KTH Matematik

Exercises in Mathematical Systems Theory

by Per Enqvist.

Division of Optimization and Systems Theory
Department of Mathematics
Royal Institute of Technology
Stockholm, Sweden

Contents

1 Linear Dynamical Systems **5**

2 Reachability and Observability **7**

3 Stability **13**

4 Realization Theory **14**

5 Feedback, Pole-assignment and Observers **17**

6 Linear-quadratic Optimal Control **22**

7 Kalman filters **25**

8 Answers to the exercises **28**

8.1 Answers to: Linear Dynamical Systems 28
8.2 Answers to: Reachability and Observability 31
8.3 Answers to: Stability 35
8.4 Answers to: Realization Theory 36
8.5 Answers to: Feedback, Pole-assignment and Observers 43
8.6 Answers to: Linear-quadratic Optimal Control 49
8.7 Answers to: Kalman filters 53

1 Linear Dynamical Systems

1.1 Consider the matrix

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}.$$

Determine e^{At}.

1.2 Compute the transition matrix $\Phi(t, s)$ for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x.$$

1.3 Compute the transition matrix $\Phi(t, s)$ for the harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

1.4 Consider the system

$$\dot{x} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} x.$$

(a) Compute the solution by the Laplace method.

(b) Compute the solution by diagonalizing A.

1.5 Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} x.$$

Compute the solution by using Jordan forms.

1.6 Show the Cayley-Hamilton theorem using a Jordan transformation.

1.7 Consider the time-varying system $\dot{x}(t) = A(t)x(t)$. Show that if $A(t)$ and $\int_s^t A(\tau)d\tau$ commute then

$$\Phi(t, s) = e^{\int_s^t A(\tau)d\tau}.$$

1.8 Let A be an arbitrary invertible constant matrix.

(a) Is the following equality always valid ?

$$\int_0^t e^{As}\, ds = A^{-1}[e^{At} - I].$$

(b) Determine a sufficient condition for the equality

$$A^{-1} = -\int_0^\infty e^{As}\,ds$$

to be valid.

1.9 Determine the transfer matrix of the system

$$\dot{x}(t) = A(t)x(t),$$

where the block matrix $A(t)$ is of the form

$$A(t) = \begin{bmatrix} f(t)B & g(t)C \\ 0 & f(t)B \end{bmatrix}.$$

The matrixes B and C commute and f, g are scalar functions.

1.10 Consider the system

$$\dot{x}(t) = \begin{bmatrix} \alpha & e^{-t} \\ -e^{-t} & \alpha \end{bmatrix} x(t).$$

Compute $\Phi(t,s)$.

1.11 Determine the transfer matrix of the system

$$\dot{x}(t) = A(t)x(t),$$

where the matrix $A(t)$ is

$$A(t) = \begin{bmatrix} \cos(t) & t \\ 0 & \cos(t) \end{bmatrix}.$$

1.12 Consider the time varying system $\dot{x}(t) = A(t)x(t), \quad t > -1$, where

$$A(t) = \begin{bmatrix} \frac{1}{1+t} & 0 \\ te^{-t} & \frac{1}{1+t} \end{bmatrix}.$$

Determine the transfer matrix $\Phi(t,s)$ for $s, t > -1$.

1.13 Consider the system

$$\dot{x}(t) = \begin{bmatrix} \cos(t) - \frac{4}{t} & \frac{-1}{t} \\ \frac{4}{t} & \cos(t) \end{bmatrix} x(t).$$

Compute $\Phi(t,s)$.

1.14 Consider the time-varying system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ t & \frac{1}{t} \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad t > 0.$$

Compute $\Phi(t,s)$.

2 Reachability and Observability

2.1 Consider the time-varying system of 1.14,

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ t & \frac{1}{t} \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad t > 0.$$

Given $x(t_0)$, what states $x(t_1)$ are reachable?

2.2 Consider a linear time invariant system of the form $\dot{x} = Ax + bu$, and assume that A is diagonal with one multiple eigenvalue. Can the system then be reachable?

2.3 Under what conditions on b is the system

$$\dot{x} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} x + bu,$$

reachable?

2.4 Consider the system

$$\dot{x} = Ax + bu, \quad \text{where } x \in \mathbb{R}^n \quad \text{and } u \in \mathbb{R}.$$

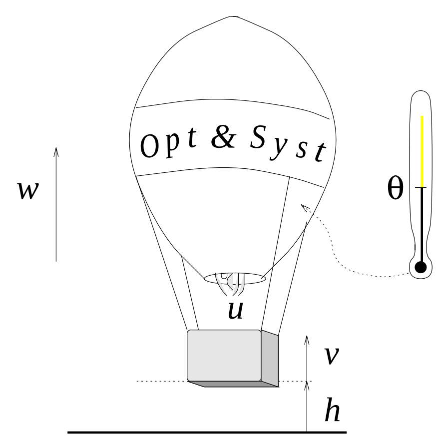
Assume the state $d \in \mathbb{R}^n$ is reachable from $c \in \mathbb{R}^n$.
Is it then possible to transfer the system from c to d in an arbitrary time ?

2.5 A rough description of the movement of a hot air balloon is

$$\begin{cases} \dot{\theta} &= -\frac{1}{\alpha}\theta + u, \\ \dot{v} &= -\frac{1}{\beta}v + \sigma\theta + \frac{1}{\beta}w, \\ \dot{h} &= v, \end{cases}$$

where

$$\begin{aligned} \theta &= \text{temperature}, \\ u &= \text{heating}, \\ v &= \text{vertical velocity}, \\ h &= \text{height}, \\ w &= \text{vertical wind velocity}. \end{aligned}$$



The parameters α, β and σ are given positive constants of the system.

(a) Assume that the wind velocity w is constant but unknown. Is it then possible to reconstruct θ and w through observations of the height h ?

(b) Assume that $w = 0$, is the system then reachable?

2.6 Consider the system Σ given by

$$\dot{x} = Ax + bu, \quad \text{where } x \in \mathbb{R}^n \quad \text{and } u \in \mathbb{R}.$$

Assume that A has n distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with linearly independent right hand side eigenvectors $\{b_1, b_2, \ldots, b_n\}$, i.e.

$$Ab_i = \lambda_i b_i, \quad i = 1, 2, \ldots, n.$$

Since $\{b_1, b_2, \ldots, b_n\}$ form a base for $\mathbb{R}^n$, there exist unique α_i:s so that

$$b = \sum_{i=1}^{n} \alpha_i b_i.$$

Show that the system is completely reachable iff $\alpha_i \neq 0$ for $i = 1, 2, \ldots, n$.

2.7 Consider the system

$$\begin{cases} \dot{x}(t) &= A(t)x(t), \\ y(t) &= Cx(t). \end{cases}$$

Denote the transfer matrix by $\Phi(s, t)$, and define the observability grammian

$$M(t_0, t_1) \triangleq \int_{t_0}^{t_1} \Phi'(t, t_0) C'(t) C(t) \Phi(t, t_0)\, dt.$$

We start the system in an unknown point $x(t_0)$ at t_0, and observe the system over $[t_0, t_1]$.

(a) Show that we can distinguish between the starting points $x(t_0) = a$, and $x(t_0) = b$ iff $b - a \notin \ker M(t_0, t_1)$.

(b) Assume now that A and C are constant. Define the observability matrix Ω, and show that $\ker M(t_0, t_1) = \ker \Omega$.

(c) Define the "quiet" subspace S as $S \triangleq \ker \Omega$. Partition the state space as $\mathbb{R}^n = V \oplus S$, where V is the orthogonal complement of S, $V = S^\perp$.

Then show that S is A-invariant, $A\mathcal{S} \subseteq \mathcal{S}$, and determine the block-matrix representation of the system corresponding to the subspaces above.

(d) Assume now that we have scalar observations, i.e. that the matrix C is a row vector.

Show that the pair $[A, C]$ is completely observable iff there exists no vector $q \neq 0$ such that

i. q is a right hand eigenvector of A.

ii. q is orthogonal to C.

2.8 Consider the complex functions $x(t,z)$, where t is time and z is a complex variable, such that for each fixed value of t the functions are entire as a variable of z. (entire=analytic in the whole complex plane)

Let us denote the class of such functions by *Hol*.

We would like to study linear systems of the form

$$\left\{\begin{array}{rcl} \frac{\partial}{\partial t}x(t,z) & = & \frac{\partial}{\partial z}x(t,z), \\ x(0,z) & = & \phi(z), \quad z \in \mathbb{C}. \end{array}\right. \qquad \dagger$$

We can think of $x(t,z)$ as an infinite dimensional vector $\bar{x}(t)$ for each t. (one coordinate for each value of z)

From this point of view and with $A = \frac{\partial}{\partial z}$ the system † assumes the familiar form

$$\left\{\begin{array}{rcl} \dot{\bar{x}}(t) & = & A\bar{x}(t), \\ \bar{x}(0) & = & \phi(z). \end{array}\right.$$

From the well known solution $x(t) = e^{At}x(0)$ of linear systems, we are led to believe that

$$x(t,z) = \left[\exp(t\partial_z)\phi\right](z), \quad z \in \mathbb{C}. \qquad \ddagger$$

(a) Define for each complex value α the operator

$$\exp(\alpha\partial_z) : Hol \mapsto Hol,$$

and show that the solution to the system † is given by ‡.

(b) Let c be a function $c : \mathbb{C} \mapsto \mathbb{C}$ and let Γ be a fixed curve in $\mathbb{C}$.
Define the mapping $C : Hol \mapsto \mathbb{C}$, by

$$Cf \triangleq \int_\Gamma c(z)f(z)\,dz.$$

Now we can define the observed system Σ by,

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t), \\ y(t) & = & Cx(t). \end{array}\right.$$

Define the "quiet" subspace S and determine whether the result

$$S = \ker\Omega, \quad \text{where} \quad \Omega = [C,\ CA,\ CA^2,\ \ldots]',$$

is valid also in the infinite case.

Note that, since the A and C operators do not have a finite dimensional matrix representation, we can not use Cayley-Hamilton to reduce the dimension of the Ω-matrix.

(c) Let Γ be the interval $[0,1]$ and let $c(z) = \delta(z)$ be the dirac function.
Is Σ completely observable ?

(d) Let Γ be the interval $[-\pi,\pi]$ and let $c(z) \equiv 1$.
Is Σ completely observable ?

2.9 Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an analytic function in the whole complex plane $\mathbb{C}$, with the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Let A be an arbitrary quadratic matrix, and define

$$f(A) \triangleq \sum_{n=0}^{\infty} c_n A^n.$$

The sum above can be shown to converge. Define the spectrum, $\sigma[A]$, as

$$\sigma[A] \triangleq \{\text{eigenvalues of } A\}.$$

The well known "spectral mapping theorem" states

$$\sigma[f(A)] = \{f(\lambda) \mid \lambda \in \sigma[A]\}.$$

(a) Show that if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$. (Half the theorem above.)

(b) Define f as

$$f(z) = \int_0^\delta e^{(\delta - t)z}\, dt,$$

and show that $f(A)$ is invertible iff A has no eigenvalue of the form

$$\lambda = k\frac{2\pi i}{\delta}, \quad k = \pm 1, \pm 2, \ldots$$

The spectral mapping theorem may be used in the proof.

(c) Consider the system Σ, given by

$$\dot{x} = Ax(t) + Bu(t),$$

where A and B are given constant matrixes.
Let δ be a fixed time interval, and consider the sampled system $\Sigma(\delta)$, defined by the state $y_k = x(k\delta)$. We let the control be constant over each interval $[k\delta, (k+1)\delta]$.
Determine the matrixes F, G in the representation

$$y_{k+1} = Fy_k + Gv_k,$$

of the sampled system $\Sigma(\delta)$. (In terms of A, B and δ.)

(d) Show that $\Sigma(\delta)$ is reachable iff the following to conditions are fulfilled.

i. The pair $[e^{\delta A}, B]$ is reachable.
ii. A has no eigenvalue λ of the form

$$\lambda = k\frac{2\pi i}{\delta}, \quad k = \pm 1, \pm 2, \ldots$$

2.10 Consider a train on a one-dimensional railway described by the equation

$$m\ddot{x} = -k\dot{x} + u,$$

where m is the mass of the train and k is a friction coefficient. Suppose that $x(t_0)$ and $\dot{x}(t_0)$ are given and that we want to reach $x(t_1) = \dot{x}(t_1) = 0$, where t_1 is a given time. In order to maximize the comfort of the passengers we want to operate the train in such a way that the integral

$$\int_{t_0}^{t_1} \ddot{x}(t)^2 \, dt,$$

is minimized. Determine the optimal control.

2.11 Consider the following problem of process control. In a cylindrical tank of cross-sectional area S there is a volume $V(t)$ of fluid having the concentration $c(t)$ of some substrate. The tank has two different inflows of constant substrate concentration, $u_1(t)$ of concentration c_1 and $u_2(t)$ of concentration c_2. The outflow of the tank is $F(t)$.

