Comparison elliptic PDE and parabolic PDE: discretization in space

Elliptic

$$\begin{cases} -u_{xx}(x) = f(x) \quad x \in (a,b) \\ \left.\begin{aligned} u(a) &= \alpha \\ u(b) &= \beta \end{aligned}\right\} \text{BC} \end{cases}$$

Parabolic

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) \\ \qquad x \in (a,b) \\ \qquad t > 0 \\ \left.\begin{aligned} u(a,t) &= \alpha(t) \\ u(b,t) &= \beta(t) \end{aligned}\right\} \text{BC} \\ u(x,0) = g(x) \quad \text{IC} \end{cases}$$

Discretize in space: $x_j = a + \Delta x j$

$j = 0, \dots, N$

$x_0 = a$

$x_N = b$

for interior points x_j, $j = 1, \dots, N-1$

$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = f_j$$

$j = 1, \dots, N-1$

$$\frac{du_j(t)}{dt} = \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\Delta x^2} + f_j(t)$$

$j = 1, \dots, N-1$

$u_j \approx u(x_j)$

$f_j = f(x_j)$

$u_j(t) \approx u(x_j, t)$

$f_j(t) = f(x_j, t)$

$u_0 = \alpha$
$u_N = \beta$

$u_0(t) = \alpha(t)$
$u_N(t) = \beta(t)$

- insert boundary values into equations $j=1$ and $j=N-1$
- move known values to RHS

u_j , $j=1,\dots,N-1$ are unknowns

obtained from solving one linear system

- $A\bar{u} = \bar{f}$

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$u_j(t)$, $j=1,\dots,N-1$ unknowns

$u_j(t)$ obtained by using ODE-method to timestep system

$$\frac{d\bar{u}}{dt} = A\bar{u}(t) + \bar{f}(t)$$

$$\bar{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix}$$

for fully discrete

problem we obtain a vector-solution per time $t = 0, \Delta t, 2\Delta t, \ldots T-\Delta t, T$ until endtime T.

fully discrete solution

$$u_j^n \approx u(x_j, t_n)$$

$$t_n = n \cdot \Delta t$$

$$\Delta t = \frac{T}{M}$$, M time steps

Let $u_j^0 = u(x_j, 0) = g(x_j)$

Matrix solution: goes in MoL

$$
\begin{array}{c}
x_1 - \\
x_2 - \\
\vdots \\
x_{N-1}
\end{array}
\begin{bmatrix}
u_1^0 & u_1^1 & \cdots & u_1^M \\
u_2^0 & u_2^1 & \cdots & u_2^M \\
\vdots & \vdots & & \vdots \\
u_{N-1}^0 & u_{N-1}^1 & \cdots & u_{N-1}^M
\end{bmatrix}
$$

$(N-1) \times (M+1)$ matrix

$\bar{u}(0) = \bar{g}(x) \quad = \bar{u}(\Delta t) \approx u(x_j, \Delta t) \quad j = 1, \dots, N-1 \qquad = \bar{u}(T)$

Local truncation error for θ-scheme

Heat equation $\quad u_t = u_{xx}$

θ-scheme:

$$u_j^{n+1} = u_j^n + \Delta t \cdot \left[\theta \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \right.$$

$$\left. + (1-\theta) \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right]$$

Define local truncation error l_j^n by inserting exact solution $u(x_j, t_n)$ into θ-scheme:

$$u(x_j, t_{n+1}) = u(x_j, t_n) +$$

$$\Delta t \Big[\theta \underbrace{\frac{u(x_{j+1}, t_{n+1}) - 2u(x_j, t_{n+1}) + u(x_{j-1}, t_{n+1})}{\Delta x^2}}_{A_{n+1}}$$

$$+ (1-\theta) \underbrace{\frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{\Delta x^2}}_{A_n}$$

$$+ \ell_j^n$$

Rewrite θ-scheme:

$$\underbrace{\frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t}}_{LHS} = \theta A_{n+1} +$$

$$(1-\theta) A_n + \frac{\ell_j^n}{\Delta t}$$

Taylor expand u around x_j and t_n to obtain precise form of ℓ_j^n

1. LHS: (T.E. w.r.t time)

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \Delta t\, u_t(x_j, t_n) + \frac{\Delta t^2}{2} u_{tt}(x_j, t_n) + O(\Delta t^3)$$

so

$$\frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} = u_t(x_j, t_n)$$

$$+ \frac{\Delta t}{2} u_{tt} + O(\Delta t^2)$$

2. A_n: (T.E w.r.t x)

$$u(x_{j\pm1}, t_n) = u(x_j, t_n) \pm \Delta x\, u_x + \frac{\Delta x^2}{2} u_{xx} \pm \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{24} u_{xxxx}$$

