Almost sure convergence

So far: (Y_n) sequence of random variables.

1.) $Y_n \xrightarrow{P} Y$ as $n \to \infty$, if $\mathbb{P}(|Y_n - Y| > \varepsilon) \to 0$ as $n \to \infty$, for every $\varepsilon > 0$.

Convergence in probability

2.) $Y_n \xrightarrow{2} Y$ as $n \to \infty$, if $\mathbb{E}[(Y_n - Y)^2] \to 0$, as $n \to \infty$.

Convergence in square mean.

We showed: $Y_n \xrightarrow{2} Y \underset{\not\Leftarrow}{\Rightarrow} Y_n \xrightarrow{P} Y$ (an application of Markov's inequality)

Almost sure convergence: $Y_n : \Omega \to \mathbb{R}$

$\omega \mapsto Y_n(\omega)$.

For fixed $\omega \in \Omega$, $Y_n(\omega)$ is a sequence of real numbers. Convergence could mean $Y_n(\omega) \to Y(\omega)$, as $n \to \infty$. "pointwise"

Introduce the event

$$C := \{\omega \in \Omega : Y_n(\omega) \to Y(\omega) \text{ as } n \to \infty\}.$$

Def: The sequence (Y_n) converges almost surely to Y as $n\to\infty$, denoted $Y_n \xrightarrow[n\to\infty]{a.s.} Y$, if

$$\mathbb{P}(C) = \mathbb{P}(\{\omega\in\Omega : Y_n(\omega)\to Y(\omega) \text{ as } n\to\infty\}) = 1.$$

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" Pointwise convergence, $\forall \omega\in C$: $Y_n(\omega)\xrightarrow[n\to\infty]{} Y(\omega)$. The complementary event of C, where convergence fails, is negligible, i.e.

$$\mathbb{P}(C^c)=0.$$

Lemma: $Y_n \xrightarrow{a.s.} Y \underset{\not\Leftarrow}{\Rightarrow} Y_n \xrightarrow{P} Y$ as $n\to\infty$

"**Proof:**" Let $\omega\in C$, then $|Y_n(\omega)-Y(\omega)|\to 0$, as $n\to\infty$.

$\mathbb{P}(A)=\mathbb{E}[\mathbb{1}_A]$

Hence $\mathbb{1}_{\{|Y_n-Y|>\varepsilon\}}(\omega)\longrightarrow 0$, as $n\to\infty$ $\forall\varepsilon>0$.

$$\lim_{n\to\infty}\mathbb{P}(|Y_n-Y|>\varepsilon) = \lim_{n\to\infty}\mathbb{E}\left[\mathbb{1}_{\{|Y_n-Y|>\varepsilon\}}(\omega)\right]$$

dominated convergence theorem (or close our eyes)

$$= \mathbb{E}\Big[\underbrace{\lim_{n\to\infty}\mathbb{1}_{\{|Y_n-Y|>\varepsilon\}}(\omega)}_{=0 \text{ if } \omega\in C}\Big]$$

$\mathbb{P}(C)=1$

$$=0.$$

Summary:

$$Y_n \xrightarrow{a.s} Y \Rightarrow Y_n \xrightarrow{P} Y \Rightarrow Y_n \xrightarrow{d} Y.$$

$$Y_n \xrightarrow{2} Y \Rightarrow Y_n \xrightarrow{P} Y$$

"convergence in distribution" (next note)

Remark: Almost sure converge and mean square convergence cannot be ordered.

Lemma: X_n, Y_n two sequences of random variables (not necessary independent)

Then:

If $X_n \xrightarrow{a.s} X$ and $Y_n \xrightarrow{a.s.} Y \Rightarrow X_n + Y_n \xrightarrow{a.s} X + Y$

and

$$X_n \cdot Y_n \xrightarrow{a.s.} X \cdot Y.$$

If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X + Y$

$$X_n \cdot X_n \xrightarrow{P} X \cdot Y$$

If $X_n \xrightarrow{2} X$ and $Y_n \xrightarrow{2} Y \Rightarrow X_n + Y_n \xrightarrow{2} X + Y$

(not valid for product) //

Bonus (rest of this note not on exam, but very interesting)

Lemma: Let $X_1, X_2, \ldots$ a sequence of independent random variables.

Then:

$$X_n \xrightarrow{a.s.} 0 \text{ as } n \to \infty$$

$$\overset{\text{iff}}{\Longleftrightarrow}$$

$$\mathbb{P}(\{|X_n| > \varepsilon, \text{ infinitely often}\}) = 0 \text{ for every } \varepsilon > 0.$$

Follows from Borel-Cantelli Lemma.

$$\overset{\text{iff}}{\Longleftrightarrow}$$

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty, \text{ for every } \varepsilon > 0 \quad //$$

Example: $X_2, X_3, \ldots$ independent.

$$\mathbb{P}(X_n = 1) = 1 - \frac{1}{n^{\cancel{2}\alpha}} \qquad \mathbb{P}(X_n = n) = \frac{1}{n^{\cancel{2}\alpha}}$$

(X_n takes the values 1 and n only)

$\alpha > 0$

$$\Rightarrow \sum_{n=2}^{\infty} \mathbb{P}(|X_n - 1| > \varepsilon) = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

$0 < \varepsilon < 1$

$$= \sum_{n=2}^{\infty} \frac{1}{n^\alpha} = \begin{cases} \text{finite if } \alpha > 1, \\ \text{diverges if } 0 < \alpha \leq 1. \end{cases}$$

$$\Rightarrow X_n \xrightarrow[n \to \infty]{a.s.} 1 \text{ if } \alpha > 1.$$

(c.f. Example 3.1. in [AG]) //

5.

Strong law of large numbers:

Thm: (X_i) a sequence of <u>pairwise independent</u> and identically distributed random variables, such that
(Etemadi, 1981)

$$\mathbb{E}|X_i| < \infty.$$

Then $\frac{1}{n}\sum_{i=1}^{n} X \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_1] = \mu.$ //

(w/o proof, harder).

Remark: <u>pairwise</u> independence means

distribution function $\rightsquigarrow$ $F_{X_i,X_j}(x,y) = F_{X_i}(x)\cdot F_{X_j}(y)$ for all $i\neq j$.

(X_i) independent $\Longrightarrow$ (X_i) pairwise independent $\Rightarrow$ (X_i) uncorrelated
(if variance exists).

Exercise: (X_i) i.i.d such that $\mathbb{E}(X_i)^4 < \infty$.

Show: $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mathbb{E}[X_i] = \mu \quad n\to\infty$ //

<u>Hint:</u> Assume $\mu = 0$, replace X_i by $\tilde{X}_i := X_i - \mu$.

Fix $\varepsilon > 0$. Then

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \varepsilon\right) = \mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| > n\varepsilon\right)$$

Markov inequality (with $r=4$)

$$\leq \frac{1}{(n\varepsilon)^4}\, \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right]$$

Next, $\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{i,j,k,l=1}^{n} \mathbb{E}\left[X_i X_j X_k X_l\right]$

use independence and $\mathbb{E}X_i = 0$.

to show that there is a constant $C < \infty$ such that

$$\leq \frac{Cn^2}{(n\varepsilon)^4} = \frac{C}{n^2\varepsilon^4}.$$

Use the Lemma on page 4 to conclude that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{a.s.} 0.$$

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