The change of fluid in the tank is

$$\dot{V}(t) = -F(t) + u_1(t) + u_2(t),$$

and the change of substrate is

$$\frac{d}{dt}(c(t)V(t)) = -c(t)F(t) + c_1u_1(t) + c_2u_2(t).$$

The outflow of the tank is given by the Bernoulli law,i.e.

$$F(t) = k\sqrt{\frac{V(t)}{S}}.$$

Suppose that $u_1(t) = F_{10}, u_2(t) = F_{20}, F(t) = F_0$ and $c(t) = c_0, V(t) = V_0$ is known to be a stationary solution of the differential equations.

(a) Linearize the system along the stationary solution and determine a linear system (approximately) describing the deviation from the stationary solution.

(b) Is the linearized system reachable?

(c) Is the linearized system observable? (Considering the linearization of $F(t)$ as the output.)

(d) What happens if $c_1 = c_2$, and what can be controlled?

2.12 Consider the classical inverted pendulum on a cart. The cart is moving in the horizontal direction and the pendulum is moving in a vertical plane through the horizontal axis. The objective is to move the cart in the horizontal direction, keeping the pendulum in the upward direction.

Let the pendulum consist of a massless rod of length L with a mass m attached at the far end, and let the mass of the cart be M. Suppose that the influence of the pendulum on the cart's motion is negligible.

Let $s(t)$ be the position of a point of reference on the cart and let $u(t)$ be the force applied to the cart in the horizontal direction.

The motion of the cart is given by

$$M\ddot{s} = u(t) - f\dot{s},$$

where f is a constant of friction. The motion of the pendulum is given by

$$L\ddot{\phi} + \ddot{s}\, cos\phi - g\, sin\phi = 0.$$

(a) Linearize the system along the solution $\phi = \dot{\phi} = s = \dot{s} = u = 0$.

(b) Is the linearized system stable?

(c) Is the linearized system reachable?

(d) Suppose that only the angle of the pendulum can be measured. Verify that the system is not observable. What is the unobservable subspace?

2.13 Consider the classical satellite problem. A satellite is modeled as a particle of unit mass moving in an inverse square law force field. The position of the satellite is given by the radius $r(t)$ and the angle $\theta(t)$. The satellite has the capability of thrusting in the radial direction with a thrust $u_1(t)$ and thrusting in the tangential direction with a thrust $u_2(t)$. The equations of motion are

$$\begin{cases} \ddot{r} &= r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\ \ddot{\theta} &= -2\frac{1}{r}\dot{r}\dot{\theta} + \frac{1}{r}u_2. \end{cases}$$

(a) Verify that with no input applied, circular orbits of the form $r(t) = \sigma, \theta(t) = \omega t$, where σ and ω are constants, are possible. What is the relation between k, σ and ω?

(b) Normalize such that $\sigma = 1$ and linearize the system along a circular orbit.

(c) Is the linearized system reachable?

(d) Is the system reachable if the radial thrust is broken? Is the system reachable if the tangential thrust is broken?

(e) Is the system observable if only r can be measured? If only θ can be measured?

2.14 Consider the harmonic oscillator in 1.3. Sample the system with sampling time h. Assume piecewise constant input and determine the discrete-time representation of the system $x(t+h) = Fx(t) + Gu(t)$.

(a) For what values of h is the sampled system not reachable?

(b) For what values of h is the sampled system not observable?

(c) Let $\omega = 1$ and $h = 2\pi$. What are F and G? Interpret.

(d) Let $\omega = 1$ and $h = \pi$. What are F and G? Interpret.

3 Stability

3.1 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

Use the Lyapunov method to determine if A is a stable matrix,

(a) in continuous time sense.

(b) in discrete time sense.

3.2 Consider

$$\dot{x} = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} x.$$

Discuss for what c the system is unstable.

3.3 Consider the matrix

$$A = \begin{bmatrix} -3 & -13 \\ \frac{1}{2} & 2 \end{bmatrix}.$$

Use the Lyapunov method to determine if A is a stable matrix,

(a) in continuous time sense.

(b) in discrete time sense.

3.4 Consider the discrete-time system

$$x(t+1) = Ax(t).$$

Show that the system is asymptotically stable if and only if $|\lambda(A)| < 1$.

3.5 Is the scalar system

$$\begin{cases} \dot{x}(t) &= x(t) + e^{2t}u(t) \\ y(t) &= e^{-2t}x(t) \end{cases}$$

input-output stable?

3.6 Consider the system of 1.13,

$$\dot{x}(t) = \begin{bmatrix} \cos(t) - \frac{4}{t} & \frac{-1}{t} \\ \frac{4}{t} & \cos(t) \end{bmatrix} x(t).$$

For large t it holds that $A(t) \approx \cos(t)I$. Does the solution of $\dot{z}(t) = \cos(t)z(t)$, where $z(t) \in \mathbb{R}^2$, behave like that of $\dot{x}(t) = A(t)x(t)$?

4 Realization Theory

4.1 Consider the transfer function

$$g(s) = \frac{2s-4}{s^3-7s+6}.$$

(a) Determine the standard reachable realization.

(b) Determine the standard observable realization.

(c) Determine a minimal realization.

4.2 A control system has the transfer function

$$R(s) = \left[\frac{1}{s^2+4s+3}, \frac{1}{s+1}\right].$$

(a) Determine the observable canonic realization.

(b) Determine the McMillan degree of the system.

4.3 Consider the transfer function

$$R(s) = \left[\frac{s}{s-1}, \frac{2s}{s^2-1}\right].$$

(a) Determine a realization.

(b) Determine the McMillan degree of the system.

4.4 Determine the McMillan degree of the system

$$R(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{s^2+4}{s^2-4} \end{bmatrix}.$$

4.5 Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s-1} \\ \frac{1}{s} & 0 \end{bmatrix}.$$

(a) Determine the standard reachable realization.

(b) Determine the standard observable realization.

(c) What is the dimension of a minimal realization?

(d) Determine a minimal realization.

4.6 Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}.$$

(a) Determine the standard reachable realization.

(b) Is the standard reachable realization minimal?

(c) Determine the standard observable realization.

(d) Is the standard observable realization minimal?

(e) What is the dimension of a minimal realization?

(f) Determine a minimal realization.

4.7 Consider the transfer matrix

$$R(s) = \left[\begin{array}{cc} \frac{1}{s-1} & \frac{2s}{s^2-1} \end{array} \right].$$

(a) Determine a realization.

(b) Determine the McMillan degree of the system.

4.8 Consider the transfer matrix

$$R(s) = \left[\begin{array}{cc} \frac{1}{s-3} & \frac{2s-5}{s^2-5s+6} \end{array} \right].$$

(a) Determine a realization.

(b) Determine the McMillan degree of the system.

4.9 Consider the transfer matrix

$$R(s) = \left[\begin{array}{cc} \frac{s-1}{s} & \frac{1}{s^2-1} \end{array} \right].$$

(a) Determine the standard observable realization.

(b) Is the realization minimal?

4.10 Consider the transfer matrix

$$R(s) = \left[\begin{array}{cc} \frac{1}{(s+1)^2} & \frac{1}{s+2} \\ 0 & \frac{1}{s^2+3s+2} \end{array} \right].$$

(a) Determine the standard reachable realization.

(b) Determine the standard observable realization.

(c) What is the dimension of a minimal realization?

4.11 Given the transfer function

$$R(s) = \left[\begin{array}{cc} \frac{1}{s^3} & \frac{1}{s^2-s} \\ \frac{1}{s^2} & 0 \end{array} \right].$$

(a) Determine a realization.

(b) Determine the McMillan degree of the system.

4.12 Consider the system

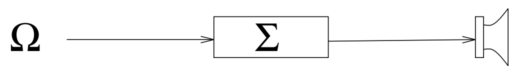
$$\left\{ \begin{array}{rcl} \dot{x} & = & \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] x + \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] u, \\ y & = & \left[\begin{array}{ccc} 1 & -1 & 1 \end{array} \right] x. \end{array} \right.$$

(a) Is the system reachable and/or observable ?

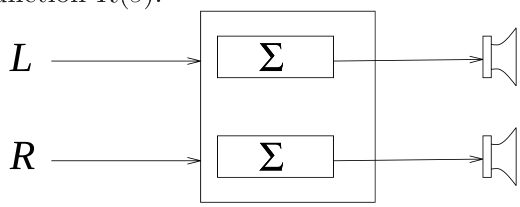
(b) Determine the transfer function of the system .

(c) Use the result of (b) to determine a minimal realization of the system.

4.13 Consider a manufacturer of amplifiers. Given is a mono-amplifier of model Σ, with the known scalar transfer function $G(s) = p(s)/q(s)$. We also know that there is no common zeros of p and q, and the degree of q is strictly greater than that of p.



We can now combine two mono-amplifiers to one stereo-amplifier, called Σ^2, with transfer function $R(s)$.



The manufacturers concern is if this production uses a minimal number of components for the implementation of the transfer function $R(s)$.
Determine the McMillan degree of the amplifier Σ^2.

4.14 Let $R(s)$ be a rational strictly proper matrix of dimension $m \times k$.

(a) Assume that $m = 1$, and that R has a representation

$$R(s) = \sum_{i=1}^{p} \frac{h_i}{s - \lambda_i},$$

where all h-vectors are nonzero and $\lambda_i \neq \lambda_j$ for $i \neq j$.
Determine a minimal realization of R explicitly.

(b) Let m be arbitrary, and assume that R has a representation

$$R(s) = \sum_{i=1}^{p} \frac{H_i}{s - \lambda_i},$$

where $\operatorname{rank} H_i = r_i, \quad i = 1, \ldots, p$ and as before $\lambda_i \neq \lambda_j$ for $i \neq j$.
Show that there is a realization of R of dimension $r = r_1 + r_2 + \ldots + r_p$.
Hint: If A is a linear mapping $A : \mathbb{R}^k \mapsto \mathbb{R}^m$ with rank r, then A can be factorized as

$$A = CB : \mathbb{R}^k \xrightarrow{B} \mathbb{R}^r \xrightarrow{C} \mathbb{R}^m.$$

5 Feedback, Pole-assignment and Observers

5.1 Given the system

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u,$$

determine the feedback that makes the eigenvalues of the closed-loop system -1 and -2. Compute the gain vector k directly without using any equivalence transformation.

5.2 Consider the system

$$\begin{cases} \dot{x} &= \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x. \end{cases}$$

(a) Verify that the system is unstable, but that it is possible to move the poles of the system by means of linear state feedback.

(b) Find a state-feedback law placing the poles in $\{-2, -1+i, -1-i\}$.

(c) Verify that the system is observable and construct an observer with poles in -2 such that the feedback law can be applied.

5.3 Consider the system

$$\begin{cases} \dot{x} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + u. \end{cases}$$

Determine an observer with poles in -1 estimating the state.

5.4 Consider the system

$$\ddot{x} = a\dot{x} + bx + u,$$

where u is the input, and x is the output signal.

(a) Determine a state space realization.

(b) Is the system reachable?

(c) Determine a state feedback so that the closed loop system has the poles $-1+i$ and $-1-i$.

(d) Is it possible to stabilize the system with an output signal feedback?

If we take the output signal $y = \dot{x}$ instead, is it then possible to determine a feedback of the output so the system is stabilized ?

5.5 Consider the problem of balancing a pointer on the tip of your finger. Suppose the pointer consists of a mass less rod of length L with a point mass m attached in the far end, and that the motion takes place in a vertical plane. Moreover, suppose that you can control the acceleration u of your fingertip instantaneously. Let $\phi(t)$ be the angle of the pointer and $s(t)$ be the position of your fingertip. The nonlinear equations of motion, according to Newton, are

$$\begin{cases} L\ddot{\phi} + \ddot{s}\,cos\phi - g\sin\phi &= 0 \\ \ddot{s} &= u. \end{cases}$$

(a) Linearize the nonlinear system along the solution $\phi(t) = s(t) = u(t) = 0$ to get a linear system (approximately) describing the deviations from the given solution.

(b) Is the linearized system stable?

(c) Is the linearized system reachable?

(d) Is it possible to stabilize the system by a control law of the form $u = kx_1$, i.e. using information about the angle only?

(e) Suppose the whole state is accessible for feedback. Find a feedback law placing the poles in $-\sqrt{\frac{g}{L}}$.

(f) Let $g = 9$ and $L = 1$ and suppose that $y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$, i.e. the angle is measured. Verify that the system is observable and construct an observer with poles in $\{-10 + 10i, -10 - 10i\}$.

5.6 Consider the system

$$\begin{cases} \dot{x} &= \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} x \end{cases}$$

(a) Is the system stable?

(b) Is the system reachable?

(c) Find a feedback gain k placing the poles in $\{-1, -1 + i, -1 - i\}$.

(d) Is the system observable?

(e) Determine a three-dimensional observer such that the control law can be used.

5.7 An input-output system described by the transfer function

$$g(s) = \frac{1}{s^3 - 4s^2 + 6s - 4},$$

is to be stabilized. Construct an observer with a triple-pole in -2 and a feedback law placing the remaining poles in $\{-1, -1 + i, -1 - i\}$.

5.8 A space shuttle follows the dynamics $\ddot{y} = u$, where y is the location and u is the force from the rockets.

(a) Find a minimal state space form of the system.

(b) What are the poles of the open loop system?

(c) Make a pole placement by state feedback so that the poles of the closed loop system are $1 + i$ and $1 - i$.

