Conditional expectation: Introduction

Example 1: Throw a symmetric/fair die twice

X_i : # of eyes in trial $i=1,2$

$Y := X_1 + X_2$.

$$\mathbb{E}[Y] \overset{\text{linearity}}{=} \mathbb{E}[X_1] + \mathbb{E}[X_2] = 2\mathbb{E}[X_1] = 2\sum_{i=1}^{6} i \cdot \tfrac{1}{6} = 7.$$

Remark: Do not need to know the distribution of Y.

$\mathbb{E}[Y]$ is the "best guess" for Y before we do the experiment.

Expected number of total dots given that X_1 takes the value x_1,

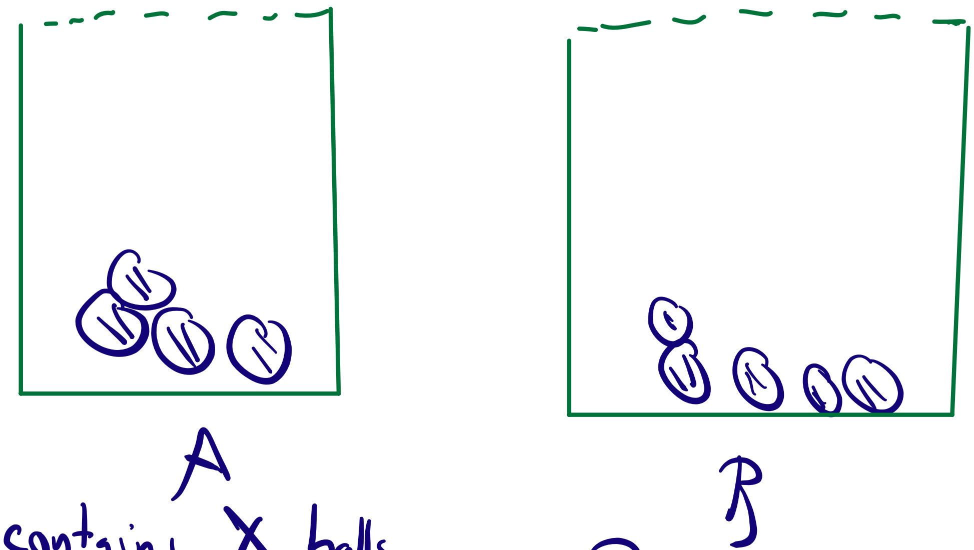
$$\mathbb{E}[Y \mid X_1 = x_1] := x_1 + \mathbb{E}[X_2] = \underline{x_1} + 3.5.$$

Conditional expectation of Y given that $X_1 = x_1$,

function of x_1.

"Best guess" for Y after the have tossed the die once.

Example 2: Consider two urns A and B.



Urn A contains X balls, urn B contains Y balls.

Assume X, Y are independent, $X \in Po(\lambda_A)$, $Y \in Po(\lambda_B)$ for fixed $\lambda_A, \lambda_B > 0$.

Find the conditional distribution of X given the total number of balls, $T = X + Y$, is equal to n. number, not r.v. here.

$$\mathbb{P}(X = k \mid T = n) = p_{X|T=n}(k) \qquad 0 \leq k \leq n$$

$$= \frac{\mathbb{P}(\{X = k\} \cap \{T = n\})}{\mathbb{P}(\{T = n\})} = \frac{\mathbb{P}(\{X = k\} \cap \{Y = n-k\})}{\mathbb{P}(\{T = n\})}$$

$T = X + Y = n$

$$\overset{\text{independence of } X \,\&\, Y}{=} \frac{\mathbb{P}(\{X = k\}) \cdot \mathbb{P}(\{Y = n-k\})}{\mathbb{P}(\{T = n\})}$$

Lemma: $X \in Po(\lambda_A)$, $Y \in Po(\lambda_B)$, X, Y independent

(useful to remember this one)

$$\Longrightarrow X + Y \in Po(\lambda_A + \lambda_B).$$

$$\mathbb{P}(X = k \mid T = n) = \frac{\cancel{e^{-\lambda_A}} \cdot \lambda_A^k}{k!} \cdot \frac{\cancel{e^{-\lambda_B}} \lambda_B^{n-k}}{(n-k)!} \cdot \frac{1}{\frac{\cancel{e^{-(\lambda_A+\lambda_B)}} (\lambda_B + \lambda_B)^n}{n!}}$$

