SF2520 — Applied numerical methods

Lecture 9

FEM theory
Scaling differential equations

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Today's lecture

- Summary of last lecture.
- FEM and other boundary conditions (1D)
- Theory for FEM
- Scaling to dimensionless form

Finite element method (FEM)

Consider elliptic equation set in Ω,

$$\begin{aligned} -\Delta u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

- Make ansatz of solution as

$$u(x) \approx \sum_{k=1}^{n} u_k \phi_k(x) =: u_h(x).$$

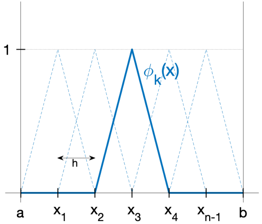
- Determine $\{u_k\}$ by Galerkin condition

$$\int_\Omega r_h(x)\phi_k(x)dx = 0, \qquad r_h(x) := -\Delta u_h(x) - f(x), \qquad k = 1, \dots, n.$$

- Gives linear system $\boldsymbol{u} = \{u_k\}$, $A = \{a_{k,\ell}\}$, $\mathbf{f} = \{f_\ell\}$.

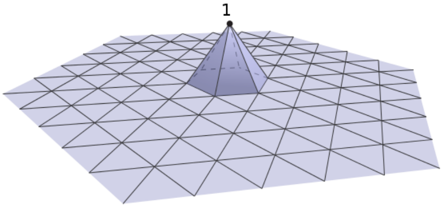
$$A\boldsymbol{u} = \boldsymbol{f}, \qquad a_{k,\ell} = \int_\Omega \nabla\phi_k(x) \cdot \nabla\phi_\ell(x)dx, \qquad f_\ell = \int_\Omega f(x)\phi_\ell(x)dx.$$

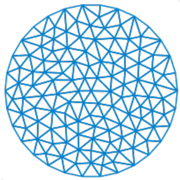
- In 1D, basis functions ϕ_k = piecewise linear hat functions,



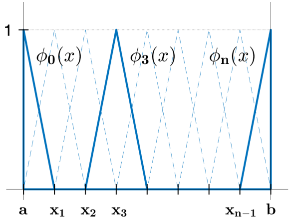
$$\phi_k(x) = \begin{cases} 1, & x = x_k, \\ 0, & x = x_j, \quad j \neq k \\ \text{linear}, & \text{otherwise.} \end{cases}$$

- In 2D, discretize by triangles and use ϕ_k = "pyramid" functions





- Second order approximation. Almost same system as FD in 1D.



Consider BVP again but with inhomogeneous (non-zero) Dirichlet conditions,

$$-u''(x) = f(x), \qquad u(a) = \alpha, \qquad u(b) = \beta.$$

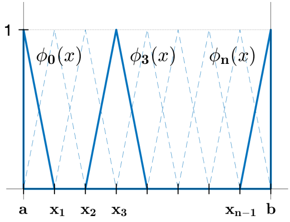
- Also use ϕ_0 and ϕ_n. Make the new ansatz

$$u_h(x) := \sum_{k=1}^{n-1} u_k \phi_k(x) + \alpha\phi_0(x) + \beta\phi_n(x).$$

  Then BC is again automatically satisfied by u_h.
- As before we require, when $\ell = 1, \ldots, n-1$,

$$0 = \int_a^b r_h(x)\phi_\ell(x)dx = -\int_a^b u_h''(x)\phi_\ell(x)dx - \int_a^b f(x)\phi_\ell(x)dx = \cdots$$
$$= \sum_{k=1}^{n-1} a_{k,\ell}u_k - f_\ell + \alpha\int_a^b \phi_0'\phi_\ell' dx + \beta\int_a^b \phi_n'\phi_\ell' dx.$$

FEM with other BC, 1D



Consider BVP again but with inhomogeneous (non-zero) Dirichlet conditions,

$$-u''(x) = f(x), \qquad u(a) = \alpha, \qquad u(b) = \beta.$$

- We get

$$\sum_{k=1}^{n-1} a_{k,\ell} u_k = f_\ell - \alpha \underbrace{\int_a^b \phi_0' \phi_\ell' dx}_{d_{0,\ell}} - \beta \underbrace{\int_a^b \phi_n' \phi_\ell' dx}_{d_{n,\ell}},$$

  where

$$d_{0,\ell} = \begin{cases} -\frac{1}{h}, & \ell = 1, \\ 0, & \ell \geq 2, \end{cases} \qquad d_{n,\ell} = \begin{cases} 0, & \ell \leq n-2, \\ -\frac{1}{h}, & \ell = n-1. \end{cases}$$