(d) Determine an observer with the poles $2 + 2i$ and $2 - 2i$.

(e) What are the poles of the system when we use the feedback in (c) on the observed states?

5.9 Consider the system

$$\tilde{y}(s) = \frac{s+1}{s^2-1}\tilde{u}(s).$$

Do not cancel the factor $s + 1$ in the transfer function above.

(a) Determine the canonical reachable realization of the system.

(b) Is the system minimal?

(c) Determine a state feedback so that the closed loop system has the double pole -1.

(d) Is it possible to stabilize the system with an output signal feedback?

5.10 Consider the discrete-time system

$$x(t+1) = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t).$$

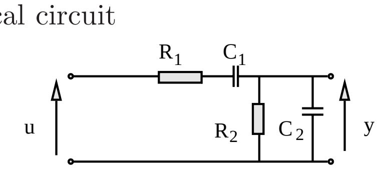
Find a linear feedback law such that for any initial state, the system returns to the origin in a fixed finite number of steps. This is called dead-beat control.

5.11 Consider the system

$$x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Determine a state feedback law $u(t) = Kx(t)$, so the closed loop system have a 6-periodic solution, but no solution with a longer period.

5.12 Consider the electrical circuit



with $C_1 = C_2 = C$ and $R_1 = R_2 = R$. Consider the circuit as an input-output system with the voltages u and y as in- and outputs.

(a) Determine a state space representation having the capacitor voltages as state variables.

(b) Is the system input-output stable?

(c) Construct an oscillator by letting $u = ky$, where k is a constant gain.

(d) What is the frequency of the oscillator?

5.13 Consider the system

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} u.$$

(a) Show that the system is unstable.

(b) Can you, using state feedback, place the poles of the closed-loop system in

i. $-2, -2, -1, -1$

ii. $-2, -2, -2, -1$

iii. $-2, -2, -2, -2$.

5.14 Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u.$$

Is the system stabilizable, and can you in particular find a feedback gain k such that the closed-loop system has a double pole in -1 and a triple pole in -2?

5.15 Let (A, B, C) be a minimal realization of SISO system with the transfer function

$$G(s) = C(sI - A)^{-1}B.$$

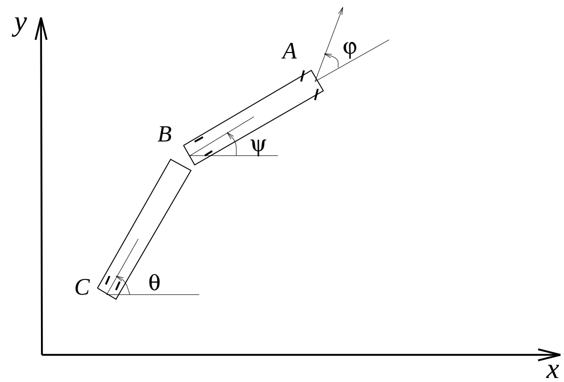
Assume that the system is in reachable canonic form. If $G(s_0) = 0$, then $s_0 \in \mathbb{C}$ is called a zero of the system.

(a) If we use a feedback $u = Kx + v$, how can the zeros of the closed loop system be altered ?

(b) What is possible to say about reachability, observability and minimality of the closed loop system ?

5.16 Consider the following model of a truck with a trailer.

Let the length of the truck and the trailer both be 1. The truck has two pairs of wheels, while the trailer just have one. We model the vehicle as two stiff rods as in the figure.



The control is modeled by a vector of length 1 describing the direction of the trucks front wheels, point B. The following nonlinear equations govern the system.

$$\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\varphi+\psi) \\ \sin(\varphi+\psi) \\ \sin(\varphi) \\ \cos(\varphi)\sin(\psi-\theta) \end{bmatrix}.$$

We would now like to model the problem of backing up along a straight line, say the x-axis.
i.e. $\bar{x}(t) = [-t, 0, 0, 0]'$, $\bar{\varphi}(t) = \pi$ is the desired solution.

(a) Determine the linearized model $\dot{z} = A(t)z + B(t)u$ that approximates the deviations $z(t)$, $u(t)$ from $\bar{x}(t)$, $\bar{\varphi}$.

Hint : The system is of the form $\dot{x} = f(x, u)$, so

$A_{i,j}(t) = \dfrac{\partial f_i}{\partial x_j}(\bar{x}(t), \bar{u}(t))$, and $B_{i,j}(t) = \dfrac{\partial f_i}{\partial u_j}(\bar{x}(t), \bar{u}(t))$.

(b) Show that the linearized system in (a) is unstable.

(c) Determine a feedback control law for the linearized system in (a) such that deviations from $y_B = \varphi = \theta = 0$ are damped out asymptotically.

Assign the poles that need to be moved to -1, and determine the feedback vector.

6 Linear-quadratic Optimal Control

6.1 Determine the control law $u(t,x)$ minimizing

$$\frac{1}{2}\int_0^T 3x^2 + \frac{1}{4}u^2\,dt + x(T)^2,$$

under the constraint $\dot{x} = 2x + u$.

6.2 Determine the control law $u(t,x)$ minimizing

$$x(1)^2 + \int_0^1 2u^2\,dt,$$

under the constraint $\dot{x}(t) = \frac{1}{2-t}x(t) + \frac{1}{2}u(t)$.

6.3 Determine the control law $u(t,x)$ minimizing

$$\int_0^1 u^2\,dt + x(1)^2,$$

subject to $\dot{x} = x + u$, $x(0) = x_0$.

6.4 Consider the scalar system

$$\begin{cases} \dot{x} &= f(t)(x+u), \\ x(0) &= x_0. \end{cases}$$

Determine the control that minimizes

$$\int_0^1 f(t)u^2(t)\,dt + x^2(1),$$

where $f(t)$ is a given function and $F(t) = \int_0^t f(s)\,ds$ is also known.

6.5 Consider the scalar system

$$\begin{cases} \dot{x} &= tx(t) + \sqrt{t}u(t), \\ x(0) &= x_0. \end{cases}$$

Determine the control that minimizes

$$\int_0^1 u^2(t)\,dt + x^2(1).$$

6.6 Determine the control law minimizing

$$\int_0^\infty \frac{5}{2}y^2 + 2u^2\,dt,$$

subject to

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{cases}$$

Verify that the closed-loop system is stable.

6.7 Consider the problem

$$\min_u \int_0^\infty (y^2 + ru^2)\, dt, \quad r > 0,$$

as the systems dynamics are given by

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases}$$

(a) Determine the optimal control.

(b) Verify that the closed loop system is stable.

6.8 Consider the problem of minimizing

$$\int_0^\infty qx^2 + ru^2\, dt,$$

subject to

$$\dot{x} = ax + bu \quad a, b \in \mathbb{R} \quad b \neq 0,\ r > 0,\ q > 0.$$

(a) Determine the optimal control law.

(b) What happens if $b = 0$ and $a > 0$?

(c) What happens if $q = 0$?

(d) What happens with the gain k when $r \to \infty$?
Determine the corresponding closed-loop system?

6.9 In the *servo problem* we want a system to follow a given reference signal $r(t)$, which need not be differentiable. As a simple example consider the problem

$$\min_u \int_0^T (x(t) - r(t))^2 + u(t)^2\, dt,$$

subject to the dynamics $\dot{x} = u$, $x(0) = x_0$.

Determine an optimal control law $\hat{u}(t, x)$.

Hint: Try an optimal-value function of the form

$$V(x, t) = x^2 p(t) + x f(t) + h(t),$$

and give differential equations with boundary values for the functions p, f, h.

6.10 Consider the following (very simple) model of the velocity of a car:

$$\begin{cases} \dot{x}(t) &= -x(t) + u(t), \\ y(t) &= x(t). \end{cases}$$

We would like to minimize the following cost function

$$\min_u \int_0^1 [(y(t) - a(t))^2 + (u(t) - b(t))^2]\, dt,$$

where $a(t)$ can be seen as a reference speed, and $b(t)$ as forces on the car. (i.e. wind, gravity.)

Hint: Transform the cost function in to a LQ-problem, using completion of squares, and seek a value function of the form $V(t,x) = x^2 p(t) + f(t)x + h(t)$.

6.11 Consider the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0.$$

(a) Assume P is a real positive **semidefinite** solution. Show that $kerP$ is A-invariant (i.e, $\forall x \in kerP,\;\; Ax \in kerP$) and $kerP \subset kerC$.

(b) Show that if (C, A) is observable, then every positive semidefinite solution P is positive definite. **Hint:** use the conclusions in (a).

7 Kalman filters

7.1 Consider the system

$$\begin{cases} x(t+1) &= Ax(t) + Bv(t), \\ y(t) &= Cx(t), \end{cases}$$

where

$$\begin{aligned} x(0) &= x_0, & \mathrm{E}\,[v(t)v'(t)] &= \delta_{t,s} I, \\ \mathrm{E}\,[x(0)x'(0)] &= P, \end{aligned}$$

and $v, x(0)$ are uncorrelated.

(a) Determine the Kalman filter for this system, and show that the normal Kalman filter converges to this one as $D \to 0$.

(b) Use the result in (a) to determine the optimal estimate filter for the system

$$\begin{cases} x(t+1) &= \frac{1}{2}x(t) + v(t), \\ y(t) &= x(t). \end{cases}$$

What is the value of $P(t)$? Interpret.

7.2 Consider the system

$$\begin{cases} x(t+1) &= Ax(t) + Bv(t), \\ y(t) &= Cx(t) + Dw(t), \end{cases}$$

where

$$\begin{aligned} x(0) &= x_0, & \mathrm{E}\,[v(t)v'(t)] &= \delta_{t,s} I, \\ \mathrm{E}\,[x(0)x'(0)] &= P, & \mathrm{E}\,[w(t)w'(t)] &= \delta_{t,s} I, \end{aligned}$$

and $v, w, x(0)$ are uncorrelated.

Determine the best linear prediction of $x(t)$ given $\{y(0), y(1), \ldots, y(t-2)\}$. This is equivalent to determining $\mathrm{E}^{H_{t-2}(y)} x(t)$.

7.3 Consider the system

$$\begin{cases} x(t+1) &= Ax(t) + Bv(t) + Gu(t), \\ y(t) &= Cx(t) + Dw(t), \end{cases}$$

where

$$\begin{aligned} x(0) &= x_0, & \mathrm{E}\,[v(t)v'(t)] &= \delta_{t,s} I, \\ \mathrm{E}\,[x(0)x'(0)] &= P, & \mathrm{E}\,[w(t)w'(t)] &= \delta_{t,s} I, \end{aligned}$$

and $v, w, x(0)$ are uncorrelated.

The control signal u_t is a linear function of $\{y(0), y(1), \ldots, y(t-1)\}$.

Determine the Kalman filter for this system.

7.4 Let y be a stochastic process given by the system

$$\begin{cases} x(t+1) &= Ax(t) + Bv(t), \\ y(t) &= Cx(t) + Dw(t), \end{cases}$$

where $x(0)$ and v, w satisfies the usual assumptions. Define the innovation process by

$$\tilde{y}(t) \stackrel{\triangle}{=} y(t) - \mathrm{E}^{H_{t-1}(y)} y(t).$$

(a) Determine a causal timevariable system, that driven by the output $\{y(0), y(1), \ldots, y(T)\}$ give the innovations $\{\tilde{y}(0), \tilde{y}(1), \ldots, \tilde{y}(T)\}$.

(b) Determine a causal timevariable system, that driven by the innovations $\{\tilde{y}(0), \tilde{y}(1), \ldots, \tilde{y}(T)\}$ give the output $\{y(0), y(1), \ldots, y(T)\}$.

7.5 Let y be a stochastic process given by the system

$$\begin{cases} x(t+1) &= Ax(t) + Bv(t), \\ y(t) &= Cx(t) + Dv(t), \end{cases}$$

where $x(0)$ and v satisfies the usual assumptions. Notice that the state- and the measured noises are the same. Determine a Kalmanfilter for this case.

7.6 Let $z(t)$ be a stochastic process with $\mathrm{E}\, z(t) = 0$. The observations of the process are disturbed by normalized white noise $w(t)$.

The problem is to estimate $z(t)$ as a linear function of the noisy observations $\{z(t-1) + w(t-1),\ z(t-2) + w(t-2), \ldots,\ z(0) + w(0)\}$, in such a way that the variance of the estimation error is minimized.

Assume $z(t)$ can be represented by

$$\begin{cases} z(t) &= Cx(t), \\ x(t+1) &= Ax(t) + Bv(t), \end{cases}$$

where $v(t)$ is normalized white noise, uncorrelated with $w(t)$. The matrixes A, B, C are known and A is stable.

Show that under these assumptions on $z(t)$, the problem can be solved recursively with finite memory, i.e. storage of all old observations is not necessary. Determine the recursion equations, and the initial values.
The Kalman gain $K(t)$ may be considered known.

