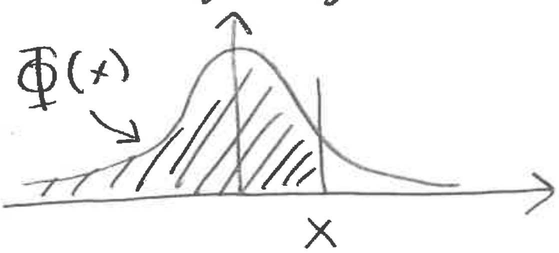
9 Repetition:

$$X \sim N(\mu,\sigma^2) \iff f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x\in\mathbb{R}.$$

Förra gången: $Z \sim N(0,1)$

E[Z] V(Z)



Z:s fördelningsfkt: $\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\,dt.$

Allmän normalfördelning

$X \in N(\mu,\sigma^2)$, $\mu\in\mathbb{R}$, $\sigma^2>0$.

Sats: $X\sim N(\mu,\sigma^2)$, $Y=aX+b$, $a,b\in\mathbb{R}$, $a\neq 0$

$\Rightarrow Y\sim N(a\mu+b, a^2\sigma^2)$.

Bevis. $F_Y(y)=P(Y\le y)=P(aX+b\le y) \overset{a>0}{=}$

$= P\left(X\le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$

$\Rightarrow f_Y(y)=F_Y'(y)=\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$

$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ ger $\underbrace{f_Y(y)=\frac{1}{\sqrt{2\pi\sigma^2}\,a}\, e^{-\frac{(y-b-a\mu)^2}{2\sigma^2 a^2}}}_{\text{täthet för } N(a\mu+b,\,a^2\sigma^2)}$

Följd: $X\sim N(\mu,\sigma^2)$, $Z=\frac{X-\mu}{\sigma} \Rightarrow Z\sim N(0,1)$

Tag $a=\frac{1}{\sigma}$ och $b=-\frac{\mu}{\sigma}$ i satsen.

Följd: $X\sim N(\mu,\sigma^2) \Rightarrow E[X]=\mu$, $V(X)=\sigma^2$.

Bevis: Låt $Z = \frac{X-\mu}{\sigma}$. Vet: $E[Z]=0$, $V(Z)=1$.

$$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{E[X]-\mu}{\sigma} = 0 \Rightarrow E[X]=\mu$$

$$V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}V(X) = 1 \Rightarrow V(X)=\sigma^2$$

Ex. $X \sim N(1,4)$.

$$P(X \le 4) = P\Big(\underbrace{\frac{X-1}{2}}_{N(0,1)} \le \underbrace{\frac{4-1}{2}}_{3/2}\Big) = \Phi(3/2) = 0.9332.$$

Ex. Rattfylleri, gräns 1‰.

Mätvärde $X \sim N(\mu, 0.013)$ ‰

Döms om $X > 1.15$. (μ: sant värde)

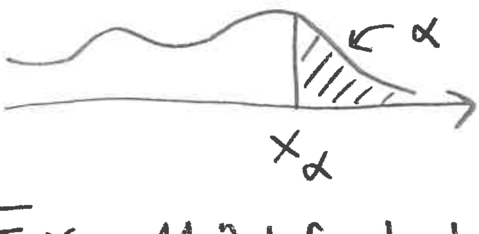
Slh att person som har 0.9‰ döms?

$$P(X > 1.15) = P\Big(\underbrace{\frac{X-0.9}{\sqrt{0.013}}}_{Z\sim N(0,1)} > \underbrace{\frac{1.15-0.9}{\sqrt{0.013}}}_{2.19}\Big) =$$

$$= 1 - P(Z \le 2.19) = 1 - \Phi(2.19) = 1 - 0.9857 = 0.0143.$$

Hitta x s.a. $P(X > x) = 0.01$.

Allmänt. Ett tal x s.a. $P(X>x)=\alpha$ kallas α-kvantil. Beteckning: x_α.



$$1 - F(x_\alpha) = \alpha \Leftrightarrow F(x_\alpha) = 1-\alpha$$

Ex. Mätfelet: Vi söker $x_{0.01}$.

