Dag 8

Kedjeregeln i flera variabler

Anmäl er till tentan.

Bonus 2 på måndag.

Bonus 3 9 okt. lörd

Tenta på campus. 14/10

Besked senast! 18/10

Sem 1 klart!

Sem 2 två veckor senare.

Sem 3 18, 19, 20 okt.

Anmäl dig till tentan!!!!

Om funktionen $f(x,y)$ är det känt att den tillhör klass C^2 samt att $f(t,t) = 5t^2$, $f(t,t^2) = t + 2t^2$ och att $f(t^2,t) = -t - 2t^2$ för t nära 0. Bestäm f:s partiella derivator upp till ordning två i origo.

$$f(t,t) = 5t^2 \Rightarrow \frac{\partial f}{\partial x}(t,t) + \frac{\partial f}{\partial y}(t,t) = 10t$$

$$f(t,t^2) = t+2t^2 \Rightarrow \frac{\partial f}{\partial x}(t,t^2) + 2t\frac{\partial f}{\partial y}(t,t^2) = 1+4t$$

$$f(t^2,t) = -t-2t^2 \Rightarrow 2t\frac{\partial f}{\partial x}(t^2,t) + \frac{\partial f}{\partial y}(t^2,t) = -1-4t$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0) = 0 \\ \underline{\frac{\partial f}{\partial x}(0,0) = 1} \\ \underline{\frac{\partial f}{\partial y}(0,0) = -1} \end{cases}$$

stämmer!
(Ingen slump)

$$\frac{\partial^2 f}{\partial x^2}(t,t) + \frac{\partial^2 f}{\partial y \partial x}(t,t) + \frac{\partial^2 f}{\partial x \partial y}(t,t) + \frac{\partial^2 f}{\partial y^2}(t,t) = 10$$

$$\frac{\partial^2 f}{\partial x^2} + 2t\frac{\partial^2 f}{\partial y \partial x} + 2t\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}2t\right) + 2\frac{\partial f}{\partial y} = 4$$

$$2t\left(\frac{\partial^2 f}{\partial x^2}2t + \frac{\partial^2 f}{\partial y \partial x}\right) + 2\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x \partial y}\cdot 2t + \frac{\partial^2 f}{\partial y^2} = -4$$

$$t=0 \begin{cases} \frac{\partial^2 f}{\partial x^2}(0,0) + 2\frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{\partial^2 f}{\partial y^2}(0,0) = 10 & * \\ \frac{\partial^2 f}{\partial x^2}(0,0) + 2\frac{\partial f}{\partial y}(0,0) = 4 \\ 2\frac{\partial f}{\partial x}(0,0) + \frac{\partial^2 f}{\partial y^2}(0,0) = -4 \end{cases}$$

$$\underline{\frac{\partial^2 f}{\partial x^2}(0,0)} = 4 - 2\frac{\partial f}{\partial y}(0,0) = 4-(-2) = \underline{6}$$

$$\underline{\frac{\partial^2 f}{\partial y^2}(0,0)} = -4 - 2\frac{\partial f}{\partial x}(0,0) = -4-2 = \underline{-6}$$

$$* \Rightarrow \cancel{6} + 2\frac{\partial^2 f}{\partial x \partial y}(0,0) - \cancel{6} = 10$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = 5$$

a) Bestäm ett linjärt variabelbyte som överför differentialuttrycket

$$2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2},$$

på standardformen $C\dfrac{\partial^2 f}{\partial u \partial v}$ genom att ansätta $u = x + ay, v = bx + y$.

b) Bestäm den allmänna lösningen till

$$2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2} = 0.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial f}{\partial v} \cdot \underbrace{\frac{\partial v}{\partial x}}_{=b} = \frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \underbrace{\frac{\partial u}{\partial y}}_{=a} + \frac{\partial f}{\partial v} \cdot \underbrace{\frac{\partial v}{\partial y}}_{=1} = a\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

Kan även skrivas

$$\boxed{\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + b\frac{\partial}{\partial v} \qquad \frac{\partial}{\partial y} = a\frac{\partial}{\partial u} + \frac{\partial}{\partial v}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(\frac{\partial f}{\partial u} + b\frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right)$$

$$= \underline{\frac{\partial^2 f}{\partial u^2} + 2b\frac{\partial^2 f}{\partial u \partial v} + b^2\frac{\partial^2 f}{\partial v^2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(a\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(a\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$

$$= \underline{a^2\frac{\partial^2 f}{\partial u^2} + 2a\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \left(\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)\left(a\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$

$$= a\frac{\partial^2 f}{\partial u^2} + (1+ab)\frac{\partial^2 f}{\partial u \partial v} + b\frac{\partial^2 f}{\partial v^2}$$

Insättning i den ursprungliga ekvationen ger:

$$2\left(\frac{\partial^2 f}{\partial u^2} + 2b\frac{\partial^2 f}{\partial u \partial v} + b^2\frac{\partial^2 f}{\partial v^2}\right) + \left(a\frac{\partial^2 f}{\partial u^2} + (1+ab)\frac{\partial^2 f}{\partial u \partial v} + b\frac{\partial^2 f}{\partial v^2}\right)$$

$$-\left(a^2\frac{\partial^2 f}{\partial u^2} + 2a\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right) = 0$$

$$(2+a-a^2)\frac{\partial^2 f}{\partial u^2} + (4b + (1+ab) - 2a)\frac{\partial^2 f}{\partial u \partial v} + (2b^2 + b - 1)\frac{\partial^2 f}{\partial v^2} = 0$$

Välj a och b så att $2+a-a^2=0$ och $2b^2+b-1=0$, t.ex. $a=-1$ och $b=\frac{1}{2}$

Variabelbytet $\begin{cases} u = x-y \\ v = \frac{1}{2}x + y \end{cases}$ överför

alltså ekvationen i

$$2\frac{\partial^2 f}{\partial u \partial v} = 0$$ med den allmänna

lösningen $f = \varphi(u) + \psi(v)$ eller i x och y:

$$f(x,y) = \varphi(x-y) + \psi\left(\tfrac{1}{2}x + y\right).$$

Låt $f(x,y)$ vara av klass C^2. Låt $u = x^2 - y^2, v = 2xy$.
Hur uttrycker vi $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ och $\dfrac{\partial^2 f}{\partial y^2}$ i termer av u, v?

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$$

$$= 2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}\right)$$

$$= 2x\frac{\partial}{\partial u}\left(\underline{2x}\frac{\partial f}{\partial u}\right) + \ldots$$

$$4x^2 \frac{\partial^2 y}{\partial u^2} + 2x \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \dots$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{pmatrix} = I$$