Mathematics of systems theory
Homework 1

Ville Wassberg
Collaborator: Klara Zimmermann

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1

a)

Solve

$$\dot{x} = \begin{pmatrix} -1 & t & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x(t),\ x(0) = (1,2,3)^T. \tag{1}$$

We can begin by solving for x_2 and x_3 quite easily, since we have following equations:

$$x_2 = x_2' \Rightarrow x_2 = C_2 e^t,\ x_2(0) = 2 \Rightarrow C_2 = 2,$$

$$-x_3 = x_3' \Rightarrow x_3 = C_3 e^{-t},\ x_3(0) = 3 \Rightarrow C_3 = 3.$$

Now setting these into the equation for x_1 one gets:

$$x_1' + x_1 = t2e^t + 3e^{-t}.$$

Finding integrating factor $\mu(t) = e^{\int 1dt} = e^t$. Thus, the new equation becomes:

$$e^T(x_1' + x_1) = 2te^{2t} + 3$$

$$\Leftrightarrow$$

$$\frac{d}{dt}(e^t x_1(t)) = 2te^{2t} + 3.$$

Thus through integration by parts we get:

$$x_1(t) = e^{-t}((t - \frac{1}{2})e^{2t} + 3t + C_1),\ x_1(0) = 1 \Rightarrow C_1 = 3/2.$$

Hence, the solution to 1 is: $x(t) = \begin{pmatrix} (t - \frac{1}{2})e^t + e^{-t}(3t + \frac{3}{2}) \\ 2e^t \\ 3e^{-t} \end{pmatrix}.$

b)

Solve

$$\dot{x} = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} x(t),\ x(0) = (x_1^0, x_2^0)^T,\ \omega > 0, \tag{2}$$

where ω is constant, show that $x_1(t) = a\sin(\omega t + b)$, where a, b are functions of $x(0)$.

Since the matrix, call it A, is time invariant e^{At} is the fundamental matrix, $\Psi(t)$, from which we can find the state transition matrix $\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0)$, since A is time invariant, and solve for $x(t) = \Phi(t, t_0)x(0)$. Since $t_0 = 0$, fundamental and the state transition matrix are equal. Thus, we need to compute e^{At}. For this we can begin by finding its eigen values and vectors. Solving $det(A - \lambda I) = 0$ we get $\lambda = \pm i\omega$. Searching for the eigen vectors we get

$$\begin{pmatrix} -i\omega & \omega \\ -\omega & -i\omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} t$$

and

$$\begin{pmatrix} i\omega & \omega \\ -\omega & i\omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} t,$$

hence

$$P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

is one possible matrix satistfying the equation $P\Lambda P^{-1} = A$, where Λ is the diagonal matrix with the eigenvalues as its entries. Calculating P^{-1} one gets:

$$P^{-1} = \frac{1}{2i} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}.$$

Thus the state transition matrix becomes:

$$\Phi(t, t_0) = Pe^{\Lambda t}P^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} ie^{i\omega t} + ie^{-i\omega t} & e^{i\omega t} - e^{-i\omega t} \\ -e^{i\omega t} + e^{-i\omega t} & ie^{i\omega t} + ie^{-i\omega t} \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Solving for $x(t) = \Phi(t, t_0)x(0)$ we get

$$x(t) = \begin{pmatrix} x_1^0\cos\omega t + x_2^0\sin\omega t \\ -x_1^0\sin\omega t + x_2^0\cos\omega t \end{pmatrix}.$$

Now if we let $\cos\theta = \frac{x_1^0}{||x(0)||}$ and $\sin\theta = \frac{x_2^0}{||x(0)||}$ we get

$$x_1(t) = x_1^0\cos\omega t + x_2^0\sin\omega t = ||x(0)||\sin(\omega t + \theta) = a\sin(\omega t + b)$$

$$\begin{cases} a = ||x(0)||, \\ b = \arcsin\frac{x_2^0}{||x(0)||}. \end{cases}$$

2

a)

