14

Ex. n myntkast

$$X_i = \begin{cases} 1 & \text{om krona i kast } i\text{, slh } p = 1/2 \\ 0 & \text{annars} \end{cases}$$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{\#\text{ kronor}}{\#\text{ kast}} = \text{andel kronor}$$

$$E[X_i] = p \quad V(X_i) = p(1-p)$$

$$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{1}^{n} X_i\right] = \frac{1}{n}\sum_{1}^{n} E[X_i] = p$$

$$V(\bar{X}_n) = V\left(\frac{1}{n}\sum_{1}^{n} X_i\right) = \frac{1}{n^2}\sum_{1}^{n} V(X_i) = \cancel{\frac{p(1-p)}{n^2}}$$

Simulering: Ser ut som att $\bar{X}_n \to p$ då $n \to \infty$.

Stora talens lag

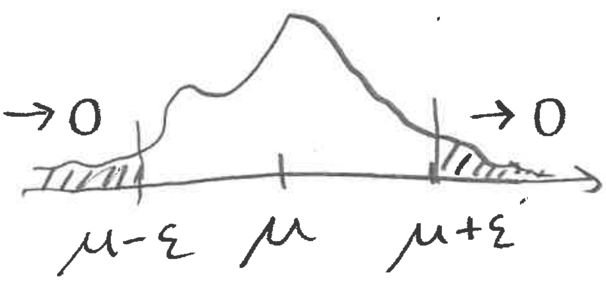
Antag $X_1, X_2, \ldots$ oberoende.

$E[X_i] = \mu$, $V(X_i) = \sigma^2$.

$\bar{X}_n = \frac{1}{n}\sum_{1}^{n} X_i$, $E[\bar{X}_n] = \mu$, $V(\bar{X}_n) = \frac{\sigma^2}{n}$

Sats. (stora talens lag)

För varje $\varepsilon > 0$ gäller att $P(|\bar{X}_n - \mu| > \varepsilon) \to 0$ då $n \to \infty$.



Ex. Myntkasten

$\mu = p = 1/2$

$P(|\bar{X}_n - \frac{1}{2}| > \varepsilon) \to 0$

$\therefore \bar{X}_n \to \frac{1}{2}$

Markovs olikhet: Om $X \geq 0$ och $a > 0$ så gäller att $P(X \geq a) \leq \frac{E[X]}{a}$.

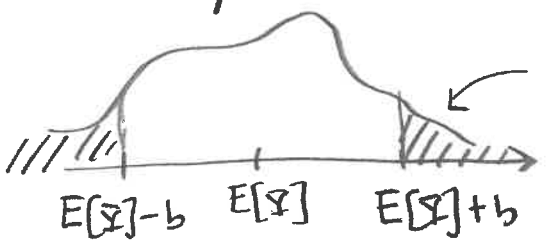
Bevis. Tag $a > 0$ och sätt $I = \begin{cases} 1 & \text{om } X \geq a \\ 0 & \text{annars} \end{cases}$

$$X \geq 0 \Rightarrow I \leq \frac{X}{a} \Rightarrow E[I] \leq \frac{E[X]}{a}$$

$$E[I] = 1 \cdot P(X \geq a) + 0 \cdot P(X < a) = P(X \geq a)$$

$$\therefore P(X \geq a) \leq \frac{E[X]}{a}$$

Chebyshevs olikhet. Y stok. var, $b > 0$. Då gäller att $P(|Y - E[Y]| \geq b) \leq \frac{V(Y)}{b^2}$



arean avtar snabbare än $V(Y)/b^2$ då $b \to \infty$.

Bevis. Välj $X = (Y - E[Y])^2$ och $a = b^2$ i Markovs olikhet.

$$E[X] = E[(Y - E[Y])^2] = V(Y) \text{ ger:}$$

$$P(\underbrace{(Y - E[Y])^2 \geq b^2}) \leq \frac{V(Y)}{b^2}$$

$$\Leftrightarrow |Y - E[Y]| \geq b$$

$$\therefore P(|Y - E[Y]| \geq b) \leq \frac{V(Y)}{b}$$

Bevis av Stora talens lag.

