13 Max och min av ett antal stok. var.

$X_1, \dots, X_n$ ober. stok. var.

$X_{max} = \max\{X_1, \dots, X_n\}$ $X_{min} = \min\{X_1, \dots, X_n\}$

Fördelning för X_{max} och X_{min}?

- $F_{X_{max}}(x) = P(X_{max} \le x) = P(X_1 \le x, \dots, X_n \le x) =$
$$= P(X_1 \le x) \cdots P(X_n \le x) = \prod_{i=1}^{n} F_{X_i}(x).$$

- $F_{X_{min}}(x) = P(X_{min} \le x) = 1 - P(X_{min} > x) =$
$$= 1 - P(X_1 > x, \dots, X_n > x) = 1 - P(X_1 > x) \cdots P(X_n > x) =$$
$$= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(x))$$

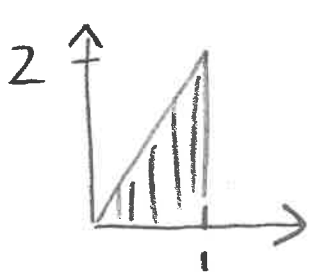
Om $X_1, \dots, X_n$ har samma fördeln. $F(x)$:

$$F_{X_{max}}(x) = (F(x))^n, \quad F_{X_{min}}(x) = 1 - (1 - F(x))^n$$

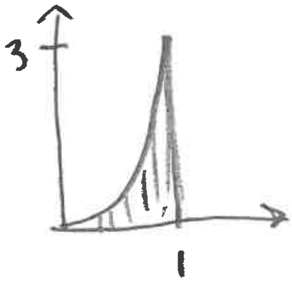
Ex. Oberoende slumptal, $X_i \sim Re(0,1)$.

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & x \in (0,1) \\ 1, & x > 1 \end{cases}$$

$$F_{X_{max}}(x) = x^n, \quad x \in (0,1) \Rightarrow f_{X_{max}}(x) = n x^{n-1}$$

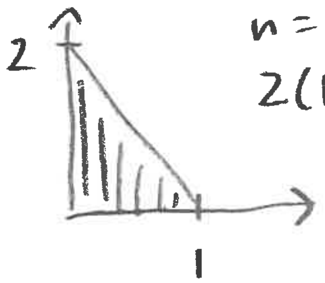


$n = 2$: $2x$

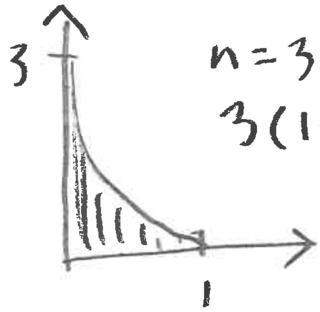


$n = 3$: $3x^2$

$$F_{X_{min}}(x) = 1-(1-x)^n, \quad x \in (0,1) \Rightarrow f_{X_{min}}(x) = n(1-x)^{n-1}$$



$n=2$: $2(1-x)$



$n=3$: $3(1-x)^2$

Ex. $X_1, \dots, X_n$ ober $Exp(\lambda)$ $\left(f(x) = \lambda e^{-\lambda x}, x \geq 0\right)$

$$F_{X_i}(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

$$F_{X_{max}}(x) = (1-e^{-\lambda x})^n$$

$$F_{X_{min}}(x) = 1-(1-(1-e^{-\lambda x}))^n = 1-e^{-n\lambda x}$$

$$\therefore X_{min} \sim Exp(n\lambda) \quad \left(E[X_{min}] = \frac{1}{n\lambda}\right)$$

Ex. 5 komponenter/maskin

X_i = livslängd komp. i maskin A (år)

Y_i = — " — B

$X_i \sim Exp(2)$ $(E[X] = 0{,}5)$.

$Y_i \sim Exp(0.4)$ $(E[X] = 2{,}5)$

A: räcker att en fungerar (parallellkoppl.)

B: alla måste fungera (seriekoppl.)

Slh. maskinen fungerar $> 0{,}5$ år?