- Therefore $A\boldsymbol{u} = \tilde{\boldsymbol{f}}$ where $\tilde{\boldsymbol{f}}$ is modified as in finite differences,

$$\tilde{\boldsymbol{f}} = \boldsymbol{f} + \frac{1}{h}\begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}.$$

FEM with other Robin BC, 1D

BVP with Robin conditions at $x = 0$,

$$-u''(x) = f(x), \qquad u'(a) = \alpha_0 u(a) + \alpha_1, \qquad u(b) = 0.$$

- Include also ϕ_0 and keep u_0 as unknown. Make the ansatz

$$u_h(x) := \sum_{k=0}^{n-1} u_k \phi_k(x). \qquad \text{(Note: BC not automatically satisfied.)}$$

- We require $r_h \perp \operatorname{span}\{\phi_k\}$ and that u_h satsify BC. Then for $0 \le \ell \le n-1$,

$$\begin{aligned}
0 &= \int_a^b r_h(x)\phi_\ell(x)dx = -\int_a^b u_h''(x)\phi_\ell(x)dx - \int_a^b f(x)\phi_\ell(x)dx \\
&= -\Big[u_h'(x)\phi_\ell(x)\Big]_a^b + \int_a^b u_h'\phi_\ell' dx - \int_a^b f\phi_\ell dx \\
&= u_h'(a)\phi_\ell(a) + \sum_{k=0}^{n-1} a_{k,\ell}u_k - f_\ell = [\alpha_0 u_h(a) + \alpha_1]\phi_\ell(a) + \sum_{k=0}^{n-1} a_{k,\ell}u_k - f_\ell \\
&= \sum_{k=0}^{n-1} a_{k,\ell}u_k - f_\ell + \begin{cases} \alpha_0 u_0 + \alpha_1 & \ell = 0, \\ 0 & \ell \ge 1. \end{cases}
\end{aligned}$$

Modifies a_{00} in matrix A and first element of $\boldsymbol{f}$.

Theory background: strong form of Poisson equation

Consider the Poisson equation

$$\begin{aligned} -\Delta u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded open set.

- This is called the *strong form* of the equation.
- A solution to (1) is required to have two continuous derivatives in Ω and be continuous upto the boundary. These are called *strong solutions* or *classical solutions*.
- For many practical problems, f or $\partial\Omega$ are not smooth enough to allow a strong solution. E.g. f may be discontinuous or even a δ-function.
- To treat more general (and physically relevant) cases the *weak* form of the Poisson equation is used.

Theory background: weak form of Poisson equation

Weak form

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) - f(x)v(x)dx = 0, \qquad \forall v \in H_0^1(\Omega).$$

Here $H_0^1(\Omega)$ is a *Sobolev space* which includes the functions v for which

$$v \in L^2(\Omega), \qquad \nabla v \in L^2(\Omega), \qquad v = 0 \text{ on } \partial\Omega.$$

- u is called a *weak* solution to the Poisson equation.
- The weak form is well-posed also for less smooth f and $\partial\Omega$.
- There are similar weak forms for other elliptic PDEs, with $H_0^1(\Omega)$ replaced by other Sobolev spaces.

Finite element method 2D

For pyramid functions $\{\phi_j\}$, let

$$\mathcal{V}_h = \text{span}\{\phi_j\} \quad \Rightarrow \quad u_h = \sum_{j=1}^{n} u_j \phi_j(x) \in \mathcal{V}_h.$$

- $\mathcal{V}_h$ = piecewise linear functions which are zero on the boundary,
- $\mathcal{V}_h \subset H_0^1(\Omega)$. (Since each $\phi_j \in H_0^1(\Omega)$.)

