KTH Matematik

Homework 3
Mathematical Systems Theory, SF2832
Passing grade: 12p

1. Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x,$$

where a is a constant.

(a) Can we always design a feedback controller $u = kx$ such that the closed-loop poles are placed in $\{-1, -1\}$? .... (1p)

(b) Is the resulting closed-loop system observable? .... (1p)

(c) Assume now that the state is not available. Design an observer when you can such that the state estimation error converges to 0 by the rate e^{-t} or faster. (2p)

2. Consider the optimal control problem

$$\min_u \int_0^{t_1} (y^2 + u^2) dt$$

subject to

$$\dot{x} = Ax + bu$$
$$y = cx$$

where,

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(a) Show that the solution $P(t)$ to the associated dynamical Riccati Equation (DRE or RE as in the compendium) is positive definite $\forall\ 0 \le t < t_1$ (you do not have to solve the DRE). .... (2p)

(b) Let $t_1 \to \infty$. Show the existence of optimal control by solving the associated ARE and verify that your solution is positive definite. .... (3p)

3. We have discussed how to use the linear adjoint system to solve the DRE

$$\dot{P} = -A^T P - PA + PBR^{-1}B^T P - C^T C$$
$$P(t_1) = S.$$

Through this exercise we will show that similar idea can be used to solve ARE. Consider the following ARE

$$A^T P + PA - PBR^{-1}B^T P + C^T C = 0,$$

and the associated

$$H = \begin{pmatrix} A & -BR^{-1}B^T \\ -C^T C & -A^T \end{pmatrix},$$

where H is a so-called Hamiltonian matrix.

(a) Show that if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue. Hint: It is not difficult to construct a matrix J such that $JHJ^{-1} = -H^T$. .... (2p)

(b) Assume that (A, B) is reachable and (C, A) is observable. Show that $\lambda = 0$ is not an eigenvalue of H. Hint: If $\lambda = 0$ is an eigenvalue, then $Hx = 0$ for some $x \neq 0$. .... (2p)

(c) Under the assumptions in (b) we can even show that H does not have any eigenvalue on the imaginary axis. Now Assume further that $[X_1^T \ X_2^T]^T$ consists of n eigenvectors associated with the negative eigenvalues of H, namely

$$H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} Z,$$

where Z is a stable matrix. Show $P = X_2 X_1^{-1}$ is a solution to the ARE if X_1 is invertible. .... (2p)

In fact, we can further show that this solution is symmetric and positive definite, but this is not required in this exercise.

4. At time $t = 1, 2, 3, \cdots$, an observation $y(t)$ is made of an unknown constant x. The observation error $y(t) - x$ is zero mean white noise with variance σ^2. Our apriori knowledge on x has variance p_0.

(a) Design a Kalman filter for the estimation of x, and express the covariance matrix $p(t) = E\{(x - \hat{x}(t))^2\}$ in terms of t, σ, p_0........................ (2p)

(b) Show $\hat{x}(t+1) = \hat{x}(t)$ as $t \to \infty$. .... (1p)

(c) What is $\hat{x}(t)$ if $\sigma^2 \to \infty$? Give a brief explanation of the result. .... (2p)