Differentierbarhet och kedjeregeln

$f(x,y)$ är diff.bar i (a,b) om det finns konstanter A_1 och A_2

så att $f(a+h,b+k)-f(a,b)=A_1h+A_2k+\sqrt{h^2+k^2}\,\rho(h,k)$

där $A_1=\frac{\partial f}{\partial x}(a,b)$ och $A_2=\frac{\partial f}{\partial y}(a,b)$ och

$$\lim_{(h,k)\to 0}\rho(h,k)=0$$

Differentierbar i (a,b) $\Rightarrow$ Kontinuerlig och partiellt deriverbar i (a,b)

Omvändningen gäller inte.

Men: $f\in C^1 \Rightarrow f$ är differentierbar

2.8c) $f(x,y)=e^{x+2y}\in C^1 \Rightarrow$ differentierbar överallt

punkten $(2,2)$

$$\left.\begin{aligned} f(2,2)&=e^6\\ f'_x(2,2)&=e^6\\ f'_y(2,2)&=2e^6\end{aligned}\right\}\quad \rho(h,k)=\frac{f(2+h,2+k)-f(2,2)-f'_x(2,2)h-f'_y(2,2)k}{\sqrt{h^2+k^2}}$$

$$=\frac{e^{2+h+4+2k}-e^6-e^6h-2e^6k}{\sqrt{h^2+k^2}}=$$

$$=\frac{e^6\left[e^{h+2k}-1-h-2k\right]}{\sqrt{h^2+k^2}}$$

$$\lim_{(h,k)\to(0,0)}\rho(h,k)=0 \Rightarrow \text{differentierbar}$$

forts.

2.8c fortsättning

$$e^{h+2k} = 1+(h+2k)+\frac{(h+2k)^2}{2}+O(h+2k)^3$$

$$\lim_{(h,k)\to(0,0)} g(h,k) = e^6 \lim_{(h,k)\to(0,0)} \frac{\left[1+h+2k+\frac{h^2}{2}+2hk+2k^2-1-h-2k + O(h+2k)^3\right)}{\sqrt{h^2+k^2}}$$

$$= e^6 \lim_{(h,k)\to(0,0)} \frac{h^2/2+2hk+2k^2+O(h+2k)^3}{\sqrt{h^2+k^2}} =$$

$$= \{\text{Polära koord.}\} = \lim_{r\to 0} \frac{r^2/2+2r^2\cos\theta\sin\theta+\frac{3}{2}r^2\sin^2\theta+(+O(r^3))}{r}$$

$$= \lim_{r\to 0} \overset{\to 0}{r}\Big[\underbrace{1/2+\sin 2\theta+\frac{3}{2}\sin^2\theta}_{|\cdot|\le M}+\overset{\to 0}{O(r)}\Big] = 0$$

$\therefore$ $f(x,y)$ differentierbar i $(2,2)$

2.17) Visa att $f(x,y)=\frac{1}{\sqrt{xy}}g\left(\frac{x}{y}\right)$ $x>0, y>0$ löser ekv.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+f=0$$

$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{y}}\left(-\frac{1}{2}x^{-3/2}\right)g(x/y)+\frac{1}{\sqrt{xy}}g'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)=$$

$$=-\frac{1}{2x\sqrt{xy}}g\left(\frac{x}{y}\right)+\frac{1}{y\sqrt{xy}}g'(x/y)$$

$$\frac{\partial f}{\partial y}=-\frac{1}{2y\sqrt{xy}}g\left(\frac{x}{y}\right)+\frac{1}{\sqrt{xy}}g'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)=-\frac{1}{2y\sqrt{xy}}g(x/y)-\frac{x}{y^2\sqrt{xy}}g'\left(\frac{x}{y}\right)$$

2.17)

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+f=$$

$$=-\frac{1}{2\sqrt{xy}}g\left(\frac{x}{y}\right)+\frac{\sqrt{x}}{y\sqrt{y}}g'\left(\frac{x}{y}\right)+\left[-\frac{1}{2\sqrt{xy}}g\left(\frac{x}{y}\right)-\frac{\sqrt{x}}{y\sqrt{y}}g'(x/y)\right]+$$

$$+\frac{1}{\sqrt{xy}}g(x/y)=-\frac{1}{\sqrt{xy}}g\left(\frac{x}{y}\right)+\frac{1}{\sqrt{xy}}g\left(\frac{x}{y}\right)=\underline{\underline{0}}$$

