

Modified LTQueue without Load-Link/Store-Conditional

1. Original LTQueue

The original algorithm is given in [1] by Prasad Jayanti and Srdjan Petrovic in 2005. LTQueue is a wait-free MPSC algorithm, with logarithmic-time complexity for both enqueue and dequeue operations. The algorithm achieves good scalability by distributing the “global” queue over n queues that are local to each enqueueer. This helps avoid contention on the global queue among the enqueueers and allows multiple enqueueers to succeed at the same time. Furthermore, each enqueue and dequeue is efficient: they are wait-free and guaranteed to complete in $\theta(\log n)$ steps where n is the number of enqueueers. This is possible due to the novel tree structure proposed by the authors.

1.1. Local queue algorithm

Each local queue in LTQueue is an SPSC that allows an enqueueer and a dequeuer to concurrently access. Every enqueueer has one such local SPSC. Only the owner enqueueer and the dequeuer can access this local SPSC.

This section presents the SPSC data structure proposed in [1]. Beside the usual enqueue and dequeue procedures, this SPSC also supports the readFront procedure, which allows the enqueueer and dequeuer to retrieve the first item in the SPSC. Notice that enqueueer and dequeuer each has its own readFront method.

Types

```
data_t = The type of data stored
node_t = The type of the linked-list's nodes
record
  val: data_t
  next: pointer to node_t
end
```

Shared variables

```
First: pointer to node_t
Last: pointer to node_t
Announce: pointer to node_t
FreeLater: pointer to node_t
Help: data_t
```

Initialization

```
First = Last = new Node()
FreeLater = new Node()
```

The SPSC always has a dummy node at the end.

The enqueueer's procedures are given as follows.

Procedure 1: spsc_enqueue(v: data_t)

```
1 newNode = new Node()
2 tmp = Last
3 tmp.val = v
4 tmp.next = newNode
5 Last = newNode
```

Procedure 2: spsc_readFront_e() **returns** data_t

```
6 tmp = First
7 if (tmp == Last) return ⊥
8 Announce = tmp
9 if (tmp != First)
10 | retval = Help
11 else retval = tmp.val
12 return retval
```

The dequeuer's procedures are given as follows.

Procedure 3: spsc_dequeue() returns data_t

```

13 tmp = First
14 if (tmp == Last) return  $\perp$ 
15 retval = tmp.val
16 Help = retval
17 First = tmp.next
18 if (tmp == Announce)
19   tmp' = FreeLater
20   FreeLater = tmp
21   free(tmp')
22 else free(tmp)
23 return retval

```

Procedure 4: spsc_readFront_d() returns data_t

```

24 tmp = First
25 if (tmp == Last)
26   return  $\perp$ 
27 return tmp.val

```

The treatment of linearizability, wait-freedom and memory safety is given in the original paper [1]. However, we can establish some intuition by looking at the procedures.

For wait-freedom, each procedure doesn't loop and doesn't wait for any other procedures to be able to complete on its own, therefore, they are all wait-free.

For memory safety, note that spsc_enqueue never accesses freed memory because the node pointed to by Last is never freed inside spsc_dequeue. spsc_readFront_e tries to read First (line 6) and announces it not to be deleted to the dequeuer (line 8), it then checks if the pointer it read is still First (line 9), if it is, then it's safe to dereference the pointer (line 11) because the dequeuer would take note not to free it (line 18), otherwise, it returns Help (line 10), which is safely placed by the dequeuer (line 16). spsc_dequeue safely reclaims memory and does not leak memory (line 18-22).

spsc_readFront_d is safe because there's only one dequeuer running at a time.

For linearizability, the linearization point of spsc_enqueue is after line 5, spsc_readFront_e is right after line 8, spsc_dequeue is right after line 17 and spsc_readFront_d is right after line 25. Additionally, all the operations take constant time no matter the size of the queue.

1.2. LTQueue algorithm

LTQueue's idea is to maintain a tree structure as in Image 1. Each enqueueer is represented by the local SPSC node at the bottom of the tree. Every SPSC node in the local queue contains data and a timestamp indicating when it's enqueued into the SPSC. For consistency in node structure, we consider the leaf nodes of the tree to be the ones that are attached to the local SPSC of each enqueueer. Every internal node contains the minimum timestamp among its children's timestamps.

Types

| | |
|--|--|
| | data_t = The type of the data to be stored in LTQueue |
| | spsc_t = The type of the local SPSC |
| | tree_t = The type of the tree constructed by LTQueue |
| | node_t = The node type of tree_t, containing a 64-bit timestamp value, packing a monotonic counter and the enqueueer's rank. |

Shared variables

| | |
|--|--|
| | counter: integer (counter supports LL and SC operations) |
| | Q: array [1..n] of spsc_t |
| | T: tree_t |

Initialization

| | |
|--|-------------|
| | counter = 0 |
|--|-------------|



Image 1: LTQueue's structure.

Procedure 5: enqueue(rank: int, value: data_t)

```

1 count = LL(counter)
2 SC(counter, count + 1)
3 timestamp = (count, rank)
4 spsc_enqueue(Q[rank], (value, timestamp))
5 propagate(Q[rank])

```

Procedure 6: dequeue() returns data_t

```

6 [count, rank] = read(root(T))
7 if (rank == ⊥) return ⊥
8 ret = spsc_dequeue(Q[rank])
9 propagate(Q[rank])
10 return ret.val

```

The followings are the timestamp propagation procedures.

Procedure 7: propagate(spsc: spsc_t)

```

11 if ¬refreshLeaf(spsc)
12 | refreshLeaf(spsc)
13 currentNode = leafNode(spsc)
14 repeat
15 | currentNode = parent(currentNode)
16 | if ¬refresh(currentNode)
17 | | refresh(currentNode)
18 until currentNode == root(T)

```

Procedure 8: refresh(currentNode: pointer to node_t)

```

19 LL(currentNode)
20 for childNode in children(currentNode)
21 | let minT be the minimum timestamp for every childNode
22 SC(currentNode, minT)

```

Procedure 9: refreshLeaf(spsc: spsc_t)

```

23 leafNode = leafNode(spsc)
24 LL(leafNode)
25 SC(leafNode, spsc_readFront(spsc))

```

Note that compare to the original paper [1], we have make some trivial modification on line 11-12 to handle the leaf node case, which was left unspecified in the original algorithm. In many ways, this modification is in the same light with the mechanism the algorithm is already using, so intuitively, it should not affect the algorithm's correctness or wait-freedom. Note that on line 25 of refreshLeaf, we omit which version of spsc_readFront it's calling, simply assume that the dequeuer and the enqueueer should call their corresponding version of spsc_readFront.