FEM is then: Find $u_h \in \mathcal{V}_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla \phi_j - f \phi_j dx = 0, \qquad j = 1, \ldots, n.$$

But since any $v \in \mathcal{V}_h$ is a linear combination of $\{\phi_j\}$, this is the same as

Finite element method

Find $u_h \in \mathcal{V}_h$ such that

$$\int_\Omega \nabla u_h(x) \cdot \nabla v(x) - f(x) v(x) dx = 0, \qquad \forall v \in \mathcal{V}_h.$$

Finite element method 2D

Finite element method

Find $u_h \in \mathcal{V}_h$ such that

$$\int_\Omega \nabla u_h(x) \cdot \nabla v(x) - f(x)v(x)dx = 0, \qquad \forall v \in \mathcal{V}_h.$$

- FEM is hence the *same* as the weak form of Poisson with $H_0^1(\Omega)$ replaced by a finite-dimensional subspace $\mathcal{V}_h \subset H_0^1(\Omega)$.
- FEM solution u_h is the best approximation of the exact solution u in the subspace $\mathcal{V}_h$, measured in "energy norm", i.e.

Céa's lemma

$$||\nabla u - \nabla u_h||_{L^2} \leq ||\nabla u - \nabla v||_{L^2}, \qquad \forall v \in \mathcal{V}_h.$$

- By taking in particular $v \in \mathcal{V}_h$ to be the linear interpolant of u, one gets error estimates

$$||\nabla u - \nabla u_h||_{L^2} \leq Ch, \qquad ||u - u_h||_{L^2} \leq Ch^2.$$

Céa's lemma proof

Since $\mathcal{V}_h \subset H_0^1(\Omega)$,

$$\int_\Omega (\nabla u_h(x) - \nabla u(x)) \cdot \nabla v(x) dx = \int_\Omega f(x)v(x) - f(x)v(x) dx = 0, \qquad \forall v \in \mathcal{V}_h.$$

Therefore, for all $v \in \mathcal{V}_h$,

$$\begin{aligned} ||\nabla u - \nabla u_h||_{L^2}^2 &= \int_\Omega (\nabla u(x) - \nabla u_h(x)) \cdot (\nabla u(x) - \nabla u_h(x)) dx \\ &= \int_\Omega (\nabla u(x) - \nabla u_h(x)) \cdot \nabla u(x) dx \\ &= \int_\Omega (\nabla u(x) - \nabla u_h(x)) \cdot (\nabla u(x) - \nabla v(x)) dx \\ &\leq \left(\int_\Omega |\nabla u(x) - \nabla u_h(x)|^2 dx \cdot \int_\Omega |\nabla u(x) - \nabla v(x)|^2 dx \right)^{1/2} \\ &= ||\nabla u - \nabla u_h||_{L^2} \cdot ||\nabla u - \nabla v||_{L^2}. \end{aligned}$$

Divide both sides by $||\nabla u - \nabla u_h||_{L^2}$.

Scaling to dimensionless form

Heat equation – physical

$$
\begin{aligned}
\rho c \frac{\partial T}{\partial t} &= k \Delta T, && \text{(in } \Omega = [0, L]^2\text{)},\\
-k \frac{\partial T}{\partial n} &= h(T - T_e), && \text{(on } \partial\Omega\text{)},\\
T &= T_0, && \text{(initially at } t = 0\text{)},
\end{aligned}
$$

Heat equation – scaled

$$
\begin{aligned}
\frac{\partial u}{\partial \tau} &= \Delta u, && \text{(in } \tilde{\Omega} = [0, 1]\text{)},\\
-\frac{\partial u}{\partial n} &= bu, && \text{(on } \partial\tilde{\Omega}\text{)},\\
u &= 1, && \text{(initially at } \tau = 0\text{)},
\end{aligned}
$$

- How to relate $T(t, x)$ and $u(\tau, y)$?
- Here $T(t, x) = T_e + T_1 u(t/t_0, x/x_0)$, for suitable choices of T_1, τ_0 and x_0.