Let

$$A = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.$$

Show that

$$\det \Phi(t, t_0) = e^{\int_{t_0}^{t} (a_{11}(s)+a_{22}(s))\, ds}. \tag{3}$$

To show this we can use the Jacobi formula which says that if A is a linear map $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, and if A is invertible then the following equation holds:

$$\frac{d}{dt} \det A(t) = (\det A(t)) \operatorname{Tr}\left\{ A^{-1}(t) \frac{dA(t)}{dt} \right\}.$$

Using the fact that $\operatorname{Tr}\{AB\} = \operatorname{Tr}\{BA\}$ for A, $B \in \mathbb{R}^{n\times n}$ and since the state transition matrix fulfills the conditions for the Jacobi formula, as well as $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$ and $\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$, we get

$$\det \Phi(t, t_0) = \frac{d}{dt} \det \Phi(t_0, t) = (\det \Phi(t_0, t)) \operatorname{Tr}\{\Phi(t_0, t)(t)A\Phi(t, t_0)\} =$$

$$\frac{d}{dt} \det \Phi(t_0, t) = (\det \Phi(t_0, t)) \operatorname{Tr}\{A\} = (\det \Phi(t_0, t)) \sum_{i=1}^{n} a_{ii}.$$

Now, solving this homogeneous ODE, we get

$$\det \Phi(t, t_0) = e^{\int_{t_0}^{t} \sum_{i=1}^{2} a_{ii}(s)\, ds},$$

which is the sought equation 3.

b)

Verify that $\Psi(t) = \Phi(t, t_0)\Psi_0\Phi^T(t, t_0)$ is the solution to $\dot{\Psi}(t) = A(t)\Psi(t) + \Psi(t)A^T(t), \quad \Psi(t_0) = \Psi_0$, where $\Phi(t, t_0)$ is the state transition matrix of $\dot{x} = A(t)x$.

Since the state transition matrix fulfills that $x = \Phi(t, t_0)x(t_0)$ we have $\Psi(t_0) = \Phi(t_0, t_0)\Psi_0\Phi^T(t_0, t_0) = \Psi_0$. Now

$$\dot{\Psi}(t) = \frac{d}{dt}(\Phi(t, t_0)\Psi_0\Phi^T(t, t_0)) = \frac{d}{dt}(\Phi(t, t_0))\Psi_0\Phi^T(t, t_0) + \Phi(t, t_0)\Psi_0\frac{d}{dt}(\Phi^T(t, t_0)) =$$

$$A(t)\Phi(t, t_0)\Psi_0\Phi^T(t, t_0) + \Phi(t, t_0)\Psi_0\Phi^T(t, t_0)A^T(t).$$

Setting $\Psi(t) = \Phi(t, t_0)\Psi_0\Phi^T(t, t_0)$ this becomes $A(t)\Psi(t) + \Psi(t)A^T(t)$, thus, $\Psi(t) = \Phi(t, t_0)\Psi_0\Phi^T(t, t_0)$ is the solution.

3

Consider the system

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x. \end{cases}$$

Show that controllability and observability of linear time-varying systems are invariant under linear transformation $\bar{x} = P(t)x$ where $P(t)$ is nonsingular and continuously differentiable for all $t \in (-\infty, \infty)$.

Inserting this into the system we get

$$\begin{cases} P^{-1}(t)\dot{x} = (A(t)P^{-1}(t) - \dot{P^{-1}}(t))\bar{x} + B(t)u, \\ y = C(t)P^{-1}(t)\bar{x}. \end{cases}$$

Now setting

$$\begin{cases} \bar{A}(t) = P(t)(A(t)P^{-1}(t) - \dot{P^{-1}}(t)) \\ \bar{B}(t) = P(t)B(t) \\ \bar{C}(t) = C(t)P^{-1}(t) \end{cases}$$

the system in terms of $\bar{x}$ becomes

$$\begin{cases} \dot{\bar{x}} = \bar{A}(t)x + \bar{B}(t)u, \\ y = \bar{C}(t)\bar{x}. \end{cases}$$

