ASSIGNMENT SF2940
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19930610-1156
VILLE WASSBERG

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## Problem 1

a) $\mathrm{Cov}(X_1, 2X_1 + X_2) = \mathrm{Cov}(X_1, 2X_1) + \underbrace{\mathrm{Cov}(X_1, X_2)}_{=0}$

$= 2\mathrm{Cov}(X_1, X_1) = 2\mathbb{E}\left[(X_1 - \underbrace{\mathbb{E}(X_1)}_{=0})(X_1 - \underbrace{\mathbb{E}[X_1]}_{=0})\right]$

$= 2\mathbb{E}[X_1^2] = 2(\mathbb{E}[X_1^2] - \underbrace{\mathbb{E}[X_1]}_{=0}^2) =$

$= 2\mathrm{Var}(X_1) = 2a^1 = \underline{\underline{2a}}$

b) $\mathrm{Var}(X_1 + X_2) = \mathbb{E}[(X_1 + X_2)^2] - \mathbb{E}[X_1 + X_2]^2 =$

$= \mathbb{E}[X_1^2 + 2X_1X_2 + X_2^2] - (\mathbb{E}[X_1] + \mathbb{E}[X_2])^2 =$

$= \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 + \mathbb{E}[X_2^2] - \mathbb{E}[X_2]^2 +$

$+ \mathbb{E}[2X_1X_2] - 2\underbrace{\mathbb{E}[X_1]}_{=0}\underbrace{\mathbb{E}[X_2]}_{=0} =$

$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathbb{E}[X_1X_2] = *$

$* \mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1X_2]$ by above in a)

$\Rightarrow \mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$

Chebyshev

c) $\lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}[(X_n - 0)^2] =$

$= \frac{1}{\varepsilon^2}\mathbb{E}[X_n^2] = \frac{1}{\varepsilon^2}(\mathbb{E}[X_n^2] - \underbrace{\mathbb{E}[X_n]^2}_{=0}) =$

$= \frac{1}{\varepsilon^2}\mathrm{Var}[X_n] = \frac{1}{\varepsilon^2}a^n$ as $n \to \infty$

$a \in (0,1) \Rightarrow \lim_{n\to\infty} \frac{1}{\varepsilon^2}a^n = \frac{1}{\varepsilon^2}\cdot 0 = 0$

## Problem 2

a) $Y_n := \frac{1}{n} X_n$

$$\lim_{n\to\infty} \mathbb{E}\left[(Y_n - \lambda)^2\right] = \lim \left(\mathbb{E}[Y_n^2] - 2\lambda \mathbb{E}[Y_n] + \mathbb{E}[\lambda^2]\right)$$

$$= \lim \frac{1}{n^2} \left(\mathbb{E}[X_n^2] - 2\lambda \mathbb{E}[X_n] + \lambda^2\right)$$

$=$

$$\mathbb{E}[X_n] = \sum_{k=0}^{\infty} k e^{-n\lambda} \frac{(n\lambda)^k}{k!} \longrightarrow 0 \text{ as } n \to \infty$$

b) $Z_n := \sqrt{n}(Y_n - \lambda)$

$$\varphi_{Z_n} = \mathbb{E}\left[e^{itZ_n}\right] = \mathbb{E}\left[e^{it\sqrt{n}(Y_n - \lambda)}\right] =$$

$$\varphi_{X_n}(t/\sqrt{n}) = \sum_{k=0}^{\infty} e^{-n\lambda} \frac{(n\lambda)^k}{k!} \cdot e^{\frac{itk}{\sqrt{n}}} = e^{-n\lambda} e^{n\lambda} \cdot e^{\frac{it}{\sqrt{n}}}$$