

Slot-queue - An optimized wait-free distributed MPSC

1. Motivation

A good example of a wait-free MPSC has been presented in [1]. In this paper, the authors propose a novel tree-structure and a min-timestamp scheme that allow both enqueue and dequeue to be wait-free and always complete in $\Theta(\log n)$ where n is the number of enqueueers.

We have tried to port this algorithm to distributed context using MPI. The most problematic issue was that the original algorithm uses load-link/store-conditional (LL/SC). To adapt to MPI, we have to propose some modification to the original algorithm to make it use only compare-and-swap (CAS). Even though the resulting algorithm pretty much preserve the original algorithm's characteristic, that is wait-freedom and time complexity of $\Theta(\log n)$, we have to be aware that this is $\Theta(\log n)$ remote operations, which is very expensive. We have estimated that for an enqueue or a dequeue operation in our initial LTQueue version, there are about $2 * \log n$ to $10 * \log n$ remote operations, depending on data placements and the current state of the LTQueue.

Therefore, to be more suitable for distributed context, we propose a new algorithm that's inspired by LTQueue, in which both enqueue and dequeue only perform a constant number of remote operations, at the cost of dequeue having to perform $\Theta(n)$ local operations, where n is the number of enqueueers. Because remote operations are much more expensive, this might be a worthy tradeoff.

2. Structure

Each enqueue will have a local SPSC as in LTQueue [1] that supports dequeue, enqueue and readFront. There's a global queue whose entries store the minimum timestamp of the corresponding enqueueer's local SPSC.



Figure 1: Basic structure of slot queue

3. Pseudocode

3.1. SPSC

The SPSC of [1] is kept in tact, except that we change it into a circular buffer implementation.

Types

```

data_t = The type of data stored
spsc_t = The type of the local SPSC

record
    First: int
    Last: int
    Capacity: int
    Data: an array of data_t of capacity
    Capacity
end

```

Shared variables

```

First: index of the first undequeued entry
Last: index of the first unenqueued entry

```

Initialization

```

First = Last = 0
Set Capacity and allocate array.

```

The procedures are given as follows.

Procedure 1: `spsc_enqueue(v: data_t)` **returns** `bool`

```

1 if (Last + 1 == First)
2   | return false
3 Data[Last] = v
4 Last = (Last + 1) % Capacity
5 return true

```

Procedure 2: `spsc_dequeue()` **returns** `data_t`

```

6 if (First == Last) return  $\perp$ 
7 res = Data[First]
8 First = (First + 1) % Capacity
9 return res

```

Procedure 3: `spsc_readFront` **returns** `data_t`

```

10 if (First == Last)
11   | return  $\perp$ 
12 return Data[First]

```

3.2. Slot-queue

The slot-queue types and structures are given as follows:

Types

data_t = The type of data stored
 timestamp_t = `uint64_t`
 spsc_t = The type of the local SPSC

Shared variables

slots: An array of `timestamp_t` with the number of entries equal the number of enqueueers
 spscs: An array of `spsc_t` with the number of entries equal the number of enqueueers
 counter: `uint64_t`

Initialization

| Initialize all local SPSCs.

| Initialize slots entries to MAX.

The enqueue operations are given as follows:

Procedure 4: `enqueue(rank: int, v: data_t)` **returns** `bool`

```

1 timestamp = FAA(counter)
2 value = (v, timestamp)
3 res = spsc_enqueue(spsc_s[rank], value)
4 if (!res) return false
5 if (!refreshEnqueue(rank, timestamp))
6   | refreshEnqueue(rank, timestamp)
7 return res

```

Procedure 5: `refreshEnqueue(rank: int, ts: timestamp_t)` **returns** `bool`

```

8 old-timestamp = slots[rank]
9 front = spsc_readFront(spsc_s[rank])
10 new-timestamp = front ==  $\perp$  ? MAX : front.timestamp
11 if (new-timestamp != ts)
12   | return true
13 return CAS(&slots[rank], old-timestamp, new-timestamp)

```

The dequeue operations are given as follows:

Procedure 6: `dequeue()` **returns** `data_t`

```

14 rank = readMinimumRank()
15 if (rank == DUMMY || slots[rank] == MAX)
16   | return  $\perp$ 
17 res = spsc_dequeue(spsc_s[rank])
18 if (res ==  $\perp$ ) return  $\perp$ 
19 if (!refreshDequeue(rank))
20   | refreshDequeue(rank)
21 return res