(A) Z_A = tid tills alla komp. går sönder

$= \max\{X_1, \dots, X_5\}$

$P(Z_A > 0.5) = 1 - P(Z_A \leq 0.5) = 1 - F_{Z_A}(0.5) =$

$= 1 - (F_{X_i}(0.5))^5 = 1 - (1 - e^{-2 \cdot 0.5})^5 \approx 0.90$

(B) Z_B = tid tills första komp. går sönder

$= \min\{Y_1, \dots, Y_5\}$

$Z_B \sim Exp(5 \cdot 0.4) = Exp(2)$

$P(Z_B > 0.5) = e^{-2 \cdot 0.5} \approx 0.37$

Felfortplantningsformlerna

X stok. var. $E[X] = \mu$, $Var\, X = \sigma^2$

$g(x)$ fkt. av x.

$E[g(X)] = ?$ $V(g(X)) = ?$

- Om fördeln. för X känd:

$$E[g(X)] = \begin{cases} \sum_i g(x_i) p(x_i) & \text{diskret} \\ \int g(x) f(x)\, dx & \text{kont.} \end{cases}$$

- $g(x) = ax + b$

$E[g(X)] = E[aX + b] = aE[X] + b = a\mu + b = g(\mu)$

$V(g(X)) = a^2 V(X) = (g'(\mu))^2 \cdot \sigma^2$

Allmänt. Taylorutv. kring $x = \mu$:

$$g(x) = g(\mu) + (x-\mu) g'(\mu) + \underbrace{\frac{(x-\mu)^2}{2!} g''(\mu) + \dots + \frac{(x-\mu)^n}{n!} g^{(n)}(\mu) + \dots}_{R(x)}$$

$g(x) \approx$ linjär $\Rightarrow R(x)$ liten

$$\Rightarrow g(x) \approx g(\mu) + (x-\mu) g'(\mu)$$

$$E[g(X)] \approx E[g(\mu) + (X-\mu) g'(\mu)] =$$

$$= g(\mu) + (E[X] - \mu) g'(\mu) = g(\mu)$$

$$V(g(X)) \approx V(g(\mu) + (X-\mu) g'(\mu)) = (g'(\mu))^2 \cdot \sigma^2$$

$\therefore$ $E[g(X)] \approx g(\mu)$

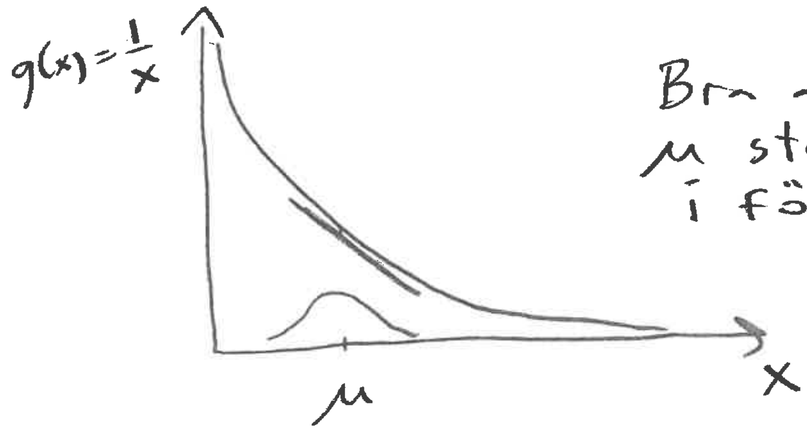
$V(g(X)) \approx (g'(\mu))^2 \cdot \sigma^2$

(Felfortpl.-formlerna för en stok. var.)

Ex. $Y = \frac{1}{X}$ $\quad g(x) = \frac{1}{x}$ $\quad g'(x) = -\frac{1}{x^2}$

$$E\left[\frac{1}{X}\right] \approx g(\mu) = \frac{1}{\mu} = \frac{1}{E[X]}$$

$$V\left(\frac{1}{X}\right) \approx (g'(\mu))^2 \cdot V(X) = \frac{V(X)}{E[X]^4}$$



Bra approximation om μ stort och σ^2 litet i förhållande till μ.

Varning: $g(x) = x^2$ ger $E[X^2] = E[X]^2$

$\Rightarrow V(X) = 0$ för alla stok. var...