Dimensions and units

- Variables and parameters in differential equations describing physical processes etc. have a dimension expressed in units.
- Example: Heat equation for $T = T(t, x)$

$$\rho c \frac{\partial T}{\partial t} = k \Delta T \ \ \text{(in } \Omega = [0, L]^2\text{)}, \quad -k \frac{\partial T}{\partial n} = h(T - T_e) \ \ \text{(on } \partial\Omega\text{)}, \quad T(0, x) = T_0,$$

	Name	Dimension	Unit
Variables	T	temperature	K
	x	length	m
	t	time	s
Parameters	ρ (density)	mass/length^3	kg/m^3
	c (specific heat capacity)	$\frac{\text{energy}}{\text{temperature} \times \text{mass}}$	J/(K·kg)
	k (thermal conduction)	$\frac{\text{power}}{\text{length} \times \text{temperature}}$	W/(m·K)
	h (heat transfer coeff.)	$\frac{\text{power}}{\text{length}^2 \times \text{temperature}}$	W/(m^2·K)
	T_e (surrounding temp.)	temperature	K
	T_0 (initial temp.)	temperature	K
	L (domain size)	length	m

Scaling example

$$\rho c \frac{\partial T}{\partial t} = k \Delta T \ \text{(in } \Omega = [0, L]^2\text{)}, \quad -k \frac{\partial T}{\partial n} = h(T - T_e) \ \text{(on } \partial\Omega\text{)}, \quad T(0, x) = T_0,$$

- Scaling to dimensionless form simplifies equation. It "removes" the units/dimensions and can reduce the number of parameters (often to just one).
- Many such scalings possible. Here one for $u = u(\tau, y)$ is

$$\frac{\partial u}{\partial \tau} = \Delta u \ \text{(in } \tilde{\Omega} = [0, 1]^2\text{)}, \quad -\frac{\partial u}{\partial n} = bu \ \text{(on } \partial\tilde{\Omega}\text{)}, \quad u(0, y) = 1.$$

- The new quantities u, τ, y and b are *dimensionless*.
- T equation has 7 parameters; u equation has 1 parameter ($= b$).
- Scaling used is $T(t, x) = T_e + T_1 u(\tau, y)$ where

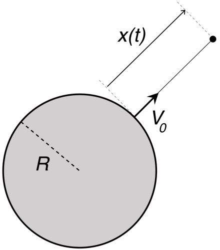
$$T_1 = T_0 - T_e, \qquad \tau = \frac{t}{t_0}, \qquad y = \frac{x}{L}, \qquad t_0 = \frac{L^2 \rho c}{k}, \qquad b = \frac{hL}{k}.$$

Dimensionless form

- The dimenionsless form makes the equations more clear and simplifies e.g. numerical computations. (Actual solution can easily be recovered from dimensionless solution.)
- Dimensionless numbers reveal what combination of original parameters actually matters, and the different physical regimes of the system.
- Some famous dimensionless numbers:
  - Reynolds number (velocity $\times$ length $/$ viscosity)
  - Biot number (heat transfer coeff $\times$ length $/$ thermal conductivity)
  - Péclet number (velocity $\times$ length $/$ diffusion coeff.)
  - Froude number (velocity $/\sqrt{\text{length} \times \text{gravitational acc}}$)
- Limiting behavior of equation for big/small dimensionless numbers is studied in asymptotic analysis.

How to scale to dimensionless form

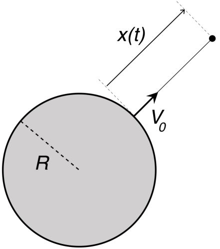
- Example: Projectile problem.



$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \quad x(0) = 0, \quad \frac{dx(0)}{dt} = v_0.$$

	Name	Dimension	Unit
Variables	x (position)	length	m
	t	time	s
Parameters	R (earth radius)	length	m
	g (gravitational const.)	length/time^2	m/s^2
	v_0 (initial velocity)	length/time	m/s

How to scale to dimensionless form



$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \quad x(0) = 0, \quad \frac{dx(0)}{dt} = v_0.$$

Scaling steps

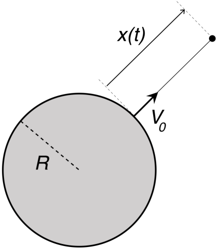
1. Introduce reference values for each variable and define the corresponding dimensionless variable:

$$\begin{aligned} y &= x/L, && L = \text{reference length}, \\ \tau &= t/T, && T = \text{reference time}, \end{aligned}$$

$\Rightarrow y, \tau$ are dimensionless.