Similarly, the proofs of LTQueue's linearizability, wait-freedom, memory-safety and logarithmic-time complexity of enqueue and dequeue operations are given in [1]. One notable technique that allows LTQueue to be both correct and wait-free is the double-refresh trick during the propagation process on line 16-17.

The idea behind the propagate procedure is simple: Each time an SPSC queue is modified (inserted/deleted), the timestamp of a leaf has

changed so the timestamps of all nodes on the path from that leaf to the root can potentially change. Therefore, we have to propagate the change towards the root, starting from the leaf (line 11-18).

The refresh procedure is by itself simple: we access all child nodes to determine the minimum timestamp in each child's subtree and try to set the current node's timestamp with the minimum timestamp using a pair of LL/SC. However, LL/SC can not always succeed so the current node's timestamp may not be updated by refresh at all. The key to fix this is to retry refresh on line 17 in case of the first refresh's failure. Later, when we prove the correctness of the modified LTQueue, we provide a formal proof of why this works. Here, for intuition, we visualize in Image 2 the case where both refresh fails but correctness is still ensures.

2. Adaption of LTQueue without load-link/store-conditional

The SPSC data structure in the original LTQueue is kept intact so one may refer to Procedure 1, Procedure 2, Procedure 3, Procedure 4 for the supported SPSC procedures.

The followings are the rewritten LTQueue's algorithm without LL/SC.



Image 2: Even though two refreshes fails, the currentNode's timestamp is still updated correctly.

The structure of LTQueue is modified as in Image 3. At the bottom **enqueuer nodes** (represented by the type `enqueuer_t`), besides the local SPSC, the minimum-timestamp among the elements in the SPSC is also stored. The **internal nodes** no longer store a timestamp but a rank of an enqueuer. This rank corresponds to the enqueuer with the minimum timestamp among the node's children's ranks. Note that if a local SPSC is empty, the minimum-timestamp of the corresponding enqueuer node is set to MAX and the corresponding leaf node's rank is set to a DUMMY rank.

Types

`data_t` = The type of the data to be stored in LTQueue

`spsc_t` = The type of the local SPSC

`rank_t` = The rank of an enqueuer

```

struct
  | value: uint32_t
  | version: uint32_t
end
timestamp_t =
struct
  | value: uint32_t

```

```

  | version: uint32_t
end
enqueuer_t =
struct
  | spsc: spsc_t
  | min-timestamp: timestamp_t
end
node_t = The node type of the tree constructed by LTQueue
struct
  | rank: rank_t
end

```

Shared variables

```

counter: uint64_t
root: pointer to node_t
enqueuers: array [1..n] of enqueuer_t

```

Initialization

```

counter = 0
construct the tree structure and set root to the root node

```



Image 3: Modified LTQueue's structure.

```

initialize every node in the tree to contain DUMMY
rank and version 0
initialize every enqueueer's timestamp to MAX
and version 0

```

Procedure 10: enqueue(rank: int, value: data_t)

```

1 timestamp = FAA(counter)
2 spsc_enqueue(enqueuers[rank].spsc,
  (value, timestamp))
3 propagate(rank)

```

Procedure 11: dequeue() returns data_t

```

4 [rank, version] = root->rank
5 if (rank == DUMMY) return  $\perp$ 
6 ret = spsc_dequeue(enqueuers[rank].spsc)
7 propagate(rank)
8 return ret.val

```

We omit the description of procedures parent, leafNode, children, leaving how the tree is constructed and children-parent relationship is determined to the implementor. The tree structure used by LTQueue is read-only so a wait-free implementation of these procedures is trivial.

Procedure 12: propagate(rank: uint32_t)

```

9 if  $\neg$ refreshTimestamp(rank)
10 | refreshTimestamp(rank)
11 if  $\neg$ refreshLeaf(rank)
12 | refreshLeaf(rank)
13 currentNode = leafNode(rank)
14 repeat
15 | currentNode = parent(currentNode)
16 | if  $\neg$ refresh(currentNode)
17 | | refresh(currentNode)
18 until currentNode == root

```

Procedure 13: refresh(currentNode: pointer to node_t)

```

19 [old-rank, old-version] = currentNode->rank
20 min-rank = DUMMY
21 min-timestamp = MAX
22 for childNode in children(currentNode)
23 | [child-rank, ...] = childNode->rank
24 | if (child-rank == DUMMY) continue
25 | child-timestamp = enqueuers[child-rank].min-timestamp
26 | if (child-timestamp < min-timestamp)
27 | | min-timestamp = child-timestamp
28 | | min-rank = child-rank
    return CAS(&currentNode->rank, [old-rank, old-version], [min-rank, old-version + 1])

```

Procedure 14: refreshTimestamp(rank: uint32_t)

```

30 [old-timestamp, old-version] = enqueuers[rank].timestamp
    front
31 = spsc_readFront(enqueuers[rank].spsc)
32 if (front ==  $\perp$ )
33 | return CAS(&enqueuers[rank].timestamp, [old-timestamp, old-version], [MAX, old-version + 1])
34 else
35 | return CAS(&enqueuers[rank].timestamp, [old-timestamp, old-version], [front.timestamp, old-version + 1])

```

Procedure 15: refreshLeaf(rank: uint32_t)

```

36 leafNode = leafNode(spsc)
37 [old-rank, old-version] = leafNode->rank
   [timestamp, ...,]
38 = enqueueers[rank].timestamp
   return CAS(&leafNode->rank, [old-rank,
39 old-version], [timestamp == MAX ? DUMMY :
   rank, old-version + 1])

```

Notice that we omit which version of `spsc_readFront` we're calling on line 32, simply assuming that the producer and each enqueueer are calling their respective version.

3. Proof of correctness

This section proves that the algorithm given in the last section is linearizable, memory-safe and wait-free.

3.1. Linearizability

Within the next two sections, we formalize what it means for an MPSC to be linearizable.

Definition of linearizability

The following discussion of linearizability is based on [2] by Herlihy and Shavit.

For a concurrent object S , we can call some methods on S concurrently. A method call on the object S is said to have an **invocation event** when it starts and a **response event** when it ends.

Definition 3.1.1 An **invocation event** is a triple $(S, t, args)$, where S is the object the method is invoked on, t is the timestamp of when the event happens and $args$ is the arguments passed to the method call.

Definition 3.1.2 A **response event** is a triple (S, t, res) , where S is the object the method is invoked on, t is the timestamp of when the event happens and res is the results of the method call.

Definition 3.1.3 A **method call** is a tuple of (i, r) where i is an invocation event and r is a response event or the special value \perp indicating that its response event hasn't happened yet. A well-formed **method call** should have a response event with a larger timestamp than its invocation event or the response event hasn't happened yet.

Definition 3.1.4 A **method call** is **pending** if its invocation event is \perp .

