(11) Förra gången:

- X och Y oberoende om

$p(x,y) = p(x)p(y)$ alla x,y (diskret)

$f(x,y) = f(x)f(y)$ — " — (kont.)

- $$E[g(X,Y)] = \begin{cases} \sum_i \sum_j g(x_i,y_j)p(x_i,y_j) \\ \iint g(x,y)f(x,y)\,dx\,dy \end{cases}$$

$\Rightarrow E[X+Y] = E[X] + E[Y]$ alltid

Sats: X och Y oberoende $\Rightarrow E[XY] = E[X]E[Y]$.

Bevis. $$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty} yf_Y(y)\underbrace{\left(\int_{-\infty}^{\infty} xf_X(x)\,dx\right)}_{E[X]}dy =$$

$$= E[X]\int_{-\infty}^{\infty} yf_Y(y)\,dy = E[X]E[Y].$$

Kovarians

Antag X och Y beroende. Hur mycket?

Sätt $\mu_x = E[X]$, $\mu_y = E[Y]$.

Def. Kovariansen mellan X och Y ges av

$$\mathrm{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)].$$

Räkneregel: $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$.

Bevis. $$\mathrm{Cov}(X,Y) = E[XY - \mu_y X - \mu_x Y + \mu_x\mu_y] =$$

$$= E[XY] - \mu_y\underbrace{E[X]}_{\mu_x} - \mu_y\underbrace{E[Y]}_{\mu_y} + \mu_x\mu_y = E[XY] - \mu_x\mu_y$$

Följd: X och Y ober. $\Rightarrow$ $Cov(X,Y)=0$.

Dock: $Cov(X,Y)=0 \not\Rightarrow$ X och Y ober.

Ex. X s.v. med $P(X=-1)=P(X=0)=P(X=1)=1/3$

$$Y=\begin{cases}0 & \text{om } X\neq 0\\ 1 & \text{annars.}\end{cases}$$

$XY\equiv 0 \Rightarrow E[XY]=0$

$E[X]=0$ $\therefore$ $Cov(X,Y)=0$. Ej oberoende!

Egenskaper:

(i) $Cov(X,Y)=Cov(Y,X)$

(ii) $Cov(X,X)=V(X)$

(iii) $Cov(aX+b,Y)=a\,Cov(X,Y)$

(iv) $Cov(\sum X_i,\sum Y_j)=\sum_i\sum_j Cov(X_i,Y_j)$

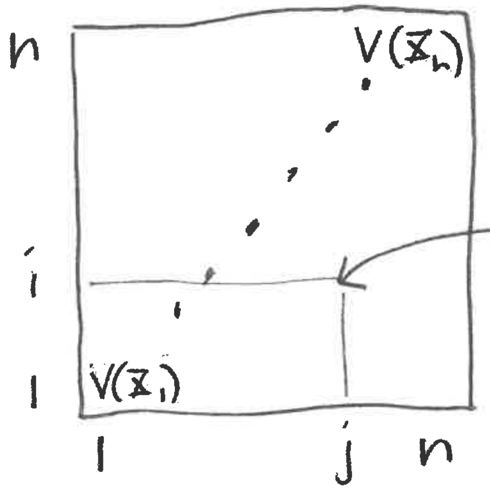
Bevis. (i) och (ii) följer från def.

(iii) $Cov(aX+b,Y)=E[(aX+b-(a\mu_x+b))(Y-\mu_y)]=$
$=aE[(X-\mu_x)(Y-\mu_y)]=a\,Cov(X,Y)$

(iv) $Cov(X_1+X_2,Y)=E[(X_1+X_2-(\mu_1+\mu_2))(Y-\mu_y)]=$
$=E[(X_1-\mu_1)(Y-\mu_y)]+E[(X_2-\mu_2)(Y-\mu_y)]=$
$=Cov(X_1,Y)+Cov(X_2,Y)$

