Characteristic function

Def: Characteristic function of a random variable X is

$i = \sqrt{-1}$.

$$\varphi_X(t) := \mathbb{E}\left[e^{itX}\right] \overset{\text{e.g.}}{=} \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx$$

Remark: $\Psi_X(t) = \mathbb{E}\left[e^{tX}\right]$ (if mgf exists) $t \mapsto it$ gives $\varphi_X(t)$.

$i = \sqrt{-1}$, $i^2 = -1$.

Upshot: $\varphi_X(t)$ exists for all random variables and all $t \in \mathbb{R}$,

$$\left|\mathbb{E}\, e^{itX}\right| \leq \mathbb{E}\underbrace{\left|e^{itX}\right|}_{\substack{=1 \text{ because } X \text{ is} \\ \text{real valued} \\ \text{and } t \in \mathbb{R}}} = \mathbb{E}[1] = 1.$$

"Price we pay": $\varphi_X(t)$ is complex valued.

Recall: $e^{iy} = \cos(y) + i\sin(y)$, "Euler formula"

$|e^{iy}| = 1$, $\forall y \in \mathbb{R}$.

②

Ex: $X \in Be(p) \Rightarrow \varphi_X(t) = q + pe^{it}$, $p+q=1$.

Ex: $X \in U(a,b)$

$$\varphi_X(t) = \int_a^b e^{itx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\frac{1}{it} e^{itx}\Big|_a^b$$

$$= \frac{1}{(b-a)it}\left(e^{itb} - e^{ita}\right)$$ (What happens when $t \to 0$?)

Lemma: X a r.v. Then

a.) $|\varphi_X(t)| \leq \varphi_X(0) = 1$

b.) $\overline{\varphi_X(t)} = \varphi_X(-t)$ complex conjugate

c.) $\varphi_X(t)$ is (uniformly) continuous //

Proof: a) $|\varphi_X(t)| = |\mathbb{E}e^{itX}| \leq \mathbb{E}\underbrace{|e^{itX}|}_{=1} = \mathbb{E}[1] = \mathbb{E}[e^0] = \varphi_X(0)$.

b.) $\overline{\varphi_X(t)} = \overline{\mathbb{E}[e^{itX}]} = \mathbb{E}[e^{-itX}]$
$= \varphi_X(-t)$.

c.) Fix t and let $|h|>0$

$$|\varphi(t+h)-\varphi(t)| = |\mathbb{E}[e^{i(t+h)X}] - \mathbb{E}[e^{itX}]|$$

$$\overset{\text{linearity}}{=} |\mathbb{E}[e^{i(t+h)X} - e^{itX}]|$$

$$= |\mathbb{E}[e^{itX}(e^{ihX}-1)]|$$

$$\overset{\text{move absolute value inside}}{\leq} \mathbb{E}[\underbrace{|e^{itX}|}_{=1} \cdot |e^{ihX}-1|]$$

Observe: $|e^{ihx}-1| \overset{\text{triangle inequality}}{\leq} \underbrace{|e^{ihx}|}_{=1} + 1 = 2$ for any $x \in \mathbb{R}$.

also $|e^{ihx}-1| \xrightarrow{h\to 0} 0$.

It follows from dominated convergence theorem (or close our eyes, or look at [AG]) that

$$\lim_{h\to 0} |\varphi(t+h)-\varphi(t)| \leq \lim_{h\to 0} \mathbb{E}[|e^{ihX}-1|] = 0$$

$\Rightarrow$ $\varphi_X(t)$ is continuous in t.

$\boxed{\text{////}}$

Thm 1: Let X and Y be random variables. If

$$\varphi_X(t) = \varphi_Y(t) \quad \text{for all } t,$$

then $$X \overset{d}{=} Y. \quad //$$

equality in distribution.

w/o proof.