```

Procedure 7: readMinimumRank() returns int

```

22 rank = length(slots)
23 min-timestamp = MAX
24 for index in 0..length(slots)
25   timestamp = slots[index]
26   if (min-timestamp < timestamp)
27     rank = index
28     min-timestamp = timestamp
29 old-rank = rank
30 for index in 0..old-rank
31   timestamp = slots[index]
32   if (min-timestamp < timestamp)
33     rank = index
34     min-timestamp = timestamp
35 return rank == length(slots) ? DUMMY :
   rank

```

Procedure 8: refreshDequeue(rank: int) returns bool

```

36 old-timestamp = slots[rank]
37 front = spsc_readFront(spsc[rank])
38 new-timestamp = front ==  $\perp$  ? MAX :
   front.timestamp
39 return CAS(&slots[rank], old-timestamp,
   new-timestamp)

```

4. Linearizability of the local SPSC

In this section, we prove that the local SPSC is linearizable.

Lemma 4.1 (*Linearizability of spsc_enqueue*) The linearization point of spsc_enqueue is right after line 2 or right after line 4.

Proof Notice that only spsc_enqueue can modify the Last shared variable and only spsc_dequeue can modify the First shared variable.

If line 2 is executed, that means $\text{Last} + 1 == \text{First}$ and the enqueue is deemed as failed. This

state can only be exited when an spsc_dequeue executes line 8. If line 2 is executed before line 8 of spsc_dequeue then it's safe that we linearize spsc_enqueue before spsc_dequeue. Therefore, line 2 is a linearization point of spsc_enqueue.

Suppose line 2 is not executed.

If line 4 hasn't been executed, the SPSC is as if no element has been enqueued. On the other hand, if line 4 is executed, the enqueue is sure to have completed and other spsc_dequeues or spsc_readFront can see this enqueue's effect. Therefore, line 4 is another linearization point of spsc_enqueue. \square

Lemma 4.2 (*Linearizability of spsc_dequeue*) The linearization point of spsc_dequeue is right after line 6 or right after line 8.

Proof Notice that only spsc_enqueue can modify the Tail shared variable and only spsc_dequeue can modify the Head shared variable.

If line 6 is executed, that means $\text{Last} == \text{First}$ and the dequeue is deemed as failed. This state can only be exited when an spsc_enqueue executes line 4. If line 6 is executed before line 4 of spsc_dequeue then it's safe that we linearize spsc_dequeue before spsc_enqueue. Therefore, line 6 is a linearization point of spsc_enqueue.

Suppose line 6 is not executed.

If line 8 hasn't been executed, the SPSC is as if no element has been dequeued. On the other hand, if line 8 is executed, the dequeue is sure to have completed and other spsc_enqueue or spsc_readFront can see this enqueue's effect. Therefore, line 8 is another linearization point of spsc_enqueue. \square

Lemma 4.3 (*Linearizability of spsc_readFront*) The linearization point spsc_readFront is right after line 11 or right after line 12.

Proof If line 11 is executed, that means spsc_readFront has just observed and effect of line 8 of spsc_dequeue or has not observed line 4 of spsc_enqueue. In this case, spsc_readFront

is linearized after this `spsc_dequeue` or before `spsc_enqueue`.

If line 12 is executed, that means `spsc_readFront` has just observed and effect of line 4 of `spsc_enqueue` or has not observed line 8 of `spsc_dequeue`. In this case, `spsc_readFront` is linearized after this `spsc_enqueue` or before `spsc_dequeue`. \square

Theorem 4.4 (*Linearizability of local SPSC*) The local SPSC is linearizable.

Proof This directly follows from Lemma 4.1, Lemma 4.2, Lemma 4.3. \square

5. ABA problem

Noticeably, we use no scheme to avoid ABA problem in Slot-queue. In actuality, ABA problem does not adversely affect our algorithm's correctness, except in the extreme case that the 64-bit global counter overflows, which is unlikely.

5.1. ABA-safety

Not every ABA problem is unsafe. We formalize in this section which ABA problem is safe and which is not.

Definition 5.1.1 A **modification instruction** on a variable v is an atomic instruction that may change the value of v e.g. a store or a CAS.

Definition 5.1.2 A **successful modification instruction** on a variable v is an atomic instruction that changes the value of v e.g. a store or a successful CAS.

