SF2520 — Applied numerical methods

Lecture 12

Hyperbolic equations

Anna Nissen
Numerical analysis
Department of Mathematics, KTH

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Today's lecture

- Hyperbolic equations
  - Recap
  - Applications
  - Characteristics, scalar case
  - Characteristics, systems
  - Boundary conditions

Hyperbolic PDEs

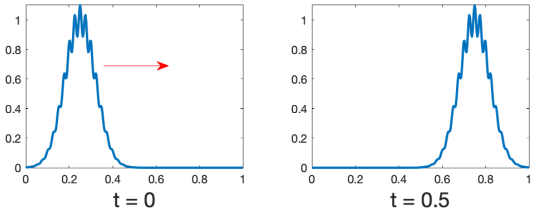
We consider the simplest hyperbolic PDE, the 1D scalar "advection" or "transport" equation for $u = u(x, t)$

$$\begin{aligned} u_t + au_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= g(x), \end{aligned}$$

where a is a constant and $g \in C^1(\mathbb{R})$.

- Model for transport and one-way wave propagation.
- Solution easy to write down,

$$u(t, x) = g(x - at) \qquad \text{(check: } u_t = -ag' \text{ and } au_x = ag'\text{)}$$



- a represents speed of propagation.

Hyperbolic PDEs

Other more complicated versions of advection equation:

- $u_t + a(x)u_x = 0,$ (variable coefficient a)
- $\boldsymbol{u}_t + A\boldsymbol{u}_x = 0,$ (system)
- $\boldsymbol{u}_t + A(x)\boldsymbol{u}_x = 0,$ (system+variable coefficient)
- $\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y = 0,$ (system, 2D)
- $\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0,$ (nonlinear)
- Combinations of the above.

Classification, requirements for hyperbolicity:

- $u_t + a(x)u_x = 0,$ a real
- $\boldsymbol{u}_t + A\boldsymbol{u}_x = 0,$ A diagonalizable with real eigenvalues
- $\boldsymbol{u}_t + A(x)\boldsymbol{u}_x = 0,$ $A(x)$ ——"—— for all x
- $\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y = 0,$ $\alpha_1 A + \alpha_2 B$ ——"—— for all α_1, α_2
- $\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0,$ $J(\boldsymbol{u})$ ——"—— for all $\boldsymbol{u}(x,t)$

  Typical case is that A, B, J are real symmetric matrices.

Applications

Acoustic waves

The wave equation in 1D is

$$u_{tt} = c^2 u_{xx},$$

where u is sound pressure deviation and c is the speed of propagation of waves.

- Can be written as a system of hyperbolic equations. Let

$$\boldsymbol{u} = \begin{pmatrix} u_t \\ u_x \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{u}_t = \begin{pmatrix} u_{tt} \\ u_{xt} \end{pmatrix} = \begin{pmatrix} c^2 u_{xx} \\ u_{xt} \end{pmatrix} \qquad \boldsymbol{u}_x = \begin{pmatrix} u_{xt} \\ u_{xx} \end{pmatrix},$$

  so that

$$\boldsymbol{u}_t - \underbrace{\begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix}}_{A} \boldsymbol{u}_x = 0.$$

- Easy to check that A is diagonalizable and eigenvalues are $\pm c$.
- Called the "first order form" of the wave equation.

Maxwell equations

For no free charges or currents we have:

$$\boldsymbol{E}_t - c^2\nabla \times \boldsymbol{B} = 0, \qquad \boldsymbol{B}_t + \nabla \times \boldsymbol{E} = 0.$$

where

$$\boldsymbol{E} = \text{electric field}\,, \qquad \boldsymbol{B} = \text{magnetic field}\,, \qquad c = \frac{1}{\sqrt{\epsilon\mu}} = \text{speed of light}$$

- Can be written as a system of hyperbolic equations.
- Note first that in 3D (with $\boldsymbol{B} = (B_1, B_2, B_3)^T$),

$$\nabla \times \boldsymbol{B} = \begin{pmatrix} \partial_y B_3 - \partial_z B_2 \\ \partial_z B_1 - \partial_x B_3 \\ \partial_x B_2 - \partial_y B_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{R_1} \boldsymbol{B}_x + \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{R_2} \boldsymbol{B}_y + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{R_3} \boldsymbol{B}_z,$$

where $R_j^T = -R_j$, skew symmetric.