2.25) Transformera $y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=xyf$ genom de nya variablerna $\begin{cases}u=x^2+y^2\\ v=e^{-x^2/2}\end{cases}$

Kedjeregeln: $\begin{cases}\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{2x}+\frac{\partial f}{\partial v}\overbrace{\frac{\partial v}{\partial x}}^{-xe^{-x^2/2}}=2x\frac{\partial f}{\partial u}-xe^{-x^2/2}\frac{\partial f}{\partial v}\\ \frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{2y}+\frac{\partial f}{\partial v}\underbrace{\frac{\partial v}{\partial y}}_{=0}=2y\frac{\partial f}{\partial u}\end{cases}$

$$y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=xyf\Rightarrow \cancel{2xy\frac{\partial f}{\partial u}}-\cancel{xy}e^{-x^2/2}\frac{\partial f}{\partial v}-\cancel{2xy\frac{\partial f}{\partial u}}=\cancel{xy}f$$

$$\therefore\ \underbrace{e^{-x^2/2}}_{v}\frac{\partial f}{\partial v}+f=0\Rightarrow\boxed{\frac{\partial f}{\partial v}+\frac{1}{v}f=0}$$

Integrerande faktor $e^{\ln v}=v\Rightarrow$

$$\Rightarrow\frac{\partial}{\partial v}(vf)=0\Rightarrow vf=\varphi(u)$$

$$\therefore\ f=\frac{1}{v}\varphi(u)\Rightarrow f(x,y)=\underline{e^{x^2/2}\varphi(x^2+y^2)}$$

2.25b)

(4)

b) $f(0,y)=y^2 \Rightarrow \varphi(y^2)=y^2 \quad \varphi(y)=y$

$\therefore \underline{f(x,y)=e^{x^2/2}(x^2+y^2)}$

2.27) $f(x_1,x_2)$ funktion på $\mathbb{R}^2$, $x_1=g(t_1,t_2), x_2=h(t_1,t_2)$

Bestäm de partiella derivatorna av

$u(t_1,t_2)=f(g(t_1,t_2),h(t_1,t_2))$ i $(1,1)$

$$\frac{\partial u}{\partial t_1}=\frac{\partial f}{\partial x_1}(g(t_1,t_2),h(t_1,t_2))\frac{\partial g}{\partial t_1}(t_1,t_2)+\frac{\partial f}{\partial x_2}(g(t_1,t_2),h(t_1,t_2))\cdot\frac{\partial h}{\partial t_1}(t_1,t_2)$$

$$\frac{\partial u}{\partial t_2}=\frac{\partial f}{\partial x_1}(g(t_1,t_2),h(t_1,t_2))\frac{\partial g}{\partial t_2}(t_1,t_2)+\frac{\partial f}{\partial x_2}(g(t_1,t_2),h(t_1,t_2))\frac{\partial h}{\partial t_2}(t_1,t_2)$$

$(t_1,t_2)=(1,1) \Rightarrow$

$$\frac{\partial u}{\partial t_1}=\underbrace{\frac{\partial f}{\partial x_1}(\underbrace{g(1,1),h(1,1)}_{(1,2)})}_{7}\underbrace{\frac{\partial g}{\partial t_1}(1,1)}_{2}+\underbrace{\frac{\partial f}{\partial x_2}(\underbrace{g(1,1),h(1,1)}_{(1,2)})}_{-4}\underbrace{\frac{\partial h}{\partial t_1}(1,1)}_{1}$$

$$=7\cdot 2-4=\underline{\underline{10}}$$

P.S.S. för $\frac{\partial u}{\partial t_2}(1,1)=7\cdot 1+(-4)\cdot 2=\underline{\underline{-1}}$

$$\boxed{\therefore \frac{\partial u}{\partial t_1}(1,1)=10, \quad \frac{\partial u}{\partial t_2}(1,1)=-1}$$