Moment generating function

Def: The moment generating function (mgf) of a random variable X is defined as

$$\psi_X(t) := \mathbb{E}\left[e^{tX}\right] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \\ \sum_k e^{tx_k} p_X(x_k) \end{cases}$$

provided there exists $h>0$ such that the expectation exists and is finite for $|t|<h$.

Ex: $X \in Bin(n,p)$. Then

$$\psi_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} \qquad p+q=1.$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k q^{n-k} = (q+pe^t)^n.$$

binomial identity

Binomial identity: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ where $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Ex: $X \in Exp(a)$, density $f_X(x) = \frac{1}{a} e^{-x/a} \mathbb{1}_{\{x>0\}}$, $a>0$.

$$\psi_X(t) = \int_0^\infty e^{tx} \frac{1}{a} e^{-x/a}\, dx = \frac{1}{a}\int_0^\infty e^{-x\overbrace{\left(\frac{1}{a}-t\right)}^{\text{assume: } >0}}\, dx$$

$$= \frac{1}{a} \frac{1}{\frac{1}{a}-t} e^{-x\left(\frac{1}{a}-t\right)} \Big|_0^\infty$$

assume that $t < \frac{1}{a}$

$$= \frac{1}{a} \frac{1}{\frac{1}{a}-t} = \frac{1}{1-ta} \quad \text{if} \quad t < \frac{1}{a}.$$

Choose $h = \frac{1}{a}$, okay for $|t| < h = \frac{1}{a}$.

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Ex: $X \in C(0,1)$ with density $f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}$, $x \in \mathbb{R}$.

Cauchy distribution

$$\frac{1}{\pi}\int_{-R}^{R} e^{tx} \frac{1}{1+x^2}\, dx \xrightarrow[\text{as } R\to\infty]{} \infty \text{ for any } t \neq 0^*!$$

$\Longrightarrow$ mgf does not exist at all.

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* For $t=0$, $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2} = 1$.

Thm 1: Let X and Y be random variables.

If there exists $h > 0$ such that

$$\psi_X(t) = \psi_Y(t) \text{ for all } |t| < h,$$

then $X \stackrel{d}{=} Y$. //

Thm 2: Let $X_1, X_2, \ldots, X_n$ be <u>independent</u> random variables, whose mgf exist for $|t| < h$, for some $h > 0$.

Set

$$S_n := \sum_{i=1}^{n} X_i.$$

Then

$$\psi_{S_n}(t) = \prod_{i=1}^{n} \psi_{X_i}(t), \quad |t| < h.$$ //

Exercise: Prove Thm 2.

Ex: $X_1 \in \mathcal{N}(0, \sigma_1^2)$, $X_2 \in \mathcal{N}(0, \sigma_2^2)$, independent.

$$\psi_{X_1}(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{-x^2}{2\sigma_1^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma_1^2} - tx\right)} dx$$

Complete the square

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma_1^2 t)^2}{2\sigma_1^2}} e^{\frac{\sigma_1^2 t^2}{2}} dx$$

$$= e^{\frac{\sigma_1^2 t^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma_1^2 t)^2}{2\sigma_1^2}} dx}_{=1 \text{ (change variables } x \to x - \sigma_1^2 t)}$$

$$= e^{\frac{\sigma_1^2 t^2}{2}}.$$

$$\psi_{X_1+X_2}(t) \overset{\text{Thm 2}}{=} \psi_{X_1}(t) \cdot \psi_{X_2}(t) = e^{\frac{\sigma_1^2 t^2}{2}} e^{\frac{\sigma_2^2 t^2}{2}} = e^{\frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}.$$

$$\overset{\text{Thm 1}}{\Longrightarrow} X_1 + X_2 \in \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$$

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Why moment generating function?

Thm 3: Let X be a r.v. whose mgf $\psi_X(t)$ exists for $|t|<h$, for some $h>0$. Then

a.) all moments of X exist, i.e. $\mathbb{E}|X|^n<\infty$, for $n\in\mathbb{N}$.

b.) $\mathbb{E}X^n = \psi_X^{(n)}(0) = \left.\dfrac{d^n}{dt^n}\psi_X(t)\right|_{t=0}$

Proof: b.) assuming X has a density, for simplicity.

$$\psi_X'(t) = \int_{-\infty}^{\infty} x e^{tx} f_X(x)\,dx$$

$$\psi_X''(t) = \int_{-\infty}^{\infty} x^2 e^{tx} f_X(x)\,dx$$

$$\vdots$$

$$\psi_X^{(n)}(t) = \int_{-\infty}^{\infty} x^n e^{tx} f_X(x)\,dx$$

choose now $t=0$

$$\psi_X^{(n)}(0) = \int_{-\infty}^{\infty} x^n f_X(x)\,dx = \mathbb{E}X^n.$$

$\blacksquare$

Remark: Taylor expansion for exponential function

$$e^{tX} = 1 + \sum_{n=1}^{\infty} \frac{(tX)^n}{n!}.$$

$$\Rightarrow \quad \psi_X(t) = \mathbb{E}[e^{tX}] = 1 + \sum_{n=1}^{\infty} t^n \frac{\mathbb{E}X^n}{n!}$$

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Ex: $X \in \mathrm{Exp}(a)$

geometric series

$$\psi_X(t) = \frac{1}{1-at} = 1 + \sum_{n=1}^{\infty} (at)^n = 1 + \sum_{n=1}^{\infty} t^n a^n.$$

for $|t| < \frac{1}{a}$

Compare coefficients in the power series in t: $\frac{\mathbb{E}X^n}{n!} = a^n$

$$\Rightarrow \quad \mathbb{E}[X^n] = n!\, a^n \quad \text{for any } n \in \mathbb{N}. \qquad //$$

Exercise: (optional) $X \in \mathcal{N}(0,\sigma^2)$. Show that

$$\mathbb{E}X^{2n+1} = 0 \qquad \mathbb{E}X^{2n} = \frac{(2n)!}{2^n n!}\sigma^{2n}.$$

Hint: $\psi_X(t) = e^{\frac{\sigma^2 t^2}{2}}$. Claim: $\psi_X^{(n)}(t) = (n-1)\sigma^2 \psi_X^{(n-2)}(t) + t\sigma^2 \psi_X^{(n-1)}(t)$, $n \geq 2$.

Proof: Check $n=2$, then induction.

Then set $t=0$ and solve for $\psi_X^{(n)}(0)$.

Geometric series:

$$\sum_{n=0}^{m-1} x^n = 1 + \sum_{n=1}^{m-1} x^n = \frac{1-x^m}{1-x}.$$

If $|x|<1$, we can send $m \to \infty$ to obtain

$$1 + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}.$$