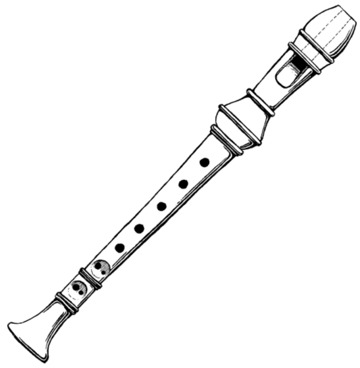
KTH Engineering Sciences

Project
Flute



In this project you will simulate the wave propagation inside a flute and determine the frequencies of the notes that one can play with it. Consider a recorder flute ("blockföjt") of length L with a number of finger holes that are used to play different notes. Let the deviation from the ambient pressure inside the flute at position x [m] and time t [s] be $P = P(t, x)$ [N/m^2]. Then a simple mathematical model for P is a variant of the the wave equation

$$\frac{\partial^2 P}{\partial t^2} - c_0^2 \frac{\partial^2 P}{\partial x^2} + D(x)P = G(x), \quad 0 \leq x \leq L, \quad t > 0,$$

with zero initial data $P(0, x) = 0$ and homogeneous Dirichlet boundary conditions

$$P(t, 0) = P(t, L) = 0.$$

Here c_0 [m/s] is the speed of sound in air.

The variable coefficient $D(x)$ models the effect of the holes. It is positive at the holes, and zero otherwise. We assume it has the following form for a single hole of diameter S [m] at $x = \hat{x}$,

$$D(x) = S\, H(x - \hat{x}), \qquad H(x) = \frac{\alpha c_0^2}{L\delta R^2} \begin{cases} 1 + \cos\left(\frac{2\pi x}{L\delta}\right), & |x| \leq L\delta/2, \\ 0, & |x| > L\delta/2, \end{cases}$$

where δ and α are dimensionless numbers representing the shape of the hole and R [m] is the radius of the flute. If there are more than one hole, the H functions are simply added together, e.g. $D(x) = S_1\, H(x - \hat{x}_1) + S_2\, H(x - \hat{x}_2)$, etc.

The forcing function G represents the (very complicated) sound generation process caused by blowing in the mouthpiece. We suppose here that it concentrates at $0 < x < w$ and is given as

$$G(x) = G_0 \frac{x}{w} e^{-4x/w},$$

for the constants w [m] and G_0 [N/(m^2·s^2)].

Background

When all finger holes are covered, such that $D \equiv 0$, the fundamental frequency f_0 of the flute is given by the lowest eigenmode of the system, i.e. the standing wave with the lowest frequency. Since we have Dirichlet boundary conditions, $f_0 = c_0/2L$. For a standard (soprano) recorder f_0 would be the note C5 (523.25 [Hz]). This is the lowest note that can be played with the recorder.

For the recorder to be a useful, one must be able to play more notes than f_0, at least all (tempered) notes in an octave above f_0, i.e. the frequencies $f_k = 2^{k/12} f_0$, with $k = 0, \ldots, 12$. One naive way to achieve that would be to place 12 very large finger holes along the flute, because if a hole at $x = \hat{x}$ is very large, the effect will be the same as cutting off the recorder at $\hat{x}$, hence effectively taking $L = \hat{x}$. In the same way as above, one would then be able to play a note with frequency $f_h = c_0/2\hat{x}$ and the holes could therefore be placed at $\hat{x}_k = 2^{-k/12} L$ for $k = 1, \ldots, 12$.

However, it is not practical with so many closely placed holes, and we do not have enough fingers to cover all of them. Instead, *small* holes are used. A small hole at $x = \hat{x}$ does not change the fundamental frequency as drastically as a large one. Moreover, other holes at $x > \hat{x}$ will still influence the frequency of the note that is played. Therefore one can use fewer holes, placed further apart, since more frequencies can be reached by different combinations of open and closed holes. In a standard recorder there are only seven holes. Finding suitable positions and sizes for the holes is not trivial, however.

Tasks

1. Let P_0 and T be reference values for the pressure P and time t. Introduce the rescaled variables u, τ and y as

$$P = uP_0, \qquad t = \tau T, \qquad x = yL,$$

   Rescale the equations to dimensionless form and show that, with appropriate choices of T and P_0 they can be written as

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial y^2} + d(y)u = g(y), \qquad 0 \le y \le 1, \qquad \tau > 0, \tag{1}$$

   with boundary and initial conditions given by

$$u(0, y) = 0, \qquad 0 \le y \le 1, \qquad u(\tau, 0) = u(\tau, 1) = 0.$$

   Moreover,

$$d(y) = sh(y - \hat{y}), \qquad h(y) = \frac{1}{\delta} \begin{cases} 1 + \cos\left(\frac{2\pi y}{\delta}\right), & |y| \le \delta/2, \\ 0, & |y| > \delta/2, \end{cases} \qquad g(y) = ye^{-\beta y}.$$

   Determine the dimensionless numbers s, β and γ as functions of S, L, c_0, α, g_0 and w.

