Multivariate random variables:

$$(\Omega, \mathcal{A}, \mathbb{P})$$

Def: n-dimensional random variable or vector $\underset{\sim}{X}$ is a measurable function from Ω to $\mathbb{R}^n$

$$\underset{\sim}{X} : \Omega \longrightarrow \mathbb{R}^n$$
$$\omega \longmapsto (X_1(\omega), X_2(\omega), \ldots, X_n(\omega))'$$

Measurable: $\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then

$$\{\omega \in \Omega : X_1(\omega) \le x_1, X_2(\omega) \le x_2, \ldots, X_n(\omega) \le x_n\}$$

belongs to $\mathcal{A}$.

Def: Joint cumulative distribution function

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) := \mathbb{P}(\underbrace{\{X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n\}})$$

event is in $\mathcal{A}$ because $\underset{\sim}{X}$ is measurable.

for $x_1, \ldots, x_n \in \mathbb{R}$.

2.

Choose $n=2$: <u>bivariate case</u>, pair of r.v. (X,Y)

$$F_{X,Y}(x,y) = \mathbb{P}(\{X \le x, Y \le y\}), \quad (x,y) \in \mathbb{R}^2$$

They are continuous r.v. if there is a <u>joint density function</u> $f_{X,Y}(x,y)$ such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,dv$$

and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \, \partial y} F_{X,Y}(x,y)$$

for every point of continuity of $f_{X,Y}$.

Moreover,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv = 1.$$

Marginal density and distribution: Pair (X,Y)

- $$\mathbb{P}(X \leq x) = F_{X,Y}(x,\infty) = \int_{-\infty}^{x} \underbrace{\left(\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv \right)}_{=: f_X(u)} du$$

  marginal density.

- $$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$ marginal distribution.

Example: Choose (X,Y) with uniform density on the unit disc

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \leq 1 \\ 0 & \text{else} \end{cases}$$ (Area of the unit disc is π)

Determine the distribution of X:

Marginal density:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \leq 1 \\ 0 & \text{else.} \end{cases}$$

From $x^2+y^2 \leq 1$ we get $y^2 \leq 1-x^2 \Rightarrow -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$

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4.

Independence: The components of a random vector are independent iff

$$F_{X_1, X_2, \dots X_n}(x_1, x_2, \dots, x_n) = \underbrace{\prod_{k=1}^{n} F_{X_k}(x_k)}_{\text{product of marginal distributions.}}$$

for $x_1, \dots, x_n \in \mathbb{R}$

Continuous case: $$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \underbrace{\prod_{k=1}^{n} f_{X_k}(x_k)}_{\text{product of marginal densities.}}$$

In the example: $$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \leq 1, \\ 0 & \text{else.} \end{cases}$$

$$f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}, \quad f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2},$$

$\Rightarrow$ joint density does not factorize: (X,Y) are dependent.