Mathematics of systems theory
Homework 2

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1

To solve the Lyapunov equation $A^T P + PA + c^T c = 0$ with

$$A = \begin{pmatrix} 1 & a \\ -1 & -2 \end{pmatrix}, \tag{1}$$

$c = (1, 0)$, we can begin by expanding the product and get:

$$A^T P + PA = \begin{pmatrix} 2p_{11} - p_{21} - p_{12} & ap_{11} - p_{12} - p_{22} \\ ap_{11} - p_{21} - p_{22} & a(p_{12} + p_{21}) - 4p_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

Since P is symmetric, we set $p_{21} = p_{12}$. This lets us determine P by solving the system of equations:

$$\begin{cases} 2(p_{11} - p_{12}) = -1 \\ ap_{11} - p_{12} - p_{22} = 0 \\ 2ap_{12} - 4p_{22} = 0. \end{cases} \tag{3}$$

Solving for p_{11}, p_{12}, p_{22}, respectively, we get:

$$\begin{cases} p_{11} = \frac{a+2}{2a-4}, \\ p_{21} = \frac{a}{a-2}, \\ p_{22} = \frac{a^2}{2a-4}, \end{cases} \tag{4}$$

which makes

$$P = \frac{1}{2a-4} \begin{pmatrix} a+2 & 2a \\ 2a & a^2 \end{pmatrix}. \tag{5}$$

To evaluate which values of a that makes P positive definite and positive semi-definite, respectively, we can inspect the product $x^T Px$. Expanding the product one gets

$$x^T Px = \frac{1}{2a-4}((2x_1 + ax_2)^2 + (a-2)x_1^2), \tag{6}$$

which one can see quite directly is positive definite for all $a > 2$, and it is not defined for $a = 2$. If $a < 2$ then the coefficient in front of x_2^2 is always negative, thus for P to be positive definite a has to be strictly larger than 2.

However, it is not trivial to see for which values of a this expression is 0, thus inspecting the eigenvalues might be the next reasonable step. Solving the equation $\det(P - \lambda\mathbf{I}) = 0$ one gets the eigenvalues:

$$\lambda = \frac{a^2 + a + \pm 2(a^4 - 2a^3 + 13a^2 + 4a + 4)^{1/2}}{4(a-2)} = \tag{7}$$
$$\frac{(a+1/2)^2 + 1 \pm ((a^2 - a)^2 + (2a+2)^2 + 8a^2)^{1/2}}{4(a-2)}.$$

There are one real value of a making one of the eigenvalues 0, $a = 0$, but then the other becomes negative, thus there is no possibility of positive semi-definiteness of P other than for all values of a making P positive definite.

2

In order to compute the standard reachable realisation from

$$R(s) = \begin{pmatrix} \frac{s+2}{s+1} \\ \frac{1}{s+3} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+3} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8}$$

we need to find the least common denominator $\chi(s)$, thus we need to manipulate $R(s)$ a bit:

$$R(s) = \begin{pmatrix} \frac{s+3}{(s+1)(s+3)} \\ \frac{(s+1)}{(s+3)(s+1)} \end{pmatrix}. \tag{9}$$

Hence, $\chi(s) = s^2 + 4s + 3$, with coefficients $a_1 = 4$, $a_2 = 3$. Letting

$$N(s) = \chi(s)R(s) = \begin{pmatrix} s+3 \\ s+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s + \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \tag{10}$$

we can now form the realisation (A, B, C, D), where $D = (1,\ 0)^T$. Starting with A we get:

$$A = \begin{pmatrix} 0 & \mathbf{I}_1 \\ -a_2\mathbf{I}_1 & -a_3\mathbf{I}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}. \tag{11}$$

And B becomes:

$$B = \begin{pmatrix} 0 \\ \mathbf{I}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12}$$

Finally, C becomes:

$$C = \begin{pmatrix} N_0 & N_1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}. \tag{13}$$

And the standard reachable realisation is the triple (A, B, C, D).

To check if the system is observable we can compute the rank of the observability matrix $\Omega = \left[C^T, (CA)^T\right]^T$ and check that it is equal to the number of states, or the dimension of A:

$$\operatorname{rank}\begin{pmatrix} 3 & 1 \\ 1 & 1 \\ -9 & -11 \\ -3 & -3 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 2 = \dim A. \tag{14}$$

Thus the system is observable.

To determine the standard observable realisation we can compute the Laurent expansion of $R(s)$ around $s = \infty$. Hence, we get the following equations:

$$\begin{cases} r_1 = \frac{1}{s+1} = \frac{1}{s}\frac{1}{1+1/s} = s^{-1}\sum_{j=0}^{\infty}(-1)^j s^{-j} = s^{-1} - s^{-2} + ... \\ r_2 = \frac{1}{s+3} = \frac{1}{s}\frac{1}{1+3/s} = s^{-1}\sum_{j=0}^{\infty}(-3)^j s^{-j} = s^{-1} - 3s^{-2} + ... \end{cases} \tag{15}$$

This lets us extract

$$R_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; R_2 = \begin{pmatrix} -1 \\ -3 \end{pmatrix}. \tag{16}$$

Thus the new A becomes:

$$A = \begin{pmatrix} 0 & \mathbf{I}_2 \\ -a_2\mathbf{I}_2 & -a_1\mathbf{I}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{pmatrix}. \tag{17}$$

B becomes:

$$B = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -3 \end{pmatrix}. \tag{18}$$

And C becomes:

$$C = [\mathbf{I}_2 \; 0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{19}$$

Thus the standard observable realisation becomes the new quadruple of the new three matrices: $(A,\, B,\, C,\, D)$.