Maxwell equations

$$\boldsymbol{E}_t - c^2\nabla \times \boldsymbol{B} = 0, \qquad \boldsymbol{B}_t + \nabla \times \boldsymbol{E} = 0.$$

- Let
$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix}.$$
- Recalling
$$\nabla \times \boldsymbol{B} = R_1\boldsymbol{B}_x + R_2\boldsymbol{B}_y + R_3\boldsymbol{B}_z,$$
we get
$$\begin{aligned} \boldsymbol{u}_t &= \begin{pmatrix} c^2\nabla \times \boldsymbol{B} \\ -\nabla \times \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} c^2(R_1\boldsymbol{B}_x + R_2\boldsymbol{B}_y + R_3\boldsymbol{B}_z) \\ -(R_1\boldsymbol{E}_x + R_2\boldsymbol{E}_y + R_3\boldsymbol{E}_z) \end{pmatrix} \\ &= \begin{pmatrix} 0 & c^2R_1 \\ -R_1 & 0 \end{pmatrix}\boldsymbol{u}_x + \begin{pmatrix} 0 & c^2R_2 \\ -R_2 & 0 \end{pmatrix}\boldsymbol{u}_y + \begin{pmatrix} 0 & c^2R_3 \\ -R_3 & 0 \end{pmatrix}\boldsymbol{u}_z \\ &=: A_1\boldsymbol{u}_x + A_2\boldsymbol{u}_y + A_3\boldsymbol{u}_z. \end{aligned}$$
- It can be verified that $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ is diagonalizable with real eigenvalues. E.g. if $c = 1$ then the A_j matrices are symmetric.

Euler equations

Let $\rho =$ density, $u =$ fluid velocity. Then 1D isentropic Euler equations read

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + \left(\rho u^2 + \kappa \rho^\gamma\right)_x &= 0. \end{aligned}$$

Here $\gamma =$ heat capacity ratio and $\kappa > 0$ a constant depending on initial data.

- Can be written as a nonlinear system of hyperbolic equations. Let $m = \rho u$ = momentum and

$$\boldsymbol{u} = \begin{pmatrix} \rho \\ m \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{u}_t + \begin{pmatrix} m \\ \frac{m^2}{\rho} + \kappa\rho^\gamma \end{pmatrix}_x =: \boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0.$$

- Jacobian of $\boldsymbol{F}$ and its eigenvalues $\lambda_\pm$ are

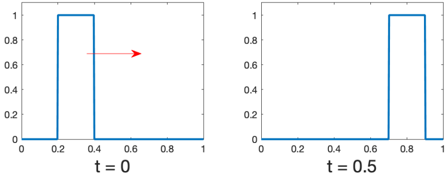
$$J(\boldsymbol{u}) = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + \kappa\gamma\rho^{\gamma-1} & \frac{2m}{\rho} \end{pmatrix}, \qquad \lambda_\pm = u \pm \sqrt{\kappa\gamma\rho^{\gamma-1}}.$$

  Hence, J is diagonalizable with real eigenvalues when $\rho > 0$.
- Linearization of the Euler equations around $u = 0$ gives wave equation with $c = \sqrt{\kappa\gamma\rho^{\gamma-1}} = \sqrt{P'(\rho)}$ where $P(\rho) = \kappa\rho^\gamma$.

Hyperbolic PDEs

Remarks:

- Solution is not getting smoother as in the parabolic case. Makes numerics more difficult.
- We usually require that initial data g is smooth, at least $C^1(\mathbb{R})$. This gives smooth solution u.
- If g is not $C^1(\mathbb{R})$ one can still define $u(x,t) = g(x - at)$ as a generalized (weak) solution. E.g a square pulse:



  This would solve the weak form of the advection equation defined similar to the elliptic case.
- In non-linear problems (e.g. Euler) the solution may become discontinuous even if g is smooth (corresponds to shock waves in Euler). Weak solutions must then always be considered.