Definition 3.1.5 A **history** is a set of well-formed **method calls**.

Definition 3.1.6 An extension of **history** H is a **history** H' such that any pending method call is given a response event.

We can define a **strict partial order** on the set of well-formed method calls:

Definition 3.1.7 \rightarrow is a relation on the set of well-formed method calls. With two method calls X and Y , we have $X \rightarrow Y \Leftrightarrow X$'s response event is not \perp and its response timestamp is not greater than Y 's invocation timestamp.

Definition 3.1.8 Given a **history** H , \rightarrow_H is a relation on H such that for two method calls X and Y in H , $X \rightarrow_H Y \Leftrightarrow X \rightarrow Y$.

Definition 3.1.9 A **sequential history** H is a **history** such that \rightarrow_H is a total order on H .

Now that we have formalized the way to describe the order of events via **histories**, we can now formalize the mechanism to determine if a **history** is valid. The easier case is for a **sequential history**:

Definition 3.1.10 For a concurrent object S , a **sequential specification** of S is a function that either returns true (valid) or false (invalid) for a **sequential history** H .

The harder case is handled via the notion of **linearizable**:

Definition 3.1.11 A history H on a concurrent object S is **linearizable** if it has an extension H' and there exists a **sequential history** H_S such that:

1. The **sequential specification** of S accepts H_S .
2. There exists a one-to-one mapping M of a method call $(i, r) \in H'$ to a method call $(i_S, r_S) \in H_S$ with the properties that:
 - i must be the same as i_S except for the time-stamp.
 - r must be the same r_S except for the time-stamp or r .
3. For any two method calls X and Y in H' , $X \rightarrow_{H'} Y \Rightarrow M(X) \rightarrow_{H_S} M(Y)$.

We consider a history to be valid if it's linearizable.

Definition of linearizable MPSC

An MPSC supports 2 **methods**:

- enqueue which accepts a value and returns nothing
- dequeue which doesn't accept anything and returns a value

An MPSC has the same **sequential specification** as a FIFO:

- dequeue returns values in the same order as they were enqueued.
- An enqueue can only be dequeued once.
- An item can only be dequeued after it's enqueued.
- If the queue is empty, dequeue returns nothing.

An MPSC places a special constraint on **the set of histories** it can produce: Any history H must not have overlapping dequeue method calls.

Definition 3.1.12 An MPSC is **linearizable** if and only if any history produced from the MPSC that does not have overlapping dequeue method calls is *linearizable* according to the *FIFO sequential specification*.

Proof of linearizability

Definition 3.1.13 An enqueue operation e is said to **match** a dequeue operation d if d returns a timestamp that e enqueues. Similarly, d is said to **match** e . In this case, both e and d are said to be **matched**.

Definition 3.1.14 An enqueue operation e is said to be **unmatched** if no dequeue operation **matches** it.

Definition 3.1.15 A dequeue operation d is said to be **unmatched** if no enqueue operation **matches** it, in other word, d returns \perp .

Theorem 3.1.1 Only the dequeuer and one enqueue can operate on an enqueue node.

Proof This is trivial. \square

We immediately obtain the following result.

Corollary 3.1.2 Only one dequeue operation and one enqueue operation can operate concurrently on an enqueue node.

Proof This is trivial. \square

Theorem 3.1.3 The SPSC at an enqueue node contains items with increasing timestamps.

Proof Each enqueue would FAA the shared counter (line 1 in Procedure 10) and enqueue into the local SPSC an item with the timestamp obtained from the counter. Applying Corollary 3.1.2, we know that items are enqueued one at a time into the SPSC. Therefore, later items are enqueued by later enqueues, which obtain increasing values by FAA-ing the shared counter. The theorem holds. \square

Definition 3.1.16 For a tree node n , the enqueue rank stored in n at time t is denoted as $rank(n, t)$.

Definition 3.1.17 For an enqueue or a dequeue op, the rank of the enqueue it affects is denoted as $rank(op)$.

Definition 3.1.18 For an enqueue whose rank is r , the min-timestamp value stored in its enqueue node at time t is denoted as $min-ts(r, t)$. If r is DUMMY, $min-ts(r, t)$ is MAX.

Definition 3.1.19 For an enqueue with rank r , the minimum timestamp among the elements between First and Last in the local SPSC at time t is denoted as $min-spvc-ts(r, t)$. If r is dummy, $min-spvc-ts(r, t)$ is MAX.

Definition 3.1.20 For an enqueue or a dequeue op, the set of nodes that it calls refresh or refreshLeaf on is denoted as $path(op)$.

Definition 3.1.21 For an enqueue or a dequeue, **timestamp-refresh phase** refer to its execution of line 9-10 in propagate (Procedure 12).

Definition 3.1.22 For an enqueue or a dequeue op, and a node $n \in path(op)$, **node- n -refresh phase** refer to its execution of line 11-12 (if n is a leaf node) or line 16-17 (if n is a non-leaf node) to refresh n 's rank in propagate (Procedure 12).

Definition 3.1.23 refreshTimestamp is said to start its **CAS-sequence** if it finishes line 30 in Procedure 14. refreshTimestamp is said to end its **CAS-sequence** if it finishes line 33 or line 35 in Procedure 14.

Definition 3.1.24 refresh is said to start its **CAS-sequence** if it finishes line 19 in Procedure 13. refresh is said to end its **CAS-sequence** if it finishes line 29 in Procedure 13.

Definition 3.1.25 refreshLeaf is said to start its **CAS-sequence** if it finishes line 37 in Procedure 15. refreshLeaf is said to end its **CAS-sequence** if it finishes line 39 in Procedure 15.

Theorem 3.1.4 For an enqueue or a dequeue op, if op modifies an enqueueer node and this enqueueer node is attached to a leaf node l , then $path(op)$ is the set of nodes lying on the path from l to the root node.

Proof This is trivial considering how propagate (Procedure 12) works. \square

Theorem 3.1.5 For any time t and a node n , $rank(n, t)$ can only be DUMMY or the rank of one of the enqueueer nodes in the subtree rooted at n .

Proof This is trivial considering how refresh and refreshLeaf works. \square

Theorem 3.1.6 If an enqueue or a dequeue op begins its **timestamp-refresh phase** at t_0 and finishes at time t_1 , there's always at least one suc-

cessful refreshTimestamp on $rank(op)$ starting and ending its **CAS-sequence** between t_0 and t_1 .

Proof If one of the two refreshTimestamps succeeds, then the theorem obviously holds.

Consider the case where both fail.

The first refreshTimestamp fails because there's another refreshTimestamp ending its **CAS-sequence** successfully after t_0 but before the end of the first refreshTimestamp's **CAS-sequence**.

