Random variables

Ex: Toss a fair coin twice: $\Omega = \{HH, HT, TH, TT\}$

$$\mathbb{P}(\{HH\}) = 1/4, \text{ etc.}$$

More convenient to work with functions $X : \Omega \to \mathbb{R}$.

For example: Let X be the total number of heads:

$$X(HH) = 2, \quad X(HT) = X(TH) = 1, \quad X(TT) = 0.$$

X is an example of a random variable.

Def: $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. A random variable, r.v., is a measurable function

$$X : \Omega \to \mathbb{R}$$
$$\omega \mapsto X(\omega)$$

Measurable means: $\forall x \in \mathbb{R}$ the event $\{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{A}$

$\Longleftrightarrow$

$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{A}$ for every $B \in \mathcal{B}(\mathbb{R})$

Borel σ-algebra.

Notation: $\{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\}$, $B \in \mathcal{B}(\mathbb{R})$

and

$$\{X \leq x\} = \{\omega \in \Omega : X(\omega) \leq x\}.$$

Example: Indicator function. Let $A \in \mathcal{A}$,

$$\chi_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

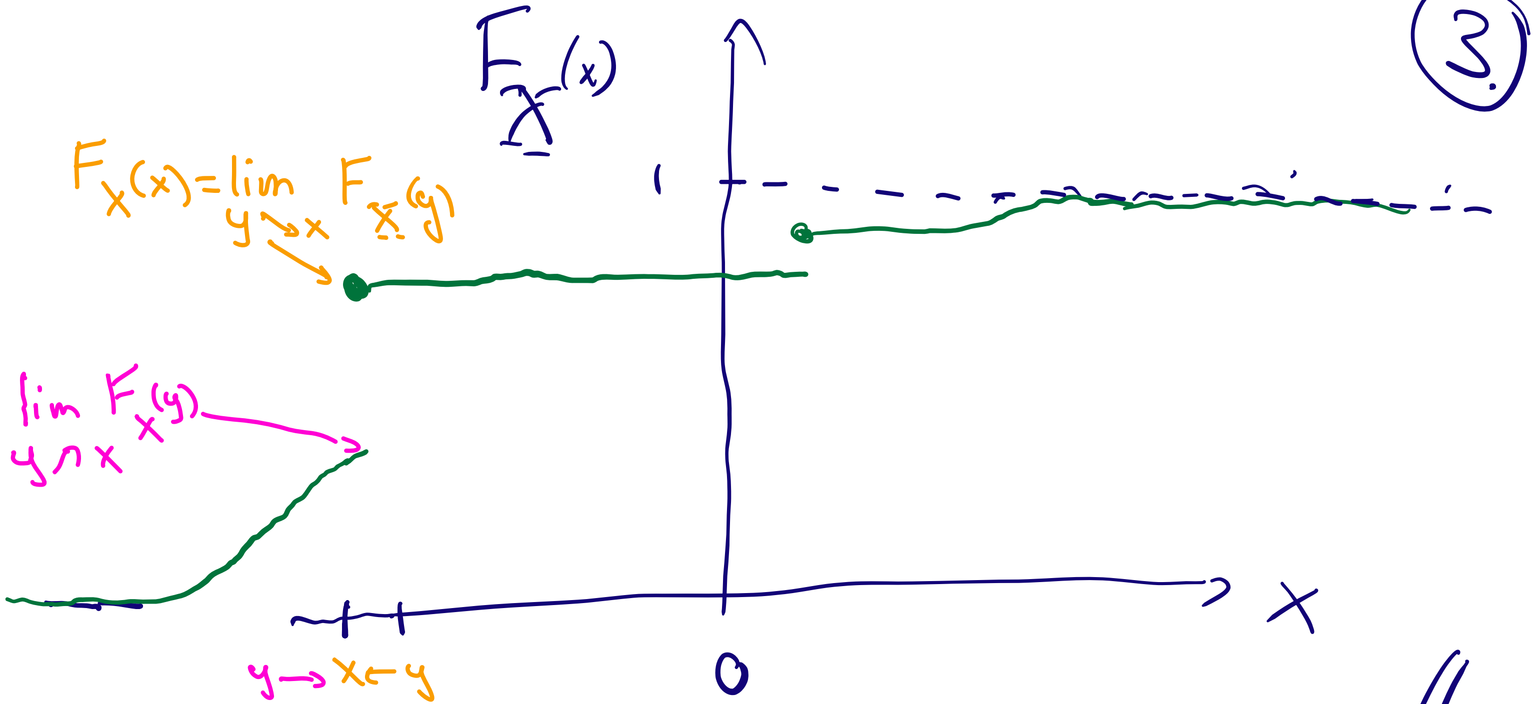
Claim: χ_A is a random variable (see exercises)