7.7 Let z be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{z\} = 0,\ E\{z^2\} = \alpha^2$). We would like to determine the value of z by a set of noisy measurements

$$y(t) = z + w(t) \text{ for } t = 0, 1, \ldots, n-1$$

where $w(t) \in N(0, \sigma^2)$ are independent of each other and of z.

a) Determine $P(n) = E\{(z - \hat{z}_n)^2\}$ as a function of α, σ, n, where $\hat{z}_n$ is the optimal estimate of z based on measurements up to time instant $n-1$. (*Hint:* note that $f(k+1)^{-1} = f(k)^{-1} + \delta \Rightarrow f(k) = \frac{1}{f(0)^{-1}+k\delta}$.) (8p)

b) Find $\hat{z}_n$ of z in terms of $y(0), y(1), \ldots, y(n-1)$. (6p)

c) What is $\hat{z}_3$ if $y(0) = 1, y(1) = 0.5, y(2) = 1, \alpha = \sigma = 2$? (3p)

d) For what values of α, σ does the best estimate $\hat{z}_n$ equal the arithmetic mean $\frac{1}{n}\sum_{t=0}^{n-1} y(t)$? (3p)

8 Answers to the exercises

8.1 Answers to: Linear Dynamical Systems

1.1 Diagonalize the matrix by determining eigenvalues and eigenvectors.
The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$. We get

$$\lambda_1 = 3, \text{ with eigenvector } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1, \text{ with eigenvector } \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Let

$$W \triangleq \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, \quad \Rightarrow \quad W^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}.$$

Now

$$e^{At} = W \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} W^{-1} = \frac{1}{4}\begin{bmatrix} e^{3t} + 3e^{-t} & 3e^{3t} - 3e^{-t} \\ e^{3t} - e^{-t} & 3e^{3t} + e^{-t} \end{bmatrix}.$$

1.2 The matrix can be diagonalized as

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and then the transition matrix is given by

$$\Phi(t,s) = \begin{bmatrix} \cosh(t-s) & \sinh(t-s) \\ \sinh(t-s) & \cosh(t-s) \end{bmatrix}.$$

1.3 The transition matrix can be calculated by the Laplace method

$$\det(sI - A) = \det \begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix} = s^2 + \omega^2 = (s + i\omega)(s - i\omega),$$

The inverse is

$$(sI - A)^{-1} = \frac{1}{(s+i\omega)(s-i\omega)}\begin{bmatrix} s & -\omega \\ -\omega & s \end{bmatrix} = \frac{1}{s+i\omega}B + \frac{1}{s-i\omega}C,$$

where

$$B = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, \quad C = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$

Now the transition matrix is given by

$$\Phi(t,s) = e^{-i\omega(t-s)}B + e^{i\omega(t-s)}C = \begin{bmatrix} \cos(\omega(t-s)) & \sin(\omega(t-s)) \\ -\sin(\omega(t-s)) & \cos(\omega(t-s)) \end{bmatrix}.$$

1.4 The solution is

$$x(t) = \frac{1}{3}\begin{bmatrix} 2e^{-t}+e^{5t} & -2e^{-t}+2e^{5t} \\ -e^{-t}+e^{5t} & e^{-t}+2e^{5t} \end{bmatrix} x_0.$$

1.5 The solution is

$$x(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}-2te^{-2t} & 4te^{-2t} \\ 0 & -te^{-2t} & 2te^{-2t}+e^{-2t} \end{bmatrix} x_0.$$

1.8 **(a)** Let $x(t) \triangleq \int_0^t e^{As}\,ds$, and $z(t) \triangleq A^{-1}[e^{At}-I]$.
Then x and z both solve the linear differential equation

$$\begin{cases} \dot{y} &= e^{At}, \\ y(0) &= 0. \end{cases}$$

Since the solution is unique, x and z must be equal.

(b) If we assume that A is stable, then $e^{At} \to 0$, and according to (a) $\int_0^t e^{As}\,ds \to -A^{-1}$.

1.9 Divide A into two parts

$$A = f(t)\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} + g(t)\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} = f(t)F + g(t)G.$$

The matrices F and G commute, so

$$\Phi(t,s) = \exp(F\int_s^t f(u)\,du)\exp(G\int_s^t g(u)\,du),$$

and

$$e^F = \begin{bmatrix} e^B & 0 \\ 0 & e^B \end{bmatrix}, \quad e^G = \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}, \quad \Rightarrow$$

$$\Phi(t,s) = \begin{bmatrix} \exp(B\int_s^t f(u)\,du) & \exp(B\int_s^t f(u)\,du)\,C\int_s^t g(u)\,du \\ 0 & \exp(B\int_s^t f(u)\,du) \end{bmatrix}.$$

1.10 We know that $\Phi(t,s) = \exp(\int_s^t A(\tau)\,d\tau)$ if $A(t)A(s) = A(s)A(t) \quad \forall t,s$.
Divide A into two parts as $A(t) = \alpha I + e^{-t}B$, where

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrices I and B commute so $\Phi(t,s) = \exp\{\alpha(t-s)\}\exp\{-(e^{-t}-e^{-s})T\}$.
Note that $T^{2n+1} = (-1)^n T$, and $T^{2n} = (-1)^n T$, and use the definition of e^{At}, then

$$\Phi(t,s) = e^{\alpha(t-s)}\begin{bmatrix} \cos(e^{-s}-e^{-t}) & \sin(e^{-s}-e^{-t}) \\ -\sin(e^{-s}-e^{-t}) & \cos(e^{-s}-e^{-t}) \end{bmatrix}.$$

1.11 Divide A into two parts

$$A = \cos(t)I + tB, \quad \text{where } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The matrices I and B commute, so

$$\Phi(t,s) = \exp((\sin(t) - \sin(s))I)\exp(\frac{1}{2}(t^2 - s^2)B),$$

and

$$\Phi(t,s) = e^{\sin t - \sin s} e^{\frac{1}{2}(t^2 - s^2)B} = e^{\sin t - \sin s} \begin{bmatrix} 1 & \frac{1}{2}(t^2 - s^2) \\ 0 & 1 \end{bmatrix}.$$

1.12 Divide A into two parts as $A(t) = te^{-t}B + \frac{1}{1+t}I$, where

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The matrices I and B commute so

$$\Phi(t,s) = \exp\{B \int_s^t ue^u \, du\} \exp\{B \int_s^t \frac{1}{1+u} \, du\}.$$

Since B is nilpotent of order 1 we have

$$e^{Bt} = I + Bt = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

We finally get

$$\begin{aligned} \Phi(t,s) &= \begin{bmatrix} 1 & 0 \\ (s+1)e^{-s} - (t+1)e^{-t} & 1 \end{bmatrix} I e^{\log(1+t) - \log(1+s)} = \\ &= \frac{(1+t)}{(1+s)} \begin{bmatrix} 1 & 0 \\ (s+1)e^{-s} - (t+1)e^{-t} & 1 \end{bmatrix}. \end{aligned}$$

1.13 $\Phi(t,s) = \exp\{\sin(t) - \sin(s)\}(\frac{s}{t})^2 \begin{bmatrix} 1 - 2\ln(t/s) & -\ln(t/s) \\ 4\ln(t/s) & 1 + 2\ln(t/s) \end{bmatrix}.$

1.14 $\Phi(t,s) = \begin{bmatrix} 1 & 0 \\ t(t-s) & t/s \end{bmatrix}.$

8.2 Answers to: Reachability and Observability

2.1 The Reachability grammian is

$$W(t_0,t_1) = (t_1 - t_0)\begin{bmatrix} 1 & \frac{1}{2}t_1(t_1-t_0) \\ \frac{1}{2}t_1(t_1-t_0) & \frac{1}{3}t_1^2(t_1-t_0)^2 \end{bmatrix},$$

and since $\det\{W(t_0,t_1)\} = \frac{1}{12}t_1^2(t_1-t_0)^2 > 0$ for all $t_1 > t_0 > 0$, we can reach any state.

2.2 The matrices can be partitioned as

$$A = \begin{bmatrix} \lambda I & 0 \\ 0 & D \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where D is a diagonal matrix, and I is a $r \times r$-matrix with $r \geq 2$.

We get

$$A^k b = \begin{bmatrix} \lambda^k b_1 \\ y \end{bmatrix} = \lambda^k \begin{bmatrix} b_1 \\ z \end{bmatrix}.$$

$\because$ Im $[b, Ab, \ldots, A^{n-1}b]$ is a subspace of the space spanned by vectors of the form

$$\begin{bmatrix} b_1 \\ z \end{bmatrix}, \quad z \in \mathbb{R}^{n-r},$$

and as such subspaces can not be the whole $\mathbb{R}^n$, the system can not be reachable.

2.3 The easiest way, is to write the system as $\dot{x} = (\lambda I + T)x + bu$, and change the state as $x = e^{\lambda t}z$. Then

$$e^{\lambda t}\dot{z} + \lambda e^{\lambda t}z = \lambda e^{\lambda t}z + bu,$$

and with the control $v(t) = e^{-\lambda t}u(t)$ we get the system

$$\dot{z} = Tz + bv, \quad \text{where } T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This systems reachability matrix is

$$\Gamma = \begin{bmatrix} b & Tb & T^2b \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & 0 \\ b_3 & 0 & 0 \end{bmatrix}$$

is full rank if $b_3 \neq 0$, and the original system is reachable if the same condition is satisfied.

2.4 No, a counter example is $\dot{x} = x$, $c = 1, d = 1$.

2.5 **(a)** Add w as a state,

$$\begin{cases} \begin{bmatrix} \dot{\theta} \\ \dot{v} \\ \dot{h} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 & 0 \\ \sigma & -\frac{1}{\beta} & 0 & \frac{1}{\beta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ v \\ h \\ w \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta & v & h & w \end{bmatrix}^T \end{cases}$$

The observability matrix is

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \sigma & -\frac{1}{\beta} & 0 & \frac{1}{\beta} \\ -(\frac{\sigma}{\alpha} + \frac{\sigma}{\beta}) & \frac{1}{\beta^2} & 0 & -\frac{1}{\beta^2} \end{bmatrix}.$$

Ω is full rank, ($\sigma \neq 0$) so all states can be reconstructed.

(b) The reachability matrix is

$$\Gamma = \begin{bmatrix} 1 & -1/\alpha & 1/\alpha^2 \\ 0 & \sigma & -\sigma(\frac{1}{\alpha} + \frac{1}{\beta}) \\ 0 & 0 & \sigma \end{bmatrix}.$$

Γ is full rank, so the system is completely reachable.

2.6 It is easy to show that

$$\Gamma = FG = \begin{bmatrix} | & & | \\ b_1 & \dots & b_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 & \lambda_1\alpha_1 & \dots & \lambda_1^{n-1}\alpha_1 \\ \vdots & \vdots & & \vdots \\ \alpha_n & \lambda_n\alpha_n & \dots & \lambda_n^{n-1}\alpha_n \end{bmatrix}.$$

Since F has full rank, we know that

$$\operatorname{rank}\Gamma = n \Leftrightarrow \operatorname{rank} G = n.$$

(a) If some $\alpha_i = 0$, row i in G will consist of zeroes, i.e. $\operatorname{rank} G < n$.
$\because$ Reachability $\quad \Rightarrow \quad \alpha_i \neq 0, \quad \forall i.$

(b) Assume that $\alpha_i \neq 0, \quad \forall i$. We want to show that $\operatorname{rank} G = n$.
Assume that $\operatorname{rank} G < n$, then there exists $y_1, \dots, y_n$ so that

$$G \begin{bmatrix} y_1 \\ | \\ y_n \end{bmatrix} = 0, \quad \begin{bmatrix} y_1 \\ | \\ y_n \end{bmatrix} \neq 0.$$

i.e.

$$\begin{cases} (\alpha_1)[y_1 + y_2\lambda_1 + \dots + y_n\lambda_1^{n-1}] = 0 \\ (\alpha_2)[y_1 + y_2\lambda_2 + \dots + y_n\lambda_2^{n-1}] = 0 \\ \vdots \qquad\qquad \vdots \\ (\alpha_n)[y_1 + y_2\lambda_n + \dots + y_n\lambda_n^{n-1}] = 0 \end{cases}$$

We have n distinct solutions to $y_1 + y_2\lambda + \dots + y_n\lambda^{n-1}$, but this is a $(n-1)-$degree polynomial, so we have a contradiction.

2.7 **(a)** See textbook.

(b) See textbook.

(c) The Block matrix representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} C_1, & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(d) i. Assume $q \perp c$, $Aq = \lambda q$. This implies that $\Omega q = 0$, hence that Ω can not have full rank.

$\therefore$ The system is not completely observable.

ii. Assume the system is not completly observable. Use the decomposition from (c) and define q as

$$q \triangleq \begin{bmatrix} 0 \\ z \end{bmatrix},$$

where z is an arbitrary eigenvector of A_{22}.

2.8 **(a)** Define

$$(e^{\alpha\partial_z} f)(z) \triangleq \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \frac{\partial^n f}{\partial z^n}(z) = f(z+\alpha).$$

where the equality comes from Taylors formula.

It follows that $x(t,z) = [\exp(t\partial_z)\phi](z) = \phi(z+t)$, which can be shown to satisfy the system †.