3

For the state space system

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{20}$$

and

$$\bar{A} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \bar{B} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \bar{C} = \begin{pmatrix} 1 & 1/2 \end{pmatrix} \tag{21}$$

we can find a mapping T by the use of the observability matrix $\Omega = (C^T, (CA)^T)^T$. Since

$$\bar{\Omega} = \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \end{pmatrix} = \begin{pmatrix} CT^{-1} \\ CAT^{-1} \end{pmatrix} = \Omega T^{-1}, \tag{22}$$

we can check if both Ω and $\bar{\Omega}$ are non-singular, which in that case would guarantee T, and then we can solve for T and verify that the product $TB = \bar{B}$. So

$$\Omega T^{-1} = \begin{pmatrix} \bar{C} \\ \bar{C}\bar{A} \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \tag{23}$$

and

$$\Omega = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

As we can see, they are both non-singular. Furthermore, $T^{-1} = \bar{\Omega}$, hence $T = \bar{\Omega}^{-1}$. Thus we can now check if $\bar{B} = \bar{\Omega}^{-1}B$:

$$\bar{\Omega}^{-1}B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \bar{B}. \tag{25}$$

Thus the two systems (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are equivalent.

For the state space system

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{26}$$

and

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \bar{C} = \begin{pmatrix} 1 & 2 \end{pmatrix} \tag{27}$$

we can do the same thing. Checking Ω:

$$\Omega = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{28}$$

we see it is singular, hence there is no non-singular transformation T relating the systems. Thus, the systems are not equivalent.

4

In order to find further conditions on the matrices A, b and c such that for any $u = Kx$ the pair $(c, A+bK)$ is observable, it is natural to attack the observable matrix

$$\bar{\Omega} = \begin{pmatrix} c \\ c(A+bK) \\ c(A+bK)^2 \\ \vdots \\ c(A+bK)^{n-1} \end{pmatrix}. \tag{29}$$

Since the system is minimal, hence observable and completely reachable, the reachability is kept under state feedback and we now want it to be observable, the new system is minimal, hence, there is by Lemma 6.14 and the state space isomorphism theorem a non-singular $n \times n$ transformation matrix T such that

$$TAT^{-1} = A + bK \triangleq \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\gamma_n & -\gamma_{n-1} & -\gamma_{n-2} & \dots & -\gamma_1 \end{pmatrix}, \tag{30}$$

where $\gamma_1, \gamma_2, \dots, \gamma_n$ are the coefficients of $\chi_{A+bK}(s) = s^n + \gamma_1 s^{n-1} + \dots + \gamma_n$. And T is unique and given by

$$T = \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix}, \tag{31}$$

where c is the unique row vector solution to $c[b, Ab, \dots, A^{n-1}b] = (0, 0, \dots, 0, 1)$. Then by the same reasoning as in problem 3, we need c in our system to equal this particular c. Also,

$$\bar{\Omega} = \begin{pmatrix} c \\ c(A+bK) \\ c(A+bK)^2 \\ \vdots \\ c(A+bK)^{n-1} \end{pmatrix} = \begin{pmatrix} cT^{-1} \\ cA^2T^{-1} \\ cA^2T^{-1} \\ \vdots \\ cA^{n-1}T^{-1} \end{pmatrix}, \tag{32}$$

when taking the matrix $A+bK$ to any finite power we can see that $(A+bk)^j$, $j = 1, 2, \dots, n-1$ always has zeros in the first column except in the last row, thus the row vector c has to have a non-zero element in the last column to make the new observability matrix keeping the full rank. Thus one condition on A, b and c making the new system observable is that $c = (c_1, c_2, \dots, c_n)$ where $c_n \neq 0$.

5

We have

$$\Gamma = \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & -2 & 1 & 3 & 0 \\ 1 & 0 & -2 & 1 & 3 & 0 & -6 & 3 \\ 0 & 0 & -1 & 1 & 2 & 0 & -5 & 3 \\ -1 & 1 & 2 & 0 & -5 & 3 & 8 & 0 \end{pmatrix}, \tag{33}$$

hence the image of Γ is:

$$\operatorname{Im}\Gamma = \operatorname{Im}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix} = \operatorname{Im}\begin{pmatrix} \mathbf{e}_2 & \mathbf{e}_4 & \mathbf{e}_1 & \mathbf{e}_3 \end{pmatrix} = \mathbb{R}^4. \tag{34}$$

Thus the system is completely reachable and we can make a pole placement for this problem, since also all complex conjugates are included.