Def: Cumulative distribution function (CDF) X a random variable:

$$F_X(x) := \mathbb{P}(\{X \leq x\}) = \mathbb{P}(\underbrace{\{\omega \in \Omega : X(\omega) \leq x\}}_{\in \mathcal{A}}) \quad \forall x \in \mathbb{R}$$

because X is measurable, thus $F_X(x)$ is well-defined

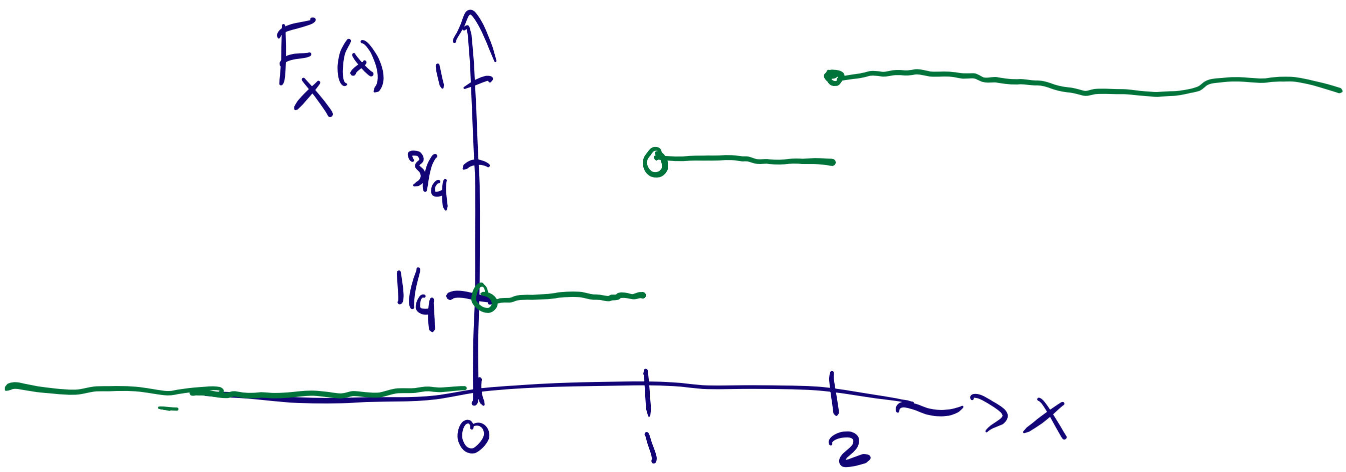
Proposition:
- $F_X(x)$ is non-decreasing
- $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.
- $F_X(x)$ is right continuous: $\lim_{y \searrow x} F_X(y) = F_X(x)$



Example: Tossing a fair coin twice: $\Omega = \{HH, HT, TH, TT\}$

X : total number of heads.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \ge 2. \end{cases}$$



④.