Since

$$\begin{cases} x = P^{-1}(t)\bar{x} \\ x = \Phi(t_0, t)x_0 \\ \bar{x} = \bar{\Phi}(t_0, t)\bar{x}_0 \end{cases}$$

we get that $\bar{\Phi}(t, t_0) = P(t)\Phi(t_0, t)P^{-1}(t)$. Thus the controllability Gramian and the observability Gramian, respectively, becomes:

$$\bar{W}(t_0, t) = \int_{t_0}^{t} \bar{\Phi}(t_0, s)\bar{B}(s)\bar{B}^T(s)\bar{\Phi}^T(t_0, s)\, ds =$$
$$\int_{t_0}^{t} P(s)\Phi(t_0, s)P^{-1}(s)P(s)B(s)B^T(s)P^T(s)(P^{-1}(s))^T\Phi^T(t_0, s)P^T(s)\, ds =$$
$$\int_{t_0}^{t} P(s)\Phi(t_0, s)B(s)B^T(s)\Phi^T(t_0, s)P^T(s)\, ds,$$

$$\bar{M}(t_0, t) = \int_{t_0}^{t} \bar{\Phi}^T(t_0, s)\bar{C}^T(s)\bar{C}(s)\bar{\Phi}(t_0, s)\, ds =$$
$$\int_{t_0}^{t} (P^{-1}(s))^T\Phi^T(t_0, s)P^T(s)(P^{-1}(s))^TC^T(s)C(s)P^{-1}(s)P(s)\Phi(t_0, s)P^{-1}(s)\, ds =$$
$$\int_{t_0}^{t} (P^{-1}(s))^T\Phi^T(t_0, s)C^T(s)C(s)\Phi(t_0, s)P^{-1}(s)\, ds.$$

Hence, if the controllability and observability Gramians are non-singular for the original system, then the controllability and observability Gramians are non-singular; since the mapping $P(t)$ is non-singular and continuously differentiable. Thus, controllability and observability are conserved under the linear mapping $P(t)$.

4

Consider the inverted pendulum as we did in the lecture (see Figure 1). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0, \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

a)

We consider $\ddot{x}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e., let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$) and show that the model you derive is both controllable and observable.

By setting $x = (\theta, \dot{\theta})^T$ and $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, then we get the system:

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ g/L & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/L \end{pmatrix} \ddot{x} \\ y = (1,\, 0)x. \end{cases}$$

Setting $\Gamma = [B, AB] = \begin{pmatrix} 0 & 1/L \\ 1/L & 0 \end{pmatrix}$ and $\Omega = [C, CA]^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we immediately see that the rank of them both are 2, which is the dimension of A, hence the system is both controllable and observable.

b)

Setting $u(t) = 0$ and using the linearized model, can we find an initial state $x_1(0) \neq 0$ and $x_2(0)$ such that $x_1(t) = 0$ for all $t \geq T$ where $T > 0$ is some finite time?

Now setting $u(t) = 0$ we can analyse the system and search for the existence of such T. Solving $det(A - \lambda I) = 0$ we get $\lambda = \pm\sqrt{\frac{g}{L}}$.

$$\begin{pmatrix} -\sqrt{\frac{g}{L}} & 1 \\ \sqrt{\frac{g}{L}} & -\sqrt{\frac{g}{L}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ \sqrt{\frac{g}{L}} \end{pmatrix} t,\ t \in \mathbb{R},$$

and

$$\begin{pmatrix} \sqrt{\frac{g}{L}} & 1 \\ \sqrt{\frac{g}{L}} & \sqrt{\frac{g}{L}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{L}} \end{pmatrix} t,\ t \in \mathbb{R}$$

hence

$$P = \begin{pmatrix} 1 & 1 \\ \sqrt{\frac{g}{L}} & -\sqrt{\frac{g}{L}} \end{pmatrix}$$

is one possible matrix satistfying the equation $P\Lambda P^{-1} = A$, where Λ is the diagonal matrix with the eigenvalues as its entries. Also,

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{\frac{g}{L}} \\ 1 & -\sqrt{\frac{g}{L}} \end{pmatrix}.$$