To find a feedback control with poles at $\{-1, -2, -3, -4\}$ we need the characteristic polynomial of $A + BK$ to be

$$\chi_{A+BK} = s^4 + \gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4 = \tag{35}$$

$$= (s+4)(s+3)(s+2)(s+1) = s^4 + 10s^3 + 50s + 24.$$

Now, computing $A + BK$ we get:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ k_{1,1} & k_{1,2} - 1 & k_{1,3} + 1 & k_{1,4} + 1 \\ 0 & 0 & 0 & 1 \\ k_{2,1} - k_{1,1} & k_{2,2} - k_{1,2} + 2 & k_{2,3} - k_{1,3} + 1 & k_{2,4} - k_{1,4} \end{pmatrix}. \tag{36}$$

Since the system is completely reachable there is a non-singular tranformation T such that the transformation fulfills

$$TAT^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma_4 & -\gamma_3 & -\gamma_2 & -\gamma_1 \end{pmatrix}. \tag{37}$$

Thus, we get

$$\begin{cases} k_{1,1} = 0, & k_{1,2} = 1, \\ k_{1,3} = 0, & k_{1,4} = -1, \\ k_{2,1} = -24, & k_{2,2} = -51, \\ k_{2,3} = -36, & k_{2,4} = -11. \end{cases} \tag{38}$$

Thus one feedback control $u = Kx$ that assigns poles at $\{-1, -2, -3, -4\}$ is:

$$u = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -24 & -51 & -36 & -11 \end{pmatrix} x. \tag{39}$$

The realisation is minimal if and only if it is reachable and observable. Thus if one can show that it is not observable, then it is not minimal. The observability matrix is:

$$\Omega = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 3 & 3 \end{pmatrix}, \tag{40}$$

which does not have full row-rank since it has a zero column. Thus, the system is not observable, hence, not minimal. To use Kalman decomposition to find a minimal realisation we can begin by dividing $\mathbb{R}^4$ into 4 subspaces:

$$\mathbb{R}^4 = V_{\bar{o}r} \oplus V_{\bar{o}\bar{r}} \oplus V_{or} \oplus V_{o\bar{r}}.$$

We also have:

$$\begin{cases} \operatorname{Im}\Gamma = V_{\bar{o}r} \oplus V_{or}, \\ \ker\Omega = V_{\bar{o}r} \oplus V_{\bar{o}\bar{r}}, \\ V_{\bar{o}r} = \ker\Omega \cap \operatorname{Im}\Gamma. \end{cases}$$

We have already computed the image of Γ, which is $\mathbb{R}^4$. The kernel of Ω is quite direct, too. We can for instance se that $\mathrm{e}_1 \in \ker\Omega$ and since the two last columns equal we also have $[0\,0\,1\,-1]^T \in \ker\Omega$, hence

$$\ker\Omega = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}. \tag{41}$$

Since $\ker\Omega \subset \operatorname{Im}\Gamma$ we have $V_{\bar{o}r} = \ker\Omega \cap \operatorname{Im}\Gamma = \ker\Omega$. Since $\operatorname{Im}\Gamma = \mathbb{R}^4 = V_{\bar{o}r} \oplus V_{or} = \ker\Omega \oplus V_{or}$ we can let $V_{or} = \operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$. Since $\operatorname{Im}\Gamma = \mathbb{R}^4 = V_{\bar{o}r} \oplus V_{\bar{o}\bar{r}} \oplus V_{or} \oplus V_{o\bar{r}}$, we can let $V_{o\bar{r}} = V_{\bar{o}\bar{r}} = \{0\}$.

Now, setting

$$T = \begin{pmatrix} V_{\bar{o}r} & V_{or} & V_{\bar{o}\bar{r}} & V_{o\bar{r}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{42}$$

and then

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{43}$$

Thus our new state space representation of the system becomes:

$$T^{-1}AT = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad CT = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}. \tag{44}$$

To deduce the dimension of $A_{2,2}, B_2, C_2$ and where they shall be extracted we can see that the dimension of the observable and reachable space is two, and the dimension of the unobservable and reachable space is two, while the others are zeroo, hence extracting the 2-by-2 A-block from the third column and row, grabbing 2-by-2 B-block from third row, and C from the third column we get that

$$A_{2,2} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{45}$$

is a minimal realisation of the system.

We can check the rank of the product of Ω and Γ to verify the dimesnions. Taking 2 linear independent rows of Ω as Ω_2 and 4 linear independent columns of Γ as Γ_2 we can check the rank of the product of them.

$$\operatorname{rank} \Omega\Gamma = \operatorname{rank} \Omega_2\Gamma_2 = \operatorname{rank} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix} = \tag{46}$$

$$\operatorname{rank} \begin{pmatrix} 1 & 0 & -2 & 1 \\ -2 & 1 & 3 & 0 \end{pmatrix} = 2.$$

Thus, we can conclude that the new minimal system plausibly is correctly derived.