Välj $Y = \overline{X}$ i Chebyshevs olikhet.

$E[\overline{X}] = \mu$, $V(\overline{X}) = \frac{\sigma^2}{n}$

$\therefore \forall \varepsilon > 0$: $P(|\overline{X} - \mu| > \varepsilon) \leq \frac{\sigma^2/n}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2} \cdot \frac{1}{n} \longrightarrow 0$,

Ex. $X_1, X_2, \ldots$ ober. $Be(p)$ dvs $X_1 = \begin{cases} 1 & \text{slh } p \\ 0 & \text{annars.} \end{cases}$

$S_n = \sum_{i=1}^{n} X_i \sim Bin(n,p)$

De Moivre visade att $Bin(n,p) \approx N(\underbrace{np}_{E[S_n]}, \underbrace{np(1-p)}_{V(S_n)})$.

Allmänt.

Antag $X_1, X_2, \ldots$ ober. lika fördelade (iid)

$E[X_i] = \mu$, $V(X_i) = \sigma^2$.

$S_n = \sum_{i=1}^{n} X_i \Rightarrow E[S_n] = n\mu$, $V(S_n) = n\sigma^2$

Normera: $Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \Rightarrow \begin{array}{l} E[Z_n] = 0 \\ V(Z_n) = 1 \end{array}$

Kom ihåg: $\Phi(x)$ = fördelningsfkt. för $N(0,1)$

Sats. (Centrala gränsvärdessatsen)

Fördeln. för Z_n går mot $N(0,1)$ då $n \to \infty$,

dvs $P(Z_n \leq x) \to \Phi(x)$.

$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \approx N(0,1) \Rightarrow \overline{X}_n - \mu \approx N(0, \frac{\sigma^2}{n})$

$\overline{X}_n \to \mu$. Avvikelsen $\overline{X}_n - \mu \approx N(0, \frac{\sigma^2}{n})$.

(stora talens lag)

Användning:

$$P(a < S_n \le b) = P\left(\frac{a-n\mu}{\sqrt{n\sigma^2}} < \underbrace{\frac{S_n - n\mu}{\sqrt{n\sigma^2}}}_{Z_n \approx N(0,1)} \le \frac{b-n\mu}{\sqrt{n\sigma^2}}\right) =$$

$$= P\left(Z_n \le \frac{b-n\mu}{\sqrt{n\sigma^2}}\right) - P\left(Z_n \le \frac{a-n\mu}{\sqrt{n\sigma^2}}\right) \approx$$

om n stort

$$\approx \Phi\left(\frac{b-n\mu}{\sqrt{n\sigma^2}}\right) - \Phi\left(\frac{a-n\mu}{\sqrt{n\sigma^2}}\right)$$

Ex. Avrundningsfel

60 tal avrundas till hela tal och summeras.

X_i = avrundningsfel tal i

Antag $X_i \sim Re(-0.5, 0.5)$

Bestäm $P\left(\left|\sum_{i=1}^{60} X_i\right| > 4\right) = P\left(\sum X_i > 4\right) + P\left(\sum X_i < -4\right) = (*)$

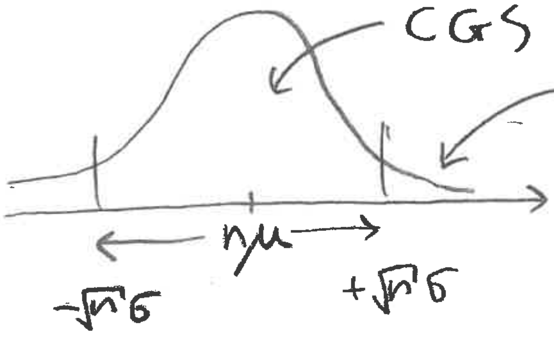
$E[X_i] = 0, \quad V(X_i) = \frac{1}{12}$

$$(*) = P\left(\frac{\sum X_i - 60\cdot 0}{\sqrt{60\cdot 1/12}} > \underbrace{\frac{4 - 60\cdot 0}{\sqrt{60\cdot 1/12}}}_{1.78}\right) + P\left(\frac{\sum X_i - 60\cdot 0}{\sqrt{60\cdot 1/12}} < \underbrace{\frac{-4-60\cdot 0}{\sqrt{60\cdot 1/12}}}_{-1.78}\right)$$

$$\approx 1 - \Phi(1.78) + \Phi(-1.78) = 2(1 - \underbrace{\Phi(1.78)}_{0.9625}) \approx 0.075$$

"Centrala"?