The second refreshTimestamp fails because there's another refreshTimestamp ending its **CAS-sequence** successfully after t_0 but before the end of the second refreshTimestamp's **CAS-sequence**. This another refreshTimestamp must start its **CAS-sequence** after the end of the first successful refreshTimestamp, else it overlaps with the **CAS-sequence** of the first successful refreshTimestamp but successful **CAS-sequences** cannot overlap. In other words, this another refreshTimestamp starts and successfully ends its **CAS-sequence** between t_0 and t_1 .

We have proved the theorem. \square

Theorem 3.1.7 If an enqueue or a dequeue begins its **node- n -refresh phase** at t_0 and finishes at t_1 , there's always at least one successful refresh or refreshLeaf on n starting and ending its **CAS-sequence** between t_0 and t_1 .

Proof This is similar to the above proof. \square

Theorem 3.1.8 For any node n , $min-ts(rank(n, t_x), t_y)$ is monotonically decreasing for $t_x, t_y \in [t_0, t_1]$ if within t_0 and t_1 , any dequeue d where $n \in path(d)$ has finished its **node- n -refresh phase**.

Proof We have the assumption that within t_0 and t_1 , all dequeue where $n \in path(d)$ has finished its **node- n -refresh phase**. Notice that if n satisfies this assumption, any child of n also satisfies this assumption.

We will prove a stronger version of this theorem: Given a node n , time t_0 and t_1 such that within

$[t_0, t_1]$, any dequeue d where $n \in \text{path}(d)$ has finished its **node- n -refresh phase**. Consider the last dequeue's **node- n -refresh phase** before t_0 (there maybe none). Take $t_s(n)$ and $t_e(n)$ to be the starting and ending time of the CAS-sequence of the last successful **n -refresh call** during this phase, or if there is none, $t_s(n) = t_e(n) = 0$. Then, $\min\text{-ts}(\text{rank}(n, t_x), t_y)$ is monotonically decreasing for $t_x, t_y \in [t_e(n), t_1]$.

Consider any enqueueer node of rank r that's attached to a satisfied leaf node. For any n' that is a descendant of n , during $t_s(n')$ and t_1 , there's no call to `spsc_dequeue`. Because:

- If an `spsc_dequeue` starts between t_0 and t_1 , the dequeue that calls it hasn't finished its **node- n' -refresh phase**.
- If an `spsc_dequeue` starts between $t_s(n')$ and t_0 , then a dequeue's **node- n' -refresh phase** must start after $t_s(n')$ and before t_0 , but this violates our assumption of $t_s(n')$.

Therefore, there can only be calls to `spsc_enqueue` during $t_s(n')$ and t_1 . Thus, $\min\text{-spsc-ts}(r, t_x)$ can only decrease from MAX to some timestamp and remain constant for $t_x \in [t_s(n'), t_1]$. (1)

Similarly, there can be no dequeue that hasn't finished its **timestamp-refresh phase** during $t_s(n')$ and t_1 . Therefore, $\min\text{-ts}(r, t_x)$ can only decrease from MAX to some timestamp and remain constant for $t_x \in [t_s(n'), t_1]$. (2)

Consider any satisfied leaf node n_0 . There can be no dequeue that hasn't finished its **node- n_0 -refresh phase** during $t_e(n_0)$ and t_1 . Therefore, any successful `refreshLeaf` on n_0 during $[t_e(n_0), t_1]$ must be called by an enqueue. Because there's no `spsc_dequeue`, this `refreshLeaf` can only set $\text{rank}(n_0, t_x)$ from DUMMY to r and this remain r until t_1 , which is the rank of the enqueueer node it's attached to. Therefore, combining with (1), $\min\text{-ts}(\text{rank}(n_0, t_x), t_y)$ is monotonically decreasing for $t_x, t_y \in [t_e(n_0), t_1]$. (3)

Consider any satisfied non-leaf node n' that is a descendant of n . Suppose during $[t_e(n'), t_1]$,

we have a sequence of successful **n' -refresh calls** that start their CAS-sequences at $t_{\text{start-0}} < t_{\text{start-1}} < t_{\text{start-2}} < \dots < t_{\text{start-k}}$ and end them at $t_{\text{end-0}} < t_{\text{end-1}} < t_{\text{end-2}} < \dots < t_{\text{end-k}}$. By definition, $t_{\text{end-0}} = t_e(n')$ and $t_{\text{start-0}} = t_s(n')$. We can prove that $t_{\text{end-i}} < t_{\text{start-(i+1)}}$ because successful CAS-sequences cannot overlap.

Due to how refresh is defined, for any $k \geq i \geq 1$:

- Suppose $t_{\text{rank-i}}(c)$ is the time refresh reads the rank stored in the child node c , so $t_{\text{start-i}} \leq t_{\text{rank-i}}(c) \leq t_{\text{end-i}}$.
- Suppose $t_{\text{ts-i}}(c)$ is the time refresh reads the timestamp stored in the enqueueer with the rank read previously, so $t_{\text{start-i}} \leq t_{\text{ts-i}}(c) \leq t_{\text{end-i}}$.
- There exists a child c_i such that $\text{rank}(n', t_{\text{end-i}}) = \text{rank}(c_i, t_{\text{rank-i}}(c_i))$. (4)
- For every child c of n' ,

$$\min\text{-ts}(\text{rank}(n', t_{\text{end-i}}), t_{\text{ts-i}}(c_i)) \leq \min\text{-ts}(\text{rank}(c, t_{\text{rank-i}}(c)), t_{\text{ts-i}}(c)).$$
 (5)

Suppose the stronger theorem already holds for every child c of n' . (6)

For any $i \geq 1$, we have $t_e(c) \leq t_s(n') \leq t_{\text{start-(i-1)}} \leq t_{\text{rank-(i-1)}}(c) \leq t_{\text{end-(i-1)}} \leq t_{\text{start-i}} \leq t_{\text{rank-i}}(c) \leq t_1$. Combining with (5), (6), we have for any $k \geq i \geq 1$,

$$\begin{aligned} & \min\text{-ts}(\text{rank}(n', t_{\text{end-i}}), t_{\text{ts-i}}(c_i)) \\ & \leq \min\text{-ts}(\text{rank}(c, t_{\text{rank-i}}(c)), t_{\text{ts-i}}(c)) \\ & \leq \min\text{-ts}(\text{rank}(c, t_{\text{rank-(i-1)}}(c)), t_{\text{ts-i}}(c)). \end{aligned}$$