Y: total number of tails: By symmetry:

$$F_Y(x) = F_X(x) \quad \forall x \in \mathbb{R}.$$

We say X and Y are <u>identically distributed</u>

<u>Notation</u>: $$X \overset{d}{=} Y$$

<u>However</u>: $X(\omega) \neq Y(\omega)$ for $\omega = \{HH\}$ or $\{TT\}$

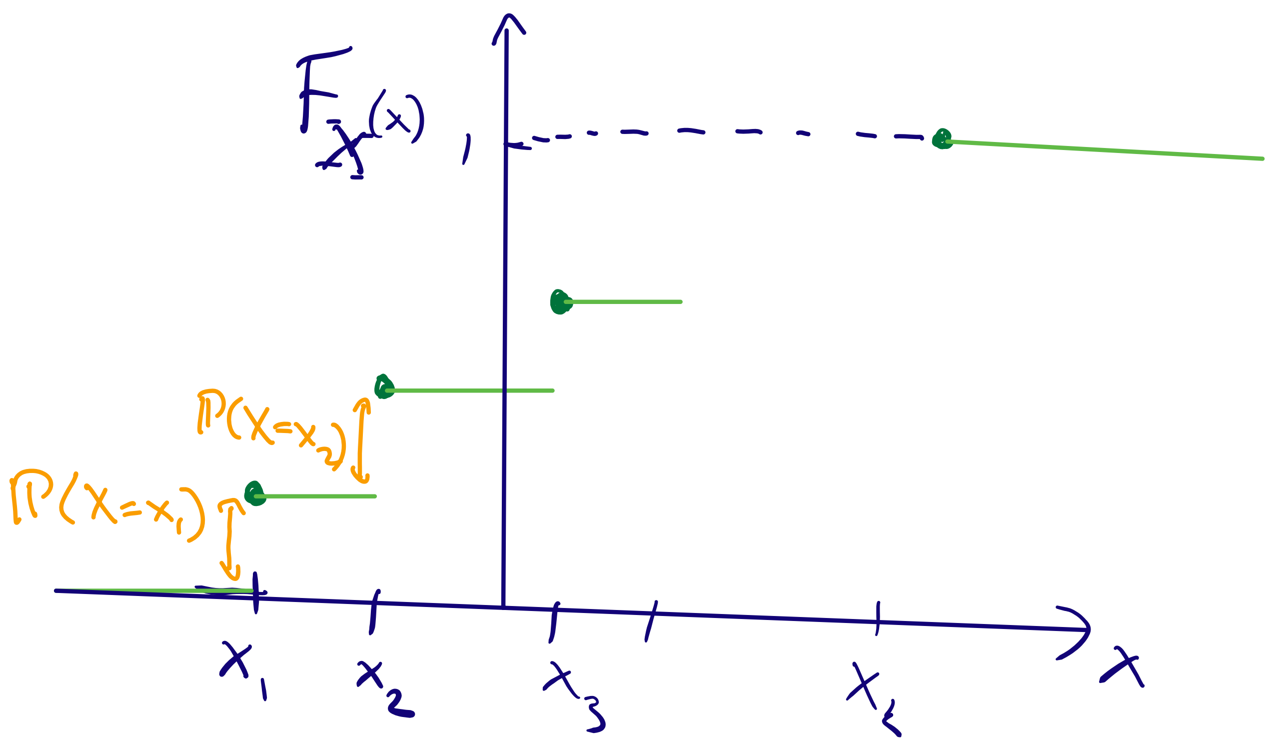
$X(\omega) = Y(\omega)$ for $\omega = \{HT\}$ or $\{TH\}$ //

Two types of distributions

<u>discrete</u>: X takes only a finite or countable number of values, say, $\{x_1, x_2, x_3, \dots\}$

$$P_X(x_i) := \mathbb{P}(\{X = x_i\})$$ probability mass function (pmf)