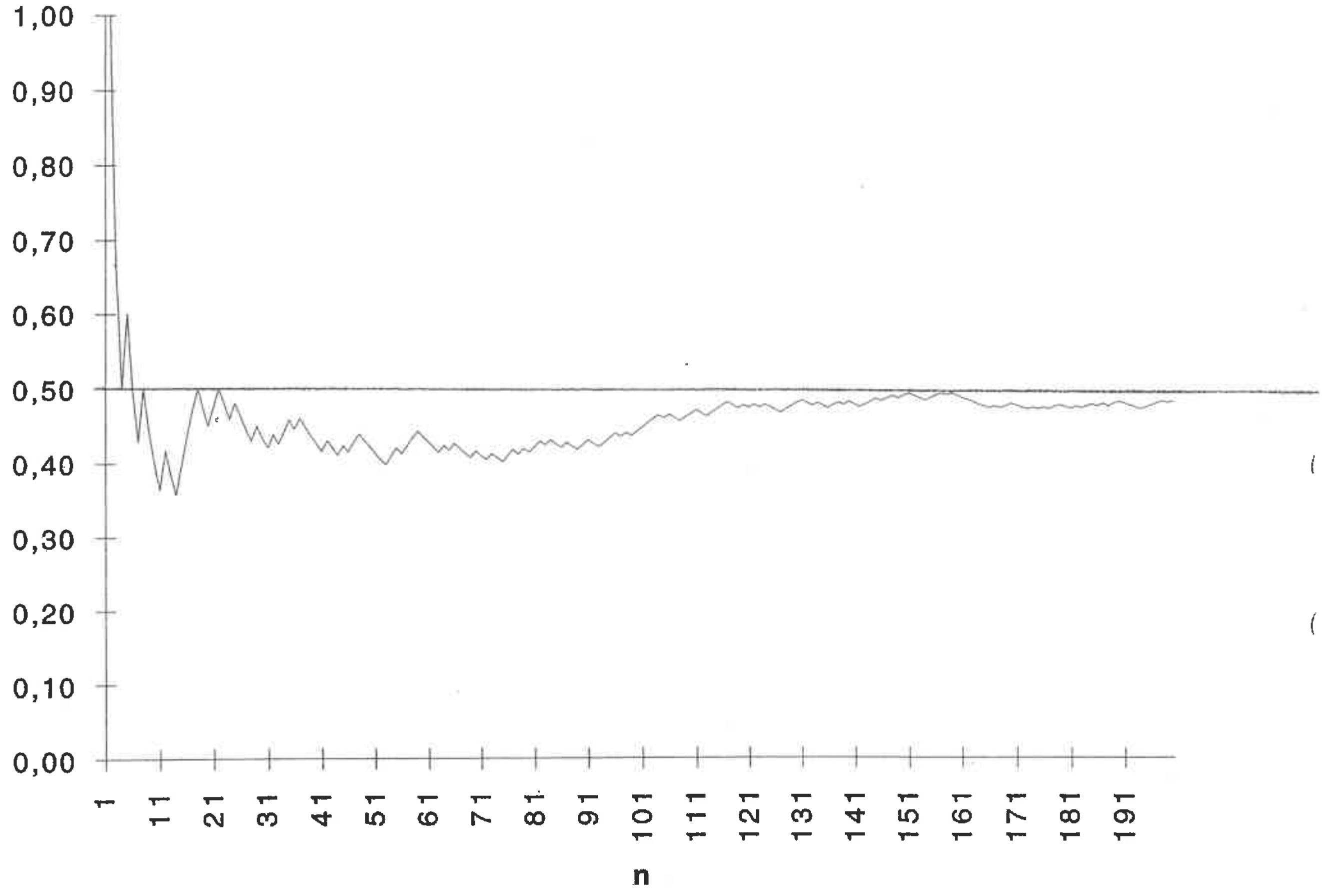
$S_n \approx N(n\mu, n\sigma^2), \quad SD(S_n) = \sqrt{n}\,\sigma$