Choose $c = c_{i-1}$ as in (4). We have for any $k \geq i \geq 1$,

$$\begin{aligned} & \min\text{-ts}(\text{rank}(n', t_{\text{end-i}}), t_{\text{ts-i}}(c_i)) \\ & \leq \min\text{-ts}(\text{rank}(c_{i-1}, t_{\text{rank-(i-1)}}(c_{i-1})), t_{\text{ts-i}}(c_{i-1})) \\ & = \min\text{-ts}(\text{rank}(n', t_{\text{end-(i-1)}}), t_{\text{ts-i}}(c_{i-1})). \end{aligned}$$

Because $t_{\text{ts-i}}(c_i) \leq t_{\text{end-i}}$ and $t_{\text{ts-i}}(c_{i-1}) \geq t_{\text{end-(i-1)}}$ and (2), we have for any $k \geq i \geq 1$,

$$\min\text{-ts}(\text{rank}(n', t_{\text{end-i}}), t_{\text{end-i}}) \leq \min\text{-ts}(\text{rank}(n', t_{\text{end-(i-1)}}), t_{\text{end-(i-1)}}). \quad (*)$$

$\text{rank}(n', t_x)$ can only change after each successful refresh, therefore, the sequence of its value is $\text{rank}(n', t_{\text{end-0}})$, $\text{rank}(n', t_{\text{end-1}})$, ..., $\text{rank}(n', t_{\text{end-k}})$. (**)

Note that if refresh observes that an enqueuee has a min-timestamp of MAX, it would never try to CAS n' 's rank to the rank of that enqueuee (line 21 and line 26 of Procedure 13). So, if refresh actually set the rank of n' to some non-DUMMY value, the corresponding enqueuee must actually has a non-MAX min-timestamp at some point. Due to (2), this is constant up until t_1 . Therefore, $\min\text{-ts}(\text{rank}(n', t_{\text{end-}i}), t))$ is constant for any $t \geq t_{\text{end-}i}$ and $k \geq i \geq 1$. $\min\text{-ts}(\text{rank}(n', t_{\text{end-}0}), t))$ is constant for any $t \geq t_{\text{end-}0}$ if there's a refresh before t_0 . If there's no refresh before t_0 , it is constant MAX. So, $\min\text{-ts}(\text{rank}(n', t_{\text{end-}i}), t))$ is constant for any $t \geq t_{\text{end-}i}$ and $k \geq i \geq 0$. (***)

Combining (*), (**), (***), we obtain the stronger version of the theorem. \square

Theorem 3.1.9 If an enqueuee e obtains a timestamp c and finishes at time t_0 and is still **unmatched** at time t_1 , then for any subrange T of $[t_0, t_1]$ that does not overlap with a dequeue, $\min\text{-ts}(\text{rank}(\text{root}, t_r), t_s) \leq c$ for any $t_r, t_s \in T$.

Proof We will prove a stronger version of this theorem: Suppose an enqueuee e obtains a timestamp c and finishes at time t_0 and is still **unmatched** at time t_1 . For every $n_i \in \text{path}(e)$, n_0 is the leaf node and n_i is the parent of n_{i-1} , $i \geq 1$. If e starts and finishes its **node- n_i -refresh phase** at $t_{\text{start-}i}$ and $t_{\text{end-}i}$ then for any subrange T of $[t_{\text{end-}i}, t_1]$ that does not overlap with a dequeue d where $n_i \in \text{path}(d)$ and d hasn't finished its **node n_i refresh phase**, $\min\text{-ts}(\text{rank}(n_i, t_r), t_s) \leq c$ for any $t_r, t_s \in T$.

If $t_1 < t_0$ then the theorem holds.

Take r_e to be the rank of the enqueuee that performs e .

Suppose e enqueues an item with the timestamp c into the local SPSC at time t_{enqueuee} . Because it's still unmatched up until t_1 , c is always in the local SPSC during t_{enqueuee} to t_1 . Therefore, $\min\text{-spsc-ts}(r_e, t) \leq c$ for any $t \in [t_{\text{enqueuee}}, t_1]$. (1)

Suppose e finishes its **timestamp refresh phase** at $t_{r\text{-ts}}$. Because $t_{r\text{-ts}} \geq t_{\text{enqueuee}}$, due to (1), $\min\text{-ts}(r_e, t) \leq c$ for every $t \in [t_{r\text{-ts}}, t_1]$. (2)

Consider the leaf node $n_0 \in \text{path}(e)$. Due to (2), $\text{rank}(n_0, t)$ is always r_e for any $t \in [t_{\text{end-}0}, t_1]$. Also due to (2), $\min\text{-ts}(\text{rank}(n_0, t_r), t_s) \leq c$ for any $t_r, t_s \in [t_{\text{end-}0}, t_1]$.

Consider any non-leaf node $n_i \in \text{path}(e)$. We can extend any subrange T to the left until we either:

- Reach a dequeue d such that $n_i \in \text{path}(d)$ and d has just finished its **node- n_i -refresh phase**.
- Reach $t_{\text{end-}i}$.

Consider one such subrange T_i .

Notice that T_i always starts right after a **node- n_i -refresh phase**. Due to Theorem 3.1.7, there's always at least one successful refresh in this **node- n_i -refresh phase**.

Suppose the stronger version of the theorem already holds for n_{i-1} . That is, if e starts and finishes its **node- n_{i-1} -refresh phase** at $t_{\text{start-}(i-1)}$ and $t_{\text{end-}(i-1)}$ then for any subrange T of $[t_{\text{end-}(i-1)}, t_1]$ that does not overlap with a dequeue d where $n_i \in \text{path}(d)$ and d hasn't finished its **node n_{i-1} refresh phase**, $\min\text{-ts}(\text{rank}(n_i, t_r), t_s) \leq c$ for any $t_r, t_s \in T$.

Extend T_i to the left until we either:

- Reach a dequeue d such that $n_i \in \text{path}(d)$ and d has just finished its **node- n_{i-1} -refresh phase**.
- Reach $t_{\text{end-}(i-1)}$.

Take the resulting range to be T_{i-1} . Obviously, $T_i \subseteq T_{i-1}$.

T_{i-1} satisfies both criteria:

- It's a subrange of $[t_{\text{end-}(i-1)}, t_1]$.
- It does not overlap with a dequeue d where $n_i \in \text{path}(d)$ and d hasn't finished its **node- n_{i-1} -refresh phase**.

Therefore, $\min\text{-ts}(\text{rank}(n_{i-1}, t_r), t_s) \leq c$ for any $t_r, t_s \in T_{i-1}$.

