Dag 4

Seminarierna

Bonus 1,
Deadline tisdag 12:00

Standardgränsvärden

$$a)\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,\quad b)\quad \lim_{x\to 0}\frac{\sin x}{x} = 1,\quad c)\quad \lim_{x\to\infty}\frac{a^x}{x} = +\infty,\quad a>1.$$

a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ $\qquad n = [x]$

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$\to e \qquad \to e \qquad \to e$

$$\left(1+\frac{1}{[x]}\right)^{[x]}\left(1+\frac{1}{[x]}\right)$$

$\to e \qquad \to 1$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$x = \frac{1}{t}$ $\qquad \lim_{t\to 0}(1+t)^{1/t} \overset{?}{=} e$ $\qquad x\to\pm\infty$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \begin{bmatrix} u=-x \\ x=-n \end{bmatrix} = \lim_{u\to\infty}\left(1-\frac{1}{u}\right)^{-u} =$$

$$= \lim_{u\to\infty}\left(\frac{u-1}{u}\right)^{-u} = \lim_{u\to\infty}\left(\frac{u-1+1}{u-1}\right)^{u} =$$

$$\lim_{u\to\infty}\left(1+\frac{1}{u-1}\right)^{u} = \lim_{u\to\infty}\left(\underbrace{\left(1+\frac{1}{u-1}\right)^{u-1}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{u-1}\right)}_{\to 1}\right) = e$$

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

$$\sin x < x < \tan x$$
$$0 < x < \frac{\pi}{2}$$

$$\lim_{x\to0}\frac{\arcsin x}{x}=1 \qquad (x=\arcsin t)$$

$$\lim_{x\to\infty}\frac{a^x}{x}=+\infty \qquad (a>1)$$

$$n=[x] \qquad a=1+p \qquad p>0$$

$$a^x=(1+p)^x \geq (1+p)^n = \ldots + \underline{\underline{\binom{n}{2}}}p^2+\ldots$$

$$\geq \frac{n(n-1)}{2}p^2 \geq \frac{(x-1)(x-2)}{2}p^2$$

Generaliserade integraler

" Som serier, fast lättare "

Integralkalkylens huvudsats.

G.I. kan vara generaliserade på flera ställen.

$$\int_0^\pi \frac{\sqrt{x}}{\sin x}\,dx,$$

generaliserad i 0 och π.

$$\int_0^\pi = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{x}}{\sin x}\,dx$$

$$\underline{\underline{f}}(x) = \frac{\sqrt{x}}{\sin x} \approx \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} = \underline{\underline{g}}(x)$$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\frac{\sqrt{x}}{\sin x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0+} \frac{x}{\sin x} = 1$$

JFK II

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{x}}{\sin x}\,dx \text{ konv} \Leftrightarrow \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{x}}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0+} \int_\varepsilon^{\frac{\pi}{2}} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0+} \left[2\sqrt{x}\right]_\varepsilon^{\frac{\pi}{2}} = 2\sqrt{\frac{\pi}{2}}.$$

Slutsats: $\int_0^{\frac{\pi}{2}} \frac{\sqrt{x}}{\sin x}\,dx$ är konv.

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{x}}{\sin x}\,dx \qquad f(x) = \frac{\sqrt{x}}{\sin x} \qquad g(x) = \frac{1}{\pi - x}$$

$\sin x = \sin(\pi - x) \approx \pi - x$ nära π.

$$\boxed{\lim_{x\to\pi-} \frac{f(x)}{g(x)}} = \lim_{x\to\pi-} \frac{\frac{\sqrt{x}}{\sin x}}{\frac{1}{\pi - x}} = \lim_{x\to\pi-} \underbrace{\sqrt{x}}_{\to\sqrt{\pi}}\, \underbrace{\frac{\pi - x}{\sin x}}_{\to 1}$$

JFKII $\int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{x}}{\sin x}\,dx$ konv $\Leftrightarrow \int_{\frac{\pi}{2}}^{\pi} \frac{dx}{\pi - x} = +\infty$

Slutsats: $\int_0^{\pi} \frac{\sqrt{x}}{\sin x}\,dx$ divergent.

$$\int_0^\infty \frac{e^{\arctan x} - 1}{x\sqrt{x}}\,dx.$$ gen: 0, ∞

$$\int_0^\infty = \int_0^1 + \int_1^\infty$$

$$e^{\arctan x} - 1 = x + O(x^2)$$

$$f(x) = \frac{e^{\arctan x} - 1}{x\sqrt{x}} \qquad g(x) = \frac{x}{x\sqrt{x}} = \frac{1}{\sqrt{x}}$$

JFK II: $\displaystyle\int_0^1 \frac{e^{\arctan x} - 1}{x\sqrt{x}}\,dx$ konv $\Leftrightarrow \displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ konv $= 2$

$$\int_1^\infty \frac{e^{\arctan x} - 1}{x\sqrt{x}}\,dx$$

JFK I: $0 \le f(x) \le g(x)$

$\int g(x)\,dx$ konv $\Rightarrow \int f(x)\,dx$ konv

$$0 \le e^{\arctan x} - 1 \le e^{\frac{\pi}{2}} - 1$$

$$\frac{e^{\arctan x} - 1}{x\sqrt{x}} \le \left(e^{\frac{\pi}{2}} - 1\right) \frac{1}{x\sqrt{x}}$$

Räcker visa att

$$\int_1^\infty \frac{dx}{x\sqrt{x}} \quad \text{konvergerar,}$$

$$= \lim_{R\to\infty} \int_1^R \frac{dx}{x^{3/2}} = \lim_{R\to\infty} \left[-\frac{1}{2} x^{-\frac{1}{2}}\right]_1^R = \frac{1}{2}$$

<u>Slutsats</u> $\displaystyle\int_0^\infty \frac{e^{\arctan x} - 1}{x\sqrt{x}}\,dx$ konv.

$$\sum_{k=1}^\infty \frac{1}{k^\alpha} \text{ konv} \iff \alpha > 1$$