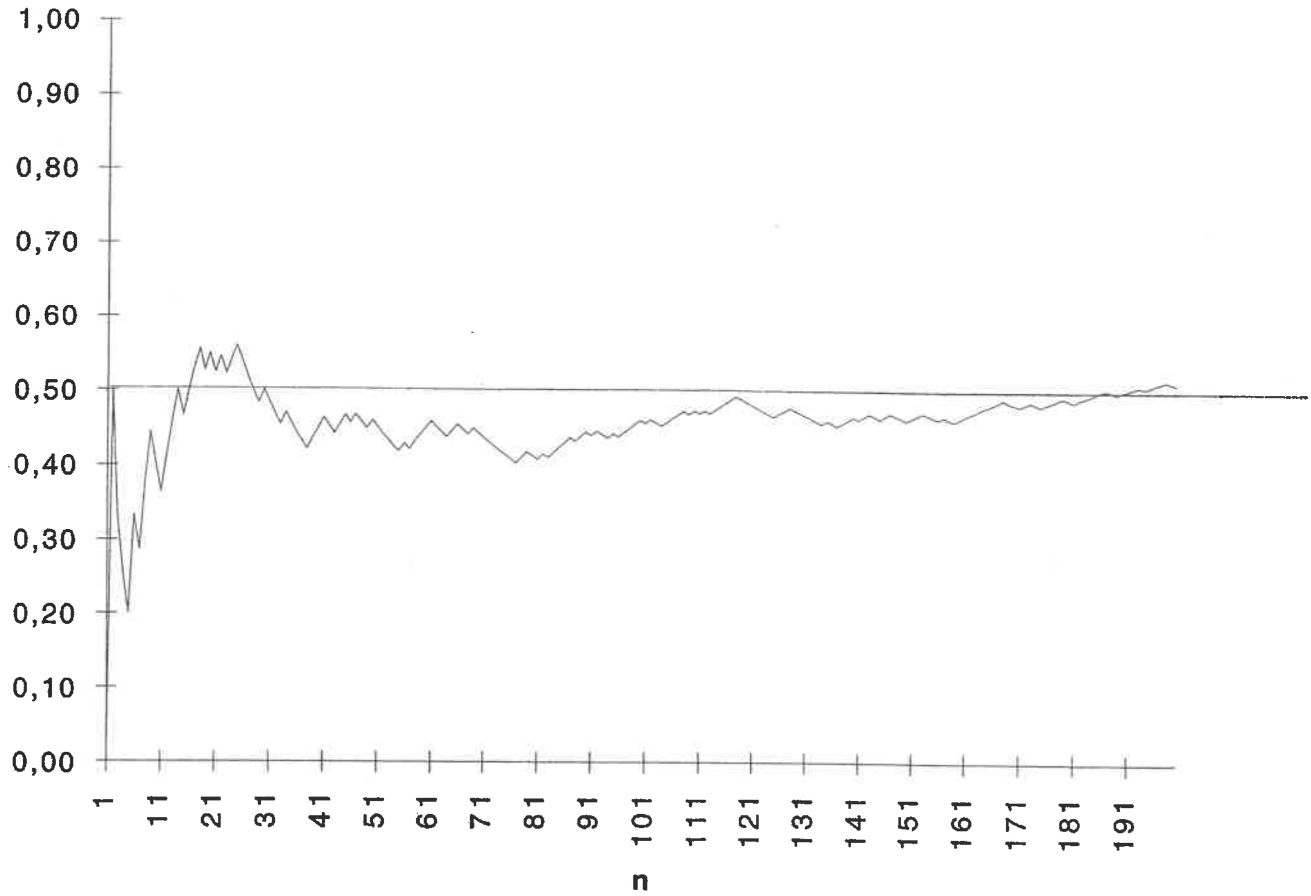


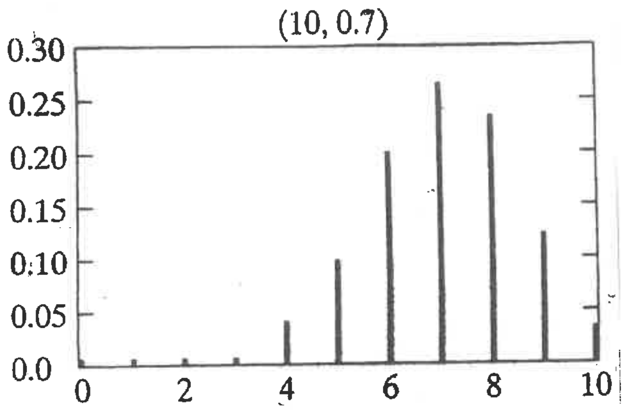
Teori för stora avvikelser

$P(S_n > n(\mu + \varepsilon))$

Rel frekv slant



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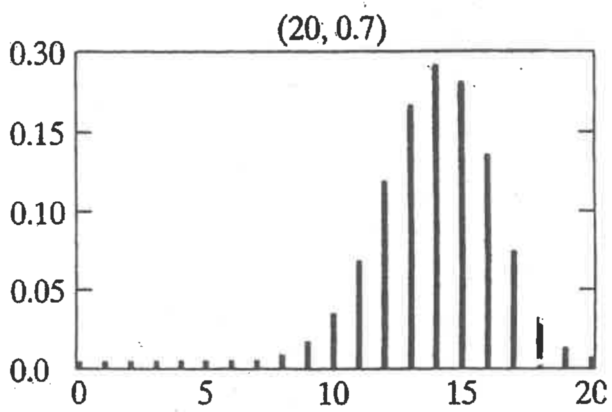
0.25
0.20
0.15
0.10
0.05
0.0
0
2
4
6
8
10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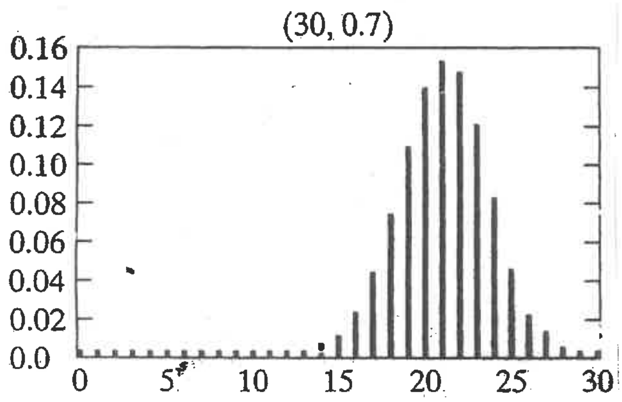
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0.06
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0.0
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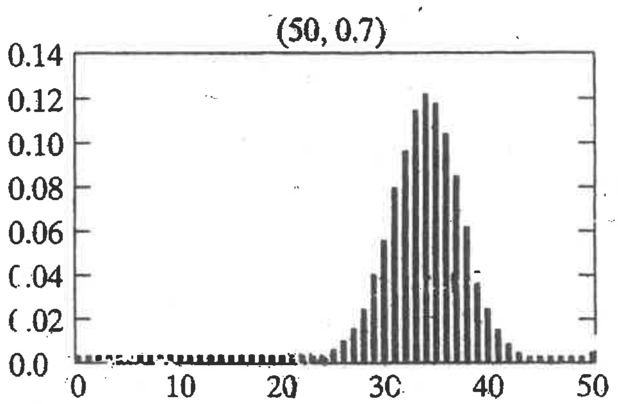
15
20
25
30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0.04
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30
40
50