(b) Define S as $S \triangleq \{\phi | y(t) \equiv 0\}$, which makes $y(t) = \int_\Gamma c(z)\phi(z+t)\,dz$.

i. Assume $y \equiv 0$. Then we must have $y^{(n)}(0) = 0$ ($\frac{\mathrm{d}^n}{\mathrm{d}t^n} y(t)$), i.e.

$$0 = \int_\Gamma c(z)\phi^{(n)}(z)\,dz = CA^n\phi,$$

$\therefore$ $\phi \in S \Rightarrow \phi \in \ker \Omega$.

ii. Assume $CA^n\phi = 0$, then for arbitrary t we have

$$\frac{t^n}{n!}\int_\Gamma c(z)\phi^{(n)}(z)\,dz = 0,$$

which summed up gives

$$\int_\Gamma c(z) \sum_{n=0}^{\infty} \frac{t^n}{n!}\phi^{(n)}(z)\,dz = 0,$$

$$\int_\Gamma c(z)\phi(z+t)\,dz = y(t) = 0,$$

$\therefore$ $\phi \in \ker\Omega \Rightarrow \phi \in S$.

$\therefore$ $S = \ker \Omega$.

(c) With

$$c(z) = \delta(z) \qquad \Rightarrow \qquad cf = f(0)$$

we have $y(t) = \phi(t)$. So $y(t) = 0 \quad \forall t \geq 0 \quad \Rightarrow \quad \phi(t) = 0 \quad \forall t \in \mathbb{R}^+$.

As ϕ is analytic we conclude that $\phi \equiv 0$ in $\mathbb{C}$.

(d) No, Σ is not completely observable, since $\phi(z) = \sin(z)$ gives the output $y \equiv 0$.

2.9 **(a)** Assume $Ax = \lambda x$, then

$$f(A)x = \sum c_n A^n x = \sum c_n \lambda^n x = f(\lambda)x.$$

(b) Integration gives $f(z) = \dfrac{e^{\delta z} - 1}{z}$.

$f(A)$ is invertible iff 0 is not an eigenvalue of $f(A)$. From (a) we conclude that 0 is an eigenvalue of $f(A)$ iff $f(\lambda) = 0$ for some $\lambda \in \sigma[A]$, that is $e^{\delta\lambda} = 1$ for some $\lambda \in \sigma[A]$. This is the case if A has an eigenvalue of the form

$$\lambda = k\frac{2\pi i}{\delta}, \quad k = \pm 1, \pm 2, \ldots$$

(c) It is well known that $x(t)$ can be solved for according to

$$x(t) = e^{A(t-s)}x(s) + \int_s^t e^{A(t-r)}Bu(r)\,dr.$$

Thus

$$\begin{aligned} y_{k+1} &= x((k+1)\delta) = e^{A\delta}x(k\delta) + \int_{k\delta}^{(k+1)\delta} e^{A((k+1)\delta - r)}Bu_k\,dr = \\ &= e^{A\delta}y_k + \int_0^\delta e^{A(\delta - r)}\,dr\,Bu_k, \end{aligned}$$

so $F = e^{A\delta}$ and $G = f(A)B$.

(d) The Reachability matrix is given by

$$R = [G,\, FG,\, \ldots,\, F^{n-1}G] = f(A)[B,\, e^{\delta A}B,\, \ldots,\, e^{\delta(n-1)A}B],$$

where we have used that A and $f(A)$ commutes. The result now follows from (b).

2.10 Assume $k = m = 1$, and let $v = -\dot{x} + u$ to transform the minimization to the standard form of minimizing the control energy.

$$\min \int_{t_0}^{t_1} v^2\,dt, \qquad \text{as} \qquad \ddot{x} = v.$$

Defining the states $x_1 = x$, and $x_2 = \dot{x}$, we get the dynamics (completely reachable system)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v.$$

and the reachability Grammian

$$W(t_0, t_1) = \begin{bmatrix} \frac{(t_1 - t_0)^3}{3} & \frac{(t_1 - t_0)^2}{2} \\ \frac{(t_1 - t_0)^2}{2} & t_1 - t_0 \end{bmatrix}.$$

Now the optimal control is given by $\hat{u}(t) = \hat{v}(t) + x_2(t)$, where

$$\hat{v}(t) = B^T \Phi^T(t_1, t) W^{-1}(t_0, t_1)\{x(t_1) - \Phi(t_1, t_0)x(t_0)\}.$$

This is a mixture of a control law and control program, but we note that

$$\hat{v}(t) = B^T \Phi^T(t_1, t) W^{-1}(t, t_1)\{\underbrace{x(t_1)}_{=0} - \Phi(t_1, t_0)x(t)\},$$

so we can write $\hat{u}(t)$ as a control law. ($\hat{u}(t) = f(t)x(t)$)

2.11 **(a)** Introduce the state variables $x_1 = V(t)$, $x_2 = c(t)$. The linearized problem is

$$\dot{z} = \begin{bmatrix} -\frac{1}{2}\theta & 0 \\ 0 & -\theta \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ \frac{c_1 - c_0}{v_0} & \frac{c_2 - c_0}{v_0} \end{bmatrix} v,$$
$$y = \begin{bmatrix} \frac{1}{2}\theta & 0 \\ 0 & 1 \end{bmatrix}$$

where

$$z_1 = x_1 - v_0, \quad v_1 = u_1 - F_0, \quad y_1 = F(t) - F_0, \quad \theta = F_0/V_0.$$

$$z_2 = x_2 - c_0, \quad v_2 = u_2 - F_1, \quad y_2 = c(t) - c_0.$$

(b) The reachability matrix should have been

$$\Gamma = \begin{bmatrix} 1 & 1 & -\frac{1}{2}\theta & -\frac{1}{2}\theta \\ \frac{c_1 - c_0}{v_0} & \frac{c_2 - c_0}{v_0} & -\theta\frac{c_1 - c_0}{v_0} & -\theta\frac{c_2 - c_0}{v_0} \end{bmatrix},$$

and if $c_1 \neq c_2$ it has full rank and the system is completely reachable.

(c) With C as above the system is observable, but with

$$C = \begin{bmatrix} \frac{1}{2}\theta & 0 \end{bmatrix} \quad \Rightarrow \quad \Omega = \begin{bmatrix} \frac{1}{2}\theta & 0 \\ -\frac{1}{4}\theta^2 & 0 \end{bmatrix},$$

we see that having y_1 as output, the system is not completely observable.

(d) If $c_1 = c_2 = \bar{c}$, then $c_0 = \bar{c}$ and

$$\Gamma = \begin{bmatrix} 1 & 1 & -2\theta & -2\theta \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is only of rank 1. (only the volume can be controlled)

8.3 Answers to: Stability

3.1 **(a)** Let the positive definite matrix $Q = I$, and solve the equation

$$A^T P + PA + I = 0,$$

Assuming $P = (p_{i,j})$, where $p_{1,2} = p_{2,1}$, we get

$$\begin{cases} 0 &= 4p_{1,2} + 1, \\ 0 &= p_{1,1} - p_{1,2} + 2p_{2,2}, \\ 0 &= 2p_{1,2} - 2p_{2,2} + 1, \end{cases} \quad \Rightarrow \quad P = \frac{1}{4}\begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}$$

It is evident that P is not positive definite, so A is not stable.

(b) Let the positive definite matrix $Q = I$, and solve the equation

$$P - A^T PA = I,$$

Assuming $P = (p_{i,j})$, where $p_{1,2} = p_{2,1}$, we get

$$\begin{cases} 1 &= p_{1,1} - 4p_{2,2}, \\ 0 &= -p_{1,2} + 2p_{2,2}, \\ 1 &= -p_{1,1} + 2p_{1,2}. \end{cases}$$

This system has an infinite number of solutions, so A is not stable.

3.2 The system is stable if $c = 0$, otherwise unstable.

3.3 **(a)** Let the positive definite matrix $Q = I$, and solve the equation

$$A^T P + PA + I = 0,$$

Assuming $P = (p_{i,j})$, where $p_{1,2} = p_{2,1}$, we get

$$\begin{cases} 0 &= -6p_{1,1} + p_{1,2} + 1, \\ 0 &= -13p_{1,1} - p_{1,2} + \frac{1}{2}p_{2,2}, \\ 0 &= -26p_{1,2} + 4p_{2,2} + 1, \end{cases} \qquad \Rightarrow \qquad P = \frac{1}{4}\begin{bmatrix} 19 & 110 \\ 110 & 714 \end{bmatrix}$$

It is evident that P is positive definite, so A is stable.

(b) Let the positive definite matrix $Q = I$, and solve the equation

$$P - A^T PA = I,$$

Assuming $P = (p_{i,j})$, where $p_{1,2} = p_{2,1}$, we get

$$\begin{cases} 1 &= -8p_{1,1} + 3p_{1,2} - \frac{1}{4}p_{2,2}, \\ 0 &= -39p_{1,1} + \frac{27}{2}p_{1,2} - p_{2,2}, \\ 1 &= -169p_{1,1} + 52p_{1,2} - 3p_{2,2}. \end{cases} \qquad \Rightarrow \qquad P = \begin{bmatrix} 19 & 86 \\ 86 & 420 \end{bmatrix}.$$

As $p_{1,1} > 0$ and $\det P > 0$ this matrix is positive definite and A is a stability matrix.

3.5 Introduce the new state $z(t) = e^{-2t}x(t)$, then $y(t) = z(t)$ and

$$\dot{z}(t) = -2e^{-2t}x(t) + e^{-2t}\dot{x}(t) = -2z(t) + z(t) + u(t) = -z(t) + u(t).$$

From this it is easy to see that the system is input-output stable.

3.6 No, because

$$z(t) = e^{\sin t - \sin s}z(s) \not\to 0 \quad \text{as} \quad t \to \infty.$$

8.4 Answers to: Realization Theory

4.1 **(a)** The reachable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} -4 & 2 & 0 \end{bmatrix} x. \end{cases}$$

(b) The observable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x. \end{cases}$$

(c) We have $g(s) = \dfrac{2(s-2)}{(s-1)(s-2)(s+3)} = \dfrac{2}{(s-1)(s+3)} = \dfrac{2}{s^2+2s-3}$.

The reachable standard form is now

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 2 & 0 \end{bmatrix} x. \end{cases}$$

and since it is observable it is minimal.

4.2 **(a)**
$$R(s) = \left[\frac{1}{(s+1)(s+3)}, \frac{1}{s+1}\right] = \left[-\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3}, \frac{1}{s+1}\right].$$

So we study the Laurent expansions

$$\frac{1}{s+1} = s^{-1} - s^{-2} + s^{-3} - s^{-4} + \dots$$

$$\frac{1}{s+3} = s^{-1} - 3s^{-2} + 9s^{-3} + \dots$$

This gives us

$$R(s) = [0, 1]\, s^{-1} + [1, -1]\, s^{-2} + [-4, 1]\, s^{-3} + \dots$$

and the characteristic polynomial

$$\chi(s) = s^2 + 4s + 3, \qquad \Rightarrow \qquad a_1 = 4,\ a_2 = 3.$$

The observable standard realization is given by

$$\begin{cases} A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{cases}$$

(b) We have $\deg \chi = 2$, and

$$H_{2-1} = H_1 = \begin{bmatrix} R_0 & R_1 \\ R_1 & R_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & -1 & -4 & 1 \end{bmatrix},$$

which have row rank 2, so the McMillan degree is 2.

4.3
$$R(s) = [1\ 0] + \left[\frac{1}{s-1}, \frac{1}{s-1} + \frac{1}{s+1}\right].$$

So $D = [1\ 0]$, and by studying the Laurent expansions

$$\frac{1}{s-1} = s^{-1} + s^{-2} + s^{-3} + \dots$$

$$\frac{1}{s+1} = s^{-1} - s^{-2} + s^{-3} - s^{-4} + \dots$$

$$\Rightarrow \qquad R_1 = [1\ 2]\,,\ R_2 = [1\ 0]\,,$$

and the characteristic polynomial

$$\chi(s) = s^2 - 1, \qquad \Rightarrow \qquad a_1 = 0,\ a_2 = -1.$$

The observable standard realization is

$$\begin{cases} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{cases}$$

The dimension of the realization $= 2 = \deg \chi$, so it is minimal and the McMillan degree is 2.

4.4 $R(s)$ is not strictly proper,

$$R(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{s^2+4}{s^2-4} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & \frac{2}{s-2} - \frac{2}{s+2} \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} = T(s) + D.$$

We want to determine $\delta(T)$, so we study the Laurent expansions

$$\frac{1}{s+2} = s^{-1} - 2s^{-2} + 4s^{-3} - 8s^{-4} + \ldots$$

$$\frac{1}{s-2} = s^{-1} + 2s^{-2} + 4s^{-3} + \ldots$$

This gives us

$$T(s) = [-1, 0]\, s^{-1} + [2, 8]\, s^{-2} + [-4, 0]\, s^{-3} + \ldots$$

and

$$\delta(T) = \operatorname{rank} H_2 = \operatorname{rank} \begin{bmatrix} T_0 & T_1 \\ T_1 & T_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & 0 & 2 & 8 \\ 2 & 8 & -4 & 0 \end{bmatrix} = 2.$$

4.5 The characteristic equation is $\chi(s) = s^3 - s^2$, and the Laurent expansion

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + R_4 s^{-4} + \ldots$$

has the coefficients

$$R_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, R_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(a) The reachable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} x. \end{cases}$$

(b) The observable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} x. \end{cases}$$

(c) The Hankel matrix H_2 has rank 3 by inspection , so a minimal realization is of dimension 3.