Definition 5.1.3 A **CAS-sequence** on a variable v is a sequence of instructions of a method m such that:

- The first instruction is a load $v_0 = \text{load}(v)$.
- The last instruction is a CAS($\&v, v_0, v_1$).
- There's no modification instruction on v between the first and the last instruction.

Definition 5.1.4 A **successful CAS-sequence** on a variable v is a **CAS-sequence** on v that ends with a successful CAS.

Definition 5.1.5 Consider a method m on a concurrent object S . m is said to be **ABA-safe** if and only if for any history of method calls produced from S , we can reorder any successful CAS-sequences by an invocation of m in the following fashion:

- If a successful CAS-sequence is part of an invocation of m , after reordering, it must still be part of that invocation.
- If a successful CAS-sequence by an invocation of m precedes another in a method invocation, after reordering, this ordering is still respected.
- Any successful CAS-sequence by an invocation of m after reordering must not overlap with a successful modification instruction on the same variable.
- After reordering, all method calls' response events on the concurrent object S stay the same.

5.2. Proof of ABA-safety

Notice that we only use CAS on:

- Line 13 of `refreshEnqueue` (Procedure 5), or an `enqueue` in general (Procedure 4).
- Line 42 of `refreshDequeue` (Procedure 8) or a `dequeue` in general (Procedure 6).

Both CAS target some slot in the `slots` array.

We apply some domain knowledge of our algorithm to the above formalism.

Definition 5.2.1 A **CAS-sequence** on a slot s of an `enqueue` that corresponds to s is the sequence of instructions from line 8 to line 13 of its `refreshEnqueue`.

Definition 5.2.2 A **slot-modification instruction** on a slot s of an `enqueue` that corresponds to s is line 13 of `refreshEnqueue`.

Definition 5.2.3 A **CAS-sequence** on a slot s of a `dequeue` that corresponds to s is the sequence of instructions from line 36 to line 42 of its `refreshDequeue`.

Definition 5.2.4 A **slot-modification instruction** on a slot s of a `dequeue` that corresponds to s is line 40 or line 42 of `refreshDequeue`.

Definition 5.2.5 A **CAS-sequence** of a dequeue/enqueue is said to **observes a slot value of** s_0 if it loads s_0 at line 8 of refreshEnqueue or line 36 of refreshDequeue.

We can now turn to our interested problem in this section.

Lemma 5.2.1 (*Concurrent accesses on a local SPSC and a slot*) Only one dequeuer and one enqueue can concurrently modify a local SPSC and a slot in the slots array.

Proof This is trivial to prove based on the algorithm's definition. \square

Lemma 5.2.2 (*Monotonicity of local SPSC timestamps*) Each local SPSC in Slot-queue contains elements with increasing timestamps.

Proof Each enqueue would FAA the global counter (line 1 in Procedure 4) and enqueue into the local SPSC an item with the timestamp obtained from the counter. Applying Lemma 5.2.1, we know that items are enqueued one at a time into the SPSC. Therefore, later items are enqueued by later enqueues, which obtain increasing values by FAA-ing the shared counter. The theorem holds. \square

Lemma 5.2.3 A refreshEnqueue (Procedure 5) can only changes a slot to a value other than MAX.

Proof For refreshEnqueue to change the slot's value, the condition on line 11 must be false. Then new-timestamp must equal to t_s , which is not MAX. It's obvious that the CAS on line 13 changes the slot to a value other than MAX. \square

Theorem 5.2.4 (*ABA safety of dequeue*) Assume that the 64-bit global counter never overflows, dequeue (Procedure 6) is ABA-safe.

Proof Consider a **successful CAS-sequence** on slot s by a dequeue d .

Denote t_d as the value this CAS-sequence observes.

Due to Lemma 5.2.1, there can only be at most one enqueue at one point in time within d .

If there's no **successful slot-modification instruction** on slot s by an enqueue e within d 's **successful CAS-sequence**, then this dequeue is ABA-safe.

Suppose the enqueue e executes the **last successful slot-modification instruction** on slot s within d 's **successful CAS-sequence**. Denote t_e to be the value that e sets s .

If $t_e \neq t_d$, this CAS-sequence of d cannot be successful, which is a contradiction.

Therefore, $t_e = t_d$.

