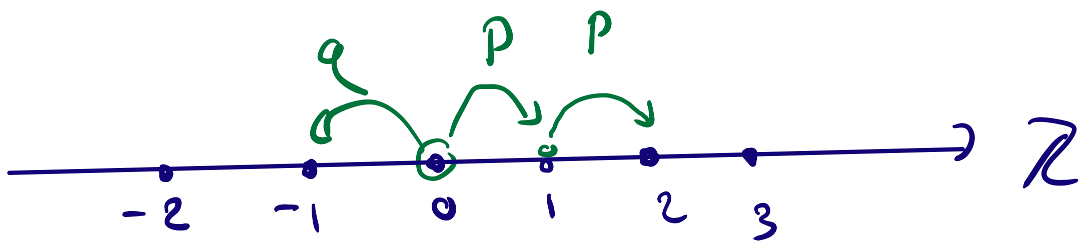
Random walks on $\mathbb{Z}$

integers

(X_i) iid random variables with $\mathbb{P}(X_i = 1) = p$.

and $\mathbb{P}(X_i = -1) = q = 1 - p$, $\quad p + q = 1$.

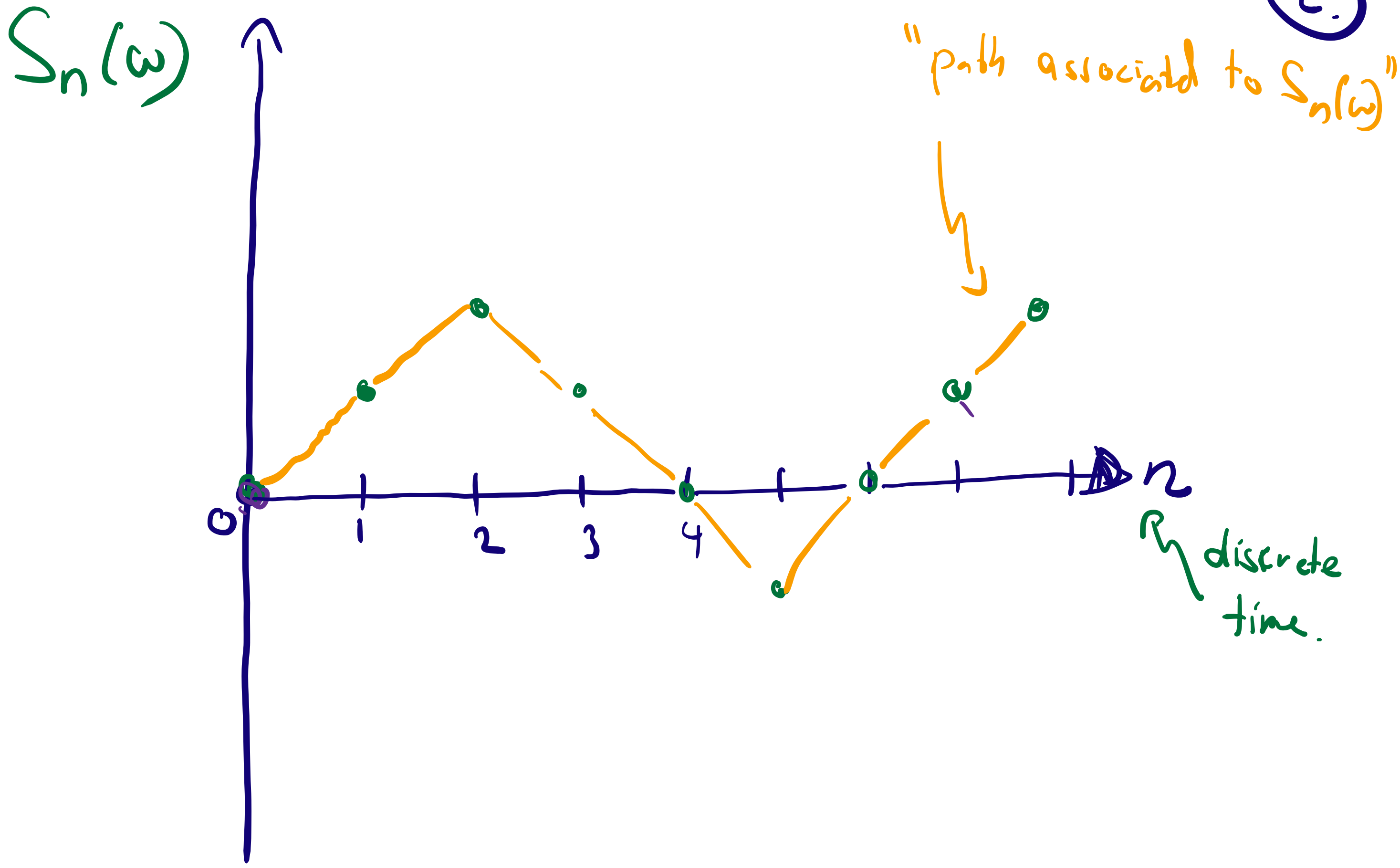
$$\begin{cases} S_n := X_1 + X_2 + \dots + X_n, & n \geq 1. \\ S_0 := 0 \end{cases}$$



"Walker who at each time step n tosses a (biased) coin, then" takes a step to the left or the right.

(S_n) is called a simple random walk; simple = steps of size 1.

If $p = q = 1/2$, symmetric simple random walk.



Extract the leading order of S_n:

<u>Thm 1</u>: (strong law of large numbers). If $\mathbb{P}(X_i=1)=p$ and $\mathbb{P}(X_i=-1)=1-p$, then

distance from the origin

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_i]=2p-1.$$

time

"asymptotic velocity". $=0$ if $p=q=\frac{1}{2}$.

Thm 2: (CLT)

$$\frac{S_n - n\mathbb{E}[X_i]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = \mathrm{Var}(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2$

$= \ldots = 4p(1-p)$

$= 1$ if $p = 1/2 = q$.

"Typical deviation of S_n from the expected position $n\mathbb{E}[X_i]$" is of order $\sqrt{n}$.

Exercise: Choose $p = q = 1/2$ for simplicity.

Let $u_n := \mathbb{P}(S_n = 0)$ Probability to be back at the origin at time step n.

Then $u_{2m+1} = 0$ (cannot come back with an odd number of steps.)

$$u_{2m} = \binom{2m}{m}\left(\frac{1}{2}\right)^{2m}, \quad m \geq 0.$$

Hint: To come back to the origin, one needs to do the same number of steps to the left as to the right.