

Slot-queue - An optimized wait-free distributed MPSC

1. Motivation

A good example of a wait-free MPSC has been presented in [1]. In this paper, the authors propose a novel tree-structure and a min-timestamp scheme that allow both enqueue and dequeue to be wait-free and always complete in $\Theta(\log n)$ where n is the number of enqueueers.

We have tried to port this algorithm to distributed context using MPI. The most problematic issue was that the original algorithm uses load-link/store-conditional (LL/SC). To adapt to MPI, we have to propose some modification to the original algorithm to make it use only compare-and-swap (CAS). Even though the resulting algorithm pretty much preserve the original algorithm's characteristic, that is wait-freedom and time complexity of $\Theta(\log n)$, we have to be aware that this is $\Theta(\log n)$ remote operations, which is very expensive. We have estimated that for an enqueue or a dequeue operation in our initial LTQueue version, there are about $2 * \log n$ to $10 * \log n$ remote operations, depending on data placements and the current state of the LTQueue.

Therefore, to be more suitable for distributed context, we propose a new algorithm that's inspired by LTQueue, in which both enqueue and dequeue only perform a constant number of remote operations, at the cost of dequeue having to perform $\Theta(n)$ local operations, where n is the number of enqueueers. Because remote operations are much more expensive, this might be a worthy tradeoff.

2. Structure

Each enqueue will have a local SPSC as in LTQueue [1] that supports dequeue, enqueue and readFront. There's a global queue whose entries store the minimum timestamp of the corresponding enqueueer's local SPSC.



Figure 1: Basic structure of slot queue

3. Pseudocode

3.1. SPSC

The SPSC of [1] is kept in tact, except that we change it into a circular buffer implementation.

Types

```

data_t = The type of data stored
spsc_t = The type of the local SPSC

record
    First: int
    Last: int
    Capacity: int
    Data: an array of data_t of capacity
    Capacity
end

```

Shared variables

```

First: index of the first undequeued entry
Last: index of the first unenqueued entry

```

Initialization

```

First = Last = 0
Set Capacity and allocate array.

```

The procedures are given as follows.

Procedure 1: `spsc_enqueue(v: data_t)` **returns** `bool`

```

1 if (Last + 1 == First)
2   | return false
3 Data[Last] = v
4 Last = (Last + 1) % Capacity
5 return true

```

Procedure 2: `spsc_dequeue()` **returns** `data_t`

```

6 if (First == Last) return  $\perp$ 
7 res = Data[First]
8 First = (First + 1) % Capacity
9 return res

```

Procedure 3: `spsc_readFront` **returns** `data_t`

```

10 if (First == Last)
11   | return  $\perp$ 
12 return Data[First]

```

3.2. Slot-queue

The slot-queue types and structures are given as follows:

Types

data_t = The type of data stored
 timestamp_t = uint64_t
 spsc_t = The type of the local SPSC

Shared variables

slots: An array of timestamp_t with the number of entries equal the number of enqueueers
 spscs: An array of spsc_t with the number of entries equal the number of enqueueers
 counter: uint64_t

Initialization

| Initialize all local SPSCs.

| Initialize slots entries to MAX.

The enqueue operations are given as follows:

Procedure 4: `enqueue(rank: int, v: data_t)` **returns** `bool`

```

1 timestamp = FAA(counter)
2 value = (v, timestamp)
3 res = spsc_enqueue(spsc_s[rank], value)
4 if (!res) return false
5 if (!refreshEnqueue(rank, timestamp))
6   | refreshEnqueue(rank, timestamp)
7 return res

```

Procedure 5: `refreshEnqueue(rank: int, ts: timestamp_t)` **returns** `bool`

```

8 old-timestamp = slots[rank]
9 front = spsc_readFront(spsc_s[rank])
10 new-timestamp = front ==  $\perp$  ? MAX : front.timestamp
11 if (new-timestamp != ts)
12   | return true
13 return CAS(&slots[rank], old-timestamp, new-timestamp)

```

The dequeue operations are given as follows:

Procedure 6: `dequeue()` **returns** `data_t`

```

14 rank = readMinimumRank()
15 if (rank == DUMMY || slots[rank] == MAX)
16   | return  $\perp$ 
17 res = spsc_dequeue(spsc_s[rank])
18 if (res ==  $\perp$ ) return  $\perp$ 
19 if (!refreshDequeue(rank))
20   | refreshDequeue(rank)
21 return res