Note that e can only set s to the timestamp of the item it enqueues. That means, e must have enqueued a value with timestamp t_d . However, by definition, t_d is read before e executes the CAS. This means another process (dequeuer/enqueueer) has seen the value e enqueued and CAS s for e before t_d . By Lemma 5.2.1, this "another process" must be another dequeuer d' that precedes d because it overlaps with e .

Because d' and d cannot overlap, while e overlaps with both d' and d , e must be the *first* enqueue on s that overlaps with d . Combining with Lemma 5.2.1 and the fact that e executes the **last successful slot-modification instruction** on slot s within d 's **successful CAS-sequence**, e must be the only enqueue that executes a **successful slot-modification instruction** on s within d 's **successful CAS-sequence**.

During the start of d 's successful CAS-sequence till the end of e , `spsc_readFront` on the local SPSC must return the same element, because:

- There's no other dequeues running during this time.
- There's no enqueue other than e running.
- The `spsc_enqueue` of e must have completed before the start of d 's successful CAS sequence, because a previous dequeuer d' can see its effect.

Therefore, if we were to move the starting time of d 's successful CAS-sequence right after e has

ended, we still retain the output of the program because:

- The CAS sequence only reads two shared values: `slots[rank]` and `spsc_readFront()`, but we have proven that these two values remain the same if we were to move the starting time of d 's successful CAS-sequence this way.
- The CAS sequence does not modify any values except for the last CAS instruction, and the ending time of the CAS sequence is still the same.
- The CAS sequence modifies `slots[rank]` at the CAS but the target value is the same because inputs and shared values are the same in both cases.

We have proved that if we move d 's successful CAS-sequence to start after the *last successful slot-modification instruction* on slot s within d 's **successful CAS-sequence**, we still retain the program's output.

If we apply the reordering for every dequeue, the theorem directly follows. \square

Theorem 5.2.5 (*ABA safety of enqueue*) Assume that the 64-bit global counter never overflows, enqueue (Procedure 4) is ABA-safe.

Proof Consider a **successful CAS-sequence** on slot s by an enqueue e .

Denote t_e as the value this CAS-sequence observes.

Due to Lemma 5.2.1, there can only be at most one enqueue at one point in time within e .

If there's no **successful slot-modification instruction** on slot s by a dequeue d within e 's **successful CAS-sequence**, then this enqueue is ABA-safe.

Suppose the dequeue d executes the *last successful slot-modification instruction* on slot s within e 's **successful CAS-sequence**. Denote t_d to be the value that d sets s .

If $t_d \neq t_e$, this CAS-sequence of e cannot be successful, which is a contradiction.

Therefore, $t_d = t_e$.

If $t_d = t_e = \text{MAX}$, this means e observes a value of MAX before d even sets s to MAX. If this MAX value is the initialized value of s , it's a contradiction, as s must be non-MAX at some point for a dequeue such as d to run. If this MAX value is set by an enqueue, it's also a contradiction, as `refreshEnqueue` cannot set a slot to MAX. Therefore, this MAX value is set by a dequeue d' . If $d' \neq d$ then it's a contradiction, because between d' and d , s must be set to be a non-MAX value before d can be run. Therefore, $d' = d$. But, this means e observes a value set by d , which violates our assumption.

Therefore $t_d = t_e = t' \neq \text{MAX}$. e cannot observe the value t' set by d due to our assumption. Suppose e observes the value t' from s set by another enqueue/dequeue call other than d .

If this "another call" is a dequeue d' other than d , d' precedes d . By Lemma 5.2.2, after each dequeue, the front element's timestamp will be increasing, therefore, d' must have set s to a timestamp smaller than t_d . However, e observes $t_e = t_d$. This is a contradiction.

Therefore, this "another call" is an enqueue e' other than e and e' precedes e . We know that an enqueue only sets s to the timestamp it obtains.

Suppose e' does not overlap with d . e' can only set s to t' if e' sees that the local SPSC has the front element as the element it enqueues. Due to Lemma 5.2.1, this means e' must observe a local SPSC with only the element it enqueues. Then, when d executes `readFront`, the item e' enqueues must have been dequeued out already, thus, d cannot set s to t' . This is a contradiction.

Therefore, e' overlaps with d .

For e' to set s to the same value as d , e' 's `spsc_readFront` must serialize after d 's `spsc_dequeue`.