2. From now on use $\beta = 50$ and $\delta = 0.01$ in (1). The dimensionless hole size s should be between zero (no hole) and ca 100 (very large hole).

   Describe a physical setup of the flute parameters L, R, α, w and S that corresponds to this case, for which also the fundamental frequency $f_0 = 523.25$ [Hz]. (This recorder will be slightly longer than a real recorder, due to the simplified boundary conditions in the model.)

3. Make a uniform discretization in space and time, with $y_j = j\Delta y$, $\tau_n = n\Delta\tau$, and let $u_j^n \approx u(\tau_n, y_j)$ be the numerical approximation. Use central difference approximations for both the space and time derivatives, as in Edsberg Section 8.2.7. It will be convenient to write and implement the scheme in matrix form as

$$\frac{\boldsymbol{u}^{n+1} - 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{\Delta\tau^2} = A\boldsymbol{u}^n - D\boldsymbol{u}^n + \boldsymbol{g}, \qquad \boldsymbol{u}^n = (u_1^n, u_2,^n \ldots, u_N^n)^T, \tag{2}$$

where $D = \mathrm{diag}(d(y_j))$ is a diagonal matrix and $\boldsymbol{g} = (g(y_1), \ldots, g(y_N))^T$. Let $u(\tau, y) \equiv 0$ for $\tau \leq 0$ (i.e. set $\boldsymbol{u}^{-1} = \boldsymbol{u}^0 \equiv 0$).

4. First do a simulation with all finger holes closed, i.e. $d(y) \equiv 0$ over a fairly long time period. The wave should have time to go back and forth in the flute multiple times. Plot the pressure at $y = 1/3$ as a function of time, i.e. the function $u(\tau, 1/3)$. Test different $\Delta\tau$ and Δy. Verify that the solution is accurate.

5. Analyze the time signal $u(\tau, 1/3)$ with Matlab's `fft` command and plot the absolute value of the components with `semilogy` to see the spectrum of the sound.

   Connnect back to the physical model. Rescale the horizontal axis in the plot such that it corresponds to the physical frequency. It should have unit Hz and go from 0 upto 2000 [Hz]. Does the first spike in the plot align with the theoretical fundamental frequency $f_0 = 523.25$ [Hz]? The peaks at higher frequencies correspond to overtones.

   Note here: The longer simulation time, the higher the resolution of the spectrum. (Why?)

6. Repeat the experiments but with one or more open holes of different sizes. How does the fundamental frequency change? Show examples. Is it true that a large hole has the same effect as cutting off the recorder?

7. There is a more direct way to compute the fundamental frequency. A standing (resonant) wave U with (scaled) frequency f and amplitude a can be written $U(\tau, y) = a(y)\exp(i2\pi f\tau)$. It satisfies the same PDE as u, but with $g \equiv 0$. Show that the amplitude then must satisfy the time-independent PDE

$$-a_{yy} + d(y)a = (2\pi f)^2 a, \qquad a(0) = a(1) = 0,$$

   and that after the same discretization as in (2),

$$(-A + D)\boldsymbol{a} = (2\pi f)^2 \boldsymbol{a}.$$

   Hence, $(2\pi f)^2$ is an eigenvalue and $\boldsymbol{a}$ an eigenfunction of the matrix $-A + D$. The smallest eigenvalue gives the fundamental frequency. Other eigenvalues give the overtones.

   Go through the cases above. Compute the eigenvalues of $-A + D$ and (after the same rescaling as above) compare with the spikes in the spectrum. Also plot the eigenfunctions for the fundamental frequencies. How do they differ from the eigenfunctions with very large, or no, holes?

8. Optional final exercises: Place seven finger holes of sizes $s_1, \ldots, s_7$ at $y = \hat{y}_1, \ldots, \hat{y}_7$ such that all the 13 tempered notes mentioned above can be played, by having suitable combinations of them open and closed[1]. Use the Matlab `sound` command to play the sound and verify with a tuning device that the frequencies are right. (Less ambitious version: Place two holes and play four notes.)

[1] You may allow two different sizes for holes 6 and 7 as in a real recorder.