- Can also include translation $y = (x - x_0)/L$, etc.
- L and T can also be thought of as new units of length and time.

How to scale to dimensionless form



$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \quad x(0) = 0, \quad \frac{dx(0)}{dt} = v_0,$$

$$y = x/L, \qquad \tau = t/T.$$

2. Rewrite differential equation in dimensionless variables,

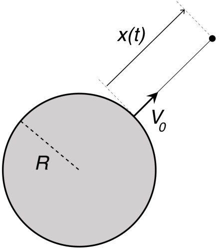
$$\frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}, \qquad x = Ly, \quad \text{etc.} \quad \Rightarrow$$

$$\frac{L}{T^2}\frac{d^2y}{d\tau^2} = -\frac{gR^2}{(Ly+R)^2}, \qquad \frac{L}{T}\frac{dy(0)}{d\tau} = v_0, \quad \Rightarrow$$

$$a\frac{d^2y}{d\tau^2} = -\frac{1}{(by+1)^2}, \qquad y(0) = 0, \quad y'(0) = c.$$

$$a = \frac{L}{T^2 g}, \qquad b = \frac{L}{R}, \qquad c = \frac{v_0 T}{L}, \qquad \text{dimensionless.}$$

How to scale to dimensionless form



Dimensionless form $(y = x/L,\ \tau = t/T)$

$$a\frac{d^2y}{d\tau^2} = -\frac{1}{(by+1)^2}, \qquad y(0) = 0, \quad y'(0) = c.$$

$$a = \frac{L}{T^2 g}, \qquad b = \frac{L}{R}, \qquad c = \frac{v_0 T}{L}.$$

3. Choose L, T that makes some of a, b, c equal 1. This can be done in several ways.

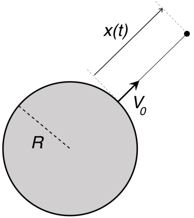
- Ex 1: $L = R,\ T = \sqrt{R/g} \quad \Rightarrow \quad a = b = 1$

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(y+1)^2}, \qquad y(0) = 0, \quad y'(0) = c = \frac{v_0}{\sqrt{gR}}.$$

- Ex 2: $L = R,\ T = R/v_0 \quad \Rightarrow \quad b = c = 1$

$$a\frac{d^2y}{d\tau^2} = -\frac{1}{(y+1)^2}, \qquad y(0) = 0, \quad y'(0) = 1, \qquad a = \frac{v_0}{Rg}.$$

How to scale to dimensionless form



Dimensionless form $(y = x/R,\ \tau = tv_0/R)$

$$a\frac{d^2y}{d\tau^2} = -\frac{1}{(y+1)^2}, \qquad y(0) = 0, \quad y'(0) = 1,$$

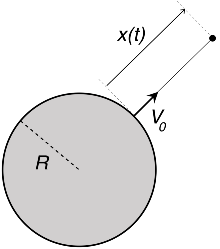
$$a = \frac{v_0}{Rg}.$$

- Here solution $y(\tau)$ only depends on τ and one parameter. All cases in original equation can be recovered by setting

$$x(t) = Ly(t/T),$$

  with T and L as chosen above.
- With more parameters and constraints one will have more dimensionless parameters. (E.g. If the time interval of interest for the projectile is specified as $t \in [0, t_0]$.)
- Many scalings possible. Choice of reference values should be tailored to solution regime of interest. Want to describe the situation at hand in the simplest way. Often a matter of taste.

Asymptotic analysis



Dimensionless form $(y = x/L,\ \tau = t/T)$

$$a\frac{d^2y}{d\tau^2} = -\frac{1}{(by+1)^2}, \qquad y(0) = 0, \quad y'(0) = c.$$

$$a = \frac{L}{T^2 g}, \qquad b = \frac{L}{R}, \qquad c = \frac{v_0 T}{L}.$$

- Want to study "extreme" regimes where a dimensionless number is very big or small. E.g. when $x/R \ll 1$ (projectile close to earth).
- Goal is to choose scaling such that the dimensionless parameters reveals what terms in the differential equation are big/small.
- Therefore, terms multiplied by dimensionless parameters should be $O(1)$.
- Then small terms can be approximated or neglected to get a simplified differential equation that describes the limiting behavior of the system.

