Probability generating function

Fundamental question: $X_1, X_2, \ldots, X_n$ independent random variables.

What is the distribution of $\sum_{i=1}^{n} X_i$?

For $n=2$, can use convolution formula, then iteratively extend to $n>2$.

In practice, not efficient, tedious, and for large n too time consuming.

Better tools are the following three transforms:

1.) Probability generating function (good for integer-valued random variables)

2.) Moment generating function, mgf

3.) Characteristic function.

<u>Def:</u> Let X be a non-negative, integer-valued random variable. The <u>probability generating function</u> of X is defined as

$$g_X(t) := \mathbb{E}\left[t^X\right] \qquad t \in \mathbb{R}$$

$$= \sum_{n=0}^{\infty} t^n \cdot \mathbb{P}(X=n).$$

$$= 1 \, \mathbb{P}(X=0) + t\,\mathbb{P}(X=1) + t^2\,\mathbb{P}(X=2) + \ldots$$

<u>Remark:</u> At least for $|t| \leq 1$, the sum is absolutely convergent, because

$$\sum_{n=0}^{\infty} |t^n| \cdot \mathbb{P}(X=n) \leq \sum_{n=0}^{\infty} \mathbb{P}(X=n) = 1.$$

In particular: $\boxed{g_X(1) = 1.}$ //

<u>Ex:</u> $X \in Be(p)$, $\mathbb{P}(X=0) = q$, $\mathbb{P}(X=1) = p$, $p+q=1$.

$$g_X(t) = \mathbb{E}\left[t^X\right] = t^0 q + t^1 p = q + tp.$$

//

Thm 1: Let X and Y be non-negative integer-valued random variables.

If $g_X = g_Y$, then $Y \overset{d}{=} X$ //

Thm 2: Let $X_1, X_2, \ldots, X_n$ be independent, non-negative integer-valued random variables. Set

$$S_n := X_1 + X_2 + \ldots + X_n.$$

Then

$$g_{S_n}(t) = \prod_{k=1}^{n} g_{X_k}(t)$$

//

Proof: Observe that $t^{X_1}, t^{X_2}, \ldots, t^{X_n}$ are independent

$$g_{S_n}(t) \overset{\text{def}}{=} \mathbb{E}\left[t^{X_1+X_2+\cdots+X_n}\right] \overset{\text{computation rules}}{=} \mathbb{E}\left[t^{X_1} \cdot t^{X_2} \cdots t^{X_n}\right]$$

$$\overset{\text{independence}}{=} \mathbb{E}\left[t^{X_1}\right] \cdot \mathbb{E}\left[t^{X_2}\right] \cdots \mathbb{E}\left[t^{X_n}\right]$$

$$\overset{\text{def}}{=} \prod_{k=1}^{2} g_{X_k}(t).$$

$\blacksquare$

Ex: • X_1, X_2 are independent, both $Be(p)$. Then 4.

$$g_{X_1+X_2}(t) = g_{X_1}(t) \cdot g_{X_2}(t) = (q+tp)^2 . \quad (*)$$

• $Y \in Bin(n,p)$, then

$$g_Y(t) = \sum_{k=0}^{n} t^k \binom{n}{k} \cdot p^k q^{n-k} \qquad p+q=1.$$

$$= \sum_{k=0}^{n} \binom{n}{k} (tp)^k q^{n-k}$$

binomial identity

$$= (q+pt)^n \quad \text{compare with } (*)$$

Choose $n=2$:

$$X_1 + X_2 \overset{d}{=} Y \in Bin(2,p)$$

//

Binomial identity: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, $a,b \in \mathbb{R}$, $n \in \mathbb{N}$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

5.

Why the name generating function?

$$g_X(t) = \sum_{n=0}^{\infty} t^n \,\mathbb{P}(X=n) \qquad t \in \mathbb{R}$$

$$= \mathbb{P}(X=0) + t\,\mathbb{P}(X=1) + t^2\,\mathbb{P}(X=2) + \dots$$

Take derivative in t

(ok, for $|t|<1$)

$$g_X'(t) = \frac{d}{dt} g_X(t) = \sum_{n=1}^{\infty} n\, t^{n-1}\,\mathbb{P}(X=n)$$

$$= \mathbb{P}(X=1) + 2t\,\mathbb{P}(X=2) + 3t^2\,\mathbb{P}(X=3) + \dots$$

$$g_X''(t) = \sum_{n=2}^{\infty} n(n-1)\, t^{n-2}\,\mathbb{P}(X=n)$$

$$= 2\,\mathbb{P}(X=2) + 3\cdot 2t\,\mathbb{P}(X=3) + 4\cdot 3t^2\,\mathbb{P}(X=4) + \dots$$

Choose now $t=0$:

$$g_X'(0) = \mathbb{P}(X=1), \quad g_X''(0) = 2\cdot 1\cdot \mathbb{P}(X=2)$$

In general:

$$\mathbb{P}(X=n) = \frac{g_X^{(n)}(0)}{n!} = \frac{1}{n!}\frac{d^n}{dt^n} g_X(t)\Big|_{t=0}.$$

Exercise: Prove now Theorem 1 using this result.

Can also compute moments of X:

$$g_X'(t) = \sum_{n=1}^{\infty} n\, t^{n-1}\, \mathbb{P}(X=n)$$

Set $t=1$, then

$$g_X'(1) = \sum_{n=1}^{\infty} n\, \mathbb{P}(X=n) = \mathbb{E}[X]$$

$$\boxed{\mathbb{E}[X] = g_X'(1)}$$

Similarly,

$$\mathrm{Var}[X] = \mathbb{E}(X-\mathbb{E}X)^2 = g_X''(1) + g_X'(1) - \left(g_X'(1)\right)^2$$

(provided that mean and second moment exist).

<u>Ex:</u> $X \in \mathrm{Bin}(n,p)$

$$\mathbb{E}[X] = g_X'(1) = \frac{d}{dt}(q+pt)^n \Big|_{t=1}$$

$$= n\,(\underbrace{q+pt}_{=1})^{n-1}\, p \Big|_{t=1}$$

$$= n \cdot p. \qquad q+p=1$$

Ex: $X \in Po(m)$.

$$g_X(t) = \sum_{k=0}^{\infty} t^k e^{-m} \frac{m^k}{k!}.$$

$$= e^{-m} \underbrace{\sum_{k=0}^{\infty} \frac{(tm)^k}{k!}}_{= e^{tm}}.$$

Taylor series of exponential function around zero.

$$= e^{m(t-1)}.$$

Exercise: • Compute $\mathbb{E}X$, $\operatorname{Var} X$.

• Find the distribution of $X_1 + X_2$ when $X_1 \in Po(m_1)$ and $X_2 \in Po(m_2)$, independent.

Answers: $\mathbb{E}X = m$, $\operatorname{Var} X = m$

$$X_1 + X_2 \in Po(m_1 + m_2)$$

//