```

Procedure 7: readMinimumRank() returns int

```

22 rank = length(slots)
23 min-timestamp = MAX
24 for index in 0..length(slots)
25   timestamp = slots[index]
26   if (min-timestamp < timestamp)
27     rank = index
28     min-timestamp = timestamp
29 old-rank = rank
30 for index in 0..old-rank
31   timestamp = slots[index]
32   if (min-timestamp < timestamp)
33     rank = index
34     min-timestamp = timestamp
35 return rank == length(slots) ? DUMMY :
   rank

```

Procedure 8: refreshDequeue(rank: int) returns bool

```

36 old-timestamp = slots[rank]
37 front = spsc_readFront(spsc[rank])
38 new-timestamp = front ==  $\perp$  ? MAX :
   front.timestamp
39 return CAS(&slots[rank], old-timestamp,
   new-timestamp)

```

4. Linearizability of the local SPSC

In this section, we prove that the local SPSC is linearizable.

Lemma 4.1 (*Linearizability of spsc_enqueue*) The linearization point of spsc_enqueue is right after line 2 or right after line 4.

Lemma 4.2 (*Linearizability of spsc_dequeue*) The linearization point of spsc_dequeue is right after line 6 or right after line 8.

Lemma 4.3 (*Linearizability of spsc_readFront*) The linearization point spsc_readFront is right after line 11 or right after line 12.

Theorem 4.4 (*Linearizability of local SPSC*) The local SPSC is linearizable.

Proof This directly follows from Lemma 4.1, Lemma 4.2, Lemma 4.3. \square

5. ABA problem

Noticeably, we use no scheme to avoid ABA problem in Slot-queue. In actuality, ABA problem does not adversely affect our algorithm's correctness, except in the extreme case that the 64-bit global counter overflows, which is unlikely.

5.1. ABA-safety

Not every ABA problem is unsafe. We formalize in this section which ABA problem is safe and which is not.

For a concurrent object S , we can call some methods on S concurrently. A method call on the object S is said to have an **invocation event** when it starts and a **response event** when it ends.

Definition 5.1.1 An **invocation event** is a triple $(S, t, args)$, where S is the object the method is invoked on, t is the timestamp of when the event happens and $args$ is the arguments passed to the method call.

Definition 5.1.2 A **response event** is a triple (S, t, res) , where S is the object the method is invoked on, t is the timestamp of when the event happens and res is the results of the method call.

Definition 5.1.3 A **method call** is a tuple of (i, r) where i is an invocation event and r is a response event or the special value \perp indicating that its response event hasn't happened yet. A well-formed **method call** should have a response event with a larger timestamp than its invocation event or the response event hasn't happened yet.

Definition 5.1.4 A **method call** is **pending** if its invocation event is \perp .

Definition 5.1.5 A **history** is a set of well-formed **method calls**.

Definition 5.1.6 A **completed history** is a **history** without pending method calls.

Definition 5.1.7 A **modification instruction** on a variable v is an atomic instruction that may change the value of v e.g. a store or a CAS.

Definition 5.1.8 A **successful modification instruction** on a variable v is an atomic instruction that changes the value of v e.g. a store or a successful CAS.

Definition 5.1.9 A **CAS-sequence** is a tuple of (T, m, v, t_s, t_e) such that:

- v is a shared variable.
- m is a method call executed by the thread T .
- $t_s < t_e$.
- t_s happens after m 's invocation event.
- t_e happens before m 's response event.
- At time t_s , $v_0 = \text{load}(v)$ is executed by m .
- At time t_e , $\text{CAS}(\&v, v_0, v_1)$ is executed by thread m .
- There's no **modification instruction** to a shared variable executed by m during $[t_s, t_e)$.

Definition 5.1.10 A **successful CAS-sequence** is a **CAS-sequence** whose CAS is executed successfully.

Definition 5.1.11 A **CAS-augmented history** is a **completed history** in which each method call is annotated with a list of **CAS-sequences** it executes.

We can define a **strict partial order** on the set of successful CAS-sequences:

Definition 5.1.12 \rightarrow is a relation on the set of successful CAS-sequences. With two CAS-sequences $X = (T_1, v_1, t_{s_1}, t_{e_1})$ and $Y = (T_2, v_2, t_{s_2}, t_{e_2})$, we have $X \rightarrow Y \Leftrightarrow v_1 = v_2 \wedge t_{e_1} < t_{s_2}$.

