# Multivariate Gaussians:

1.

**Def:** $\vec{X} = (X_1, X_2, \dots, X_n) \in \mathcal{N}(\underset{n}{\vec{\mu}}, \Lambda)$

mean vector, $\vec{\mu} \in \mathbb{R}^n$.

covariance matrix: $n$ by $n$

if its characteristic function is

$$\varphi_{\vec{X}}(\vec{t}) = e^{i\vec{t}'\cdot\vec{\mu} - \frac{1}{2}\vec{t}'\Lambda\vec{t}} \qquad \forall \vec{t} \in \mathbb{R}^n$$

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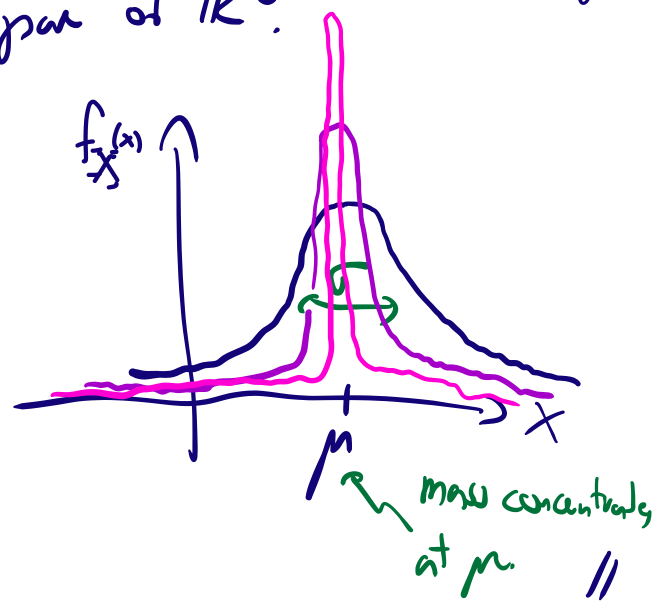
**Remark:** If $\det \Lambda = 0$ then the distribution is <u>singular</u>, meaning it concentrates on a subspace of $\mathbb{R}^n$.

Ex: $n=1$, $\mathcal{N}(\mu, \underset{\Lambda}{\underset{\shortparallel}{\sigma^2}})$

Take $\sigma \to 0$

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2.

<u>Thm</u> (linear transformation) $B$ an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, deterministic.

Let $\vec{Y} := B\vec{X} + \vec{b}$.

Then $\vec{Y} \in \mathcal{N}(B\cdot\vec{\mu} + \vec{b}, B\Lambda B')$. //

<u>Proof</u>: Choose $\vec{\mu} = \vec{b} = 0$ for simplicity.

$$\varphi_{\vec{Y}}(\vec{t}) = \mathbb{E}\left[e^{i\vec{t}'\cdot\vec{Y}}\right] = \mathbb{E}\left[e^{i\vec{t}'(B\vec{X})}\right]$$

Note that

$$\vec{t}'(B\vec{X}) = (B'\vec{t})'\vec{X}$$ (properties of scalar product; write it out in components)

$$= \mathbb{E}\left[e^{i(B'\vec{t})'\vec{X}}\right] = \varphi_{\vec{X}}(B'\vec{t})$$

$$= e^{-\frac{1}{2}(B'\vec{t})'\Lambda(B'\vec{t})}$$

$$= e^{-\frac{1}{2}\vec{t}'\cdot(B\Lambda B')\vec{t}}$$

Hence $\vec{Y} \in \mathcal{N}(0, B\Lambda B')$. ∎

**Thm**: Let $\vec{X}$ be a Gaussian vector. Then the components of $\vec{X} = (X_1, X_2, \ldots, X_n)$ are <u>independent</u> <u>if and only</u> if they are uncorrelated, i.e.

$$\lambda_{ij} = \mathrm{Cov}(X_i, X_j) = 0 \quad \text{for all } i \neq j$$

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Special property of Gaussian vectors.

**Proof**:

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Then

$$\varphi_{\vec{X}}(\vec{t}) = e^{i\vec{t}\cdot\vec{\mu} - \frac{1}{2}\vec{t}\,\Lambda\,\vec{t}}$$

$$= e^{i\sum_{k=1}^{n} t_k \mu_k - \frac{1}{2}\sum_{k=1}^{n} t_k^2 \lambda_k}$$

$$= \prod_{k=1}^{n} \underbrace{e^{i t_k \mu_k - \frac{1}{2} t_k^2 \lambda_k}}_{\text{characteristic function}}$$

$$= \prod_{k=1}^{n} \varphi_{X_k}(t_k).$$

**Upshot**: Characteristic function factorizes.

$$\overset{\text{"Fact of life"}}{\Longrightarrow} X_i\text{'s are independent.}$$

**Exercise**: Let $X_1$ and $X_2$ be independent standard Gaussian. Show that $X_1 + X_2$ and $X_1 - X_2$ are independent. //

## Density of Gaussian vectors

Let $\vec{X} \in \mathcal{N}(\vec{\mu}, \Lambda)$ with $\det \Lambda > 0$ (non-singular case).