Because e' and e cannot overlap, while d overlaps with both e' and e , d must be the *first* dequeue on s that overlaps with e . Combining with Lemma 5.2.1

and the fact that d executes the *last successful slot-modification instruction* on slot s within e 's *successful CAS-sequence*, d must be the only dequeue that executes a *successful slot-modification instruction* within e 's *successful CAS-sequence*.

During the start of e 's successful CAS-sequence till the end of d , `spsc_readFront` on the local SPSC must return the same element, because:

- There's no other enqueues running during this time.
- There's no dequeue other than d running.
- The `spsc_dequeue` of d must have completed before the start of e 's successful CAS sequence, because a previous enqueue e' can see its effect.

Therefore, if we were to move the starting time of e 's successful CAS-sequence right after d has ended, we still retain the output of the program because:

- The CAS sequence only reads two shared values: `slots[rank]` and `spsc_readFront()`, but we have proven that these two values remain the same if we were to move the starting time of e 's successful CAS-sequence this way.
- The CAS sequence does not modify any values except for the last CAS/store instruction, and the ending time of the CAS sequence is still the same.
- The CAS sequence modifies `slots[rank]` at the CAS but the target value is the same because inputs and shared values are the same in both cases.

We have proved that if we move e 's successful CAS-sequence to start after the *last successful slot-modification instruction* on slot s within e 's *successful CAS-sequence*, we still retain the program's output.

If we apply the reordering for every enqueue, the theorem directly follows. \square

Theorem 5.2.6 (ABA safety) Assume that the 64-bit global counter never overflows, Slot-queue is ABA-safe.

Proof This follows from Theorem 5.2.5 and Theorem 5.2.4. \square

6. Linearizability of Slot-queue

Definition 6.1 For an enqueue or dequeue op , $rank(op)$ is the rank of the enqueue whose local SPSC is affected by op .

Definition 6.2 For an enqueue whose rank is r , the value stored in its corresponding slot at time t is denoted as $slot(r, t)$.

Definition 6.3 For an enqueue with rank r , the minimum timestamp among the elements between First and Last in its local SPSC at time t is denoted as $min_spsc_ts(r, t)$.

Definition 6.4 For an enqueue, **slot-refresh phase** refer to its execution of line 5-6 of Procedure 4.

Definition 6.5 For a dequeue, **slot-refresh phase** refer to its execution of line 19-20 of Procedure 6.

Definition 6.6 For a dequeue, **slot-scan phase** refer to its execution of line 24-34 of Procedure 7.

Definition 6.7 An enqueue operation e is said to **match** a dequeue operation d if d returns a timestamp that e enqueues. Similarly, d is said to **match** e . In this case, both e and d are said to be **matched**.

Definition 6.8 An enqueue operation e is said to be **unmatched** if no dequeue operation **matches** it.

Definition 6.9 A dequeue operation d is said to be **unmatched** if no enqueue operation **matches** it, in other word, d returns \perp .

We prove some algorithm-specific results first, which will form the basis for the more fundamental results.

Lemma 6.1 If an enqueue e begins its **slot-refresh phase** at time t_0 and finishes at time t_1 , there's always at least one successful

refreshEnqueue or refreshDequeue on $rank(e)$ starting and ending its **CAS-sequence** between t_0 and t_1 .

Proof If one of the two refreshEnqueues succeeds, then the lemma obviously holds.

Consider the case where both fail.

The first refreshEnqueue fails because there's another refreshDequeue executing its **slot-modification instruction** successfully after t_0 but before the end of the first refreshEnqueue's **CAS-sequence**.

The second refreshEnqueue fails because there's another refreshDequeue executing its **slot-modification instruction** successfully after t_0 but before the end of the second refreshEnqueue's **CAS-sequence**. This another refreshDequeue must start its **CAS-sequence** after the end of the first successful refreshDequeue, due to Lemma 5.2.1. In other words, this another refreshDequeue starts and successfully ends its **CAS-sequence** between t_0 and t_1 .

We have proved the theorem. \square

Lemma 6.2 If a dequeue d begins its **slot-refresh phase** at time t_0 and finishes at time t_1 , there's always at least one successful refreshEnqueue or refreshDequeue on $rank(d)$ starting and ending its **CAS-sequence** between t_0 and t_1 .

Proof This is similar to the above lemma. \square

Lemma 6.3 Given a rank r , if an enqueue e on r that obtains the timestamp c completes at t_0 and is still unmatched by t_1 , then $slot(r, t) \leq c$ for any $t \in [t_0, t_1]$.