Definition 5.1.13 Given a **CAS-augmented history** H , \rightarrow_H is a relation on the CAS-sequences

within H such that for two sequences X and Y in H , $X \rightarrow_H Y \Leftrightarrow X \rightarrow Y$.

Definition 5.1.14 Given a **CAS-augmented history** H . **ABA problem** is said to occur in H with method type M if there exists two distinct **successful CAS-sequences** $(T_1, m_1, v, t_{s_1}, t_{e_1})$ and $(T_2, m_2, v, t_{s_2}, t_{e_2})$ such that:

- m_1 is an invocation of M .
- $[t_{s_1}, t_{e_1}]$ overlaps with $[t_{s_2}, t_{e_2}]$.

Definition 5.1.15 A method type M is said to be **ABA-safe** if for any possible **CAS-augmented history** H :

- **ABA problem** does not occur in H with M .
- We can define a history H' such that:
 - **ABA problem** does not occur in H' with M .
 - There exists a one-to-one mapping M from a method call of H to a method call of H' .
 - There exists a one-to-one mapping M_c from a CAS-sequence of H to a CAS-sequence of H' such that for a CAS-sequence X of H that has m as its method, $M_c(X)$ has $M(m)$ as its method call.
 - For any two CAS sequences X and Y , $X \rightarrow_H Y \Rightarrow M(X) \rightarrow_{H'} M(Y)$.

5.2. Proof of ABA-safety

Notice that we only use CAS on:

- Line 13 of refreshEnqueue (Procedure 5), or an enqueue in general (Procedure 4).
- Line 42 of refreshDequeue (Procedure 8) or a dequeue in general (Procedure 6).

Both CAS target some slot in the slots array.

We apply some domain knowledge of our algorithm to the above formalism.

Definition 5.2.1 A **CAS-sequence** on a slot s of an enqueue that corresponds to s is the sequence of instructions from line 8 to line 13 of its refreshEnqueue.

Definition 5.2.2 A **CAS-sequence** on a slot s of a dequeue that corresponds to s is the sequence

of instructions from line 36 to line 39 of its refreshDequeue.

Definition 5.2.3 A **CAS-sequence** of a dequeue/enqueue is said to execute a **slot-modification instruction** on line 13 of refreshEnqueue or on line 39 of refreshDequeue.

Definition 5.2.4 A **CAS-sequence** of a dequeue/enqueue is said to **observes a slot value of s_0** if it executes line 8 of refreshEnqueue or line 36 of refreshDequeue.

We can now turn to our interested problem in this section.

Lemma 5.2.1 (*Concurrent accesses on a local SPSC and a slot*) Only one dequeuer and one enqueue can concurrently modify a local SPSC and a slot in the slots array.

Proof This is trivial to prove based on the algorithm's definition. \square

Lemma 5.2.2 (*Monotonicity of local SPSC timestamps*) Each local SPSC in Slot-queue contains elements with increasing timestamps.

Proof Each enqueue would FAA the global counter (line 1 in Procedure 4) and enqueue into the local SPSC an item with the timestamp obtained from the counter. Applying Lemma 5.2.1, we know that items are enqueued one at a time into the SPSC. Therefore, later items are enqueued by later enqueues, which obtain increasing values by FAA-ing the shared counter. The theorem holds. \square

Lemma 5.2.3 A refreshEnqueue (Procedure 5) can only changes a slot to a value other than MAX.

Proof For refreshEnqueue to change the slot's value, the condition on line 11 must be false. Then new-timestamp must equal to ts, which is not MAX. It's obvious that the CAS on line 13 changes the slot to a value other than MAX. \square

Theorem 5.2.4 (*ABA safety of dequeue*) Assume that the 64-bit global counter never overflows, dequeue (Procedure 6) is ABA-safe.

Proof Consider a **successful CAS-sequence** on slot s by a dequeue d .

Denote t_d as the value this CAS-sequence observes.

Due to Lemma 5.2.1, there can only be at most one enqueue at one point in time within d .

If there's no **successful slot-modification instruction** on slot s by an enqueue e within d 's **successful CAS-sequence**, then this dequeue is ABA-safe.

Suppose the enqueue e executes the *last successful slot-modification instruction* on slot s within d 's **successful CAS-sequence**. Denote t_e to be the value that e sets s .

If $t_e \neq t_d$, this CAS-sequence of d cannot be successful, which is a contradiction.

Therefore, $t_e = t_d$.

