Conditional expectation: continued

(X,Y) continuous random variables, joint density

$$f_{X,Y}(x,y).$$

- Conditional density of Y given that $X=x$, $x\in\mathbb{R}$

$$f_{Y|X=x}(y) := \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,v)\,dv}$$

function of x and y; and

$$\int_{-\infty}^{\infty} f_{Y|X=x}(y)\,dy = 1 \quad \text{(density)}$$

- Conditional expectation of Y given that $X=x$ is

$$\mathbb{E}[Y|X=x] := \int_{-\infty}^{\infty} y\, f_{Y|X=x}(y)\,dy = h(x)$$

function of x!

- Conditional expectation of Y given X

$$\mathbb{E}[Y|X](\omega) = h(X(\omega)) \quad \text{random variable} \quad \omega\in\Omega$$

Theorem 1: (Law of total expectation).

Suppose that $\mathbb{E}|Y| < \infty$. Then

$$\underbrace{\mathbb{E}[\underbrace{\mathbb{E}[Y|X]}_{\text{random variable}}]}_{\text{number}} = \underbrace{\mathbb{E}[Y]}_{\text{number}}.$$

Proof:

$$\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x] f_X(x)\,dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} y f_{Y|X=x}(y)\,dy\right) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \frac{f_{X,Y}(x,y)}{\cancel{f_X(x)}}\,dy\,\cancel{f_X(x)}\,dx.$$

Fubini thm

change order of the integrals.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y \Big(\underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}_{\text{marginal density } f_Y(y)}\Big)\,dy$$

$$= \int_{-\infty}^{\infty} y f_Y(y)\,dy = \mathbb{E}[Y] \qquad \blacksquare$$

Thm 2: Let X and Y be r.v and g a function.
Then,

1.) $\mathbb{E}[g(X)Y \mid X] = g(X) \cdot \mathbb{E}[Y \mid X]$

2.) $\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$ if X and Y are independent //

Proof: exercise.

Interpretation:

1.) By conditioning on X, we mean that X is known, so $g(X)$ is a constant and can be pulled out of the conditional expectation.

2.) Y has nothing to do with X, so the best prediction is its expected value $\mathbb{E}[Y]$.

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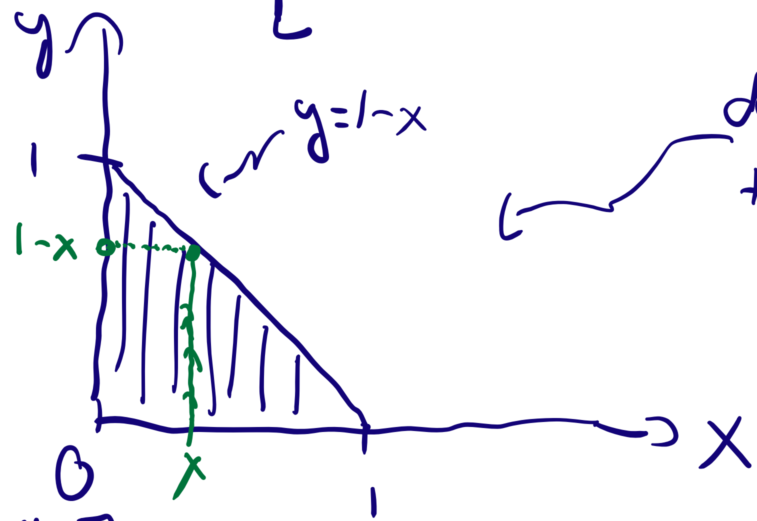
Example: (AG: 2.6.9)

X, Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} 6x & \text{if } x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0 & \text{else} \end{cases}$$



density vanishes outside this triangle.

<u>Find $E[Y|X]$.</u>

$$E[Y|X=x] \overset{\text{definition}}{=} \int_{\mathbb{R}} y\, f_{Y|X=x}(y)\, dy \overset{\text{definition}}{=} \int_0^1 y \frac{f_{X,Y}(x,y)}{f_X(x)}\, dy$$

Compute: $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, dy = \int_0^{1-x} 6x\, dy = 6x \int_0^{1-x} 1\, dy = 6x(1-x)$

So

$$\int_0^1 y \frac{f_{X,Y}(x,y)}{f_X(x)}\, dy = \int_0^{1-x} y \frac{\cancel{6x}}{\cancel{6x}(1-x)}\, dy = \frac{1}{1-x} \int_0^{1-x} y\, dy$$

$$= \frac{1}{1-x} \frac{(1-x)^2}{2} = \frac{1-x}{2}.$$

<u>Answer:</u> $E[Y|X] = \dfrac{1-X}{2}$

5.

Determine $\mathbb{E}[X \mid Y] = \frac{2}{3}(1-Y)$