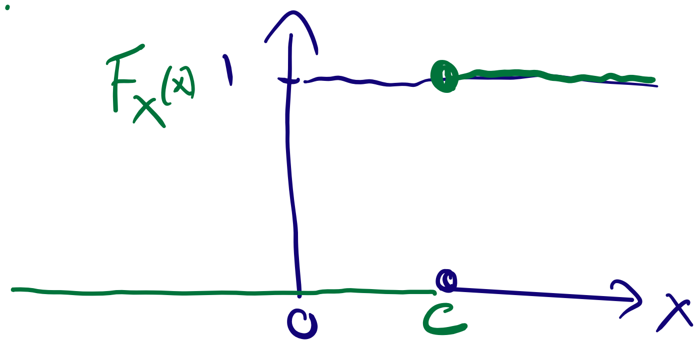
$$X_n \xrightarrow{2} X$$

Thm: $X_1, X_2, \ldots$ a sequence of random variables. Let $c \in \mathbb{R}$. Then

$$X_n \xrightarrow{P} c \iff X_n \xrightarrow{d} \delta(c)$$

number
not a random variable.

Shorthand for $X \in \delta(c)$.



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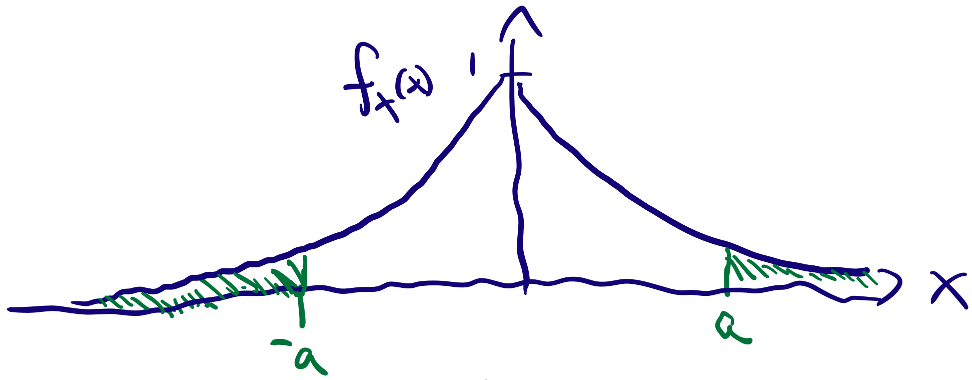
Proof: Exercise, page 157 in [AG].

⑥

Laplace distribution with parameter $\lambda = 1$.

Example: $X \in \mathcal{L}(1)$, $\quad f_X(x) = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$



$X_1, X_2, \ldots$ a sequence of random variables defined by

$$X_{2n} := X, \quad X_{2n-1} := -X, \quad n = 1, 2, 3, \ldots$$

($2n$: even, $2n-1$: odd)

Note $X_{2n} \overset{d}{=} X_{2n-1}$, because f_X is symmetric about zero.

In particular, $X_n \xrightarrow{d} X$, as $n \to \infty$. $\left(F_{X_n}(x) = F_X(x) \right)$

But, $\mathbb{P}(|X| > a) > 0$, for any $a > 0$.

Hence, for any $\varepsilon > 0$:

$$\mathbb{P}(|X_n - X| > \varepsilon) = \begin{cases} 0 & \text{for } n \text{ even, because } X_n = X. \\ \mathbb{P}(|X| > \frac{\varepsilon}{2}) > 0 & \text{for } n \text{ odd.} \end{cases}$$

indep of n

Because $X_n = -X$, so $|X_n - X| = 2|X|$ and hence $\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}(2|X| > \varepsilon)$ for n odd.

$$\Longrightarrow X_n \cancel{\xrightarrow{P}} X$$

X_n does not converge in probability. //