Asymptotic analysis – choosing scaling

Example: Projectile close to earth, $x(t)/R \ll 1$.

- Since
$$a\frac{d^2y}{d\tau^2} = -\frac{1}{(by+1)^2}, \qquad y(0)=0, \quad y'(0)=c.$$
we want to scale the system so that y and y'' are both $O(1)$.
- When $x/R \ll 1$ we have
$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} = -\frac{g}{(x/R+1)^2} \approx -g \quad \Rightarrow \quad \frac{d^2y}{d\tau^2} = \frac{T^2}{L}\frac{d^2x}{dt^2} \approx -\frac{T^2 g}{L}.$$
Therefore choose $T^2 = L/g$!
- Similarly one can check that $x \in [0, v_0^2/g]$ for $0 \le t \le v_0/g$. Therefore
$$y = \frac{x}{L} \in \left[0, \frac{v_0^2}{gL}\right] \qquad \text{for} \qquad 0 \le \tau \le \frac{v_0}{gT} = \sqrt{\frac{v_0^2}{gL}} \qquad \Rightarrow \text{choose } L = \frac{v_0^2}{g}.$$
- This gives $a = c = 1$ and
$$\frac{d^2y}{d\tau^2} = -\frac{1}{(by+1)^2}, \qquad y(0)=0, \quad y'(0)=1, \qquad b = \frac{L}{R} \ll 1.$$

Asymptotic analysis – studying limiting behavior

With the choice above we have

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(\varepsilon y+1)^2}, \qquad y(0)=0, \quad y'(0)=1, \qquad \varepsilon = \frac{L}{R} \ll 1.$$

and also that $y'', y \sim O(1)$ for $\tau \sim O(1)$.

- Since $\varepsilon y = O(\varepsilon)$ we can approximate RHS by Taylor expansion

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(\varepsilon y+1)^2} = -1 + 2\varepsilon y - 3\varepsilon^2 y^2 + \cdots$$

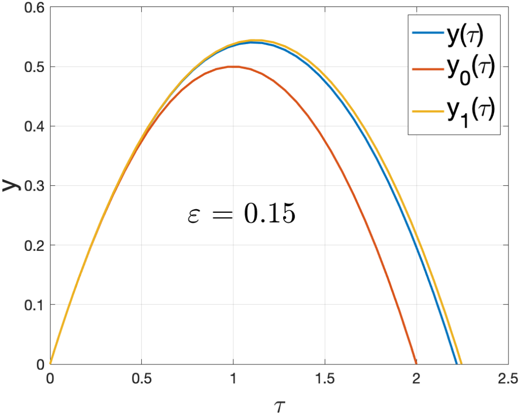
- Keeping only the leading term gives the usual close earth approximation of constant gravity (a parabola in τ)

$$\frac{d^2y}{d\tau^2} = -1 \quad \Rightarrow \quad y(\tau) = \tau - \frac{\tau^2}{2}.$$

- Keeping two terms gives a first order corrected solution,

$$\frac{d^2y}{d\tau^2} = -1 + 2\varepsilon y \quad \Rightarrow \quad y(\tau) = \frac{(\sqrt{2\varepsilon}-1)e^{\sqrt{2\varepsilon}\tau} - (\sqrt{2\varepsilon}+1)e^{-\sqrt{2\varepsilon}\tau} + 2}{4\varepsilon}.$$

Asymptotic analysis, example when $\varepsilon = 0.15$



$$y(\tau) = \text{Solution of } \frac{d^2y}{d\tau^2} = -\frac{1}{(\varepsilon y + 1)^2},$$

$$y_0(\tau) = \tau - \frac{\tau^2}{2}, \qquad y_1(\tau) = \frac{(\sqrt{2\varepsilon} - 1)e^{\sqrt{2\varepsilon}\tau} - (\sqrt{2\varepsilon} + 1)e^{-\sqrt{2\varepsilon}\tau} + 2}{4\varepsilon}.$$