SF2520 — Applied numerical methods

Lecture 11

Parabolic equations,
Intro hyperbolic equations

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2023-10-11

Today's lecture

- Numerical methods for parabolic equations
  - Recap
  - Convergence and accuracy of MoL
  - FEM for parabolic equations
- Hyperbolic equations, intro

Numerical methods for parabolic PDEs, recap

Want to construct numerical methods for heat equation

$$\begin{aligned} u_t - u_{xx} &= f, & x \in (a,b), \quad t > 0, \\ u(x,0) &= g(x), & x \in (a,b), \\ u(a,t) = \alpha(t), \quad u(b,t) &= \beta(t), & t > 0. \end{aligned}$$

- Semi-discretization, "method of lines" (just discretize in space)

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} + \boldsymbol{b}(t), \qquad \boldsymbol{u}(0) = \boldsymbol{g}, \qquad \text{where } A \text{ as in elliptic case,}$$

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}, \quad \boldsymbol{b}(t) = \begin{pmatrix} f(x_1,t) + \frac{\alpha(t)}{\Delta x^2} \\ f(x_2,t) \\ \vdots \\ f(x_{N-2},t) \\ f(x_{N-1},t) + \frac{\beta(t)}{\Delta x^2} \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_{N-2}) \\ g(x_{N-1}) \end{pmatrix}.$$

- Solve ODEs with standard ODE method

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t(A\boldsymbol{u}^n + \boldsymbol{b}(t_n)), \qquad \boldsymbol{u}_0 = \boldsymbol{g}.$$

Stability

- Absolutely stability requirement gives the CFL condition:

$$\frac{\Delta t}{\Delta x^2} < C \quad \text{for explicit ODE methods.}$$

- Severe condition due to stiffness of semi-discretized parabolic equations.
- Implicit methods therefore preferred.
  Examples:

$$(I - \Delta t A)\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t \boldsymbol{b}(t_{n+1}), \qquad \text{(Implicit Euler)},$$

$$\left(I - \frac{1}{2}\Delta t A\right)\boldsymbol{u}^{n+1} = \left(I + \frac{1}{2}\Delta t A\right)\boldsymbol{u}^n + \frac{1}{2}\Delta t\Big(\boldsymbol{b}(t_n) + \boldsymbol{b}(t_{n+1})\Big), \qquad \text{(Crank-Nicolson)}.$$

- Need to solve one sparse linear system per time step (tridiagonal in 1D, block tridiagonal in 2D). LU factorization recommended. Still cheaper than explicit methods, since fewer steps can be taken.

Convergence and accuracy

After semi-discretization we have,

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} + \boldsymbol{b}(t), \qquad \boldsymbol{u}(0) = \boldsymbol{g},$$

or, elementwise,

$$\frac{du_j(t)}{dt} = \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\Delta x^2} + f(x_j, t), \qquad j = 1, \ldots, N-1.$$

- ODE theory only gives convergence when $\Delta t \to 0$ for a fixed Δx, and then just to a solution of the semi-discrete problem.
- We want convergence when $\Delta t \to 0$ and $\Delta x \to 0$ at the same time!
- Usually one studies the limit $\Delta x \to 0$ when a fixed relation like $\Delta t = \alpha \Delta x^2$ or $\Delta t = \beta \Delta x$ is maintained for the time step. The paramaters α and β are chosen such that the method is absolutely stable.

Local truncation error

After semi-discretization we have,

$$\frac{du_j(t)}{dt} = \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\Delta x^2} + f(x_j, t), \qquad j = 1, \ldots, N-1.$$

- We consider now the fully discrete scheme, with

$$u_j^n \approx u(x_j, t_n), \qquad t_n = n\Delta t.$$

Example: Explicit Euler

$$u_j^{n+1} = u_j^n + \Delta t \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \Delta t\, f(x_j, t_n), \qquad j = 1, \ldots, N-1.$$

- Define the local truncation error (LTE), denoted ℓ_j^n ,as the residual when the exact solution is entered into the scheme.

Example: Explicit Euler

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \Delta t \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{\Delta x^2} + \Delta t\, f(x_j, t_n) + \underbrace{\ell_j^n}_{LTE}.$$

Note: ℓ_j^n depends on both Δt and Δx.

Local truncation error and order

After semi-discretization we have,

$$\frac{du_j(t)}{dt} = \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\Delta x^2} + f(x_j, t), \qquad j = 1, \dots, N-1.$$

- Terminology:
  - If $\ell_j^n = O(\Delta t(\Delta t^p + \Delta x^q))$ we say that the method is order p in time and order q in space.
  - If $p \geq 1$ and $q \geq 1$ the method is consistent.
- We consider the so called θ-scheme:

$$u_j^{n+1} = u_j^n + \Delta t \left[\theta \left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + f(x_j, t_{n+1}) \right) + (1-\theta) \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + f(x_j, t_n) \right) \right] \qquad j = 1, \dots, N-1.$$

- Derivations.

