Dag 7

Medelvärdessatsen

$f(x)$ kontinuerlig på $[a, b]$
$f(x)$ deriverbar på $]a, b[$
Då finns $\xi \in]a, b[$ så att

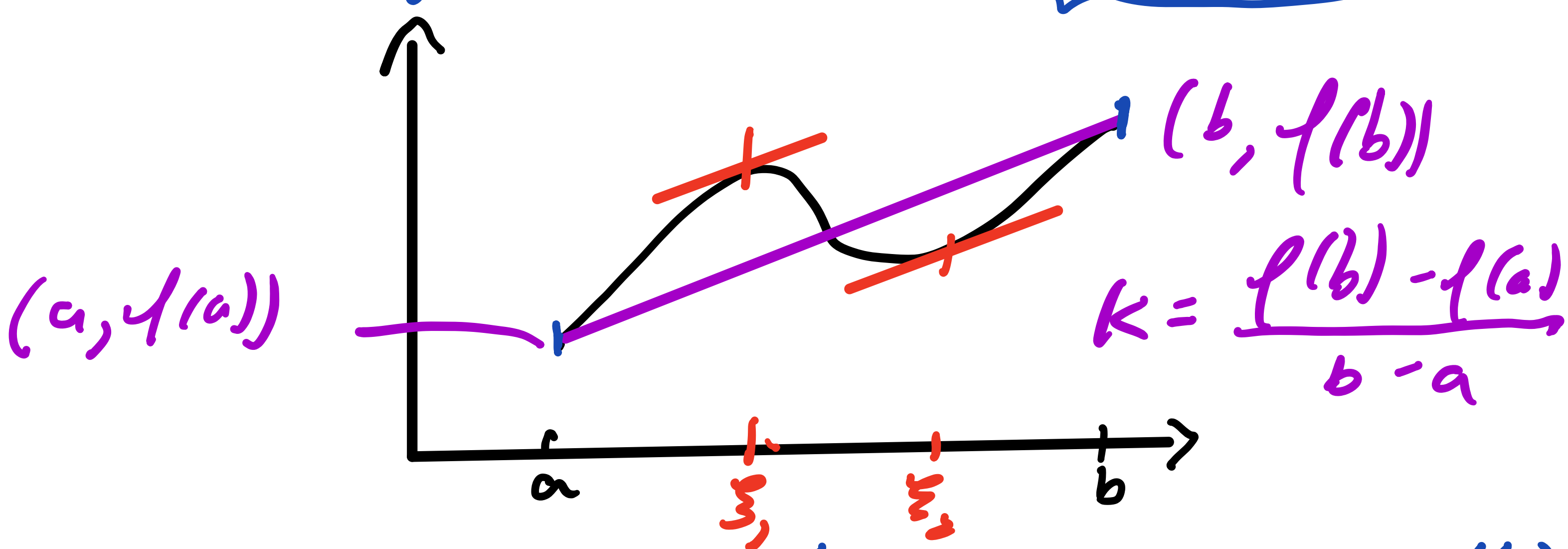
$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

$$\| \quad f(b) - f(a) = f'(\xi)(b - a)$$

$$\xi = a + \theta(b - a) \qquad 0 < \theta < 1$$

Bevis: Bygger på Rolles sats.

$$g(x) = f(x) - \underbrace{f(a) - \frac{f(b) - f(a)}{b - a}(x - a)}$$



$g(a) = g(b) \; (= 0) \qquad R.S. \Rightarrow \exists \xi, g'(\xi).$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$0 = g'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a}$$

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

$$\xleftarrow{\text{M.S.}}$$

$$\frac{f(b)-f(a)}{b-a} \quad \xrightarrow{\text{der.}} \quad f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

SATS f deriverbar på $]a,b[$. Antas också att $f'(x) \geq 0$ på $]a,b[$. Då är $f(x)$ växande på $]a,b[$.

B: $a < x_1 < x_2 < b$ Vill visa att $f(x_1) \leq f(x_2)$

$$f(x_2) - f(x_1) = \underbrace{f'(\xi)}_{\geq 0}\underbrace{(x_2 - x_1)}_{>0} \geq 0.$$

V.S.B.

Differentialkalkyl i flera variabler.

1 var: Deriverbarhet $\Leftrightarrow$ existens av en entydig tangent.