Thm 2: Let $X_1, X_2, \dots, X_n$ be <u>independent</u> random variables, and set

$$S_n := \sum_{k=1}^{n} X_k .$$

Then

$$\varphi_{S_n}(t) = \prod_{k=1}^{n} \varphi_{X_k}(t) \qquad \forall t \in \mathbb{R} \quad //$$

Exercise: Prove Thm 2.

Some more examples:

- $X \in Exp(a)$, $a > 0$. Then $\varphi_X(t) = \dfrac{1}{1 - ait}$

  Observe: Since a and t are real, we have no singularity.

- $\boxed{X \in \mathcal{N}(0,1) \text{ then } \varphi_X(t) = e^{-t^2/2}}$ ← Good to remember!

Thm 3: Let X be a random variable. If $\mathbb{E}|X|^n < \infty$ for some $n = 1, 2, \ldots$ then

a.) $$\varphi_X^{(k)}(0) = \frac{d^k}{dt^k} \varphi_X(t) \Big|_{t=0} = (i)^k \mathbb{E}[X^k].$$

for all $k = 1, 2, \ldots, n$.

b.) $$\varphi_X(t) = 1 + \sum_{k=1}^{n} \mathbb{E}[X^k] \cdot \frac{(it)^k}{k!} + o(|t|^n)$$

as $t \to 0$. //

Little-o notation: For a function $f(t)$, $f(t) = o(|t|^n)$ means

$$\lim_{t \to 0} \frac{f(t)}{|t|^n} = 0$$

"$|f(t)|$ is much smaller than $|t|^2$ as t becomes small."

"Proof": Follows from Taylor expansion of e^{itx} to order n:

$$e^{itx} = 1 + \sum_{k=1}^{n} \frac{(itx)^k}{k!} + o(|tx|^n)$$

and use then $\mathbb{E}|X|^k < \infty$ for all $k \le n$. ∎

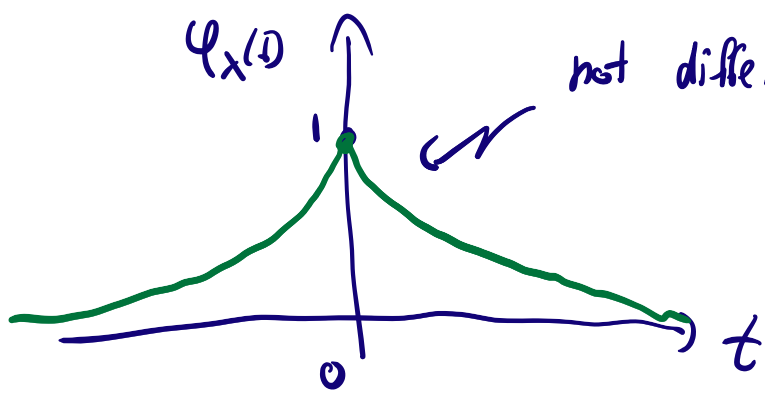
Remark: $\varphi_X(t) = 1 + it\,\mathbb{E}[X] - \frac{t^2}{2}\mathbb{E}[X^2] + o(t^2)$

as $t \to 0$.

Example: $X \in C(0,1)$. We know that the moment generating function does not exist. Expected value and moments do not exist.

- Fourier transform
- Complex analysis (residue thm)

$$\varphi_X(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{itx}\,\frac{1}{1+x^2}\,dx = e^{-|t|}, \quad t \in \mathbb{R}.$$



not differentiable at $t=0$.

Remark: X a random variable

$\varphi_X(t)$ is real $\iff$ the distribution of X is symmetric

$X \overset{d}{=} -X \iff F_X(x) = F_{-X}(x)$

Ex: $X \in \mathcal{N}(\mu, \sigma^2)$ then $\varphi_X(t) = e^{i\mu t - \frac{t^2\sigma^2}{2}}$

real if $\mu = 0$.

Exercise: X a r.v. Let $a, b \in \mathbb{R}$. Show that

$\varphi_{aX+b}(t) = e^{ibt}\varphi_X(at)$ //