Local truncation error and order

- Terminology:
  - If $\ell_j^n = O(\Delta t(\Delta t^p + \Delta x^q))$ we say that the method is order p in time and order q in space.
  - If $p \geq 1$ and $q \geq 1$ the method is consistent.

- Taylor expansion gives

$$\ell_j^n = O(\Delta t(\Delta t + \Delta x^2)) \text{ for Explicit Euler } (\theta = 0), \text{ i.e. } p = 1,\ q = 2,$$

$$\ell_j^n = O(\Delta t(\Delta t + \Delta x^2)) \text{ for Implicit Euler } (\theta = 1), \text{ i.e. } p = 1,\ q = 2,$$

$$\ell_j^n = O(\Delta t(\Delta t^2 + \Delta x^2)) \text{ for Crank-Nicolson } (\theta = \tfrac{1}{2}), \text{ i.e. } p = q = 2.$$

Lax equivalence theorem

One can finally prove convergence using the fundamental theorem in numerical analysis of finite difference methods:

Theorem (Lax equivalence theorem)

For a well-posed, linear evolution PDE, a consistent finite difference method is convergent if and only if *it is stable, i.e.:*

$$\textit{consistency} \quad + \quad \textit{stability} \qquad \Leftrightarrow \qquad \textit{convergence}$$

Convergence of global error is one order lower than the LTE in Δt*,*

$$|u_j^n - u(x_j, t_n)| = O(\ell_j^n/\Delta t) = O(\Delta t^p + \Delta x^q).$$

- The stability definition may be different for different PDEs/methods/convergence modes. A typical definition is:

$$\sum_j (u_j^n)^2 \leq C \sum_j (u_j^0)^2, \qquad \text{with } C \text{ independent of } \Delta x,\, \Delta t \text{ and } n \leq \frac{T}{\Delta t}.$$

  For MoL, absolute stability of the ODE method is a necessary condition.
- Holds also for (nonlinear) ODEs and then often called the *Dahlquist equivalence theorem* (after the famous KTH professor).

Accuracy, conclusions

- Method converges as $O(\Delta t^p + \Delta x^q)$ when order p in time and order q in space.
- If $\Delta t = \alpha \Delta x^2$ the total order is $\min(2p, q)$ in Δx
- If $\Delta t = \beta \Delta x$ the total order is $\min(p, q)$ in Δx
- Thus,
  - Central differences + Explicit Euler
    - Order 1 in time, 2 in space.
    - Order 2 in Δx if $\Delta t = \alpha \Delta x^2$ (needed for stability, expensive).
  - Central differences + Implicit Euler
    - Order 1 in time, 2 in space.
    - Order 1 in Δx if $\Delta t = \beta \Delta x$ (allowed, unconditionally stable).
  - Crank-Nicolson
    - Order 2 in time, 2 in space.
    - Order 2 in Δx if $\Delta t = \beta \Delta x$ (allowed, unconditionally stable).

FEM for parabolic PDEs

Consider

$$\begin{aligned} u_t - u_{xx} &= f, && x \in (a,b), \quad t > 0, \\ u(x,0) &= g(x), && x \in (a,b), \\ u(a,t) = 0, \quad u(b,t) &= 0, && t > 0. \end{aligned}$$

We will solve this based on a semi-discretization with FEM.

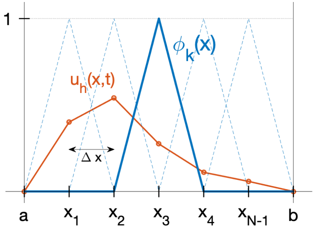
- Discretize the interval

$$x_k = a + k\Delta x, \qquad \Delta x = \frac{b-a}{N}.$$

- Make a time-dependent ansatz using hat functions

$$u_h(x,t) := \sum_{k=1}^{N-1} u_k(t)\phi_k(x) \approx u(x,t),$$

- $u_h(a,t) = u_h(b,t) = 0$ so BC automatically satisified.



- Let $r_h(x,t)$ be the residual,

$$\begin{aligned} r_h(x,t) &= \partial_t u_h(x,t) - \partial_{xx} u_h(x,t) - f(x,t) \\ &= \sum_{k=1}^{N-1} u_k'(t)\phi_k(x) - \sum_{k=1}^{N-1} u_k(t)\phi_k''(x) - f(x,t). \end{aligned}$$

- Galerkin principle for fixed t: Find $\{u_k(t)\}$ such that

$$\int_a^b r_h(x,t)\phi_\ell(x)dx = 0, \qquad \ell = 1,\ldots,N-1.$$

- We get, for $\ell = 1,\ldots,N-1$,

$$\begin{aligned} 0 &= \sum_{k=1}^{N-1} u_k' \int_a^b \phi_k\phi_\ell dx - \sum_{k=1}^{N-1} u_k \int_a^b \phi_k''\phi_\ell dx - \int_a^b f\phi_\ell dx \\ &= \sum_{k=1}^{N-1} u_k' \underbrace{\int_a^b \phi_k\phi_\ell dx}_{m_{\ell,k}} + \sum_{k=1}^{N-1} u_k \underbrace{\int_a^b \phi_k'\phi_\ell' dx}_{-a_{\ell,k}} - \underbrace{\int_a^b f\phi_\ell dx}_{f_\ell(t)}. \end{aligned}$$

FEM for parabolic PDEs, cont.