$$F_X(x) = \sum_{y \le x} P_X(y) \qquad x \in \mathbb{R}$$

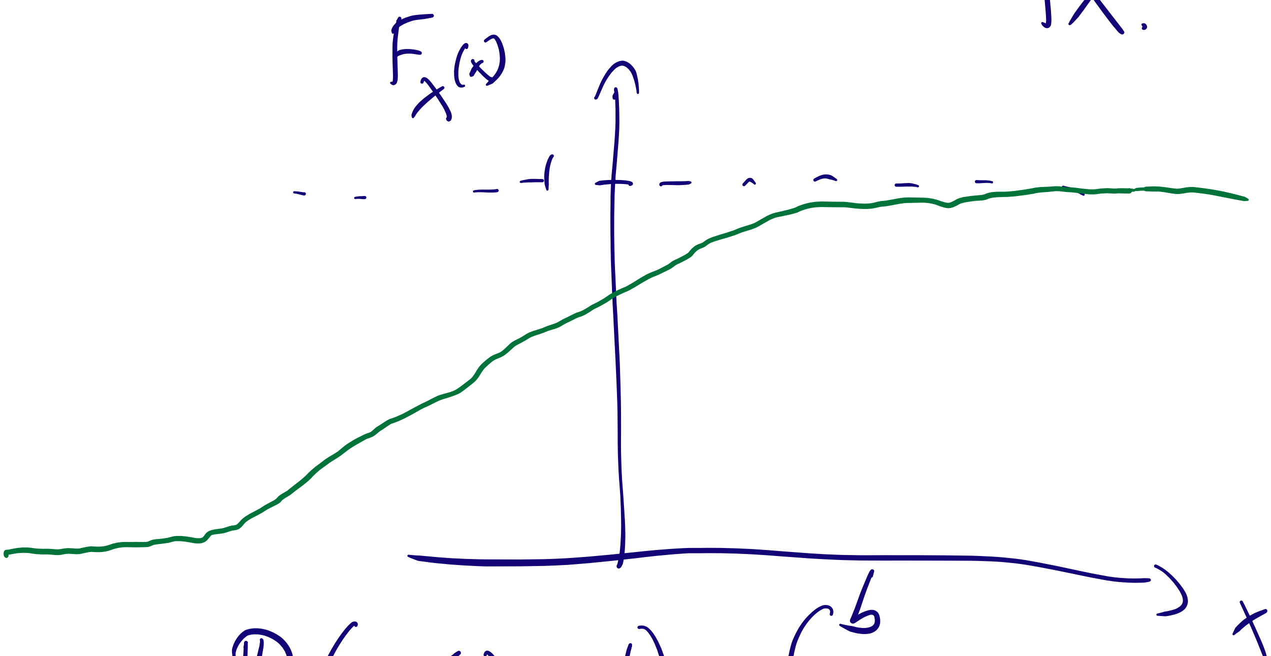


$$\mathbb{P}(X = x_i) = F_X(x_i) - \lim_{x \nearrow x_i} F_X(x)$$

absolutely continuous: density function f_X : ⑥

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

and $F_X'(x) = f_X(x)$ for all x that are continuity points of f_X.



$$\mathbb{P}(a < X \leq b) = \int_a^b f_X(y)\,dy.$$

$$= F_X(b) - F_X(a)$$

Expectation and moments of r.v

Expected value:
$$\mathbb{E}X := \begin{cases} \sum_k x_k \cdot p_X(x_k) & \text{if } X \text{ is discrete.} \\ \int_{-\infty}^{\infty} x f_X(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

provided that sum and integral are absolutely convergent, e.g.

$$\sum_k |x_k| p_X(x_k) < \infty.$$

Property: • $\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$. (linearity)

m-th moment $(m \geq 0)$

$$\mathbb{E}X^m = \begin{cases} \sum_k x_k^m \cdot p_X(x_k) \\ \int_{\mathbb{R}} x^m f_X(x)\,dx \end{cases},$$

provided sum or integral are absolutely convergent.

$$\operatorname{Var} X := \mathbb{E}\Big(\underbrace{(X - \mathbb{E}X)^2}_{\geq 0}\Big) \geq 0$$

$$= \mathbb{E}(X^2) - (\mathbb{E}X)^2 \geq 0$$

$$\Longrightarrow \mathbb{E} X^2 \geq (\mathbb{E}X)^2$$ (special case of Jensen's inequality).

<u>Exercise:</u> Assume that $\operatorname{Var} X = 0$, what can be said about the distribution of X?