Proof Take t' to be the time e 's `spsc_enqueue` takes effect.

By Lemma 6.1, there must be a successful refresh call that observes the effect of `spsc_enqueue` happening at t'' , $t'' \in [t', t_0]$.

By the same reasoning as in Theorem 5.2.6, any successful slot-modification instructions happening after t'' must observe the effect of

`spsc_enqueue`. However, because e is never matched between t'' and t_1 , the timestamp c is in the local SPSC the whole timespan $[t'', t_1]$. Therefore, any slot-modification instructions during $[t'', t_1]$ must set the slot's value to some value not greater than c . \square

We now look at the more fundamental results.

Theorem 6.4 If an enqueue e precedes another dequeue d , then either:

- d isn't matched.
- d matches e .
- e matches d' and d' precedes d .
- d matches e' and e' precedes e .
- d matches e' and e' overlaps with e .

Proof If d doesn't match anything, the theorem holds. If d matches e , the theorem also holds. Suppose d matches e' , $e' \neq e$.

If e matches d' and d' precedes d , the theorem also holds. Suppose e matches d' such that d precedes d' or is unmatched. (1)

Suppose e obtains a timestamp of c and e' obtains a timestamp of c' .

Due to (1), at the time d starts, e has finished but it is still unmatched. By the way Procedure 7 is defined and by Lemma 6.3, d would find a slot that stores a timestamp that is not greater than the one e enqueues. In other word, $c' \leq c$. But $c' \neq c$, then $c' < c$. Therefore, e cannot precede e' , otherwise, $c < c'$.

So, either e' precedes or overlaps with e . The theorem holds. \square

Lemma 6.5 If d matches e , then either e precedes or overlaps with d .

Proof If d precedes e , none of the local SPSCs can contain an item with the timestamp of e . Therefore, d cannot return an item with a timestamp of e . Thus d cannot match e .

Therefore, e either precedes or overlaps with d . \square

Theorem 6.6 If a dequeue d precedes another enqueue e , then either:

- d isn't matched.
- d matches e' such that e' precedes or overlaps with e and $e' \neq e$.

Proof If d isn't matched, the theorem holds.

Suppose d matches e' . By Lemma 6.5, either e' precedes or overlaps with d . Therefore, $e' \neq e$. Furthermore, e cannot precede e' , because then d would precede e' .

We have proved the theorem. \square

Theorem 6.7 If an enqueue e_0 precedes another enqueue e_1 , then either:

- Both e_0 and e_1 aren't matched.
- e_0 is matched but e_1 is not matched.
- e_0 matches d_0 and e_1 matches d_1 such that d_0 precedes d_1 .

Proof if e_1 is not matched, the theorem holds.

Suppose e_1 matches d_1 . By Lemma 6.5, either e_1 precedes or overlaps with d_1 .

Suppose the contrary, e_0 is unmatched or e_0 matches d_0 such that d_1 precedes d_0 , then when d_1 starts, e_0 is still unmatched.

If e_0 and e_1 targets the same rank, it's obvious that d_1 must prioritize e_0 over e_1 . Thus d_1 cannot match e_1 .

If e_0 targets a later rank than e_1 , d_1 cannot find e_1 in the first scan, because the scan is left-to-right, and if it finds e_1 it would later find e_0 that has a lower timestamp. Suppose d_1 finds e_1 in the second scan, that means d_1 finds $e' \neq e_1$ and e' 's timestamp is larger than e_1 's, which is larger than e_0 's. Due to the scan being left-to-right, e' must target a later rank than e_1 . If e' also targets a later rank than e_0 , then in the second scan, d_1 would have prioritized e_0 that has a lower timestamp. Suppose e' targets an earlier rank than e_0 but later than e_1 . Because e_0 's timestamp is larger than e' 's, it must precede or overlap with e . Similarly, e_1 must precede or overlap with e . Because e' targets an earlier rank than e_0 , e_0 's **slot-refresh phase** must finish after e' 's. That means e_1 must start

after e' 's **slot-refresh phase**, because e_0 precedes e_1 . But then, e_1 must obtain a timestamp larger than e' , which is a contradiction.

