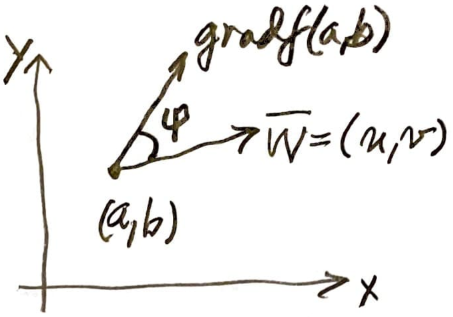
Räkneövning 27/9 – Analys A

Riktningsderivata i punkten (a,b) i riktning $\overline{w}$.



$$f'_{\overline{w}}(a,b) = \operatorname{grad} f(a,b) \cdot \frac{\overline{w}}{|\overline{w}|}$$

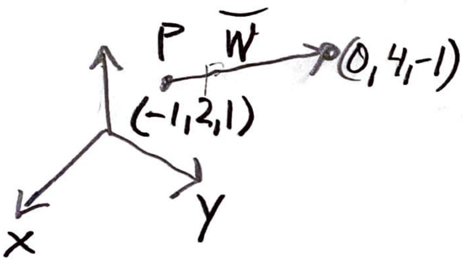
; maximal då $\varphi = 0$, dvs i gradientens riktning. Maximalt värde på $f'_{\overline{w}}(a,b)$ är

$$|\operatorname{grad} f(a,b)| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

ÖPB2 2.32) Beräkna riktningsderivatan till

$$f(x,y,z) = \frac{(x^2+y^2)z}{2-z^2}$$

i $P=(-1,2,1)$ mot $(0,4,-1)$



$$\overline{w} = (0,4,-1)-(-1,2,1) = (1,2,-2)$$

$$|\overline{w}| = \sqrt{1+4+4} = 3$$

$$f'_{\overline{w}}(P) = \operatorname{grad} f\big|_P \cdot \frac{\overline{w}}{|\overline{w}|} = \frac{1}{3}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_P \cdot (1,2,-2)$$

$$\frac{\partial f}{\partial x} = \frac{2xz}{2-z^2} \Rightarrow \frac{\partial f}{\partial x}\Big|_{P=(-1,2,1)} = \underline{\underline{-2}}$$

$$\frac{\partial f}{\partial y} = \frac{2yz}{2-z^2} \Rightarrow \frac{\partial f}{\partial y}\Big|_{P=(-1,2,1)} = \underline{\underline{4}}$$

$$\frac{\partial f}{\partial z} = \frac{(2-z^2)(x^2+y^2)-(x^2+y^2)z(-2z)}{(2-z^2)^2} = \frac{(x^2+y^2)(2+z^2)}{(2-z^2)^2} \Rightarrow$$

$$\frac{\partial f}{\partial z}\Big|_{P=(-1,2,1)} = \underline{\underline{15}}\ ; \quad f'_{\overline{w}}(P) = \frac{1}{3}(-2,4,15)\cdot(1,2,-2) = \frac{1}{3}(-2+8-30) = \underline{\underline{-8}}$$

$\therefore$ Riktningsderivatan är -8

ÖPB2 (2)

2.52 Lös diff.ekv. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=x^2+y^2$ där

$u(x,y)$ bara beror av $r=\sqrt{x^2+y^2}$, dvs. $u(x,y)=f(r)$.

Lösning: $\frac{\partial u}{\partial x}=f'(r)\frac{\partial r}{\partial x}$; $\frac{\partial u}{\partial y}=f'(r)\frac{\partial r}{\partial y}$ (PB2 sid 61+)

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r} \text{ och } \frac{\partial r}{\partial y}=\frac{y}{r}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right)=\frac{1}{r}f'(r)+x\frac{\partial}{\partial x}\left(\frac{1}{r}f'(r)\right)$$

$$=\frac{1}{r}f'(r)+x\frac{d}{dr}\left(\frac{1}{r}f'(r)\right)\underbrace{\frac{\partial r}{\partial x}}_{x/r}=$$

$$=\frac{1}{r}f'(r)+x\left(-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right)\cdot\frac{x}{r}=$$

$$=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$$

P.s.s. $\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}f'(r)-\frac{y^2}{r^3}f'(r)+\frac{y^2}{r^2}f''(r)$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{r}f'(r)-\frac{\overbrace{(x^2+y^2)}^{r^2}}{r^3}f'(r)+\frac{\overbrace{(x^2+y^2)}^{r^2}}{r^2}f''(r)=$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=x^2+y^2=r^2 \Rightarrow$$

$$\Rightarrow \boxed{f''(r)+\frac{1}{r}f'(r)=r^2}$$

2.52 forts.