Thus the state transition matrix becomes

$$\Phi(t,0) = e^{At} = Pe^{\Lambda t}P^{-1} = \frac{1}{2}\begin{pmatrix} e^{\sqrt{\frac{g}{L}}t} + e^{-\sqrt{\frac{g}{L}}t} & \sqrt{\frac{L}{g}}\left(e^{\sqrt{\frac{g}{L}}t} - e^{-\sqrt{\frac{g}{L}}t}\right) \\ \sqrt{\frac{g}{L}}\left(e^{\sqrt{\frac{g}{L}}t} - e^{-\sqrt{\frac{g}{L}}t}\right) & e^{\sqrt{\frac{g}{L}}t} + e^{-\sqrt{\frac{g}{L}}t} \end{pmatrix}.$$

Now since $x(t) = \Phi(t, t_0)x(0)$ and if $x(0) = (a,\, b)^T$, $a \neq 0$, then

$$x_1(t) = \frac{1}{2}\left(a\left(e^{\sqrt{\frac{g}{L}}t} + e^{-\sqrt{\frac{g}{L}}t}\right) + b\sqrt{\frac{L}{g}}\left(e^{\sqrt{\frac{g}{L}}t} - e^{-\sqrt{\frac{g}{L}}t}\right)\right)$$

Thus, making $x_1(t) = 0\,\forall t > T$, is not possible without setting $a = 0$.

5

Now consider an inverted pendulum with an oscillatory base. The following equation describes the motion of the pendulum around the equilibrium $\theta = 0, \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

a)

We consider $\ddot{z}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system.

Putting this into a system we get:

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ g/L & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ x_1/L \end{pmatrix} u \\ y = (1,\, 0)x. \end{cases}$$

b)

Is the model you derive in (a) controllable?

Assuming that $\theta \approx 0$ we get that $\Gamma = [B, AB] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which does not have full rank, hence the system is not controllable.

c)

Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab (or any other software of your choice) simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is "slow" and when the oscillation is "fast". Take $L = 1, \theta(0) = 0.02, \dot{\theta}(0) = 0$.

Calculating $\ddot{z} = -\omega^2 * 0.1\sin(\omega t)$ we get $\ddot{\theta} = \frac{1}{L}(g - \omega^2 * 0.1\sin(\omega t))\sin(\theta)$. Then by writing the code in Matlab as shown in 1 we get that the motion is kind of vibrating as ω increases, as seen in figure 2.

```
%% Homework 1 Mathematics of Systems theory
% Constants
L = 1;
g = 9.82;
omegavec = [5, 1000];
% Initial conditions
theta0 = 0.02;
dtheta0 = 0;
x0 = [theta0; dtheta0];

% Function defining the derivative of x
dx = @(t, x, omega) [x(2); (g - omega^2*0.1*sin(omega*t))*sin(x(1))/L];

i = 1;
for omega = omegavec
[T, X] = ode23(@(t, x)dx(t, x, omega), [0 0.05], x0);
subplot(1,2,i)
plot(T, X(:, 2))
title("\omega = " + omega)
i = i + 1;
end
```

Figure 1: The Matlab code for following ' picture

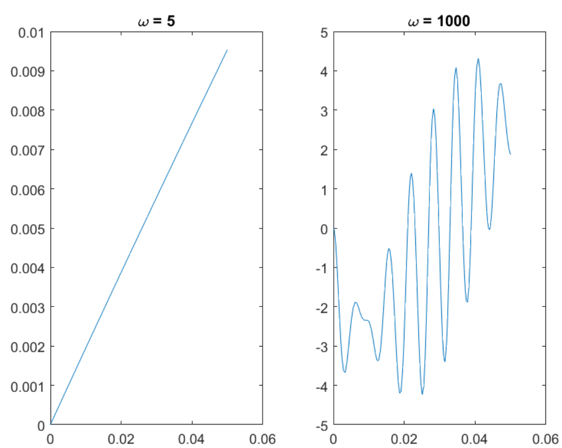

Figure 2: Motion begins to vibrate/oscillate as ω increases, around $\theta = 0$.