(d) Using Ho's algorithm, we get

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} x. \end{cases}$$

4.6 The characteristic equation is $\chi(s) = s^2 + 3s + 2$, and the Laurent expansion

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + R_4 s^{-4} + \dots$$

has the coefficients

$$R_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, R_2 = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}, R_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, R_4 = \begin{bmatrix} -1 & -2 \\ -7 & -8 \end{bmatrix}.$$

(a) The reachable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 2 & 4 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix} x. \end{cases}$$

(b) The Hankel matrix H_2 has rank 3 by inspection , so the realization is not minimal.

(c) The observable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & -2 \\ -1 & -2 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x. \end{cases}$$

(d) No.

(e) 3, se (b) above.

(f) Using Ho's algorithm, we get

$$\begin{cases} \dot{x} &= \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x. \end{cases}$$

4.7 **(a)** The observable standard realization is given by

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases}$$

(b) The Hankel matrix is

$$H = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix},$$

and $\operatorname{rank} H = 2$, so $\delta(R) = 2$.

4.8 **(a)** The observable standard realization is given by

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases}$$

(b) The realization is minimal, because $n = 2$ and $\chi(s)$ is of degree 2, so the McMillan degree is 2.

4.9 **(a)** The observable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} u. \end{cases}$$

(b) Check Reachability.

$$\Gamma = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x,$$

is full rank, so it is minimal.

4.10 The characteristic equation is $\chi(s) = (s+1)^2(s+3) = s^3 + 4s^2 + 5s + 2$,

(a) The observable standard realization is given by

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -5 & 0 & -4 & 0 \\ 0 & -2 & 0 & -5 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} x. \end{cases}$$

(b) the Laurent expansion

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + R_4 s^{-4} + R_5 s^{-5} \dots$$

has the coefficients

$$R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, R_3 = \begin{bmatrix} -2 & 4 \\ 0 & -3 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 3 & -8 \\ 0 & 7 \end{bmatrix}, R_3 = \begin{bmatrix} -4 & 16 \\ 0 & -15 \end{bmatrix}.$$

so by inspection the Hankel matrix H_3 has rank 4, therefore the dimension of a minimal realization is 4.

4.11 As

$$\begin{aligned} \tfrac{1}{s^2-s} &= \tfrac{1}{s-1} - \tfrac{1}{s} = s^{-1}(s^{-1} + s^{-2} + s^{-3} + \dots) - s^{-1} = \\ &= -s^{-1} + s^{-2} + s^{-3} + s^{-4} + \dots \end{aligned}$$

the Laurent expansion

$$R(s) = R_1 s^{-1} + R_2 s^{-2} + R_3 s^{-3} + R_4 s^{-4} + \dots$$

has the coefficients

$$R_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, R_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\chi(s) = s^3(s-1) = s^4 - s^3 \quad \Rightarrow \quad a_1 = -1, a_2 = 0, a_3 = 0, a_4 = 0.$$

(a) The observable standard form is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b)
$$H_4 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $\operatorname{rank} H_4 = 4$, so the McMillan degree of the system is 4.

4.12 **(a)** We get

$$\Gamma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

and since both matrices has rank 2, the system is neither observable nor reachable.

(b) We get $R(s) = C(sI - A)^{-1}B = \frac{1}{s+1} + \frac{1}{s-1} = \frac{2s}{s^2-1}$.

(c) The reachable standard form is

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 2 \end{bmatrix} x. \end{cases}$$

Since the system is observable (and reachable) the system is minimal.

4.13 Let $n = \deg q(s) = \delta(G)$, and $G(s) = r_1 s^{-1} + r_2 s^{-2} + \ldots$. We know that

$$\operatorname{rank} \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & & \\ \vdots & & \ddots & \\ r_n & & & r_{2n-1} \end{bmatrix} = n,$$

and now since $R(s) = [G(s),\ G(s)]$, we have

$$\operatorname{rank} \begin{bmatrix} r_1 & 0 & \cdots & r_n & 0 \\ 0 & r_1 & \cdots & 0 & r_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_n & 0 & \cdots & r_{2n-1} & 0 \\ 0 & r_n & \cdots & 0 & r_{2n-1} \end{bmatrix} = 2 \operatorname{rank} \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & & \\ \vdots & & \ddots & \\ r_n & & & r_{2n-1} \end{bmatrix}.$$

The McMillan degree $\delta(R) = 2\delta(G)$, so the realization is minimal.

4.14 **(a)** A realization of $R(s)$ is given by

$$\begin{cases} A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}, \quad B = \begin{bmatrix} - & h_1 & - \\ & \vdots & \\ - & h_p & - \end{bmatrix}, \\ C = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}. \end{cases}$$

Since the dimension $= p = \deg \chi$, the realization is minimal.

(b) Let H_i be $H_i = C_i B_i$, where C_i is $m \times r_i$, and B_i is $r_i \times k$. Then

$$\begin{cases} A = \begin{bmatrix} \lambda_1 I_{r_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_p I_{r_p} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_p \end{bmatrix}, \\ C = \begin{bmatrix} C_1 & \dots & C_p \end{bmatrix}. \end{cases}$$

is a realization. (I_n is the $n \times n$ identity matrix).

8.5 Answers to: Feedback, Pole-assignment and Observers

5.1 The feedback is $u = -\begin{bmatrix} 4 & 1 \end{bmatrix} x$.

5.2 **(a)** The characteristic equation is

$$\chi_A(s) = \det \begin{bmatrix} s-1 & -2 & 0 \\ -2 & s-3 & 0 \\ 0 & 0 & s-1 \end{bmatrix} = (s-1)(s^2-4s-1),$$

which have two real positive zeros, so the system is unstable.
The Reachability matrix Γ is

$$\begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 8 \\ 1 & 2 & 3 \end{bmatrix}.$$

and since it has full rank, the system is reachable.

(b) We have $\chi_A(s) = s^3 - 5s^2 + 3s + 1$, and we want $\chi_{A+bk}(s) = s^3 + 4s^2 + 6s + 4$, so $g = \begin{bmatrix} -3 & -3 & -9 \end{bmatrix}$.

$$t\Gamma = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow t = \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad T = \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 5 & 8 & -1 \end{bmatrix},$$

The feedback is then chosen as

$$k = gT = \frac{1}{4}\begin{bmatrix} -51 & -78 & 15 \end{bmatrix}.$$

(c) The system is observable since

$$\Omega = \begin{bmatrix} 1 & 10 & 1 \\ 1 & 2 & 1 \\ 5 & 8 & 1 \end{bmatrix},$$

is full rank.

(d) The dynamics of the error is given by $\dot{\tilde{x}} = (A - LC)\tilde{x}$. The characteristic equation is

$$\chi_{A-LC} = \chi_{A^T - C^T L^T} = \{K = -L^T\} = \chi_{A^T + C^T K},$$

and we have a pole-assignment problem for the dual system. Since $A = A^T$ and $C^T = b$ the system is "self dual" and we can use T from (b). We want $\chi_{A-LC} = (\lambda+2)^3 = \lambda^3 + 6\lambda^2 + 12\lambda + 8$, so $g = \begin{bmatrix} -7 & -9 & -11 \end{bmatrix}$. The observer is now given by

$$-L^T = K = gT = \frac{1}{4}\begin{bmatrix} -71 & -106 & 27 \end{bmatrix}.$$

5.3 Consider the observer $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du)$. The system for the estimate error $\tilde{x} = x - \hat{x}$, is

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = \ldots = (A - LC)\tilde{x}.$$

Pole-assignment problem for (A^T, C^T). We have $\chi_{A^T}(s) = s^2 - 2s + 1$, and we want $\chi_{A^T - C^T L^T}(s) = s^2 + 2s + 1$.

$$\Gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad t\Gamma = [0,\ 1] \qquad \Rightarrow \qquad t = [0,\ 1].$$

We get

$$T = \begin{bmatrix} t \\ tA^T \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \Rightarrow \qquad k = gT = [-4,\ -4] = -L^T.$$

The system

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u + Ly = \begin{bmatrix} -3 & 1 \\ -4 & 1 \end{bmatrix}\hat{x} + \begin{bmatrix} -3 \\ -4 \end{bmatrix}u + \begin{bmatrix} 4 \\ 4 \end{bmatrix}y,$$

give the estimate.

5.4 **(a)** Let $x_1 = \dot{\phi}$, and $x_2 = \phi$ be the states. The realization is then given by

$$\begin{cases} A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{cases}$$

(b) To determine if the system is reachable, we consider

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix},$$

which has rank 2, so the system is reachable.

(c) Let $k = [k_1, k_2]$, then

$$\chi_{A+Bk}(s) = \det(sI - A - Bk) = s^2 - (a + k_1)s - b - k_2.$$

We want the characteristic equation to be $s^2 + 2s + 2$, so $k_1 = -2 - a$, and $k_2 = -2 - b$ give the right poles.

(d) We have

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which have full rank, so the system is observable.
$\therefore$ we have a minimal system, so a stable output feedback exists.
Consider now the case $\tilde{C} = [1\ 0]$, and we have

$$\Omega = \begin{bmatrix} \tilde{C} \\ \tilde{C}A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}.$$

It is full rank as long as $b \neq 0$, and we can then determine the poles of the closed loop system.

5.5 **(a)** Linearization gives the system $\ddot{\phi} = \frac{g}{L}\phi - \frac{1}{L}u$. With the states $x_1 = \phi$, $x_2 = \dot{\phi}$, the state space model is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix} u.$$

(b) Since the characteristic equation $\chi_A(s) = s^2 - \frac{g}{L}$, there are positive eigenvalues of A and the linearized system is unstable.

(c) Since $\Gamma = [B\ AB] = \begin{bmatrix} 0 & -\frac{1}{L} \\ -\frac{1}{L} & 0 \end{bmatrix}$ is full rank, the linearized system is reachable.

(d) Let $y = Cx = [1\ 0]x$, then

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so the system is observable. Since the linearized system is minimal, it is possible to stabilize it.

(e) $t\Gamma = [0\ 1] \quad \Rightarrow \quad t = [-L\ 0]$ so $T = \begin{bmatrix} t \\ tA \end{bmatrix} = \begin{bmatrix} -L & 0 \\ 0 & -L \end{bmatrix}$

and the feedback is

$$K = gT = \begin{bmatrix} \frac{g}{L} & 2\sqrt{\frac{g}{L}} \end{bmatrix} \begin{bmatrix} -L & 0 \\ 0 & -\sqrt{gL} \end{bmatrix} = \begin{bmatrix} -g & -2\sqrt{gL} \end{bmatrix}.$$

5.6 **(a)** $\chi_A(s) = \det(sI - a) = (s-1)(s^2+2s+2)$ has a pole in $+1$, so the system is unstable.

(b) Since $\det \Gamma \neq 0$ the system is reachable.

(c) We have $\chi_A(s) = s^3 - 3s^2 - 2$, and we want $\chi_{A+bk}(s) = s^3 - s^2 + 4s + 2$, so $g = \begin{bmatrix} -4 & -4 & -2 \end{bmatrix}$.

$$t\Gamma = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow t = \frac{1}{4}\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}, \quad T = \frac{1}{4}\begin{bmatrix} 0 & 0 & -1 \\ -2 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix},$$

The feedback is then chosen as

$$k = gT = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}.$$

(d) Since $\det \Omega \neq 0$ the system is observable.

(e) Choose the poles of the observer so that the dynamics are faster than for the feedback, for example $-2, -2+2i, -2-2i$.

5.7 The reachable standard realization is given by

$$\begin{cases} A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \end{cases}$$

We construct an observer that delivers the estimate $\hat{x}(t)$ of the state $x(t)$.

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)).$$

Choose L so that the eigenvalues of $A - LC$ are all -2.

$$\Rightarrow \quad L = \begin{bmatrix} 10 \\ 46 \\ 136 \end{bmatrix}.$$

Let $u = k\hat{x} + v$ be the feedback, and assign the poles so we get the characteristic equation $\chi_{A+Bk}(s) = (s+1)(s+1-i)(s+1+i)$. Standard calculations give $k = [-6,\ 2,\ -7]$.

5.8 **(a)** A minimal representation is given by

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases}$$

(b) The characteristic polynomial $\chi_A(s) = s^2, \quad \Rightarrow \quad \lambda = 0.$

(c) We get

$$A + BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix},$$

and this matrix has the characteristic polynomial

$$\chi_{A+BK} = s(s-k_2) - k_1 = s^2 - k_2 s - k_1.$$

We would like it to be

$$(s+1-i)(s+1+i) = s^2 + 2s + 2 \quad \Rightarrow \quad k_1 = k_2 = -2.$$

(d) We get

$$A + LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} l_1 & 1 \\ l_2 & 0 \end{bmatrix},$$

and this matrix has the characteristic polynomial

$$\chi_{A-LC} = s(s-l_1) - l_2 = s^2 - l_1 s - l_2.$$

We would like it to be

$$(s+2-2i)(s+2+2i) = s^2 + 4s + 8 \quad \Rightarrow \quad l_1 = -4,\ l_2 = -8.$$

(e) The separation principle says thatthe characteristic equation for the total system is given by $\chi_{A+BK}\chi_{A-LC}$, so the poles are in $1 \pm i$, and $2 \pm 2i$.