Integrerande faktor $e^{\ln r} = r \Rightarrow$

$$rf''(r) + f'(r) = r^3 \Rightarrow \frac{d}{dr}\left[rf'(r)\right] = r^3 \Rightarrow$$

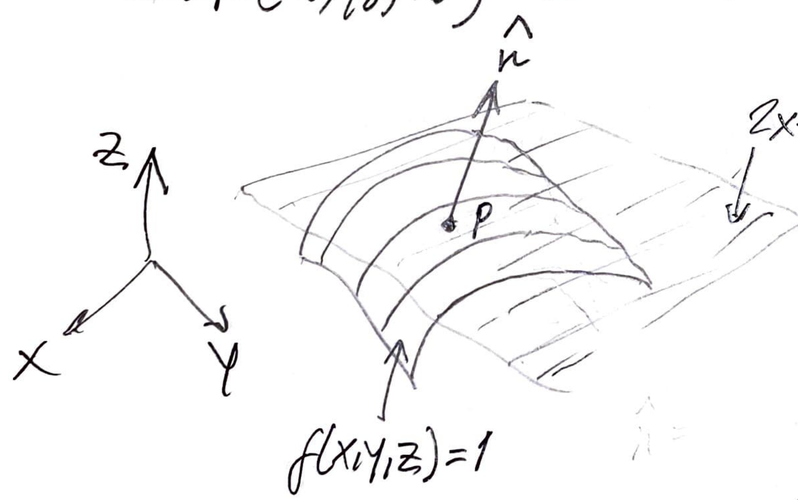
$$\Rightarrow \{\text{integrera}\} \Rightarrow rf'(r) = \frac{r^4}{4} + C \Rightarrow f'(r) = \frac{r^3}{4} + \frac{C}{r}$$

$$\therefore \boxed{f(r) = \frac{r^4}{16} + C\ln r + D}$$

ÖPB2 2.86

Bestäm konstanten c, så att planet $2x+2y+z=c$ tangerar ytan $f(x,y,z) = x+y^2+z^4 = 1$.

Lösning: Låt $P=(x_0, y_0, z_0)$ vara tangeringspunkten.



planets normal $= \underline{(2,2,1)}$

funktionsytans normal i P: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)_P =$

$$= (1, 2y, 4z^3)\Big|_{P=(x_0,y_0,z_0)} = \underline{(1, 2y_0, 4z_0^3)}$$

Planets och funktionsytans normal parallella i P.

nästa blad

[illegible]
2.86 forts

$\therefore k(2,2,1) = (1, 2y_0, 4z_0^3) \Rightarrow$

$k = 1/2$, $y_0 = k = 1/2$, $4z_0^3 = k = \frac{1}{2} \Rightarrow z_0 = \frac{1}{2}$

$\therefore P = (x_0, y_0, z_0) = (x_0, 1/2, 1/2)$

P ligger både i planet och på funktionsytan

$$\Rightarrow \begin{cases} 2x_0 + 2 \cdot y_0 + z_0 = C \\ x_0 + y_0^2 + z_0^4 = 1 \end{cases} \Rightarrow \begin{cases} 2x_0 + 1 + 1/2 = C & (1) \\ x_0 + \frac{1}{4} + \frac{1}{16} = 1 & (2) \end{cases}$$

$(2) \Rightarrow x_0 = 11/16$

$(1) \Rightarrow C = 2 \cdot 11/16 + 1 + 1/2 = \frac{11+8+4}{8} = \underline{\underline{\frac{23}{8}}}$

$$\boxed{\therefore \; C = 23/8}$$