Note that e can only set s to the timestamp of the item it enqueues. That means, e must have enqueued a value with timestamp t_d . However, by definition, t_d is read before e executes the CAS. This means another process (dequeuer/enqueue) has seen the value e enqueued and CAS s for e before t_d . By Lemma 5.2.1, this "another process" must be another dequeuer d' that precedes d because it overlaps with e .

Because d' and d cannot overlap, while e overlaps with both d' and d , e must be the *first* enqueue on s that overlaps with d . Combining with Lemma 5.2.1 and the fact that e executes the *last successful slot-modification instruction* on slot s within d 's **successful CAS-sequence**, e must be the only enqueue that executes a **successful slot-modification instruction** on s within d 's **successful CAS-sequence**.

During the start of d 's successful CAS-sequence till the end of e , spsc_readFront on the local SPSC must return the same element, because:

- There's no other dequeues running during this time.
- There's no enqueue other than e running.
- The `spsc_enqueue` of e must have completed before the start of d 's successful CAS sequence, because a previous dequeuer d' can see its effect.

Therefore, if we were to move the starting time of d 's successful CAS-sequence right after e has ended, we still retain the output of the program because:

- The CAS sequence only reads two shared values: `slots[rank]` and `spsc_readFront()`, but we have proven that these two values remain the same if we were to move the starting time of d 's successful CAS-sequence this way.
- The CAS sequence does not modify any values except for the last CAS instruction, and the ending time of the CAS sequence is still the same.
- The CAS sequence modifies `slots[rank]` at the CAS but the target value is the same because inputs and shared values are the same in both cases.

We have proven that if we move d 's successful CAS-sequence to start after the **last successful slot-modification instruction** on slot s within d 's **successful CAS-sequence**, we still retain the program's output. Furthermore, we only move the starting time of a CAS-sequence, therefore, the strict partial order defined above is not changed.

If we apply the reordering for every dequeue, the theorem directly follows. \square

Theorem 5.2.5 (*ABA safety of enqueue*) Assume that the 64-bit global counter never overflows, enqueue (Procedure 4) is ABA-safe.

Proof Consider a **successful CAS-sequence** on slot s by an enqueue e .

Denote t_e as the value this CAS-sequence observes.

Due to Lemma 5.2.1, there can only be at most one enqueue at one point in time within e .

If there's no **successful slot-modification instruction** on slot s by a dequeue d within e 's **successful CAS-sequence**, then this enqueue is ABA-safe.

Suppose the dequeue d executes the **last successful slot-modification instruction** on slot s within e 's **successful CAS-sequence**. Denote t_d to be the value that d sets s .

If $t_d \neq t_e$, this CAS-sequence of e cannot be successful, which is a contradiction.

Therefore, $t_d = t_e$.

If $t_d = t_e = \text{MAX}$, this means e observes a value of MAX before d even sets s to MAX. If this MAX value is the initialized value of s , it's a contradiction, as s must be non-MAX at some point for a dequeue such as d to run. If this MAX value is set by an enqueue, it's also a contradiction, as `refreshEnqueue` cannot set a slot to MAX. Therefore, this MAX value is set by a dequeue d' . If $d' \neq d$ then it's a contradiction, because between d' and d , s must be set to be a non-MAX value before d can be run. Therefore, $d' = d$. But, this means e observes a value set by d , which violates our assumption.

Therefore $t_d = t_e = t' \neq \text{MAX}$. e cannot observe the value t' set by d due to our assumption. Suppose e observes the value t' from s set by another enqueue/dequeue call other than d .

If this "another call" is a dequeue d' other than d , d' precedes d . By Lemma 5.2.2, after each dequeue, the front element's timestamp will be increasing, therefore, d' must have set s to a timestamp smaller than t_d . However, e observes $t_e = t_d$. This is a contradiction.

Therefore, this "another call" is an enqueue e' other than e and e' precedes e . We know that an enqueue only sets s to the timestamp it obtains.

Suppose e' does not overlap with d . e' can only set s to t' if e' sees that the local SPSC has the front element as the element it enqueues. Due to Lemma 5.2.1, this means e' must observe a local SPSC with only the element it enqueues. Then,

when d executes `readFront`, the item e' enqueues must have been dequeued out already, thus, d cannot set s to t' . This is a contradiction.

Therefore, e' overlaps with d .

For e' to set s to the same value as d , e' 's `spsc_readFront` must serialize after d 's `spsc_dequeue`.

