Convergence by transforms: Central limit theorem

Recall: convergence in distribution: $X_n \xrightarrow{d} X$ if

$$F_{X_n}(x) \to F_X(x) \text{ for all continuity points of } F_X.$$

Example: (X_n), $X_n \in Bin(n, \frac{\lambda}{n})$, $\lambda > 0$,

$$\frac{\lambda}{n} = p, \quad p+q=1$$

$$\varphi_{X_n}(t) = (q + pe^{it})^n = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^{it}\right)^n$$

$$= \left(1 + \frac{\lambda}{n}(e^{it} - 1)\right)^n$$

Euler: $\left(1 + \frac{y}{n}\right)^n \xrightarrow[n\to\infty]{} e^y$.

$$\xrightarrow[\text{Euler identity}]{n\to\infty} e^{\lambda(e^{it}-1)} = \varphi_{Po(\lambda)}(t).$$

This suggests that

$$X_n \xrightarrow{d} X \text{ with } X \in Po(\lambda).$$

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<u>Thm</u> (Continuity theorem) X and $X_1, X_2, \ldots$ random variables and suppose that

$$\varphi_{X_n}(t) \longrightarrow \varphi_X(t) \qquad \forall t \in \mathbb{R}$$

Then $X_n \xrightarrow{d} X$ as $n \to \infty$ //

<u>Remark</u>: If $\varphi_{X_n}(t) \to \varphi(t)$ with $\varphi(t)$ <u>some</u> function continuous at $t=0$,

then there is a random variable X such that

$$X_n \xrightarrow{d} X \quad \text{and} \quad \varphi(t) = \varphi_X(t).$$

The converse is also true: If $X_n \xrightarrow{d} X$, then $\varphi_{X_n}(t) \to \varphi_X(t)$ $\forall t \in \mathbb{R}$. //

Thm (Central limit thm). (X_k) i.i.d. random variables with $\mathbb{E}[X_k] = \mu$ and $\operatorname{Var} X_k = \sigma^2 < \infty$.

Then,

$$\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n} X_k - \mu\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2),$$

or equivalently

$$\frac{1}{\sqrt{n}\,\sigma}\left(\sum_{k=1}^{n} X_k - n\mu\right) \xrightarrow{d} \mathcal{N}(0,1)$$

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Proof: Let $Z_n := \dfrac{\left(\sum_{k=1}^{n} X_k - n\mu\right)}{\sqrt{n}\,\sigma}$. It suffices to prove by the continuity theorem that $\varphi_{Z_n}(t) \to \varphi_Z(t)$, with $Z \in \mathcal{N}(0,1)$.

Can assume that $\mu = 0$ and $\sigma = 1$ (why?)

$$\varphi_{Z_n}(t) = \mathbb{E}\left[e^{it\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k}\right] \overset{\text{independence}}{=} \prod_{k=1}^{n} \underbrace{\mathbb{E}\left[e^{i\frac{t}{\sqrt{n}} X_k}\right]}_{= \varphi_{X_k}\left(\frac{t}{\sqrt{n}}\right)}$$

$$\overset{\text{identically distributed}}{=} \left(\varphi_{X_1}\left(\tfrac{t}{\sqrt{n}}\right)\right)^n.$$

But X_1 has finite variance $(=1)$, so we have the expansion

$i^2=-1$

$$\varphi_{X_1}\left(\frac{t}{\sqrt{n}}\right) = 1 + \frac{it}{\sqrt{n}}\underbrace{E[X_1]}_{=\mu=0} + \frac{(it)^2}{2n}\underbrace{E[X^2]}_{=\sigma^2=1} + o\left(\frac{t^2}{n}\right)$$

as $n \to \infty$.

little-o notation

See notes about characteristic function.

$$= 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right) = e^{\log\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)}.$$

Used that $y = e^{\log y}$. (here $\log = \ln$ natural logarithm)

So
$$\left(\varphi_{X_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n = e^{n\log\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)}.$$

Taylor expansion $\log(1+z) = z + o(z)$ as $|z| \to 0$

$$= e^{n\left(-\frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)}$$

$$= e^{-t^2/2 + n\, o\left(\frac{t^2}{n}\right)}.$$

but $n \cdot o\left(\frac{t^2}{n}\right) \to 0$ as $n \to \infty$ for any fixed t. Moreover,

$$\xrightarrow{n\to\infty} e^{-t^2/2} = \varphi_X(t), \quad X \in N(0,1) \quad \blacksquare$$

5.

<u>Exercise</u>: Prove the following weak law of large numbers:

(X_k) i.i.d. r.v. with $\mathbb{E} X_k = \mu$.

Show that

$$\overline{X}_n := \frac{1}{n} \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{P} \mu.$$

<u>Hint</u>: Use characteristic function to show convergence in distribution to $\delta(\mu)$.

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