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Problem 1

$$f_{X,Y} = \begin{cases} 21y^2, & \text{if } 0<y<x^2<1 \\ 0, & \text{else} \end{cases}$$

a) Marg. density f_X.

By convolution formula

$$f_X = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty} 21y^2 \mathbb{1}_{\{0<y<x^2<1\}}\,dy =$$

$$= \int_0^{x^2} 21y^2\,dy = 21\left[y^3/3\right]_0^{x^2} = \underline{\underline{7x^6}}$$

$$\Rightarrow \underline{\underline{f_X(x)}} = \begin{cases} 7x^6, & \text{if } 0<y<x^2<1 \\ 0, & \text{else.} \end{cases}$$

b) By conditional density formula:

$$f_{Y|X=x}(y) = \frac{f_{X,Y}}{f_X} = \frac{21y^2}{7x^6} = \underline{3\frac{y^2}{x^6}} \qquad \Rightarrow \underline{\underline{f_{Y|X} = \begin{cases} 3\frac{y^2}{x^6} & 0<y<x^2<1 \\ 0, & \text{else} \end{cases}}}$$

c) $$\mathbb{E}[Y|X] = \int_{-\infty}^{\infty} y f_{Y|X=x}\,dy = \frac{3}{x^6}\int_{-\infty}^{\infty} y\cdot y^2 \mathbb{1}_{\{0<y<x^2<1\}}\,dy =$$

$$= \frac{3}{x^6}\left[\frac{y^4}{4}\right]_0^{x^2} = \frac{1}{4x^6}\cdot x^8 = \underline{\underline{3\frac{x^2}{4}}} \text{ if } 0<y<x^2<1$$

0 else

Problem 2

$$f_X(x) = \begin{cases} 2x, & \text{if } 0<x<1 \\ 0, & \text{else} \end{cases}$$

$$Y|X=x \in U(0,x)$$

a) $Y|X=x \in U(0,x) \Rightarrow f_{Y|X=x}(y) = 1/x.$

$$\Rightarrow \mathbb{E}[Y|X] = \int_{-\infty}^{\infty} y \cdot \frac{1}{x} \mathbb{1}_{\{0<y<x\}} dy =$$

$$= \frac{1}{x}\int_0^x y\,dy = \frac{1}{x}\left[y^2/2\right]_0^x = \underline{\underline{x/2}} \quad \text{if } 0<x<1 \text{ and } 0<y<x$$

By law of tot. exp.! $\underline{\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]} \Rightarrow$

$$\Rightarrow \mathbb{E}[x] = \mathbb{E}[x/2] = \int y\, x/2\, \mathbb{1}_{\{0<y<x\}} dy =$$

$$= \frac{x}{2}\int_0^x y\,dx = \frac{x}{2}\left[y^2/2\right]_0^x = \underline{\underline{x^3/4}}, \text{ if } 0<x<1$$

b) M.G.F.: $\psi_Y(t) := \mathbb{E}[e^{tY}] = \mathbb{E}[\mathbb{E}[e^{tY}|X]] =$

$$= \begin{bmatrix} \mathbb{E}[e^{tY}] = \int y e^{ty} dy = \\ = \left[\frac{y}{t}e^{ty}\right]_0^x - \int_0^x \frac{e^{ty}}{t} dy = \frac{1}{t}\left[xe^{tx} - \left(\frac{e^{tx}}{t} - \frac{1}{t}\right)\right] \end{bmatrix} =$$

$$= \mathbb{E}\Big[\underbrace{\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2} + \frac{1}{t^2}}_{u}\Big] = \int_0^x yu\,dy =$$

$$= \frac{x^2}{2}\left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2} + \frac{1}{t^2}\right), \quad 0<y<x<1$$