Consider the last successful refresh on n_i ending not after T_i , take $t_{s'}$ and $t_{e'}$ to be the start and

end time of this refresh's CAS-sequence. Because right at the start of T_i , a **node- n_i -refresh phase** just ends, this refresh must be within this **node- n_i -refresh phase**. (4)

This refresh's CAS-sequence must be within T_{i-1} . This is because right at the start of T_{i-1} , a **node- n_{i-1} -refresh phase** just ends and $T_{i-1} \supseteq T_i$, T_{i-1} must cover the **node- n_i -refresh phase** whose end T_i starts from. Combining with (4), $t_{s'} \in T_{i-1}$ and $t_{e'} \in T_i$. (5)

Due to how refresh is defined and the fact that n_{i-1} is a child of n_i :

- t_{rank} is the time refresh reads the rank stored in n_{i-1} , so that $t_{s'} \leq t_{rank} \leq t_{e'}$. Combining with (5), $t_{rank} \in T_{i-1}$.
- t_{ts} is the time refresh reads the timestamp from that rank $t_{s'} \leq t_{ts} \leq t_{e'}$. Combining with (5), $t_{ts} \in T_{i-1}$.
- There exists a time t' , $t_{s'} \leq t' \leq t_{e'}$, $\min-ts(rank(n_i, t_{e'}), t') \leq \min-ts(rank(n_{i-1}, t_{rank}), t_{ts})$. (6)

From (6) and the fact that $t_{rank} \in T_{i-1}$ and $t_{ts} \in T_{i-1}$, $\min-ts(rank(n_i, t_{e'}), t') \leq c$.

There shall be no spsc_dequeue starting within $t_{s'}$ till the end of T_i because:

- If there's an spsc_dequeue starting within T_i , then T_i 's assumption is violated.
- If there's an spsc_dequeue starting after $t_{s'}$ but before T_i , its dequeue must finish its **node- n_i -refresh phase** after $t_{s'}$ and before T_i . However, then $t_{e'}$ is no longer the end of the last successful refresh on n_i not after T_i .

Because there's no spsc_dequeue starting in this timespan, $\min-ts(rank(n_i, t_{e'}), t_{e'}) \leq \min-ts(rank(n_i, t_{e'}), t') \leq c$.

If there's no dequeue between $t_{e'}$ and the end of T_i whose **node- n_i -refresh phase** hasn't finished, then by Theorem 3.1.8, $\min-ts(rank(n_i, t_r), t_s)$ is monotonically decreasing for any t_r, t_s starting from $t_{e'}$ till the end of T_i . Therefore, $\min-ts(rank(n_i, t_r), t_s) \leq c$ for any $t_r, t_s \in T_i$.

Suppose there's a dequeue whose **node- n_i -refresh phase** is in progress some time between $t_{e'}$ and the end of T_i . By definition, this dequeue must finish it before T_i . Because $t_{e'}$ is the time of the last successful refresh on n_i before T_i , $t_{e'}$ must be within the **node- n_i -refresh phase** of this dequeue and there should be no dequeue after that. By the way $t_{e'}$ is defined, technically, this dequeue has finished its **node- n_i -refresh phase** right at $t_{e'}$. Therefore, similarly, we can apply Theorem 3.1.8, $\min-ts(rank(n_i, t_r), t_s) \leq c$ for any $t_r, t_s \in T_i$.

By induction, we have proved the stronger version of the theorem. \square

Corollary 3.1.10 If an enqueue e obtains a timestamp c and finishes at time t_0 and is still **unmatched** at time t_1 , then for any sub-range T of $[t_0, t_1]$ that does not overlap with a dequeue, $\min-spsc-ts(rank(root, t_r), t_s) \leq c$ for any $t_r, t_s \in T$.

Proof Call t_{start} and t_{end} to be the start and end time of T .

Applying Theorem 3.1.9, we have that $\min-ts(rank(root, t_r), t_s) \leq c$ for any $t_r, t_s \in T$.

Fix t_r so that $rank(root, t_r) = r$. We have that $\min-ts(r, t) \leq c$ for any $t \in T$.

$\min-ts(r, t)$ can only change due to a successful refreshTimestamp on the enqueuer node with rank r . Consider the last successful refreshTimestamp on the enqueuer node with rank r not after T . Suppose that refreshTimestamp reads out the minimum timestamp of the local SPSC at $t' \leq t_{start}$.

Therefore, $\min-ts(r, t_{start}) = \min-spsc-ts(r, t') \leq c$.

We will prove that after t' until t_{end} , there's no spsc_dequeue on r running.

Suppose the contrary, then this spsc_dequeue must be part of a dequeue. By definition, this dequeue must start and end before t_{start} , else it violates the assumption of T . If this spsc_dequeue

starts after t' , then its `refreshTimestamp` must finish after t' and before t_{start} . But this violates the assumption that the last `refreshTimestamp` not after t_{start} reads out the minimum timestamp at t' .

Therefore, there's no `spsc_dequeue` on r running during $[t', t_{end}]$. Therefore, $min_spsc_ts(r, t)$ remains constant during $[t', t_{end}]$ because it's not MAX.

In conclusion, $min_spsc_ts(r, t) \leq c$ for $t \in [t', t_{end}]$.

We have proved the theorem. \square

Theorem 3.1.11 Given a rank r . If within $[t_0, t_1]$, there's no uncompleted enqueues on rank r and all matching dequeues for any completed enqueues on rank r has finished, then $rank(n, t) \neq r$ for every node n and $t \in [t_0, t_1]$.

Proof If n doesn't lie on the path from root to the leaf node that's attached to the enqueue node with rank r , the theorem obviously holds.

Due to Corollary 3.1.2, there can only be one enqueue and one dequeue at a time at an enqueue node with rank r . Therefore, there is a sequential ordering among the enqueues and a sequential ordering within the dequeues. Therefore, it's sensible to talk about the last enqueue before t_0 and the last matching dequeue d before t_0 .

Since all of these dequeues and enqueues work on the same local SPSC and the SPSC is linearizable, d must match the last enqueue. After this dequeue d , the local SPSC is empty.

When d finishes its **timestamp-refresh phase** at $t_{ts} \leq t_0$, due to Theorem 3.1.6, there's at least one successful `refreshTimestamp` call in this phase. Because the last enqueue has been matched, $min_ts(r, t) = \text{MAX}$ for any $t \in [t_{ts}, t_1]$.

Similarly, for a leaf node n_0 , suppose d finishes its **node- n_0 -refresh phase** at $t_{r-0} \geq t_{ts}$, then $rank(n_0, t) = \text{DUMMY}$ for any $t \in [t_{r-0}, t_1]$. (1)

For any non-leaf node $n_i \in path(d)$, when d finishes its **node- n_i -refresh phase** at t_{r-i} , there's at least one successful refresh call during this phase. Suppose this refresh call starts and ends at $t_{start-i}$ and t_{end-i} . Suppose $rank(n_{i-1}, t) \neq r$ for $t \in [t_{r-(i-1)}, t_1]$. By the way refresh is defined after this refresh call, n_i will store some rank other than r . Because of (1), after this up until t_1 , r never has a chance to be visible to a refresh on node n_i during $[n_{i-1}, t]$. In other words, $rank(n_i, t) \neq r$ for $t \in [t_{r-i}, t_1]$.