Then,

$$f_{\vec{X}}(\vec{x}) = \left(\frac{1}{2\pi}\right)^{n/2} \cdot \frac{1}{\sqrt{\det \Lambda}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})' \Lambda^{-1} (\vec{x}-\vec{\mu})}$$

matrix inverse of $\Lambda$

$\vec{x} \in \mathbb{R}^n$

<u>Proof</u>: Let $\vec{Y} \in \mathcal{N}(0, \mathbb{1})$ then by independence we know the density of $\vec{Y}$.

$$f_{\vec{Y}}(\vec{y}) \overset{(*)}{=} \prod_{k=1}^{n} f_{Y_k}(y_k) = \left(\frac{1}{2\pi}\right)^{n/2} e^{\overbrace{-\frac{1}{2}\sum_{k=1}^{n} y_k^2}^{= -\frac{1}{2}\vec{y}'\vec{y}}}$$

By linear transformation we have

$\Lambda = \Lambda'$ symmetric

$\Lambda^{1/2} = (\Lambda^{1/2})'$

$$\vec{X} = \Lambda^{1/2}\vec{Y} + \vec{\mu} \in \mathcal{N}(\vec{\mu}, \underbrace{\Lambda^{1/2}\mathbb{1}\Lambda^{1/2}}_{=\Lambda}) = \mathcal{N}(\vec{\mu}, \Lambda)$$

Transformation theorem for densities

$$\vec{Y} = \Lambda^{-1/2}(\vec{X} - \vec{\mu}).$$

Remark: $\Lambda^{-1/2}$ exists because $\det \Lambda > 0 \Rightarrow \det(\Lambda^{-1/2}) = (\det \Lambda)^{-1/2}$.
(linear algebra)

Thm 2.1, Chapter 1, [AG] "Transformation theorem".

$$\Rightarrow f_{\vec{X}}(\vec{x}) = f_{\vec{Y}}\left(\Lambda^{-1/2}(\vec{x}-\vec{\mu})\right) \left|\det \Lambda^{-1/2}\right|$$

$$\overset{(*)}{=} \frac{1}{\sqrt{\det \Lambda}} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left(\Lambda^{-1/2}(\vec{x}-\vec{\mu})\right)'\left(\Lambda^{-1/2}(\vec{x}-\vec{\mu})\right)}$$

$$= \frac{1}{\sqrt{\det \Lambda}} \cdot \frac{1}{(2\pi)^{n/2}} \cdot e^{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Lambda^{-1}(\vec{x}-\vec{\mu})}$$

$\blacksquare$

<u>Example:</u> Bivariate Gaussian $(X_1, X_2)$.

$$\vec{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$\rho$: "correlation coefficient".

We have $|\rho|<1$, $\sigma_1>0$, $\sigma_2>0$ iff $\det \Lambda > 0$.

To find the joint density, we need to invert the matrix $\Lambda$.

Cramer's rule: $$\Lambda^{-1} = \frac{1}{\det \Lambda}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \overset{[AG]}{=} \cdots = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}$$

$$\Rightarrow f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left(\left(\frac{x_1}{\sigma_1}\right)^2 - \underbrace{2\rho\frac{x_1\cdot x_2}{\sigma_1\sigma_2}}_{=0} + \left(\frac{x_2}{\sigma_2}\right)^2\right)}$$

So <u>if</u> $\rho=0$, i.e. $\Lambda$ diagonal, then

$$= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \quad \text{independence}$$

$\blacksquare$

Exercise: Bivariate Gaussian, $\vec{\mu} = \begin{pmatrix}0\\0\end{pmatrix}$

just computed.

$$f_{X_2|X_1=x_1}(x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)}$$

Gaussian $\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{x_1^2}{2\sigma_1^2}}$

$\vdots$ [AG], p. 127

$$= \frac{1}{\sqrt{2\pi\sigma_2^2}\sqrt{1-\rho^2}} e^{-\frac{1}{2\underbrace{\sigma_2^2(1-\rho^2)}}\left(x_2-\underbrace{\rho\frac{\sigma_2}{\sigma_1}x_1}\right)^2}$$

Can read off:

$$\begin{cases} \mathbb{E}[X_2 \mid X_1 = x_1] = \rho\frac{\sigma_2}{\sigma_1}x_1 \\ \mathrm{Var}[X_2 \mid X_1 = x_1] = \sigma_2^2(1-\rho^2) \end{cases},$$

that is

$$\begin{cases} \mathbb{E}[X_2 \mid X_1] = \rho\frac{\sigma_1}{\sigma_1}X_1, \\ \mathrm{Var}[X_2 \mid X_1] = \sigma_2^2(1-\rho^2) \end{cases}$$

⚠ These nice formulas apply for bivariate Gaussian, but are in general not correct.