Först: $Z \sim N(0,1)$

α-kvantilen betecknas här λ_α.

λ_α löser alltså $\Phi(\lambda_\alpha) = 1-\alpha$.

Tabell: $\lambda_{0.01} = 2.3263$

$\therefore P(Z > 2.3263) = 0.01$

Ex. Mätfelet.

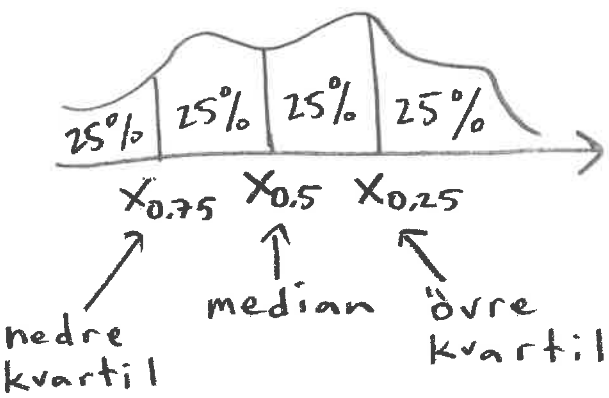
$$P(X > x_{0.01}) = P\Big(\underbrace{\frac{X-0.9}{\sqrt{0.013}}}_{Z\sim N(0,1)} > \frac{x_{0.01}-0.9}{\sqrt{0.013}}\Big) = 0.01$$

$$\therefore \frac{x_{0.01}-0.9}{\sqrt{0.013}} = \lambda_{0.01} = 2.3263$$

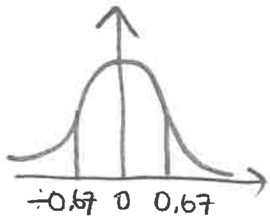
$$\Rightarrow x_{0.01} = 0.9 + \sqrt{0.013}\cdot 2.3263 \approx 1.17$$

$$P(Z > \lambda_\alpha) = \alpha \iff P(\underbrace{\sigma Z + \mu}_{N(\mu,\sigma^2)} > \underbrace{\sigma\lambda_\alpha + \mu}_{\substack{\alpha\text{-kvantil} \\ \text{för } N(\mu,\sigma^2)}}) = \alpha$$

Allmänt för slh.fördelningar:



Ex. $Z \sim N(0,1)$



Median $= 0$

Övre kvartil $= 0.67$

Nedre — " — $= -0.67$

$X \sim N(\mu, \sigma^2)$

Median $= \mu$

Övre kvartil $= \mu + 0.67\sigma$

Nedre — " — $= \mu - 0.67\sigma$

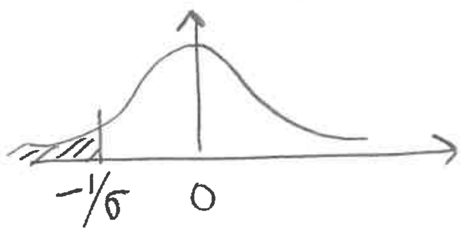
Ex. X = prisförändring per dag hos aktie

$X \sim N(0, \sigma^2)$

Köp på morgonen, sälj på kvällen.

$P(\text{förlora mer än 1 kr}) = P(X < -1) =$

$= P(\underbrace{\frac{X-0}{\sigma}}_{Z \sim N(0,1)} < \frac{-1}{\sigma}) = \Phi(-\frac{1}{\sigma}) = 1 - \Phi(\frac{1}{\sigma})$



$\sigma = 1$ ger $1 - 0.8413 = 0.1587$

$\sigma = 2$ ger $1 - 0.6915 = 0.3085$

Bestäm σ s.a. $P(\text{förlora mer än 1 kr}) \leq 5\%$.

$1 - \Phi(\frac{1}{\sigma}) \leq 0.05 \iff \Phi(\frac{1}{\sigma}) \geq 0.95 \iff$

$\frac{1}{\sigma} \geq \chi_{0.05} = 1.645 \iff \sigma \leq \frac{1}{1.645} = 0.6079$