Deriverbarhet $\Leftrightarrow$ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ existerar.

Dessa aspekter är ekvivalenta!

flera var: Geometriska sidan:

$z = f(x,y)$. Existens av tangentplan. (differentierbarhet)

Analytiska sidan:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

SATS Om f:s partiella derivator är kontinuerliga ($f \in C^1$), då är f differentierbar.

Kedjeregeln: f differentierbar

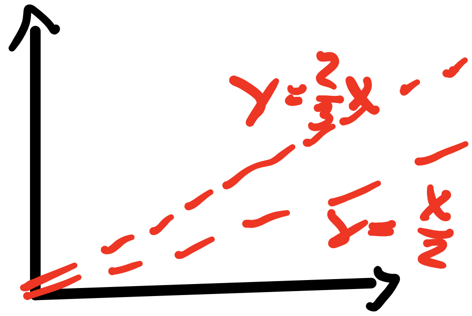
$$\frac{d}{dt}\big(f(g(t),h(t))\big) = \frac{\partial f}{\partial x}(g(t),h(t))\cdot g'(t) + \frac{\partial f}{\partial y}(\ ,\ )h'(t)$$

$x = 3t \quad y = 2t \quad y = \frac{2}{3}x$

Antag att $f(x,y)$ är en differentierbar funktion sådan att

$$f(2t,t) = \underline{3t + 4t^2} \quad \text{och} \quad f\underline{(3t,2t) = \sin} t.$$

Bestäm f:s partiella derivator i origo.



$$\frac{d}{dt} f(2t,t) = \frac{\partial f}{\partial x}(,)\cdot 2 + \frac{\partial f}{\partial y}(,)\cdot 1$$

$$t = 0: \; 2\frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0)$$

$$\frac{d}{dt}(f(2t,t)) = \frac{d}{dt}(3t + 4t^2) = \underline{3 + 8t}$$

$$\boxed{2\frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0) = 3}$$

$$\frac{d}{dt} f(3t,2t) = 3\frac{\partial f}{\partial x}(,) + 2\frac{\partial f}{\partial y}(,)$$

$$\parallel \quad \frac{d}{dt}(\sin t) = \cos t$$

$$t = 0 \quad \boxed{3\frac{\partial f}{\partial x}(0,0) + 2\frac{\partial f}{\partial y}(0,0) = 1}$$

$$② - 2 \times ①: \quad \underbrace{3\frac{\partial f}{\partial x}(0,0) - 4\frac{\partial f}{\partial x}(0,0)}_{-\frac{\partial f}{\partial x}(0,0) = -5} = 1 - 2\cdot 3 = -5$$

$$\frac{\partial f}{\partial x}(0,0) = 5,$$

$$\frac{\partial f}{\partial y}(0,0) = -7.$$

Bestäm den lösning f till den partiella differentialekvationen

$$x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0,$$

för vilken $f = e^{-2y}$ då $x = 1$, t ex genom att införa de nya variablerna $u = xe^{y}$ och $v = xe^{-y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = e^{y}\frac{\partial f}{\partial u} + e^{-y}\frac{\partial f}{\partial v}$$

$$\frac{\partial u}{\partial x} = e^{y} \qquad \frac{\partial v}{\partial x} = e^{-y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = xe^{y}\frac{\partial f}{\partial u} - xe^{-y}\frac{\partial f}{\partial v}$$

$$\frac{\partial u}{\partial y} = xe^{y} \qquad \frac{\partial v}{\partial y} = -xe^{-y}$$

$$x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = x\left(e^{y}\frac{\partial f}{\partial u} + \cancel{e^{-y}\frac{\partial f}{\partial v}}\right) + xe^{y}\frac{\partial f}{\partial u} - \cancel{xe^{-y}\frac{\partial f}{\partial v}}$$

$$2\cancel{xe^{y}}\frac{\partial f}{\partial u} = 0 \iff \frac{\partial f}{\partial u} = 0$$

$$f = \varphi(v)$$

i x,y-variablerna: $f(x,y) = \varphi(xe^{-y})$

$$e^{-2y} = f(1,y) = \varphi(e^{-y})$$

$\varphi(t) = t^2$ $\qquad f(x,y) = (xe^{-y})^2 = x^2e^{-2y}$