By induction, we obtain the theorem. \square

Theorem 3.1.12 If an enqueue e precedes another dequeue d , then either:

- d isn't matched.
- d matches e .
- e matches d' and d' precedes d .
- d matches e' and e' precedes e .
- d matches e' and e' overlaps with e .

Proof If d doesn't match anything, the theorem holds. If d matches e , the theorem also holds. Suppose d matches e' , $e' \neq e$.

If e matches d' and d' precedes d , the theorem also holds. Suppose e matches d' such that d precedes d' or is unmatched. (1)

Suppose e obtains a timestamp of c and e' obtains a timestamp of c' .

Because e precedes d and because an MPSC does not allow multiple dequeues, from the start of d at t_0 until after line 4 of dequeue (Procedure 11) at t_1 , e has finished and there's no dequeue running that has *actually performed* `spsc_dequeue`. Also by t_0 and t_1 , e is still unmatched due to (1).

Applying Corollary 3.1.10, $min_spsc_ts(rank(root, t_x), t_y) \leq c$ for $t_x, t_y \in [t_0, t_1]$. Therefore, d reads out a rank r such that $min_spsc_ts(r, t) \leq c$ for $t \in [t_0, t_1]$. Consequently, d dequeues out a value with a timestamp not greater than c . Because d matches e' , $c' \leq c$. However, $e' \neq e$ so $c' < c$.

This means that e cannot precede e' , because if so, $c < c'$.

Therefore, e' precedes e or overlaps with e . \square

Lemma 3.1.13 If d matches e , then either e precedes or overlaps with d .

Proof If d precedes e , none of the local SPSCs can contain an item with the timestamp of e . Therefore, d cannot return an item with a timestamp of e . Thus d cannot match e .

Therefore, e either precedes or overlaps with d . \square

Theorem 3.1.14 If a dequeue d precedes another enqueue e , then either:

- d isn't matched.
- d matches e' such that e' precedes or overlaps with e and $e' \neq e$.

Proof If d isn't matched, the theorem holds.

Suppose d matches e' . Applying Lemma 3.1.13, e' must precede or overlap with d . In other words, d cannot precede e' .

If e precedes or is e' , then d must precede e' , which is contradictory.

Therefore, e' must precede e or overlap with e . \square

Theorem 3.1.15 If an enqueue e_0 precedes another enqueue e_1 , then either:

- Both e_0 and e_1 aren't matched.
- e_0 is matched but e_1 is not matched.
- e_0 matches d_0 and e_1 matches d_1 such that d_0 precedes d_1 .

Proof If both e_0 and e_1 aren't matched, the theorem holds.

Suppose e_1 matches d_1 . By Lemma 3.1.13, either e_1 precedes or overlaps with d_1 .

If e_0 precedes d_1 , applying Theorem 3.1.12 for d_1 and e_0 :

- d_1 isn't matched, contradictory.
- d_1 matches e_0 , contradictory.
- e_0 matches d_0 and d_0 precedes d_1 , the theorem holds.
- d_1 matches e_1 and e_1 precedes e_0 , contradictory.
- d_1 matches e_1 and e_1 overlaps with e_0 , contradictory.

If d_1 precedes e_0 , applying Theorem 3.1.14 for d_1 and e_0 :

- d_1 isn't matched, contradictory.
- d_1 matches e_1 and e_1 precedes or overlaps with e_0 , contradictory.

Consider that d_1 overlaps with e_0 , then d_1 must also overlap with e_1 . Call r_1 the rank of the enqueue that performs e_1 . Call t to be the time d_1 atomically reads the root's rank on line 4 of dequeue (Procedure 11). Because d_1 matches e_1 , d_1 must read out r_1 at t_1 .

If e_1 is the first enqueue of rank r_1 , then t must be after e_1 has started, because otherwise, due to Theorem 3.1.11, r_1 would not be in *root* before e_1 .

If e_1 is not the first enqueue of rank r_1 , then t must also be after e_1 has started. Suppose the contrary, t is before e_1 has started:

- If there's no uncompleted enqueue of rank r_1 at t and they are all matched by the time t , due to Theorem 3.1.11, r_1 would not be in *root* at t . Therefore, d_1 cannot read out r_1 , which is contradictory.
- If there's some unmatched enqueue of rank r_1 at t , d_1 will match one of these enqueues instead because:
 - There's only one dequeue at a time, so unmatched enqueues at t remain unmatched until d_1 performs an `spsc_dequeue`.
 - Due to Corollary 3.1.2, all the enqueues of rank r_1 must finish before another starts. Therefore, there's some unmatched enqueue of rank r_1 finishing before e_1 .
 - The local SPSC of the enqueue node of rank r_1 is serializable, so d_1 will favor one of these enqueues over e_1 .

Therefore, t must happen after e_1 has started. Right at t , no dequeue is actually modifying the LTQueue state and e_0 has finished. If e_0 has been matched at t then the theorem holds. If e_0 hasn't been matched at t , applying Theorem 3.1.9, d_1 will favor e_0 over e_1 , which is a contradiction.

We have proved the theorem. \square

Theorem 3.1.16 If a dequeue d_0 precedes another dequeue d_1 , then either:

- d_0 isn't matched.
- d_1 isn't matched.
- d_0 matches e_0 and d_1 matches e_1 such that e_0 precedes or overlaps with e_1 .

Proof If d_0 isn't matched or d_1 isn't matched, the theorem holds.

Suppose d_0 matches e_0 and d_1 matches e_1 .

Suppose the contrary, e_1 precedes e_0 . Applying Theorem 3.1.12:

- Both e_0 and e_1 aren't matched, which is contradictory.
- e_1 is matched but e_0 is not matched, which contradictory.
- e_1 matches d_1 and e_0 matches d_0 such that d_1 precedes d_0 , which is contradictory.

Therefore, the theorem holds. \square

Theorem 3.1.17 The modified LTQueue algorithm is linearizable.

Proof Suppose some history H produced from the modified LTQueue algorithm.

If H contains some pending method calls, we can just wait for them to complete (because the algorithm is wait-free, which we will prove later). Therefore, now we consider all H to contain only completed method calls. So, we know that if a dequeue or an enqueue in H is matched or not.

If there are some unmatched enqueues, we can append dequeues sequentially to the end of H until there's no unmatched enqueues. Consider one such H' .

We already have a strict partial order $\rightarrow_{H'}$ on H' .

Because the queue is MPSC, there's already a total order among the dequeues.