We have thus

$$\sum_{k=1}^{N-1} u_k' m_{\ell,k} = \sum_{k=1}^{N-1} u_k a_{\ell,k} + f_\ell, \qquad \ell = 1, \dots, n,$$

or in matrix form,

$$M\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} + \boldsymbol{b}(t),$$

where

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}, \qquad \boldsymbol{b}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_{N-1}(t) \end{pmatrix},$$

$$\underbrace{A = \frac{1}{\Delta x}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}}_{\text{"stiffness matrix"}}, \qquad \underbrace{M = \frac{\Delta x}{6}\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}}_{\text{"mass matrix"}}.$$

FEM for parabolic PDEs, cont.

Initial data: $u(x,0) = g(x)$

- At the initial data we let the residual $r_h(x,t)$ be
$$r_h(x,t) = u_h(0,x) - g(x).$$
- Galerkin principle then gives
$$0 = \sum_{k=1}^{N-1} u_k(0) \underbrace{\int_a^b \phi_k \phi_\ell dx}_{m_{\ell,k}} - \underbrace{\int_a^b g\phi_\ell dx}_{g_\ell}.$$
- In matrix form
$$M\boldsymbol{u}(0) = \boldsymbol{g}, \qquad \boldsymbol{g} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{N-1} \end{pmatrix}.$$
- Final form of semi-discretization with FEM is thus
$$M\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} + \boldsymbol{b}(t), \qquad M\boldsymbol{u}(0) = \boldsymbol{g}.$$

  Note: Both A and M sparse, tridiagonal in 1D.

Hyperbolic PDEs

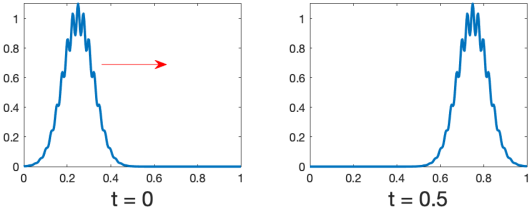
We consider the simplest hyperbolic PDE, the 1D "advection" or "transport" equation for $u = u(x, t)$

$$\begin{aligned} u_t + a u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= g(x), \end{aligned}$$

where a is a constant and $g \in C^1(\mathbb{R})$.

- Model for transport and one-way wave propagation.
- Solution easy to write down,

$$u(t, x) = g(x - at) \qquad \text{(check: } u_t = -ag' \text{ and } a u_x = ag'\text{)}$$



- a represents speed of propagation.

Hyperbolic PDEs

Other more complicated versions of advection equation:

- $u_t + a(x)u_x = 0$, (variable coefficient a)
- $\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$, (system)
- $\boldsymbol{u}_t + A(x)\boldsymbol{u}_x = 0$, (system+variable coefficient)
- $\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y = 0$, (system, 2D)
- $\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0$, (nonlinear)
- Combinations of the above.

Classification, requirements for hyperbolicity:

- $u_t + a(x)u_x = 0$, a real
- $\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$, A diagonalizable with real eigenvalues
- $\boldsymbol{u}_t + A(x)\boldsymbol{u}_x = 0$, $A(x)$ ———"——— for all x
- $\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y = 0$, $\alpha_1 A + \alpha_2 B$ ———"——— for all α_1, α_2
- $\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0$, $J(\boldsymbol{u})$ ———"——— for all $\boldsymbol{u}(x, t)$

  Typical case is that A, B, J are real symmetric matrices.

Acoustic waves

The wave equation in 1D is

$$u_{tt} = c^2 u_{xx},$$

where u is sound pressure deviation and c is the speed of propagation of waves.

- Can be written as a system of hyperbolic equations. Let

$$\boldsymbol{u} = \begin{pmatrix} u_t \\ u_x \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{u}_t = \begin{pmatrix} u_{tt} \\ u_{xt} \end{pmatrix} = \begin{pmatrix} c^2 u_{xx} \\ u_{xt} \end{pmatrix} \qquad \boldsymbol{u}_x = \begin{pmatrix} u_{xt} \\ u_{xx} \end{pmatrix},$$

  so that

$$\boldsymbol{u}_t \underbrace{- \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix}}_{A} \boldsymbol{u}_x = 0.$$

- Easy to check that A is diagonalizable and eigenvalues are $\pm c$.
- Called the "first order form" of the wave equation.