Cramér-Slutsky Thm

Setup: (X_n), (Y_n) sequences of random variables such that

$$X_n \xrightarrow{P} X \quad \text{and} \quad Y_n \xrightarrow{P} Y, \quad \text{as } n \to \infty.$$

then

$$X_n + Y_n \xrightarrow[n\to\infty]{P} X + Y$$ (also okay for $\overset{a.s}{2}$ convergence)

⚠ In general wrong for convergence in distribution.

Easy case: $(X_n), X$ and $(Y_n), Y$ are independent, then

$$X_n + Y_n \xrightarrow{d} X + Y \quad \text{as } n \to \infty.$$

Proof:

$$\varphi_{X_n+Y_n}(t) \overset{\text{independence}}{=} \varphi_{X_n}(t) \cdot \varphi_{Y_n}(t) \xrightarrow[n\to\infty]{} \varphi_X(t) \cdot \varphi_Y(t)$$

two sequences of numbers

$$= \varphi_{X+Y}(t)$$

independence

Thm (Cramér - Slutsky)

a number!

$$X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{P} a \quad \text{as } n \to \infty$$

$a \in \mathbb{R}$.

then

$$X_n + Y_n \xrightarrow{d} X + a$$

$$X_n \cdot Y_n \xrightarrow{d} X \cdot a \qquad \text{as } n \to \infty.$$

$$\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{a} \quad \text{for } a \neq 0.$$

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Proof: Recall from the notes the following estimate: Z, T random variables. $x \in \mathbb{R}$, $\varepsilon > 0$. Then,

$$F_T(x) = \mathbb{P}(T \leq x) \leq \mathbb{P}(Z \leq x + \varepsilon) + \mathbb{P}(|Z - T| > \varepsilon)$$

Choose $T = X_n + Y_n$, $Z = X_n + a$

$$F_{X_n + Y_n}(x) \leq \mathbb{P}(X_n + a \leq x + \varepsilon) + \mathbb{P}(|Y_n - a| > \varepsilon)$$

$$= \underbrace{\mathbb{P}(X_n \leq x - a + \varepsilon)}_{= F_{X_n}(x - a + \varepsilon)} + \underbrace{\mathbb{P}(|Y_n - a| > \varepsilon)}_{\to 0 \text{ as } n \to \infty} \quad Y_n \xrightarrow{P} a$$

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Take $n \to \infty$

$$\limsup_{n\to\infty} F_{X_n+Y_n}(x) \leq F_X(x-a+\varepsilon)$$

for $x-a+\varepsilon \in C(F_X)$

set of continuity points of F_X.

Similarly,

$$F_X(x-a-\varepsilon) \leq \liminf_{n\to\infty} F_{X_n+Y_n}(x) \leq \limsup_{n\to\infty} F_{X_n+Y_n}(x)$$

$$\leq F_X(x-a+\varepsilon)$$

for $x-a-\varepsilon \in C(F_X)$

If $x-a \in C(F_X)$, take $\varepsilon \to 0$, we get

$$\lim_{n\to\infty} F_{X_n+Y_n}(x) = F_X(x-a)$$

$$= F_{X+a}(x)$$

So $X_n + Y_n \xrightarrow{d} X + a$

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Example: (X_n) iid with $\mathbb{E}[X_1]=0$

$\mathbb{E}[X_1^2]=\sigma^2$, $\sigma^2>0$.

Claim:

$$\sqrt{n}\,\frac{X_1+X_2+\dots+X_n}{X_1^2+X_2^2+\dots+X_n^2} \xrightarrow{d} \mathcal{N}\left(0,\frac{1}{\sigma^2}\right)$$

Solution:

$$\frac{\overbrace{\frac{1}{\sqrt{n}}(X_1+X_2+\dots+X_n)}^{\text{use the CLT}}}{\underbrace{\frac{1}{n}(X_1^2+X_2^2+\dots+X_n^2)}_{\text{use the LLN}}} \qquad \frac{1}{\sqrt{n}}(X_1+X_2+\dots+X_n) \xrightarrow[n\to\infty]{d} Y, \quad Y\in\mathcal{N}(0,\sigma^2)$$

$$\frac{1}{n}(X_1^2+X_2^2+\dots+X_n^2) \xrightarrow[n\to\infty]{P} \mathbb{E}[X_1^2]=\sigma^2$$

$$\xrightarrow[n\to\infty]{d}^{\text{Cramér-Slutsky}} \frac{Y}{\sigma^2} \in \mathcal{N}\left(0,\frac{\sigma^2}{\sigma^4}\right)=\mathcal{N}\left(0,\frac{1}{\sigma^2}\right) \qquad //$$

next week