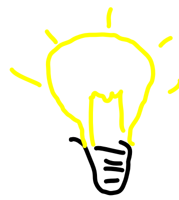
Functions of random variables: $g : \mathbb{R} \to \mathbb{R}$ 'nice', X a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. Then

$$Y := g(X)$$

is a random variable, too. ("nice": Borel measurable).

Example 1: Let X be uniformly distributed on $(0,1)$.
Choose $Y = X^2$.

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \in (0,1) \\ 1 & \text{if } x \geq 1 \end{cases}.$$



Y takes values between 0 and 1.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(X \leq \sqrt{y})$$
$$= F_X(\sqrt{y})$$

To find the density, we take the derivative:

$$f_Y(y) = \frac{d}{dy} F_X(\sqrt{y})) = \begin{cases} \frac{d\sqrt{y}}{dy} = \frac{1}{2\sqrt{y}}, & \text{if } 0 < y < 1 \\ 0, & \text{else.} \end{cases} \quad //$$

Bivariate & continuous case: (X_1, X_2) have joint density

$f_{X_1,X_2}(x_1,x_2)$ that is concentrated on some domain

$S \subset \mathbb{R}^2$ (vanishes outside of S).

Let now g be a bijection from S to some set

$T \subset \mathbb{R}^2$ (so the inverse of g exists uniquely).

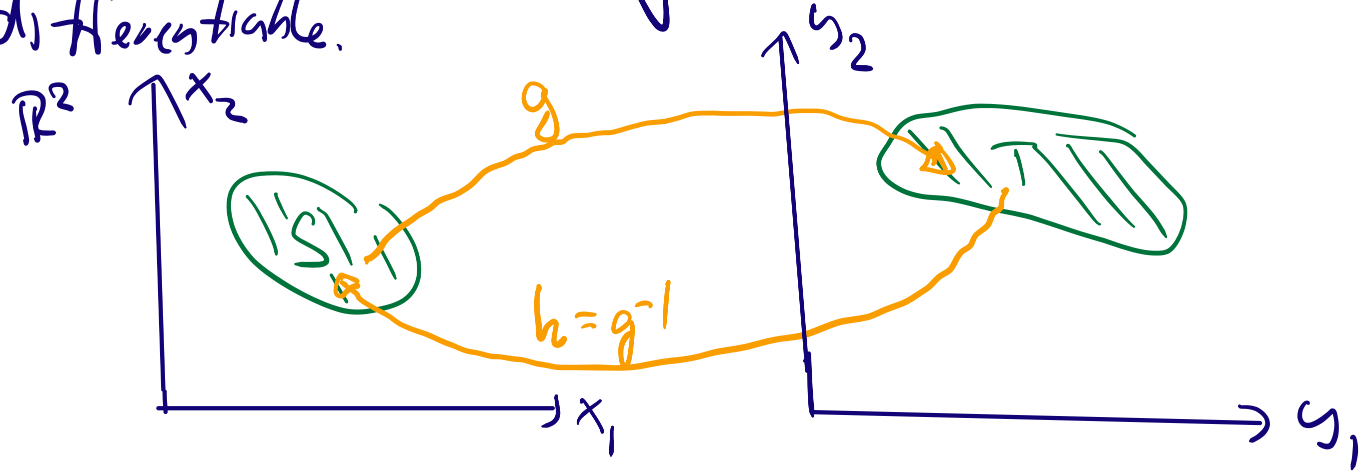
Consider

$$Y = g(X).$$

In components: $Y_1 = g_1(X_1, X_2)$

$$Y_2 = g_2(X_1, X_2).$$

Assume in addition that g and its inverse are continuously differentiable.