$$A_n = u_{xx}(x_j, t_n) + \underbrace{\frac{\Delta x^2}{12} u_{xxxx}(\xi, t_n)}_{O(\Delta x^2)}$$

$$x_{j-1} < \xi < x_{j+1}$$

3. A_{n+1}: (T.E w.r.t x)

$$A_{n+1} = u_{xx}(x_j, t_{n+1}) + O(\Delta x^2)$$

(T.E w.r.t t)

$$A_{n+1} = u_{xx}(x_j, t_n) + \Delta t \underbrace{u_{xxt}(x_j, t_n)}_{= u_{tt}(x_j, t_n)}$$

$$+ \Theta(\Delta x^2 + \Delta t^2)$$

Since $u_t = u_{xx}$

we have $u_{tt} = u_{xxt}$

Combine 1, 2, 3 :

At $x = x_j$, $t = t_n$:

$$\underline{u_t} + \frac{\Delta t}{2} u_{tt} + \Theta(\Delta t^2)$$

$$= \underline{\theta u_{xx}} + \theta \Delta t\, u_{tt}$$

$$+ \underline{(1-\theta) u_{xx}} + \Theta(\Delta x^2 + \Delta t^2)$$

$$+ \frac{e_j^n}{\Delta t}$$

$u_t = u_{xx}$ so

$$\frac{\ell_j^n}{\Delta t} = \Delta t\left(\frac{1}{2} - \theta\right) u_{tt}$$

$$+ O\left(\Delta x^2 + \Delta t^2\right)$$

$$\ell_j^n = \begin{cases} O(\Delta t \cdot (\Delta x^2 + \Delta t^2)), & \theta = \frac{1}{2} \\ O(\Delta t \cdot (\Delta x^2 + \Delta t)), & \theta \neq \frac{1}{2} \end{cases}$$

FEM for parabolic PDES

Ansatz:

$$u_h(x,t) = \sum_{k=1}^{N-1} u_k(t)\,\phi_k(x)$$

Galerkin principle for fixed t:

find $\{u_k(t)\}$ such that

$$(*) \int_a^b r_h(x,t)\,\phi_l(x)\,dx = 0$$

$$l = 1, \dots, N-1$$

Insert $r_h(x,t) = \frac{\partial}{\partial t} u_h(x,t)$
$- \partial_{xx} u_h(x,t) - f(x,t)$

and ansatz $u_N(x,t)$ in $\boxed{*}$

$$\int_a^b \left(\sum_{k=1}^{N-1} u_k'(t)\phi_k(x) - \sum_{k=1}^{N-1} u_k(t)\phi_k''(x) - f(x,t) \right) \phi_\ell(x)\, dx =$$

$$\sum_{k=1}^{N-1} u_k'(t) \int_a^b \phi_k(x)\phi_\ell(x)\, dx$$

$$- \sum_{k=1}^{N-1} u_k(t) \underbrace{\int_a^b \phi_k''(x)\phi_\ell(x)\, dx}$$

Integration by parts, bdy terms vanishes because $\phi_k(a) =$

$$- \int_a^b f(x,t)\phi_\ell(x)\, dx$$

$\phi_k(b) = 0$

$$= \sum_{k=1}^{N-1} u_k'(t) \underbrace{\int_a^b \phi_k(x)\, \phi_\ell(x)\, dx}_{m_{\ell,k} \text{ - elements in mass matrix}}$$

$$+ \sum_{k=1}^{N-1} u_k(t) \underbrace{\int_a^b \phi_k'(x)\, \phi_\ell'(x)\, dx}_{-a_{\ell,k} \text{ - elements in stiffness matrix}}$$

$$- \underbrace{\int_a^b f(x,t)\, \phi_\ell(x)\, dx}_{f_\ell(t)}$$

$$\sum_{k=1}^{N-1} u_k' \, m_{l,k} = u_1' \, m_{l,1} + u_2' \, m_{l,2}$$

$$+ \cdots \; u_{N-1}' \, m_{l,N-1}$$

row l in M multiplied with $\frac{d\bar{u}}{dt}$

$$M \frac{d\bar{u}}{dt} = \begin{bmatrix} m_{11} & m_{12} & m_{1,3} & \cdots & m_{1,N-1} \\ m_{21} & m_{22} & - & - & m_{2,N-1} \\ & & & & \\ & & & & \\ m_{N-1,1} & \cdots & & & \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_{N-1}' \end{bmatrix}$$

Wave equation

$$\underbrace{\begin{pmatrix} u_{tt} \\ u_{xt} \end{pmatrix}}_{\bar{U}_t} - \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} u_{xt} \\ u_{xx} \end{pmatrix}}_{\bar{U}_x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\bar{U} = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$$

Eigenvalues:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -c^2 \\ -1 & \lambda \end{vmatrix}$$

$$= \lambda^2 - c^2 = 0, \quad \lambda_{1,2} = \pm c$$