We will extend $\rightarrow_{H'}$ to a strict total order $\Rightarrow_{H'}$ on H' as follows:

- If $X \rightarrow_{H'} Y$ then $X \Rightarrow_{H'} Y$. (1)
- If a dequeue d matches e then $e \Rightarrow_{H'} d$. (2)
- If a dequeue d_0 matches e_0 and another dequeue matches e_1 such that $d_0 \Rightarrow_{H'} d_1$ then $e_0 \Rightarrow_{H'} e_1$. (3)
- If a dequeue d overlaps with an enqueue e but does not match e , $d \Rightarrow_{H'} e$. (4)

We will prove that $\Rightarrow_{H'}$ is a strict total order on H' . That is, for every pair of different method calls X and Y , either exactly one of these is true $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ and for any X , $X \not\Rightarrow_{H'} X$.

It's obvious that $X \not\Rightarrow_{H'} X$.

If X and Y are dequeues, because there's a total order among the dequeues, either exactly one of these is true: $X \rightarrow_{H'} Y$ or $Y \rightarrow_{H'} X$. Then due to (1), either $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$. Notice that we cannot obtain $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ from (2), (3), or (4).

Therefore, exactly one of $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ is true. (*)

If X is dequeue and Y is enqueue, in this case (3) cannot help us obtain either $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$, so we can disregard it.

- If $X \rightarrow_{H'} Y$, then due to (1), $X \Rightarrow_{H'} Y$. By definition, X precedes Y , so (4) cannot apply. Applying Theorem 3.1.14, either
 - X isn't matched, (2) cannot apply. Therefore, $Y \nRightarrow_{H'} X$.
 - X matches e' and $e' \neq Y$. Therefore, X does not match Y , or (2) cannot apply. Therefore, $Y \nRightarrow_{H'} X$.

Therefore, in this case, $X \Rightarrow_{H'} Y$ and $Y \nRightarrow_{H'} X$.

- If $Y \rightarrow_{H'} X$, then due to (1), $Y \Rightarrow_{H'} X$. By definition, Y precedes X , so (4) cannot apply. Even if (2) applies, it can only help us obtain $Y \Rightarrow_{H'} X$. Therefore, in this case, $Y \Rightarrow_{H'} X$ and $X \nRightarrow_{H'} Y$.
- If X overlaps with Y :
 - If X matches Y , then due to (2), $Y \Rightarrow_{H'} X$. Because X matches Y , (4) cannot apply. Therefore, in this case $Y \Rightarrow_{H'} X$ but $X \nRightarrow_{H'} Y$.
 - If X does not match Y , then due to (4), $X \Rightarrow_{H'} Y$. Because X doesn't match Y , (2) cannot apply. Therefore, in this case $X \Rightarrow_{H'} Y$ but $Y \nRightarrow_{H'} X$.

Therefore, exactly one of $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ is true. ($**$)

If X is enqueue and Y is enqueue, in this case (2) and (4) are irrelevant:

- If $X \rightarrow_{H'} Y$, then due to (1), $X \Rightarrow_{H'} Y$. By definition, X precedes Y . Applying Theorem 3.1.15,
 - Both X and Y aren't matched, then (3) cannot apply. Therefore, in this case, $Y \nRightarrow_{H'} X$.
 - X is matched but Y is not matched, then (3) cannot apply. Therefore, in this case, $Y \nRightarrow_{H'} X$.
 - X matches d_x and Y matches d_y such that d_x precedes d_y , then (3) applies and we obtain $X \Rightarrow_{H'} Y$.

Therefore, in this case, $X \Rightarrow_{H'} Y$ but $Y \nRightarrow_{H'} X$.

- If $Y \rightarrow_{H'} X$, this case is symmetric to the first case. We obtain $Y \Rightarrow_{H'} X$ but $X \nRightarrow_{H'} Y$.
- If X overlaps with Y , because in H' , all enqueues are matched, then, X matches d_x and d_y . Because d_x either precedes or succeeds d_y , Applying (3), we obtain either $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ and there's no way to obtain the other.

Therefore, exactly one of $X \Rightarrow_{H'} Y$ or $Y \Rightarrow_{H'} X$ is true. ($***$)

From ($*$), ($**$), ($***$), we have proved that $\Rightarrow_{H'}$ is a strict total ordering that is consistent with $\rightarrow_{H'}$. In other words, we can order method calls in H' in a sequential manner. We will prove that this sequential order is consistent with FIFO semantics:

- An item can only be dequeued once: This is trivial as a dequeue can only match one enqueue.
- Items are dequeued in the order they are enqueued: Suppose there are two enqueues e_1, e_2 such that $e_1 \Rightarrow_{H'} e_2$ and suppose they match d_1 and d_2 . Then we have obtained $e_1 \Rightarrow_{H'} e_2$ either because:
 - (3) applies, in this case $d_1 \Rightarrow_{H'} d_2$ is a condition for it to apply.
 - (1) applies, then e_1 precedes e_2 , by Theorem 3.1.15, d_1 must precede d_2 , thus $d_1 \Rightarrow_{H'} d_2$.

Therefore, if $e_1 \Rightarrow_{H'} e_2$ then $d_1 \Rightarrow_{H'} d_2$.

- An item can only be dequeued after it's enqueued: Suppose there is an enqueue e matched by d . By (2), obviously $e \Rightarrow_{H'} d$.
- If the queue is empty, dequeues return nothing. Suppose a dequeue d such that any $e \Rightarrow_{H'} d$ is all matched by some d' and $d' \Rightarrow_{H'} d$, we will prove that d is unmatched. By Lemma 3.1.13, d can only match an enqueue e_0 that precedes or overlaps with d .
 - If e_0 precedes d , by our assumption, it's already matched by another dequeue.
 - If e_0 overlaps with d , by our assumption, $d \Rightarrow_{H'} e_0$ because if $e_0 \Rightarrow_{H'} d$, e_0 is already matched by another d' . Then, we can only obtain this because (4) applies, but then d does not match e_0 .

Therefore, d is unmatched.

In conclusion, $\Rightarrow_{H'}$ is a way we can order method calls in H' sequentially that conforms to FIFO semantics. Therefore, we can also order method calls in H sequentially that conforms to FIFO semantics as we only append dequeues sequentially to the end of H to obtain H' .

We have proved the theorem. \square

3.2. ABA problem

ABA problem is unlikely to occur, because any CAS usage increases the value of a monotonic tag.

3.3. Memory safety

Memory allocation and deallocation are performed only in the SPSC data structure. Since [1] has proved that it's memory-safe, the modified algorithm is also memory-safe.

3.4. Wait-freedom

All the procedures enqueue, dequeue, refresh, refreshTimestamp, refreshLeaf, propagate are wait-free because there's no possibility of infinite loops and a process waiting for another process.

References

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