LEMMA 6.1.4. *Let $b \in \mathbb{R}^n$ and let (A, b) be completely reachable, i.e. $\langle A|\,\mathrm{Im}\ b\rangle = \mathbb{R}^n$. Then there is a nonsingular $n \times n$ matrix T such that*

$$TAT^{-1} = F \triangleq \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \quad Tb = h \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $a_1, a_2, \cdots, a_n$ are the coefficients of

$$\mathcal{X}_A(s) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

The transformation T is unique and is given by

$$T = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} \tag{50}$$

where c is the unique row vector solution to

$$c[b, Ab, \cdots, A^{n-1}b] = (0, 0, \cdots, 0, 1). \tag{51}$$

Proof: Since (A, b) is completely reachable, there is a unique solution to (51). The system (51) can be written

$$\begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

or equivalently, $Tb = h$. To see that T is nonsingular, we show that the rows are linearly independent. To this end, let $\alpha_1, \alpha_2, \dots \alpha_n$ be real numbers such that

$$\alpha_1 c + \alpha_2 cA + \cdots + \alpha_n cA^{n-1} = 0,$$

multiply from the right by b and use (51). This yields $\alpha_n = 0$. Then multiply by Ab which yields $\alpha_{n-1} = 0$ and then by $A^2 b$ which yields $\alpha_{n-2} = 0$ etc. Since therefore $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$, the rows of T are linearly independent. Set $T^{-1} = [s_1, s_2, \cdots, s_n]$ where s_i are the columns. Then

$$TT^{-1} = \begin{bmatrix} cs_1 & cs_2 & \dots & cs_n \\ cAs_1 & cAs_2 & \dots & cAs_n \\ \vdots & \vdots & \vdots & \\ cA^{n-1}s_1 & cA^{n-1}s_2 & \dots & cA^{n-1}s_n \end{bmatrix}$$