Characteristics

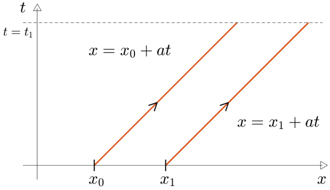
Solution to

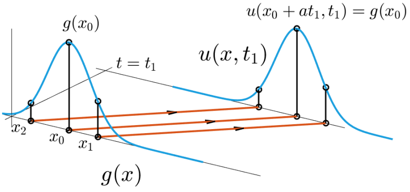
$$u_t + au_x = 0, \qquad u(x,0) = g(x),$$

is

$$u(x,t) = g(x - at) \qquad \text{or} \qquad u(x_0 + at, t) = g(x_0) \quad \forall x_0.$$

- This means that $u(x,t)$ is constant along the lines $x =$constant$+at$ in the (x,t)-plane.
- These lines are called characteristics.
- In a 3D plot we can visualize the solution as the initial data propagates, staying constant along the characteristics.





Characteristics, generalization

Want to find similar description for variable coefficients,

$$u_t + a(x)u_x = 0, \qquad u(x,0) = g(x).$$

- Define the characteristic X as the curve in the (x,t) plane given by the ODE
$$\frac{dX}{dt} = a(X), \qquad X(0) = x_0.$$
To tell characteristics with different starting points apart, we write $X = X(t; x_0)$ for the curve starting in x_0.
- Then solution u is constant along X, since
$$\begin{aligned}\frac{d}{dt}u(X(t),t) &= u_t(X(t),t) + \frac{dX}{dt}u_x(X(t),t)\\ &= u_t(X(t),t) + a(X(t))u_x(X(t),t) = 0.\end{aligned}$$
- Moreover, the constant is given by initial data,
$$u(t, X(t;x_0)) = u(0, X(0;x_0)) = u(0,x_0) = g(x_0).$$
- Hence, $u(t,x) = g(X^{-1}(t,x))$.

Characteristics

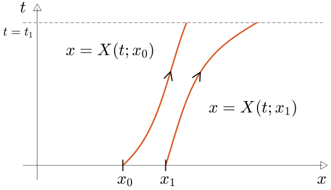
For

$$u_t + a(x)u_x = 0, \qquad u(x,0) = g(x),$$

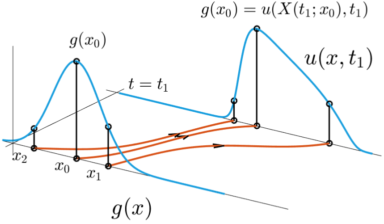
- The solution $u(t,x)$ is constant along the characteristics $x = X(t; x_0)$, in the (x,t)-plane, where

$$\frac{dX}{dt} = a(X), \qquad X(0) = x_0.$$

  (Note: If a =constant then $X = x_0 + at$ as before.)



- In a 3D plot we can visualize the solution as the initial data propagates, staying constant along the characteristics, $u(X(t; x_0)) = g(x_0)$.



Conclusions

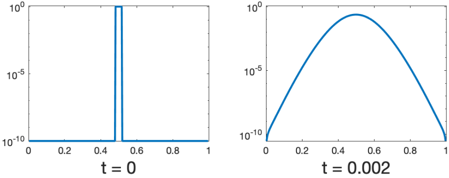
- Solution $u(X(t; x_0), t)$ only depends on inital data in one point: $g(x_0)$ (not the rest!). In general, for hyperbolic problems $u(x, t)$ depends in initial data in a bounded domain called the domain of dependence for (x, t).
- Conversely, initial data value $g(x_0)$ only influences solution in one point at t (and in a bounded domain in general).
- *Information propagates along characteristics*, and, assuming the coefficient a is bounded,

$$\left|\frac{dX(t; x_0)}{dt}\right| = |a(X(t))| < \infty,$$

information propagates with finite speed.

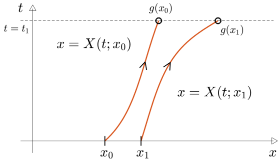
Conclusions

- Finite speed of propagation characterizes all hyperbolic PDEs.
- Not true for parabolic PDEs: solution at time $t > 0$ depends on all of $g(x)$. *Information propagates with infinite speed.*

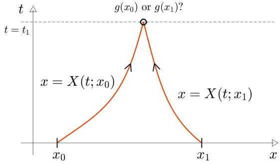


(log scale)

- Characteristics cannot cross for linear equations (since solutions of ODEs are unique) $\Rightarrow$ solution is well-defined by characteristics.
- In non-linear equations they can cross! $\Rightarrow$ solution not well-defined $\Rightarrow$ discontinuous solution.



vs



Systems of hyperbolic PDEs

Consider now a system of hyperbolic PDEs in 1D,

$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ \boldsymbol{u}(x,0) &= \boldsymbol{g}(x), \end{aligned}$$

where $\boldsymbol{u}(x,t) \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$.