Följd. $V(\sum_{i=1}^n X_i)=\sum_{i=1}^n V(X_i)+2\sum\sum_{i<j} Cov(X_i,X_j)$

Bevis. $V(\sum_{i=1}^n X_i)\overset{(ii)}{=}Cov(\sum_{i=1}^n X_i,\sum_{j=1}^n X_j)\overset{(iv)}{=}$
$=\sum_{i=1}^n\sum_{j=1}^n Cov(X_i,X_j)=(*)$



Kovariansmatris

$Cov(\bar{X}_i, \bar{X}_j)$

$$(*) = \sum_{i=1}^{n} V(\bar{X}_i) + \underbrace{\sum\sum_{i \neq j} Cov(\bar{X}_i, \bar{X}_j)}_{2\sum\sum_{i<j} Cov(\bar{X}_i, \bar{X}_j)}$$

(eftersom $Cov(\bar{X}_i, \bar{X}_j) = Cov(\bar{X}_j, \bar{X}_i)$)

Följd. $\bar{X}_1, \ldots, \bar{X}_n$ ober. $\Rightarrow V\left(\sum_{i=1}^{n} \bar{X}_i\right) = \sum_{i=1}^{n} V(\bar{X}_i)$.

Ex. $\bar{X}_1, \ldots, \bar{X}_n$ ober. $Be(p)$

$V(\bar{X}_i) = pq$

$\bar{Y} = \sum_1^n \bar{X}_i \sim Bin(n, p)$

$V(\bar{Y}) = \sum_1^n V(\bar{X}_i) = npq$

Ex. $\bar{X}_1, \ldots, \bar{X}_r$ ober. $ffg(p)$

$V(\bar{X}_i) = \frac{q}{p^2}$

$\bar{Y} = \sum_1^r \bar{X}_i \sim NegBin(r, p)$

$V(\bar{Y}) = \sum_1^n V(\bar{X}_i) = \frac{rq}{p^2}$

Ex. $\bar{X}_1, \ldots, \bar{X}_n$ ober. $E[\bar{X}_i] = \mu$, $V(\bar{X}_i) = \sigma^2$

$\overline{\bar{X}}_n = \frac{1}{n}(\bar{X}_1 + \ldots + \bar{X}_n)$

Förra gången: $E[\bar{X}] = \mu$

$$V(\bar{X}) = \frac{1}{n^2}\sum_{1}^{n} V(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

Korrelation

Ex. X och Y stok. var., pris i kr.

$100X$, $100Y$: pris i öre

$$Cov(100X, 100Y) = 100\, Cov(X, 100Y) = 100^2 Cov(X, Y)$$

Men: samma beroende.

Skal-fritt mått på beroende:

Def. Korrelationskoefficienten ges av

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}}.$$

Ex. $$\rho(100X, 100Y) = \frac{Cov(100X, 100Y)}{\sqrt{V(100X)\cdot V(100Y)}} =$$

$$= \frac{100^2 Cov(X,Y)}{\sqrt{100^2 V(X)\cdot 100^2 V(Y)}} = \frac{Cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}} = \rho(X,Y)$$

Egenskaper (boken s. 123):

(i) $-1 \leq \rho(X,Y) \leq 1$

(ii) $\rho = 1 \Leftrightarrow Y = aX + b$, a, b konstanter, $a > 0$.

(iii) $\rho = -1 \Leftrightarrow Y = -aX + b$, $a > 0$.

(iv) X och Y ober. $\Rightarrow \rho(X,Y) = 0$.

$\not\Leftarrow$

Ex. Kovarians/korrelation mäter linjärt beroende:

X stok. var. med symmetrisk fördeln. kring origo $(f_X(x) = f_X(-x))$.

$Y = X^2$

$$\mathrm{Cov}(X,Y) = E[X \cdot X^2] - \underbrace{E[X]}_{=0} E[Y] = E[X^3] = 0$$

$$\left(\int_{-\infty}^{\infty} x^3 f_X(x)\,dx = 0 \right)$$

$\therefore$ Cov = korr = 0

Men: beroende.