Random vectors:

transpose

$$\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = (X_1, X_2, \dots, X_n)'$$

Notation:

scalar product

1 by 1 matrix

$$\vec{X}' \cdot \vec{X} = (X_1, X_2, \dots X_n)\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = X_1^2 + X_2^2 + \dots + X_n^2$$

$$\vec{X} \cdot \vec{X}' = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}(X_1, \dots, X_n) = \begin{pmatrix} X_1^2 & X_1 X_2 & X_1 X_3 & \dots \\ X_2 X_1 & X_2^2 & \dots & \\ \vdots & & \ddots & \end{pmatrix}$$

nxn matrix

$$= (X_i \cdot X_k)_{i,k=1}^{n}.$$

Cumulative joint distribution of $\vec{X}$

$$F_{\vec{X}}(\vec{x}) = \mathbb{P}(X_1 < x_1, X_2 < x_2, \dots, X_n < x_n).$$

Mean vector $\vec{\mu}_{\vec{X}} = \mathbb{E}[\vec{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \dots, \mathbb{E}[X_n])'$

Covariance matrix:

$$\Lambda_{\vec{X}} := \mathbb{E}\left[(\vec{X}-\vec{\mu})\cdot(\vec{X}-\vec{\mu})'\right] \quad n \text{ by } n \text{ matrix}$$
$$= (\lambda_{ik})_{i,k=1}^{n} .$$

$$\lambda_{ij} = \mathbb{E}\left[(X_i - \mathbb{E}X_i)(X_k - \mathbb{E}X_k)\right] = \lambda_{jk} .$$

Properties of $\Lambda_{\vec{X}}$:

- $\Lambda = \Lambda'$ symmetric/self-adjoint $\lambda_{ik} = \lambda_{ki}$
- Non-negative definite, i.e. $\forall \vec{y} \in \mathbb{R}^n$

$$\vec{y}\,'\Lambda\vec{y} \geq 0 .$$

Proof: $\vec{y}\,'\Lambda\vec{y} = \vec{y}\,'\mathbb{E}\left[(\vec{X}-\vec{\mu})(\vec{X}-\vec{\mu})'\right]\vec{y}$

$$= \mathbb{E}\left[(\underbrace{\vec{y}\,'\cdot(\vec{X}-\vec{\mu})}_{\geq 0})^2\right] \geq 0$$

$\blacksquare$

- $\det \Lambda \geq 0$
- $\lambda_{ij}^2 \leq \lambda_{ii}\cdot\lambda_{jj}$ (Cauchy-Schwarz)

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Proposition: (linear transformation): $\vec{X}$ an n-dimensional random vector with mean $\vec{\mu}$ and covariance Λ.

Let B be an $m \times n$ deterministic matrix and $\vec{b} \in \mathbb{R}^m$ (deterministic)

m-dimensional vector!

Then $$\vec{Y} = B\vec{X} + \vec{b}$$

has mean vector

$$E[\vec{Y}] = B \cdot \vec{\mu}_X + \vec{b},$$

and covariance matrix ($m \times m$ matrix)

$$\Lambda_{\vec{Y}} = B \Lambda B'. \qquad //$$

Proof: exercise AG p. 120.

Multivariate normal/Gaussian distributions

Def: $\vec{X}$ has a multivariate Gaussian distribution with mean vector $\vec{\mu}$ and covariance Λ, denoted $\vec{X} \in N(\vec{\mu}, \Lambda)$, if the characteristic function is given by

Scalar product

$\vec{t}'\vec{X} = \sum_{i=1}^{n} t_i X_i$

$$\varphi_{\vec{X}}(\vec{t}) = E\left[e^{i\vec{t}'\cdot\vec{X}}\right] = e^{i\vec{t}'\vec{\mu} - \frac{1}{2}\vec{t}'\Lambda\vec{t}},$$

for every $\vec{t} \in \mathbb{R}^n$ //

Thm: (Cramér-Wold device) $\vec{X}$ has a multivariate normal distribution $N(\vec{\mu}, \Lambda)$ if and only if

$$\vec{a}'\cdot\vec{X} = \sum_{i=1}^{n} a_i X_i$$

has a normal distribution for <u>all</u> $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ //

Proof: See A6.

Example: Bivariate Gaussian.

$$(X_1, X_2)' \in \mathcal{N}(\vec{\mu}, \Lambda) \quad \text{with} \quad \vec{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $$\Lambda = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

1.) Show that $-1 \le \rho \le 1$, else Λ is not a covariance matrix.

- $\det \Lambda \geq 0$ here $\det \Lambda = 1 - \rho^2 \overset{!}{\geq} 0 \Rightarrow -1 \le \rho \le 1$.
- Alternatively, $\lambda_{ij}^2 \overset{\text{Cauchy-Schwarz}}{\le} \lambda_{ii} \cdot \lambda_{jj} = 1 \Rightarrow -1 \le \rho \le 1$.

2.) Set $Y := X_1 - X_2$. Then $Y \in \mathcal{N}(0, 2 - 2\rho)$.

By Cramér-Wold device $Y = (1, -1) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ with $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$= X_1 - X_2$

is normal.

We have $\mathbb{E}\, Y = \mathbb{E}\, X_1 - \mathbb{E}\, X_2 = 0$

$$\operatorname{Var} Y \overset{\mathcal{N}}{=} \vec{a}' \Lambda \vec{a} = (1, -1) \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1, -1) \begin{pmatrix} 1 - \rho \\ \rho - 1 \end{pmatrix}$$

$$= 1 - \rho + 1 - \rho$$

$$= 2 - 2\rho.$$

Proposition linear transformation with $B = \vec{a}'$ and $\vec{b} = 0$