- If A is diagonalizable and has real eigenvalues this is a hyperbolic system.
- Then, with eigenvalues λ_k and eigenvectors $\boldsymbol{e}_k$, let

$$A = S\Lambda S^{-1}, \qquad S = \begin{pmatrix} | & & | \\ \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_d \\ | & & | \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}.$$

- Defining $\boldsymbol{v}(x,t) := S^{-1}\boldsymbol{u}(x,t)$ gives

$$\boldsymbol{v}_t = S^{-1}\boldsymbol{u}_t = -S^{-1}A\boldsymbol{u}_x = -S^{-1}S\Lambda(S^{-1}\boldsymbol{u})_x = -\Lambda\boldsymbol{v}_x,$$

  so that $\boldsymbol{v}$ satisfies,

$$\boldsymbol{v}_t + \Lambda\boldsymbol{v}_x = 0, \qquad \boldsymbol{v}(x,0) = S^{-1}\boldsymbol{g}(x).$$

Systems of hyperbolic PDEs

After diagonalization we obtained

$$\mathbf{v}_t + \Lambda \mathbf{v}_x = 0, \qquad x \in \mathbb{R}, \quad t > 0,$$
$$\mathbf{v}(x, 0) = S^{-1}\mathbf{g}(x),$$

where $\Lambda \in \mathbb{R}^{d\times d}$ is diagonal with eigenvalues on diagonal.

- We get d uncoupled scalar advection equations for the components of $\mathbf{v} = (v_1, \ldots, v_d)^T$,

$$\frac{\partial v_k}{\partial t} + \lambda_k \frac{\partial v_k}{\partial x} = 0, \qquad k = 1, \ldots, d.$$

- Setting $\mathbf{w} = S^{-1}\mathbf{g} = (w_1, \ldots, w_d)^T$ we can write the solutions as

$$v_k(x, t) = w_k(x - \lambda_k t)$$

and

$$\mathbf{u}(x, t) = S\mathbf{v}(x, t) = \sum_{k=1}^{d} v_k(x, t)\mathbf{e}_k = \sum_{k=1}^{d} w_k(x - \lambda_k t)\mathbf{e}_k.$$

Example (Acoustic wave equation, first order form)

$$\boldsymbol{u}_t - \underbrace{\begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix}}_{A} \boldsymbol{u}_x = 0, \qquad \boldsymbol{u}(x,0) = \boldsymbol{g}(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}.$$

- For A we have eigenvalues and eigenvectors:

$$\boldsymbol{e}_1 = \begin{pmatrix} c \\ -1 \end{pmatrix}, \qquad \lambda_1 = c, \qquad \boldsymbol{e}_2 = \begin{pmatrix} c \\ 1 \end{pmatrix}, \qquad \lambda_2 = -c.$$

- Then the modes are given by,

$$S\boldsymbol{w} = \begin{pmatrix} c & c \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

which has the solution

$$w_1 = \frac{1}{2c}(g_1 - cg_2), \qquad w_2 = \frac{1}{2c}(g_1 + cg_2), \qquad \text{(note: } \boldsymbol{g} = w_1\boldsymbol{e}_1 + w_2\boldsymbol{e}_2\text{)}.$$

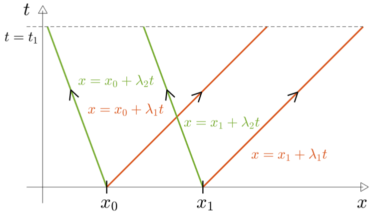
- The full solution is then: $u(x,t) = w_1(x-ct)\boldsymbol{e}_1 + w_2(x+ct)\boldsymbol{e}_2$, i.e. propagation of waves with speed c backwards and forwards (c.f. d'Alembert's solution formula).