Because e' and e cannot overlap, while d overlaps with both e' and e , d must be the *first* dequeue on s that overlaps with e . Combining with Lemma 5.2.1 and the fact that d executes the **last successful slot-modification instruction** on slot s within e 's **successful CAS-sequence**, d must be the only dequeue that executes a **successful slot-modification instruction** within e 's **successful CAS-sequence**.

During the start of e 's successful CAS-sequence till the end of d , `spsc_readFront` on the local SPSC must return the same element, because:

- There's no other enqueues running during this time.
- There's no dequeue other than d running.
- The `spsc_dequeue` of d must have completed before the start of e 's successful CAS sequence, because a previous enqueue e' can see its effect.

Therefore, if we were to move the starting time of e 's successful CAS-sequence right after d has ended, we still retain the output of the program because:

- The CAS sequence only reads two shared values: `slots[rank]` and `spsc_readFront()`, but we have proven that these two values remain the same if we were to move the starting time of e 's successful CAS-sequence this way.
- The CAS sequence does not modify any values except for the last CAS/store instruction, and the ending time of the CAS sequence is still the same.
- The CAS sequence modifies `slots[rank]` at the CAS but the target value is the same because inputs and shared values are the same in both cases.

We have proven that if we move e 's successful CAS-sequence to start after the **last successful slot-modification instruction** on slot s within e 's **successful CAS-sequence**, we still retain the program's output. Furthermore, we only move the starting time of a CAS-sequence, therefore, the strict partial order defined above is not changed.

If we apply the reordering for every enqueue, the theorem directly follows. \square

Theorem 5.2.6 (ABA safety) Assume that the 64-bit global counter never overflows, Slot-queue is ABA-safe.

Proof This follows from Theorem 5.2.5 and Theorem 5.2.4. \square

6. Linearizability of Slot-queue

Definition 6.1 For an enqueue or dequeue op , $rank(op)$ is the rank of the enqueue whose local SPSC is affected by op .

Definition 6.2 For an enqueue whose rank is r , the value stored in its corresponding slot at time t is denoted as $slot(r, t)$.

Definition 6.3 For an enqueue with rank r , the minimum timestamp among the elements between `First` and `Last` in its local SPSC at time t is denoted as $min-spsc-ts(r, t)$.

Definition 6.4 For an enqueue, **slot-refresh phase** refer to its execution of line 5-6 of Procedure 4.

Definition 6.5 For a dequeue, **slot-refresh phase** refer to its execution of line 19-20 of Procedure 6.

Definition 6.6 For a dequeue, **slot-scan phase** refer to its execution of line 24-34 of Procedure 7.

Definition 6.7 An enqueue operation e is said to **match** a dequeue operation d if d returns a timestamp that e enqueues. Similarly, d is said to **match** e . In this case, both e and d are said to be **matched**.

Definition 6.8 An enqueue operation e is said to be **unmatched** if no dequeue operation **matches** it.

Definition 6.9 A dequeue operation d is said to be **unmatched** if no enqueue operation **matches** it, in other word, d returns \perp .

We prove some algorithm-specific results first, which will form the basis for the more fundamental results.

Lemma 6.1 If an enqueue e begins its **slot-refresh phase** at time t_0 and finishes at time t_1 , there's always at least one successful refreshEnqueue or refreshDequeue on $rank(e)$ starting and ending its **CAS-sequence** between t_0 and t_1 .

Proof If one of the two refreshEnqueues succeeds, then the lemma obviously holds.

Consider the case where both fail.

The first refreshEnqueue fails because there's another refreshDequeue executing its **slot-modification instruction** successfully after t_0 but before the end of the first refreshEnqueue's **CAS-sequence**.

The second refreshEnqueue fails because there's another refreshDequeue executing its **slot-modification instruction** successfully after t_0 but before the end of the second refreshEnqueue's **CAS-sequence**. This another refreshDequeue must start its **CAS-sequence** after the end of the first successful refreshDequeue, due to Lemma 5.2.1. In other words, this another refreshDequeue starts and successfully ends its **CAS-sequence** between t_0 and t_1 .

We have proved the theorem. \square

Lemma 6.2 If a dequeue d begins its **slot-refresh phase** at time t_0 and finishes at time t_1 , there's always at least one successful refreshEnqueue or refreshDequeue on $rank(d)$ starting and ending its **CAS-sequence** between t_0 and t_1 .