Suppose e_0 targets an earlier rank than e_1 . If d_1 finds e_1 in the first scan, then in the second scan, d_1 would have prioritize e_0 's timestamp. Suppose d_1 finds e_1 in the second scan and during the first scan, it finds $e' \neq e_1$ and e' 's timestamp is larger than e_1 's, which is larger than e_0 's. Due to how the second scan is defined, e' targets a later rank than e_1 , which targets a later rank than e_0 . Because during the second scan, e_0 is not chosen, its **slot-refresh phase** must finish after e' 's. Because e_0 precedes e_1 , e_1 must start after e' 's **slot-refresh phase**, so it must obtain a larger timestamp than e' , which is a contradiction.

Therefore, by contradiction, e_0 must be matched and e_0 matches d_0 such that d_0 precedes d_1 . \square

Theorem 6.8 If a dequeue d_0 precedes another dequeue d_1 , then either:

- d_0 isn't matched.
- d_1 isn't matched.
- d_0 matches e_0 and d_1 matches e_1 such that e_0 precedes or overlaps with e_1 .

Proof If either d_0 isn't matched or d_1 isn't matched, the theorem holds.

Suppose d_0 matches e_0 and d_1 matches e_1 .

If e_1 precedes e_0 , applying Theorem 6.7, we have e_1 matches d_1 and e_0 matches d_0 such that d_1 precedes d_0 . This is a contradiction.

Therefore, e_0 either precedes or overlaps with e_1 . \square

Theorem 6.9 (*Linearizability of Slot-queue*) Slot-queue is linearizable.

Proof Suppose some history H produced from the Slot-queue algorithm.

If H contains some pending method calls, we can just wait for them to complete (because the algorithm is wait-free, which we will prove later). Therefore, now we consider all H to contain only

completed method calls. So, we know that if a dequeue or an enqueue in H is matched or not.

If there are some unmatched enqueues, we can append dequeues sequentially to the end of H until there's no unmatched enqueues. Consider one such H' .

We already have a strict partial order $\rightarrow_{H'}$ on H' .

Because the queue is MPSC, there's already a total order among the dequeues.

We will extend $\rightarrow_{H'}$ to a strict total order $\Rightarrow_{H'}$ on H' as follows:

- If $X \rightarrow_{H'} Y$ then $X \Rightarrow_{H'} Y$. (1)
- If a dequeue d matches e then $e \Rightarrow_{H'} d$. (2)
- If a dequeue d_0 matches e_0 and another dequeue matches e_1 such that $d_0 \Rightarrow_{H'} d_1$ then $e_0 \Rightarrow_{H'} e_1$. (3)
- If a dequeue d overlaps with an enqueue e but does not match e , $d \Rightarrow_{H'} e$. (4)

We will prove that $\Rightarrow_{H'}$ is a strict total order on H' . That is, for every pair of different method calls X and Y , either exactly one of these is true $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ and for any X , $X \not\Rightarrow_{H'} X$.

It's obvious that $X \not\Rightarrow_{H'} X$.

If X and Y are dequeues, because there's a total order among the dequeues, either exactly one of these is true: $X \rightarrow_{H'} Y$ or $Y \rightarrow_{H'} X$. Then due to (1), either $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$. Notice that we cannot obtain $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ from (2), (3), or (4).

Therefore, exactly one of $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ is true. (*)

If X is dequeue and Y is enqueue, in this case (3) cannot help us obtain either $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$, so we can disregard it.

- If $X \rightarrow_{H'} Y$, then due to (1), $X \Rightarrow_{H'} Y$. By definition, X precedes Y , so (4) cannot apply. Applying Theorem 6.6, either
 - X isn't matched, (2) cannot apply. Therefore, $Y \not\Rightarrow_{H'} X$.
 - X matches e' and $e' \neq Y$. Therefore, X does not match Y , or (2) cannot apply. Therefore, $Y \not\Rightarrow_{H'} X$.

Therefore, in this case, $X \Rightarrow_{H'} Y$ and $Y \not\Rightarrow_{H'} X$.