5.9 (a) The reachable standard realization is given by

$$\begin{cases} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \end{cases}$$

(b) We know the system is reachable, but

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

has only rank 1, so the system is not minimal.

(c) The system is on reachable form. We have the characteristic equation $\chi_A(s) = s^2 - 1$, and we want $\chi_A(s) = s^2 + 2s + 1$.
Let $g = [\alpha_2 - \gamma_2, \alpha_1 - \gamma_1] = [0 - 2, -1 - 1] = -2[1, 1]$.
The feedback $u(t) = Kx(t) = gTx(t) = gIx(t) = -2[1, 1]x(t)$ give the right poles.

(d) The system is not minimal, so a stable outsignal feedback is not certain to exist. But, since $u(t) = -2[1, 1]x(t) = -2y(t)$ this feedback can be used.

(The unobservable pole is in -1 to start with.)

5.10 The closed loop system has the solution $x(k + N) = (A + bk)^N x(k)$, so if $(A + bk)^N = 0$ we have dead-beat control. If we have $\chi_{A+bk}(\lambda) = \lambda^3$, then Cayley-Hamilton say $(A + bk)^3 = 0$, i.e. $N = 3$.

$\because$ Assign all poles to zero.
We have

$$\Gamma = \begin{bmatrix} 1 & 4 & 13 \\ 1 & 3 & 10 \\ 0 & -1 & -5 \end{bmatrix}, \quad \Rightarrow \quad \begin{aligned} &\det(\lambda I - A) = \lambda^3 - 4\lambda^2 + 9, \\ &\text{so let } g = \begin{bmatrix} 9 & 0 & -4 \end{bmatrix}. \end{aligned}$$

Now

$$t\Gamma = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow t = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix}, \quad T = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & 3 & -3 \end{bmatrix},$$

and the dead-beat feedback is $k = gT = \frac{1}{2}\begin{bmatrix} -5 & -3 & 3 \end{bmatrix}$.

5.11 Let $\bar{A} \triangleq A + BK$. We want $\bar{A}^6 x(0) = x(0)$, so $\bar{A}^6$ should have eigenvalue 1. If λ is an eigenvalue of $\bar{A}$, then $\bar{A}^6 x(0) = \lambda^6 x(0)$, so $\lambda^6 = 1$ give a 6-periodic solution.

Let $\lambda = re^{i\phi}$, then $r^6 e^{i6\phi} = 1$. A solution is $\lambda = e^{i\phi/3}$, and we must also have complex conjugate $e^{-i\phi/3}$ as an eigenvalue. The third eigenvalue must be real and different from ± 1 and 0 to avoid longer periods.

Let $\lambda_3 = 1/2$, then the desired equation is

$$(\lambda - e^{i\pi/3})(\lambda - e^{-i\pi/3})(\lambda - 1/2) = \lambda^3 - 3/2\,\lambda^2 + 3\lambda + 2.$$

As $\chi_A(s) = s^3 + 4s^2 + 3s + 2$, we can use $K = [5/2,\ 3/2,\ 11/2]$.

5.12 **(a)** Introduce the states x_1= voltage over C_1, and x_2= voltage over C_2.

$$\begin{cases} \dot{x} &= \frac{1}{RC}\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} x + \frac{1}{RC}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{cases}$$

(b) We have to check if A is stability matrix. The eigenvalues of A have negative real part, so A is stable.

(c) Assign the poles of $A+BK$ to the imaginary axis, with the state feedback $u = [0,\ k]x$.

$$\det \begin{bmatrix} s+1 & 1-k \\ 1 & s+2-k \end{bmatrix} = s^2 + s(3-k) + 1 = 0,$$

and $k = 3$ gives imaginary roots.

(d) The eigenvalues $\lambda = \pm\frac{i}{RC}$ determine the frequency.

5.13 **(a)** $\chi(s) = (s-2)^2(s+1)^2$ has two real positive poles.

(b) The system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z \\ x_4 \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} z \\ x_4 \end{bmatrix} + \begin{bmatrix} \tilde{b} \\ 0 \end{bmatrix} u.$$

Since

$$\begin{bmatrix} \tilde{b} & \tilde{A}\tilde{b} & \tilde{A}^2\tilde{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}.$$

is full rank, and x_4 obviously is an uncontrollable state, we have a decomposition $\mathcal{R} \oplus V$ of the state space.

Using state feedback, we get $\chi_{A+bk}(s) = \chi_{\tilde{A}+\tilde{b}\tilde{k}}(s)(s+1)$. and we can not change the pole -1, but the others can be assigned arbitrarily.

(the system is stabilizable, since the uncontrollable state is stable.)

i. Yes.

ii. Yes.

iii. No.

5.14 The system is not reachable, since $\operatorname{rank}\Gamma = 4$. It is easily seen that x_2 is an uncontrollable state, but that this state is stable. ($\dot{x}_2 = -2x_2$)

Since the dimension of the controllable subspace is 4, the other states are reachable, and the system is stablizable.

The pole -2 of the uncontrollable state can not be changed, but the others can be changed arbitrarily, so it is possible to get that pole configuration.

5.15 **(a)** Let $G(s) = \frac{p(s)}{q(s)}$. It is easy to confirm that the transfer function of the closed loop system is $G_K(s) = \frac{p(s)}{q_K(s)}$, where $q_K(s)$ is the characteristic polynomial of $A + BK$.

The only way to change the zeros is by cancellation, i.e. to determine the feedback so that we get the same poles as the zeros, which then cancel.

(b) According to a well known theorem, reachability is not influenced by feedback. Cancellation of zeros and poles reduces the degree of G_K so $(A+BK, B, C)$ can not be a minimal realization. Then $(C, A+BK)$ can not be observable.

If no cancellation takes place, the system is still minimal and thus observable.

5.16 **(a)** The matrices A, B are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}.$$

(b) The characteristic equation $\det(\lambda I - A) = \lambda^3(\lambda - 1)$ has a solution with positive real part, so the system is unstable.

(c) We do not need to consider the state x_B, so we study the subsystem

$$\tilde{A} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \tilde{B} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$$

with the characteristic equation $\chi_{\tilde{A}} = \lambda^2(\lambda - 1) = \lambda^3 - \lambda^2$. The desired equation is $(\lambda + 1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$, so $g = \begin{bmatrix} -1 & -3 & -4 \end{bmatrix}$.

Now

$$\tilde{\Gamma} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad t\Gamma = [0,\ 0,\ 1] \quad \Rightarrow \quad t = [-1,\ 1,\ 1].$$

We get

$$T = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \Rightarrow \quad k = gT = [1,\ 3,\ -8].$$

8.6 Answers to: Linear-quadratic Optimal Control

6.1 The Riccati equation is

$$\begin{cases} \dot{p} &= -4p - 3/2 + 8p^2, \\ p(T) &= 1, \end{cases}$$

Let $p(t) = k\frac{\dot{v}(t)}{v(t)}$, then

$$\dot{p} = k\frac{\ddot{v}}{v} - k\frac{\dot{v}^2}{v^2} = -4k\frac{\dot{v}}{v} + 8k^2\frac{\dot{v}^2}{v^2} - \frac{3}{2}.$$

Choose k so that $-k = 8k^2$, i.e. $k = -1/8$.

$$-\frac{1}{8}\frac{\ddot{v}}{v} = -4(-\frac{1}{8})\frac{\dot{v}}{v} - \frac{3}{2} \qquad \Rightarrow \qquad \ddot{v} + 4\dot{v} - 12v = 0.$$

This linear DE has the solution $v(t) = Ae^{2t} + Be^{-6t}$, and inserted in the expression for p and with use of the boundary value the optimal control is

$$\hat{u}(x,t) = -8px = -2\frac{15e^{8(T-t)}+1}{5e^{8(T-t)}-1}x(t).$$

6.2 The Riccati equation is

$$\begin{cases} \dot{p} &= -\frac{2}{2-t}p + \frac{1}{8}p^2, \\ p(1) &= 1, \end{cases} .$$

Let $p(t) = k(t)\dfrac{\dot{v}(t)}{v(t)}$, then

$$\dot{p} = \dot{k}\frac{\dot{v}}{v} + k\frac{\ddot{v}}{v} - k\frac{\dot{v}^2}{v^2} = -\frac{2}{2-t}k\frac{\dot{v}}{v} + \frac{1}{8}k^2\frac{\dot{v}^2}{v^2}.$$

Choose k so that $-8k = k^2$, i.e. $k = -8$ and $\dot{k} = 0$.

$$-8\frac{\ddot{v}}{v} = -\frac{2}{2-t}(-8)\frac{\dot{v}}{v} \qquad \Rightarrow \qquad \ddot{v} + \frac{2}{2-t}\dot{v} = 0.$$

This linear DE has the solution $v(t) = A(t-2)^3 + B$, and inserted in the expression for p and with use of the boundary value the optimal control is

$$\hat{u}(x,t) = -\frac{1}{4}px = \frac{24(2-t)^2}{(2-t)^3+23}x(t).$$

6.3 The Riccati equation is

$$\begin{cases} \dot{p} &= -2p + p^2, \\ p(1) &= 1, \end{cases} \qquad \Rightarrow \qquad p(t) = \frac{2}{e^{2(t-1)}+1}.$$

so the optimal control is $\hat{u}(x,t) = -2x(e^{2(t-1)}+1)^{-1}$.

6.4 Identify the terms of the LQC setup.

$$A = B = R = f,\ Q = 0,\ S = 1,\ \Rightarrow\ \dot{p} = -2Ap + \frac{B^2}{R}p^2 - Q = f(p^2 - p).$$

$$\frac{\dot{p}}{p-2} - \frac{\dot{p}}{p} = 2f(t) \qquad \Rightarrow \qquad \ln|p-2| - \ln|p| = F(t) + C_1,$$

From which we get $p(t) = 2(1 - C_2e^{F(t)})^{-1}$. The boundary condition $p(1) = 1$ implies that $C_2 = e^{-F(1)}$. The optimal control is now

$$\hat{u}(t) = -Bpx = -\frac{2f(t)}{1+e^{F(t)-F(1)}}x(t).$$

6.5 Identify the terms of the LQC setup.

$$A = t,\ B = \sqrt{t},\ R = 1,\ Q = 0,\ S = 1, \quad \Rightarrow \quad \dot{p} = tp^2 - 2tp.$$

$$\frac{2\dot{p}}{p(p-2)} = -\frac{\dot{p}}{p} + \frac{\dot{p}}{p-2} = 2t, \quad \Rightarrow \quad -\ln|p| + \ln|p-2| = t^2 + C_1.$$

From which we get $p(t) = 2(1 - C_2 e^{t^2})^{-1}$. The boundary condition $p(1) = 1$ implies that $C_2 = -e^{-1}$. The optimal control is now

$$\hat{u}(t) = -Bpx = -\frac{2\sqrt{t}}{1 + e^{t^2-1}} x(t).$$

6.6 The system is minimal, so a unique positive definite solution of the ARE will exist. With

$$Q = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 2,$$

the ARE $-A^T P - PA + PBR^{-1}B^T P - Q = 0$, gives the system

$$\begin{cases} 0 &= -\frac{5}{2} - 2P_{12} + \frac{1}{2}P_{12}^2, \\ 0 &= -P_{22} - P_{11} + P_{12} + \frac{1}{2}P_{12}P_{22}, \\ 0 &= -2P_{12} + 2P_{22} + \frac{1}{2}P_{22}^2, \end{cases}$$

with the positive definite solution

$$P = \begin{bmatrix} 2 + 3\sqrt{6} & 5 \\ 5 & -2 + 2\sqrt{6} \end{bmatrix}.$$

and the optimal control $\hat{u} = -\frac{5}{2}x_1 + (1 - \sqrt{6})x_2$.

According to a known theorem, lemma 8.5, the closed loop system is stable. To verify, we see that $(A - BR^{-1}B^T P)$, has a double pool in $\lambda = -\sqrt{6}/2$.

6.7 **(a)** The ARE gives the system

$$\begin{cases} 1 &= \frac{1}{r}P_{12}^2, \\ P_{11} - 10P_{12} &= \frac{1}{r}P_{12}P_{22}, \\ 2P_{12} - 20P_{22} &= \frac{1}{r}P_{22}^2, \end{cases}$$

with the positive definite solution

$$P = \begin{bmatrix} \sqrt{100r + 2\sqrt{r}} & \sqrt{r} \\ \sqrt{r} & -10r + \sqrt{100r^2 + 2r\sqrt{r}} \end{bmatrix}.$$

and the optimal control

$$\hat{u} = \frac{1}{\sqrt{r}}x_1 + (\sqrt{100 + \frac{2}{\sqrt{r}}} - 10)x_2.$$

(b) The closed loop system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{r}} & -\sqrt{100+\frac{2}{\sqrt{r}}} \end{bmatrix} = \hat{A}x.$$

The eigenvalues of $\hat{A}$ have negative real parts, so the closed loop system is stable.