Proof This is similar to the above lemma. \square

Lemma 6.3 Given a rank r and a dequeue d that begins its **slot-scan phase** at time t_0 and finishes at time t_1 . If d finds that $slot(r, t') = s_0 \neq \text{MAX}$ for some time t' such that $t_0 \leq t' \leq t_1$, then $slot(r, t) = s_0 \neq \text{MAX}$ for any t such that $t' \leq t \leq t_1$.

Proof Denote s_r as the slot of rank r .

$slot(r, t') = s_0 \neq \text{MAX}$ because some processes have executed a successful slot-modification instruction on s_r to set it to s_0 .

Take op to be the enqueue/dequeue that executes the last successful slot-modification instruction on s_r before t' . By definition, op set s_r to s_0 .

Any dequeue before d would have finished before t_0 , and thus its **slot-fresh phase**. By Lemma 6.2, for each dequeue before d , there must be some successful refresh call whose `spsc_readFront` observes the state of the local SPSC after d 's `spsc_dequeue`. By definition, op 's refresh call ended after all of these successful refresh call. By Theorem 5.2.6, the effect is as if op starts after all of these successful refresh call. Therefore, op can be treated as if it has seen the local SPSC after any of the previous dequeues' `spsc_dequeue` calls. In other words, op has set s_r to the front element's timestamp after it has observed all previous `spsc_dequeue` before d . During t_0 to t_1 , there's no `spsc_dequeue`. Therefore, from after op 's successful refresh call until t_1 , there is no new `spsc_dequeue` that can be observed. Any refresh calls after op until t_1 can only observe new `spsc_enqueues`, but because op set s_r to a non-MAX value, their corresponding refreshEnqueues cannot affect s_r . Therefore, the lemma holds. \square

Lemma 6.4 Given a rank r and a dequeue d that begins its **slot-scan phase** at time t_0 and finishes at time t_1 . If d finds that $slot(r, t') = \text{MAX}$ for some time t' such that $t_0 \leq t' \leq t_1$, then $slot(r, t) \neq \text{MAX}$ for any t such that $t_0 \leq t \leq t'$.

Proof Because during d 's **slot-scan phase**, no other dequeue can run and enqueues can only set a slot to non-MAX, if d finds that $slot(r, t') =$

MAX for some time t' such that $t_0 \leq t' \leq t_1$, then $slot(r, t) \neq \text{MAX}$ for any t such that $t_0 \leq t \leq t'$.

The theorem holds. \square

Lemma 6.5 If at time t , no enqueue or dequeue is running on the slot of rank r , then $min\text{-}spsc\text{-}ts(r, t) = slot(r, t)$.

Proof Denote op as the enqueue/dequeue that executes the last successful slot-modification instruction before t . \square

We now look at the more fundamental results.

Lemma 6.6 If d matches e , then either e precedes or overlaps with d .

Proof If d precedes e , none of the local SPSCs can contain an item with the timestamp of e . Therefore, d cannot return an item with a timestamp of e . Thus d cannot match e .

Therefore, e either precedes or overlaps with d . \square

Theorem 6.7 If an enqueue e precedes another dequeue d , then either:

- d isn't matched.
- d matches e .
- e matches d' and d' precedes d .
- d matches e' and e' precedes e .
- d matches e' and e' overlaps with e .

Proof \square

Lemma 6.8 If d matches e , then either e precedes or overlaps with d .

Proof \square

Theorem 6.9 If a dequeue d precedes another enqueue e , then either:

- d isn't matched.
- d matches e' such that e' precedes or overlaps with e and $e' \neq e$.

Proof \square

Theorem 6.10 If an enqueue e_0 precedes another enqueue e_1 , then either:

- Both e_0 and e_1 aren't matched.
- e_0 is matched but e_1 is not matched.
- e_0 matches d_0 and e_1 matches d_1 such that d_0 precedes d_1 .

Proof \square

Theorem 6.11 If a dequeue d_0 precedes another dequeue d_1 , then either:

- d_0 isn't matched.
- d_1 isn't matched.
- d_0 matches e_0 and d_1 matches e_1 such that e_0 precedes or overlaps with e_1 .

Proof \square

Theorem 6.12 (*Linearizability of Slot-queue*) Slot-queue is linearizable.

7. Wait-freedom

The algorithm is trivially wait-free as there is no possibility of infinite loops.

8. Memory-safety

The algorithm is memory-safe: No memory deallocation happens and accesses are only made on allocated memory.

References

- [1] P. Jayanti and S. Petrovic, "Logarithmic-time single deleter, multiple inserter wait-free queues and stacks," 2005, *Springer-Verlag*. doi: 10.1007/11590156_33.