- If $Y \rightarrow_{H'} X$, then due to (1), $Y \Rightarrow_{H'} X$. By definition, Y precedes X , so (4) cannot apply. Even if (2) applies, it can only help us obtain $Y \Rightarrow_{H'} X$. Therefore, in this case, $Y \Rightarrow_{H'} X$ and $X \not\Rightarrow_{H'} Y$.
- If X overlaps with Y :
 - If X matches Y , then due to (2), $Y \Rightarrow_{H'} X$. Because X matches Y , (4) cannot apply. Therefore, in this case $Y \Rightarrow_{H'} X$ but $X \not\Rightarrow_{H'} Y$.
 - If X does not match Y , then due to (4), $X \Rightarrow_{H'} Y$. Because X doesn't match Y , (2) cannot apply. Therefore, in this case $X \Rightarrow_{H'} Y$ but $Y \not\Rightarrow_{H'} X$.

Therefore, exactly one of $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ is true. (**)

If X is enqueue and Y is enqueue, in this case (2) and (4) are irrelevant:

- If $X \rightarrow_{H'} Y$, then due to (1), $X \Rightarrow_{H'} Y$. By definition, X precedes Y . Applying Theorem 6.7,
 - Both X and Y aren't matched, then (3) cannot apply. Therefore, in this case, $Y \not\Rightarrow_{H'} X$.
 - X is matched but Y is not matched, then (3) cannot apply. Therefore, in this case, $Y \not\Rightarrow_{H'} X$.
 - X matches d_x and Y matches d_y such that d_x precedes d_y , then (3) applies and we obtain $X \Rightarrow_{H'} Y$.

Therefore, in this case, $X \Rightarrow_{H'} Y$ but $Y \not\Rightarrow_{H'} X$.

- If $Y \rightarrow_{H'} X$, this case is symmetric to the first case. We obtain $Y \Rightarrow_{H'} X$ but $X \not\Rightarrow_{H'} Y$.
- If X overlaps with Y , because in H' , all enqueues are matched, then, X matches d_x and d_y . Because d_x either precedes or succeeds d_y , Applying (3), we obtain either $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ and there's no way to obtain the other.

Therefore, exactly one of $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ is true. (***)

From (*), (**), (***), we have proved that $\Rightarrow_{H'}$ is a strict total ordering that is consistent with $\rightarrow_{H'}$. In other words, we can order method calls in H' in a sequential manner. We will prove that this sequential order is consistent with FIFO semantics:

- An item can only be dequeued once: This is trivial as a dequeue can only match one enqueue.
- Items are dequeued in the order they are enqueued: Suppose there are two enqueues e_1, e_2 such that $e_1 \Rightarrow_{H'} e_2$ and suppose they match d_1 and d_2 . Then we have obtained $e_1 \Rightarrow_{H'} e_2$ either because:
 - (3) applies, in this case $d_1 \Rightarrow_{H'} d_2$ is a condition for it to apply.
 - (1) applies, then e_1 precedes e_2 , by Theorem 6.7, d_1 must precede d_2 , thus $d_1 \Rightarrow_{H'} d_2$.

Therefore, if $e_1 \Rightarrow_{H'} e_2$ then $d_1 \Rightarrow_{H'} d_2$.

- An item can only be dequeued after it's enqueued: Suppose there is an enqueue e matched by d . By (2), obviously $e \Rightarrow_{H'} d$.
- If the queue is empty, dequeues return nothing. Suppose a dequeue d such that any $e \Rightarrow_{H'} d$ is all matched by some d' and $d' \Rightarrow_{H'} d$, we will prove that d is unmatched. By Lemma 6.5, d can only match an enqueue e_0 that precedes or overlaps with d .
 - If e_0 precedes d , by our assumption, it's already matched by another dequeue.
 - If e_0 overlaps with d , by our assumption, $d \Rightarrow_{H'} e_0$ because if $e_0 \Rightarrow_{H'} d$, e_0 is already matched by another d' . Then, we can only obtain this because (4) applies, but then d does not match e_0 .

Therefore, d is unmatched.

In conclusion, $\Rightarrow_{H'}$ is a way we can order method calls in H' sequentially that conforms to FIFO semantics. Therefore, we can also order method calls in H sequentially that conforms to FIFO semantics as we only append dequeues sequentially to the end of H to obtain H' .

We have proved the theorem. \square

7. Wait-freedom

The algorithm is trivially wait-free as there is no possibility of infinite loops.

8. Memory-safety

The algorithm is memory-safe: No memory deallocation happens and accesses are only made on allocated memory.

References

- [1] P. Jayanti and S. Petrovic, "Logarithmic-time single deleter, multiple inserter wait-free queues and stacks," 2005, *Springer-Verlag*. doi: 10.1007/11590156_33.