6.8 **(a)** The ARE gives $p = \frac{r}{b^2}(a + \sqrt{a^2 + b^2(q/r)})$, so

$$\hat{u} = -\frac{1}{b}(a + \sqrt{a^2 + b^2(q/r)})x.$$

(b) The system is not reachable if $b = 0$, and since $x(t) = e^{at}x_0$ the integral gets infinite if $a > 0$. That problem is ill posed.

(c) With $q = 0$ we have no dependence on the state in the optimality functional, so the optimal control is $u \equiv 0$.

Note that in (a), insertion of $q = 0$ does not give $u \equiv 0$, this is because $y = \sqrt{q}x \equiv 0$ so our system is not minimal.

(d) As $r \to \infty$ the gain goes to $(-a + |a|)/b$, and the closed loop system is

$$\dot{x} = \begin{cases} -ax & a \geq 0, \\ ax & a < 0, \end{cases} = -|a|x.$$

6.9 Let $V(x,t) = x^2p(t) + xf(t) + h(t)$. Since V is the optimal value function $V(x,T) = 0$, and we get the conditions $p(T) = f(T) = h(T) = 0$.

Then $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t),t) = 2xup + uf + x^2\dot{p} + x\dot{f} + \dot{h}$, so

$$\begin{aligned} J(u) &= V(x_0,0) + \int_0^T (x-r)^2 + u^2 + 2xup + uf + x^2\dot{p} + x\dot{f} + \dot{h}\,dt \\ &= V(x_0,0) + \int_0^T [(u + px + \frac{1}{2}f)^2 + x^2(1 + \dot{p} - p^2) + \\ &\qquad + x(-2r + \dot{f} - pf) + (r^2 + \dot{h} - \frac{1}{4}f^2)]\,dt. \end{aligned}$$

Solving the equations

$$\begin{cases} \dot{p} &= p^2 - 1, \\ \dot{f} &= pf + 2r, \\ \dot{h} &= \frac{1}{4}f^2 - r^2. \end{cases}$$

the optimal control is given by $\hat{u} = -p(t)x(t) - \frac{1}{2}f(t)$.

6.10 We assume $V = px^2 + fx + h$, so we get

$$\begin{aligned} \dot{V} &= \dot{p}x^2 + 2x\dot{x}p + \dot{f}x + f\dot{x} + \dot{h} = \\ &= \dot{p}x^2 + 2x(-x+u)p + \dot{f}x + f(-x+u) + \dot{h} = \\ &= x^2(\dot{p} - 2p) + 2upx + \dot{f}x + fu - fx + \dot{h}. \end{aligned}$$

This makes

$$J(u) - v(0, x_0) = \int_0^1 \underbrace{((x-a)^2 + (u-b)^2 - \dot{V})}_{K} \, dt,$$

where

$$\begin{aligned} K &= (x-a)^2 + (u-b)^2 - (x^2(\dot{p} - 2p) + 2upx + \dot{f}x + fu - fx + \dot{h}) = \\ &= (u - \frac{1}{2}f - b - px)^2 + x^2(1 - \dot{p} + 2p - p^2) + \\ &\quad +x(-2a - \dot{f} + f - fp - 2bp) + a^2 - \frac{1}{4}f^2 - fb + \dot{h}. \end{aligned}$$

Now let p, f, h solve the equations

$$\begin{cases} \dot{p} &= p^2 - 2p - 1, & p(1) = 0, & \text{Riccati equation} \\ \dot{f} &= (1-p)f - 2a - 2bp, & f(1) = 0, & \text{Linear equation in } f\text{,as } p \text{ known} \\ \dot{h} &= a^2 - \frac{1}{4} - fb, & h(1) = 0, & \text{Integrable as } p, f \text{ known.} \end{cases}$$

The optimal control is given by $\hat{u} = \frac{1}{2}f + b + px$.

6.11 Suppose $x \in KerP$. Multiplying both sides of the ARE by x:

$$PAx + C^TCx = 0,$$

similarly $x^TC^TCx = 0$. Therefore $x \in KerC$, which implies $kerP \subset kerC$. Furthermore, this leads to that $PAx = 0$. Thus $kerP$ is A-invariant.

When (C, A) is observable, the only A-invariant subspace in $KerC$ (unobservable subspace) is $\{0\}$. Thus, $kerP = \{0\}$.

8.7 Answers to: Kalman filters

7.1 **(a)** The recursion of the Kalman estimate is

$$\begin{aligned} \hat{x}_{t+1} &= \mathrm{E}^{H_t(y)} x_{t+1} = \mathrm{E}^{H_t(y)}(Ax_t + Bv_t) = \\ &= \mathrm{E}^{H_{t-1}(y)}(Ax_t + Bv_t) + \mathrm{E}^{[\tilde{y}]}(Ax_t + Bv_t) = A\hat{x}_t + K_t\tilde{y}_t. \end{aligned}$$

We need

$$\mathrm{E}[\tilde{y}_t\tilde{y}_t^T] = CP_tC^T,$$

$$\mathrm{E}[x_t\tilde{y}_t^T] = P_tC^T.$$

The projection theorem now gives us

$$K_t = AP_tC^T(CP_tC^T)^{-1}.$$

For the recursion of P we use the expression

$$\tilde{x}_{t+1} = (A - K_tC)\tilde{x}_t + Bv_t,$$

to obtain

$$P_{t+1} = (A - K_tC)P_t(A - K_tC)T + BB^T.$$

Insert K_t from above,

$$P_{t+1} = AP_tA^T - AP_tC^T(CP_tC^T)^{-1}CP_tA^T + BB^T.$$

(b) With the given data, the recursion for P is

$$P_{t+1} = \frac{1}{4}P_t - \frac{1}{4}\frac{P_t^2}{P_t} + BB^T = BB^T,$$

and the Kalman gain is

$$K_t = AP_tC^T(CP_tC^T)^{-1} = \frac{A}{C} = \frac{1}{2}.$$

The optimal estimate $\hat{x}_{t+1} = \frac{1}{2}y_t$. (Naturally, since we have no measure noise.)

7.2 Use the Kalman filter,

$$\begin{aligned} z(t+2) &= \mathrm{E}^{H_t(y)}x(t+2) = \mathrm{E}^{H_t(y)}(Ax(t+1) + Bv(t+1)) = \\ &= \{v(t+1) \perp H_t(y)\} = A\ \mathrm{E}^{H_t(y)}x(t+1) = A\hat{x}(t+1), \end{aligned}$$

where $\hat{x}$ is the state estimate given by the normal Kalman filter.

7.3 We want a recursion for $\hat{x}_t = \mathrm{E}^{H_{t-1}(y)}x_t$.

$$\begin{aligned} \hat{x}_{t+1} &= \mathrm{E}^{H_t(y)}x_{t+1} = A\ \mathrm{E}^{H_{t-1}(y)}x_t + A\ \mathrm{E}^{[\tilde{y}]}x_t + B\ \mathrm{E}^{H_t(y)}v_t + G\ \mathrm{E}^{H_t(y)}u_t \\ &= A\hat{x}_t + K_t\tilde{y}_t + Gu_t. \end{aligned}$$

The Kalman gain is by standard calculations

$$K_t = AP_tC^T(CP_tC^T + DD^T)^{-1}, \quad \text{where } P_t = \mathrm{E}\tilde{x}_t\tilde{x}_t'.$$

Since $\tilde{x}_t = x_t - \hat{x}_t = A\tilde{x}_t + K_t\tilde{y}_t$, the recursion for P_t is the same as the standard case without a control signal.

7.4 **(a)** The innovation process is given by

$$\tilde{y}(t) = y(t) - \mathrm{E}^{H_{t-1}(y)}y(t) = y(t) - C\ \mathrm{E}^{H_{t-1}(y)}x(t) = y(t) - C\hat{x}(t),$$

where $\hat{x}(t)$ is given by Kalman.
The innovations are given by the system

$$\begin{cases} \hat{x}(t+1) = (A - K(t)C)\hat{x}(t) + K(t)y(t), & \hat{x}(0) = 0, \\ \tilde{y}(t) = -C\hat{x}(t) + y(t), \end{cases}$$

driven by the output signal.

(b) The output signal is given by the system

$$\begin{cases} \hat{x}(t+1) = A\hat{x}(t) + K(t)\tilde{y}(t), \\ y(t) = C\hat{x}(t) + \tilde{y}(t). \end{cases}$$

driven by the innovations.

7.5 The Kalman estimate is given by

$$\hat{x}_{t+1} = \mathrm{E}^{H_t(y)}x_{t+1} = \mathrm{E}^{H_t(y)}(Ax_t + Bv_t) =$$

$$= \mathrm{E}^{H_{t-1}(y)}(Ax_t + Bv_t) + \mathrm{E}^{[\tilde{y}]}(Ax_t + Bv_t) = A\hat{x}_t + K_t\tilde{y}_t.$$

We need

$$\mathrm{E}[\tilde{y}_t\tilde{y}_t^T] = \{\tilde{x}_t \perp v_t\} = CP_tC^T + DD^T,$$

$$\mathrm{E}[(Ax_t + Bv_t)\tilde{y}_t^T] = \ldots = AP_tC^T + BD^T.$$

The projection theorem now gives us

$$K_t = (AP_tC^T + BD^T)(CP_tC^T + DD^T)^{-1}.$$

For the recursion of P we use the expression

$$\tilde{x}_{t+1} = (A - K_tC)\tilde{x}_t + (B - K_tD)v_t,$$

to obtain

$$P_{t+1} = (A - K_tC)P_t(A - K_tC)^T + (B - K_tD)(B - K_tD)^T.$$

K_t can be inserted from above and depends on P_t.

7.6 Let the observations be $y(t) \triangleq z(t) + w(t)$, and define $H_t(y)$ as usual. What we are looking for is the projection of $z(t)$ on $H_{t-1}(y)$, and since $z(t) = Cx(t)$ we have

$$\hat{z}(t) = \mathrm{E}^{H_{t-1}(y)} z(t) = C\ \mathrm{E}^{H_{t-1}(y)} x(t) = c\hat{x}(t).$$

Use the Kalman filter on

$$\begin{cases} x(t+1) &= Ax(t) + Bv(t), \\ y(t) &= Cx(t) + w(t), \end{cases}$$

to get $\hat{x}(t)$ and then $\hat{z}(t) = C\hat{x}(t)$.

The recursive equations are given by

$$\begin{cases} \hat{x}(t+1) &= (A - K(t)C)\hat{x}(t) + K(t)y(t), \\ \hat{x}(0) &= 0, \end{cases}$$

and

$$\begin{cases} P(t+1) &= AP(t)A^T - AP(t)C^T[CP(t)C^T + I]^{-1}CP(t)A^T + BB^T, \\ P(0) &= P_0. \end{cases}$$

The initial value P_0 is given by

$$P(0) = \mathrm{E}\ [x(0) - \hat{x}(0)][x(0) - \hat{x}(0)]^T = \{\hat{x}(0) = 0\} = \mathrm{E}\ x(0)x^T(0),$$

so P_0 solves the Liapunov equation $P = APA^T + I$, and since A is stable the solution is unique.

7.7 One way to solve this is to use the Kalman filter for the system described above with $z = z(t)$ for $t = 0, \ldots$

$$\begin{aligned} z(t+1) &= z(t) \\ y(t) &= z(t) + \sigma v(t), \end{aligned}$$

where v(t) is normalized white noise, and initially $P(0) = E(z(0)^2) = \alpha^2$. Namely, $A = 1, B = 0, C = 1, D = \sigma$. Then the Kalman filter is given by

$$\hat{x}(t+1) = \hat{x}(t) + K(t)(y(t) - \hat{x}(t)) = \hat{x}(t)(1 - K(t)) + K(t)y(t),$$

where $K(t) = \frac{P(t)}{P(t)+\sigma^2}$. The update equation for $P(t)$ is

$$P(t+1) = P(t) - \frac{P(t)^2}{P(t) + \sigma 2} = \frac{1}{P(t)^{-1} + \sigma^{-2}}.$$

Solving the recursion, using that $P(0) = \alpha^2$, we get

$$P(t) = \frac{1}{\alpha^{-2} + t\sigma^{-2}},$$

which equals $P(t) = E\{(\hat{z}_n - z)2\}$, and answers a. The Kalman gain is then

$$K(t) = \frac{1}{1 + t + \alpha^{-2}\sigma^2}.$$

We see that

$$\begin{aligned} \hat{z}(1) &= K(0)y(0) \\ \hat{z}(2) &= (1 - K(1))K(0)y(0) + K(1)y(1) = (y(0) + y(1))K(1) \\ \hat{z}(3) &= (1 - K(2))K(1)(y(0) + y(1)) + K(2)y(2) = (y(0) + y(1) + y(2))K(2) \end{aligned}$$

and so on, by noting that $K(n) = (1 - K(n))K(n-1)$. I.e.,

$$\hat{z}(n) = K(n-1) \sum_{t=0}^{n-1} y(t).$$

This answers b). In c) we get by plugging in the values

$$\hat{z}(3) = \frac{1 + 0.5 + 1}{4} = 0.625.$$

To answer d), $K(n-1) = \frac{1}{n}$ whenever $\alpha^{-2}\sigma^2 = 0$. I.e., when $\sigma = 0$ or $\alpha = \infty$.