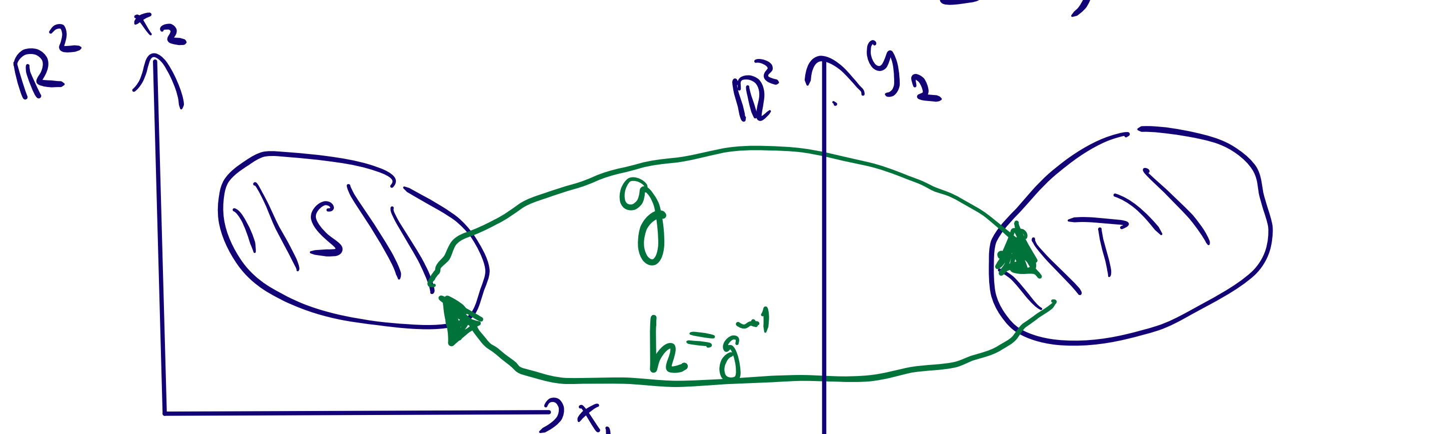
<u>Transformation theorem</u>: The joint density of $Y = g(X)$ is given by

⚠ absolut value

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} f_{X_1,X_2}(h_1(y_1,y_2), h_2(y_1,y_2)) \cdot |J| & \text{for } y \in T, \\ 0 & \text{else} \end{cases}$$

where $h = (h_1, h_2)$ is the <u>inverse</u> of g, and J is the Jacobian

$$J = \det \begin{pmatrix} \frac{\partial h_1(y_1,y_2)}{\partial y_1} & \frac{\partial h_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial h_2(y_1,y_2)}{\partial y_1} & \frac{\partial h_2(y_1,y_2)}{\partial y_2} \end{pmatrix}$$



J: Jacobian of h. //

(4)

<u>Proof</u>: Notation: $h(B) := \{x \in \mathbb{R}^2 : g(x) \in B\}$ for any $B \subset \mathbb{R}^2$.

Then $\mathbb{P}(\underline{Y} \in B) = \mathbb{P}(\underline{X} \in h(B)) = \int_{h(B)} f_{\underline{X}}(x)\, d^2x$.

inverse of g

Change of variables: $y = g(x)$, $x = h(y)$, $d^2x = |J|\, d^2y$.

$$\mathbb{P}(Y \in B) = \int_{h(B)} f_{\underline{X}}(x)\, d^2x = \int_B \underbrace{f_{\underline{X}}(h_1(y_1, y_2), h_2(y_1, y_2))\, |J|}_{\text{density of } Y}\, d^2y$$

$\blacksquare$

5.

Exercise: Let X_1 and X_2 be independent standard Gaussian r.v. $\mathcal{N}(0,1)$. Show that

$$X_1+X_2 \quad , \quad X_1-X_2$$

are independent $\mathcal{N}(0,2)$ r.v. //

Convolution formula: X_1 and X_2 be independent with densities $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$.

Find the density of X_1+X_2.

Trick:
$$g_1(x_1,x_2) := x_1+x_2 = y_1$$
$$g_2(x_1,x_2) := x_1 \quad = y_2$$

Find the inverse:
$$x_1 = y_2$$
$$x_2 = y_1 - y_2$$

Jacobian $J = \det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = -1$

So $|J| = 1$.

$$f_{(X_1+X_2),X_1}(y_1,y_2) = f_{X_1,X_2}\big(\overset{h_1}{y_2},\ \overset{h_2}{y_1-y_2}\big)\,\overset{1}{\overset{\shortparallel}{|J|}}$$

$$\overset{\text{independence}}{=} f_{X_1}(y_2)\cdot f_{X_2}(y_1-y_2)$$

Pass to the marginal density:

$$f_{(X_1+X_2)}(y_1) = \int_{-\infty}^{\infty} f_{X_1}(y_2)\, f_{X_2}(y_1-y_2)\, dy_2$$

Convolution formula

$$= \int_{-\infty}^{\infty} f_{X_1}(y_1-y_2)\cdot f_{X_2}(y_2)\, dy_2 .$$

<u>Exercise:</u> X continuous one-dimensional r.v. $Y = X^2 = g(X)$.

g is not injective. Show that

$$f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}} .$$

<u>Hint:</u> Apply the transformation thm twice:
$S_1 = (-\infty, 0]$ and $S_2 = (0,\infty)$.

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