$\binom{n}{k}$ "n choose k"

$$= \frac{n!}{k!(n-k)!} \frac{\lambda_A^k}{(\lambda_A+\lambda_B)^k} \frac{\lambda_B^{n-k}}{(\lambda_A+\lambda_B)^{n-k}}$$

$p_A := \frac{\lambda_A}{\lambda_A + \lambda_B}$; $p_B := \frac{\lambda_B}{\lambda_A + \lambda_B}$, note $p_A + p_B = 1$.

$$= \binom{n}{k} p_A^k (1 - p_A)^{n-k} \in Bin(n, p_A).$$

$$= P_{X \mid T = n}(k).$$

"Expected number of balls in urn A given T=n" (4.)

$$\mathbb{E}[X \mid T=n] := \sum_{k=0}^{\infty} k \cdot P(X=k \mid T=n) = \underline{n} \cdot p_A$$

function of n.

use $\mathrm{Bin}(n, p_A)$ Check!

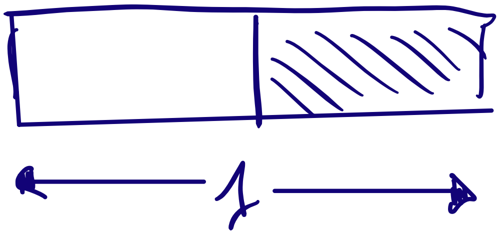
<u>Def</u>: X, Y discrete jointly distributed random variables.

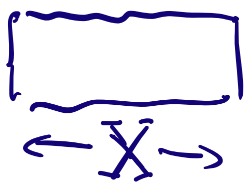
The <u>conditional expectation of Y</u> given <u>$X=x$</u> is

$$\mathbb{E}[Y \mid X=x] := \sum_y y \cdot P_{Y \mid X=x}(y)$$

Provided that the sum is absolutely convergent. //

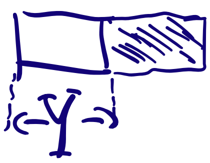
Example 3: A stick of length 1 is broken at a random point, uniformly distributed over the stick





length $X \in U(0,1)$

Do it again



Find the expected value of the remaining piece.

Idea: $X \in U(0,1)$. If $X=x$ then Y is uniformly distributed on the interval $(0,x)$.

$$Y \mid X=x \in U(0,x)$$

Conditional density of Y given $X=x$:

$$f_{Y|X=x}(y) = \begin{cases} \frac{1}{x} & 0<y<x \\ 0 & \text{else.} \end{cases}$$

Conditional expectation of Y given that $X=x$.

Clearly: $\mathbb{E}[X] = \frac{1}{2}$, $\mathbb{E}[Y \mid X=x] = \frac{x}{2}$

function of x.

6.

Def: X, Y continuous and jointly distributed.

The conditional expectation of Y given that $X=x$ is

$$\mathbb{E}[Y \mid X=x] := \int_{-\infty}^{\infty} y \, f_{Y \mid X=x}(y)\, dy.$$

provided the integral is absolutely convergent. //

"is a function of x"

Exercise: X, Y, Y_1, Y_2 random variables, $a, b \in \mathbb{R}$

- $\mathbb{E}[aY_1 + bY_2 \mid X=x] = a\,\mathbb{E}[Y_1 \mid X=x] + b\,\mathbb{E}[Y_2 \mid X=x]$.
- g a function: $\mathbb{R}^2 \to \mathbb{R}$

  $\mathbb{E}[g(X,Y) \mid X=x] = \mathbb{E}[g(x,Y) \mid X=x]$
- $\mathbb{E}[Y \mid X=x] = \mathbb{E}[Y]$ if X and Y are independent //

In all three examples we had that

$$\mathbb{E}[Y \mid X = x] = h(x) \text{ for some function } h.$$

We can look at it as a random variable $h(X)$.

$$h(X) := \mathbb{E}[Y \mid X]$$

Conditional expectation of Y given X

$\boxed{\text{Conditional expectation is a random variable}}$