Systems of hyperbolic PDEs

We have obtained the general solution

$$\boldsymbol{u}(x,t) = \sum_{k=1}^{d} w_k(x - \lambda_k t)\boldsymbol{e}_k, \qquad \boldsymbol{w} = S^{-1}\boldsymbol{g} = \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix}$$

- w_k are the different (eigen)modes of the solution ($w_k\boldsymbol{e}_k$ is an eigenvector of A).
- The eigenvalue λ_k represents the speed of propagation of the corresponding eigenmode w_k. Thus, several different speeds in general.
- There are now d families of characteristics: the lines $x =$constant $+\lambda_k t$, with $k = 1, \ldots, d$, propagating initial data in different directions.

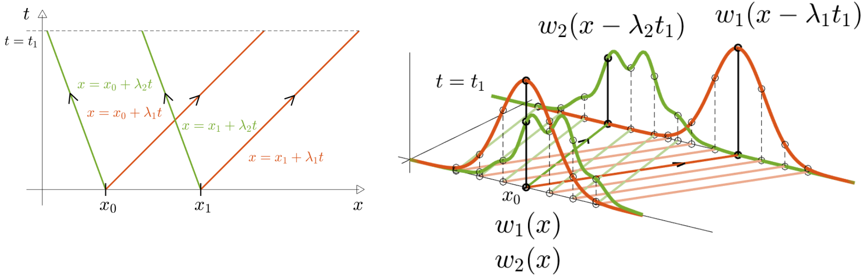


Systems of hyperbolic PDEs

Suppose we have a system of two equations. Then

$$\boldsymbol{u}(x,t) = w_1(x-\lambda_1 t)\boldsymbol{e}_1 + w_2(x-\lambda_2 t)\boldsymbol{e}_2, \qquad S\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \boldsymbol{g}.$$

- We have two families of characteristics with modes w_1 and w_2.



- Solution $u(x,t)$ depends on $\boldsymbol{g}(x)$ evaluated in $x - \lambda_1 t$ and $x - \lambda_2 t$, i.e. two points. This is the domain of dependence for (x,t). In general it consists of d points for a system of d equations.

Boundary conditions

Consider now advection equation in a bounded domain,

$$\begin{aligned} u_t + au_x &= 0, \qquad x \in (0,1), \quad t > 0, \\ u(x,0) &= g(x). \end{aligned}$$

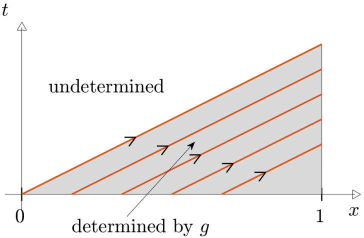
How to choose boundary conditions at $x = 0$ and $x = 1$?

- Setting boundary conditions more tricky for hyperbolic equations than parabolic or elliptic.
- Observation
  If $a > 1$ the solution is already uniquely determined at $x = 1$ and $t < 1/a$ since
  $$u(1,t) = g(1 - at).$$
  $\Rightarrow$ If we specify a boundary condition $u(1,t) = \beta$ the problem would not have a solution in general! (Same for $x = 0$ if $a < 0$.)
- Cannot specify boundary condition at $x = 1$ when $a > 0$!

Boundary conditions

Consider
$$u_t + au_x = 0, \qquad x \in (0,1), \quad t > 0,$$
$$u(x,0) = g(x).$$

- Initial data g propagates along lines with slope a. Part of the solution is completely determined by it.
- Rest must be given by boundary conditions.



- $x = 0$ called inflow boundary (if $a > 0$)
  $x = 1$ called outflow boundary (if $a > 0$)
- Rule:
  - Must specify BC at inflow (at $x = 0$ if $a > 0$)
  - Cannot specify BC at outflow (at $x = 1$ if $a > 0$)
- BC can be any of the usual type: Dirichlet, Neumann, Robin. (Periodic also possible here.)

Boundary conditions, summary

Consider

$$
\begin{aligned}
u_t + a u_x &= 0, \qquad x \in (0,1), \quad t > 0, \\
u(x,0) &= g(x).
\end{aligned}
$$

- For scalar equations:
  - If $a > 0$, give one BC at $x = 0$.
  - If $a < 0$, give one BC at $x = 1$.
- For systems of equations, BC selection is determined by the signs of the eigenvalues:
  - # positive eigenvalues = # BC at $x = 0$.
  - # negative eigenvalues = # BC at $x = 1$.
- **Example:** Acoustic wave equation
  Here we have the two eigenvalues $\lambda = \pm c$, i.e. one of each sign.
  $\Rightarrow$ should set one BC at $x = 0$ and one at $x = 1$.